On Derived Equivalences of K3 Surfaces in Positive Characteristic

Tanya Kaushal Srivastava

Received: October 8, 2018
Revised: May 20, 2019

Communicated by Takeshi Saito

Abstract. For an ordinary K3 surface over an algebraically closed field of positive characteristic we show that every automorphism lifts to characteristic zero. Moreover, we show that the Fourier-Mukai partners of an ordinary K3 surface are in one-to-one correspondence with the Fourier-Mukai partners of the geometric generic fiber of its canonical lift. We also prove that the explicit counting formula for Fourier-Mukai partners of the K3 surfaces with Picard rank two and with discriminant equal to minus of a prime number, in terms of the class number of the prime, holds over a field of positive characteristic as well. We show that the image of the derived autoequivalence group of a K3 surface of finite height in the group of isometries of its crystalline cohomology has index at least two. Moreover, we provide a conditional upper bound on the kernel of this natural cohomological descent map. Further, we give an extended remark in the appendix on the possibility of an F-crystal structure on the crystalline cohomology of a K3 surface over an algebraically closed field of positive characteristic and show that the naive F-crystal structure fails in being compatible with inner product.

2010 Mathematics Subject Classification: 14F05, 14F30, 14J50, 14J28, 14G17
Keywords and Phrases: Derived equivalences, K3 surfaces, automorphisms, positive characteristic

Acknowledgement

The results contained in this article are a part of PhD thesis written under the supervision of Prof. Dr. Hélène Esnault. I owe her special gratitude for guidance, support, continuous encouragement and inspiration she always provided me. I thank Berlin Mathematical School for the PhD fellowship. I am greatly thankful to Michael Groechenig, Vasudevan Srinivas, François Charles, Christian Liedtke, Daniel Huybrechts, Martin Olsson, Max Lieblich, Lenny Taelman and Sofia Tirabassi for many mathematical discussions and suggestions. Finally, I would like to thank the referee for careful reading and valuable suggestions.

1 Introduction

The derived category of coherent sheaves on a smooth projective variety was first studied as a geometrical invariant by Mukai in the early 1980's. In case the smooth projective variety has an ample canonical or anti-canonical bundle, Bondal-Orlov [12] proved that, if two such varieties have equivalent bounded derived categories of coherent sheaves, then they are isomorphic. However, in general this is not true. The bounded derived category of coherent sheaves is not an isomorphism invariant. Mukai [52] showed that for an Abelian variety over $\mathbb{C}$, its dual has equivalent bounded derived category. Moreover, in many cases it can be shown that the dual of an Abelian variety is not birational to it, which implies that derived categories are not even birational invariants, see [30, Chapter 9]. Similarly, Mukai showed in [53] that for K3 surfaces over $\mathbb{C}$, there are non-isomorphic K3 surfaces with equivalent derived categories. This led to the natural question of classifying all derived equivalent varieties.

For K3 surfaces, the case of interest to us, this was completed over $\mathbb{C}$ in late 1990's by Mukai and Orlov ([53, Theorem 1.4], [59, Theorem 1.5]) using Hodge theory along with the Global Torelli Theorem (see [4, VIII Corollary 11.2], [36, Theorem 7.5.3]). As a consequence, it was shown that there are only finitely many non-isomorphic K3 surfaces with equivalent bounded derived categories (see Proposition 2.28) and a counting formula was also proved by Hosono et. al in [29]. On the other hand, for K3 surfaces over a field of positive characteristic, a partial answer to the classification question was first given by Lieblich-Olsson [45] (see Theorem 2.37) in early 2010's. They showed that there are only finitely many non-isomorphic K3 surfaces with equivalent bounded derived categories. We remark here that due to unavailability of a positive characteristic version of the global Torelli Theorem for K3 surfaces of finite height, it is currently not feasible to give a complete cohomological description of derived equivalent K3 surfaces. However, a description in terms of moduli spaces was given by Lieblich-Olsson. We also point out here that the proofs of these results go via lifting to characteristic zero and thus use the Hodge theoretic description given by Mukai and Orlov. Furthermore, Lieblich-Olsson [46] also proved the derived version of the Torelli theorem using the Crystalline Torelli theorem for supersingular K3 surfaces.

In this article, we study the above question in more details for the case of K3 surfaces over an algebraically closed field of positive characteristic. We show that the number of isomorphism classes of ordinary K3 surfaces which are derived equivalent to a chosen ordinary K3 surface is the same as the number of isomorphism classes of K3 surfaces in characteristic 0 derived equivalent to the canonical lift of our chosen ordinary K3 (Theorem 4.11). This result should be seen as an evidence to the long held belief that the number of Fourier-Mukai partners behaves well with respect to deformation to characteristic zero. Moreover, we show that the geometric reformulations [29, Question I' and II'] of questions of Gauss on the behavior of class numbers can be extended to include K3 surfaces over algebraically closed fields of characteristic p (Theorem 4.14). This, we hope, will provide with more ways to answer the questions of Gauss posed in 1801 [29, Question I and II] on the class number $h(p)$ of the real quadratic field $\mathbb{Q}(\sqrt{p})$ for a prime number $p \equiv 1 \mod 4$:
Question I: Are there infinitely many primes p such that the class number $h(p)$ is 1?
Recall that for imaginary quadratic fields there are only finitely many primes with class number 1, namely -1, -2, -3, -7, -11, -19, -43, -67, -163 (see [55, Chapter 1 page 37]).
Question II: Is there a sequence of primes $p_1, p_2, \ldots$ such that $h(p_k) \to \infty$?

Using the counting formula for derived equivalent K3 surfaces, we can reformulate the above questions as:
Question I': Are there infinitely many isomorphism classes of K3 surfaces over an algebraically closed field of positive characteristic or over $\mathbb{C}$ with Picard rank 2 and discriminant $-q$ for distinct primes q, such that it has no non-isomorphic K3 surfaces derived equivalent to it?
Question II': Is there a sequence of K3 surfaces over an algebraically closed field of positive characteristic or over $\mathbb{C}$ with Picard rank 2 and discriminant $-q$ for distinct primes q such that the number of K3 surfaces derived equivalent to it tends to infinity ?

Meanwhile in 1990's another school of thought inspired by string theory in physics led Kontsevich [41] to propose the homological mirror symmetry conjecture which states that the bounded derived category $D^b(X)$ of coherent sheaves of a projective variety X is equivalent (as a triangulated category) to the bounded derived category $D^bFuk(\check{X}, \beta)$ of the Fukaya category $Fuk(\check{X}, \beta)$ of a mirror $\check{X}$ with its symplectic structure β. Moreover, the symplectic automorphisms of $\check{X}$ induce derived autoequivalences of $D^b(X)$. This provided a natural motivation for the study of the derived autoequivalence group.

For K3 surfaces X over $\mathbb{C}$, the structure of the group of derived autoequivalences was analyzed by Ploog in [63], Hosono et al. in [28] and Huybrechts, et al. in [32]. They showed that the image of $Aut(D^b(X))$ under the homomorphism

$$Aut(D^b(X)) \to O_{Hodge}(\widetilde{H(X,\mathbb{Z})}),$$

where $O_{Hodge}(\widetilde{H(X,\mathbb{Z})})$ is the group of Hodge isometries of the Mukai lattice of X, has index 2. However, the kernel of this map has a description only in the special case when the Picard rank of X is 1, given by [6].

In the spirit of the question on the structure of derived autoequivalence group of K3 surfaces, we show that this group for K3 surfaces over algebraically closed fields of positive characteristic displays similar behavior as a K3 surface over $\mathbb{C}$. More precisely, let X be a K3 surface of finite height over an algebraically closed field k of characteristic $p > 3$ and let $W(k)$ be the Witt ring with K its field of fraction. Then any derived autoequivalence induces naturally an automorphism of F-isocrystals on $\mathcal{H}^*_{crys}(X/K)$.

Theorem 1.1 (cf. Theorem 3.18) *The image of $Auteq(D^b(X))$ in $Aut(\mathcal{H}^*_{crys}(X/K))$ has index at least 2.*

This is exactly similar to the behavior of K3 surfaces over $\mathbb{C}$ as remarked above. Moreover, for general K3 surfaces we expect that the kernel of the natural map $Auteq(D^b(X)) \to Aut(\mathcal{H}^*_{crys}(X/K))$ will embed in the kernel of a Picard rank 1 lift of it (see Proposition 3.21). This provides us with a possible approach to proof of Bridgeland's conjecture ([14, Conjecture 1.2]) for K3 surfaces over $\mathbb{C}$, by first specializing any K3 surface over $\mathbb{C}$ with good reduction to characteristic p and then embedding back the kernel in a Picard rank 1 lift to characteristic zero. This will be undertaken in future work.
As a consequence of studying derived autoequivalences, we prove that for ordinary K3 surfaces every automorphism lifts to characteristic zero, which should be seen as adhering to the general philosophy that ordinary K3 surfaces behave just like complex K3 surfaces.

Here is a brief outline of the article. In Section 2 we recall the notion of height of a K3 surface over a field of positive characteristic, the results on lifting K3 surfaces from characteristic p to characteristic 0, the moduli spaces of stable sheaves on a K3 surface and derived equivalences on K3 surfaces. We end this section by proving that height of a K3 surface remains invariant under derived equivalences (Lemma 2.39). In Section 3, we address the question on the group of derived autoequivalences for K3 surfaces of finite height. We show that the image of the derived autoequivalence group of a K3 surface of finite height in the group of isometries of its crystalline cohomology has index at least two (Theorem 3.18). Moreover, we provide a conditional upper bound on the kernel of this natural cohomological descent map (Proposition 3.21). In Section 4, we count the number of Fourier-Mukai partner for an ordinary K3 surface (Theorem 4.11) along with showing that the automorphism group lifts to characteristic 0 (Theorem 4.5). We also prove that the explicit counting formula for Fourier-Mukai partners of the K3 surfaces with Picard rank two and with discriminant equal to minus of a prime number, in terms of the class number of the prime, holds over a field of positive characteristic as well (Theorem 4.14). In Appendix A, we define an F-crystal structure and show that this integral structure is preserved by derived equivalences but its compatibility with intersection pairing fails.

1.1 CONVENTIONS AND NOTATIONS

For a perfect field k of positive characteristic p, $W(k)$ will be its ring of Witt vectors. For any cohomology theory $H^*_{\dots}(...)$, we will denote the dimension of the cohomology groups $H^i_{\dots}(\dots)$ as $h^i_{\dots}(\dots)$. We will implicitly assume that the cardinality of $K := Frac(W(k))$ and its algebraic closure $\bar{K}$ are not bigger than that of $\mathbb{C}$, this will allow us to choose an embedding $\bar{K} \hookrightarrow \mathbb{C}$ which we will use in our arguments to transfer results from characteristic 0 to characteristic p. See also Remarks 2.35 and 2.29. Moreover at times, we will put the condition of characteristic $p > 3$ as at many places we may have denominators in factors of 2 and 3, like in the definition of Chern characters for K3 surfaces, and these will become invertible in $W(k)$ due to our assumption on the characteristic.

2 PRELIMINARIES ON K3 SURFACES AND DERIVED EQUIVALENCES

We recall the notion height of a K3 surface over a field of positive characteristic through its F-crystal, which gives a subclass of K3 surfaces with finite height or infinite height called supersingular K3 surfaces. For an introduction to Brauer group of K3 surfaces and the definition of height via the Brauer groups see [36] and [49]. Both definitions turn out to be equivalent (see, for example, [49, Prop. 6.17]).

Let k be an algebraically closed field of positive characteristic, $W(k)$ its ring of Witt vectors and $Frob_W$ the Frobenius morphism of $W(k)$ induced by the Frobenius automorphism of k. Note that $Frob_W$ is a ring homomorphism and induces an automorphism of the fraction fields $K := Frac(W(k))$, denoted as $Frob_K$. We begin by recalling the notion of F-isocrystal and F-crystals which we will use later to stratify the moduli of K3 surfaces.

DEFINITION 2.1 [F-(iso)crystal] An *F-crystal* (M, ϕ_M) over k is a free W-module M of finite rank together with an injective $Frob_W$-linear map $\phi_M : M \to M$, that is, ϕ_M is additive, injective and satisfies

$$\phi_M(r \cdot m) = Frob_W(r) \cdot \phi_M(m) \text{ for all } r \in W(k), m \in M.$$

An *F-isocrystal* (V, ϕ_V) is a finite dimensional K-vector space V together with an injective $Frob_K$-linear map $\phi_V : V \to V$.

A *morphism* $u : (M, \phi_M) \to (N, \phi_N)$ *of F-crystals* (resp. *F-isocrystals*) is a $W(k)$-linear (resp. K-linear) map $M \to N$ such that $\phi_N \circ u = u \circ \phi_M$. An *isogeny* of F-crystals is a morphism $u : (M, \phi_M) \to (N, \phi_N)$ of F-crystals, such that the induced map $u \otimes Id_K : M \otimes_{W(k)} K \to N \otimes_{W(k)} K$ is an isomorphism of F-isocrystals.

EXAMPLES:

1. The trivial crystal: $(W, Frob_W)$.

2. This is the case which will be of most interest to us:
Let X be a smooth and proper variety over k. For any n, take the free $W(k)$ module M to be $H^n := H^n_{crys}(X/W(k))/torsion$ and ϕ_M to be the Frobenius F^*. The Poincaré duality induces a perfect pairing

$$\langle -, - \rangle : H^n \times H^{2dim(X)-n} \to H^{2dim(X)} \cong W$$

which satisfies the following compatibility with Frobenius

$$\langle F^*(x), F^*(y) \rangle = p^{dim(X)} Frob_W(\langle x, y \rangle),$$

where $x \in H^n$ and $y \in H^{2dim(X)-n}$. As $Frob_W$ is injective, we have that F^* is injective. Thus, (H^n, F^*) is an F-crystal. We will denote the F-isocrystal $H^n_{crys}(X/W) \otimes K$ by $H^n_{crys}(X/K)$.

3. The F-isocrystal $K(1) := (K, Frob_K/p)$. Similarly, one has the F-isocrystal $K(n) := (K, Frob_K/p^n)$ for all $n \in \mathbb{Z}$. Moreover, for any F-isocrystal V and $n \in \mathbb{Z}$, we denote by $V(n)$ the F-isocrystal $V \otimes K(n)$.

Recall that the category of F-crystals over k up to isogeny is semi-simple and the simple objects are the F-crystals:

$$M_\alpha = ((\mathbb{Z}_p[T])/(T^s - p^r)) \otimes_{Z_p} W(k), (\text{mult. by } T) \otimes Frob_W),$$

for $\alpha = r/s \in \mathbb{Q}_{\geq 0}$ and r, s non-negative coprime integers. This is a theorem of Dieudonné–Manin [18], [50]. Note that the rank of the F-crystal M_α is s. We call α the SLOPE of the F-crystal M_α.

DEFINITION 2.2 Let (M, ϕ) be an F-crystal over k and let

$$(M, \phi) \sim^{isogeny} \oplus_{\alpha \in \mathbb{Q}_{\geq 0}} M_\alpha^{n_\alpha}$$

be its decomposition up to isogeny. Then the elements of the set

$$\{\alpha \in \mathbb{Q}_{\geq 0} | n_\alpha \neq 0\}$$

are called the *slopes* of (M, ϕ). For every slope α of (M, ϕ), the integer $\lambda_\alpha := n_\alpha \cdot rank_W M_\alpha$ is called the *multiplicity* of the slope α.

REMARK 2.3 In case (M, ϕ) is an F-crystal over a perfect field k (rather than being algebraically closed as assumed above), we define its slope and multiplicities to be that of the F-crystal $(M, \phi) \otimes_{W(k)} W(\bar{k})$, where $\bar{k}$ is an algebraic closure of k.

We still keep our assumption of k being an algebraically closed field of positive characteristic.
The above classification result of Dieudonné-Manin is more general. Any F-isocrystal V with bijective ϕ_V is isomorphic to a direct sum of F-isocrystals

$$(V_\alpha := K[T]/(T^s - p^r), (\text{mult. by } T) \otimes Frob_K),$$

for $\alpha = r/s \in \mathbb{Q}$. The dimension of V_α is s and we call α the *slope* of V_α.

DEFINITION 2.4 [Height] The *height* of a K3 surface X over k is the sum of multiplicities of slope strictly less than 1 part of the F-crystal $H^2_{crys}(X/W)$. In other words, the dimension of the subspace of slope strictly less than one of the F-isocrystal $H^2_{crys}(X/K)$, which is $\dim(H^2_{crys}(X/K)_{[0,1)} := \oplus_{\alpha_i<1} V_{\alpha_i}^{n_{\alpha_i}})$.

If for a K3 surface X the $\dim(H^2_{crys}(X/K)_{[0,1)}) = 0$, then we say that the height of X is infinite. Supersingular K3 surfaces (i.e., K3 surfaces with infinite height) also have an equivalent description that their Picard rank is 22 (see [49, Theorem 4.8]). We will be discussing more about F-crystals later in Appendix A. On the other hand, we have ordinary K3 surfaces.

DEFINITION 2.5 [Ordinary K3 surface] A K3 surface X over a perfect field k of positive characteristic is called *ordinary* if the height of X is 1.

They also have equivalent description via height of Brauer group, see [57, Lemma 1.3].

2.1 LIFTING K3 SURFACES

We state the theorem by Deligne about lifting K3 surfaces which will be used a lot in the theorems that follow. Let X_0 be a K3 surface over a field k of characteristic $p > 0$.

DEFINITION 2.6 [Lift of a K3 surface] A *lift* of a K3 surface X_0 to characteristic 0 is a smooth projective scheme X over R, where R is a discrete valuation ring such that $R/\mathfrak{m} = k$, $K := \text{Frac}(R)$ is a field of characteristic zero, the generic fiber of X, denoted X_K, is a K3 surface and the special fiber is X_0.

THEOREM 2.7 (DELIGNE [16] THEOREM 1.6, COROLLARY 1.7, 1.8) *Let X_0 be a K3 surface over a field k algebraically closed of characteristic $p > 0$. Let L_0 be an ample line bundle on X_0. Then there exists a finite extension T of $W(k)$, the Witt ring of k, such that there exists a deformation of X_0 to a smooth proper scheme X over T and an extension of L_0 to an ample line bundle L on X.*

Consider the situation where we have a lift of a K3 surface, i.e., let X_0 be a K3 surface over a field of characteristic $p > 0$ and X a lift over $S = \text{Spec}(R)$

as defined above. The de Rham cohomology of X/S, $H^*_{DR}(X/S)$ is equipped with a filtration induced from the Hodge to de Rham spectral sequence:

$$E_1^{i,j} = H^j(X, \Omega^i_{X/S}) \Rightarrow H^*_{DR}(X/S)$$

For a construction of this spectral sequence, see [21, III-0 11.2]. We call this filtration on $H^2_{DR}(X/S)$ the Hodge filtration. Using the comparison isomorphism between the crystalline cohomology of the special fiber and the de Rham cohomology of X [9, 7.26.3],

$$H^i_{crys}(X_0/W(k)) \otimes R \cong H^i_{DR}(X/S),$$

we get a filtration on the crystalline cohomology, also called *the Hodge filtration*. This Hodge filtration on the crystalline cohomology depends on the choice of a lift of X_0.
In case X is an ordinary K3 surface, it admits a special lift to characteristic zero called cananical lift and it has the following Picard preserving property:

Proposition 2.8 ([57], Proposition 1.8) *For X an ordinary K3 surface, there exists a canonical lift X_{can} with the property that any line bundle on X lifts uniquely to X_{can}.*

Moreover, in [68, Theorem C], Taelman proved a criterion to determine when a lift of an ordinary K3 surface is going to a be a cananical lift. We will be using this criterion.

2.2 Moduli Space of Sheaves

Next we discuss about the Moduli space of sheaves on a K3 surface as these spaces turn out to play a very important role in the theory of derived equivalences of K3 surfaces. We introduce the moduli stack of sheaves on a K3 surface and show that it's a μ_r-Gerbe under some numerical conditions. We will try to keep the exposition here characteristic independent and in case of characteristic restrictions we will mention them as necessary. Moreover, in the case of a K3 surface defined over a field we will not assume the field to be algebraically closed and in general, for a relative K3 surface, we will work with a spectrum of a mixed characteristic discrete valuation ring as the base scheme. The main references for this section are [44, Section 2.3.3] and [45, Section 3.15]. We refer the reader to [24], for a comparison between the moduli stack point of view and that of more classical moduli functors. For an introduction to theory of gerbes we refer the reader to [58].
Before proceeding to the definition of moduli stacks of sheaves that we will be working with, let us also recall the notion of (Gieseker) semistability for coherent sheaves (for details see [31, Section 1.2]): Let X be a projective scheme over a field k. The Euler characteristic of a coherent sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) = \sum(-1)^i h^i(X, \mathcal{F})$. If we fix an ample line bundle $\mathcal{O}(1)$ on X, then

the Hilbert polynomial $P(\mathcal{F})$ given by $n \mapsto \chi(\mathcal{F} \otimes \mathcal{O}(n))$ can be uniquely written in the form

$$P(\mathcal{F}, n) = \sum_{i=0}^{dim(\mathcal{F})} \alpha_i(\mathcal{F}) m^i / i!,$$

with integral coefficients $\alpha_i(\mathcal{F})$. We denote by $p(\mathcal{F}, n) := P(\mathcal{F}, n)/\alpha_{dim(\mathcal{F})}(\mathcal{F})$, the *reduced Hilbert polynomial of* $\mathcal{F}$.

Definition 2.9 [Semistability] A coherent sheaf $\mathcal{F}$ of dimension d is *semistable* if $\mathcal{F}$ has no nontrivial proper subsheaves of strictly smaller dimension and for any subsheaf $\mathcal{E} \subset \mathcal{F}$, one has $p(\mathcal{E}) \leq p(\mathcal{F})$. It is called *stable* if for any proper subsheaf the inequality is strict.

Remark 2.10 The ordering on polynomials is the ordering on the coefficients.

Definition 2.11 [Mukai vector] For a smooth projective X over k, given a perfect complex $E \in D(X)$, where $D(X)$ is the derived category of coherent sheaves on X, we define the *Mukai vector* of E to be

$$v(E) := ch(E)\sqrt{td_X} \in A^*(X)_{num,\mathbb{Q}}.$$

Here, $ch(-)$ denotes the Chern class map, td_X is the Todd genus and $A^*(X)_{num,\mathbb{Q}}$ is the numerical Chow group of X with rational coefficients.

For X a K3 surface over k, the Mukai vector of a complex is given by (see [30, Chapter 10]):

$$v(E) = (\mathrm{rank}(E), c_1(E), \mathrm{rank}(E) + c_1(E)^2/2 - c_2(E)).$$

Let X be a projective scheme over k and h an ample line bundle.

Definition 2.12 [Moduli Stack] The *moduli stack of semistable sheaves*, denoted $\mathfrak{M}_h^{ss}$, is defined as follows:

$$\begin{aligned} \mathfrak{M}_h^{ss} :& (Sch/k) \to (\text{groupoids}) \\ & S \mapsto \{\mathcal{F} | \mathcal{F} \text{ an } S\text{-flat coherent sheaf on } X \times S \text{ with semistable fibers.}\} \end{aligned}$$

Similarly, the *moduli stack of stable sheaves* can be defined by replacing semistable above with stable and we denote it by $\mathfrak{M}_h^s$.

If we fix a vector $v \in A^*(X)_{num,\mathbb{Q}}$, we get an open and closed substack $\mathfrak{M}_h^{ss}(v)$ classifying semistable sheaves on X with Mukai vector v.
The following result has been proved by Lieblich [44], for the more general case of moduli of twisted sheaves. Restricting to the case of semistable sheaves without any twisting a simpler argument is given in [67, Theorem 2.30].

Theorem 2.13 *The stack $\mathfrak{M}_h^{ss}$ is an algebraic stack and the stack $\mathfrak{M}^{ss}(v)$ is an algebraic substack of finite type over k.*

Remark 2.14 Recall that the Mukai vector v for a sheaf on a K3 surface determines its Hilbert polynomial and its rank as well.

Moreover, the stack $\mathfrak{M}_h^{ss}(v)$ contains an open substack of geometrically stable points (see Footnote 3) denoted $\mathfrak{M}_h^s(v)$.

Theorem 2.15 *The algebraic stack $\mathfrak{M}_h^s(v)$ admits a coarse moduli space.*

For a proof see [44, Lemma 2.3.3.3, Prop. 2.3.3.4] or [67, Theorem 2.34].

Theorem 2.16 (Mukai-Orlov) *Let X be a K3 surface over a field k.*

1. *Let $v \in A^*(X)_{num,\mathbb{Q}}$ be a primitive element with $v^2 = 0$ (with respect to the Mukai pairing[1]) and positive degree 0 part[2]. Then $\mathfrak{M}_h^{ss}(v)$is non-empty.*
2. *If, in addition, there is a complex $P \in D(X)$ with Mukai vector v' such that $< v, v' >= 1$, then every semistable sheaf with Mukai vector v is locally free and geometrically stable[3], in which case $\mathfrak{M}_h^{ss}(v)$ is a μ_r-gerbe for some r, over a smooth projective surface $M_h(v)$ such that the associated $\mathbb{G}_m$-gerbe is trivial[4].*

Remark 2.17

1. Note that the triviality of the $\mathbb{G}_m$-gerbe is equivalent to the existence of a universal bundle over $X \times M_h(v)$, also see [45, Remark 3.19].
2. See [31, Remark 6.1.9] for a proof that under the assumption of the above Theorem part (2), any semistable sheaf is locally free and geometrically stable.

Proof: The non-emptiness follows from [36, Chapter 10 Theorem 2.7] and [45, Remark 3.17]. For the construction of the universal bundle, one has to ,in the end, actually use GIT again. For a proof see [36, Chapter 10 Proposition 3.4] and [31, Theorem 4.6.5] (this is from where we have the numerical criteria, in particular, also see [31, Corollary 4.6.7]). □

We generalize our moduli stack to the relative setting. Let X_S be a flat projective scheme over S with an ample line bundle h. (The case of $S = \mathrm{Spec}(R)$ for R a discrete valuation ring of mixed characteristic, will be of most interest to us.)

[1]The Mukai pairing is just an extension of the intersection pairing, defined as follows: let $(a_1, b_1, c_1) \in A^*(X)_{num,\mathbb{Q}}$ and $(a_2, b_2, c_2) \in A^*(X)_{num,\mathbb{Q}}$, then the Mukai pairing is $< (a_1, b_1, c_1), (a_2, b_2, c_2) >= b_2 \cdot b_1 - a_1 \cdot c_2 - a_2 \cdot c_1 \in A^2(X)_{num,\mathbb{Q}}$.

[2]The degree zero part just means the $A^0(X)_{num,\mathbb{Q}}$ term in the representation of the Mukai vector in $A^*(X)_{num,\mathbb{Q}}$.

[3]A coherent sheaf $\mathcal{F}$ is *geometrically stable* if for any base field extension l/k, the pullback $\mathcal{F} \otimes_k l$ along $X_l = X \times_k \mathrm{Spec}(l) \to X$ is stable.

[4]We will denote this moduli space later as $M_X(v)$ to lay emphasis that it is the moduli space of stable sheaves over X.

Definition 2.18 [Relative Moduli Stack] The *relative moduli stack of semi-stable sheaves*, denoted $\mathfrak{M}_h^{ss}$, is defined as follows:

$$\begin{aligned}\mathfrak{M}_h^{ss} :&(Sch/S) \to (\text{groupoids})\\ &T \mapsto \{\mathcal{F}|\mathcal{F}\ T\text{-flat coherent sheaf on } X \times_S T \text{ with semistable fibers}\}.\end{aligned}$$

The *relative moduli stack of stable sheaves* can be defined similarly and we denoted it by $\mathfrak{M}_h^s$.

The following theorem shows the existence of the fine moduli space for the relative moduli stack, when X_R is a relative K3 surface over a mixed characteristic discrete valuation ring, under some numerical conditions. Recall that the condition of flatness is going to be always satisfied in our relative K3's case by definition as they are smooth. The relative stack can be provided to be an algebraic stack using arguments similar to the ones used for proving Theorem 2.13. Moreover, all the results above about the moduli stack hold also for the relative stack. So, there exists a coarse moduli space (Compare from footnote 1 in [36, Chapter 10] or [31, Theorem 4.3.7] , the statement there is actually weaker as we do not ask for morphism of k-schemes, which is not going to be possible for mixed characteristic case. So, for the mixed characteristic case one replaces, in the GIT part of the proof, the quot functor by its relative functor, which is representable in this case as well [56, Theorem 5.1]). Moreover, the non-emptiness results also remain valid in mixed characteristic setting and we have:

Theorem 2.19 (Fine relative Moduli Space) *Let X_V be a relative K3 surface over a mixed characteristic discrete valuation ring V with X as a special fiber over* $\mathrm{Spec}(k)$

1. *Let $v \in A^*(X)_{num,\mathbb{Q}}$ be a primitive element with $v^2 = 0$ (with respect to the Mukai pairing) and positive degree 0 part. Then, $\mathfrak{M}_h^{ss}(v)$, the sub-moduli stack of $\mathfrak{M}_h^{ss}$ with fixed Mukai vector v, is non-empty.*

2. *If, in addition, there is a complex $P \in D(X_V)$ with Mukai vector v' such that $< v, v' >= 1$, then every semistable sheaf with Mukai vector v is locally free and stable, in which case $\mathfrak{M}_h^{ss}(v)$ is a μ_r-gerbe for some r, over a smooth projective surface $M_h(v)$ such that the associated $\mathbb{G}_m$-gerbe is trivial.*

Note that in the mixed characteristic setting, for any complex $E_V \in D^b(X_V)$ we define its Mukai vector to be just the Mukai vector of $E := E_V \otimes_V k$ in $A^*(X)_{num,\mathbb{Q}}$. This definition makes sense as $X_V \to V$ is flat.
With this we conclude our exposition on moduli stacks and spaces of sheaves.

2.3 Derived Equivalences of K3 Surfaces

We now give a summary of selected results on derived equivalences of a K3 surfaces for both positive characteristic and characteristic zero. We begin by a general discussion on derived equivalences and then specialize to different characteristics. Let X be a K3 surface over a field k and let $D^b(X)$ be the bounded derived category of coherent sheaves of X. We refer the reader to [30] for a quick introduction to derived categories and the textbooks [20], [39] for details.

DEFINITION 2.20 Two K3 surfaces X and Y over k are said to be *derived equivalent* if there exists an exact equivalence $D^b(X) \simeq D^b(Y)$ of the derived categories as triangulated categories[5].

DEFINITION 2.21 [Fourier-Mukai Transform] For a perfect complex $\mathcal{P} \in D^b(X \times Y)$, the *Fourier-Mukai transform* is a functor of the derived categories which is defined as follows:

$$\begin{aligned} \Phi_P : D^b(X) &\to D^b(Y) \\ \mathcal{E} &\mapsto \mathbb{R}p_{Y*}((p_X^*\mathcal{E}) \otimes^{\mathbb{L}} \mathcal{P}), \end{aligned}$$

where p_X, p_Y are the projections from $X \times Y$ to the respective X and Y.

For details on the properties of Fourier-Mukai transform see [30, Chapter 5]. Note that not every Fourier-Mukai transform induces an equivalence. The only general enough criteria available to check whether the Fourier Mukai transform induces a derived equivalence is by Bondol-Orlov, see for example, [36, Chapter 16 Lemma 1.4, Proposition 1.6 and Lemma 1.7]. In case the Fourier-Mukai transform is an equivalence, we have the following definition:

DEFINITION 2.22 A K3 surface Y is said to be a *Fourier Mukai partner* of X if there exists a Fourier-Mukai transform between $D^b(X)$ and $D^b(Y)$ which is an equivalence. We denote by $FM(X)$ the set of isomorphism classes of Fourier Mukai Partners of X and by $|FM(X)|$ the cardinality of the set, which is called the *Fourier Mukai number* of X.

We state here the most important result in the theory of Fourier-Mukai transforms and derived equivalences.

THEOREM 2.23 (ORLOV, [30] THEOREM 5.14) *Every equivalence of derived categories for smooth projective varieties is given by a Fourier Mukai transform. More precisely, let X and Y be two smooth projective varieties and let*

$$F : D^b(X) \to D^b(Y)$$

be a fully faithful exact functor. If F admits right and left adjoint functors, then there exists an object $P \in D^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to Φ_P.

[5] We don't need to start with Y being a K3 surface, this can be deduced as a consequence by the existence of an equivalence on the level of derived categories of varieties, see [30, Chapter 4, 6 and 10] and [5, Chapter 2] for the properties preserved by derived equivalences. However, note that Orlov's Representability Theorem 2.23 is used in some proofs.

Remark 2.24 This theorem allows us to restrict the collection of derived equivalences to a smaller and more manageable collection of Fourier-Mukai transforms, which will be studied via cohomological descent.

Any Fourier Mukai transform, Φ_P, descends from the level of the derived categories to various cohomological theories ($H^*_{\cdots}(\ )$), as

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\mathcal{E} \mapsto \mathbb{R}p_{Y*}((\mathbb{L}p_X^*\mathcal{E}) \otimes^{\mathbb{L}} P)} & D^b(Y) \\ \Big\downarrow ch(\ )\sqrt{td_X} & & \Big\downarrow ch(\ )\sqrt{td_Y} \\ H^*(X) & \xrightarrow{\alpha \mapsto p_{Y*}\left((p_X^*\alpha)\cdot ch(P)\sqrt{td_{X\times Y}}\ \right)} & H^*(Y), \end{array}$$

where $ch(\ )$ is the total Chern character and td_X is the Todd genus of X. This descent provides a way to study the Fourier Mukai partners of X using cohomological methods. For details see [30, Section 5.2] and [45, Section 2].

In characteristic 0 (mostly over $\mathbb{C}$, see remark 2.35 below), we will use the singular cohomology along with p/l-adic/étale cohomology and in characteristic $p > 0$, we will use crystalline cohomology or l-adic etale cohomology. In the mixed characteristic setting, we will be frequently using a different combination of cohomologies along with their comparison theorems from p-adic Hodge theory.

Remark 2.25 The Orlov Representability Theorem 2.23 works only for smooth projective varieties, so when we work with relative schemes we will restrict from the collection of derived equivalences and work only with the subcollection of Fourier-Mukai transforms.

Over the field of complex numbers, Mukai and Orlov provide the full description of the set $FM(X)$ as:

Theorem 2.26 (Mukai [53], Theorem 1.4 and Theorem 1.5, [59]) *Let X be a K3 surface over $\mathbb{C}$. Then the following are equivalent:*

1. *There exists a Fourier-Mukai transform $\Phi : D^b(X) \cong D^b(Y)$ with kernel $\mathcal{P}$.*
2. *There exists a Hodge isometry $f : \tilde{H}^*(X,\mathbb{Z}) \to \tilde{H}^*(Y,\mathbb{Z})$, where $\tilde{H}^*(\ ,\mathbb{Z})$ is the singular cohomology of the corresponding analytic space and is compared with the de Rham cohomology of the algebraic variety X which comes with a Hodge filtrations and Mukai pairing* [6]*.*
3. *There exists a Hodge isometry $f : T(X) \simeq T(Y)$ between their transcendental lattices.*

[6] The Mukai pairing is just an extension of the intersection pairing, defined as follows: let $(a_1, b_1, c_1) \in \tilde{H}^*(X,\mathbb{Z})$ and $(a_2, b_2, c_2) \in \tilde{H}^*(X,\mathbb{Z})$, then the Mukai pairing is $< (a_1, b_1, c_1), (a_2, b_2, c_2) >= b_2 \cdot b_1 - a_1 \cdot c_2 - a_2 \cdot c_1 \in H^4(X,\mathbb{Z})$.

4. *Y is a two dimensional fine compact moduli space of stable sheaves on X with respect to some polarization on X, i.e., $Y \cong M_X(v)$ for some Mukai vector $v \in A^*(X)_{num,\mathbb{Q}}$ (cf. Definition 2.11).*

5. *There is an isomorphism of Hodge structures between $H^2(M_X(v),\mathbb{Z})$ and $v^\perp/\mathbb{Z}v$ which is compatible with the cup product pairing on $H^2(M_X(v),\mathbb{Z})$ and the bilinear form on $v^\perp/\mathbb{Z}v$ induced by that on the Mukai lattice $\tilde{H}^*(X,\mathbb{Z})$.*

The following result is the étale version of the Mukai-Orlov cohomological version of decription of derived equivalences of K3 surfaces over $\mathbb{C}$.

PROPOSITION 2.27 (P-ADIC ÉTALE COHOMOLOGY VERSION) *If X and Y are derived equivalent K3 surfaces, then there is an isomorphism between $H^2_{ét}(M_X(v),\mathbb{Z}_p)$ and $v^\perp/\mathbb{Z}_p v$, (see footnote [7]), which is compatible with the cup product pairing on $H^2_{ét}(M_X(v),\mathbb{Z}_p)$ and the bilinear form on $v^\perp/\mathbb{Z}_p v$ induced by that on the Mukai lattice $\tilde{H}^*(X,\mathbb{Z}_p)$, where p is a prime number and $\mathbb{Z}_p$ is the ring of p-adic integers.*

PROOF: This follows from Artin's Comparison Theorem [22, Tome III, Exposé 11, Théorème 4.4] between étale and singular cohomology and the theorem above. □

PROPOSITION 2.28 ([36] PROPOSITION 3.10) *Let X be a complex projective K3 surface, then X has only finitely many Fourier-Mukai partners, i.e.,*

$$|FM(X)| < \infty.$$

REMARK 2.29 The above result is also true for any algebraically closed field of characteristic 0. Indeed, if X and Y are two K3 surfaces over a field K algebraically closed and characteristic 0, we have $X \cong Y \Leftrightarrow X_{\mathbb{C}} \cong Y_{\mathbb{C}}$. One way is obvious via base change and for the other direction we just need to show that every isomorphism $X_{\mathbb{C}} \cong Y_{\mathbb{C}}$ comes from an isomorphism $X \cong Y$. To define an isomorphism only finitely many equations are needed, so we can assume that the isomorphism is defined over A, a finitely generated K-algebra (take A to be the ring $K[a_1,\ldots,a_n]$, where a_i are the finitely many coefficients of the finitely many equations defining our isomorphism). Thus, we have have our isomorphism defined over an affine scheme, $X_A \cong Y_A$, where $X_A := X \times_K \operatorname{Spec}(A)$ (resp. $Y_A := Y \times_K \operatorname{Spec}(A)$). As K is algebraically closed, any closed point $t \in \operatorname{Spec}(A)$ has residue field K. Now taking a K-rational point will give us our required isomorphism.

[7] We are abusing the notation here: The Mukai vector is now considered as an element of $H^*_{ét}(X,\mathbb{Z}_p)$ and $v^\perp$ is the orthogonal complement of v in $H^*_{ét}(X,\mathbb{Z}_p)$ with respect to Mukai pairing. Thus, $v^\perp$ is a $\mathbb{Z}_p$ lattice. Then we mod out this lattice by the $\mathbb{Z}_p$ module generated by v.

This gives us a natural injection:

$$\begin{aligned} FM(X) &\hookrightarrow FM(X_{\mathbb{C}}) \\ Y &\mapsto Y_{\mathbb{C}}. \end{aligned}$$

Hence, we have $|FM(X)| \leq |FM(X_{\mathbb{C}})| < \infty$.

Let $S = NS(X)$ be the Néron-Severi lattice of X. The following theorem gives us the complete counting formula for Fourier-Mukai partners of a K3 surface.

Theorem 2.30 (Counting formula [29]) *Let $\mathcal{G}(S) = \{S_1 = S, S_2, \ldots S_m\}$ be the set of isomorphism classes of lattices with same signature and discriminant as S. Then*

$$|FM(X)| = \sum_{j=1}^{m} |Aut(S_j)\backslash Aut(S_j^*/S_j)/O_{Hdg}(T(X))| < \infty.$$

The relation with the class number $h(p)$ of $\mathbb{Q}(\sqrt{-p})$, for a prime p, is:

Theorem 2.31 ([29] Theorem 3.3) *Let the rank $NS(X) = 2$ for X, a K3 surface, then $\det NS(X) = -p$ for some prime p, and $|FM(X)| = (h(p)+1)/2$.*

Remark 2.32 The surjectivity of period map [36, Theorem 6.3.1] along with [36, Corollary 14.3.1] implies that there exists a K3 with Picard rank 2 and discriminant $-p$, for each prime p (see [29, Remark after Theorem 3.3]).

We now describe the known results about the derived autoequivalence group $Aut(D^b(X))$ for a K3 surface over $\mathbb{C}$. Observe that Theorem 2.26 implies that we have the following natural map of groups:

$$Aut(X) \hookrightarrow Aut(D^b(X)) \to O_{Hdg}(\tilde{H}^*(X,\mathbb{Z})).$$

The following theorem gives a description of the second map:

Theorem 2.33 ([29], [63]) *Let φ be a Hodge isometry of the Mukai lattice $\tilde{H}^*(X,\mathbb{Z})$ of a K3 surface X, i.e. $\varphi \in O_{Hdg}(\tilde{H}^*(X,\mathbb{Z}))$. Then there exists an autoequivalence*

$$\Phi_E : D^b(X) \to D^b(X) \tag{1}$$

with $\Phi_E^H = \varphi \circ (\pm id_{H^2}) : \tilde{H}^*(X,\mathbb{Z}) \to \tilde{H}^*(X,\mathbb{Z})$. In particular, the index of image

$$Aut(D^b(X)) \to O_{Hdg}(\tilde{H}^*(X,\mathbb{Z})) \tag{2}$$

is at most 2.

On the other hand, it has been shown that

THEOREM 2.34 ([32]) *The cone-inversion Hodge isometry $id_{H^0 \oplus H^4} \oplus -id_{H^2}$ on $\tilde{H}^*(X,\mathbb{Z})$ is not induced by any derived auto-equivalence. In particular, the index of image*

$$Aut(D^b(X)) \to O_{Hdg}(\tilde{H}^*(X,\mathbb{Z})) \tag{3}$$

is exactly 2.

REMARK 2.35 [36, 16.4.2] The above results have been shown for K3 surfaces over $\mathbb{C}$ only but the results are valid for K3 surfaces over any algebraically closed field of characteristic 0, in the sense made precise below. The argument goes as follows: We reduce the case of $char(k) = 0$ to the case of $\mathbb{C}$. We begin by making the observation that every K3 surface X over a field k is defined over a finitely generated subfield k_0, i.e., there exists a K3 surface X_0 over k_0 such that $X := X_0 \times_{k_0} k$. Similarly, if $\Phi_P : D^b(X) \to D^b(Y)$ is a Fourier Mukai equivalence, then there exists a finitely generated field k_0 such that X, Y and P are defined over k_0. Moreover, the k_0- linear Fourier-Mukai transform induced by P_0, $\Phi_{P_0} : D^b(X_0) \to D^b(Y_0)$ will again be a derived equivalence (use, for example, the criteria [30, Proposition 7.1] to check this.).
Now assume that k_0 is algebraically closed. Note that any Fourier-Mukai kernel which induces an equivalence $\Phi_{P_0} : D^b(X_0) \xrightarrow{\sim} D^b(X_0)$ is rigid, i.e. $\mathrm{Ext}^1(P_0, P_0) = 0$ (see [36, Proposition 16.2.1]), thus any Fourier-Mukai equivalence

$$\Phi_P : D^b(X_0 \times_{k_0} k) \xrightarrow{\sim} D^b(X_0 \times_{k_0} k)$$

descends to k_0 (see for example [36, Lemma 17.2.2] for the case of line bundles, the general case follows similarly[8]). Hence, for a K3 surface X_0 over the algebraic closure k_0 of a finitely generated field extension of $\mathbb{Q}$ and for any choice of an embedding $k_0 \hookrightarrow \mathbb{C}$, which always exists, one has

$$Aut(D^b(X_0 \times_{k_0} k)) \cong Aut(D^b(X_0)) \cong Aut(D^b(X_0 \times_{k_0} \mathbb{C})).$$

In this sense, for K3 surfaces over algebraically closed fields k with $char(k) = 0$, the situation is identical to the case of complex K3 surfaces.

We can now write down the following exact sequence: For X a projective complex $K3$ surface one has

$$0 \to \mathrm{Ker} \to \mathrm{Aut}(D^b(X)) \to \mathrm{O_{Hdg}}(\tilde{H}^*(X,\mathbb{Z}))/\{\mathfrak{i}\} \to 0, \tag{4}$$

[8]In the general case we sketch the proof: Use the moduli stack of simple universally gluable perfect complexes over $X_0 \times X_0/k_0$, denoted $s\mathcal{D}_{X_0 \times X_0/k_0}$, as defined in [45, Section 5]. From the arguments following the definition, it is an algebraic stack which admits a coarse moduli algebraic space $sD_{X_0 \times X_0/k_0}$. Note that for any k_0 point P_0 which induces an equivalence, the local dimension of the coarse moduli space is zero as the tangent space is a subspace of $\mathrm{Ext}^1(P_0, P_0) = 0$ (see, for example, [43, 3.1.1] or proof of [45, Lemma 5.2]) and the coarse moduli space is also smooth. The smoothness follows from the fact that the deformation of the complex is unobstructed (see, for example, [1, Tag 03ZB and Tag 02HX]) in equi-characteristic case as one always has a trivial deformation. Indeed, let A be any Artinian local k-algebra, then pullback along the structure morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ gives a trivial deformation of $X \times X$ and also a trivial deformation of any complex on $X \times X$. Thus, we can repeat the argument as in [36, Lemma 17.2.2] as now the image of the classifying map $f : \mathrm{Spec}(A) \to sD_{X_0 \times X_0/k_0}$ is constant (In the notation of [36, Lemma 17.2.2]).

where $\tilde{H}^*(X, \mathbb{Z})$ is the cohomology lattice with Mukai pairing and extended Hodge structure, and $\mathrm{O}_{\mathrm{Hdg}}(-)$ is the group of Hodge isometries, $\mathfrak{i}$ is the cone inversion isometry $\mathrm{Id}_{H^0 \oplus H^4} \oplus -\mathrm{Id}_{H^2}$.

Remark 2.36 The structure of the kernel of this map has been described only in the special case of a projective complex $K3$ surface with $\mathrm{Pic}(X) = 1$ in [6]. (For a discussion about the results in non-projective case see [33].) However, Bridgeland in [14, Conjecture 1.2] has conjectured that this kernel can be described as the fundamental group of an open subset of $H^{1,1} \otimes \mathbb{C}$. Equivalently, the conjecture says that the connected component of the stability manifold (see [13], [14] for the definitions) associated to the collection of the stability conditions on $D^b(X)$ covering an open subset of $H^{1,1} \otimes \mathbb{C}$ is simply connected. The equivalence of the two formulations follows from a result of Bridgeland ([14, Theorem 1.1]), which states that the kernel acts as the group of deck transformations of the covering of an open subset of $H^{1,1} \otimes \mathbb{C}$ by a connected component of the stability manifold. Bayer and Bridgeland [6] have verified the conjecture in the special cases of $\mathrm{Pic}(X) = 1$ (see [33] for the non-projective case).

Lastly, we state the main results on derived equivalences of K3 surfaces over an algebraically closed field of positive characteristic known so far. For generalizations of some results to non-algebraically closed fields of positive characteristic see [69].
In case, $char(k) = p > 2$, Lieblich-Olsson [45], proved the following:

Theorem 2.37 ([45], Theorem 1.1) *Let X be a K3 surface over an algebraically closed field k of positive characteristic $\neq 2$.*

1. *If Y is a smooth projective k-scheme with $D^b(X) \cong D^b(Y)$, then Y is a K3 surface isomorphic to a fine moduli space of stable sheaves.*
2. *There exists only finitely many smooth projective k-schemes Y with $D^b(X) \cong D^b(Y)$. If X has rank $NS(X) \geq 12$, then $D^b(X) \cong D^b(Y)$ implies that $X \cong Y$. In particular, any supersingular K3 surface is determined up to isomorphism by its derived category.*

Remark 2.38 One of the open questions is to have a cohomological criteria for derived equivalent K3 surfaces over a field of positive characteristic like we have in characteristic 0 where Hodge theory and Torelli Theorems were available. However, as there is no crystalline Torelli Theorem for non-supersingular K3 surfaces over a field of positive characteristic and the naive F-crystal (see Appendix) fails to be compatible with inner product, the description in terms of F-crystals is not yet possible. Even though one has crystalline Torelli Theorem for supersingular K3 surfaces, it is essentially not providing any more information as there are no non-trivial Fourier-Mukai partners of a supersingular K3 surface. However, Lieblich-Olsson proved a derived Torelli Theorem using the Ogus Crystalline Torelli Theorem [62], see [46, Theorem 1.2].

2.4 Height is a Derived Invariant

Let us already show here that height of a K3 surface is a derived invariant. This will allow us to stay within a subclass of K3 surfaces while checking derived equivalences.

Lemma 2.39 *Height of a K3 surface X over an algebraically closed field of characteristic $p > 3$ is a derived invariant.*

Proof: Recall that the height of a K3 surface X is given by the dimension of the subspace $H^2_{crys}(X/K)_{[0,1)}$ of the F-isocrystal $H^2_{crys}(X/K)$. Now note that the Frobenius acts on the one dimensional isocrystals $H^0_{crys}(X/K)(-1)$ and $H^4_{crys}(X/K)(1)$ (Tate twisted) as multiplication by p (see Appendix below for this computation). This implies that the slope of these F-isocrystals is exactly one. Thus, the F-isocrystal

$$H^*_{crys}(X/K) := H^0_{crys}(X/K)(-1) \oplus H^2_{crys}(X/K) \oplus H^4_{crys}(X/K)(1)$$

has the same subspace of slope of dimension strictly less than one as that of the F-isocrystal $H^2_{crys}(X/K)$, i.e., $H^*_{crys}(X/K)_{[0,1)} = H^2_{crys}(X/K)_{[0,1)}$.
Note that any derived equivalence of X and Y preserves the F-isocrystal $H^*_{crys}(-/K)$, i.e., if $\Phi_P : D^b(X) \simeq D^b(Y)$ is a derived equivalence of two K3 surfaces X and Y, then the induced map on the F-isocrystals

$$\Phi^*_P : H^*_{crys}(X/K) \to H^*_{crys}(Y/K)$$

is an isometry. Thus, for the height of Y given by $\dim(H^2_{crys}(Y/K)_{[0,1)})$ we have

$$\begin{aligned}\dim(H^2_{crys}(Y/K)_{[0,1)}) &= \dim(H^*_{crys}(Y/K)_{[0,1)}) \\ &= \dim(H^*_{crys}(X/K)_{[0,1)}) \\ &= \dim(H^2_{crys}(X/K)_{[0,1)}) = \text{height of } X\end{aligned}$$

Hence the result. □

Remark 2.40
1. In characteristic 0, there is no notion of height but in this case the Brauer group itself is a derived invariant of a K3 surface, as $Br(X) \cong Hom(T(X), \mathbb{Q}/\mathbb{Z})$, where $T(X)$ is the transcendental lattice.
2. On the other hand, the Picard lattice is not a derived invariant in any characteristic, though it trivially remains invariant in the case of K3 surfaces which do not have non-trivial Fourier-Mukai partners.

3 Derived Autoequivalences of K3 Surfaces in Positive Characteristic

In this section, we compare the deformation of an automorphism as a morphism and as a derived autoequivalence and show that for K3 surfaces these deformations are in one-to-one correspondence. Then we discuss Lieblich-Olsson's results on lifting derived autoequivalences. Then we use these lifting results to prove results on the structure of the group of derived autoequivalences of a K3 surface of finite height over a field of positive characteristic.

3.1 Obstruction to Lifting Derived Autoequivalences

Let X be a projective variety over an algebraically closed field k of positive characteristic p, $W(k)$ its ring of Witt vectors and $\sigma : X \to X$ an automorphism of X. Then this automorphism will induce an equivalence of derived categories

$$\sigma^* : D^b(X) \xrightarrow{\cong} D^b(X)$$

and it is easy to check that derived equivalence σ^* is also represented by the Fourier-Mukai transform $\Phi_{\mathcal{O}_{\Gamma(\sigma)}}$, where $\mathcal{O}_{\Gamma(\sigma)}$ is the pushforward of the structure sheaf of the graph of σ to $X \times X$ and is considered as a coherent sheaf in $D^b(X \times X)$. This representation of σ provides us with another way of deforming it as a perfect complex in $D^b(X \times X)$, other than just as a morphism. A priori these two ways of deformations are not equivalent but for K3 surfaces they turn out to be so and we will exploit this equivalence of deformations later to prove that every automorphism of ordinary K3 surfaces lift.

We begin by recalling the classical result that for a variety the infinitesimal deformation of a closed sub-variety with a vanishing $H^1(X, \mathcal{O}_X)$ as a closed subscheme is determined by the deformation of its (pushforward of) structure sheaf as a coherent sheaf on $X \times X$. We then use this result to show that on a K3 surface we can lift an automorphism as a automorphism if and only if we can lift it as a perfect complex in the derived category.

Remark 3.1 For a K3 surface this result can also be seen using [45, Proposition 7.1] and the p-adic criterion of lifting automorphisms on K3 surfaces [19, Remark 6.5].

Let X and σ be as above.

Definition 3.2 For any Artin local $W(k)$-algebra A with residue field k, an *infinitesimal deformation of X over A* is a proper and flat scheme X_A over A such that the following square is cartesian:

$$\begin{array}{ccc} X & \longrightarrow & X_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

In case X is smooth, we ask X_A to be smooth over A as well. In this case, X_A is automatically flat over A.

Consider the following two deformation functors:

$$\begin{aligned} F_{aut} :& (\text{Artin local } W(k)\text{-algebras with residue field } k) \to (Sets) \\ & A \mapsto \{\text{Lifts of automorphism } \sigma \text{ to } A\}, \end{aligned} \tag{5}$$

where by lifting of automorphism σ over A we mean that there exists an infinitesimal deformation X_A of X and an automorphism $\sigma_A : X_A \to X_A$ which reduces to σ, i.e., we have the following commutative diagram:

$$\begin{array}{ccc} X_A & \xrightarrow{\sigma_A} & X_A \\ \uparrow & & \uparrow \\ X & \xrightarrow{\sigma} & X. \end{array}$$

This is the *deformation functor of an automorphism as a morphism*. Now consider the *deformation functor of an automorphism as a coherent sheaf* defined as follows:

$$\begin{aligned} F_{coh} :& (\text{Artin local } W(k)\text{-algebras with residue field } k) \to (Sets) \\ & A \mapsto \{\text{Deformations of } \mathcal{O}_{\Gamma(\sigma)} \text{ to } A\}/iso, \end{aligned} \tag{6}$$

where by deformations of $\mathcal{O}_{\Gamma(\sigma)}$ to A we mean that there exists an infinitesimal deformation Y_A of $Y := X \times X$ over A and a coherent sheaf $\mathcal{F}_A$, which is a deformation of the coherent sheaf $\mathcal{O}_{\Gamma(\sigma)}$ and $\mathcal{O}_{\Gamma(\sigma)}$ is considered as a coherent sheaf on $X \times X$ via the closed embedding $\Gamma(\sigma) \hookrightarrow X \times X$. Isomorphisms are defined in the obvious way.

Remark 3.3 Note that there are more deformations of $X \times X$ than the ones of the shape $X_A \times_A X'_A$, where X_A and X'_A are deformations of X over A. From now we make a choice of this deformation (Y_A) to be $X_A \times X_A$. Also see Remark [3.14] and compare from Theorem [3.7] and Remark [3.15] below.

Let X be a smooth projective scheme over k and for A an Artin local $W(k)$-algebra assume that there exists an infinitesimal lift of X to X_A. Note that such a lift may not always exist but for the case of K3 surfaces of finite height it does, see [47, Corollary 4.2] and Theorem 2.7. However, for supersingular K3 surfaces, the lift does not exists over all Artin local rings but in some cases it does exist by Theorem 2.7. Observe that there is a natural transformation $\eta : F_{aut} \to F_{coh}$ given by

$$\begin{aligned} \eta_A : F_{aut}(A) &\longrightarrow F_{coh}(A) \\ (\sigma_A : X_A \to X_A) &\mapsto \mathcal{O}_{\Gamma(\sigma_A)}/X_A \times X_A. \end{aligned} \tag{7}$$

Theorem 3.4 *The natural transformation $\eta : F_{aut} \to F_{coh}$ between the deformation functors is an isomorphism for varieties with $H^1(X, \mathcal{O}_X) = 0$.*

We provide an algebraic proof by constructing a deformation-obstruction long exact sequence connecting the two functors. The proof follows from the following more general proposition 3.6, substituting $X \times X$ for Y and taking the embedding i to be the graph of the automorphism σ. To use proposition 3.6 we need the following lemma.

LEMMA 3.5 (CF. [27] LEMMA 24.8) *To give an infinitesimal deformation of an automorphism $f : X \to X$ over X_A it is equivalent to give an infinitesimal deformation of the graph Γ_f as a closed subscheme of $X \times X$.*

PROOF: To any deformation f_A of f we associate its graph Γ_{f_A}, which gives a closed subscheme of $X_A \times X_A$. It is an infinitesimal deformation of Γ_f. Conversely, given a deformation Z of Γ_f over A, the projection $p_1 : Z \hookrightarrow X_A \times_A X_A \to X_A$ gives an isomorphism after tensoring with k. From flatness (see, for example, EGA IV, Corollary 17.9.5) of Z over A it follows that p_1 is an isomorphism, and so Z is the graph of $f_A = p_2 \circ p_1^{-1}$. □

The following proposition is certainly known to the experts but we were unable to find a proof in literature, so we wrote one for reader's convenience.

PROPOSITION 3.6 *(Cf. [27, Ex 19.1]) Let $i : X \hookrightarrow Y$ be a closed embedding with X integral and projective scheme of finite type over k. Then there exists a long exact sequence*

$$0 \to H^0(\mathcal{N}_X) \to Ext^1_Y(\mathcal{O}_X, \mathcal{O}_X) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{N}_X) \to Ext^2_Y(\mathcal{O}_X, \mathcal{O}_X) \to \dots, \tag{8}$$

where $\mathcal{N}_X$ is the normal bundle of X.

PROOF: Consider the short exact sequence given by the closed embedding i

$$0 \to I \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0. \tag{9}$$

Apply the global Hom contravariant functor $\mathrm{Hom}_Y(-, i_*\mathcal{O}_X)$ to the above short exact sequence and we get the following long exact sequence from [26, III Proposition 6.4],

$$0 \to \mathrm{Hom}_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \mathrm{Hom}_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) \to \mathrm{Hom}_Y(I, i_*\mathcal{O}_X) \to \mathrm{Ext}^1_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \mathrm{Ext}^1_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) \to \mathrm{Ext}^1_Y(I, i_*\mathcal{O}_X) \to \mathrm{Ext}^2_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \dots.$$

Now note that we can make the following identifications

1. $\mathrm{Hom}_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \cong k$ as X is integral and projective.
2. $\mathrm{Hom}_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) = H^0(\mathcal{O}_X) = k$ using [26, III Propostion 6.3 (iii), Lemma 2.10] and the fact that X is connected.

3. As any injective endomorphism of a field is an automorphism, we can modify the long exact sequence as follows:

$$0 \to \mathrm{Hom}_Y(I, i_*\mathcal{O}_X) \to \mathrm{Ext}^1_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \mathrm{Ext}^1_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) \to \dots .$$

4. $\mathrm{Hom}_Y(I, i_*\mathcal{O}_X) \cong \mathrm{Hom}_X(i^*I, \mathcal{O}_X)$ using ajunction formula on page 110 of [26]. Moreover, using [26, III, Proposition 6.9], we have

$$\mathrm{Hom}_X(i^*I, \mathcal{O}_X) = \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{H}om_X(i^*I, \mathcal{O}_X)),$$

and using the discussion in [1, Tag 01R1], we have $\mathcal{H}om_X(i^*I, \mathcal{O}_X) = \mathcal{N}_X$. Thus, putting this together with [26, III Proposition 6.3 (iii) and Lemma 2.10], we get

$$\mathrm{Hom}_Y(I, i_*\mathcal{O}_X) \cong H^0(\mathcal{N}_X).$$

5. Note that again using [26, III Proposition 6.3 (iii) and Lemma 2.10], we get

$$\mathrm{Ext}^1_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) \cong H^1(\mathcal{O}_X).$$

6. Note that using the adjunction for Hom sheaves we have:

$$i_*\mathcal{N}_X = i_*\mathcal{H}om_X(i^*I, \mathcal{O}_X) \cong \mathcal{H}om_Y(I, i_*\mathcal{O}_X).$$

Thus, $H^1(\mathcal{N}_X) := H^1(X, \mathcal{N}_X) = H^1(Y, i_*\mathcal{N}_X)$ using [26, III Lemma 2.10]. To compute $H^1(Y, i_*\mathcal{N}_X)$, we choose an injective resolution of $i_*\mathcal{O}_X$ as an $\mathcal{O}_Y$-module $0 \to \mathcal{O}_X \to \mathcal{J}^\bullet$. From [23, Proposition 4.1.3], we know that $\mathcal{H}om_Y(I, \mathcal{J}^i)$ are flasque sheaves and so we can compute the cohomology group using this flasque resolution. Hence,

$$H^i = \frac{Ker(\mathrm{Hom}_Y(I, \mathcal{J}^i) \to \mathrm{Hom}_Y(I, \mathcal{J}^{i+1}))}{Im(\mathrm{Hom}_Y(I, \mathcal{J}^{i-1}) \to \mathrm{Hom}_Y(I, \mathcal{J}^i))} = \mathrm{Ext}^i_Y(I, i_*\mathcal{O}_X).$$

Thus, putting all of the above observations together, we get our required long exact sequence. □

Proof: [Proof of Theorem 3.4:] Note that the obstruction spaces for the functors F_{aut} and F_{coh} are $H^1(\mathcal{N}_X)$ and $\mathrm{Ext}^2_Y(\mathcal{O}_X, \mathcal{O}_X)$ respectively. See, for example, [27, Theorem 6.2, Theorem 7.3] and Lemma 3.5 above. The same results give us the tangent spaces for the functors F_{aut} and F_{coh} and they are $H^0(\mathcal{N}_X)$ and $\mathrm{Ext}^1_Y(\mathcal{O}_X, \mathcal{O}_X)$. Now using Proposition 3.6 along with our assumption of vanishing $H^1(X, \mathcal{O}_X)$ one has that the obstruction space of F_{aut} is a subspace of the obstruction of F_{coh} and this inclusion sends one obstruction class to the other one. Therefore, the obstruction to lifting the automorphism as a morphism vanishes if and only if the obstruction to lifting the automorphism as a sheaf vanishes. Moreover, the isomorphism of tangent spaces implies that the number of lifts in both cases is same. □

This shows that for projective varieties with vanishing $H^1(X, \mathcal{O}_X)$, one doesn't have extra deformations of automorphisms as a sheaf. Note that we could still ask for deformations as a perfect complex but since the perfect complex we start with is a coherent sheaf any deformation of it as a perfect complex will also have only one non-zero coherent cohomology sheaf. Indeed, this follows from the fact that deformations cannot grow cohomology sheaves ,as if $F_A^\bullet$ is the deformation of $\mathcal{O}_X$ over A such that $H^1(F_A^\bullet) \neq 0$ (to simplify our argument we are assuming $F_A^\bullet$ is bounded above at level 1, i.e., $F_A^i = 0 \; \forall i > 1$), then we can replace this complex in the derived category by a complex like

$$\ldots \to F_A^{-1} \to Ker(F_A^0 \to F_A^1) \xrightarrow{0} H^1(F_A^\bullet) \to 0.$$

Then reducing to special fiber gives that $H^1(F_A^\bullet) \otimes_A k = 0$, but this will only happen if $H^i(F_A^\bullet) = 0$. Moreover, as we are in the derived category, we can show that the deformed perfect complex is then quasi isomorphic to a coherent sheaf. Indeed, the quotient map to the non-zero coherent cohomology sheaf provides the quasi-isomorphism. This shows that there are no extra deformations as a perfect complex as well. Hence, an automorphism σ on a projective variety X with vanishing $H^1(X, \mathcal{O}_X)$ lifts if and only if the derived equivalence it induces, $\Phi_{\mathcal{O}_{\Gamma(\sigma)}} : D^b(X) \to D^b(X)$, lifts as a Fourier-Mukai transform.
Now we state the two theorems proved by Lieblich-Olsson which give a criteria to lifting perfect complexes.

Theorem 3.7 ([45] Theorem 6.3) *Let X and Y be two K3 surfaces over an algebraically closed field k, and $P \in D^b(Y \times X)$ be a perfect complex inducing an equivalence $\Phi : D^b(Y) \to D^b(X)$ on the derived categories. Assume that the induced map on cohomology (see below) satisfies:*

1. $\Phi(1, 0, 0) = (1, 0, 0)$,
2. *the induced isometry $\kappa : Pic(Y) \to Pic(X)$ sends C_Y, the ample cone of Y, isomorphically to either C_X or $-C_X$, the $(-)$ample cone of X.*

Then there exists an isomorphism of infinitesimal deformation functors $\delta : Def_X \to Def_Y$ such that

1. $\delta^{-1}(Def_{(Y,L)}) = Def_{(X,\Phi(L))}$;
2. *for each augmented Artinian W-algebra $W \to A$ and each $(X_A \to A) \in Def_X(A)$, there is an object $P_A \in D^b(\delta(X_A) \times_A X_A)$ reducing to P on $Y \times X$.*

Theorem 3.8 ([45], Theorem 7.1) *Let k be a perfect field of characteristic $p > 0$, W be the ring of Witt vectors of k, and K be the field of fractions of W. Fix K3 surfaces X and Y over k with lifts X_W/W and Y_W/W. These lifts induce corresponding Hodge filtrations via de Rham cohomology on the crystalline cohomology of the special fibers. Denote by $F^1_{Hdg}(X) \subset H^2(X/K) \subset H^*(X/K)$*

and $F^1_{Hdg}(Y) \subset H^2(Y/K) \subset H^*(Y/K)$ *(similarly for* $F^2_{Hdg}(-)$*), where* $H^*(X/K)$ *and* $H^*(Y/K)$ *are the corresponding Mukai F-isocrystals. Suppose that* $P \in D^b(X \times Y)$ *is a kernel whose associated functor* $\Phi : D^b(X) \to D^b(Y)$ *is fully faithful. If*

$$\Phi : H^*(X/K) \to H^*(Y/K)$$

sends $F^1_{Hdg}(X)$ *to* $F^1_{Hdg}(Y)$ *and* $F^2_{Hdg}(X)$ *to* $F^2_{Hdg}(Y)$*, then* P *lifts to a perfect complex* $P_W \in D^b(X_W \times_W Y_W)$*.*

REMARK 3.9 Note that however, infinitesimally the hodge filtration is not preserved. We have the same counterexamples as in the case of infinitesimal integral variational Hodge conjecture: take a line bundle such that $\mathcal{L}^{\otimes p} \neq \mathcal{O}_X$, then we have the Chern character of $\mathcal{L}^{\otimes p}$ is 0 as $p.ch(\mathcal{L}) = 0$, so it lies in the correct Hodge level, but it need not lift. For example: see [10, Lemma 3.10].

REMARK 3.10 Note that the lifted kernel also induces an equivalence. Indeed, for a K3 surface fully faithful Fourier-Mukai functor of derived categories is an equivalence (see [30, Proposition 7.6]) and so we can also lift the Fourier-Mukai kernel of the inverse equivalence. Then the composition of the equivalence we started with and its inverse will give us a lift of the identity as an derived autoequivalence. But using the fact that the $\mathrm{Ext}^1_{X\times X}(P,P) = 0$ (see [46, Lemma 3.7 (ii)]) for any kernel inducing an equivalence, we get that the lift of the identity is unique and is the identity itself. Thus, the lifted Fourier-Mukai functor is an equivalence.

COROLLARY 3.11 *Take* P *to be* $\mathcal{O}_{\Gamma(\sigma)}$*, where* $\sigma : X \to X$ *is an automorphism of a K3 surface* X *over* k*. Then the following are equivalent*

1. *P lifts to an autoequivalence of $D^b(X_W)$*
2. *σ lifts to an automorphism of X_W*
3. *$\Phi_P : H^*(X/K) \to H^*(X/K)$ preserves the Hodge filtration.*

However, we see that we can still lift it as an isomorphism as follows:

THEOREM 3.12 (WEAK LIFTING OF AUTOMORPHISMS) *Let* $\sigma : X \to X$ *be an automorphism of a K3 surface* X *defined over an algebraically closed field* k *of characteristic* p*. There exists a smooth projective model* X_R/R*, where* R *is a discrete valuation ring that is a finite extension of* $W(k)$*, with* X_K *its generic fiber such that there is a* P_R*, a perfect complex in* $D^b(X_R \times Y_R)$*, reducing to* $\mathcal{O}_{\Gamma(\sigma)}$ *on* $X \times X$*, where* Y_R *is another smooth projective model abstractly isomorphic to* X_R *(see Remark [3.13] below).*

PROOF: We divide the proof into 3 steps:

1. Lifting Kernels Infinitesimally: Note that $\Phi_{\mathcal{O}_{\Gamma(\sigma)}}$ is a strongly filtered derived equivalence, i.e.,

$$\Phi^*_{\mathcal{O}_{\Gamma(\sigma)}} = \sigma^* : H^i_{crys}(X/W) \xrightarrow{\sim} H^i_{crys}(X/W)$$

is an isomorphism which preserves the gradation of crystalline cohomology. Choose a projective lift of X to characteristic zero along with a lift of H_X. It always exists as proved by Deligne [16], i.e., a projective lift (X_V, H_{X_V}) of (X, H_X) over V a discrete valuation ring, which is a finite extension of $W(k)$, the Witt ring over k. Let $V_n := V/\mathfrak{m}^n$ for $n \geq 1$, $\mathfrak{m}$ the maximal ideal of V and let K denote the fraction field of V. Then, for each n, using the lifting criterion above, there exists a polarized lift $(X'_n, H_{X'_n})$ over V_n and a complex $P_n \in D_{Perf}(X_n \times X'_n)$ lifting $\mathcal{O}_{\Gamma(\sigma)}$.

2. Applying the Grothendieck Existence Theorem for perfect complexes: By the classical Grothendieck Existence Ttheorem [26, II.9.6], the polarized formal scheme $(\varprojlim X'_n, \varprojlim H_{X'_n})$ is algebraizable. So, there exists a projective lift $(X', H_{X'})$ over V that is the formal completion of $(X'_n, H_{X'_n})$. Now using the Grothendieck Existence Theorem for perfect complexes (see [43, Proposition 3.6.1]) the formal limit of (P_n) is algebraizable and gives a complex $P_V \in D_{Perf}(X_V \times X'_V)$. In particular, P_V lifts $\mathcal{O}_{\Gamma(\sigma)}$ and using Nakayama's lemma, P_V induces an equivalence.

3. Now apply the global Torelli Theorem to show that the two models are isomorphic: For any field extension K' over K, the generic fiber complex $P_{K'} \in D^b(X_{K'} \times X'_{K'})$ induces a Fourier-Mukai equivalence $\Phi_{P_{K'}} : D(X_{K'}) \to D(X'_{K'})$. Using Bertholet-Ogus isomorphisms [10], we see that $\Phi_{K'}$ preserves the gradation on de Rham cohomology of $X_{K'}$. Fix an embedding of $K' \hookrightarrow \mathbb{C}$ gives us a filtered Fourier Mukai equivalence

$$\Phi_{P_{\mathbb{C}}} : D^b(X_{K'} \times \mathbb{C}) \to D^b(X'_{K'} \times \mathbb{C}),$$

which in turn induces an Hodge isometry of integral lattices:

$$H^2(X_{K'} \times \mathbb{C}, \mathbb{Z}) \xrightarrow{\sim} H^2(X'_{K'} \times \mathbb{C}, \mathbb{Z}),$$

using Theorem 2.26 and the fact that a filtered equivalence preserves the grading. This implies that $X_{K'} \times \mathbb{C} \cong X'_{K'} \times \mathbb{C}$, which after taking a finite extension V' of V gives that the generic fiber are isomorphic $X_{K'} \cong X'_{K'}$ (we abuse notation to still denote the fraction field of V' by K'). And since the polarization was lifted along, this gives actually a map of polarized K3 surfaces denoted by $f_{K'} : (X_{K'}, H_{X_{K'}}) \xrightarrow{\sim} (X'_{K'}, H_{X'_{K'}})$.

Now we can conclude that the the generic fibers are isomorphic as well by forgetting the polarization. So now we need to show that the models are isomorphic, i.e., $X_{V'} \cong X'_{V'}$, which will follow from [54, Theorem 2]. □

REMARK 3.13 Note that even though the generic fibers are isomorphic which indeed implies that the models are abstractly isomorphic (via the Matsusaka-Mumford Theorem) but not as models of the special fiber as the isomorphism will not be the identity on the special fiber, just for the simple reason that we started with different polarizations on the special fibers.

REMARK 3.14 This dependence on the choice of the lift X_A of X and the ability to find another lift Y_A can be seen as a reformulation of the formula stated in [35, Theorem on Page 2].

REMARK 3.15 The above results can be rephrased to say that in the moduli space of lifts of $X \times X$ we cannot always deform the automorphism in the direction of $X_A \times X_A$ but can do so always in the direction of some $X_A \times Y_A$, where X_A and the automorphism determine Y_A uniquely.

Next, we discuss the structure of the derived autoequivalence group of a K3 surface of finite height.

3.2 The Cone Inversion Map

Let X be a K3 surface over k of finite height with $char(k) = p > 3$.

DEFINITION 3.16 The *positive cone* $\mathcal{C}_X \subset NS(X)_{\mathbb{R}}$ is the connected component of the set $\{\alpha \in NS(X) | (\alpha)^2 > 0\}$ that contains one ample class (or equivalently, all of them).

DEFINITION 3.17 [Cone Inversion map] Let $\mathcal{C}_X$ be the positive cone, the *cone inversion map* on the cohomology is the map that sends the positive cone $\mathcal{C}_X$ to $-\mathcal{C}_X$.

Explicitly, in characteristic 0, we define the map to be $(-id_{H^2}) \oplus id_{H^0 \oplus H^4}$: $\tilde{H}^*(X, \mathbb{Z}) \to \tilde{H}^*(X, \mathbb{Z})$, where $\tilde{H}^*(X, \mathbb{Z})$ is the Mukai lattice ([30, Section 10.1]). Note that the cone inversion map is a Hodge isometry. In characteristic $p > 3$, we define the map to be $(-id_{H^2}) \oplus id_{H^0 \oplus H^4} : H^*_{crys}(X/K) \to H^*_{crys}(X/K)$, where $H^*_{crys}(X/K)$ is the Mukai F-isocrystal (see appendix A). Note that the cone inversion map preserves the Hodge Filtration on $H^2_{crys}(X/K)$.
(In characteristic 0, the following proposition is proved in [32] with the Mukai F-crystal replaced with Mukai lattice.).

THEOREM 3.18 *The image of $Aut(D^b(X))$ in $Aut(\mathcal{H}^*_{crys}(X/K))$ has index at least 2, where $\mathcal{H}^*_{crys}(X/K)$ is the Mukai F-isocrystal.*

We prove the above proposition by showing that the cone inversion map on the cohomology does not come from any derived auto-equivalence. The proof is done by contradiction, we assume that such an auto-equivalence exists, then lift the kernel of the derived auto-equivalence to char 0, and then we use the results of [32], to get a contradiction that this does not happen.
Recall that we have the following diagram of descend to cohomology of a Fourier-Mukai transform Φ_P, for $P \in D^b(X \times Y)$:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\mathcal{E} \mapsto \mathbb{R}p_{Y*}(p_X^*\mathcal{E}) \otimes^{\mathbb{L}} P)} & D^b(X) \\ \big\downarrow ch(\ ) & & \big\downarrow ch(\ ) \\ CH^*(X) & \longrightarrow & CH^*(X) \\ \big\downarrow \circ\sqrt{td_X} & & \big\downarrow \circ\sqrt{td_Y} \\ H^*(X) & \xrightarrow{\alpha \mapsto p_{Y*}((p_X^*\alpha)\cdot ch(P)\sqrt{td_{X\times Y}}\ )} & H^*(X), \end{array}$$

where $ch(-)$ is the Chern character and td_- is the Todd genus.

Proof: [Proof of Theorem 3.18] Assume that the cone inversion map is induced by a derived auto-equivalence. Then using Orlov's representability Theorem ([59], [60]), we know that this derived auto-equivalence is a Fourier-Mukai transform and we denote the kernel of the transform by $\mathcal{E}$. Since $\mathcal{E}$ induces the cone inversion map and this map preserves the Hodge filtration on the crystalline cohomology, using Theorem 3.8, we know that we can lift the perfect complex $\mathcal{E}$ to a perfect complex $\mathcal{E}_W$ in $D^b(X_W \times X_W)$, where X_W is the lift of X as in [47, Corollary 4.2]. Note that the lifted complex also induces a derived equivalence. Indeed, using Nakayama's lemma we see that the adjunction maps $\Delta_*\mathcal{O}_{X_W} \to \mathcal{E}_W \circ \mathcal{E}_W^\vee$ and $\mathcal{E}_W \circ \mathcal{E}_W^\vee \to \Delta_*\mathcal{O}_{Y_W}$ are quasi-isomorphisms. Moreover, since we have $H^*_{crys}(X/W) \cong H^*_{DR}(X_W/W)$, we know that the lifted complex induces again the cone inversion map on the cohomology. It also follows that for any field extension K'/K, the generic fiber complex $\mathcal{E}_{K'} \in D^b(X_{K'} \times_{K'} X_{K'})$ induces a Fourier Mukai equivalence $\Phi : D^b(X_{K'}) \to D^b(X_{K'})$. Choosing an embedding $K \hookrightarrow \mathbb{C}$ (see our conventions [1.1]) yields a Fourier-Mukai equivalence $D^b(X_K \otimes \mathbb{C}) \to D^b(X_K \otimes \mathbb{C})$ which induces the cone inversion map on $\tilde{H}^*(X, \mathbb{Z})$. This is a contradiction as in characteristic zero this does not happen, see [32] for a proof. □

We now make an interesting observation about the kernel of the map:

Corollary 3.19 *Let X be a K3 surface over k, an algebraically closed field of positive characteristic. Then the kernel of the natural map*

$$0 \to Ker \to Aut(D^b(X)) \to Aut(H^*_{crys}(X/K))$$

lifts. More precisely, assume that X_V be a lift of X over V, a mixed characteristic discrete valuation ring with residue field k, then every derived autoequivalence in the kernel of the map above lifts as an autoequivalence of the derived category of X_V.

Proof: This is clear as any autoequivalence in the kernel induces the identity automorphism on the cohomology which is bound to respect every Hodge filtration on the F-isocrystal and then we use Theorem 3.8. □

This allows us to give at least an upper bound on the kernel as follows: Let X be a K3 surface over an algebraically closed field of characteristic $p > 2$.

Choose a lift of X, denoted as X_R, such that the Picard rank of the geometric generic fiber is 1. There always exists such a lift as shown by Esnault-Oguiso in [19, Theorem 4.1].
Let $\Phi_P : D^b(X) \to D^b(X)$ be a Fourier-Mukai autoequivalence induced by $P \in D^b(X \times X)$ that belong to the kernel of the natural map

$$Aut(D^b(X)) \to Aut(H^*_{crys}(X/K)).$$

We will denote the kernel of this map as Ker_X. Now using [46, Lemma 3.7 (ii)] we see that the set of infinitesimal deformations of the kernel P is a singleton set, which in turn implies that the lift of P to $X_R \times X_R$ (this was just the corollary 3.19) is unique.
Next, note that the fiber of the lift of P over the geometric generic point of R, denoted as $P_{\bar{K}}$, also belongs to the kernel of the natural map (again base changed to $\mathbb{C}$ using the embedding $\bar{K} \subset \mathbb{C}$)

$$Aut(D^b(X_{\mathbb{C}})) \to O_{Hdg}(\tilde{H}^*(X_{\mathbb{C}}, \mathbb{Z})),$$

denoted as $Ker_{X_{\mathbb{C}}}$. Indeed, this follows from the base change on cohomology and Berthelot-Ogus's isomorphism [10]. Let us assume that $\Phi_{P_{\mathbb{C}}}$ does not induces the identity on the singular cohomology of $X_{\mathbb{C}}$ and hence, using the following natural commutative diagram

$$\begin{array}{ccc} H^*(X_{\mathbb{C}}, \mathbb{C}) & \longrightarrow & H^*(X_{\mathbb{C}}, \mathbb{C}) \\ \downarrow \cong & & \downarrow \cong \\ H^*_{DR}(X_{\mathbb{C}}) & \longrightarrow & H^*_{DR}(X_{\mathbb{C}}), \end{array}$$

$\Phi_{P_{\mathbb{C}}}$ also does not induces the identity on the de Rham cohomology of $X_{\mathbb{C}}$. As the autoequivalence $\Phi_{P_{\mathbb{C}}}$ is just the base change of $\Phi_{P_{\bar{K}}}$ we see that the map induced by $\Phi_{P_{\bar{K}}}$ on the de Rham cohomology of $X_{\bar{K}}$ is not the identity. Now again $\Phi_{P_{\bar{K}}}$ comes via base change from Φ_{P_K} so it is not the identity on de Rham cohomology of X_K, now using the Berthelot-Ogus's isomorphism it does not induce the identity on the crystalline cohomology of X but this is not possible as it is a lift of an autoequivalence which induces the identity on the crystalline cohomology.
This gives us the following map

$$Ker_X \to Ker_{X_{\mathbb{C}}}$$
$$\Phi_P \mapsto \Phi_{P_{\mathbb{C}}}$$

with the kernel consisting of those autoequivalences which lift to the identity on the geometric generic fiber.

Remark 3.20 We expect that in the case of a general polarized K3 surface, the set of autoequivalences which lift to the identity on the geometric generic fiber will contain only the identity itself.

Now, using the Picard rank 1 lift of Esnault-Oguiso, we see that there is a subgroup of Ker_X inside $Ker_{X_{\mathbb{C}}}$. And the kernel $Ker_{X_{\mathbb{C}}}$ has been described in [6, Theorem 1.4]. Thus, we have shown that

Proposition 3.21 *Let X be a K3 surface over k, an algebraically closed field of characteristic $p > 3$, and $X_R \to \operatorname{Spec}(R)$ be a Picard rank one lift of X with $X_{\mathbb{C}}$ the base change to $\mathbb{C}$ of the geometric generic fiber of X_R. Here, R is mixed characteristic discrete valuation ring with residue field k. Then $Ker_{X_{\mathbb{C}}}$ contains a subgroup of Ker_X. Moreover, in the case the set of autoequivalences which lift to the identity on the geometric generic fiber contains only the identity itself, then $Ker_X \subset Ker_{X_{\mathbb{C}}}$.*

4 Counting Fourier-Mukai Partners in Positive Characteristic

In this last section, we count the number of Fourier-Mukai partners of an ordinary K3 surface, in terms of the Fourier-Mukai partners of the geometric generic fiber of its canonical lift. Moreover, we prove that any automorphism of ordinary K3 surfaces lifts to its canonical lift. We start with comparing the Fourier-Mukai partners of a K3 surface over a field of positive characteristic with that of the geometric generic fiber of its lift to characteristic zero. Then we restrict to ordinary K3 surfaces and give a few consequences to lifting automorphisms of ordinary K3 surfaces. Moreover, we give a sufficient condition on derived autoequivalences of an ordinary K3 surface so that they lift to the canonical lift. Lastly, we show that the class number counting formula (compare from Theorem 2.31) also holds for K3 surfaces over a characteristic p field.

Let X (resp. Y) be a regular proper scheme with $D^b(X)$ (resp. $D^b(Y)$) its bounded derived category. Recall that we say that Y is a *Fourier-Mukai partner* of X if there exists a perfect complex $\mathcal{P} \in D^b(X \times Y)$ such that the following map is an equivalence of derived categories:

$$\begin{aligned} \Phi_P : D^b(X) &\xrightarrow{\cong} D^b(Y) \\ \mathcal{Q} &\mapsto \mathbb{R}p_{Y*}((p_X^*\mathcal{Q}) \otimes^{\mathbb{L}} \mathcal{P}), \end{aligned} \tag{10}$$

where p_X (resp. p_Y) is the projection from $X \times Y$ to X (resp. Y).

We want to count the number of Fourier-Mukai partners of a K3 surface in positive characteristic. We will do this by lifting the K3 surface to characteristic 0 and then counting the Fourier-Mukai partners of the geometric generic fibers. For this we will show that the specialization map for Fourier-Mukai partners defined below is injective and surjective:

$$\begin{aligned} \{\text{FM partners of } X_{\bar{K}}\} &\to \{\text{FM partners of X}\} \\ M_{X_{\bar{K}}}(v) &\mapsto M_X(v). \end{aligned} \tag{11}$$

Here, X is a K3 surface of finite height over k an algebraically closed field of characteristic $p > 3$, $X_{\bar{K}}$ is the geometric generic fiber of X_W, which is a Picard

preserving lift of X, and $M_X(v)$ (resp. $M_{X_{\bar{K}}}(v)$, $M_{X_W}(v)$) is the (fine) moduli space of stable sheaves with Mukai vector v on X (resp. $X_{\bar{K}}$, X_W). Note that from now on we will fix one such lift of X. Such a lift always exists by [47, Corollary 4.2] for K3 surfaces of finite height. On the other hand, Theorem 2.37 shows that supersingular K3 surfaces have no nontrivial Fourier-Mukai partners, so from now we restrict to the case of K3 surfaces of finite height.
To show that the map (11) is well defined, we need the following lemma:

Lemma 4.1 ((Potentially) Good reduction) *([46, Theorem 5.3]) Let V be a discrete valuation ring with a fraction field K, a field of characteristic 0, and residue field k of characteristic p such that there is a K3 surface X_K over K with good reduction, then all the Fourier-Mukai partners of $X_{\bar{K}}$ have good reduction possibly after a finite extension of K.*

Thus for any Fourier-Mukai partner of $X_{\bar{K}}$ which is of the form $M_{X_{\bar{K}}}(v)$ is a geometric generic fiber of $M_{X_V}(v)/V$, where V is a finite (algebraic) extension of $W(k)$. Note that the residue field of V is still k as k is algebraically closed. Now using functoriality of the moduli functor we note that the special fiber of $M_{X_V}(v)$ is $M_X(v)$. This is a Fourier-Mukai partner of X (see, for example, 2.37). Thus, the map (11) is well-defined.

Proposition 4.2 (Lieblich-Olsson [45]) *The specialization map (11) above is surjective.*

Proof: From [45, Theorem 3.16], note that all Fourier-Mukai partners of X are of the form $M_X(v)$. Moreover, one can always assume v to be of the form (r, l, s) where l is the Chern class of a line bundle and r is prime to p (see [45, Lemma 8.1]). (Note that we take the Mukai vector here in the respective Chow groups rather than cohomology groups). Then since we have chosen our lift X_W of X to be Picard preserving, we can also lift the Mukai vector to (r_W, l_W, s_W), again denoted by v, and this gives a FM partner of X_W, namely $M_{X_W}(v)$, and taking the geometric generic fiber of it gives a Fourier-Mukai partner of $X_{\bar{K}}$. □

Remark 4.3 Note that the $Pic(X_{\bar{K}}) \cong Pic(X)$, i.e., the specialization map is an isomorphism. This is essentially due to the fact that k is algebraically closed and every line bundle on X lifts uniquely to X_W as $Ext^1(L, L) = H^1(X, \mathcal{O}_X) = 0$ for $L \in Pic(X)$, under which the set of infinitesimal deformations of the line bundle L is a torsor.

Remark 4.4 Note that the argument above already implies that the number of Fourier-Mukai partners of a K3 surface over an algebraically closed field of characteristic $p > 3$ is finite. This argument was given by Lieblich-Olsson in [45].

Injectivity: We need to show that if $M_X(v) \cong X$, then $M_{X_W}(v) \cong X_W$. For this statement we will restrict to the case of ordinary K3 surfaces.
Before proving injectivity we prove that the automorphisms of an ordinary K3 surface lift always to characteristic zero.

Theorem 4.5 *Every isomorphism $\varphi : X \to Y$ of ordinary K3 surfaces over an algebraically closed field of characteristic p lifts to an isomorphism of the canonical lift of the ordinary K3's $\varphi_W : X_{can} \to Y_{can}$. In particular, every automorphism of X lifts to an automorphism of X_{can}.*

Remark 4.6 Note that the above statement is stronger than the tautological statement: If X and X' are two isomorphic ordinary $K3$ surfaces over a perfect field k, then their canonical lifts are isomorphic.

Remark 4.7 This statement should be compared with the results of Esnault-Oguiso [19, Theorems 5.1, 6.4 and 7.5], who constructed automorphisms which do not lift to characteristic 0.

Proof: [Proof of Theorem 4.5] Let $\varphi : X \to Y$ be an isomorphism of ordinary K3 surfaces. Consider the graph of this isomorphism as a coherent sheaf (or even as a perfect complex) on the product $X \times Y$, then from Theorem 3.4 the deformation of isomorphism as a morphism and as a sheaf are equivalent so we use Theorem 3.7 to construct a lifting of the isomorphism for the canonical lift X_{can} of X. As isomorphisms preserve the ample cone, the induced Fourier-Mukai transform satisfies the assumptions of Theorem 3.7. Note that the Lieblich-Olsson lifting of perfect complexes allows us to be only able to choose the lifting of X and then it constructs a unique lifting Y' of Y to which the perfect complex lifts. So, now the only remaining statement to show is that Y' is the canonical lift of Y. This follows from the criteria of canonical lift [68, Theorem C] and the observation that the isomorphism between $\varphi_{\bar{K}} : X_{can,\bar{K}} \to Y'_{\bar{K}}$ induces an isomorphism of Galois module on the second p-adic étale cohomology. This isomorphism of Galois modules provides us with the required decomposition of $H^2_{et}(Y'_{\bar{K}}, \mathbb{Z}_p)$, which shows that Y' is the canonical lift of Y. □

Remark 4.8 This gives a fixed point of the δ functor constructed by [45] (see Theorem 3.7).

Corollary 4.9 *Every isomorphism of ordinary K3 surfaces over an algebraically closed field of characteristic p preserves the Hodge filtration induced by the canonical lift. In particular, every automorphism of an ordinary K3 surface over an algebraically closed field of characteristic p preserves the Hodge filtration induced by the canonical lift.*

Proof: This follows from 4.5 and [19, Remark 6.5]. □

Theorem 4.10 *Let X be an ordinary K3 surface, then the canonical lift of the moduli space of stable sheaves with a fixed Mukai vector is the moduli space of stable sheaves with the same Mukai vector on the canonical lift:*

$$(M_X(v))_{can} \cong M_{X_{can}}(v). \tag{12}$$

Proof: We use the criteria for canonical lift [68, Theorem C] to show that $M_{X_{can}}(v)$ is indeed the canonical lift of $M_X(v)$. To use the criteria, we note that

$$\begin{aligned} H^2_{\acute{e}t}(M_{X_{can}}(v)_{\bar{K}}, \mathbb{Z}_p) &= v^{\perp}/v\mathbb{Z}_p \\ &\subset H^0_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus H^2_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus H^4_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p), \end{aligned}$$

where the orthogonal complement is taken with respect to the extended pairing on the étale Mukai lattice. As X_{can} is the canonical lift of X, we have the following decomposition of

$$H^2_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) = M^0_X \oplus M^1_X(-1) \oplus M^2_X(-2)$$

as Galois modules. We define the decomposition of $H^2_{\acute{e}t}(M_{X_{can}}(v)_{\bar{K}}, \mathbb{Z}_p) = M^0 \oplus M^1(-1) \oplus M^2(-2)$ as Galois modules, where

$$\begin{aligned} M^0 &= M^0_X \\ M^2 &= M^2_X \\ M^1 &= H^0_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus H^4_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus (v^{\perp}/v\mathbb{Z}_p \cap M^1_X). \end{aligned} \tag{13}$$

The last relation above holds using Proposition 2.27 and the fact that $H^0_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p)$ and $H^4_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p)$ are orthogonal to M^1_X. □
Now, we finally prove the injectivity.

Theorem 4.11 *If X is an ordinary K3 surface over an algebraically closed field of char p, then the number of FM partners of X are the same as the number of Fourier-Mukai partners of the geometric generic fiber of the canonical lift of X over W.*

Proof: From the discussion in the Chapter 4 Section 4.2, we see that all that is left to show is the injectivity of the specialization map on the set of Fourier-Mukai partners. That is, we need to show that if $M_X(v)$ is isomorphic to X, then the lifts of both of them are also isomorphic $X_{can} \cong M_{X_{can}}(v)$. This follows from the definition of canonical lifts and Theorem 4.10 that $M_{X_{can}}(v)$ is the canonical lift of $M_X(v)$. □

Corollary 4.12 *Let X be an ordinary K3 surface over k, then the derived autoequivalences satisfying the assumptions of Theorem 3.7 lift uniquely to a derived autoequivalence of X_{can}.*

Proof: The argument is going to be similar to the one used to show that every automorphism lifts, but now we will use the proof of Theorem 4.10. Let $\mathcal{P} \in D^b(X \times X)$ induce a derived autoequivalence on X, then, using Theorem 3.7, there exists an X'/W such that we can lift $\mathcal{P}$ to a kernel $\mathcal{P}_W \in D^b(X_{can} \times X')$. Now we need to show that X' is just X_{can}. Note that $(\mathcal{P}_W)_{\bar{K}}$ gives a derived equivalence between $D^b(X_{can,\bar{K}}) \cong D^b(X'_{\bar{K}})$, this implies that X' is isomorphic to some moduli space of stable sheaves with Mukai vector v, $M_{X_{can,\bar{K}}}(v)$. Now by functoriality of the moduli spaces, we have $M_{X_{can,\bar{K}}}(v) \cong M_{X_{can}}(v)_{\bar{K}}$ and by Theorem 4.10, we have $M_{X_{can}}(v)_{\bar{K}} \cong M_X(v)_{can,\bar{K}}$. This implies that we get the required decomposition of the second p-adic integral étale cohomology of $X'_{\bar{K}}$, which using [68, Theorem C] gives us the result. □

Corollary 4.13 *Every autoequivalence of an ordinary K3 surface that satisfies the assumptions of Theorem 3.7 preserves the Hodge filtration induced by the canonical lift.*

Proof: Follows from the corollary above and Theorem 3.8. □

4.1 The Class Number Formula

Lastly, we give the corresponding class number formula in characteristic p to corollary 2.31.

Theorem 4.14 *Let X be a $K3$ surface of finite height over an algebraically field of positive characteristic (say $q > 3$). If the Néron-Severi lattice of X has rank 2 and determinant $-p$ (p and q can also be same), then the number of Fourier-Mukai partners of X is $(h(p)+1)/2$.*

Proof: We lift X to characteristic 0 using the Lieblich-Maulik Picard preserving lift and then base changing to the geometric generic fiber to get $X_{\bar{K}}$. Choose an embedding of $\bar{K}$ to $\mathbb{C}$ (complex numbers) and base change to $\mathbb{C}$, to get $X_{\mathbb{C}}$. Now, from Proposition 4.2, we get that every Fourier-Mukai partner of X lifts to a Fourier-Mukai partner of $X_{\mathbb{C}}$. So, we just need to show that if any Fourier-Mukai partner, say $Y_{\mathbb{C}}$, of $X_{\mathbb{C}}$ reduces $\mod q$ to an isomorphic $K3$ surface, say Y, to X, then it is isomorphic to $X_{\mathbb{C}}$. This follows from noting that if $Y_{\mathbb{C}}$ becomes isomorphic $\mod q$, then the Picard lattices of $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are isomorphic. The number of Fourier Mukai partners of $X_{\mathbb{C}}$ with isomorphic Picard lattices is given by the order of the quotient of the orthogonal group of discriminant group of $NS(X_{\mathbb{C}})$ by the Hodge isometries of the transcendental lattice (cf. Theorem 2.30), but in this case the discriminant group of $NS(X_{\mathbb{C}}) = \mathbb{Z}/p$ so the orthogonal group is just $\pm id$ and there is always $\pm id$ in the hodge isometries, so we get the quotient to be a group of order 1. Thus the result. □

Remark 4.15 Note that the Picard lattice $Pic(X_K)$ and $Pic(X_{\bar{K}})$ are indeed isomorphic as after reduction we are over an algebraically closed field and the line bundles lift uniquely as Pic^0_X is trivial for a K3 surface.

A F-crystal on Crystalline Cohomology

In this appendix, we analyze the possibility of having a "naive" F-crystal structure on the Mukai isocrystal of a K3 surface. We begin by recalling a few results about crystalline cohomology and the action of Frobenius on it, for details we refer to [1, Tag 07GI and Tag 07N0], [8], [10], [49, Section 1.5].

Let X be a smooth and proper variety over a perfect field k of positive characteristic p. Let $W(k)$ (resp. $W_m(k)$) be the associated ring of (resp. truncated) Witt vectors with the field of fraction K. Let us denote by $Frob_k : k \to k$; $x \mapsto x^p$, the Frobenius morphism of k, which induces a ring homomorphism $Frob_W : W(k) \to W(k)$, by functoriality, and there exists an additive map $V : W(k) \to W(k)$ such that $p = V \circ Frob_W = Frob_W \circ V$. Thus, $Frob_W$ is injective. For any $m > 0$, we have cohomology groups $H^*_{crys}(X/W_m(k))$. These are finitely generated $W_m(k)$-modules. Taking the inverse limit of these groups gives us the crystalline cohomology:

$$H^n_{crys}(X/W(k)) := \varprojlim H^n_{crys}(X/W_m(k)).$$

It has the following properties as a Weil cohomology theory:

1. $H^n_{crys}(X/W(k))$ is a contravariant functor in X and the groups are finitely generated as $W(k)$-modules. Moreover, $H^n_{crys}(X/W(k))$ is 0 if $n < 0$ or $n > 2dim(X)$.

2. Poincaré Duality: The cup-product induces a perfect pairing:
$$\frac{H^n_{crys}(X/W(k))}{torsion} \times \frac{H^{2dim(X)-n}_{crys}(X/W(k))}{torsion} \to H^{2dim(X)}_{crys}(X/W(k)) \cong W(k). \tag{14}$$

3. $H^n_{crys}(X/W(k))$ defines an integral structure on $H^n_{crys}(X/W(k)) \otimes_{W(k)} K$.

4. If there exists a proper lift of X to $W(k)$, that is, a smooth and proper scheme $X_W \to \mathrm{Spec}(W(k))$ such that its special fiber is isomorphic to X. Then we have, for each n,
$$H^n_{DR}(X_W/W(k)) \cong H^n_{crys}(X/W(k)).$$

5. Consider the commutative square given by absolute Frobenius:
$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ k & \xrightarrow{Frob_k} & k. \end{array}$$
This, by the functoriality of the crystalline cohomology, gives us a $Frob_W$-linear endomorphism on $H^i(X/W)$ of $W(k)$-modules, denoted by F^*. Moreover, F^* is injective modulo the torsion, i.e.,
$$F^* : H^i(X/W)/torsion \to H^i(X/W)/torsion$$

is injective.

Theorem A.1 (Crystalline Riemann-Roch) *Let X and Y be smooth varieties over k, a field of characteristic p, and $f : X \to Y$ be a proper map. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{f_*} & K_0(Y) \\ \downarrow{\scriptstyle ch(\ ).td_X} & & \downarrow{\scriptstyle ch(\ ).td_Y} \\ \oplus_i H^{2i}_{crys}(X/K) & \xrightarrow{f_*} & \oplus_i H^{2i}_{crys}(Y/K), \end{array}$$

i.e., $ch(f_*\alpha).td_Y = f_*(ch(\alpha).td_X) \in \oplus_i H^i_{crys}(Y/K)$ for all $\alpha \in K_0(X)$, where $K_0(X)$ is the Grothendieck group of coherent sheaves on X.

Remark A.2 The map f_* does not preserve the cohomological grading but does preserve the homological grading, i.e., if the dimensions of X and Y are n and m respectively, then we have the following commutative square:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{f_*} & K_0(Y) \\ \downarrow{\scriptstyle ch(\ ).td_X} & & \downarrow{\scriptstyle ch(\ ).td_Y} \\ \oplus_i H^{2i}_{crys}(X/K) & \xrightarrow{f_*} & \oplus_i H^{2i+(n-m)}_{crys}(Y/K), \end{array}$$

and here the grading is respected. If X and Y are K3 surfaces, then $n = m = 2$ and we do not have to worry about this remark, as then the usual cohomological grading is preserved.

Next we state a few main results about the compatibility of the Frobenius action with the various relations :

Proposition A.3 (Crystalline Künneth Formula) *Let X, Y be proper and smooth varieties over k. Then there is a canonical isomorphism in $D(W)$, the derived category of W modules, given as follows:*

$$\mathbb{R}\Gamma(X/W) \otimes^{\mathbb{L}}_W \mathbb{R}\Gamma(Y/W) \cong \mathbb{R}\Gamma(X \times_k Y/W),$$

yielding exact sequences

$$0 \to \oplus_{p+q=n}(H^p(X/W) \otimes H^q(Y/W)) \to H^n(X \times Y/W) \to$$
$$\to \oplus_{p+q=n+1} Tor_1^W(H^p(X/W), H^q(Y/W)) \to 0.$$

For a proof see [8, Chapitre 5, Théorème 4.2.1] C and [37, Section 3.3].

Remark A.4 Note that in the case of K3 surfaces the torsion is zero, so we have the following isomorphism:

$$\oplus_{p+q=n}(H^p(X/W) \otimes H^q(Y/W)) \xrightarrow{\sim} H^n(X \times Y/W).$$

The action of Frobenius gives the following map:

$$\begin{array}{ccc} F^*H^n(X\times Y/W) & \hookrightarrow & H^n(X\times Y/W) \\ \downarrow\cong & & \downarrow\cong \\ \oplus(F^*H^p(X/W)\otimes F^*H^q(Y/W)) & \hookrightarrow & \oplus(H^p(X/W)\otimes H^q(Y/W)), \end{array}$$

where the direct sum is over all $p+q=n$.

PROPOSITION A.5 *The Künneth formula is compatible with the Frobenius action in the following way:*
Let $\gamma \in H^n(X\times Y/W)$ be written (uniquely) as $\gamma = \sum \alpha_p \otimes \beta_q$, then

$$F^*\gamma = F^*\alpha_p \otimes F^*\beta_q,$$

where $\alpha_p \in H^p(X/W)$ and $\beta_q \in H^q(Y/W)$.

Let p_X(resp. p_Y) denote the projection $X\times Y \to X$ (resp. $X\times Y\to Y$).

PROPOSITION A.6 *The Frobenius has the following compatibility with the projection morphism:*

$$p_X^*(F^*(\alpha)) = F^*(p_X^*\alpha).$$

Similarly, for the other projection p_Y.

Let the denote the cup-product as follows:

$$H^i(X/W)\times H^j(X/W) \to H^{i+j}(X/W)$$

given by

$$(\alpha,\beta)\mapsto \alpha\cup\beta.$$

PROPOSITION A.7 *The Frobenius action is compatible with the cup-product in the following way:*

$$F^*(\alpha\cup\beta) = F^*(\alpha)\cup F^*(\beta).$$

Moreover, the Poincaré duality induces a perfect pairing as in relation [14]

$$< -,- >: \frac{H^n}{torsion}\times\frac{H^{2dim(X)-n}}{torsion}\to H^{2dim(X)}\cong W(k)$$

which satisfies the following compatibility with Frobenius:

$$< F^*(x), F^*(y) >= p^{dim(X)}Frob_W(< x,y >), \tag{15}$$

where $n\in[0,2dim(X)]$.

Now we define an F-crystal (see Definition 2.1) structure on the Mukai F-isocrystal of crystalline cohomology for a K3 surface.

Let X be a K3 surface over an algebraically closed field k of characteristic $p > 3$. Let $ch = ch_{cris} : K(X) \to H^{2*}(X/K)$ be the crystalline Chern character and ch^i the $2i-th$ component of ch. Reducing to the case of a line bundle via the splitting principle, we see that the Frobenius φ_X acts in the following manner on the Chern character of a line bundle E:

$$\varphi_X(ch^i(E)) = p^i ch^i(E). \tag{16}$$

We normalize the Frobenius action on the F-isocrystal $H^*(X/K)$ using the Tate twist to get the *Mukai F-isocrystal* $\oplus_i H^i(X/K)(i-1)$.

We make the following observation, which shows that how the Frobenius action works on $H^4_{crys}(X/W)$, i.e., we will compute $\varphi_X([1])$. Note that for a perfect field k of characteristic p, Serre [65, Theorem 8 on page 43] showed that the Witt ring W(k) has p as its uniformizer. Now for $H^4_{crys}(X/K)(1)$ the action of Frobenius is given by φ_X/p. But note that $ch^2(E) = 1/2(c_1^2(E) - 2c_2(E))$, for $E \in K(X)$, where $c_i(E)$ are the Chern classes of E, and as the intersection paring is even for a K3 surface, this is integral, i.e., $ch^2(E) \in H^4(X/W)$. This along with the fact that $rank_W(H^4(X/W)) = 1$ implies that $ch^2(E) = up^n[1]$, where $u \in W^\times, p$ is the characteristic of k and [1] is the generator of $H^4(X/W)$ as a $W-$module. Hence, we have

$$\begin{aligned} \varphi_X(ch^2(E)) = \varphi_X(up^n[1]) &= \sigma(up^n)\varphi_X([1]) \text{ (via semi-linearity)} \\ &= \sigma(u)p^n\varphi_X([1]) \text{ (as } \sigma \text{ is a ring map)} \\ &= p^2 \cdot ch^2(E) = p^2 up^n[1]. \end{aligned}$$

On the other hand, from the equation 16 above, we have $\varphi_X(ch^2(E)) = p^2 up^n[1]$. This gives us that

$$\varphi_X([1]) = u(\sigma(u))^{-1}p^2[1],$$

where $u(\sigma(u))^{-1} \in W^\times$ as σ is a ring map. Therefore, we have the Frobenius action on $H^4(X/W) \otimes K(1)$ given by $\varphi'_X([1]) = u(\sigma(u))^{-1}p[1]$. Thus, it indeed has a F-crystal inducing this F-isocrystal given by $(H^4(X/W), \varphi'_X)$. We remark that we are implicitly using the fact that $A \otimes_K K \cong A$, for any K-module A. Note that the Mukai vector of a sheaf P in $D^b(X)$ for a K3 surface X is by definition the class

$$v(P) = ch(P)\sqrt{td(X)} = (v_0(P), v_1(P), v_2(P)) \in \mathcal{H}^*_{crys}(X/W).$$

Indeed, we have $c_1(X) = 0$ and $2 = \chi(X, \mathcal{O}_X) = td_{2,X}$, which gives us that the Todd genus $td_X = (1, 0, 2)$ and thus $\sqrt{td_X} = (1, 0, 1)$. This then implies that

$$v(P) = (rk(P), c_1(P), rk(P) + c_1^2(P)/2 - c_2(P)).$$

Note that the intersection pairing on $H^2_{crys}(X/W)$ is even, which gives us the above conclusion as $c_i(P) \in H^{2i}_{crys}(X/W)$ (see [11]).

Lemma A.8 *The Mukai vector of any object $P \in D^b(X \times Y)$ is a F-crystal cohomology class.*

Proof: (cf. [53]) Note that from the definition of the F-crystal structure we just need to show that $ch(P) \in H^*_{crys}(X \times Y/W)$ as the square root of the Todd genus for a K3 surface is computed as follows:

$$\sqrt{td_{X\times Y}} = p_1^* \sqrt{td_X} p_2^* \sqrt{td_Y} = p_1^*(1,0,1).p_2^*(1,0,1).$$

We write the exponential chern character as follows:

$$ch(P) = (rk(P), c_1(P), 1/2(c_1^2(P) - 2c_2(P)), ch_3(P), ch_4(P))$$

where

$$ch^3(P) = 1/6(c_1^3(P) - 3c_1c_2 + 3c_3(P))$$

and

$$ch^4(P) = 1/24(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4).$$

Note that if $char(k) \neq 2,3$, then $2,3$ are invertible in $W(k)$, so $ch(P) \in H^*_{crys}(X \times Y/W)$ as again we know $c_i(P) \in H^{2i}_{crys}(X \times Y/W)$. □

Remark A.9 Thus, it makes sense to talk about the descent of a Fourier-Mukai transform to the F-crystal level but note that the new Frobenius structure on $H^4(X/W)(1)$ fails to be compatible with the intersection pairing as defined in Theorem A.7. This causes the failure of existence of an F-crystal structure on the Mukai-isocrystal and also the failure to have a cohomological criteria of derived equivalences of K3 surfaces with crystalline cohomology.

References

[1] de Jong A. et al., *Stacks Project*, http://stacks.math.columbia.edu, 2018.

[2] Artin M., *Versal deformation and algebraic stacks*, Inventiones Math. 27, 165-189, 1974.

[3] Artin M., Mazur B., *Formal groups arising from algebraic varieties*, Ann. Sc. Éc. Norm. Sup. 4e série 10, 87-132, 1977.

[4] Barth B., Hulek K., Peters C. A. M., Van De Ven A., *Compact complex surfaces*, A Series of Modern Surveys in Mathematics 4, Springer, 2003.

[5] Bartocci C., Bruzzo U., Ruipérez D. H., *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, Progress in Mathematics 276, Birkhäuser, 2009.

[6] Bayer A., Bridgeland T., *Derived automorphism groups of $K3$ surfaces of Picard rank 1*, Duke Math. J. 166, no. 1, 75-124, 2017.

[7] Beilinson A. A., Bernstein J., Deligne P., *Faisceaux pervers*, Astérisque, Soc. Math. France 100, 5-171, 1982.

[8] Berthelot P., *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math. 407, Springer, 1974.

[9] Berthelot P., Ogus A.,*Notes on crystalline cohomology*, Annals of Math. Lecture Notes, Princeton University Press, 1978.

[10] Berthelot P., Ogus A., *F-isocrystals and de Rham cohomology. I.* Inventiones Mathematicae 72, 159-200, 1983.

[11] Berthelot P., Illusie L., *Classes de Chern en cohomologie cristalline*, C.R. Acad Sci. Series A 270, 1695-1697, 1750-1752, 1970.

[12] Bondal A., Orlov D., *Reconstruction of a variety from the derived category and groups of autoequivalences*, Comp. Math. 125, 327-344, 2001.

[13] Bridgeland T., *Stability conditions on triangulated categories*, Annals of Mathematics, Second Series 166, no. 2, 317-345, 2007.

[14] Bridgeland T., *Stability conditions on $K3$ surfaces*, Duke Math. J. 141, no. 2, 241-291, 2008.

[15] Bridgeland T., *Spaces of stability conditions*, Algebraic geometry, Seattle 2005. Part 1, 1-21, Proc. Sympos. Pure Math. 80, Amer. Math. Soc., Providence, RI, 2009.

[16] Deligne P., *Relevement des surfaces K3 en characteristique nulle*, In: Surfaces Algebriques, Seminar Orsay, 1976-78. Lecture Notes in Math. 868, 58-79, Springer, 1981.

[17] Deligne P., Illusie L., *Cristaux ordinaires et coordonées canoniques*, In: Surfaces Algebriques, Seminar Orsay, 1976-78. Lecture Notes in Math. 868, Springer, 1981.

[18] Dieudonné J., *Lie groups and Lie hyperalgebras over a field of characteristic $p > 0$. IV*, American Journal of Mathematics 77 (3), 429âĂŞ452, 1955.

[19] Esnault H., Oguiso K., *Non-liftability of automorphism groups of K3 surface in positive characteristic*, Math. Ann. 363, 1187-1206, 2015.

[20] Gelfand S., Manin Y., *Methods of homological algebra*, Springer Monographs in Mathematics, 1997.

[21] Grothendieck A., *Éléments de géométrie algébrique*, III, Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math. 11, 1961.

[22] Grothendieck A., *Théorie des topos et cohomologie étale des schémas*, SGA 4, Tome III, Lecture Notes in Mathematics 305, 1972.

[23] Grothendieck A., *Sur quelques points d'algèbre homologique*, I. Tohoku Math. J., Second Series 9, 119-221, 1957.

[24] Gómez T., *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. 111 (1), 1-31, 2001.

[25] Hartshorne R., *On the De Rham cohomology of algebraic varieties*, Publications Mathematiques de l'IHÉS 45, 5-99, 1975.

[26] Hartshorne R., *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer, 1977.

[27] Hartshorne R., *Deformation theory*, Graduate Texts in Mathematics 257, Springer, 2010.

[28] Hosono S., Lian B. H., Oguiso K., Yau S.-T., *Autoequivalences of derived category of a K3 surface and monodromy transformations*, J. Alg. Geom. 13, 513-545, 2004.

[29] Hosono S., Lian B. H., Oguiso K., Yau S.-T., *Fourier-Mukai number of a K3 surface*, CRM Proc. Lecture Notes 38, 177-192, 2004.

[30] Huybrechts D., *Fourier Mukai transforms in algebraic geometry*, Oxford Science Publication, 2006.

[31] Huybrechts D., Lehn M., *The geometry of moduli spaces of sheaves*, Second edition, Cambridge Mathematical Library, Cambridge University Press, 2010.

[32] Huybrechts D., Marci E., Stellari P., *Derived equivalences of K3 surfaces and orientation*, Duke Math. J. 149, 461-507, 2009.

[33] Huybrechts D., Marci E., Stellari P., *Stability conditions for generic K3 categories*, Comp. Math. 144, 134-162, 2008.

[34] Huybrechts D., *Introduction to stability conditions*. In: Moduli Spaces, Cambridge University Press, edited by Brambila-Paz, García-Prada, Newstead and Thomas. Lectures at the Newton Institute January 2011. London Mathematical Society Lecture Note Series 411, 179-229, 2014.

[35] Huybrechts D., Richard T., *Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes*, Math. Ann. 346, 545-569, 2013.

[36] Huybrechts D., *Lectures on K3 surfaces*, Cambridge University Press, 2016.

[37] Illusie L., *Report on crystalline cohomology.* In: Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), American Mathematical Society, Providence, R.I., 459-478, 1975.

[38] Katz N., *Slope filtration of F-crystals*, Astérisque 63, 113-164, 1979.

[39] Kashiwara M., Schapira P., *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften 292, Springer, 1990.

[40] Knutson D., *Algebraic spaces*, Lecture Notes in Mathematics 203, 1971.

[41] Kontsevich M., *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians (Zürich 1994), Birkhäuser, 120-139, 1995.

[42] Langer A., *Semistable sheaves in positive characteristic*, Annals of Math. 159, 251-276, 2004.

[43] Lieblich M., *Moduli of complexes on a proper morphism*, J. Algebraic Geometry 15, 175-206, 2006.

[44] Lieblich M., *Moduli of twisted sheaves*, Duke Math. J. 138, 23-118, 2007.

[45] Lieblich M., Olsson M., *Fourier Mukai partners of K3 surfaces in positive characteristic*, Annales Scientifiques de lâĂŽENS 48, fascicule 5, 1001-1033, 2015.

[46] Lieblich M., Olsson M., *A stronger derived Torelli theorem for K3 surfaces*, In: Bogomolov F., Hassett B., Tschinkel Y. (eds), Geometry Over Nonclosed Fields. Simons Symposia. Springer, Cham, 2017.

[47] Lieblich M., Maulik D., *A note on the cone conjecture for K3 surfaces in positive characteristic*, Math. Res. Lett. 25, no. 6, 1879-1891, 2018.

[48] Liedtke C., Matsumoto Y., *Good reduction of K3 surfaces*, Compos. Math. 154, 1-35, 2018.

[49] Liedtke C., *Lectures on supersingular K3 surfaces and the crystalline Torelli theorem*, In: K3 surfaces and their moduli, Progress in Mathematics 315, Birkhäuser, 171-325, 2016.

[50] Manin Y., *Theory of commutative formal groups over fields of finite characteristic*, Uspekhi Mat. Nauk SSSR 18(6 (114)), 3-90, 1963.

[51] Matsumoto Y., *Good reduction criterion for K3 surfaces*, Math. Z. 279, no. 1-2, 241-266, 2015.

[52] Mukai S., *Duality between $D(X)$ and $D(\hat{X})$ with its applications to Picard sheaves*, Nagoya Math. J. 81, 153-175, 1981.

[53] Mukai S., *On the moduli space of bundles on K3 surfaces I.*, In: Vector bundles on algebraic varieties, Pap. Colloq., Bombay 1984, Stud. Math., Tata Inst. Fundam. Res. 11, 341-413, 1984.

[54] Matsusaka T., Mumford D., *Two theorems on deformations of polarized varities*, Amer. J. Math. 86, 668-684, 1964.

[55] Neukirch J., *Algebraic number theory*, Grundlehren der mathematischen Wissenschaften 322, Springer, 1999.

[56] Nitsure N., *Construction of Hilbert and Quot schemes*, Fundamental algebraic geometry: GrothendieckâĂŹs FGA explained, Mathematical Surveys and Monographs 123, American Mathematical Society, 105âĂŞ137, 2005.

[57] Nygaard N., *Tate conjecture for ordinary K3 surfaces over finite fields*, Inv. Math. 74, 213-237, 1983.

[58] Olsson M., *Algebraic spaces and stacks*, Colloquium Publications 62, American Mathematical Society, 2016.

[59] Orlov D., *On equivalences of derived categories and K3 surfaces.* J. Math Sci. (New York) 84, 1361-1381, 1997.

[60] Orlov D., *Derived categories and coherent sheaves and equivalences between them*, Russian Math. Surveys 58, 511-591, 2003.

[61] Ogus A., *Supersingular K3 crystals*, Journées de Géométrie Algébraique de Rennes Vol. II, Astérisque 64, 3-86, 1979.

[62] Ogus A., *A crystalline Torelli theorem of supersingular K3 surfaces*, Arithmetic and Geometry II, Progress in Mathematics 36, 361-394, Birkhäuser, 1983.

[63] Ploog D., *Group of autoequivalences of derived categories of smooth projective varieties*, PhD thesis, Freie Universität Berlin, 2005.

[64] Rudakov A. N., Shaferevich I. R., *Inseparable morphisms of algebraic surfaces*, Izv. Akad. Nauk SSSR 40, 1269-1307 (1976).

[65] Serre J.P., *A course in arithmetic*, Graduate Text in Mathematics 7, Springer, 1973.

[66] Sernesi E., *Deformation of algebraic varieties*, Grundlehren der mathematischen Wissenschaften 334, Springer, Berlin, Heidelberg, 2006.

[67] Srivastava T.K., *On derived equivalences of K3 surfaces in positive characteristic*, PhD Thesis, Freie Universität Berlin, 2018.

[68] Taelman L., *Ordinary K3 surfaces over finite fields*, preprint: https://arxiv.org/abs/1711.09225, 2017.

[69] Ward M., *Arithmetic properties of the derived category for Calabi-Yau varieties*, PhD thesis, University of Washington, 2014.

Tanya Kaushal Srivastava
Institute of Science and Technology (IST) Austria
Am Campus 1
3400 Klosterneuburg
Austria