Erratum for "The Dirac Operator with Mass $m_0 \geq 0$: Non-Existence of Zero Modes and of Threshold Eigenvalues"

Cf. Documenta Mathematica, vol. 20 (2015), p. 37-64

Hubert Kalf, Takashi Okaji, and Osanobu Yamada

Received: July 25, 2017
Revised: August 10, 2017

Communicated by Heinz Siedentop

Abstract. We give a proof of Theorem 2.1 in [KOY, p.39], namely of the following assertion. Let $Q\colon \boldsymbol{R}^n \to \boldsymbol{C}^{N\times N}$ be measurable with

$$\sup_{x\in\boldsymbol{R}^n} |x||Q(x)| \leq C \quad \text{for some} \quad 0 < C < \frac{n-1}{2}.$$

Then any solution $u \in H^1_{\text{loc}}(\boldsymbol{R}^n)^N \cap L^2(\boldsymbol{R}^n, r^{-1}dx)^N$ of $(\alpha\cdot p+Q)u = 0$ is identically zero.

2010 Mathematics Subject Classification: Primary 35P15; Secondary 81Q10.
Keywords and Phrases: Dirac operators, virial theorem, threshold eigenvalue, zero mode.

Our original argument in [KOY, p.42 f.] started with the claim that there exists a sequence of functions (u_j) in $C_0^\infty(\boldsymbol{R}^n)^N$ with

$$r^{-1/2}u_j \to r^{-1/2}u, \quad r^{1/2}(\alpha p)u_j \to r^{1/2}(\alpha p)u \tag{1}$$

in $L^2(\boldsymbol{R}^n)^N$. When Fritz Gesztesy (Baylor, TX) and Michael Pang (Columbia, MO) asked us how to find such a sequence, we realised that the proof had to proceed somewhat differently. We thank Professors Gesztesy and Pang for making us aware that the claim (1) should be avoided.

Proof of Theorem 2.1. The argument in [KOY, p.43] yields

$$\int_{\boldsymbol{R}^n} r\,|\alpha p\,\varphi|^2 \geq \left(\frac{n-1}{2}\right)^2 \int_{\boldsymbol{R}^n} \frac{|\varphi|^2}{r} \quad (\varphi \in C_0^\infty(\mathbb{R}^n)^N). \tag{2}$$

Let $\chi \in C_0^\infty([0,\infty))$ be a cutoff function with $0 \leq \chi \leq 1$ and

$$\chi(r) = 1 \quad (0 \leq r \leq 1), \qquad \chi(r) = 0 \quad (r \geq 2).$$

Let $\ell \in \boldsymbol{N}$ and $\chi_\ell(r) := \chi(r/\ell)$. Our assumption that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)^N$ implies $\chi_\ell u \in H^1(\mathbb{R}^n)^N$. Using the Friedrichs mollifiers, we can find a sequence (u_j^ℓ) in $C_0^\infty(\boldsymbol{R}^n)^N$ with support in a ball of radius $2\ell+1$ around the origin such that

$$\|u_j^\ell - \chi_\ell u\|_{L^2(\mathbb{R}^n)^N} + \|(\alpha p)u_j^\ell - (\alpha p)(\chi_\ell u)\|_{L^2(\mathbb{R}^n)^N} \longrightarrow 0$$

as $j \to \infty$. In view of (2) we have

$$\begin{aligned} (2\ell+1)\int_{\boldsymbol{R}^n} |(\alpha p)(u_j^\ell - u_k^\ell)|^2 &\geq \int_{\boldsymbol{R}^n} r\left|(\alpha p)\,(u_j^\ell - u_k^\ell)\right|^2 \\ &\geq \left(\frac{n-1}{2}\right)^2 \int_{\boldsymbol{R}^n} \frac{|u_j^\ell - u_k^\ell|^2}{r}, \end{aligned}$$

i.e., $(u_j^\ell/\sqrt{r})$ is a Cauchy sequence in $L^2(\mathbb{R}^n)^N$. The limit g^ℓ satisfies

$$\int (g^\ell)^t \bar{\varphi} = \lim_{j\to\infty} \int \frac{(u_j^\ell)^t}{\sqrt{r}}\bar{\varphi} = \int \frac{\chi_\ell u^t}{\sqrt{r}}\bar{\varphi}.$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)^N$. Hence $g^\ell = r^{-1/2}\chi_\ell u$ and

$$\int_{\boldsymbol{R}^n} r\,|\alpha p\,(\chi_\ell u)|^2 \geq \left(\frac{n-1}{2}\right)^2 \int_{\boldsymbol{R}^n} \chi_\ell^2 \frac{|u|^2}{r}. \tag{3}$$

Finally, from our assumption $r^{-1/2}u \in L^2(\boldsymbol{R}^n)^N$ we have $r^{1/2}(\alpha p)u \in L^2(\boldsymbol{R}^n)^N)$ and the existence of a subsequence of (χ_ℓ) (denoted with the same letter again) such that

$$\begin{aligned} \sqrt{r}\alpha p\,(\chi_\ell u) &= \sqrt{r}\chi_\ell\,\alpha p\,u - i\frac{\sqrt{r}}{\ell}\chi'(r/\ell)\alpha_r u \\ &\to \sqrt{r}\,(\alpha p)\,u \end{aligned}$$

as $\ell \to \infty$. From (3) we finally obtain

$$C^2 \int_{\boldsymbol{R}^n} \frac{|u|^2}{r} \geq \int_{\boldsymbol{R}^n} r|\alpha p\,u|^2 \geq \left(\frac{n-1}{2}\right)^2 \int_{\boldsymbol{R}^n} \frac{|u|^2}{r},$$

which proves $u = 0$

References

[KOY] Kalf, H., Okaji, T. and Yamada, O., The Dirac operator with mass $m_0 \geq 0$: Non-existence of zero modes and of threshold eigenvalues, Documenta Math., 20 (2015), 37–64.

Hubert Kalf
Mathematisches Institut
Universität München
Theresienstr. 39
80333 München
Germany
hubert.kalf@mathematik.
uni-muenchen.de

Takashi Okaji
Department of Mathematics
Graduate School of Science
Kyoto University
Kyoto 606-8502
Japan
okaji@math.kyoto-u.ac.jp

Osanobu Yamada
Faculty of Science and Engineering
Ritsumeikan University
Kusatsu, Shiga 525-8577
Japan
yamadaos@se.ritsumei.ac.jp