The Mean Curvature of Transverse Kähler Foliations

Seoung Dal Jung and Ken Richardson

Received: November 19, 2018
Revised: February 12, 2019

Communicated by Christian Bär

Abstract. We study properties of the mean curvature one-form and its holomorphic and antiholomorphic cousins on a transverse Kähler foliation. If the mean curvature of the foliation is automorphic, then there are some restrictions on basic cohomology similar to that on Kähler manifolds, such as the requirement that the odd basic Betti numbers must be even. However, the full Hodge diamond structure does not apply to basic Dolbeault cohomology unless the foliation is taut.

2010 Mathematics Subject Classification: 53C12, 53C21, 53C55, 57R30, 58J50
Keywords and Phrases: Riemannian foliation, transverse Kähler foliation, Lefschetz decomposition, mean curvature

1 Introduction

Let $\mathcal{F}$ be a foliation on a closed, smooth manifold M. A Riemannian foliation is a foliation such that the normal bundle $Q = TM/T\mathcal{F}$ is endowed with a holonomy-invariant metric g_Q. This metric can always be extended to a metric g on M that is called bundle-like, characterized by the property that the leaves of $\mathcal{F}$ are locally equidistant. The basic forms of $(M, \mathcal{F})$ are locally forms on the leaf space; that is, they are forms ϕ satisfying $X \lrcorner \phi = X \lrcorner d\phi = 0$ for any vector X tangent to the leaves, where $X \lrcorner$ denotes the interior product with X. The set of basic forms yields a differential complex and is used to define

This paper was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (NRF-2018R1A2B2002046). This work was also supported by a grant from the Simons Foundation (Grant Number 245818 to Ken Richardson).

basic de Rham cohomology groups $H_B^*(\mathcal{F})$. For Riemannian foliations, these groups have finite rank, and their ranks are topological invariants ([13]). The basic Laplacian Δ_B is a version of the Laplace operator that preserves the basic forms. Many researchers have studied basic forms and the basic Laplacian on Riemannian foliations. It is well-known ([2], [12], [27], [34]) that on a closed oriented manifold M with a transversely oriented Riemannian foliation $\mathcal{F}$, $H_B^r(\mathcal{F}) \cong \mathcal{H}_B^r(\mathcal{F}) = \ker\Delta_B^r$.

The basic component κ_B of the mean curvature one-form of the foliation is always closed, and its cohomology class $\xi = [\kappa_B] \in H_B^1(\mathcal{F})$ is invariant of the choice of bundle-like metric; this was proved by in [2], and ξ is called the Álvarez class. Poincaré duality holds for the basic cohomology of a Riemannian foliation $(M, \mathcal{F}, g_Q)$ if and only if the Álvarez class is trivial, if and only if $(M, \mathcal{F})$ is taut, meaning that there exists a metric for which the leaves of the foliation are (immersed) minimal submanifolds.

In this paper, we consider foliations that admit a transverse, holonomy-invariant complex structure, and in particular we consider holonomy-invariant Hermitian metrics on Q that may or may not be Kähler. The question is whether the standard Kähler manifold structures on Dolbeault cohomology such as the Hard Lefschetz Theorem, the dd_c-Lemma, and formality apply to the basic cohomology of transverse Kähler foliations. The basic Dolbeault cohomologies $H_B^{r,s}(\mathcal{F})$ and $H_{\partial_B\overline{\partial}_B}^{r,s}(\mathcal{F})$ can be defined as usual using only the transverse holomorphic structure. The Hodge diamond structure does sometimes occur for basic Dolbeault cohomology for Kähler foliations, but it turns out that two properties of the mean curvature are crucial:

1. Does the class $\eta = \left[\partial_B\kappa_B^{0,1}\right] \in H_{\partial_B\overline{\partial}_B}^{1,1}(\mathcal{F})$ vanish? This class automatically vanishes if the Álvarez class ξ vanishes, but it is possible for the $\xi \neq 0$ while $\eta = 0$ (see Example 9.2). The class is nontrivial if and only if the $\partial_B\overline{\partial}_B$-Lemma fails to hold when applied directly to $\partial_B\kappa_B^{0,1}$. The class η is an invariant of the transverse complex structure (Theorem 4.3).

2. Is the mean curvature $H = \kappa_B^{\#}$ automorphic (that is, does its flow preserve the transverse complex structure)?

The condition (2) is equivalent to $(\kappa_B^{0,1})^{\#} = H^{1,0}$ being a transverse holomorphic vector field — that locally it has the form

$$H^{1,0} = \sum_j H_j^{1,0}(z)\partial_{z_j}$$

in the transverse holomorphic coordinates, with each $H_j^{1,0}(z)$ being a transverse holomorphic function. Condition (1) is similar; only in that case each $H_j^{1,0}(z)$ is required to be a transverse antiholomorphic function.

This paper is organized as follows. In Section 2, we review the known properties of the mean curvature and basic Laplacian for Riemannian foliations. In Section

3, we investigate transverse Hermitian structures on foliations with bundle-like metrics. In Proposition 3.2 and Theorem 3.3, we show that the holomorphic and antiholomorphic basic components $\kappa_B^{1,0}$ and $\kappa_B^{0,1}$ of mean curvature are ∂_B-closed, $\overline{\partial}_B$-closed and represent basic Dolbeault cohomology classes in $H^{1,0}_{\partial_B}(\mathcal{F})$ and $H^{0,1}_{\overline{\partial}_B}(\mathcal{F})$, respectively, that are invariant under the choices of bundle-like metric and transverse metric that is compatible with a given transverse holomorphic structure. In fact, in Proposition 3.6, we show that the metrics can be chosen so that κ, $\kappa^{1,0}$, and $\kappa^{0,1}$ are basic forms. When the foliation is transversally Kähler, the metrics can be chosen so that κ, $\kappa^{1,0}$, and $\kappa^{0,1}$ are simultaneously basic, Δ_B-harmonic, $\Box_B$-harmonic, and $\overline{\Box}_B$-harmonic, respectively (see Proposition 3.10; the $\Box_B$ and $\overline{\Box}_B$ are the ∂_B and $\overline{\partial}_B$ Laplacians, respectively).

In Section 4, we show that for transverse Hermitian foliations, the form $\partial_B\kappa_B^{0,1}$ is $\partial_B\overline{\partial}_B$-closed and generates a class $\eta = \left[\partial_B\kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B\overline{\partial}_B}(\mathcal{F})$ that is invariant of the choices of bundle-like metric and compatible transverse metric (Theorem 4.3). If the foliation is not taut and is transversally Hermitian, Proposition 4.8 implies that if $\partial_B\kappa_B^{0,1} = 0$, then $[\kappa_B]$ and $[J\kappa_B]$ are linearly independent cohomology classes in $H^1_B(\mathcal{F})$, so that $\dim H^1_B(\mathcal{F}) \geq 2$ in this case. In Proposition 4.10, we derive formulas for $\Box_B$ and $\overline{\Box}_B$ that are valid for all transverse Hermitian foliations; for example,

$$\overline{\Box}_B = \Delta^Q_{\bar\partial} + \bar\partial_B \circ H^{0,1}\lrcorner + H^{0,1}\lrcorner \circ \bar\partial_B,$$

where $H^{0,1} = \left(\kappa_B^{1,0}\right)^{\#}$ and where $\Delta^Q_{\bar\partial}$ is the Dolbeault Laplacian on the local quotients of foliation charts. In Corollary 4.10, we show for example that if the foliation is transversely Kähler, then $\partial_B\kappa_B^{0,1} = 0$ if and only if $\overline{\Box}_B = \Delta^Q_{\bar\partial} + \nabla_{H^{0,1}}$.

In Section 5, we investigate the properties of the operator L, exterior product with the transverse Kähler form. As a result, we show in Lemma 5.9 that for transverse Kähler foliations,

$$\Delta_B = \Box_B + \overline{\Box}_B + \mathrm{Re}\left(\partial_B H^{0,1}\lrcorner + H^{0,1}\lrcorner\, \partial_B\right).$$

As a consequence (Corollary 5.10),

$$\dim\left(\mathcal{H}^j_B(\mathcal{F}) \cap \Omega^{r,s}_B(\mathcal{F})\right) \leq \dim H^{r,s}_B(\mathcal{F})$$

for all transverse Kähler foliations. Then it turns out that the Hard Lefschetz Theorem for basic cohomology holds if and only if the class $\eta = 0$ (Theorem 5.11 and Corollary 5.12). Further, for transverse Kähler foliations, $\eta = 0$ if and only if $\xi = 0$, which is false in general, and this condition in turn implies that the metric can be chosen so that $\kappa = \kappa_B = 0$.

In Section 6, we investigate the condition that the mean curvature κ_B is automorphic, meaning its flow preserves the transverse holomorphic structure.

The basic Laplacian satisfies $\Delta_B = \square_B + \overline{\square}_B$ if and only if it preserves the (r,s)-type of form if and only if the mean curvature is automorphic (Theorem 6.5 and Corollary 6.6).

In Section 7, we show that the dd_c-Lemma of Kähler manifolds works only for taut Kähler foliations (Lemma 7.3). In Section 8, Theorem 8.1 shows on transverse Kähler foliations that if the mean curvature is automorphic, then symmetry of a version of the Hodge diamond follows, and then we also have

$$\dim H_B^j(\mathcal{F}) = \sum_{r+s=j;\ r,s\geq 0} \dim \mathcal{H}_{\Delta_B}^{r,s}(\mathcal{F}).$$

However, the full power of the Hodge diamond with restrictions to basic Dolbeault cohomology follows from the Hard Lefschetz theorem, which applies only if the foliation is taut.

In Section 9, we provide examples of nontaut Kähler foliations and calculate their cohomologies. Also in this section, we show that for a nontaut foliation, it is possible for one transverse Hermitian structure to be Kähler with $\eta \neq 0$ and mean curvature not automorphic and for another transverse Hermitian structure to be nonKähler with $\eta = 0$ and mean curvature automorphic. These examples manifest another interesting property of nontaut transverse Kähler foliations; the Kähler form ω always yields a transverse volume form ω^n that is exact, and the Kähler form itself may be exact.

Foliations that admit a transverse Kähler structure have been studied by many researchers, but primarily in the case when the foliation is taut ($\kappa = 0$ for some metric). For example, Sasaki manifolds are not Kähler but admit transverse Kähler structures on the characteristic foliation, which is homologically oriented. Since the mean curvature vanishes, many Kähler manifold facts apply to the basic Dolbeault cohomology (see [5, Section 2], [4, Proposition 7.2.3], [39]). The authors in [6] prove the hard Lefschetz theorem for compact Sasaki manifolds, which again is a simple case of the results of this paper with $\kappa = 0$. The cosymplectic manifold case is treated in [8]. A. El Kacimi proved in [11, Section 3.4] that the standard facts about Kähler manifolds and their cohomology carry over to basic cohomology in the homologically orientable (taut) case. Also, L. A. Cordero and R. A. Wolak [9] studied basic cohomology on taut transverse Kähler foliations by using the differential operator Δ_T, which is different from Δ_B ($\mathcal{F}$ is minimal if and only if $\Delta_T = \Delta_B$). We note other recent work on transverse Kähler foliations in [22], [20], [16], [28], [24].

2 Properties of the mean curvature for Riemannian foliations

Let $(M, g_Q, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian foliation of codimension q with compact leaf closures. Here, g_Q is a holonomy invariant metric on the normal bundle $Q = TM/T\mathcal{F}$, meaning that $\mathcal{L}_X g_Q = 0$ for all $X \in T\mathcal{F}$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Next, let g_M be a bundle-like metric on M adapted to g_Q. This means that if $T\mathcal{F}^\perp$ is the g_M-orthogonal complement to $T\mathcal{F}$ in TM and $\sigma : Q \to T\mathcal{F}^\perp$ is the canonical bundle isomorphism, then $g_Q = \sigma^* \left(g_M|_{T\mathcal{F}^\perp} \right)$. We do not assume that M is compact, but we assume it is complete with finite volume.

In this section, we review some known results for this Riemannian foliation setting. Let ∇ be the transverse Levi-Civita connection on the normal bundle Q, which is torsion-free and metric with respect to g_Q [37]. Let R^Q and Ric^Q be the curvature tensor and the transversal Ricci operator of $\mathcal{F}$ with respect to ∇, respectively. The mean curvature vector τ of $\mathcal{F}$ is given by

$$\tau = \sum_{i=1}^{p} \pi(\nabla^M_{f_i} f_i), \tag{1}$$

where $\{f_i\}_{i=1,\cdots,p}$ is a local orthonormal basis of $T\mathcal{F}$ and $\pi : TM \to Q$ is natural projection. Then the *mean curvature form* κ is defined by

$$\kappa(X) = g_Q(\tau, \pi(X)) \tag{2}$$

for any tangent vector $X \in \Gamma(TM)$. An r-form ϕ is basic if and only if $X \lrcorner\, \phi = 0$ and $\mathcal{L}_X \phi = 0$ for any $X \in \Gamma(T\mathcal{F})$, where $X \lrcorner$ denotes the interior product. Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r -forms*. The foliation $\mathcal{F}$ is said to be *minimal* if $\kappa = 0$. We note that Rummler's formula (from [35]) for the mean curvature is

$$d\chi_{\mathcal{F}} = -\kappa \wedge \chi_{\mathcal{F}} + \varphi_0 \text{ with } \chi_{\mathcal{F}} \wedge *\varphi_0 = 0, \tag{3}$$

where $\chi_{\mathcal{F}} := f_1^\flat \wedge ... \wedge f_p^\flat$ is the *characteristic form*, the leafwise volume form, and $*$ is the Hodge star operator associated to g_M; we assumed M is oriented to make the property of φ_0 easier to state.

The exterior derivative d maps $\Omega^r_B(\mathcal{F})$ to $\Omega^{r+1}_B(\mathcal{F})$, and the resulting cohomology groups are called the basic cohomology groups: for $r \geq 0$,

$$H^r_B(\mathcal{F}) = \frac{\ker \left(d|_{\Omega^r_B(\mathcal{F})} \right)}{\mathrm{Im} \left(d|_{\Omega^{r-1}_B(\mathcal{F})} \right)}.$$

These groups are smooth invariants of the foliation and do not depend on the bundle-like metric and also do not even depend on the smooth foliation structure (see [13]).

The metric g_M induces a natural metric on $\Lambda^* T^* M$ and L^2 metric on $L^2\Omega^*(M)$. Let $L^2\Omega^*_B(\mathcal{F})$ denote the closure of $\Omega^*_{B,0}(\mathcal{F})$, the space of compactly supported basic forms, in $L^2\Omega^*(M)$.

Proposition 2.1. *(Proved in [34] for the closed manifold case) Let $(M, g_Q, \mathcal{F})$ be a Riemannian foliation with compact leaf closures and bundle-like metric. The orthogonal projection $P : L^2\Omega^*(M) \to L^2\Omega^*_B(\mathcal{F})$ maps smooth forms to smooth basic forms. For all $\alpha \in L^2\Omega^*(M)$, $P(\alpha)(x)$ is computed by an integral over the leaf closure containing x and only depends on the values of α on that leaf closure.*

Proof. The proof in [34] applies in this slightly more general case, where it is not assumed that M is compact. □

Now we recall the transversal star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ given by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}),$$

where $*$ is the Hodge star operator associated to g_M; this is actually well-defined as long as $(M, \mathcal{F})$ is transversely oriented. Trivially, $\bar{*}^2\phi = (-1)^{r(q-r)}\phi$ for any $\phi \in \Omega_B^r(\mathcal{F})$. Let ν be the transversal volume form; that is, $*\nu = \chi_{\mathcal{F}}$ as long as M is oriented. Then the pointwise inner product $\langle \cdot, \cdot \rangle$ on $\Lambda^r Q^*$ is defined by $\langle \phi, \psi \rangle \nu = \phi \wedge \bar{*}\psi$. The global inner product on $L^2\Omega_B^r(\mathcal{F})$ is

$$\ll \phi, \psi \gg = \int_M \langle \phi, \psi \rangle \mu_M = \int_M \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}},$$

where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form with respect to g_M.
In what follows, let $\kappa_B = P\kappa$. Also, let

$$d_B = d|_{\Omega_B^*(\mathcal{F})}, \quad d_T = d_B - \epsilon(\kappa_B), \tag{4}$$

where $\epsilon(\alpha)\psi = \alpha \wedge \psi$ for any $\alpha \in \Omega_B^1(\mathcal{F})$. The interior product $v \lrcorner$ of $v \in Q \cong T\mathcal{F}^\perp$ on differential forms satisfies

$$v \lrcorner = \epsilon \left(v^\flat \right)^*,$$

where $*$ denotes the pointwise adjoint.

PROPOSITION 2.2. *(In [2], [34] for the compact case; [24, Prop. 2.1]) Let $(M, g_Q, \mathcal{F})$ be a Riemannian foliation with compact leaf closures and bundle-like metric. The formal adjoint operators δ_B and δ_T of d_B and d_T with respect to $\ll \cdot, \cdot \gg$ on basic forms are given by*

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*} d_T \bar{*} \phi = \left(\delta_T + \kappa_B^\sharp \lrcorner \right) \phi, \quad \delta_T \phi = (-1)^{q(r+1)+1} \bar{*} d_B \bar{*} \phi,$$

on basic r-forms ϕ.

LEMMA 2.3. *The transversal divergence satisfies*

$$\delta_T = -\sum_{a=1}^{q} (E_a \lrcorner) \nabla_{E_a},$$

where the sum is over a local orthonormal frame $\{E_a\}$ of Q.

Proof. It follows from the fact that δ_T is locally the pullback of the ordinary divergence on the local quotients of foliation charts. □

PROPOSITION 2.4. *(Proved in [2] for the closed manifold case) Let $(M, g_Q, \mathcal{F})$ be a Riemannian foliation with compact leaf closures and bundle-like metric. The form $d\kappa_B = 0$, and κ_B determines a class in $H^1_B(\mathcal{F})$ that is independent of the choice of g_M or of g_Q.*

Proof. The proof in [2] is primarily a calculation confined to a neighborhood of a leaf closure, so that it applies in this slightly more general case. For the sake of exposition, we show the proof that κ_B is closed: We have

$$\delta_B = \delta_T + \kappa_B^\sharp \lrcorner$$

where δ_T is the divergence on the local quotient manifolds in the foliation charts. In particular, δ_T only depends on g_Q. Thus,

$$\delta_T^2 = 0,$$

and also

$$d_B^2 = 0.$$

Taking adjoints with respect to basic forms, from the three equations above we have

$$d_T = \delta_T^* = d_B - \epsilon(\kappa_B)$$
$$d_T^2 = 0, \ \ (\delta_B)^2 = 0.$$

Then

$$\begin{aligned} d_T(1) &= (d_B - \epsilon(\kappa_B))(1) \\ &= -\kappa_B \end{aligned}$$

and

$$\begin{aligned} d_B\kappa_B &= (d_T + \epsilon(\kappa_B))\kappa_B \\ &= d_T\kappa_B \\ &= d_T(-d_T 1) = 0. \end{aligned}$$

□

PROPOSITION 2.5. *([10])Let $(M, g_Q, \mathcal{F})$ be a Riemannian foliation on a closed manifold. Then there exists a bundle-like metric compatible with g_Q such that κ is a basic form; that is, $\kappa = \kappa_B$.*

PROPOSITION 2.6. *([29] and [30]) Let $(M, g_Q, \mathcal{F})$ be a Riemannian foliation on a closed manifold. Then there exists a bundle-like metric compatible with g_Q such that κ is basic harmonic; that is, $\kappa = \kappa_B$ and $\delta_B\kappa = 0$.*

COROLLARY 2.7. *Let $(M, g_Q, \mathcal{F})$ be a Riemannian foliation on a closed manifold, and let α be any element of the class $[\kappa_B] \in H^1_B(\mathcal{F})$. Then there exists a bundle-like metric compatible with g_Q such that $\kappa = \alpha$. The representative α corresponding to a bundle-like metric such that $\alpha = \kappa$ is basic harmonic is uniquely determined. For that metric, κ is the element of $[\kappa_B]$ of minimum L^2-norm.*

Proof. Given any bundle-like metric with basic mean curvature κ as in Proposition 2.5, any element of $[\kappa] = [\kappa_B]$ is of the form $\kappa + df$ for some basic function f. If $p = \dim \mathcal{F}$, multiplying the leafwise metric by $e^{-(2/p)f}$ yields a new characteristic form $\chi'_{\mathcal{F}} = e^{-f}\chi_{\mathcal{F}}$ so that the new mean curvature form from (3) satisfies

$$\begin{aligned} -\kappa' \wedge \chi'_{\mathcal{F}} + \varphi'_0 &= d\left(\chi'_{\mathcal{F}}\right) \\ &= -df \wedge \chi'_{\mathcal{F}} + e^{-f} d\chi_{\mathcal{F}} \\ &= -\left(\kappa + df\right) \wedge \chi'_{\mathcal{F}} + \varphi'_0. \end{aligned}$$

The second part comes from the proof in [29], where the volume form $\nu \wedge \chi'_{\mathcal{F}}$ is uniquely determined (up to rescaling, which does not change κ). The third part comes from the fact that if $\delta_B \kappa = d\kappa = 0$,

$$\begin{aligned} \ll \kappa + df, \kappa + df \gg &= \ll \kappa, \kappa \gg + 2 \ll df, \kappa \gg + \ll df, df \gg \\ &= \ll \kappa, \kappa \gg + 2 \ll f, \delta_B \kappa \gg + \ll df, df \gg \\ &= \ll \kappa, \kappa \gg + \ll df, df \gg . \end{aligned}$$

□

The *basic Laplacian* Δ_B is the operator on basic forms defined as

$$\Delta_B = \delta_B d_B + d_B \delta_B.$$

We define the operator Δ_T on basic forms as the corresponding Laplacian on the local quotient manifolds. Specifically,

$$\Delta_T = \delta_T d_B + d_B \delta_T.$$

The operator Δ_T is not essentially self-adjoint on the space of basic forms, but the operator Δ_B is.

LEMMA 2.8. *The basic Laplacian is the restriction of the operator*

$$\Delta_B = \Delta_T + \mathcal{L}_{\kappa_B^\#}.$$

Proof. From Proposition 2.2,

$$\begin{aligned} \Delta_B &= \left(\delta_T + \kappa_B^\# \lrcorner\right) d_B + d_B \left(\delta_T + \kappa_B^\# \lrcorner\right) \\ &= \Delta_T + \left(\kappa_B^\# \lrcorner\right) d_B + d_B \left(\kappa_B^\# \lrcorner\right). \end{aligned}$$

The result follows from Cartan's formula for the Lie derivative. □

3 Properties of the mean curvature for transverse Hermitian foliations

We now suppose that $(M, \mathcal{F})$ is a foliation of codimension $2n$ and is endowed with a holonomy-invariant transverse complex structure $J : Q \to Q$ and a holonomy-invariant Hermitian metric on the complexified normal bundle; we call such a foliation a *transverse Hermitian foliation*. So in particular the foliation is Riemannian. When it is convenient, we will also refer to the bundle map $J' : TM \to TM$ defined by $J'(v) = J(\pi(v))$ and abuse notation by denoting $J = J'$. In what follows, we use notation similar to [24].
For $Q^C = Q \otimes \mathbb{C}$, we let

$$Q^{1,0} = \{Z \in Q^C | JZ = iZ\}, \quad Q^{0,1} = \{Z \in Q^C | JZ = -iZ\}.$$

Elements of $Q^{1,0}$ and $Q^{0,1}$ are called *complex normal vector fields of type* $(1,0)$ and $(0,1)$, respectively. We have $Q^C = Q^{1,0} \oplus Q^{0,1}$ and

$$Q^{1,0} = \{X - iJX |\ X \in Q\}, \quad Q^{0,1} = \{X + iJX |\ X \in Q\}.$$

Let Q_C^* be the real dual of Q^C; at each $x \in M$, $(Q_C^*)_x$ is set of $\mathbb{C}$-linear maps from Q_x^C to $\mathbb{C}$. Letting $\Lambda_C Q^*$ denote ΛQ_C^*, we decompose $\Lambda_C^1 Q^* = Q_{1,0} \oplus Q_{0,1}$, where the sub-bundles $Q_{1,0}$ and $Q_{0,1}$ are given by

$$\begin{aligned} Q_{1,0} &= \{\xi \in \Lambda_C^1 Q^* |\ \xi(Z) = 0,\ \forall Z \in Q^{0,1}\}, \\ Q_{0,1} &= \{\xi \in \Lambda_C^1 Q^* |\ \xi(Z) = 0,\ \forall Z \in Q^{1,0}\}. \end{aligned}$$

Also

$$Q_{1,0} = \{\theta + iJ\theta |\ \theta \in Q^*\}, \quad Q_{0,1} = \{\theta - iJ\theta |\ \theta \in Q^*\},$$

where $(J\theta)(X) := -\theta(JX)$ for any $X \in Q$ and is extended linearly. Let $\Omega_B^{r,s}(\mathcal{F})$ be the set of the basic forms of type (r,s), the smooth sections of $\Lambda_C^{r,s} Q^*$, which is the subspace of $\Lambda_C Q^*$ spanned by $\xi \wedge \eta$, where $\xi \in \Lambda^r Q_{1,0}$ and $\eta \in \Lambda^s Q_{0,1}$. We choose $\{E_a, JE_a\}_{a=1,\cdots,n}$ so that it is a local orthonormal basic frame; we call it a *J-basic frame*) of Q. Let $\{\theta^a, J\theta^a\}_{a=1,\cdots,n}$ be the local dual frame of Q^*. We set $V_a = \frac{1}{\sqrt{2}}(E_a - iJE_a)$ and $\omega^a = \frac{1}{\sqrt{2}}(\theta^a + iJ\theta^a)$, so that

$$\omega^a(V_b) = \bar{\omega}^a(\bar{V}_b) = \delta_{ab},\ \omega^a(\bar{V}_b) = \bar{\omega}^a(V_b) = 0.$$

The frame $\{V_a\}$ is a local orthonormal basic frame field of $Q^{1,0}$, a *normal frame field of type* $(1,0)$, and $\{\omega^a\}$ is the corresponding dual coframe field.
The following 2-form ω is nondegenerate. Letting $\theta^{n+a} = J\theta^a$ for $a = 1, ..., n$,

$$\begin{aligned} \omega &= -\frac{1}{2}\sum_{a=1}^{2n} \theta^a \wedge J\theta^a \\ &= -\frac{1}{2}\left(\sum_{a=1}^{n} \theta^a \wedge J\theta^a + \sum_{a=1}^{n} J\theta^a \wedge J^2\theta^a\right) \\ &= -\sum_{a=1}^{n} \theta^a \wedge J\theta^a. \end{aligned}$$

In the event that ω is closed, this is the Kähler form, and the foliation is *transversely Kähler*.
We define $\partial_B|_{\Omega_B^{r,s}(\mathcal{F})} = \Pi^{r+1,s}\ d_B|_{\Omega_B^{r,s}(\mathcal{F})}$, where $\Pi^{r,s} : \Omega_B^{r+s}(\mathcal{F}) \to \Omega_B^{r,s}(\mathcal{F})$ is the projection, and similarly $\overline{\partial}_B|_{\Omega_B^{r,s}(\mathcal{F})} = \Pi^{r,s+1}\ d_B|_{\Omega_B^{r,s}(\mathcal{F})}$. Similarly, we define ∂_T and $\bar{\partial}_T$, using d_T from (4). We now write $\kappa_B = \kappa_B^{1,0} + \kappa_B^{0,1}$, with

$$\kappa_B^{1,0} = \frac{1}{2}(\kappa_B + iJ\kappa_B) \in \Omega_B^{1,0}(\mathcal{F}),\ \kappa_B^{0,1} = \overline{\kappa_B^{1,0}} \in \Omega_B^{0,1}(\mathcal{F}).$$

Let $H = \kappa_B^{\#}$ be the basic mean curvature vector field, and let

$$H^{1,0} \ :\ = \left(\kappa_B^{1,0}\right)^* = \left(\overline{\kappa_B^{1,0}}\right)^{\#} = \frac{1}{2}(\kappa_B^{\#} - iJ\kappa_B^{\#}) \in \Gamma\left(Q^{1,0}\right), \tag{5}$$

$$H^{0,1} \ :\ = \overline{H^{1,0}} \in \Gamma\left(Q^{1,0}\right). \tag{6}$$

In what follows, we extend the definitions of exterior product and interior product linearly to complex vectors and differential forms. Observe that $V\lrcorner$ is by definition the adjoint of $\epsilon\left(V^\flat\right)$ on real vector fields, but on complex vector fields the following holds. If v, w are real tangent vectors,

$$\begin{aligned}(v + iw)\lrcorner &= (v\lrcorner) + i\,(w\lrcorner) = \left(\epsilon\left(v^\flat\right) - i\epsilon\left(w^\flat\right)\right)^* \\ &= \left(\epsilon\left(v^\flat - iw^\flat\right)\right)^*,\end{aligned}$$

so that for complex vectors V,

$$(V\lrcorner)^* = \epsilon\left(\overline{V}^\flat\right),\ \ \left(\epsilon\left(V^\flat\right)\right)^* = \overline{V}\lrcorner.$$

Now, let $\langle\cdot,\cdot\rangle$ be a Hermitian inner product on $\Lambda_C^{r,s}(\mathcal{F})$ induced by the transverse Hermitian structure, and let $\bar{*} : \Lambda_C^{r,s}(\mathcal{F}) \to \Lambda_C^{n-s,n-r}(\mathcal{F})$ be the star operator defined by

$$\phi \wedge \bar{*}\bar{\psi} = \langle\phi,\psi\rangle\nu,$$

where ν is the transverse volume form. Then for any $\psi \in \Lambda_C^{r,s}(\mathcal{F})$,

$$\overline{\bar{*}\psi} = \bar{*}\bar{\psi}, \quad \bar{*}^2\psi = (-1)^{r+s}\psi.$$

Then for complex vectors V it follows that

$$\bar{*}\epsilon\left(V^\flat\right)\bar{*} = \overline{V}\lrcorner,\ \bar{*}\,(V\lrcorner)\,\bar{*} = -\epsilon\left(\overline{V}^\flat\right).$$

Now, by applying the projections to $\Pi^{*,*}$ to (4), we have

$$\partial_T = \partial_B - \epsilon(\kappa_B^{1,0}), \quad \bar{\partial}_T = \bar{\partial}_B - \epsilon(\kappa_B^{0,1}).$$

Then, since $(M, \mathcal{F}, J)$ is transversely holomorphic,

$$d_B = \partial_B + \bar{\partial}_B,\ d_T = \partial_T + \bar{\partial}_T.$$

Taking L^2 adjoints with respect to basic forms of the formulas above, we have

$$\delta_B = \partial_B^* + \bar{\partial}_B^*,\ \delta_T = \partial_T^* + \bar{\partial}_T^*.$$

Proposition 3.1. *[24, Prop. 3.6] Let $(M, g_Q, \mathcal{F})$ be a transverse Hermitian foliation with compact leaf closures and bundle-like metric. The formal adjoint operators ∂_B^* and ∂_T^* of ∂_B and ∂_T with respect to $\ll \cdot, \cdot \gg$ on basic forms are given by*

$$\partial_B^* \phi = -\bar{*} \partial_T \bar{*} \phi = \left(\partial_T^* + H^{1,0} \lrcorner \right) \phi, \quad \partial_T^* \phi = -\bar{*} \partial_B \bar{*} \phi,$$

on basic r-forms ϕ.

Again, we note that ∂_T^* is the holomorphic divergence on the local quotient manifolds in the foliation charts, and it only depends on the transverse metric and holomorphic structure.
Thus,

$$\partial_T^{*2} = 0.$$

Also, since ∂_B is the same as the holomorphic differential on the local quotient manifold in the foliation charts,

$$\partial_B^2 = 0.$$

Taking adjoints with respect to basic forms, from the three equations above we have

$$\partial_T := (\partial_T^*)^* = \partial_B - \epsilon \left(\kappa_B^{1,0} \right)$$

$$\partial_T^2 = 0, \ \ (\partial_B^*)^2 = 0.$$

Then

$$\partial_T (1) = \left(\partial_B - \epsilon(\kappa_B^{1,0}) \right) (1) = -\kappa_B^{1,0} \tag{7}$$

and

$$\partial_B \kappa_B^{1,0} = \left(\partial_T + \epsilon(\kappa_B^{1,0}) \right) \kappa_B^{1,0} = \partial_T \kappa_B^{1,0} = \partial_T (-\partial_T 1) = 0.$$

It also follows that

$$\partial_T \kappa_B^{1,0} = 0.$$

Similarly,

$$\bar{\partial}_B \kappa_B^{0,1} = \bar{\partial}_T \kappa_B^{0,1} = 0.$$

Since $d\kappa_B = 0$, we have

$$0 = \left(\partial_B + \bar{\partial}_B \right) \left(\kappa_B^{1,0} + \kappa_B^{0,1} \right) = \partial_B \kappa_B^{0,1} + \bar{\partial}_B \kappa_B^{1,0}.$$

So

$$\mathrm{Re} \left(\partial_B \kappa_B^{0,1} \right) = 0.$$

We summarize these results in the following Proposition.

PROPOSITION 3.2. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with a holonomy-invariant transverse complex structure and transverse Hermitian metric. Given a bundle-like metric for $(M, \mathcal{F})$ compatible with the Hermitian metric, let $\kappa_B = \kappa_B^{1,0} + \kappa_B^{0,1}$ be the basic component of the mean curvature one-form, with $\kappa_B^{1,0} = \frac{1}{2}(\kappa_B + iJ\kappa_B) \in \Omega_B^{1,0}$, $\kappa_B^{0,1} = \overline{\kappa_B^{1,0}}$. Then*

$$\begin{aligned} \partial_B \kappa_B^{1,0} &= \bar{\partial}_B \kappa_B^{0,1} = 0; \ \mathrm{Re}\left(\partial_B \kappa_B^{0,1}\right) = 0; \\ \partial_T \kappa_B^{1,0} &= \bar{\partial}_T \kappa_B^{0,1} = 0 \end{aligned}$$

We do not expect that $\partial_B \kappa_B^{0,1} = -\bar{\partial}_B \kappa_B^{1,0} \in \Omega_B^{1,1}(\mathcal{F})$ would be in general zero for any metric. Consider Example 9.1. In the next section we will examine this form more closely.

THEOREM 3.3. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures with a holonomy-invariant transverse complex structure and transverse Hermitian metric. Given a bundle-like metric for $(M, \mathcal{F})$ compatible with the Hermitian metric, let $\kappa_B = \kappa_B^{1,0} + \kappa_B^{0,1}$ be the basic component of the mean curvature one-form, with $\kappa_B^{1,0} \in \Omega_B^{1,0}$, $\kappa_B^{0,1} = \overline{\kappa_B^{1,0}}$. Then the cohomology classes of $\kappa_B^{1,0}$ and $\kappa_B^{0,1}$ in $H_{\partial_B}^{1,0}(\mathcal{F})$ and $H_{\bar{\partial}_B}^{0,1}(\mathcal{F})$, respectively, are invariant with respect to the choice of bundle-like metric and transverse metric compatible with the holomorphic structure.*

Proof. By [2] (Proposition 2.4), any change of compatible bundle-like metric and transverse metric changes κ_B to $\kappa_B' = \kappa_B + df$ for some basic function f. Then $\left(\kappa_B^{1,0}\right)' = \frac{1}{2}(\kappa_B + iJ\kappa_B) + \frac{1}{2}(df + iJdf)$. Using local coordinates, one can show that on real-valued basic functions $\partial_B = \Pi^{1,0} d$ where $\Pi^{1,0} : \Omega_B^1(\mathcal{F}) \to \Omega_B^{1,0}(\mathcal{F})$ is the projection $\alpha \mapsto \frac{1}{2}(\alpha + iJ\alpha)$, we have $\left(\kappa_B^{1,0}\right)' = \kappa_B^{1,0} + \partial_B f$. Thus $\left[\kappa_B^{1,0}\right] \in H_{\partial_B}^{1,0}(\mathcal{F})$ is independent of the choice of the metric choices. The analogous proof for $\kappa_B^{0,1}$ is similar. ☐

REMARK 3.4. *As in Corollary 2.7, we can may multiply the metric along the leaves by a conformal factor to yield any possible element $\left(\kappa_B^{1,0}\right)' \in \left[\kappa_B^{1,0}\right]$.*

REMARK 3.5. *Because $\kappa_B^{1,0} = \frac{1}{2}(\kappa_B + iJ\kappa_B)$ and $\kappa_B = \kappa_B^{1,0} + \overline{\kappa_B^{1,0}}$, M is taut (i.e. κ_B is d_B-exact) if and only if $\left[\kappa_B^{1,0}\right]$ is trivial.*

PROPOSITION 3.6. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures with a holonomy-invariant transverse complex structure and transverse Hermitian metric. Then there exists a bundle-like metric compatible with the transverse Hermitian metric, such that $\kappa^{1,0}$ and $\kappa^{0,1}$ are basic forms; that is, $\kappa^{1,0} = \kappa_B^{1,0}$ and $\kappa^{0,1} = \kappa_B^{0,1}$.*

Proof. By [10] (Proposition 2.5), there exists a bundle-like metric that does not change the transverse structure such that $\kappa = \kappa_B$. Now apply the projections $\Pi^{1,0}$ and $\Pi^{0,1}$. □

Let

$$\square_B = \partial_B^* \partial_B + \partial_B \partial_B^*, \ \overline{\square}_B = \bar{\partial}_B^* \bar{\partial}_B + \bar{\partial}_B \bar{\partial}_B^*$$

On a closed manifold M, it is clear that a basic form α satisfies $\square_B \alpha = 0$ if and only if $\partial_B \alpha = 0$ and $\partial_B^* \alpha = 0$. Also, if $\alpha \in \Omega_B^{r,0}(\mathcal{F})$, automatically $\bar{\partial}_B^* \alpha = 0$, so that $\partial_B^* \alpha = 0$ implies that $\delta_B \alpha = \left(\partial_B^* + \bar{\partial}_B^*\right)\alpha = 0$, where δ_B is the adjoint of d restricted to basic complex-valued forms.

Lemma 3.7. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation on a closed manifold. Then for any complex-valued basic function f,*

$$\delta_T d_B f = 2\partial_T^* \partial_B f = 2\bar{\partial}_T^* \bar{\partial}_B f.$$

In particular, the operator $\partial_T^* \partial_B$ is a real operator on functions.

Proof. Since δ_T, ∂_T^* and $\bar{\partial}_T^*$ correspond to the divergences d^*, ∂^*, $\bar{\partial}^*$ on the local quotient manifold, this Lemma follows directly from the local fact that $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$ on Kähler manifolds. □

With our string of successes of projecting using $\Pi^{r,s}$, one would hope that an analogue of Proposition 2.6 can be found just as easily. However, as the following remark shows, we are not so lucky.

Remark 3.8. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation on a closed manifold with a holonomy-invariant transverse complex structure and transverse Hermitian metric. By Proposition 2.6, choose g_M to be a bundle-like metric on M compatible with the transverse Hermitian structure and chosen so that the mean curvature κ is basic harmonic — that is, so that $\kappa = \kappa_B$, $\delta_B \kappa = 0$. Then observe that*

$$\begin{aligned} 0 &= \delta_B \kappa \\ &= \left(\partial_B^* + \bar{\partial}_B^*\right)\left(\kappa^{1,0} + \kappa^{0,1}\right) \\ &= \partial_B^* \kappa^{1,0} + \overline{\partial_B^* \kappa^{1,0}} = 2\mathrm{Re}\left(\partial_B^* \kappa^{1,0}\right). \end{aligned}$$

Hence, Proposition 2.6 gives us no control over the imaginary part of $\partial_B^* \kappa^{1,0}$. On the other hand, suppose we are able to find a bundle-like metric g_M such that

$$\partial_B^* \kappa^{1,0} = 0.$$

Then by the calculation above, $0 = \delta_B \kappa$, so in fact κ is basic harmonic. However, by Corollary 2.7, the leafwise volume form $\chi_{\mathcal{F}}$ is determined up to a constant scale factor, so the form κ is uniquely determined.

From the discussion in the remark above we at least have the following.

Proposition 3.9. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation on a closed manifold with a holonomy-invariant transverse complex structure and transverse Hermitian metric. Suppose that there exists a bundle-like metric compatible with the transverse structure such that $\partial_B^* \kappa^{1,0} = 0$. Then the mean curvature κ is basic harmonic, and κ is the unique element of $[\kappa] \in H_B^1(\mathcal{F})$ with this property.*

If $(M, \mathcal{F}, J, g_Q)$ is transversely Kähler, then the situation of the previous proposition always occurs.

Proposition 3.10. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation on a closed manifold. Then there exists a bundle-like metric compatible with the Kähler structure such that κ is basic harmonic; that is, $\kappa = \kappa_B$ and $\delta_B \kappa = 0$. For that same metric,*

$$\partial_B^* \kappa^{1,0} = \delta_B \kappa^{1,0} = 0,$$

so that also

$$\square_B \kappa^{1,0} = 0, \; \bar{\partial}_B^* \kappa^{0,1} = \delta_B \kappa^{0,1} = 0, \; \overline{\square}_B \kappa^{0,1} = 0.$$

Proof. Let the bundle-like metric be chosen as in Proposition 2.6, so that $\kappa = \kappa_B$ and $\delta_B \kappa = 0$. Since the foliation is transversely Kähler, $\partial_T^* \partial_B$ on functions is a real operator, as is its adjoint $\partial_B^* \partial_T$, by Lemma 3.7. But then

$$\partial_B^* \partial_T (1) = -\partial_B^* \kappa_B^{1,0}$$

is a real-valued function, so that

$$\begin{aligned}
\partial_B^* \kappa_B^{1,0} &= \frac{1}{2}\left(\partial_B^* \kappa_B^{1,0} + \overline{\partial_B^* \kappa_B^{1,0}}\right) \\
&= \frac{1}{2}\left(\partial_B^* \kappa_B^{1,0} + \bar{\partial}_B^* \kappa_B^{0,1}\right) \\
&= \frac{1}{2}\left(\delta_B \kappa_B^{1,0} + \delta_B \kappa_B^{0,1}\right) = \frac{1}{2}\delta_B \kappa_B = 0.
\end{aligned}$$

□

Remark 3.11. *After examining the proof in [29] (Proposition 2.6), it does not appear that Proposition 3.10 is true in the more general transverse Hermitian foliation case. Finding such a metric is tantamount to finding a smooth, positive basic function ψ such that*

$$\partial_B^* \partial_T \psi = 0.$$

From the original proof, there is a ψ that is unique up to a multiplicative constant such that

$$\mathrm{Re}\left(\partial_B^* \partial_T \psi\right) = 0.$$

However, it seems unlikely that the imaginary part would also vanish for this ψ.

For future use, we recall the Hodge theorem for basic Dolbeault cohomology.

Theorem 3.12. *(Proved in [11, Théorème 3.3.3] for general tranversely Hermitian foliations on a compact manifold, stated in general in [24, Theorem 3.21]) Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation on a compact Riemannian manifold with bundle-like metric g_M. Then*

$$\Omega_B^{r,s}(\mathcal{F}) \cong \mathcal{H}_B^{r,s} \oplus \mathrm{Im}\bar{\partial}_B \oplus \mathrm{Im}\bar{\partial}_B^*,$$

where $\mathcal{H}_B^{r,s} = \mathrm{Ker}\overline{\Box}_B$ is finite dimensional. Moreover, $\mathcal{H}_B^{r,s} \cong H_B^{r,s}$.

4 Another class in basic Dolbeault cohomology

Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures with a holonomy-invariant transverse complex structure and transverse Hermitian metric. By Theorem 3.3, the cohomology classes of $\kappa_B^{1,0}$ and $\kappa_B^{0,1}$ in $H^{1,0}_{\partial_B}(\mathcal{F})$ and $H^{0,1}_{\bar{\partial}_B}(\mathcal{F})$, respectively, are invariant with respect to the choice of compatible bundle-like metric. Observe that we may obtain an additional invariant basic Dolbeault cohomology class from the transverse holomorphic structure. Note that for any transversely holomorphic foliation, $\partial_B\bar{\partial}_B = -\bar{\partial}_B\partial_B$, and $\left(\partial_B\bar{\partial}_B\right)^2 = 0$, so that

$$\Omega^{0,0}(\mathcal{F}) \overset{\partial_B\bar{\partial}_B}{\to} \Omega^{1,1}(\mathcal{F}) \overset{\partial_B\bar{\partial}_B}{\to} \Omega^{2,2}(\mathcal{F}) \overset{\partial_B\bar{\partial}_B}{\to} \dots \overset{\partial_B\bar{\partial}_B}{\to} \Omega^{n,n}(\mathcal{F})$$

forms a differential complex, and so that the cohomology $H^{j,j}_{\partial_B\bar{\partial}_B}(\mathcal{F})$ is well-defined.

Also, observe that, with $*$ denoting the adjoint with respect to basic forms,

$$\Delta_{\partial_B\bar{\partial}_B} := \left(\partial_B\bar{\partial}_B\right)^* \partial_B\bar{\partial}_B + \partial_B\bar{\partial}_B\left(\partial_B\bar{\partial}_B\right)^*.$$

Also, for $\varphi \in \Omega_B^{j,j}(\mathcal{F})$, $\Delta_{\partial_B\bar{\partial}_B}\varphi = 0$ if and only if

$$\partial_B\bar{\partial}_B\varphi = 0 \text{ and } \partial_B^*\bar{\partial}_B^*\varphi = 0.$$

We get the resulting Hodge theorem, since $\Delta_{\partial_B\bar{\partial}_B}$ is strongly elliptic on basic forms on a Riemannian foliation.

Proposition 4.1. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Hermitian foliation on a compact Riemannian manifold with bundle-like metric g_M. Then*

$$\Omega_B^{r,r}(\mathcal{F}) \cong \mathcal{H}^{r,r}_{\partial_B\bar{\partial}_B} \oplus \mathrm{Im}\partial_B\bar{\partial}_B \oplus \mathrm{Im}\partial_B^*\bar{\partial}_B^*,$$

where $\mathcal{H}^{r,r}_{\partial_B\bar{\partial}_B} = \ker\left(\Delta_{\partial_B\bar{\partial}_B}\big|_{\Omega^{r,r}(\mathcal{F})}\right) \cong H^{r,r}_{\partial_B\bar{\partial}_B}(\mathcal{F})$ is finite dimensional.

Remark 4.2. *The $\Delta_{\partial_B\bar{\partial}_B}$-harmonic form representatives in the classes in $H^{r,r}_{\partial_B\bar{\partial}_B}(\mathcal{F})$ are precisely those with minimum L^2-norm. The usual proof works: if α is one such $\Delta_{\partial_B\bar{\partial}_B}$-harmonic (r,r)-form and $\beta \in \Omega_B^{r-1,r-1}(\mathcal{F})$, then*

$$\begin{aligned}
& \ll \alpha + \partial_B\bar{\partial}_B\beta, \alpha + \partial_B\bar{\partial}_B\beta \gg \\
= & \ll \alpha, \alpha \gg + 2\mathrm{Re} \ll \partial_B\bar{\partial}_B\beta, \alpha \gg + \ll \partial_B\bar{\partial}_B\beta, \partial_B\bar{\partial}_B\beta \gg \\
= & \ll \alpha, \alpha \gg + 2\mathrm{Re} \ll \beta, \left(\partial_B\bar{\partial}_B\right)^* \alpha \gg + \ll \partial_B\bar{\partial}_B\beta, \partial_B\bar{\partial}_B\beta \gg \\
= & \ll \alpha, \alpha \gg + \ll \partial_B\bar{\partial}_B\beta, \partial_B\bar{\partial}_B\beta \gg \geq \ll \alpha, \alpha \gg .
\end{aligned}$$

THEOREM 4.3. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures with a holonomy-invariant transverse complex structure and transverse Hermitian metric. For a given compatible bundle-like metric, let $\kappa_B^{1,0}$ and $\kappa_B^{0,1}$ be the corresponding basic components of the mean curvature* 1*-form* κ. *Then the form $\partial_B \kappa_B^{0,1}$ is $\partial_B \bar{\partial}_B$-closed, and its cohomology class in $H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ is invariant of the choice of transverse metric and bundle-like metric. A similar result is true for $\left[\bar{\partial}_B \kappa_B^{1,0}\right] = -\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$.*

Proof. Since $0 = d_B^2 = \left(\partial_B + \bar{\partial}_B\right)^2$, $\bar{\partial}_B \left(\partial_B \kappa_B^{0,1}\right) = -\partial_B \left(\bar{\partial}_B \kappa_B^{0,1}\right) = 0$, so $\partial_B \kappa_B^{0,1}$ is $\bar{\partial}_B$-closed and thus $\partial_B \bar{\partial}_B$-closed. By 3.3, any other choice of compatible transverse metric and bundle-like metric yields $\left(\kappa_B^{0,1}\right)' = \kappa_B^{0,1} + \bar{\partial}_B f$ for some complex-valued function f. Then

$$\begin{aligned} \partial_B \left(\kappa_B^{0,1}\right)' &= \partial_B \kappa_B^{0,1} + \partial_B \bar{\partial}_B f \\ &= \partial_B \kappa_B^{0,1} + \bar{\partial}_B \partial_B (-f) . \end{aligned}$$

Applying conjugation, we get a similar result for $\bar{\partial}_B \kappa_B^{1,0}$. □

REMARK 4.4. *The reader may wonder why the class of $\partial_B \kappa_B^{0,1}$ in $H^{1,1}_{\bar{\partial}_B}(\mathcal{F})$ is not considered, as the same proof shows that this class is an invariant of the choice of transverse metric and bundle-like metric. However, this class is always zero, because $\partial_B \kappa_B^{0,1} = -\bar{\partial}_B \kappa_B^{1,0}$ is always $\bar{\partial}$-exact. However, $\partial_B \kappa_B^{0,1}$ is not always $\partial_B \bar{\partial}_B$-exact. See Example 9.2 for a nontaut Riemannian foliation with two different transverse holomorphic structures, and in one case $\partial_B \kappa_B^{0,1}$ is not $\partial_B \bar{\partial}_B$-exact and in the other case $\partial_B \kappa_B^{0,1}$ is $\partial_B \bar{\partial}_B$-exact, and in both cases κ_B is not d_B-exact.*

REMARK 4.5. *By Proposition 3.2, $\partial_B \kappa_B^{0,1}$ and $\bar{\partial}_B \kappa_B^{1,0}$ are pure imaginary forms.*

REMARK 4.6. *If $(M, \mathcal{F})$ as above is a taut foliation, i.e. κ_B is d-exact, then the class $\left[\partial_B \kappa_B^{0,1}\right]$ is trivial because $\kappa_B^{0,1}$ is $\bar{\partial}_B$-exact. However, the converse is false; see Example 9.2.*

REMARK 4.7. *It is clear from the proof of Theorem 4.3 that we may modify the metric along the leaves of the foliation so that any purely imaginary element of the class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ may be realized, since by multiplying the leafwise metric by a conformal factor results in an arbitrary real-valued f such that $\partial_B \left(\kappa_B^{0,1}\right)' = \partial_B \kappa_B^{0,1} + \partial_B \bar{\partial}_B f$. That is, if for some complex-valued function h, $\partial_B \left(\kappa_B^{0,1}\right)' = \partial_B \kappa_B^{0,1} + \partial_B \bar{\partial}_B h$, then since $\partial_B \left(\kappa_B^{0,1}\right)'$ and $\partial_B \kappa_B^{0,1}$ are pure imaginary, we have*

$$\partial_B \bar{\partial}_B h = -\overline{\partial_B \bar{\partial}_B h} = -\bar{\partial}_B \partial_B \overline{h} = \partial_B \bar{\partial}_B \overline{h} = \partial_B \bar{\partial}_B \left(\mathrm{Re}\,(h)\right),$$

so that h may always be taken to be real.

Proposition 4.8. *Let $(M, \mathcal{F}, J, g_Q)$ be a nontaut foliation with basic harmonic mean curvature κ_B on a compact manifold with a holonomy-invariant transverse complex structure and transverse Hermitian metric. Then $\partial_B \kappa_B^{0,1} = 0$ if and only if $d\,(J\kappa_B) = 0$. If $\partial_B \kappa_B^{0,1} = 0$, then $[\kappa_B]$ and $[J\kappa_B]$ provide two linearly independent basic cohomology classes, so that $\dim H_b^1(\mathcal{F}) \geq 2$.*

Proof. The condition $\partial_B \kappa_B^{0,1} = 0$ is equivalent to

$$0 = d\kappa_B^{0,1} = \frac{1}{2}\left(d\kappa_B - idJ\kappa_B\right),$$

which is equivalent to $d\,(J\kappa_B) = 0$. Next, suppose that $J\kappa_B \neq \kappa_B$ are in the same cohomology class, so that $J\kappa_B - \kappa_B = df$ for some nonzero exact 1-form. Observe that $\ll \kappa_B, J\kappa_B \gg$ is necessarily zero since J is an isometry, and also $\ll \kappa_B, df \gg = 0$ since κ_B is basic harmonic. But then κ_B is orthogonal to itself because $\kappa_B = J\kappa_B - df$, which is possible only if $\kappa_B = 0$. The conclusion follows. □

We will see that the differential form $\partial_B \kappa_B^{0,1}$ has particular significance for Kähler foliations.

Lemma 4.9. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures, a holonomy-invariant transverse complex structure and transverse Hermitian metric. Let $V \in \Gamma_B\left(Q^{1,0}\right)$, so that $V^\flat \in \Omega_B^{0,1}(\mathcal{F})$. Then the following are equivalent.*

1. $\partial_B V^\flat = 0$.
2. $\partial_B \circ V \lrcorner + V \lrcorner \circ \partial_B = \nabla_V$ *as an operator on locally defined basic differential forms.*
3. $\nabla_Z V = 0$ *for all* $Z \in \Gamma\left(Q^{1,0}\right)$.
4. *Conjugates of the above statements.*

Proof. We assume that at the point we are evaluating the operators, the local bases $\{V_a\}$ and $\{\omega_a\}$ are chosen so that all covariant derivatives vanish (we can do that because these are locally basic sections of Q and Q^*). Statement (1) is equivalent to

$$\sum \omega^a \wedge \nabla_{V_a} V^\flat = 0.$$

Since $\nabla_{V_a} V^\flat$ is type $(0,1)$, this is equivalent to $\nabla_{V_a} V^\flat = 0$ for all a, which is equivalent to $\nabla_Z V^\flat = 0$ for all $Z \in Q^{1,0}$, i.e. statement (3). Next, assume (3): Then $\nabla_{V_a} V = 0$ $(1 \leq a \leq n)$. Then for any $\phi \in \Omega_B^{r,s}(\mathcal{F})$,

$$\begin{aligned}\partial_B\left(V \lrcorner\, \phi\right) &= \sum_a \omega^a \wedge \left(\nabla_{V_a} V\right) \lrcorner\, \phi + \sum_a \omega^a \wedge V \lrcorner \nabla_{V_a} \phi \\ &= -V \lrcorner\, \bar{\partial}_B \phi + \nabla_V \phi,\end{aligned}$$

since $\omega^a \wedge\ (V \lrcorner) + V \lrcorner (\omega^a \wedge) = \langle V, V_a \rangle$. Next, consider (2). If V satisfies (2), then $V = \sum f_b V_b$ for some basic functions f_b. Then for all b,

$$\begin{aligned} \partial_B V \lrcorner \omega^b + V \lrcorner \partial_B \omega^b &= \partial_B f_b - \sum_a f_b \omega^a \wedge (V_b \lrcorner) \nabla_{V_a} \omega^b + f_b \nabla_{V_b} \omega^b = \partial_B f_b \\ &= \sum_a (V_a f_b)\, \omega^a = 0, \end{aligned}$$

So at the point in question, (2) implies that $V_a f_b = 0$ for all a, b at that point. On the other hand, (3) is equivalent to

$$\begin{aligned} \nabla_{V_a} V &= \sum (V_a f_b)\, V_b + \sum f_b \nabla_{V_a} V_b \\ &= \sum (V_a f_b)\, V_b = 0 \end{aligned}$$

for all a if and only if $V_a f_b$ for all a, b as well. □

Proposition 4.10. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures with a holonomy-invariant transverse complex structure and transverse Hermitian metric. Then*

$$\Box_B = \Delta^Q_\partial + \partial_B \circ H^{1,0} \lrcorner + H^{1,0} \lrcorner \circ \partial_B,$$

where $\Delta^Q_\partial = \partial^*_T \partial_B + \partial_B \partial^*_T$ is the ∂-Laplacian on differential forms on the local quotients of foliation charts. Similarly,

$$\overline{\Box}_B = \Delta^Q_{\bar\partial} + \bar\partial_B \circ H^{0,1} \lrcorner + H^{0,1} \lrcorner \circ \bar\partial_B$$

Proof. From Proposition 3.1,

$$\Box_B = \left(\partial^*_T + H^{1,0} \lrcorner\right) \partial_B + \partial_B \left(\partial^*_T + H^{1,0} \lrcorner\right).$$

□

Corollary 4.11. *Let $(M, \mathcal{F}, J, g_Q)$ be a foliation with compact leaf closures with a holonomy-invariant transverse complex structure, transverse Hermitian metric, and compatible bundle-like metric. Then the following are equivalent:*

1. $\Box_B = \Delta^Q_\partial + \nabla_{H^{1,0}}$ *as operators on locally defined basic differential forms.*
2. $\overline{\Box}_B = \Delta^Q_{\bar\partial} + \nabla_{H^{0,1}}$ *as operators on locally defined basic differential forms.*
3. $\partial_B \kappa^{0,1}_B = 0$.

Proof. Apply Lemma 4.9 to $V = H^{1,0}$, $V^\flat = \kappa^{0,1}_B$ and Proposition 4.10. □

Remark 4.12. *The condition $\partial_B \kappa^{0,1}_B = 0$ is equivalent to the condition that if we write*

$$H^{1,0} = \sum_j H^{1,0}_j(z) \partial_{z_j}$$

in local transverse coordinates, then the functions $H^{1,0}_j(z)$ are antiholomorphic.

Theorem 4.13. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation with compatible bundle-like metric. If the class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ is trivial, there exists a bundle-like metric such that as operators on basic differential forms,*

$$\square_B = \overline{\square}_B - i\nabla_{JH}.$$

Proof. If $(M, \mathcal{F}, J, g_Q)$ is a transverse Kähler foliation with compatible bundle-like metric so that $\Delta^Q_{\bar{\partial}} = \Delta^Q_{\partial}$, the class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ is trivial if and only if there exists a compatible bundle-like metric such that $\partial_B \kappa_B^{0,1} = 0$. By Corollary 4.11, $\partial_B \kappa_B^{0,1} = 0$ implies that on basic forms,

$$\begin{aligned}
\square_B &= \Delta^Q_{\partial} + \nabla_{H^{1,0}} \\
&= \Delta^Q_{\bar{\partial}} + \nabla_{H^{1,0}} \\
&= \Delta^Q_{\bar{\partial}} + \nabla_{H^{0,1}} + \nabla_{H^{1,0} - H^{0,1}} \\
&= \overline{\square}_B + \nabla_{-iJH}.
\end{aligned}$$

□

Corollary 4.14. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation with compatible bundle-like metric. Suppose that the $\partial_B \kappa_B^{0,1} = 0$. Then*

$$\begin{aligned}
\ker\left(\overline{\square}_B|_{\Omega_B^{r,s}}\right) \cap \ker\left(\square_B|_{\Omega_B^{r,s}}\right) &= \ker\left(\overline{\square}_B|_{\Omega_B^{r,s}}\right) \cap \ker\left(\nabla_{JH}|_{\Omega_B^{r,s}}\right) \\
&= \ker\left(\square_B|_{\Omega_B^{r,s}}\right) \cap \ker\left(\nabla_{JH}|_{\Omega_B^{r,s}}\right).
\end{aligned}$$

Proof. This follows directly from Theorem 4.13. □

Note that the hypothesis on $\partial_B \kappa_B^{0,1}$ is needed, as Example 9.1 shows.

5 Lefschetz decompositions

We begin with some notation. Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation with compatible bundle-like metric, with associated Kähler form ω. Let $L : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+2}(\mathcal{F})$ and $\Lambda : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-2}(\mathcal{F})$ be given by

$$L(\phi) = \omega \wedge \phi, \quad \Lambda(\phi) = \omega \lrcorner\, \phi,$$

respectively, where $(\beta_1 \wedge \beta_2)\lrcorner = \beta_2^\sharp \lrcorner \beta_1^\sharp \lrcorner$ for any basic 1-forms $\beta_i (i = 1, 2)$. It follows that $\langle L\phi, \psi\rangle = \langle \phi, \Lambda\psi\rangle$ and $\Lambda = (-1)^j \bar{*} L \bar{*}$ on basic j-forms. For $X \in Q$, from [19] we have

$$\begin{aligned}
&[L, X\lrcorner] = \epsilon(JX^b), \quad [\Lambda, \epsilon(X^b)] = -(JX)\lrcorner, \\
&[L, \epsilon(X^b)] = [\Lambda, X\lrcorner] = 0.
\end{aligned} \tag{8}$$

The formulas above extend is exactly the same way to complex vectors X. We extend the complex structure J to $\Omega_B^r(\mathcal{F})$ by the formula

$$J\phi = \sum_{a=1}^{2n} J\theta^a \wedge E_a \lrcorner\, \phi.$$

This formula is consistent with $(J\theta)(X) = -\theta(JX)$ for one-forms θ, and for instance $(Jv)^\flat = Jv^\flat$ for vectors v. The operator $J : \Omega_B^{r,s}(\mathcal{F}) \to \Omega_B^{r,s}(\mathcal{F})$ is skew-Hermitian: $\langle J\phi, \psi\rangle + \langle \phi, J\psi\rangle = 0$, and $J\phi = i\,(s-r)\,\phi$ for any $\phi \in \Omega_B^{r,s}(\mathcal{F})$. This is not the same as the operator C induced from the pullback J^* used often in Kähler geometry.
We quote some known results as follows.

PROPOSITION 5.1. *[19, Proposition 3.3] If $(M, \mathcal{F}, J, g_Q)$ is a transverse Kähler foliation on a compact Riemannian manifold with bundle-like metric g_M,*

$$[L, J] = [\Lambda, J] = [L, d_B] = [\Lambda, \delta_B] = 0.$$

COROLLARY 5.2. *[19, Proposition 3.4]With the same hypothesis,*

$$[L, \partial_B] = [L, \bar\partial_B] = [\Lambda, \partial_B^*] = [\Lambda, \bar\partial_B^*] = 0, \tag{9}$$
$$[L, \partial_B^*] = -i\bar\partial_T,\ [L, \bar\partial_B^*] = i\partial_T,\ [\Lambda, \partial_B] = -i\bar\partial_T^*,\ [\Lambda, \bar\partial_B] = i\partial_T^*. \tag{10}$$

REMARK 5.3. *All equations above in Proposition 5.1 and Corollary 5.2 continue to hold if we exchange the operators $(\cdot)_B$ and $(\cdot)_T$. These results were shown in [11, Lemma 3.4.4] in the minimal foliation case, when $(\cdot)_B = (\cdot)_T$.*

PROPOSITION 5.4. *If $(M, \mathcal{F}, J, g_Q)$ is a transverse Kähler foliation on a compact Riemannian manifold with bundle-like metric g_M, we have*

$$\left[\overline{\Box}_B, L\right] = i\ \epsilon\left(\bar\partial_B \kappa_B^{1,0}\right)$$

as operators on basic forms. Similarly,

$$[\Box_B, L] = i\ \epsilon\left(\bar\partial_B \kappa_B^{1,0}\right).$$

Proof. By the corollary above, we have

$$\begin{aligned}
\overline{\Box}_B L &= \bar\partial_B^*\bar\partial_B L + \bar\partial_B\bar\partial_B^* L \\
&= \bar\partial_B^* L\bar\partial_B + \bar\partial_B L\bar\partial_B^* - i\bar\partial_B\bar\partial_T \\
&= L\bar\partial_B^*\bar\partial_B + L\bar\partial_B\bar\partial_B^* - i\bar\partial_B\partial_T - i\partial_T\bar\partial_B \\
&= L\overline{\Box}_B - i\left(\bar\partial_B\partial_B + \partial_B\bar\partial_B - \bar\partial_B\epsilon\left(\kappa_B^{1,0}\right) - \epsilon\left(\kappa_B^{1,0}\right)\bar\partial_B\right) \\
&= L\overline{\Box}_B + i\epsilon\left(\bar\partial_B\kappa_B^{1,0}\right).
\end{aligned}$$

The second part follows from noticing that $\bar\partial_B\kappa_B^{1,0}$ is pure imaginary and from taking conjugates. ∎

Lemma 5.5. *We have the following identities.*

1. $[\Lambda, L] = \sum (n-r) P_r$ *as an operator on basic forms, where* $P_r : \Omega_B^*(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ *is the projection.*
2. $[\sum (n-r) P_r, \Lambda] = 2\Lambda$.
3. $[\sum (n-r) P_r, L] = -2L$.

Proof. If $\alpha \in \Omega_B^r(\mathcal{F})$,

$$[\Lambda, L]\,\alpha = \sum_{a,b=1}^{n} JE_b \lrcorner E_b \lrcorner \theta^a \wedge J\theta^a \wedge \alpha - \theta^a \wedge J\theta^a \wedge JE_b \lrcorner E_b \lrcorner \alpha \cdot$$

It is easy to see that for a simple r form $\alpha = \theta^{i_1} \wedge ... \wedge \theta^{i_{r_1}} \wedge J\theta^{j_1} \wedge ... \wedge J\theta^{j_{r_2}}$, the term $\omega \lrcorner \omega \wedge$ will contribute τ_1, the number of a such that $E_a \lrcorner \alpha = 0$ and $JE_a \lrcorner \alpha = 0$, and the second term $\omega \wedge \omega \lrcorner$ will contribute $-\tau_2$, the number of b such that $JE_b \lrcorner E_b \lrcorner \alpha \neq 0$. All other contributions cancel between the two terms. Then by counting we see that $n = r_1 + r_2 + \tau_1 - \tau_2 = r + (\tau_1 - \tau_2)$. Equation (1) follows.
On the other hand, since $\Lambda\alpha \in \Omega_B^{r-2}(\mathcal{F})$ for $\alpha \in \Omega_B^r(\mathcal{F})$, we have by (1)

$$\begin{aligned} [[\Lambda, L], \Lambda]\,\alpha &= [\Lambda, L]\,\Lambda\alpha - \Lambda\,[\Lambda, L]\,\alpha \\ &= (n-r+2)\,\Lambda\alpha - (n-r)\,\Lambda\alpha = 2\Lambda\alpha, \end{aligned}$$

proving (2). Taking adjoints, we obtain (3). □

Letting $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the generators of $\mathfrak{sl}_2(\mathbb{C})$, we note that the relations are

$$[X, Y] = A, \;\; [A, X] = 2X, \;\; [A, Y] = -2Y.$$

Lemma 5.6. *The maps* $X \mapsto L$, $Y \mapsto \Lambda$, $A \mapsto \sum (n-r) P_r$ *induces an* $\mathfrak{sl}_2(\mathbb{C})$ *representation on the fibers of* $\Omega_B^*(\mathcal{F})$.

Proof. The relations are easily checked using the Lemma above. □

In what follows, we call an element $\xi \in \Lambda^* Q^*$ primitive if

$$\Lambda\xi = 0.$$

Corollary 5.7. *Each fiber of the bundle* $\Lambda^* Q^*$ *decompose into irreducible representations of* $\mathfrak{sl}_2(\mathbb{C})$, $\Lambda^* Q^* = \bigoplus_{0 \leq k \leq n} V_k$, *where each* V_k *of dimension* $k+1$ *has the form*

$$V_k = \mathbb{C}\alpha + \mathbb{C}L\alpha + ... + \mathbb{C}L^k\alpha,$$

where $\alpha \in (\ker \Lambda) \cap \Lambda^{n-k} Q^*$ *is primitive,* $L^r\alpha \in (Q^*)^{n-k+2r}$ *for* $0 \leq r \leq k$.

Proof. Direct application of the $\mathfrak{sl}_2(\mathbb{C})$ representation theory. □

By the Kähler conditions that $\nabla J = 0$ and $d\omega = 0$, the tensor field Λ is parallel and has constant rank on $\Omega_B^r(\mathcal{F})$. Hence its kernel $\ker \Lambda \subseteq \Omega_B^r(\mathcal{F})$ is a parallel subbundle of $\Omega_B^r(\mathcal{F})$. We let

$$\Omega_{B,P}^r(\mathcal{F}) = \Gamma_B(\ker \Lambda \cap \Lambda^r Q^*) \subseteq \Omega_B^r(\mathcal{F}).$$

denote the space of primitive basic forms.

Proposition 5.8. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation of codimension $2n$ on a compact Riemannian manifold with bundle-like metric g_M. We have the following.*

1. *$\Omega_{B,P}^r(\mathcal{F}) = 0$ if $r > n$.*
2. *If $\alpha \in \Omega_{B,P}^r(\mathcal{F})$, then $L^j\alpha \neq 0$ for $0 \leq j \leq n-r$ and $L^k\alpha = 0$ for $k > n-r$.*
3. *The map $L^k : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+2k}(\mathcal{F})$ is injective for $0 \leq k \leq n-r$.*
4. *The map $L^k : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+2k}(\mathcal{F})$ is surjective for $k = n-r$.*
5. *$\Omega_B^r(\mathcal{F}) = \bigoplus_{k\geq 0} L^k \Omega_{B,P}^{r-2k}(\mathcal{F})$.*

Proof. We apply the Lemma and Corollary above to get (1) and (2) immediately. Statement (3) follows from (2). For (4), note that pointwise the transverse Hodge star $\bar{*}$ is an isomorphism from $\Omega_B^r(\mathcal{F}) \to \Omega_B^{2n-r}(\mathcal{F})$, so the bundles have the same rank. Thus L^{n-r} is a vector bundle isomorphism from $\Omega_B^r(\mathcal{F}) \to \Omega_B^{2n-r}(\mathcal{F})$, so for all $\beta \in \Omega_B^{2n-r}(\mathcal{F})$, $(L^{n-r})^{-1}\beta \in \Omega_B^r(\mathcal{F})$ gets mapped to β, so L^k is surjective for $k = n-r$.
Statement (5) follows from the fact that every $\mathfrak{sl}_2(\mathbb{C})$ representation is a direct sum of irreducible representations. □

Lemma 5.9. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation with compact leaf closures and a compatible bundle-like metric. Then*

$$\Delta_B = \square_B + \overline{\square}_B + \partial_B H^{0,1}\lrcorner + H^{0,1}\lrcorner\, \partial_B + \bar{\partial}_B H^{1,0}\lrcorner + H^{1,0}\lrcorner\, \bar{\partial}_B.$$

Proof. Since $d_B^2 = d_T^2 = 0$, $\partial_B^2 = \bar{\partial}_B^2 = \partial_B\bar{\partial}_B + \bar{\partial}_B\partial_B = 0$ and $\partial_T^2 = \bar{\partial}_T^2 = \partial_T\bar{\partial}_T + \bar{\partial}_T\partial_T = 0$. By direct calculation, we have

$$\begin{aligned}
\Delta_B &= \square_B + \overline{\square}_B + (\bar{\partial}_B\partial_B^* + \partial_B^*\bar{\partial}_B) + (\partial_B\bar{\partial}_B^* + \bar{\partial}_B^*\partial_B) \\
&= \square_B + \overline{\square}_B + \left(\bar{\partial}_B\left(\partial_T^* + H^{1,0}\lrcorner\right) + \left(\partial_T^* + H^{1,0}\lrcorner\right)\bar{\partial}_B\right) \\
&\quad + \left(\partial_B\left(\bar{\partial}_T^* + H^{0,1}\lrcorner\right) + \left(\bar{\partial}_T^* + H^{0,1}\lrcorner\right)\partial_B\right) \\
&= \square_B + \overline{\square}_B + \bar{\partial}_T^*\partial_B + \partial_B\bar{\partial}_T^* + \partial_T^*\bar{\partial}_B + \bar{\partial}_B\partial_T^* \\
&\quad + \bar{\partial}_B H^{1,0}\lrcorner + H^{1,0}\lrcorner\bar{\partial}_B + \partial_B H^{0,1}\lrcorner + H^{0,1}\lrcorner\partial_B
\end{aligned}$$

Observe that the term $\bar{\partial}_T^*\partial_B + \partial_B\bar{\partial}_T^*$ is just the term $\bar{\partial}^*\partial + \partial\bar{\partial}^*$ on the local quotient manifolds of the foliation charts, and also $\partial_T^*\bar{\partial}_B + \bar{\partial}_B\partial_T^*$ is $\partial^*\bar{\partial} + \bar{\partial}\partial^*$. The sum of these two terms is the same as $\Delta - \overline{\square} - \square$ on the foliation chart quotients, which is zero since $(M, \mathcal{F}, J, g_Q)$ is transversely Kähler. □

Let $\mathcal{H}_B^j(\mathcal{F}) = \ker\left(\Delta_B^j\right) := \ker\left(\Delta_B|_{\Omega_B^j(\mathcal{F})}\right)$.

Corollary 5.10. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation of codimension $2n$ on a compact manifold. Then for $0 \le j \le 2n$, $r, s \ge 0$ such that $r + s = j$,*

$$\dim\left(\mathcal{H}_B^j(\mathcal{F}) \cap \Omega_B^{r,s}(\mathcal{F})\right) \le \dim H_B^{r,s}(\mathcal{F}).$$

Proof. From the Lemma above, for any (r, s)-form ϕ, we have $\Delta_B\phi \cap \Omega_B^{r,s} = \left(\square_B\phi + \overline{\square}_B\phi\right)$, so

$$\ll \Delta_B\phi, \phi \gg = \ll \square_B\phi, \phi \gg + \ll \overline{\square}_B\phi, \phi \gg .$$

Since Δ_B, $\square_B$, $\overline{\square}_B$ are nonnegative operators, if $\phi \in \mathcal{H}_B^j(\mathcal{F}) \cap \Omega_B^{r,s}(\mathcal{F})$, then $\overline{\square}_B\phi = 0$. The result follows from the Hodge theorem. □

Theorem 5.11. *(*Hard Lefschetz Theorem. *Proved in [11, Théoréme 3.4.6] for the case of minimal transverse Kähler foliations) Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation of codimension $2n$ on a compact Riemannian manifold with bundle-like metric g_M. Suppose that the class $\left[\partial_B\kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B\bar{\partial}_B}(\mathcal{F})$ is trivial. Then the Hard Lefschetz Theorem holds for basic Dolbeault cohomology. That is, the map*

$$L^k : H_B^r(\mathcal{F}) \to H_B^{r+2k}(\mathcal{F})$$

is injective for $0 \le k \le n - r$ and surjective for $k \ge n - r$, $k \ge 0$. Furthermore,

$$H_B^r(\mathcal{F}) = \bigoplus_{k\ge 0} L^k H_{B,P}^{r-2k}(\mathcal{F}), \tag{11}$$

$$H_B^{r,s}(\mathcal{F}) = \bigoplus_{k\ge 0} L^k H_{B,P}^{r-k,s-k}(\mathcal{F}). \tag{12}$$

Proof. If $\left[\partial_B\kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B\bar{\partial}_B}(\mathcal{F})$ is trivial, we first modify the leafwise metric as in Remark 4.7 without changing the transverse Kähler structure, so that $\partial_B\kappa_B^{0,1} = 0$. By Proposition 5.4, in the new metric, $\left[L, \overline{\square}_B + \square_B\right] = 0$, so that by Lemma 5.9, we have

$$\begin{aligned}
[L, \Delta_B] &= \left[L, \partial_B H^{0,1}\lrcorner\right] + \left[L, H^{0,1}\lrcorner\,\partial_B\right] + \left[L, \bar{\partial}_B H^{1,0}\lrcorner\right] + \left[L, H^{1,0}\lrcorner\,\bar{\partial}_B\right] \\
&= \partial_B\left[L, H^{0,1}\lrcorner\right] + \left[L, H^{0,1}\lrcorner\right]\partial_B + \bar{\partial}_B\left[L, H^{1,0}\lrcorner\right] + \left[L, H^{1,0}\lrcorner\right]\bar{\partial}_B \\
&= -i\left\{\partial_B\epsilon\left(\kappa_B^{1,0}\right) + \epsilon\left(\kappa_B^{1,0}\right)\partial_B\right\} + i\left\{\overline{\partial}_B\epsilon\left(\kappa_B^{0,1}\right) + \epsilon\left(\kappa_B^{0,1}\right)\overline{\partial}_B\right\}, \\
&= -i\epsilon\left(\partial_B\kappa_B^{1,0}\right) + i\epsilon\left(\overline{\partial}_B\kappa_B^{0,1}\right) = 0,
\end{aligned}$$

using (9), (8), and the fact that

$$\left(JH^{0,1}\right)^{\flat} = -i\kappa_B^{1,0}, \ \left(JH^{1,0}\right)^{\flat} = i\kappa_B^{0,1},$$

which follows from (5). The first statement, (11), and (12) follow from this calculation, the fact that $\left[L, \overline{\Box}_B\right] = \left[L, \Box_B\right] = 0$, and Proposition 5.8. □

Corollary 5.12. *Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation of codimension $2n$ on a compact Riemannian manifold with bundle-like metric g_M. Then the following are equivalent:*

1. *The class $[\kappa_B] \in H_B^1(\mathcal{F})$ is trivial; that is, $(M, \mathcal{F})$ is taut.*
2. *The class $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ is trivial.*
3. *The Hard Lefschetz Theorem holds for basic Dolbeault cohomology.*

Proof. (1) clearly implies (2), and (2) implies (3) by Theorem 5.11. Suppose that (3) holds. Then $L^n : H_B^0(\mathcal{F}) \to H_B^{2n}(\mathcal{F})$ is an isomorphism. If $(M, \mathcal{F})$ is not taut, $H_B^0(\mathcal{F})$ is nonzero and $H_B^{2n}(\mathcal{F}) = \{0\}$, a contradiction. Thus, $(M, \mathcal{F})$ must be taut, so (1) holds. □

Remark 5.13. *On nonKähler transverse Hermitian foliations, it is quite possible for $(M, \mathcal{F})$ to be nontaut and for $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ to be trivial, even zero. See the Examples section.*

Remark 5.14. *The corollary implies that tautness for transverse Kähler foliations is characterized by the weaker condition that $\left[\partial_B \kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B \bar{\partial}_B}(\mathcal{F})$ is trivial. Also, it tell us that if the class is nontrivial for a nontaut transverse Hermitian foliation, that foliation does not admit a transverse Kähler structure. Thus, the Hard Lefschetz Theorem in [11] cannot be generalized to nontaut transverse Kähler foliations.*

Remark 5.15. *Since $\partial_B \kappa_B^{0,1}$ is ∂_B-exact and $\overline{\partial}_B$-closed and d-closed, the class $\left[\partial_B \kappa_B^{0,1}\right]$ measures the failure of the classical $\partial\overline{\partial}$-lemma (or dd_c-Lemma) to hold in the case of transverse Kähler foliations, specificially applied to the mean curvature form. Thus, in general we do not expect the basic cohomology to be formal or to satisfy the typical properties of that of ordinary Kähler manifolds. In Section 7, we find sufficient conditions for the transverse dd_c-Lemma to hold.*

6 Case of automorphic mean curvature

The set of foliate vector fields is

$$V(\mathcal{F}) = \{Y \in \Gamma(TM) : [X, Y] \in \Gamma(T\mathcal{F}) \text{ for all } X \in \Gamma(T\mathcal{F})\},$$

and it consists of the set of vector fields whose flows preserve $\mathcal{F}$. For any $X \in V(\mathcal{F})$, $\pi(X)$ is a basic section of Q, meaning that $\nabla_v \pi(X) = 0$ for every v in $T\mathcal{F}$. We say that a vector field $Y \in V(\mathcal{F})$ is TRANSVERSELY AUTOMORPHIC if $\mathcal{L}_Y J = 0$, so that $[Y, J\pi(X)] = J\pi[Y, X]$ for all $X \in V(\mathcal{F})$. Such vector fields are infinitesimal automorphisms of the foliation that preserve the transverse complex structure. Sometimes we also refer to the image $\pi(Y) \in \Gamma(Q)$ as being transversely automorphic, because the property only depends on the properties of $\pi(Y)$.

For a complex basic normal vector field $Z \in \Gamma_B Q^{1,0}$, we say Z is TRANSVERSELY HOLOMORPHIC if $\nabla_{\bar{V}} Z = 0$ for $\bar{V} \in Q^{0,1}$. This is equivalent to Z being a basic vector field that can be expressed as a holomorphic vector field in the transverse variables of the local foliation charts. The following results have been previously proved.

Lemma 6.1. *[24, Proposition 3.3]Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation with compatible bundle-like metric. The field $X \in V(\mathcal{F})$ is transversely automorphic if and only if $\nabla_{JY} \pi(X) = J\nabla_Y \pi(X)$ for all $Y \in V(\mathcal{F})$.*

Lemma 6.2. *[24, Proposition 3.3]Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation with compatible bundle-like metric. The field $X \in V(\mathcal{F})$ is transversely automorphic if and only if the complex normal vector field $Z = \pi(X) - iJ\pi(X) \in \Gamma_B Q^{1,0}$ is transversely holomorphic. A complex basic normal vector field $W \in \Gamma_B Q^{1,0}$ is transversely holomorphic if and only if the field $W + \overline{W} \in \Gamma_B Q$ is transversely automorphic.*

As in the last section, we use local orthonormal basic frames $\{V_a\}$ for $Q^{1,0}$ and $\{\omega_a\}$ for $\left(Q^{1,0}\right)^* = Q_{1,0}$.

Proposition 6.3. *[24, Lemma 3.15]Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Hermitian foliation. Let $Z \in \Gamma_B Q^{1,0}$. Then the following are equivalent.*

1. *Z is transversely holomorphic.*
2. *Z satisfies $\bar{\partial}_B Z \lrcorner + Z \lrcorner \bar{\partial}_B = 0$.*

Remark 6.4. *It is interesting to determine the relationship between the condition above and the condition $\bar{\partial}_B Z^\flat = 0$. Note that if $Z \in \Gamma_B Q^{1,0}$, $Z^\flat \in \Omega_B^{0,1}(\mathcal{F})$, and if $\bar{\partial}_B Z^\flat = 0$, then making the usual choices of frame (with covariant derivatives vanishing at the point in question) we have*

$$\begin{aligned} 0 &= \sum_{a,b} \bar{\omega}^a \wedge \nabla_{\bar{V}_a} \left(\langle Z, V_b \rangle \bar{\omega}^b \right) \\ &= \sum_{a,b} \left(\bar{V}_a \langle Z, V_b \rangle \right) \bar{\omega}^a \wedge \bar{\omega}^b \\ &= \sum_{a,b} \left\langle \nabla_{\bar{V}_a} Z, V_b \right\rangle \bar{\omega}^a \wedge \bar{\omega}^b, \end{aligned}$$

so that $\bar{\partial}_B Z^\flat = 0$ *is equivalent to* $\left\langle \nabla_{\bar{V}_a} Z, V_b \right\rangle = \left\langle \nabla_{\bar{V}_b} Z, V_a \right\rangle$ *for all* a, b. *So it is definitely the case that if* Z *is transversely holomorphic, then* $\bar{\partial}_B Z^\flat = 0$, *but the converse is false in general. (In the Examples section, we will see cases where* $H^{1,0}$ *is not transversely holomorphic but where (as always)* $\overline{\partial}_B \kappa^{0,1} = \overline{\partial}_B \left(H^{1,0}\right)^\flat = 0$.*)*

We would now like to apply these results about automorphic vector fields to the mean curvature.

Theorem 6.5. *Let* $(M, \mathcal{F}, J, g_Q)$ *be a transverse Kähler foliation with compact leaf closures and a compatible bundle-like metric. Then the mean curvature of* $(M, \mathcal{F})$ *is automorphic if and only if*

$$\Delta_B = \Box_B + \overline{\Box}_B.$$

Proof. Suppose the mean curvature is automorphic. Then by Lemma 6.2 and Proposition 6.3,

$$\bar{\partial}_B H^{1,0} \lrcorner + H^{1,0} \lrcorner\, \bar{\partial}_B = 0,$$

and also by conjugating, $\partial_B H^{0,1} \lrcorner + H^{0,1} \lrcorner\, \partial_B = 0$. By the formula in Lemma 5.9, $\Delta_B = \Box_B + \overline{\Box}_B$.
Conversely, suppose that $\bar{\partial}_B H^{1,0} \lrcorner + H^{1,0} \lrcorner\, \bar{\partial}_B + \partial_B H^{0,1} \lrcorner + H^{0,1} \lrcorner\, \partial_B = 0$. Applying this operator to ω^b, we obtain as in the proof of Proposition 6.3 that

$$\sum_a (\nabla_{\bar{V}_a} H^{1,0} \lrcorner\, \omega^b) \bar{\omega}^a + 0 = 0,$$

so that $\nabla_{\bar{V}_a} H^{1,0} = 0$ for all a. Thus, $H^{1,0}$ is transversely holomorphic, making the mean curvature automorphic. □

Another consequence of Lemma 5.9 and Proposition 6.3 is the following.

Corollary 6.6. *Let* $(M, \mathcal{F}, J, g_Q)$ *be a transverse Kähler foliation with compact leaf closures and a compatible bundle-like metric. Then the mean curvature of* $(M, \mathcal{F})$ *is automorphic if and only if* Δ_B *preserves the* (r, s) *type of a form.*

Corollary 6.7. *Let* $(M, \mathcal{F}, J, g_Q)$ *be a transverse Kähler foliation of codimension* $2n$ *on a compact manifold such that the mean curvature is automorphic. Then for* $0 \le j \le 2n$,

$$\dim \left(\mathcal{H}_B^j (\mathcal{F}) \right) \le \sum_{r+s=j;\ r,s \ge 0} \dim H_B^{r,s} (\mathcal{F}).$$

Proof. From the theorem and corollary above, under these conditions, for any differential j-form ϕ,

$$\ll \Delta_B \phi, \phi \gg = \ll \Box_B \phi, \phi \gg + \ll \overline{\Box}_B \phi, \phi \gg .$$

Since Δ_B, $\Box_B$, $\overline{\Box}_B$ are nonnegative operators, if $\phi \in \mathcal{H}_B^j (\mathcal{F})$, then $\overline{\Box}_B \phi = 0$ as well, so each (r, s) component of ϕ is $\overline{\Box}_B$-harmonic. The result follows from the Hodge theorem. □

7 The dd_c Lemma

We would like to use the power of the dd_c Lemma from Kähler geometry to use in our setting. In the foliation setting, we will need some assumptions. Let $(M,\mathcal{F},J,g_Q)$ be a transverse Kähler foliation of codimension $2n$ on a compact manifold. First we extend the almost complex structure J by pullback to the operator

$$C:\Omega_B^*(\mathcal{F})\to\Omega_B^*(\mathcal{F}).$$

Note that

$$C=\sum_{0\le a,b\le n} i^{a-b}P_{a,b},$$

where $P_{a,b}:\Omega_B^*(\mathcal{F})\to\Omega_B^{a,b}(\mathcal{F})$ is the projection. Then $C^*=C^{-1}=\sum_{0\le a,b\le n} i^{b-a}P_{a,b}$.

For $0\le k\le 2n$, let $d_c:\Omega_B^k(\mathcal{F})\to\Omega_B^{k+1}(\mathcal{F})$ be defined by

$$d_c=i\left(\bar\partial_B-\partial_B\right)=C^*dC=C^{-1}dC.$$

Note that d_c is a real operator, and its adjoint with respect to basic forms is

$$d_c^*=C^*\delta_BC=C^{-1}\delta_BC.$$

Note that

$$dd_c|_{\Omega_B^*(\mathcal{F})}=2i\partial_B\bar\partial_B=-2i\bar\partial_B\partial_B=-d_cd|_{\Omega_B^*(\mathcal{F})}.$$

Let

$$\Delta_{d_c}=d_cd_c^*+d_c^*d_c=C^{-1}\Delta_BC.$$

Lemma 7.1. *Let $(M,\mathcal{F},J,g_Q)$ be a transverse Kähler foliation with compact leaf closures and a compatible bundle-like metric, such that the mean curvature of $(M,\mathcal{F})$ is automorphic. Then*

$$\Delta_B=C^{-1}\Delta_BC=\Delta_{d_c}.$$

Proof. By Corollary 6.6, Δ_B preserves the type of differential form, so that C is just multiplication by a scalar on forms of type (r,s). The result follows. □

Lemma 7.2. *Let $(M,\mathcal{F},J,g_Q)$ be a transverse Kähler foliation with compact leaf closures and a compatible bundle-like metric, such that the mean curvature of $(M,\mathcal{F})$ is automorphic. Then $\partial_B\kappa^{0,1}=0$ if and only if*

$$\delta_Bd_c+d_c\delta_B=\nabla_{JH},$$

and this is true if and only if M is taut and

$$\delta_Bd_c+d_c\delta_B=0.$$

Proof. We have

$$\begin{aligned}
\delta_B d_c + d_c \delta_B &= i\left(\partial_T^* + \bar{\partial}_T^* + H^{1,0}\lrcorner + H^{0,1}\lrcorner\right)\left(\bar{\partial}_B - \partial_B\right) \\
&\quad + i\left(\bar{\partial}_B - \partial_B\right)\left(\partial_T^* + \bar{\partial}_T^* + H^{1,0}\lrcorner + H^{0,1}\lrcorner\right) \\
&= i\left(\partial_T^* + \bar{\partial}_T^*\right)\left(\bar{\partial}_B - \partial_B\right) + i\left(\bar{\partial}_B - \partial_B\right)\left(\partial_T^* + \bar{\partial}_T^*\right) \\
&\quad + i\left(H^{1,0}\lrcorner + H^{0,1}\lrcorner\right)\left(\bar{\partial}_B - \partial_B\right) \\
&\quad + i\left(\bar{\partial}_B - \partial_B\right)\left(H^{1,0}\lrcorner + H^{0,1}\lrcorner\right) \\
&= i\left(\overline{\Box}_T - \Box_T + \partial_T^*\bar{\partial}_B + \bar{\partial}_B\partial_T^* - \bar{\partial}_T^*\partial_B - \partial_B\bar{\partial}_T^*\right) \\
&\quad + i(H^{1,0}\lrcorner\bar{\partial}_B + \bar{\partial}_B H^{1,0}\lrcorner + H^{0,1}\lrcorner\bar{\partial}_B + \bar{\partial}_B H^{0,1}\lrcorner \\
&\quad - H^{1,0}\lrcorner\partial_B - \partial_B H^{1,0}\lrcorner - H^{0,1}\lrcorner\partial_B - \partial_B H^{0,1}\lrcorner).
\end{aligned}$$

By Corollary 5.2, $-i\bar{\partial}_T^* = [\Lambda, \partial_B]$, so $-i\left(\bar{\partial}_T^*\partial_B + \partial_B\bar{\partial}_T^*\right) = \partial_B\Lambda\partial_B - \partial_B\Lambda\partial_B = 0$, and similarly $\partial_T^*\bar{\partial}_B + \bar{\partial}_B\partial_T^* = 0$. Since the foliation is transversely Kähler, $\overline{\Box}_T - \Box_T = 0$. By Lemma 6.2 and Proposition 6.3,

$$\begin{aligned}
\bar{\partial}_B H^{1,0}\lrcorner + H^{1,0}\lrcorner\,\bar{\partial}_B &= 0, \\
\partial_B H^{0,1}\lrcorner + H^{0,1}\lrcorner\,\partial_B &= 0,
\end{aligned}$$

so that

$$\begin{aligned}
\delta_B d_c + d_c \delta_B &= H^{0,1}\lrcorner\bar{\partial}_B + \bar{\partial}_B H^{0,1}\lrcorner - H^{1,0}\lrcorner\partial_B - \partial_B H^{1,0}\lrcorner \\
&= 2i\mathrm{Im}\left(H^{1,0}\lrcorner\partial_B + \partial_B H^{1,0}\lrcorner\right) = \nabla_{JH},
\end{aligned}$$

by Lemma 4.9 and Proposition 3.2, since $\partial_B\left(H^{1,0}\right)^\flat = \partial_B\kappa^{0,1} = 0$. The rest follows by Corollary 5.12. ☐

Because of this Lemma, we do not expect the dd_c Lemma from Kähler geometry to hold in our setting, except in the special case when the mean curvature is zero, since $\delta_B d_c + d_c\delta_B = 0$ is needed strongly. For this case, we prove the dd_c Lemma in the usual way.

LEMMA 7.3. *(dd_c LEMMA) Let $(M, \mathcal{F}, J, g_Q)$ be a taut, transverse Kähler foliation on a compact manifold, with a compatible bundle-like metric. Suppose that $\alpha \in \Omega_B^k(\mathcal{F})$ is d_c-exact and d-closed. Then there exists a form $\beta \in \Omega_B^{k-2}(\mathcal{F})$ with $\alpha = dd_c\beta$.*

Proof. If $\alpha = d_c\gamma$, we write $\gamma = d\tau + \eta + \delta_B\xi$ by the Hodge decomposition, with η basic harmonic. By hypothesis, η is ∂_B and $\overline{\partial}_B$-closed and thus d_c-closed as well, so

$$\begin{aligned}
d_c\gamma &= d_c\left(d\tau + \eta + \delta_B\xi\right) \\
&= d_c d\tau + d_c\delta_B\xi \\
&= dd_c\left(-\tau\right) + d_c\delta_B\xi.
\end{aligned}$$

By the Lemma above,

$$\begin{aligned} 0 &= dd_c\gamma = dd_c\delta_B\xi \\ &= -d\delta_B d_c\xi, \end{aligned}$$

so

$$0 = \langle d\delta_B d_c\xi, d_c\xi\rangle = \|\delta_B d_c\xi\|^2 = \|d_c\delta_B\xi\|^2,$$

so the equation above is

$$\alpha = d_c\gamma = dd_c(-\tau).$$

□

From this it follows that the basic cohomology is formal, as in the case of ordinary cohomology of Kähler manifolds.

8 The Hodge diamond

On a transverse Kähler foliation with compact leaf closures and compatible bundle-like metric, by Corollary 6.6 the basic Laplacian Δ_B preserves the (r,s) type of form if and only if the mean curvature is automorphic. For the purposes of what follows, we will consider the case when the basic mean curvature is automorphic, and we consider the Δ_B-harmonic forms of type (r,s). Let

$$\mathcal{H}^{r,s}_{\Delta_B}(\mathcal{F}) = \{\alpha \in \Omega^{r,s}_B(\mathcal{F}) : \Delta_B\alpha = 0\}.$$

Theorem 8.1. *(*Hodge Diamond Theorem*) Let $(M,\mathcal{F},J,g_Q)$ be a transverse Kähler foliation of codimension $2n$ on a compact manifold. If there exists a compatible bundle-like metric such that the mean curvature of $(M,\mathcal{F})$ is automorphic, then the spaces $\mathcal{H}^{r,s}_{\Delta_B}(\mathcal{F})$ and basic cohomology groups have the following structure:*

1. *(*Hodge symmetry*) For all r,s such that $0 \le r \le s \le n$, $\mathcal{H}^{r,s}_{\Delta_B}(\mathcal{F}) \cong \mathcal{H}^{s,r}_{\Delta_B}(\mathcal{F})$.*
2. *For all j such that $0 \le j \le 2n$, $\dim H^j_B(\mathcal{F}) = \sum_{r+s=j;\ r,s\ge 0} \dim \mathcal{H}^{r,s}_{\Delta_B}(\mathcal{F})$.*
3. *$\dim H^r_B(\mathcal{F})$ is even if r is odd, and $\dim H^1_B(\mathcal{F}) = 2\dim \mathcal{H}^{1,0}_{\Delta_B}(\mathcal{F})$ is a topological invariant.*
4. *If in addition the class $\left[\partial_B\kappa_B^{0,1}\right] \in H^{1,1}_{\partial_B\bar{\partial}_b}(\mathcal{F})$ is trivial, then the Álvarez class $[\kappa_B] \in H^1_B(\mathcal{F})$ is trivial, the spaces $\mathcal{H}^{r,s}_{\Delta_B} \cong H^{r,s}_B(\mathcal{F})$ have the following structure:*
   - *(a) The map $L^k : H^r_B(\mathcal{F}) \to H^{r+2k}_B(\mathcal{F})$ is injective for $0 \le k \le n-r$ and surjective for $k \ge n-r$, $k \ge 0$.*

(b) *We have* $H_B^r(\mathcal{F}) = \bigoplus_{k\geq 0} L^k H_{B,P}^{r-2k}(\mathcal{F})$, $H_B^{p,q}(\mathcal{F}) = \bigoplus_{k\geq 0} L^k H_{B,P}^{p-k,q-k}(\mathcal{F})$.

(c) *(*Kodaira-Serre duality*) For all* r,s *such that* $0 \leq r \leq s \leq n$, $H_B^{r,s}(\mathcal{F}) \cong H_B^{n-r,n-s}(\mathcal{F})$.

(d) *All of the* $\mathcal{H}_{\Delta_B}^{r,s}(\mathcal{F})$ *in (1),(2),(3) above may be replaced with* $H_B^{r,s}(\mathcal{F})$.

5. *If the class* $\left[\partial_B \kappa_B^{0,1}\right] \in H_{\partial_B \bar{\partial}_b}^{1,1}(\mathcal{F})$ *is nontrivial, then (4a), (4b), (4c) are false.*

Proof. By Theorem 6.5, Δ_B is a real operator, so that (1) holds by conjugation. By Corollary 6.6, Δ_B-harmonic forms correspond exactly to sums of Δ_B-harmonic forms of type (r,s), so that (2) follows, and (1) and (2) imply (3). By the proof of the Hard Lefschetz theorem Theorem 5.11, $[L, \Delta_B] = 0$ when $\partial_B \kappa_B^{0,1} = 0$, so we choose the leafwise metric so that this equation holds, and this change does not alter the dimensions of the harmonic forms (and certainly not the basic cohomology groups). Then (4a) and (4b) follow as in the Hard Lefschetz theorem. Corollary 5.12 implies $[\kappa_B] \in H_B^1(\mathcal{F})$ is trivial, so we modify the bundle-like metric (without changing the cohomology groups) so that $\kappa_B = \kappa = 0$, and then Theorem 4.13 and Lemma 5.9 imply in addition that $\Delta_B = 2\overline{\square}_B = 2\square_B$. Then (4c) and (4d) follow. Statement (5) is a consequence of Corollary 5.12. ☐

Remark 8.2. *Items (4a) through (4d) of the theorem above were essentially already known, because the minimal foliation case was shown in [11].*

Remark 8.3. *The theorem above gives topological obstructions to the existence of transverse Kähler foliations with automorphic mean curvature, and further obstructions if we require that* $\left[\partial_B \kappa_B^{0,1}\right] \in H_{\partial_B \bar{\partial}_b}^{1,1}(\mathcal{F})$ *is trivial.*

9 Examples

Example 9.1. Note that in contrast to the situation of a Kähler form on an ordinary manifold, it is possible that ω is a trivial class in basic cohomology. This always happens when we consider nontaut codimension 2 foliations. We consider the Carrière example from [7]. Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ of trace strictly greater than 2. We denote respectively by v_1 and v_2 unit eigenvectors associated with the eigenvalues λ and $\frac{1}{\lambda}$ of A with $\lambda > 1$ irrational. Let the hyperbolic torus $\mathbb{T}_A^3$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m,t) to $(A(m), t+1)$. The flow generated by the vector field V_2 is a Riemannian foliation with bundle-like metric (letting (x,s,t) denote local coordinates in the v_2 direction, v_1 direction, and $\mathbb{R}$ direction, respectively)

$$g = \lambda^{-2t} dx^2 + \lambda^{2t} ds^2 + dt^2.$$

Note that the mean curvature of the flow is $\kappa = \kappa_B = \log(\lambda)\,dt$, since $\chi_{\mathcal{F}} = \lambda^{-t}dx$ is the characteristic form and $d\chi_{\mathcal{F}} = -\log(\lambda)\,\lambda^{-t}dt \wedge dx = -\kappa \wedge \chi_{\mathcal{F}}$. Then an orthonormal frame field for this manifold is $\{X = \lambda^t \partial_x, S = \lambda^{-t}\partial_s, T = \partial_t\}$ corresponding to the orthonormal coframe $\{X^* = \chi_{\mathcal{F}} = \lambda^{-t}dx, S^* = \lambda^t ds, T^* = dt\}$. Then, letting J be defined by $J(S) = T, J(T) = -S$, the Nijenhuis tensor

$$N_J(S,T) = [S,T] + J\left([JS,T] + [S,JT]\right) - [JS,JT]$$

clearly vanishes, so that J is integrable. (This is also easy to see with other means.)
The corresponding transverse Kähler form is seen to be $\omega = T^* \wedge S^* = \lambda^t dt \wedge ds = d(\frac{1}{\log\lambda}S^*)$, an exact form in basic cohomology. From the above,

$$\begin{aligned}\kappa &= \kappa_B = \log(\lambda)\,dt = \log(\lambda)\,T^* \\ S^* &= \lambda^t ds,\ Z^* = \frac{1}{2}\left(S^* + iT^*\right) = \frac{1}{2}\left(\lambda^t ds + idt\right),\end{aligned}$$

so

$$\begin{aligned}\kappa_B &= -i\left(\log\lambda\right)Z^* + i\left(\log\lambda\right)\bar{Z}^* \\ &= -i\left(\log\lambda\right)\frac{1}{2}\left(\lambda^t ds + idt\right) + i\left(\log\lambda\right)\bar{Z}^*.\end{aligned}$$

Then

$$\begin{aligned}\kappa_B^{1,0} &= -i\log(\lambda)\,Z^* = -\frac{i}{2}\left(\log\lambda\right)\left(\lambda^t ds + idt\right) \\ \bar{\partial}_B\kappa_B^{1,0} &= d\kappa_B^{1,0} = \frac{i}{2}\left(\log\lambda\right)^2\lambda^t ds \wedge dt \\ &= \frac{i}{2}\left(\log\lambda\right)^2 S^* \wedge T^* \\ &= \left(\log\lambda\right)^2 \bar{Z}^* \wedge Z^*.\end{aligned}$$

It is impossible to change the metric so that this is zero. The reason is that from [2] the mean curvature κ_B' for any other compatible bundle-like metric would satisfy $\kappa_B' = \kappa_B + df$ for some real basic function f, which would imply that $(\kappa_B^{1,0})' = \kappa_B^{1,0} + \partial_B f$, and $\partial_B f = Z(f)Z^*$. Since f is a periodic function of t alone, this is $\partial_B f = -i(\partial_t f)\,Z^*$. Then in that case

$$\begin{aligned}\bar{\partial}_B(\kappa_B^{1,0})' &= d(\kappa_B^{1,0} - i(\partial_t f)\,Z^*) \\ &= d\left(-\frac{i}{2}\left(\log\lambda + 2\partial_t f\right)\left(\lambda^t ds + idt\right)\right) \\ &= \left(\frac{i}{2}(\log\lambda)^2 + i\partial_t^2 f + i(\log\lambda)\partial_t f\right)\lambda^t ds \wedge dt \\ &= \left((\log\lambda)^2 + 2\partial_t^2 f + 2(\log\lambda)\partial_t f\right)\bar{Z}^* \wedge Z^*\end{aligned}$$

Since the term in parentheses is never zero for any periodic function f, we conclude that $\bar{\partial}\kappa_B^{1,0}$ is a nonzero multiple of $\bar{Z}^* \wedge Z^*$ for any compatible bundle-like metric. This is not surprising, because this being zero would imply $(M, \mathcal{F})$ is taut by Corollary 5.12.

For later use, we compute that basic Dolbeault cohomology in this example. One can easily verify that

$$\begin{aligned} H_B^{0,0} = \ker \overline{\partial}_B^{0,0} &= \operatorname{span}\{1\} \\ H_B^{1,0} = \ker \overline{\partial}_B^{1,0} &= \{0\} \\ H_B^{0,1} = \frac{\ker \overline{\partial}_B^{0,1}}{\operatorname{im} \overline{\partial}_B^{0,0}} &= \operatorname{span}\{S^* - iT^*\} \\ H_B^{1,1} = \frac{\Omega_B^{1,1}}{\operatorname{im} \overline{\partial}_B^{1,0}} &= \{0\}, \end{aligned}$$

where the last equality is true because one can show that every element of $\Omega_B^{1,1}$ is $\overline{\partial}_B$-exact. Now we compute $H^{*,*}_{\partial_B \overline{\partial}_B}(\mathcal{F})$: because $\partial_B \overline{\partial}_B f = -\frac{1}{4}\Delta_B f \; dz \wedge d\overline{z}$ integrates to zero, we have

$$H^{0,0}_{\partial_B \overline{\partial}_B}(\mathcal{F}) \cong \mathbb{C}; \; H^{1,1}_{\partial_B \overline{\partial}_B}(\mathcal{F}) = \operatorname{span}\{\bar{Z}^* \wedge Z^*\} \cong \mathbb{C},$$

and as expected, $\left[\partial_B \kappa_B^{0,1}\right]$ is nonzero and thus a generator.

Then observe that the ordinary basic cohomology Betti numbers for this foliation are $h_B^0 = h_B^1 = 1$, $h_B^2 = 0$, we see that the basic Dolbeault Betti numbers satisfy

$$h_B^{0,0} = h_B^{0,1} = 1, \quad h_B^{1,0} = h_B^{1,1} = 0.$$

So even though it is true that

$$h_B^j = \sum_{r+s=j} h_B^{r,s},$$

and the foliation is transversely Kähler, we also have (with $n = 1$)

$$h_B^{r,s} \neq h_B^{s,r}, \quad h_B^{r,s} \neq h_B^{n-r,n-s}.$$

Theorem 8.1 tells us that the mean curvature is not automorphic. We can also verify this directly:

$$\begin{aligned} H^{1,0} &= \frac{i \log \lambda}{2} Z, \\ \left(\overline{\partial}_B H^{1,0} \lrcorner + H^{1,0} \lrcorner \overline{\partial}_B\right) \kappa_B^{1,0} &= \overline{\partial}_B \left(\frac{(\log \lambda)^2}{2} \right) + H^{1,0} \lrcorner \, (\log \lambda)^2 \, \bar{Z}^* \wedge Z^* \\ &= -\frac{i}{2} (\log \lambda)^3 \, \bar{Z}^* \neq 0. \end{aligned}$$

Another way to see this is to choose local transverse holomorphic coordinates. The reader may check that if we choose

$$x_0 = -(\log\lambda)\, s;\ y_0 = \lambda^{-t}$$

and let $z_0 = x_0 + iy_0$, then

$$\partial_{x_0} = -\frac{1}{\log\lambda}\partial_s,\ \partial_{y_0} = -\frac{\lambda^t}{\log\lambda}\partial_t;\ dx_0 = -(\log\lambda)\, ds,\ dy_0 = -(\log\lambda)\,\lambda^{-t}dt;$$
$$J(\partial_{x_0}) = \partial_{y_0},\ J(\partial_{y_0}) = -\partial_{x_0};\ dz_0 = -(\log\lambda)\left(ds + i\lambda^{-t}dt\right),$$

and so

$$\kappa_B^{1,0} = \frac{i}{2y_0}dz_0,$$

which is clearly not a holomorphic one-form.
The exactness of the basic Kähler form causes the Kodaira-Serre argument, the Lefschetz theorem, the Hodge diamond ideas to fail. Thus, for a nontaut, transverse Kähler foliation, it is not necessarily true that the odd basic Betti numbers are even, and the basic Dolbeault numbers do not have the same kinds of symmetries as Dolbeault cohomology on Kähler manifolds. Also, this example shows that the even degree basic cohomology groups are not always nonzero, as is the case for ordinary cohomology for symplectic manifolds (and thus all Kähler manifolds).

Example 9.2. We now consider the product foliation on the product manifold $M = T_A^3 \times T_A^3$. We will put two different transverse Hermitian structures on M, and the cohomological properties of the two transverse structures are different. In both cases we have fixed the product metric.

1. First, we consider the product of the two transverse holomorphic structures on each copy of T_A^3 separately. A simple calculation shows that the foliation is transversely Kähler, nontaut. The mean curvature is not automorphic, and the class $\left[\partial_B\kappa_B^{0,1}\right]$ on M is nontrivial. The Betti numbers are

$$\begin{aligned} h_B^0 &= 1,\ h_B^1 = 2,\ h_B^2 = 1, \\ h_B^{0,0} &= 1,\ h_B^{0,1} = 2,\ h_B^{0,2} = 1, \\ h_{\partial_B\overline{\partial}_B}^{0,0} &= 1,\ h_{\partial_B\overline{\partial}_B}^{1,1} = 2,\ h_{\partial_B\overline{\partial}_B}^{1,1} = 1, \end{aligned}$$

with all the other Betti numbers zero.

2. Next, instead we use the following transverse complex structure. Using the same notation as in Example 9.1 but using subscripts 1 and 2 to refer to the different copies of T_A^3 in the product, we define

$$J'(U_1) = U_2;\ J'(U_2) = -U_1,$$

where U denotes one of the unit normal vector fields S or T. We then have that the form ω is

$$\omega = \lambda^{-t_1-t_2} dx_2 \wedge dx_1 + \lambda^{t_1+t_2} ds_2 \wedge ds_1 + dt_2 \wedge dt_1,$$

which is clearly not closed, so the new transverse Hermitian structure is not Kähler. The foliation is the same as before, so it is again not taut. The mean curvature is

$$\begin{aligned} \kappa_B &= (\log \lambda)(dt_1 + dt_2), \\ \kappa_B^{1,0} &= \frac{1}{2}(\kappa_B + iJ'\kappa_B) = \frac{1}{2}(\log\lambda)(1-i)(dt_1 + idt_2). \end{aligned}$$

This vector field is clearly holomorphic, and we also have

$$\overline{\partial}_B \kappa_B^{1,0} = d\kappa_B^{1,0} = 0,$$

so that with this new holomorphic structure, the $\partial_B\overline{\partial}_B$-class $\left[\partial_B \kappa_B^{0,1}\right]$ is trivial (even though the foliation is not taut). The Betti numbers now satisfy

$$\begin{aligned} h_B^0 &= 1,\ h_B^1 = 2,\ h_B^2 = 1, \\ h_B^{0,0} &= 1,\ h_B^{0,1} = 1 = h_B^{1,0},\ h_B^{1,1} = 2,\ h_B^{2,0} = h_B^{0,2} = 1, \\ h_{\partial_B\overline{\partial}_B}^{0,0} &= 1,\ h_{\partial_B\overline{\partial}_B}^{1,1} = 1, \end{aligned}$$

with all other Betti numbers zero.

This set of examples shows that it is possible for the class $\left[\partial_B \kappa_B^{0,1}\right]$ to be trivial for some transverse holomorphic structures and to be nontrivial in others. But if this is the case, by Corollary 5.12 it must be nontrivial when the structure is transversely Kähler.

REFERENCES

[1] H. Ait Haddou, *Foliations and Lichnerowicz basic cohomology*, Int. Math. Forum 2 (2007), no. 49-52, 2437-2446.

[2] J. A. Álvarez-López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10 (1992), 179–194.

[3] A. Banyaga, *Some properties of locally conformal symplectic structures*, Comment. Math. Helv. 77 (2002), no. 2, 383-398.

[4] C. P. Boyer and K. Galicki, *Sasakian geometry*. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[5] C. P. Boyer, K. Galicki, and M. Nakamaye, *On the geometry of Sasakian-Einstein 5-manifolds*, Math. Ann. 325 (2003), no. 3, 485-524.

[6] B. Cappelletti-Montano, A. de Nicola, and I. Yudin, *Hard Lefschetz Theorem for Sasakian manifolds*, J. Diff. Geom. 101 (2015), 47-66.

[7] Y. Carrière, *Flots riemanniens*, Astérisque 116 (1984), 31–52.

[8] D. Chinea, M. de León, and J. C. Marrero, *Topology of cosymplectic manifolds*, J. Math. Pures Appl. (9) 72 (1993), no. 6, 567-591.

[9] L. A. Cordero and R. A. Wolak, *Properties of the basic cohomology of transversely Kähler foliations*, Rendiconti del Circolo Matematico di Palermo, Serie II, 40 (1991), 177-188.

[10] D. Domínguez, *Finiteness and tenseness theorems for Riemannian foliations*, Amer. J. Math. 120 (1998), no. 6, 1237-1276.

[11] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. 73 (1990), no. 1, 57-106.

[12] A. El Kacimi-Alaoui and G. Hector, *Décomposition de Hodge basique pour un feuilletage riemannien*, Ann. Inst. Fourier 36 (1986), no. 3, 207-227.

[13] A. El Kacimi-Alaoui and M. Nicolau, *On the topological invariance of the basic cohomology*, Math. Ann. 295 (1993), no. 4, 627-634.

[14] O. Goertsches, H. Nozawa, and D. Töben, *Rigidity and vanishing of basic Dolbeault cohomology of Sasakian manifolds*, J. Symplectic Geom. 14 (2016), no. 1, 31-70.

[15] G. Habib and K. Richardson, *Modified differentials and basic cohomology for Riemannian foliations*, J. Geom. Anal. 23 (2013), no. 3, 1314-1342.

[16] G. Habib and L. Vezzoni, *Some remarks on Calabi-Yau and hyper-Kähler foliations*, Differential Geom. Appl. 41 (2015), 12-32.

[17] J. Hebda, *Curvature and focal points in Riemannian foliations*, Indiana Univ. Math. J. 35 (1986), no. 2, 321-331.

[18] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39 (2001), 253-264.

[19] S. D. Jung and M. J. Jung, *Transversally holomorphic maps between Kähler foliations*, J. Math. Anal. Appl. 416 (2014), 683-697.

[20] S. D. Jung and H. Liu, *Transversal infinitesimal automorphisms on Kähler foliations*, Bull. Aust. Math. Soc. 86 (2012), 405-415.

[21] S. D. Jung and J. S. Pak, *Some vanishing theorems on Kähler foliation*, Comm. Korean Math. Soc. 11 (1996), 767-781.

[22] S. D. Jung and J. S. Pak, *Some vanishing theorems on complete Kähler foliation*, Acta Math. Hungar. 77 (1997), no. 1-2, 15-28.

[23] S. D. Jung and K. Richardson, *Transverse conformal Killing forms and a Gallot-Meyer theorem for foliations*, Math. Z. 270 (2012), 337-350.

[24] S. D. Jung and K. Richardson, *Basic Dolbeault cohomology and Weitzenböck formulas on transversely Kähler foliations*, preprint.

[25] F. W. Kamber and Ph. Tondeur, *Duality for Riemannian foliations*, Proc. Sympos. Pure Math., Amer. Math. Soc. 40 (1983), Part I, 609-618.

[26] F. W. Kamber and Ph. Tondeur, *Duality theorems for foliations*, Asterisque 116 (1984), 108-116.

[27] F. W. Kamber and Ph. Tondeur, *De Rham-Hodge theory for Riemannian foliations*, Math. Ann. 277 (1987), 415-431.

[28] Y. Kordyukov, M. Lejmi, and P. Weber, *Seiberg-Witten invariants on manifolds with Riemannian foliations of codimension 4*, J. Geom. Phys. 107 (2016), 114-135.

[29] P. March, M. Min-Oo, and E. A. Ruh, *Mean curvature of Riemannian foliations*, Canad. Math. Bull. 39 (1996), 95-105

[30] A. Mason, *An application of stochastic flows to Riemannian foliations*, Houston J. Math. 26 (2000), 481-515.

[31] M. Min-Oo, E. Ruh, and Ph. Tondeur, *Vanishing theorems for the basic cohomology of Riemannian foliations*, J. Reine Angew. Math. 415 (1991), 167-174.

[32] A. Moroianu and L. Ornea, *Eigenvalue estimates for the Dirac operator and harmonic 1-forms of constant length*, C. R. Math. Acad. Sci. Paris 338 (2004), 561-564.

[33] S. Nisikawa and Ph. Tondeur, *Transversal infinitesimal automorphisms for harmonic Kähler foliations*, Tohoku Math. J. 40 (1988), 599-611.

[34] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. 118 (1996), 1249–1275.

[35] H. Rummler, *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts*, Comm. Math. Helv. 54 (1979), 224-239.

[36] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New York, 1988.

[37] Ph. Tondeur, *Geometry of foliations*, Birkhäuser Verlag, 1997.

[38] I. Vaisman, *Remarkable operators and commutation formulas on locally conformal Kähler manifolds*, Compositio Math. 40 (1980), no. 3, 287-299.

[39] R. Wolak, *Sasakian structures: a foliated approach*, arXiv:1605.04163 [math.DG].

[40] S. Yorozu and T. Tanemura, *Green's theorem on a foliated Riemannian manifold and its applications*, Acta Math. Hungarica 56 (1990), 239-245.

Seoung Dal Jung
Department of Mathematics
Jeju National University
Jeju 690-756
Republic of Korea
sdjung@jejunu.ac.kr

Ken Richardson
Department of Mathematics
Texas Christian University
Fort Worth, Texas 76129
USA
k.richardson@tcu.edu