Non-Commutative L-Functions
for p-Adic Representations
over Totally Real Fields

Malte Witte

ABSTRACT. We prove a unicity result for the Non-Commutative L-Functions for p-Adic Representations over Totally Real Fields-functions appearing in the non-commutative Iwasawa main conjecture over totally real fields. We then consider continuous representations ρ of the absolute Galois group of a totally real field F on adic rings in the sense of Fukaya and Kato. Using our unicity result, we show that there exists a unique sensible definition of a non-commutative L-function for any such ρ that factors through the Galois group of a possibly infinite totally real extension. We also consider the case of CM-extensions and discuss the relation with the equivariant main conjecture for realisations of abstract 1-motives of Greither and Popescu.

1 Introduction

Let F_∞/F be an admissible p-adic Lie extension (in the sense of [Kak13]) of a totally real field F that is unramified over an open dense subscheme U of the spectrum X of the algebraic integers of F and write $G \coloneqq \operatorname{Gal}(F_\infty/F)$ for its Galois group. We further assume that p is invertible on U. The non-commutative main conjecture of Iwasawa theory for F_∞/F predicts the existence of a non-commutative p-adic L-function $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_p(1))$ living in the first algebraic

K-group $\mathrm{K}_1(\mathbb{Z}_p[[G]]_S)$ of the localisation at Venjakob's canonical Ore set S of the profinite group ring

$$\mathbb{Z}_p[[G]] \coloneqq \varprojlim_{\substack{H \lhd G \\ \text{open}}} \mathbb{Z}_p[[G/H]].$$

This L-function is supposed to satisfy the following two properties:

(1) For reasons that become more transparent in Section 9, we denote by $f_! f^* \mathbb{Z}_p(1)$ the étale sheaf on U corresponding to the first Tate twist of the Galois module $\mathbb{Z}_p[[G]]^\sharp$, on which an element σ of the absolute Galois group Gal_F acts by right multiplication with σ^{-1}. Then $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_p(1))$ is a characteristic element for the total complex $\mathrm{R}\Gamma_{\mathrm{c}}(U, f_! f^* \mathbb{Z}_p(1))$ of étale cohomology with proper support with values in this sheaf.

(2) It interpolates the values of the complex L-functions $L_{X-U}(\rho, s)$ for all Artin representations ρ factoring through G.

We refer to Theorem 14.1 for a more precise formulation.
Under the assumption that

(a) $p \neq 2$,

(b) the Iwasawa μ-invariant of any totally real field is zero,

the non-commutative main conjecture is now a theorem, first proved by Ritter and Weiss [RW11]. Almost simultaneously, Kakde [Kak13] published an alternative proof, building upon unpublished work of Kato [Kat06] and the seminal article [Bur15] by Burns. The book [CSSV13] is a comprehensive introduction to Kakde's work. The vanishing of the μ-invariant is still an open conjecture. We refer to [Mih16] for a recent attempt to settle it.
It turns out that properties (1) and (2) are not sufficient to guarantee the uniqueness of $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_p(1))$. It is only well-determined up to an element of a subgroup

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]]) \subset \mathrm{K}_1(\mathbb{Z}_p[[G]]_S).$$

The first objective of this article is to eradicate this indeterminacy. Under the assumptions (a) and (b) we show in Theorem 14.2 that if one lets F_∞ vary over all admissible extensions of F and requires a natural compatibility property for the elements $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_p(1))$, there is indeed a unique choice of such a family. In the course of their formulation of a very general version of the equivariant Tamagawa number conjecture, Fukaya and Kato introduced in [FK06] a certain class of coefficient rings which we call adic rings for short. This class includes among others all finite rings, the Iwasawa algebras of p-adic Lie groups and power series rings in a finite number of indeterminates. Our second objective concerns continuous representations ρ of the absolute Galois group Gal_F over some adic $\mathbb{Z}_p$-algebra Λ. Assume that ρ is *smooth at* ∞ in the sense that it

factors through the Galois group of some (possibly infinite) totally real extension of F unramified over U. As a consequence of Theorem 14.2, we show in Theorem 14.4 and Corollary 14.9 that there exists a unique sensible assignment of a non-commutative *L*-function

$$\mathcal{L}_{F_\infty/F}(U, \rho(1)) \in \mathrm{K}_1(\Lambda[[G]]_S)$$

to any such ρ. In the sequel [Wit] to the present article, we will use our result to prove the existence of the ζ-isomorphism for ρ as predicted by Fukaya's and Kato's central conjecture [FK06, Conj. 2.3.2].

In fact, Corollary 14.9 applies more generally to perfect complexes $\mathcal{F}^\bullet$ of Λ-adic sheaves on U which are smooth at ∞. Moreover, we consider *L*-functions $\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet)$ for the total derived pushforward $\mathrm{R}\, k_* \mathcal{F}^\bullet$, where $k \colon U \to W$ is the open immersion into another dense open subscheme W of X. The extension F_∞/F may be ramified over $W - U$, but places over p remain excluded from W. We also prove the existence of a dual non-commutative *L*-function $\mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)$ such that $\mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)^{-1}$ is a characteristic element for the complex $\mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet)$ and satisfies the appropriate interpolation property. If ρ is a continuous representation as above and $\check{\rho}$ the dual representation over the opposite ring Λ^{op}, then $\mathcal{L}^\oplus_{F_\infty/F}(U, \rho)$ is the image of $\mathcal{L}_{F_\infty/F}(U, \check{\rho}(1))$ under the canonical isomorphism

$$\mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S) \cong \mathrm{K}_1(\Lambda[[G]]_S).$$

As we explain in Corollary 15.3, all of this can be easily extended to the case that F_∞ is a CM field. In this form, our main result also includes non-commutative generalisations of the type of main conjectures that are treated in [GP15] by Greither and Popescu.

We give a short overview on the content of the article. A central input to the proof of Theorem 14.2 is Section 2, in which we show that $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$ vanishes for sufficiently large extensions F_∞/F. The results of this section apply not only to admissible extensions and might be useful in other contexts as well. In Section 3 we recall the essence of the K-theoretic machinery behind the formulation of the main conjecture. We also add some new material, which is needed in the later sections. In Section 4 we prove a technical result that will later be used to relate the cohomology of $f_! f^* \mathcal{F}^\bullet$ over U to the cohomology of $\mathcal{F}^\bullet$ over the integral closure of U in the cyclotomic extension of F. Section 5 contains the construction of the isomorphism $\mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S) \cong \mathrm{K}_1(\Lambda[[G]]_S)$ on the level of Waldhausen categories. First, we explain in full generality how to identify the K-groups of a biWaldhausen category with those of its opposite category and show that this identification is compatible with localisation sequences. Then, we specialise to the case of rings and explain how to identify the opposite category of the category of strictly perfect complexes over $\Lambda[[G]]$ with the category of strictly perfect complexes over $\Lambda^{\mathrm{op}}[[G]]$. Section 6 contains an investigation of the base change properties

of certain splittings of the boundary map

$$d \colon \mathrm{K}_1(\mathbb{Z}_p[[G]]_S) \to \mathrm{K}_0(\mathbb{Z}_p[[G]], S),$$

extending results from [Bur09] and [Wit13b]. With the help of these splittings we are able to produce characteristic elements with good functorial properties, which we call non-commutative algebraic L-functions, following Burns. Section 7 recalls the definition of a Waldhausen category modeling the derived category of perfect complexes of Λ-adic sheaves and the explanation of the property of being smooth at ∞. In Section 8 we consider local and global duality theorems for smooth Λ-adic sheaves. In Section 9 we recall the notion of admissible extensions and the definition of the complexes $f_! f^* \mathcal{F}^\bullet$ induced by the procovering $f \colon U_{F_\infty} \to U$. We then show in Section 10 that our hypothesis (b) on the vanishing of the μ-invariant implies that for any perfect complex of Λ-adic sheaves $\mathcal{F}^\bullet$ smooth at ∞ and any admissible extension F_∞/F, the complexes $\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1))$ have S-torsion cohomology. We also prove a unconditional local variant thereof. This local variant permits us to introduce the notion of non-commutative Euler factors by producing canonical characteristic elements for the complexes $\mathrm{R}\,\Gamma(x, i^* \mathrm{R}\,k_* \mathcal{F}^\bullet(1))$ for any closed point $i \colon x \to W$ of W. Comparing Euler factors with the non-commutative algebraic L-functions of these complexes, we obtain certain elements in $\mathrm{K}_1(\Lambda[[G]])$, which we call local modification factors. In the same way, we also introduce the notion of dual non-commutative Euler factors by producing canonical characteristic elements for the complexes $\mathrm{R}\,\Gamma(x, i^! k_! \mathcal{F}^\bullet)$ and the corresponding dual local modification factors. The investigation of the Euler factors and local modification factors is carried out in Section 11 and Section 12, first in general, then in the special case of the cyclotomic extension. This is followed by a short reminder on L-functions of Artin representations in Section 13.

Section 14 contains the main results of this article. We use Kakde's non-commutative L-functions and the non-commutative algebraic L-function of the complex $\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathbb{Z}_p(1))$ to define global modification factors. Changing the open dense subscheme U is reflected by adding or removing local modification factors. This compatibility allows us to pass to field extensions with arbitrarily large ramification. We can then use the results of Section 2 to prove the uniqueness of the family of modification factors for all pairs (U, F_∞) with F_∞/F admissible and unramified over U. The corresponding non-commutative L-functions are the product of the global modification factors and the non-commutative algebraic L-functions. We then extend in Theorem 14.4 the definition of global modification factors to Λ-adic sheaves smooth at ∞ by requiring a compatibility under twists with certain bimodules. In the same way, we construct global dual modification factors in Theorem 14.5. In Theorem 14.7 we show that the global modification factors are also compatible under changes of the base field F. The non-commutative L-functions for the complexes $\mathcal{F}^\bullet$ are then defined as the product of the algebraic L-function and the modification factors. Corollary 14.9 subsumes the transformation properties of the non-commutative L-functions.

In Section 15 we extend our results to the case of CM-extensions of F. In the following Section 16 we consider the cohomology of the complexes featuring in our main conjecture. In the final Section 17 we establish the link between our conjecture and the formulation of Greither and Popescu. In an appendix we prove a technical result on localisations in polynomial rings with not necessarily commutative coefficients.

Notational conventions

All rings in this article will be associative with identity; a module over a ring will always refer to a left unitary module. If R is a ring, R^{op} will denote the opposite ring and $R^{\times}$ its group of units. We will sometimes write $f \circlearrowleft M$ for an endomorphism f of an object M. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ have their usual meanings. For a prime number p, $\mathbb{Z}_p$ denotes the ring of p-adic integers and $\mathbb{Q}_p$ its fraction field. We write $:=$ to denote the definition of a symbol, reserving the symbol $=$ for expressing an identity. Isomorphisms are denoted by $\cong$, weak equivalences and quasi-isomorphisms by $\sim$. Cofibrations and quotient maps in Waldhausen categories are denoted by $\rightarrowtail$ and $\twoheadrightarrow$. Graded objects are denoted by $P^{\bullet}$ or $P_{\bullet}$, with P^n and P_n referring to the component in degree n, respectively.

Acknowledgements

2   On the first special K-group of a profinite group algebra

Let p be a fixed prime number. In this section only, p is allowed to be even. For any profinite group G, we write $\mathfrak{N}(G)$ for its lattice of open normal subgroups and $G_r \subset G$ for subset of p-regular elements, i.e. the union of all q-Sylow-subgroups for all primes $q \neq p$. Note that

$$G_r = \varprojlim_{U \in \mathfrak{N}(G)} (G/U)_r$$

is a closed subset of G. The group G acts continuously on G_r by conjugation. For any profinite G-set S we write $\mathbb{Z}_p[[S]]$ for the compact G-module which is freely generated by S as compact $\mathbb{Z}_p$-module.

We want to analyse the completed first special K-group

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{SK}_1(\mathbb{Z}_p[G/U])$$

of the profinite group algebra

$$\mathbb{Z}_p[[G]] \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathbb{Z}_p[G/U].$$

Note that $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ is a subgroup of the completed first K-group

$$\widehat{\mathrm{K}}_1(\mathbb{Z}_p[[G]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{K}_1(\mathbb{Z}_p[G/U]).$$

If G has an open pro-p-subgroup which is topologically finitely generated, then

$$\widehat{\mathrm{K}}_1(\mathbb{Z}_p[[G]]) = \mathrm{K}_1(\mathbb{Z}_p[[G]])$$

by [FK06, Prop. 1.5.3]. In the case that G is a pro-p p-adic Lie group a thorough analysis of $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ has been carried out in [SV13]. Note in particular that there are examples of torsionfree p-adic Lie groups with non-trivial first special K-group. Some of the results of *loc. cit.* can certainly be extended to the case that G admits elements of order prime to p. We will not pursue this further. Instead, we limit ourselves to the following results relevant to our application.

Recall from [Oli88, Thm. 10.12] that there is a canonical surjective homomorphism

$$\mathrm{H}_2(G, \mathbb{Z}_p[[G_r]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]]).$$

where

$$\mathrm{H}_2(G, \mathbb{Z}_p[[G_r]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{H}_2(G/U, \mathbb{Z}_p[(G/U)_r])$$

denotes the second continuous homology group of $\mathbb{Z}_p[[G_r]]$. We write $X(G_r) \coloneqq \mathrm{Map}(G_r, \mathbb{Q}_p/\mathbb{Z}_p)$ for the Pontryagin dual of $\mathbb{Z}_p[[G_r]]$, such that the Pontryagin dual of $\mathrm{H}_2(G, \mathbb{Z}_p[[G_r]])$ is $\mathrm{H}^2(G, X(G_r))$.

Lemma 2.1. *Let $G = H \rtimes \Gamma$ be a semi-direct product of a finite normal subgroup $H \subset G$ and $\Gamma \cong \mathbb{Z}_p$. Then $\mathrm{H}^2(G, X(G_r))$ and $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ are finite.*

Proof. Note that $X(G_r) = X(H_r)$ is of finite corank over $\mathbb{Z}_p$. The Hochschild-Serre spectral sequence induces an exact sequence

$$0 \to \mathrm{H}^1(\Gamma, \mathrm{H}^1(H, X(H_r))) \to \mathrm{H}^2(G, X(H_r)) \to \mathrm{H}^0(\Gamma, \mathrm{H}^2(H, X(H_r))) \to 0$$

where both $\mathrm{H}^1(H, X(H_r))$ and $\mathrm{H}^2(H, X(H_r))$ are finite p-groups. The lemma is an immediate consequence. □

We are interested in the following number theoretic situation. Assume that K is an algebraic number field and K_∞ is a $\mathbb{Z}_p$-extension of K. In particular K_∞/K is unramified in the places (including the archimedean places) of K that do not lie over p. Let L_∞ be a finite extension of K_∞ which is Galois over K. Write

$$\begin{aligned}
G &\coloneqq \mathrm{Gal}(L_\infty/K), \\
H &\coloneqq \mathrm{Gal}(L_\infty/K_\infty), \\
\Gamma &\coloneqq \mathrm{Gal}(K_\infty/K)
\end{aligned}$$

for the corresponding Galois groups. We fix a splitting $\Gamma \to G$ such that we may write G as the semi-direct product of H and Γ and let L be the fixed field of a p-Sylow subgroup of G containing Γ. Write $L^{(p)}$ for the maximal Galois p-extension of L inside a fixed algebraic closure $\overline{K}$ of K. Note that $L^{(p)} = L_\infty^{(p)}$ is the subfield of $\overline{K}$ fixed by the closed subgroup $\mathrm{Gal}_{L^{(p)}}$ generated by all q-Sylow subgroups of the absolute Galois group Gal_L for all primes $q \neq p$. Hence, $\mathrm{Gal}_{L^{(p)}} \subset \mathrm{Gal}_{L_\infty}$ is a characteristic subgroup and therefore, $L^{(p)}/K$ is a Galois extension. The following is an adaption of the proof of [FK06, Prop. 2.3.7].

Proposition 2.2. *Set $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(p)}/K)$. Then $\mathrm{H}^2(\mathcal{G}, X(\mathcal{G}_r)) = \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}]]) = 0$.*

Proof. Note that the projection $\mathcal{G} \to G$ induces a canonical isomorphism $X(\mathcal{G}_r) = X(H_r)$ and that $X(H_r)$ is of finite corank over $\mathbb{Z}_p$. We have

$$\mathrm{H}^i(\mathrm{Gal}(L^{(p)}/L), X(H_r)) = \mathrm{H}^i(\mathrm{Gal}_L, X(H_r))$$

for all i according to [NSW00, Cor. 10.4.8] applied to the class of p-groups and the set of all places of L. Moreover, $\mathrm{H}^2(\mathrm{Gal}_L, X(H_r)) = 0$ as a consequence of the fact that $\mathrm{H}^2(\mathrm{Gal}_F, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for any number field F [FK06, Prop. 2.3.7, Claim].
Since $[L : K]$ is prime to p, the restriction map

$$\mathrm{H}^2(\mathcal{G}, X(H_r)) \to \mathrm{H}^2(\mathrm{Gal}(L^{(p)}/L), X(H_r))$$

is split injective. In particular, $\mathrm{H}^2(\mathcal{G}, X(H_r)) = 0$ as claimed. □

Note that if L_∞/K is unramified in a real place of K and $p \neq 2$, then $L^{(p)}/K$ is unramified in this real place as well. For the sake of completeness we also deal with the case $p = 2$ and consider for a set of real places Σ of K such that L_∞/K is unramified over Σ the maximal subfield $L_{\Sigma^c}^{(2)}$ of $L^{(2)}$ which is unramified over Σ. Note that $L_{\Sigma^c}^{(2)}/K$ is still Galois over K.

PROPOSITION 2.3. *Set* $\mathcal{G} \coloneqq \operatorname{Gal}(L_{\Sigma^c}^{(2)}/K)$. *Then* $\operatorname{H}^2(\mathcal{G}, X(\mathcal{G}_r)) = \widehat{\operatorname{SK}}_1(\mathbb{Z}_2[[\mathcal{G}]]) = 0$.

Proof. Let L' be the subfield fixed by the intersection of the centre of G with Γ and let $Y \coloneqq \operatorname{Map}(\operatorname{Gal}(L'/K), X(H_r))$ be the induced module. We obtain a canonical surjection $Y \to X(H_r)$ with kernel Z. For any discrete $\mathcal{G}$-module A we have

$$\operatorname{H}^3(\mathcal{G}, A) = \bigoplus_{v \in \Sigma_{\mathbb{R}}^c} \operatorname{H}^3(\operatorname{Gal}_{K_v}, A)$$

where v runs through set $\Sigma_{\mathbb{R}}^c$ of real places of K not in Σ and $\operatorname{Gal}_{K_v} = \mathbb{Z}/2\mathbb{Z}$ denotes the Galois group of the corresponding local field $K_v = \mathbb{R}$ [NSW00, Prop. 10.6.5]. By the proof of the ($p = 2$)-case in [FK06, Prop. 2.3.7, Claim] we have $\operatorname{H}^2(\operatorname{Gal}_{K_v}, X(H_r)) = 0$ such that

$$\operatorname{H}^3(\mathcal{G}, Z) \to \operatorname{H}^3(\mathcal{G}, Y)$$

is injective and hence,

$$\operatorname{H}^2(\operatorname{Gal}(L_{\Sigma^c}^{(2)}/L'), X(H_r)) \cong \operatorname{H}^2(\mathcal{G}, Y) \to \operatorname{H}^2(\mathcal{G}, X(H_r))$$

is a surjection. Moreover, $\operatorname{Gal}_{L'}$ acts trivially on $X(H_r)$ such that it suffices to show that

$$\operatorname{H}^2(\operatorname{Gal}(L_{\Sigma^c}^{(2)}/L'), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

By the proof of [NSW00, Thm. 10.6.1] we obtain an exact sequence

$$0 \to \operatorname{H}^1(\operatorname{Gal}(L_{\Sigma^c}^{(2)}/L')) \to \operatorname{H}^1(\operatorname{Gal}(L^{(2)}/L')) \to$$
$$\bigoplus_{v \in \Sigma_{\mathbb{R}}^c(L')} \operatorname{H}^1(\operatorname{Gal}_{L'_v}) \to \operatorname{H}^2(\operatorname{Gal}(L_{\Sigma^c}^{(2)}/L')) \to \operatorname{H}^2(\operatorname{Gal}(L^{(2)}/L'))$$

where we have omitted the coefficients $\mathbb{Q}_2/\mathbb{Z}_2$ and $\Sigma_{\mathbb{R}}^c(L')$ denotes the real places of L' lying over $\Sigma_{\mathbb{R}}^c$. But

$$\operatorname{H}^2(\operatorname{Gal}(L^{(2)}/L')) = \operatorname{H}^2(\operatorname{Gal}_{L'}) = 0$$

by [NSW00, Cor. 10.4.8] and [Sch79, Satz 1.(ii)]. Moreover, L' is dense in the product of its real local fields such that for each real place v of L', we find an element a in L' which is negative with respect to v and positive with respect to all other real places. The element of $\operatorname{H}^1(\operatorname{Gal}(L^{(2)}/L'))$ corresponding via Kummer theory to a square root of a maps to the non-trivial element of

$\mathrm{H}^1(\mathrm{Gal}_{L'_v}) = \mathbb{Z}/2\mathbb{Z}$ and to the trivial element for all other real places. This shows that

$$\mathrm{H}^1(\mathrm{Gal}(L_{\Sigma^c}^{(2)}/L')) \to \bigoplus_{v \in \Sigma_{\mathbb{R}}^c(L')} \mathrm{H}^1(\mathrm{Gal}_{L'_v})$$

must be surjective. $\hfill \square$

COROLLARY 2.4. *Let K_∞/K be a $\mathbb{Z}_p$-extension of a number field K and L_∞/K_∞ be a finite extension such that L_∞/K is Galois with Galois group G. Assume further that L_∞/K is unramified in a (possibly empty) set of real places Σ of K. Then there exists a finite extension L'_∞/L_∞ such that*

  (i) *$[L'_\infty : L_\infty]$ is a power of p,*

  (ii) *L'_∞/K is Galois with Galois group G',*

  (iii) *L'_∞/K is unramified over Σ,*

  (iv) *The canonical homomorphism $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G']]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ is the zero map.*

In particular, L'_∞ may be chosen to be totally real if L_∞ is totally real.

Proof. With L as above, set $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(p)}/K)$ if $p \neq 2$ and $\mathcal{G} \coloneqq \mathrm{Gal}(L_{\Sigma^c}^{(2)}/K)$ if $p = 2$ and set $\mathcal{H} \coloneqq \ker \mathcal{G} \to \mathrm{Gal}(K_\infty/K)$. According to Lemma 2.1, $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ is finite and so, the image of

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}]]) = \varprojlim_{U \in \mathfrak{N}(\mathcal{G})} \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}/U \cap \mathcal{H}]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$$

will be equal to the image of $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}/U_0 \cap \mathcal{H}]])$ for some $U_0 \in \mathfrak{N}(\mathcal{G})$. We let L'_∞ be the fixed field of $U_0 \cap \mathcal{H}$. Then L'_∞ clearly satisfies (i), (ii), and (iii). Since $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathcal{G}]]) = 0$ by Prop. 2.2 and Prop. 2.3, it also satisfies (iv). $\hfill \square$

Remark 2.5. The extension L'_∞/K will be unramified outside a finite set of primes, but we cannot prescribe the ramification locus. However, assume L_∞/K is unramified outside the set S of places of K and that the Leopoldt conjecture holds for every finite extension F of K inside the maximal p-extension $L_S^{(p)}$ which is unramified outside S, i. e. that

$$\mathrm{H}^2(\mathrm{Gal}(L_S^{(p)}/F), \mathbb{Q}_p/\mathbb{Z}_p) = 0.$$

Then the same method of proof shows that we can additionally chose L'_∞ to lie in $L_S^{(p)}$.

3 K-THEORY OF ADIC RINGS

For the convenience of the reader, we repeat the essentials of the K-theoretic framework introduced in [Wit14].

The formulation of the non-commutative Iwasawa main conjecture involves certain K-groups. We are mainly interested in the first K-group of a certain class of rings introduced by Fukaya and Kato in [FK06]. It consists of those rings Λ such that for each $n \geq 1$ the n-th power of the Jacobson radical $\mathrm{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_{n \geq 1} \Lambda / \mathrm{Jac}(\Lambda)^n.$$

By definition, Λ carries a natural profinite topology. We will write $\mathfrak{I}_\Lambda$ for the set of open two-sided ideals of Λ, partially ordered by inclusion. In extension of the definition for commutative rings [Gro60, Ch. 0, Def. 7.1.9], these rings should be called *compact adic rings*. We will call these rings *adic rings* for short, as in [Wit14]. We do not intend to insinuate any relation to Huber's more recent concept of adic spaces with this denomination.

Classically, the first K-group of Λ may be described as the quotient of the group

$$\mathrm{Gl}_\infty(\Lambda) \coloneqq \varinjlim_{d} \mathrm{Gl}_d(\Lambda)$$

by its commutator subgroup, but for the formulation of the main conjecture, it is more convenient to follow the constructions of higher K-theory. Among the many roads to higher K-theory, Waldhausen's S-construction [Wal85] turns out to be particularly well-suited for our purposes.

To construct the K-groups of Λ, one can simply apply the S-construction to the category of finitely generated, projective modules over Λ, but the true beauty of Waldhausen's construction is that we can choose among a multitude of different Waldhausen categories that all give rise to the same K-groups. Below, we will study a number of different Waldhausen categories whose K-theory agrees with that of Λ.

We recall that for any ring R, a complex $M^\bullet$ of R-modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of right R-modules, the total complex $(N \otimes_R M)^\bullet$ is acyclic. In particular, any bounded above complex of flat R-modules is *DG*-flat. The notion of *DG*-flatness can be used to define derived tensor products without this boundedness condition. Unbounded complexes will turn up naturally in our constructions. As usual, the complex $M^\bullet$ is called *strictly perfect* if M^n is finitely generated and projective for all n and $M^n = 0$ for almost all n. A complex of R-modules is a *perfect* complex if it is quasi-isomorphic to a strictly perfect complex.

DEFINITION 3.1. For any ring R, we write $\mathbf{SP}(R)$ for the Waldhausen category of strictly perfect complexes and $\mathbf{P}(R)$ for the Waldhausen category of perfect complexes of left R-modules. In both categories, the weak equivalences are

given by quasi-isomorphisms. The cofibrations in $\mathbf{P}(R)$ are all degree-wise injections, the cofibrations in $\mathbf{SP}(R)$ are the degree-wise split injections.

It is a standard fact resulting from the Waldhausen approximation theorem [TT90, 1.9.1] that the inclusion functor $\mathbf{SP}(R) \to \mathbf{P}(R)$ induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}(R)) \cong \mathrm{K}_n(\mathbf{P}(R))$$

between the Waldhausen K-groups of these categories. Moreover, they agree with the Quillen K-groups $\mathrm{K}_n(R)$ of R by the Gillet-Waldhausen theorem [TT90, Thm. 1.11.2, 1.11.7].

If S is another ring and $M^\bullet$ is a complex of S-R-bimodules which is strictly perfect as complex of S-modules, then the tensor product with $M^\bullet$ is a Waldhausen exact functor from $\mathbf{SP}(R)$ to $\mathbf{SP}(S)$ and hence, it induces homomorphisms $\mathrm{K}_n(R) \to \mathrm{K}_n(S)$. Note, however, that the tensor product with $M^\bullet$ does not give a Waldhausen exact functor from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, as it does not preserve weak equivalences nor cofibrations. In the context of homological algebra, this problem can be solved by passing to the derived category, but there is no general recipe how to construct the K-groups of R on the basis of the derived category alone. As a consequence, in order to view certain homomorphisms between K-groups as being induced from a Waldhausen exact functor, one has make a suitable choice of the underlying Waldhausen categories.

If Λ is an adic ring, we will mainly work with the following variant taken from [Wit14]. Its main advantage is that it works well with our later definition of adic sheaves in Section 7 and that it allows a direct construction of most of the relevant Waldhausen exact functors.

DEFINITION 3.2. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ satisfying the following conditions:

1. for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a *DG*-flat perfect complex of Λ/I-modules,

2. for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \to P_J^\bullet$$

   induces an isomorphism of complexes

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I : P_I^\bullet \to Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective and the system $(\mathrm{coker}\, f_I)$ is in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

DEFINITION 3.3. Let Λ' be another adic ring and $M^\bullet$ a complex of Λ'-Λ-bimodules which is strictly perfect as complex of Λ'-modules. We define $\Psi_{M^\bullet}$ to be the following Waldhausen exact functor

$$\Psi_{M^\bullet} : \mathbf{PDG}^{\mathrm{cont}}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'), \qquad P^\bullet \to \left(\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet \right)_{I \in \mathfrak{I}_{\Lambda'}}.$$

If $P^\bullet$ is a strictly perfect complex of Λ-modules, we may identify it with the system

$$(\Lambda/I \otimes_\Lambda P^\bullet)_{I \in \mathfrak{I}_\Lambda}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. By [Wit14, Prop. 3.7], the corresponding Waldhausen exact functor

$$\mathbf{SP}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}(\Lambda)) \cong \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$$

between the K-groups of the Waldhausen categories. Hence, $\mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$ also coincide with the Quillen K-groups of the adic ring Λ and the homomorphism

$$\Psi_{M^\bullet} : \mathrm{K}_n(\Lambda) \to \mathrm{K}_n(\Lambda')$$

induced by the Waldhausen exact functor $\Psi_{M^\bullet}$ coincides with the homomorphism induced by

$$\mathbf{SP}(\Lambda) \to \mathbf{SP}(\Lambda'), \qquad P^\bullet \mapsto (M \otimes_\Lambda P)^\bullet.$$

The essential point in this observation is that $\mathfrak{I}_\Lambda$ is a countable set and that all the transition maps φ_{IJ} are surjective such that passing to the projective limit

$$\varprojlim_{I \in \mathfrak{I}_\Lambda} P_I^\bullet$$

is a Waldhausen exact functor from $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ to the Waldhausen category $\mathbf{P}(\Lambda)$ of perfect complexes of Λ-modules. We write

$$\mathrm{H}^i((P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) := \mathrm{H}^i(\varprojlim_{I \in \mathfrak{I}_\Lambda} P_I^\bullet)$$

for its cohomology groups and note that

$$\mathrm{H}^i((P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) = \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{H}^i(P_I^\bullet)$$

[Wit08, Prop. 5.2.3].

We will also need to consider localisations of certain adic rings: Note that for any adic $\mathbb{Z}_p$-algebra Λ and any profinite group G such that G has an open pro-p-subgroup which is topologically finitely generated, the profinite group algebra $\Lambda[[G]]$ is again an adic ring [Wit14, Prop. 3.2]. Assume further that $G = H \rtimes \Gamma$ is the semi-direct product of a closed normal subgroup H which is itself topologically finitely generated and a subgroup Γ which is isomorphic to $\mathbb{Z}_p$. We set

$$S := S_{\Lambda[[G]]} :=$$

$$\{f \in \Lambda[[G]] \mid \Lambda[[G]] \xrightarrow{\cdot f} \Lambda[[G]] \text{ is perfect as complex of } \Lambda[[H]]\text{-modules}\} \tag{3.1}$$

and call it *Venjakob's canonical Ore set*. We may generalise the results of [CFK$^+$05, §2] as follows.

LEMMA 3.4. *Let*
$$P^\bullet \colon P^{-1} \xrightarrow{\partial} P^0$$
be a complex of length 2 in $\mathbf{SP}(\Lambda[[G]])$. *Then the following are equivalent:*

1. $P^\bullet$ *is perfect as a complex of* $\Lambda[[H]]$-*modules.*

2. P^{-1} *and* P^0 *are isomorphic as* $\Lambda[[G]]$-*modules and* $\mathrm{H}^0(P^\bullet)$ *is finitely generated as* $\Lambda[[H]]$-*module.*

3. $\mathrm{H}^{-1}(P^\bullet) = 0$ *and* $\mathrm{H}^0(P^\bullet)$ *is finitely generated as* $\Lambda[[H]]$-*module.*

4. $\mathrm{H}^{-1}(P^\bullet) = 0$ *and* $\mathrm{H}^0(P^\bullet)$ *is finitely generated and projective as* $\Lambda[[H]]$-*module.*

Proof. Clearly, (4) implies (1).
We prove that (1) implies (2). Assume that $P^\bullet$ is perfect as complex of $\Lambda[[H]]$-modules. Then the class of $P^\bullet$ is trivial in $\mathrm{K}_0(\Lambda[[G]])$ by [Wit13b, Cor. 3.3]. As $\Lambda[[G]]$ is compact and semi-local, $\mathrm{K}_0(\Lambda[[G]])$ is the free abelian group over the isomorphism classes of indecomposable, projective $\Lambda[[G]]$-submodules of $\Lambda[[G]]$. Hence, P^{-1} and P^0 must be isomorphic. Moreover, as $P^\bullet$ is quasi-isomorphic to a strictly perfect complex of $\Lambda[[H]]$-modules, the highest non-vanishing cohomology group of $P^\bullet$ is a finitely presented $\Lambda[[H]]$-module.
We prove that (2) implies (3). It is sufficient to show that
$$\mathrm{H}^{-1}(\Lambda/I[[G/U]] \otimes_{\Lambda[[G]]} P^\bullet) = 0$$
for every open two-sided ideal I of Λ and every open subgroup U of H that is normal in G. Hence, we may assume that Λ and H are finite. Then ∂ is a homomorphism of the torsion $\mathbb{Z}_p[[\Gamma]]$-modules P^{-1} and P^0. As the two modules are isomorphic over $\mathbb{Z}_p[[\Gamma]]$ and $\mathrm{coker}\,\partial$ is finite, ∂ must be a pseudo-isomorphism. Hence, $\ker\partial$ ist finite, as well. But P^{-1} is finitely generated and projective as $\Lambda[[\Gamma]]$-module and therefore, it has no finite $\Lambda[[\Gamma]]$-submodules. We conclude that ∂ is injective.
We prove that (3) implies (4). Note that
$$\Lambda/I[[H/U]] \otimes_{\Lambda[[H]]} \mathrm{H}^0(P^\bullet) \cong \Lambda/I[[G/U]] \otimes_{\Lambda[[G]]} \mathrm{H}^0(P^\bullet)$$
$$\cong \mathrm{H}^0(\Lambda/I[[G/U]] \otimes_{\Lambda[[G]]} P^\bullet) \cong \mathrm{H}^0(\Lambda/I[[H/U]] \otimes_{\Lambda[[H]]} P^\bullet)$$
for any $I \in \mathfrak{I}_\Lambda$ and any open subgroup $U \subset H$ which is normal in G. We conclude that $\mathrm{H}^0(P^\bullet)$ is finitely generated and projective as $\Lambda[[H]]$-module if and only if $\mathrm{H}^0(\Lambda/I[[H/U]] \otimes_{\Lambda[[H]]} P^\bullet)$ is finitely generated and projective as $\Lambda/I[[H/U]]$-module for every I and U. Hence, one may reduce to the case that Λ and H are finite. By replacing G by an appropriate open subgroup of G containing H, we may assume that Γ is central in G, such that we may identify $\Lambda[[G]]$

with the power series ring $\Lambda[[H]][[t]]$ over $\Lambda[[H]]$ in one indeterminate t. For any finite right $\Lambda[[H]]$-module N, the $\mathbb{Z}_p[[t]]$-module $N \otimes_{\Lambda[[H]]} P^{-1}$ cannot contain non-trivial finite $\mathbb{Z}_p[[t]]$-submodules. Moreover, P^{-1} and P^0 are flat $\Lambda[[H]]$-modules such that $P^\bullet$ is a flat resolution of $\mathrm{H}^0(P^\bullet)$ as $\Lambda[[H]]$-module. Hence, we have

$$\mathrm{Tor}_i^{\Lambda[[H]]}(N, \mathrm{H}^0(P^\bullet)) = 0$$

for $i > 1$ and

$$\mathrm{Tor}_1^{\Lambda[[H]]}(N, \mathrm{H}^0(P^\bullet)) \subset N \otimes_{\Lambda[[H]]} P^0$$

is a finite $\mathbb{Z}_p[[t]]$-submodule. Therefore,

$$\mathrm{Tor}_1^{\Lambda[[H]]}(N, \mathrm{H}^0(P^\bullet)) = 0$$

and $\mathrm{H}^0(P^\bullet)$ is finite and projective. $\qquad\qquad\square$

Lemma 3.5. *If Λ and H are finite and $\gamma \in \Gamma$ is a topological generator of Γ, then*

$$T \coloneqq \{(\gamma - 1)^n \mid n \in \mathbb{N}\}$$

is a left and right denominator set in $\Lambda[[G]]$ consisting of left and right non-zero divisors in the sense of [GW04, Ch. 10] such that the left and right localisation

$$\Lambda[[G]]_T.$$

exists. Moreover, S is equal to the set of elements of $\Lambda[[G]]$ that become units in $\Lambda[[G]]_T$. In particular, S is also a left and right denominator set and

$$\Lambda[[G]]_S = \Lambda[[G]]_T.$$

Proof. Set $t \coloneqq \gamma - 1$. Viewing $\Lambda[[G]]$ as a skew power series ring over $\Lambda[[H]]$ in t, it is clear that left and right multiplication with t^n on $\Lambda[[G]]$ is injective with finite cokernel.

According to Lemma 3.4 we have $s \in S$ if and only if $\Lambda[[G]]/\Lambda[[G]]s$ is finite. In particular, we have $T \subset S$. Considering $\Lambda[[G]]/\Lambda[[G]]s$ as a finite $\mathbb{Z}_p[[t]]$-module we see that it is annihilated by a power of t. We conclude that there exists an integer $n \geq 0$ such that for any $a \in \Lambda[[G]]$ there exists a $b \in \Lambda[[G]]$ such that

$$t^n a = bs.$$

Applying this to elements of $T \subset S$, we see that T and S are left denominator set consisting of left and right non-zero divisors such that all elements of S are units in $\Lambda[[G]]_T = \Lambda[[G]]_S$.

Applying the same arguments to $s \in \Lambda[[G]]$ with $\Lambda[[G]]/s\Lambda[[G]]$ finite, we see that T is also a right denominator set.

Assume that $s \in \Lambda[[G]]$ becomes a unit in $\Lambda[[G]]_T$. Then kernel and cokernel of

$$\Lambda[[G]] \xrightarrow{\cdot s} \Lambda[[G]]$$

are annihilated by powers of t. Considering $\Lambda[[G]]$ as a finitely generated $\mathbb{Z}_p[[t]]$-module annihilated by a power of p, we conclude that the cokernel is finite, which implies that $s \in S$. Since T is a right denominator set, the same is then true for

$$ S = \Lambda[[G]] \cap \Lambda[[G]]_T^\times. $$

$\square$

LEMMA 3.6. *Assume that $\Lambda[[H]]$ is noetherian. Then:*

1. $S = \{f \in \Lambda[[G]] \mid \Lambda[[G]]/\Lambda[[G]]f \text{ is a f.g. left } \Lambda[[H]]\text{-module}\}.$

2. $S = \{f \in \Lambda[[G]] \mid \Lambda[[G]]/f\Lambda[[G]] \text{ is a f.g. right } \Lambda[[H]]\text{-module}\}.$

3. *S is a left and right denominator set consisting of left and right non-zero divisors.*

4. *A perfect complex of left $\Lambda[[G]]$-modules is perfect as complex of $\Lambda[[H]]$-modules if and only its cohomology groups are S-torsion.*

Proof. Lemma 3.4 implies that the elements of S are right non-zero divisors and that (1) holds. Under the assumption that $\Lambda[[H]]$ is noetherian, we know by [Wit13b, Cor. 2.21] that S is a left denominator set. Assertion (4) follows from [Wit13b, Thm. 2.18]. Write $(\Lambda[[G]])^{\mathrm{op}}$ and Λ^{op} for the opposite rings of $\Lambda[[G]]$ and Λ, respectively. Consider the ring isomorphism

$$ \sharp \colon (\Lambda[[G]])^{\mathrm{op}} \to \Lambda^{\mathrm{op}}[[G]] $$

that maps $g \in G$ to g^{-1}. To prove the remaining assertions, it is sufficient to show that $\sharp$ maps $S_{\Lambda[[G]]} \subset (\Lambda[[G]])^{\mathrm{op}}$ to $S_{\Lambda^{\mathrm{op}}[[G]]}$.

If Λ and H are finite and $\gamma \in \Gamma$ is a topological generator, then $\sharp$ maps $t \coloneqq \gamma - 1$ to $t' \coloneqq \gamma^{-1} - 1$ and hence, it maps $T \coloneqq \{t^n \mid n \in \mathbb{N}\}$ to $T' \coloneqq \{t'^n \mid n \in \mathbb{N}\}$. Using Lemma 3.5 for T and T', we conclude that $\sharp(S_{\Lambda[[G]]}) = S_{\Lambda^{\mathrm{op}}[[G]]}$.

In the general case, write

$$ \Lambda^{\mathrm{op}}[[G]] = \varprojlim_{U,I} \Lambda^{\mathrm{op}}/I[[G/H \cap U]] $$

where the limit runs over all open two-sided ideals I of Λ and all open normal subgroups U of G and note that $\Lambda^{\mathrm{op}}[[G]] \xrightarrow{\cdot s^\sharp} \Lambda^{\mathrm{op}}[[G]]$ is perfect over $\Lambda^{\mathrm{op}}[[H]]$ if and only if $(\Lambda/I)^{\mathrm{op}}[[G/H \cap U]] \xrightarrow{\cdot s^\sharp} (\Lambda/I)^{\mathrm{op}}[[G/H \cap U]]$ is perfect over the finite ring $(\Lambda/I)^{\mathrm{op}}[[H/H \cap U]]$ for all I and U. $\square$

For general Λ and H, the set S is no longer a left or right denominator set, as the following example shows.

Example 3.7. Assume that either $\Lambda = \mathbb{F}_p$ is the finite field with p elements and H is the free pro-p group on two topological generators with trivial action of Γ or $\Lambda = \mathbb{F}_p\langle\langle x, y \rangle\rangle$ is the power series ring in two non-commuting indeterminates x, y and H is trivial. In both cases, $\Lambda[[G]] = \mathbb{F}_p\langle\langle x, y \rangle\rangle[[t]]$ is the power series ring over $\mathbb{F}_p\langle\langle x, y \rangle\rangle$ with t commuting with x and y and the set S is the set of those power series $f(x, y, t)$ with $f(0, 0, t) \neq 0$. Set $s := x - t \in S$. If S were a left denominator set, we could find

$$a := \sum_{i=0}^{\infty} a_i t^i \in \mathbb{F}_p\langle\langle x, y \rangle\rangle[[t]], \quad b := \sum_{i=0}^{\infty} b_i t^i \in S$$

such that $as = by$, i.e.

$$a_0 x = b_0 y, \qquad a_i x - a_{i-1} = b_i y \quad \text{for } i > 0.$$

The only solution for this equation is $a = b = 0$, which contradicts the assumption $b \in S$.

Nevertheless, using Waldhausen K-theory, we can still give a sensible definition of $\mathrm{K}_1(\Lambda[[G]]_S)$ even if $\Lambda[[G]]_S$ does not exist.

Definition 3.8. We write $\mathbf{SP}^{w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{SP}(\Lambda[[G]])$ of strictly perfect complexes of $\Lambda[[G]]$-modules which are perfect as complexes of $\Lambda[[H]]$-modules.

We write $w_H\mathbf{SP}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{SP}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{SP}^{w_H}(\Lambda[[G]])$.

The same construction also works for $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$:

Definition 3.9. We write $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ of objects $(P_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ such that

$$\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} P_J^\bullet$$

is a perfect complex of $\Lambda[[H]]$-modules.

We write $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.

From the approximation theorem [TT90, 1.9.1] and [Wit14, Prop. 3.7] we conclude that

$$\mathrm{K}_n(\mathbf{SP}^{w_H}(\Lambda[[G]])) \cong \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])),$$
$$\mathrm{K}_n(w_H\mathbf{SP}(\Lambda[[G]])) \cong \mathrm{K}_n(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

We may then set for all $n \geq 0$

$$\begin{aligned}
\mathrm{K}_n(\Lambda[[G]], S) &\coloneqq \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])), \\
\mathrm{K}_{n+1}(\Lambda[[G]]_S) &\coloneqq \mathrm{K}_{n+1}(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])).
\end{aligned} \tag{3.2}$$

If $\Lambda[[H]]$ is noetherian, these groups agree with their usual definition [Wit14, § 4].
A closely related variant of $\mathbf{SP}^{w_H}(\Lambda[[G]])$ is the following Waldhausen category.

DEFINITION 3.10. Let $\mathbf{SP}(\Lambda[[H]], G)$ be the Waldhausen category of complexes of $\Lambda[[G]]$-modules which are strictly perfect as complexes of $\Lambda[[H]]$-modules. Cofibrations are the injective morphisms with cokernel in $\mathbf{SP}(\Lambda[[H]], G)$; the weak equivalences are given by the quasi-isomorphisms.

In other words, $\mathbf{SP}(\Lambda[[H]], G)$ is the Waldhausen category of bounded complexes over the exact category of $\Lambda[[G]]$-modules which are finitely generated and projective as $\Lambda[[H]]$-modules and hence, the groups $\mathrm{K}_n(\mathbf{SP}(\Lambda[[H]], G))$ agree with the Quillen K-groups of this exact category. Unfortunately, we cannot prove in general that $\mathrm{K}_n(\mathbf{SP}(\Lambda[[H]], G))$ agrees with $\mathrm{K}_n(\Lambda[[G]], S)$. However, we shall see below that we always have a surjection

$$\mathrm{K}_0(\mathbf{SP}(\Lambda[[H]], G)) \to \mathrm{K}_0(\Lambda[[G]], S).$$

This is sufficient for our applications.

LEMMA 3.11. *Let $P^\bullet$ be a complex of projective compact $\Lambda[[G]]$-modules that is bounded above. Assume that there exists a bounded above complex $K^\bullet$ of finitely generated, projective $\Lambda[[H]]$-modules that is quasi-isomorphic to $P^\bullet$ as complex of $\Lambda[[H]]$-modules. Then there exists in the category of complexes of $\Lambda[[G]]$-modules an injective endomorphism*

$$\psi\colon \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet \to \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet$$

and a quasi-isomorphism

$$f\colon P^\bullet \to \operatorname{coker} \psi.$$

such that $\operatorname{coker} \psi$ is a bounded above complex of $\Lambda[[G]]$-modules which are finitely generated and projective as $\Lambda[[H]]$-modules.
In particular, if $P^\bullet$ is perfect as complex of $\Lambda[[H]]$-modules, then $P^\bullet$ is perfect as complex of $\Lambda[[G]]$-modules and $\operatorname{coker} \psi$ is in $\mathbf{SP}(\Lambda[[H]], G)$.

Proof. Since $K^\bullet$ is a bounded above complex of finitely generated projective $\Lambda[[H]]$-modules, there exists a quasi-isomorphism $\alpha\colon K^\bullet \to P^\bullet$ of complexes of $\Lambda[[H]]$-modules, which is automatically continuous for the compact topologies on $K^\bullet$ and $P^\bullet$. Every projective compact $\Lambda[[G]]$-module is also projective in the category of compact $\Lambda[[H]]$-modules. Hence, there exists a weak equivalence $\beta\colon P^\bullet \to K^\bullet$ in the category of complexes of compact $\Lambda[[H]]$-modules

such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to the identity. Fix a topological generator $\gamma \in \Gamma$ and set

$$g \colon K^\bullet \to K^\bullet, \qquad x \mapsto \beta(\gamma\alpha(x)),$$
$$\psi \colon \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet \to \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet, \qquad \lambda \otimes x \mapsto \lambda \otimes x - \lambda\gamma^{-1} \otimes g(x).$$

Then ψ is a $\Lambda[[G]]$-linear complex morphism. Moreover, $\operatorname{coker}\psi$ is finitely generated over $\Lambda[[H]]$ in each degree. Indeed, if we set $t \coloneqq \gamma - 1$ and let $(e_1, \dots, e_m)$ denote a generating system of the $\Lambda[[H]]$-module K^n in degree n, then $(t^k \otimes e_i)_{k \in \mathbb{N}_0, i=1,\dots,m}$ is a topological generating system of $\Lambda[[G]] \otimes_{\Lambda[[H]]} K^n$ over $\Lambda[[H]]$. But

$$t^k \otimes v = t^{k-1} \otimes (g(v) - v) + \psi(\gamma t^{k-1} \otimes v)$$

for all $v \in K^n$, such that $\operatorname{coker}\psi$ is already generated by the images of $1 \otimes e_1, \dots, 1 \otimes e_m$.

From Lemma 3.4 we conclude that ψ is injective and that $\operatorname{coker}\psi$ is finitely generated and projective over $\Lambda[[H]]$ in each degree. Set $Q^\bullet \coloneqq \operatorname{coker}\psi$. Since $P^\bullet$ is a bounded above complex of projective compact $\Lambda[[G]]$-modules, there exists a quasi-isomorphism f completing the homotopy-commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet & \xrightarrow{\lambda\hat{\otimes}x \mapsto \lambda\hat{\otimes}x - \lambda\gamma^{-1}\hat{\otimes}\gamma x} & \Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet & \xrightarrow{\lambda\hat{\otimes}x \mapsto \lambda x} & P^\bullet & \longrightarrow & 0 \\
& & {\scriptstyle \sim}\downarrow{\scriptstyle \mathrm{id}\hat{\otimes}\beta} & & {\scriptstyle \sim}\downarrow{\scriptstyle \mathrm{id}\hat{\otimes}\beta} & & {\scriptstyle \sim}\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet & \xrightarrow{\psi} & \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet & \longrightarrow & Q^\bullet & \longrightarrow & 0
\end{array}
$$

in the category of complexes of compact $\Lambda[[G]]$-modules. Here, $\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet$ denotes the completed tensor product. The exactness of the first row follows from [Wit13b, Prop. 2.4]. If we can choose $K^\bullet$ to be a strictly perfect complex of $\Lambda[[H]]$-modules, then $P^\bullet$ is also quasi-isomorphic to the cone of ψ, which is strictly perfect as complex of $\Lambda[[G]]$-modules. Moreover, $\operatorname{coker}\psi$ is a bounded complex and hence, an object of $\mathbf{SP}(\Lambda[[H]], G)$. $\qquad\square$

PROPOSITION 3.12. *Let $\gamma \in \Gamma$ be a topological generator. The functor*

$$C_\gamma \colon \mathbf{SP}(\Lambda[[H]], G) \to \mathbf{SP}^{w_H}(\Lambda[[G]]),$$

$$P^\bullet \mapsto \operatorname{Cone}(\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\lambda\otimes p \mapsto \lambda\otimes p - \lambda\gamma^{-1}\otimes\gamma p} \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet)$$

is well defined and Waldhausen exact. It induces a surjection

$$C_\gamma \colon \mathrm{K}_0(\mathbf{SP}(\Lambda[[H]], G)) \to \mathrm{K}_0(\mathbf{SP}^{w_H}(\Lambda[[G]]))$$

which is independent of the choice of γ.

Proof. From [Wit13b, Prop. 2.4] we conclude that

$$0 \to \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\mathrm{id} - (\cdot \gamma^{-1} \otimes \gamma \cdot)} \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\lambda \otimes p \mapsto \lambda p} P^\bullet \to 0$$

is an exact sequence of complexes of $\Lambda[[G]]$-modules for any $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$. In particular, the strictly perfect complex of $\Lambda[[G]]$-modules $C_\gamma(P^\bullet)$ is quasi-isomorphic to $P^\bullet$ in the category of complexes of $\Lambda[[G]]$-modules and therefore perfect as complex of $\Lambda[[H]]$-modules. Thus, $C_\gamma(P^\bullet)$ is an object of $\mathbf{SP}^{w_H}(\Lambda[[G]])$. The Waldhausen exactness of the functor C_γ follows easily from the Waldhausen exactness of the cone construction.
Consider the Waldhausen category $\mathbf{P}^{w_H}(\Lambda[[G]])$ of perfect complexes of $\Lambda[[G]]$-modules which are also perfect as complexes of $\Lambda[[H]]$-modules. The approximation theorem [TT90, 1.9.1] implies that the inclusion

$$\iota \colon \mathbf{SP}^{w_H}(\Lambda[[G]]) \to \mathbf{P}^{w_H}(\Lambda[[G]])$$

induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}^{w_H}(\Lambda[[G]])) \cong \mathrm{K}_n(\mathbf{P}^{w_H}(\Lambda[[G]]))$$

for all n. The functorial quasi-isomorphism $C_\gamma(P^\bullet) \xrightarrow{\sim} P^\bullet$ in $\mathbf{P}^{w_H}(\Lambda[[G]])$ implies that the homomorphism of K-groups induced by $\iota \circ C_\gamma$ agrees with the homomorphism induced by the inclusion $\iota' \colon \mathbf{SP}(\Lambda[[H]], G) \to \mathbf{P}^{w_H}(\Lambda[[G]])$. Since $\mathrm{K}_0(\mathbf{P}^{w_H}(\Lambda[[G]]))$ is generated by the quasi-isomorphism classes of complexes in $\mathbf{P}^{w_H}(\Lambda[[G]])$, we deduce from Lemma 3.11 that ι' induces a surjection

$$\mathrm{K}_0(\mathbf{SP}(\Lambda[[H]], G)) \to \mathrm{K}_0(\mathbf{P}^{w_H}(\Lambda[[G]])).$$

$\square$

In the light of Proposition 3.12, we will write

$$[P^\bullet] \coloneqq [C_\gamma(P^\bullet)] \in \mathrm{K}_0(\Lambda[[G]], S)$$

for any $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$.

Remark 3.13. In order to deduce from the approximation theorem (applied to the opposite categories) that C_γ induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}(\Lambda[[H]], G)) \cong \mathrm{K}_n(\mathbf{SP}^{w_H}(\Lambda[[G]]))$$

for all n, it would suffice to verify that for every complex $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$ and every morphism $f \colon K^\bullet \to P^\bullet$ in $\mathbf{P}^{w_H}(\Lambda[[G]])$, there exists a morphism $f' \colon Q^\bullet \to P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$ and a quasi-isomorphism $w \colon K^\bullet \xrightarrow{\sim} Q^\bullet$ in $\mathbf{P}^{w_H}(\Lambda[[G]])$ such that $f = f' \circ w$.
Thanks to a result of Muro and Tonks [MT08], the groups $\mathrm{K}_0(\mathbf{W})$ and $\mathrm{K}_1(\mathbf{W})$ of any Waldhausen category $\mathbf{W}$ can be described as the cokernel and kernel of a homomorphism

$$\partial \colon \mathcal{D}_1(\mathbf{W}) \to \mathcal{D}_0(\mathbf{W}) \tag{3.3}$$

between two nil-2-groups (i. e. $[a,[b,c]] = 1$ for any three group elements a, b, c) that are given by explicit generators and relations in terms of the structure of the underlying Waldhausen category. As additional structure, there exists a pairing

$$\mathcal{D}_0(\mathbf{W}) \times \mathcal{D}_0(\mathbf{W}) \to \mathcal{D}_1(\mathbf{W}), \qquad (A, B) \mapsto \langle A, B \rangle$$

satisfying

$$\partial \langle A, B \rangle = B^{-1} A^{-1} B A,$$
$$\langle \partial a, \partial b \rangle = b^{-1} a^{-1} b a,$$
$$\langle A, B \rangle \langle B, A \rangle = 1,$$
$$\langle A, BC \rangle = \langle A, B \rangle \langle A, C \rangle.$$

In other words, $\mathcal{D}_\bullet(\mathbf{W})$ is a stable quadratic module in the sense of [Bau91]. In particular, $X \in \mathcal{D}_0(\mathbf{W})$ operates from the right on $a \in \mathcal{D}_1(\mathbf{W})$ via

$$a^X := a \langle X, \partial a \rangle.$$

More explicitly, $\mathcal{D}_0(\mathbf{W})$ is the free nil-2-group generated by the objects of $\mathbf{W}$ different from the zero object, while $\mathcal{D}_1(\mathbf{W})$ is generated by all weak equivalences and exact sequences in $\mathbf{W}$ subject to the following list of relations:
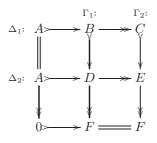
(R1) $\partial[\alpha] = [B]^{-1}[A]$ for a weak equivalence $\alpha \colon A \xrightarrow{\sim} B$,

(R2) $\partial[\Delta] = [B]^{-1}[C][A]$ for an exact sequence $\Delta \colon A \rightarrowtail B \twoheadrightarrow C$.

(R3) $\langle [A], [B] \rangle = [B \rightarrowtail A \oplus B \twoheadrightarrow A]^{-1}[A \rightarrowtail A \oplus B \twoheadrightarrow B]$ for any pair of objects A, B.

(R4) $[0 \rightarrowtail 0 \twoheadrightarrow 0] = 1_{\mathcal{D}_1}$,

(R5) $[\beta\alpha] = [\beta][\alpha]$ for weak equivalences $\alpha \colon A \xrightarrow{\sim} B$, $\beta \colon B \xrightarrow{\sim} C$,

(R6) $[\Delta'][\alpha][\gamma]^{[A]} = [\beta][\Delta]$ for any commutative diagram

(R7) $[\Gamma_1][\Delta_1] = [\Delta_2][\Gamma_2]^{[A]}$ for any commutative diagram

[MT07, Def. 1.2], [Wit14, Def. A.4].

In particular, $K_0(\Lambda[[G]], S)$ is the abelian group generated by the symbols $[P^\bullet]$ with $P^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ modulo the relations

$$[P^\bullet] = [Q^\bullet] \qquad \text{if } P^\bullet \text{ and } Q^\bullet \text{ are weakly equivalent,}$$
$$[P_2^\bullet] = [P_1^\bullet] + [P_3^\bullet] \qquad \text{if } 0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0 \text{ is an exact sequence.}$$

If $f\colon P^\bullet \to P^\bullet$ is an endomorphism which is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, we can assign to it a class $[f \circlearrowright P^\bullet]$ in $K_1(\Lambda[[G]])$. By the classical definition of the first K-group as factor group of the general linear group it is clear that these classes generate $K_1(\Lambda[[G]])$. It then follows from the splitting of the K-theory localisation sequence [Wit13b, Cor. 3.3] that the classes $[f \circlearrowright P^\bullet]$ of endomorphisms $f\colon P^\bullet \to P^\bullet$ which are weak equivalences in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ generate $K_1(\Lambda[[G]]_S)$. The relations that are satisfied by these generators can be read off from the above relations for $\mathcal{D}_1\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.

Remark 3.14. Some authors prefer the theory of determinant functors and Deligne's category of virtual objects [Del87] as an alternative model for the 1-type of the K-theory spectrum. We refer to [MTW15] for the precise connection of the two approaches.

Let Λ and Λ' be two adic $\mathbb{Z}_p$-algebras and $G = H \rtimes \Gamma$, $G = H' \rtimes \Gamma'$ be profinite groups, such that H and H' contain open, topologically finitely generated pro-p subgroups and $\Gamma \cong \mathbb{Z}_p \cong \Gamma'$. Suppose that $K^\bullet$ is a complex of $\Lambda'[[G']]$-$\Lambda[[G]]$-bimodules, strictly perfect as complex of $\Lambda'[[G']]$-modules and assume that there exists a complex $L^\bullet$ of $\Lambda'[[H']]$-$\Lambda[[H]]$-bimodules, strictly perfect as complex of $\Lambda'[[H']]$-modules, and a quasi-isomorphism of complexes of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet.$$

Here,

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \coloneqq \varprojlim_{I \in \mathfrak{I}_{\Lambda'[[G']]}} \varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} L/IL^\bullet \otimes_{\Lambda[[H]]} \Lambda[[G]]/J$$

denotes the *completed tensor product*.

In the above situation, the Waldhausen exact functor

$$\Psi_{K^\bullet}\colon \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']]) \tag{3.4}$$

takes objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ to objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_{H'}}(\Lambda'[[G']])$ and weak equivalences of $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ to weak equivalences of $w_{H'}\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']])$ [Wit14, Prop. 4.6]. Hence, it also induces homomorphisms between the corresponding K-groups. In particular, this applies to the following examples:

Example 3.15. [Wit14, Prop. 4.7]

1. Assume $G = G'$, $H = H'$. For any complex $P^\bullet$ of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, let $K^\bullet$ be the complex

$$P[[G]]^{\delta^\bullet} := \Lambda'[[G]] \otimes_{\Lambda'} P^\bullet$$

   of $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodules with the right G-operation given by the diagonal action on both factors. This applies in particular for any complex $P^\bullet$ of Λ'-Λ-bimodules, strictly perfect as complex of Λ'-modules and equipped with the trivial G-operation.

2. Assume $\Lambda = \Lambda'$. Let $\alpha \colon G \to G'$ be a continuous homomorphism such that α maps H to H' and induces a bijection of G/H and G'/H'. Let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$.

3. Assume that G' is an open subgroup of G and set $H' := H \cap G'$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the complex concentrated in degree 0 given by the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]]$.

Example 3.16. The assumptions in Example 3.15.(2) are in fact stronger than necessary. We may combine it with the following result. Assume that G is an open subgroup of G' such that $H := H' \cap G = H'$ and $\Gamma = (\Gamma')^{p^n}$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$. Fix a topological generator $\gamma' \in \Gamma'$ and let $L^\bullet$ be the $\Lambda[[H]]$-$\Lambda[[H]]$-sub-bimodule of $\Lambda[[G']]$ generated as left $\Lambda[[H]]$-module by $1, \gamma', (\gamma')^2, \ldots, (\gamma')^{p^n-1}$. Then $L^\bullet$ is a strictly perfect complex of $\Lambda[[H]]$-modules concentrated in degree 0 and the canonical map

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet, \qquad \ell \hat{\otimes} \lambda \mapsto \ell\lambda$$

is an isomorphism of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules such that [Wit14, Prop. 4.6] applies. In combination with Example 3.15.(2) this implies that any continuous group homomorphism $\alpha \colon G \to G'$ such that $\alpha(G) \not\subset H'$ induces Waldhausen exact functors between all three variants of the above Waldhausen categories.

Example 3.17. As a special case of Example 3.15.(1), assume that $\Lambda = \mathbb{Z}_p$ and that ρ is some continuous representation of G on a finitely generated and projective Λ'-module. Let $\rho^\sharp$ be the Λ'-$\mathbb{Z}_p[[G]]$-bimodule which agrees with ρ as Λ'-module, but on which $g \in G$ acts from the right by the left operation of g^{-1} on ρ. We thus obtain Waldhausen exact functors

$$\Phi_\rho := \Psi_{\Lambda'[[\Gamma]]} \circ \Psi_{\rho^\sharp[[G]]^\delta} \tag{3.5}$$

from all three variants of the Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(\mathbb{Z}_p[[G]])$ to the corresponding variant of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[\Gamma]])$. If Λ' is a commutative adic $\mathbb{Z}_p$-algebra, then the image of

$$\left[\mathbb{Z}_p[[G]] \xrightarrow{\cdot g} \mathbb{Z}_p[[G]] \right] \in \mathrm{K}_1(\mathbb{Z}_p[[G]]), \quad g \in G,$$

under the composition of Φ_ρ with

$$\det \colon \mathrm{K}_1(\Lambda'[[\Gamma]]) \xrightarrow{\cong} \Lambda'[[\Gamma]]^\times$$

is $\bar{g}\det(\rho(g))^{-1}$, where $\bar{g}$ denotes the image of g under the projection $G \to \Gamma$. Note that this differs from [CFK+05, (22)] by a sign. So, our evaluation at ρ corresponds to the evaluation at the representation dual to ρ in terms of the cited article.

4 A property of S-torsion complexes

In this section, we prove Proposition 4.1, which is an abstract generalisation of [Wit13a, Prop. 2.1]. We will apply this proposition later in Section 10. With the notation of the previous section, fix a topological generator $\gamma \in \Gamma$ and set $t \coloneqq \gamma - 1$. Assume for the moment that Λ is a finite $\mathbb{Z}_p$-algebra and that H is a finite group. By Lemma 3.5, we have

$$\Lambda[[G]]_S = \varinjlim_{n \geq 0} \Lambda[[G]]t^{-n}$$

as $\Lambda[[G]]$-modules.
Assume that $p^{i+1} = 0$ in Λ. Then

$$\binom{p^{n+i}}{k} = 0$$

in Λ whenever $p^n \nmid k$. Hence,

$$\gamma^{p^{n+i}} - 1 = (t+1)^{p^{n+i}} - 1 = t^{p^n} \sum_{k=1}^{p^i} \binom{p^{n+i}}{kp^n} t^{p^n(k-1)},$$

$$t^{p^{n+i}} = (\gamma - 1)^{p^{n+i}} - (1-1)^{p^{n+i}} = \sum_{k=1}^{p^i} \binom{p^{n+i}}{kp^n}(\gamma^{kp^n} - 1)(-1)^{p^n(p^i-k)}$$

$$= (\gamma^{p^n} - 1) \sum_{k=1}^{p^i} \binom{p^{n+i}}{kp^n}(-1)^{p^n(p^i-k)} \sum_{\ell=0}^{k-1} \gamma^{\ell p^n}$$

and therefore,

$$\Lambda[[G]]_S = \varinjlim_{n \geq 0} \Lambda[[G]](\gamma^{p^n} - 1)^{-1}.$$

Since H was assumed to be finite, the same is true for the automorphism group of H. We conclude that γ^{p^n} is a central element of G and $\Gamma^{p^n} \subset G$ a central subgroup for all $n \geq n_0$ and n_0 large enough. Set

$$N_n \coloneqq \sum_{k=0}^{p-1} \gamma^{p^n k}.$$

The homomorphism

$$\Lambda[[G]](\gamma^{p^n} - 1)^{-1} \to \Lambda[[G/\Gamma^{p^n}]], \qquad \lambda(\gamma^{p^n} - 1)^{-1} \mapsto \lambda + \Lambda[[G]](\gamma^{p^n} - 1)$$

induces an isomorphism $\Lambda[[G]](\gamma^{p^n}-1)^{-1}/\Lambda[[G]] \cong \Lambda[[G/\Gamma^{p^n}]]$ such that the diagram

$$
\begin{array}{ccc}
\Lambda[[G]](\gamma^{p^n}-1)^{-1}/\Lambda[[G]] & \overset{\subset}{\longrightarrow} & \Lambda[[G]](\gamma^{p^{n+1}}-1)^{-1}/\Lambda[[G]] \\
\downarrow{\cong} & & \downarrow{\cong} \\
\Lambda[[G/\Gamma^{p^n}]] & \xrightarrow{\;\cdot N_n\;} & \Lambda[[G/\Gamma^{p^{n+1}}]]
\end{array}
$$

commutes. Hence, we obtain an isomorphism of (left and right) $\Lambda[[G]]$-modules

$$
\Lambda[[G]]_S/\Lambda[[G]] \cong \varinjlim_n \Lambda[[G/\Gamma^{p^n}]].
$$

We note that this isomorphism may depend on the choice of the topological generator γ.

For any strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$-modules, we thus obtain an exact sequence

$$
0 \to P^\bullet \to \Lambda[[G]]_S \otimes_{\Lambda[[G]]} P^\bullet \to \varinjlim_n \Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet \to 0.
$$

If $P^\bullet$ is also perfect as a complex of $\Lambda[[H]]$-modules such that the cohomology of $P^\bullet$ is S-torsion by Lemma 3.6, then we conclude that there exists an isomorphism

$$
P^\bullet[1] \cong \varinjlim_n \Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet
$$

in the derived category of complexes of $\Lambda[[G]]$-modules. In particular, the righthand complex is perfect as complex of $\Lambda[[G]]$-modules and of $\Lambda[[H]]$-modules. This signifies that its cohomology modules

$$
\mathrm{H}^k(\varinjlim_n \Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet) \cong \varinjlim_n \mathrm{H}^k(\Lambda[[G/\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet) \cong \mathrm{H}^{k+1}(P^\bullet)
$$

are finite as abelian groups.

We now drop the assumption that Λ and H are finite. Let $I \subset J$ be two open ideals of Λ and $U \subset V$ be the intersections of two open normal subgroups of G with H. Then the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda/I[[G/U]] & \longrightarrow & \Lambda/I[[G/U]]_S & \longrightarrow & \varinjlim_n \Lambda/I[[G/U\Gamma^{p^n}]] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda/J[[G/V]] & \longrightarrow & \Lambda/J[[G/V]]_S & \longrightarrow & \varinjlim_n \Lambda/J[[G/V\Gamma^{p^n}]] & \longrightarrow & 0
\end{array}
$$

commutes and the downward pointing arrows are surjections. Tensoring with $P^\bullet$ and passing to the inverse limit we obtain the exact sequence

$$
0 \to P^\bullet \to \varprojlim_{I,U} \Lambda/I[[G/U]]_S \otimes_{\Lambda[[G]]} P^\bullet \to \varprojlim_{I,U} \varinjlim_n \Lambda/I[[G/U\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet \to 0.
$$

If $P^\bullet$ is also perfect as a complex of $\Lambda[[H]]$-modules, then complex in the middle is acyclic and we obtain again an isomorphism

$$P^\bullet[1] \cong \varprojlim_{I,U} \varinjlim_{n} \Lambda/I[[G/U\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet$$

in the derived category of complexes of $\Lambda[[G]]$-modules and hence, isomorphisms of $\Lambda[[G]]$-modules

$$\mathrm{H}^{k+1}(P^\bullet) \cong \varprojlim_{I,U} \varinjlim_{n} \mathrm{H}^k(\Lambda/I[[G/U\Gamma^{p^n}]] \otimes_{\Lambda[[G]]} P^\bullet).$$

Here, we use that the modules in the projective system on the righthand side are finite and thus $\varprojlim$-acyclic.

Finally, assume that $(Q_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ is a complex in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. Then we can find a strictly perfect complex of $\Lambda[[G]]$-modules $P^\bullet$ and a weak equivalence

$$f \colon (\Lambda[[G]]/J \otimes_{\Lambda[[G]]} P^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}} \to (Q_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$$

in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda)$ [Wit08, Cor. 5.2.6]. Moreover, this complex $P^\bullet$ will also be perfect as a complex of $\Lambda[[H]]$-modules. For $I \in \mathfrak{I}_\Lambda$, U the intersection of an open normal subgroup of G with H and a positive integer n such that Γ^{p^n} is central in G/U we set

$$J_{I,U,n} \coloneqq \ker \Lambda[[G]] \to \Lambda/I[[G/U\Gamma^{p^n}]],$$

such that the $J_{I,U,n}$ form a final subsystem in $\mathfrak{I}_{\Lambda[[G]]}$. We conclude:

PROPOSITION 4.1. *For* $(Q_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ *in* $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ *there exists an isomorphism*

$$\mathrm{R}\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} Q_J^\bullet[1] \cong \mathrm{R}\varprojlim_{I,U} \varinjlim_{n} Q_{J_{I,U,n}}^\bullet$$

in the derived category of $\Lambda[[G]]$*-modules and isomorphisms of* $\Lambda[[G]]$*-modules*

$$\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} \mathrm{H}^{k+1}(Q_J^\bullet) \cong \varprojlim_{I,U} \varinjlim_{n} \mathrm{H}^k(Q_{J_{I,U,n}}^\bullet).$$

Remark 4.2. For any $(Q_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ we obtain in the same way a distinguished triangle

$$\mathrm{R}\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} Q_J^\bullet \to \mathrm{R}\varprojlim_{I,U} \left(\Lambda/I[[G/U]]_S \otimes_{\Lambda/I[[G/U]]}^{\mathbb{L}} \mathrm{R}\varprojlim_{n} Q_{J_{I,U,n}}^\bullet \right) \to \mathrm{R}\varprojlim_{I,U} \varinjlim_{n} Q_{J_{I,U,n}}^\bullet$$

in the derived category of complexes of $\Lambda[[G]]$-modules.

5 DUALITY ON THE LEVEL OF K-GROUPS

Assume that $\mathbf{W}$ is a biWaldhausen category in the sense of [TT90, Def. 1.2.4]. In particular, the opposite category $\mathbf{W}^{\mathrm{op}}$ is a biWaldhausen category as well and there are natural isomorphisms

$$I \colon \mathrm{K}_n(\mathbf{W}) \cong \mathrm{K}_n(\mathbf{W}^{\mathrm{op}}), \tag{5.1}$$

simply because the topological realisations of the bisimplicial sets $N.w\mathbf{S}.\mathbf{W}$ and $N.w\mathbf{S}.\mathbf{W}^{\mathrm{op}}$ resulting from Waldhausen's S-construction agree [TT90, §1.5.5]. However, the obvious identifications

$$N_m w\mathbf{S}_n\mathbf{W} \cong N_m w\mathbf{S}_n\mathbf{W}^{\mathrm{op}}$$

respect the boundary and degeneracy maps only up to reordering, so that we do not obtain an isomorphism of the bisimplicial sets themselves.
In order to understand the isomorphism (5.1) in terms of the presentation of $\mathrm{K}_1(\mathbf{W})$ given by (3.3), we will construct a canonical isomorphism

$$I \colon \mathcal{D}_\bullet(\mathbf{W}) \to \mathcal{D}_\bullet(\mathbf{W}^{\mathrm{op}}).$$

For any morphism $\alpha \colon A \to B$ in $\mathbf{W}$, write $\alpha^{\mathrm{op}} \colon B \to A$ for the corresponding morphism in the opposite category $\mathbf{W}^{\mathrm{op}}$. Further, note that by the definition of biWaldhausen categories, if $A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathbf{W}$, then the dual sequence $C \rightarrowtail B \twoheadrightarrow A$ is exact in $\mathbf{W}^{\mathrm{op}}$. We then set

$$
\begin{aligned}
I([A]) &\coloneqq [A] && \text{for objects } A \text{ in } \mathbf{W}, \\
I([f \colon A \xrightarrow{\sim} B]) &\coloneqq [f^{\mathrm{op}} \colon B \xrightarrow{\sim} A]^{-1} && \text{for weak equivalences } f, \\
I([A \rightarrowtail B \twoheadrightarrow C]) &\coloneqq [C \rightarrowtail B \twoheadrightarrow A]\langle [A], [C]\rangle && \text{for exact sequences } A \rightarrowtail B \twoheadrightarrow C.
\end{aligned}
$$

PROPOSITION 5.1. *For any biWaldhausen category $\mathbf{W}$, the above assignment defines an isomorphism of stable quadratic modules*

$$I \colon \mathcal{D}_\bullet(\mathbf{W}) \to \mathcal{D}_\bullet(\mathbf{W}^{\mathrm{op}}).$$

Proof. It is sufficient to check that I respects the relations (R1)–(R7) in the definition of $\mathcal{D}_\bullet(\mathbf{W})$. This is a straight-forward, but tedious exercise. $\qquad\square$

Next, we investigate in how far I respects the boundary homomorphism of localisation sequences. For this, we consider the same situation as in [Wit14, Appendix], but with all Waldhausen categories replaced by biWaldhausen categories. Assume that $\mathbf{wW}$ is a biWaldhausen category with weak equivalences $\mathbf{w}$ that is saturated and extensional in the sense of [TT90, Def. 1.2.5, 1.2.6]. Let $\mathbf{vW}$ be a the same category with the same notion of fibrations and cofibrations, but with a coarser notion of weak equivalences $\mathbf{v} \subset \mathbf{w}$ and let $\mathbf{vW^w}$ denote the full biWaldhausen subcategory of $\mathbf{vW}$ consisting of those objects which are weakly equivalent to the zero object in $\mathbf{wW}$. We assume that Cyl

is a cylinder functor in the sense of [Wit14, Def. A.1] for both $\mathbf{wW}$ and $\mathbf{vW}$ and that it satisfies the cylinder axiom for $\mathbf{wW}$. We will write Cone and Σ for the associated cone and shift functors, i.e.

$$\mathrm{Cone}(\alpha) \coloneqq \mathrm{Cyl}(\alpha)/A \qquad \text{for any morphism } \alpha\colon A \to B,$$
$$\Sigma A \coloneqq \mathrm{Cone}(A \to 0) \qquad \text{for any object } A.$$

Further, we assume that CoCyl is a cocylinder functor for both $\mathbf{wW}$ and $\mathbf{vW}$ in the sense that the opposite functor $\mathrm{CoCyl}^{\mathrm{op}}$ is a cylinder functor for $\mathbf{wW}^{\mathrm{op}}$ and $\mathbf{vW}^{\mathrm{op}}$. Again, we assume that $\mathrm{CoCyl}^{\mathrm{op}}$ satisfies the cylinder axiom for $\mathbf{wW}^{\mathrm{op}}$. We will write CoCone and CoΣ for the associated cocone and coshift functors.

Recall from [Wit14, Thm. A.5] that the assignment

$$d(\Delta) \coloneqq 0$$
$$\text{for every exact sequence } \Delta \text{ in } \mathbf{wW},$$
$$d(\alpha) \coloneqq -[\mathrm{Cone}(\alpha)] + [\mathrm{Cone}(\mathrm{id}_A)] \qquad (5.2)$$
$$\text{for every weak equivalence } \alpha\colon A \xrightarrow{\sim} A' \text{ in } \mathbf{wW}$$

defines a homomorphism $d\colon \mathcal{D}_1(\mathbf{wW}) \to \mathrm{K}_0(\mathbf{vW^w})$ such that the sequence

$$\mathrm{K}_1(\mathbf{vW}) \to \mathrm{K}_1(\mathbf{wW}) \xrightarrow{d} \mathrm{K}_0(\mathbf{vW^w}) \to \mathrm{K}_0(\mathbf{vW}) \to \mathrm{K}_0(\mathbf{wW}) \to 0$$

is exact.

LEMMA 5.2. *For every weak equivalence* $\alpha\colon A \to B$ *in* $\mathbf{wW}$,

$$d(\alpha) = -[\mathrm{CoCone}(\mathrm{id}_B)] + [\mathrm{CoCone}(\alpha)]$$

in $\mathrm{K}_0(\mathbf{vW^w})$.

Proof. We first assume that A and B are objects of $\mathbf{vW^w}$. Then

$$B \rightarrowtail \mathrm{Cone}(\alpha) \twoheadrightarrow \Sigma A,$$
$$A \rightarrowtail \mathrm{Cone}(\mathrm{id}_A) \twoheadrightarrow \Sigma A,$$

are exact sequences in $\mathbf{vW^W}$. Hence,

$$d(\alpha) = -[B] - [\Sigma A] + [\Sigma A] + [A] = -[B] + [A] \qquad (5.3)$$

in $\mathrm{K}_0(\mathbf{vW^w})$.

For a general weak equivalence $\alpha\colon A \to B$ in $\mathbf{wW}$, the natural morphism
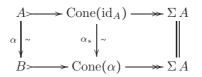
$$\mathrm{Cone}(\alpha) \to 0$$

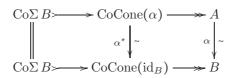is a weak equivalence in $\mathbf{wW}$ by the cylinder axiom. The commutative square

$$\begin{array}{ccc} A & =\!=\!= & A \\ \Big\| & & \alpha \Big\downarrow \sim \\ A & \xrightarrow[\alpha]{\sim} & B \end{array}$$

induces by the functoriality of the cone and shift functor a commutative diagram with exact rows

$$
\begin{array}{ccccc}
A & \longrightarrow & \mathrm{Cone}(\mathrm{id}_A) & \longrightarrow & \Sigma A \\
{\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & & {\scriptstyle\alpha_*}\downarrow{\scriptstyle\sim} & & \Vert \\
B & \longrightarrow & \mathrm{Cone}(\alpha) & \longrightarrow & \Sigma A
\end{array}
$$

where all downward pointing arrows are weak equivalences in $\mathbf{wW}$. Dually, we also obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Co}\Sigma\, B & \longrightarrow & \mathrm{CoCone}(\alpha) & \longrightarrow & A \\
\Vert & & {\scriptstyle\alpha^*}\downarrow{\scriptstyle\sim} & & {\scriptstyle\alpha}\downarrow{\scriptstyle\sim} \\
\mathrm{Co}\Sigma\, B & \longrightarrow & \mathrm{CoCone}(\mathrm{id}_B) & \longrightarrow & B
\end{array}
$$

where all downward pointing arrows are weak equivalences in $\mathbf{wW}$.
From (R6) and (5.3) we conclude

$$
-[\mathrm{Cone}(\alpha)] + [\mathrm{Cone}(\mathrm{id}_A)] = d(\alpha_*) = d(\alpha)
$$
$$
= d(\alpha^*) = -[\mathrm{CoCone}(\mathrm{id}_B)] + [\mathrm{CoCone}(\alpha)]
$$

as desired. □

Remark 5.3. By basically the same argument, one also sees that d is independent of the choice of the particular cylinder functor.

Proposition 5.4. *With the notation as above, the diagram*

$$
\begin{array}{ccc}
\mathcal{D}_1(wW) & \xrightarrow{\;\;I\;\;} & \mathcal{D}_1(wW^{\mathrm{op}}) \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\
\mathrm{K}_0(vW^w) & \xrightarrow{\;\;I\;\;} & \mathrm{K}_0((vW^w)^{\mathrm{op}})
\end{array}
$$

commutes.

Proof. This is a direct consequence of the definition of I and Lemma 5.2. □

If R is any ring and $P^\bullet$ is a strictly perfect complex of left R-modules, then

$$
(P^\bullet)^{*_R} \coloneqq \mathrm{Hom}_R(P^{-\bullet}, R)
$$

is a strictly perfect complex of left modules over the opposite ring R^{op} and

$$
\mathbf{SP}(R)^{\mathrm{op}} \to \mathbf{SP}(R^{\mathrm{op}}) \qquad P^\bullet \mapsto (P^\bullet)^{*_R}
$$

is a Waldhausen exact equivalence of categories. We omit the R from $\star_R$ if it is clear from the context. By composing with the homomorphisms I, we obtain isomorphisms

$$\mathrm{K}_n(R) \cong \mathrm{K}_n(R^{\mathrm{op}}).$$

Note that the isomorphism $\mathrm{K}_1(R) \to \mathrm{K}_1(R^{\mathrm{op}})$ corresponds to the isomorphism induced by the group isomorphism

$$\mathrm{Gl}_\infty(R) \to \mathrm{Gl}_\infty(R^{\mathrm{op}}), \qquad A \mapsto (A^t)^{-1}$$

that maps a matrix A to the inverse of its transposed matrix.

If S is a second ring and $M^\bullet$ a complex of R-S-bimodules which is strictly perfect as complex of R-modules, then $(M^\bullet)^{\star_R}$ is a complex of R^{op}-S^{op}-bimodules which is strictly perfect as complex of R^{op}-modules and there exists for any complex $P^\bullet$ in $\mathbf{SP}(S)$ a canonical isomorphism

$$(M^{\star_R} \otimes_{S^{\mathrm{op}}} P^{\star_S})^\bullet \cong ((M \otimes_S P)^\bullet)^{\star_R}. \tag{5.4}$$

Hence, we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}_n(S) & \xrightarrow[\cong]{} & \mathrm{K}_n(S^{\mathrm{op}}) \\
{\scriptstyle M^\bullet}\Big\downarrow & & \Big\downarrow{\scriptstyle (M^\bullet)^{\star_R}} \\
\mathrm{K}_n(R) & \xrightarrow[\cong]{} & \mathrm{K}_n(R^{\mathrm{op}})
\end{array}
$$

We now return to our previous setting: As before, Λ is an adic ring and $G = H \rtimes \Gamma$ is a profinite group such that H contains an open, topologically finitely generated pro-p subgroup and $\Gamma \cong \mathbb{Z}_p$.

DEFINITION 5.5. We define

$$\sharp : \Lambda[[G]]^{\mathrm{op}} \to \Lambda^{\mathrm{op}}[[G]], \qquad a \mapsto a^\sharp,$$

to be the ring homomorphism that is the identity on the coefficients and maps $g \in G$ to g^{-1} and write $\Lambda^{\mathrm{op}}[[G]]^\sharp$ for $\Lambda^{\mathrm{op}}[[G]]$ considered as $\Lambda^{\mathrm{op}}[[G]]$-$\Lambda[[G]]^{\mathrm{op}}$-bimodule via $\sharp$. Further, we let

$$\circledast : \mathbf{SP}(\Lambda[[G]])^{\mathrm{op}} \to \mathbf{SP}(\Lambda^{\mathrm{op}}[[G]]), \qquad P^\bullet \mapsto \Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda[[G]]^{\mathrm{op}}} (P^\bullet)^*.$$

denote the resulting Waldhausen exact equivalence of categories. We also write

$$\circledast : \mathrm{K}_n(\Lambda[[G]]) \to \mathrm{K}_n(\Lambda^{\mathrm{op}}[[G]]) \tag{5.5}$$

for the homomorphisms obtained by composing I from (5.1) with the homomorphism

$$\mathrm{K}_n(\mathbf{SP}(\Lambda[[G]])^{\mathrm{op}}) \to \mathrm{K}_n(\mathbf{SP}(\Lambda^{\mathrm{op}}[[G]]))$$

induced by the Waldhausen exact functor $\circledast$.

Remark 5.6. The author does not know wether it is possible to produce an extension of $\circledast$ to a Waldhausen exact functor $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])^{\mathrm{op}} \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}}[[G]])$ inducing the same homomorphisms on K-theory. This would avoid some technicalities that we need to deal with later on.

LEMMA 5.7. *Assume that $K^{\bullet}$ is in $\mathbf{SP}(\Lambda[[G]])$.*

1. *Let Λ' be another adic $\mathbb{Z}_p$-algebra. For any complex $P^{\bullet}$ of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, set*

$$(P^{\bullet})^{*,\natural} \coloneqq (P^{\bullet})^{*_\Lambda} \otimes_{\Lambda[[G]]^{\mathrm{op}}} (\Lambda[[G]]^{\mathrm{op}})^{\natural}$$

such that $(P^{\bullet})^{*,\natural}$ is a complex of Λ'^{op}-$\Lambda^{\mathrm{op}}[[G]]$-bimodules, with $g \in G$ acting by $(g^{-1})^*$. With $P[[G]]^{\delta^{\bullet}}$ as in Example 3.15,

$$(\Psi_{P[[G]]^{\delta^{\bullet}}}(K^{\bullet}))^{\circledast} \cong \Psi_{(P^{\bullet})^{*_\Lambda,\natural}[[G]]^{\delta}}((K^{\bullet})^{\circledast}).$$

2. *Let $G' = H' \rtimes \Gamma'$ be another profinite group such that H' contains an open, topologically finitely generated pro-p subgroup and $\Gamma' \cong \mathbb{Z}_p$. Let $\alpha\colon G \to G'$ be a continuous homomorphism such that $\alpha(G) \notin H'$. Consider $\Lambda[[G']]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then*

$$(\Psi_{\Lambda[[G']]}(K^{\bullet}))^{\circledast} \cong \Psi_{\Lambda^{\mathrm{op}}[[G']]}((K^{\bullet})^{\circledast}).$$

3. *Assume that G' is an open subgroup of G and set $H' \coloneqq H \cap G'$. Consider $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then*

$$(\Psi_{\Lambda[[G]]}(K^{\bullet}))^{\circledast} \cong \Psi_{\Lambda^{\mathrm{op}}[[G]]}((K^{\bullet})^{\circledast}).$$

Proof. Using the canonical isomorphism (5.4), it remains to notice that

$$\Lambda'^{\mathrm{op}}[[G]]^{\natural} \otimes_{\Lambda'[[G]]^{\mathrm{op}}} (P[[G]]^{\delta^{\bullet}})^{*_{\Lambda'[[G]]}} \cong (P^{\bullet})^{*,\natural}[[G]]^{\delta} \otimes_{\Lambda^{\mathrm{op}}[[G]]} \Lambda^{\mathrm{op}}[[G]]^{\natural}$$

as complexes of $\Lambda'^{\mathrm{op}}[[G]]$-$\Lambda[[G]]^{\mathrm{op}}$-bimodules to prove (1). The other two parts are straightforward. $\square$

PROPOSITION 5.8. *The functor $\circledast$ extends to Waldhausen exact equivalences*

$$\circledast\colon (w_H\mathbf{SP}(\Lambda[[G]]))^{\mathrm{op}} \to w_H\mathbf{SP}(\Lambda^{\mathrm{op}}[[G]]),$$
$$\circledast\colon (\mathbf{SP}^{w_H}(\Lambda[[G]]))^{\mathrm{op}} \to \mathbf{SP}^{w_H}(\Lambda^{\mathrm{op}}[[G]])$$

and hence, it induces a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_1(\Lambda[[G]]) & \longrightarrow & \mathrm{K}_1(\Lambda[[G]]_S) & \stackrel{d}{\longrightarrow} & \mathrm{K}_0(\Lambda[[G]], S) & \longrightarrow & 0 \\
& & \cong \downarrow \circledast & & \cong \downarrow \circledast & & \cong \downarrow \circledast & & \\
0 & \longrightarrow & \mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]) & \longrightarrow & \mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S) & \stackrel{d}{\longrightarrow} & \mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]], S) & \longrightarrow & 0
\end{array}
$$

with exact rows.

Proof. The exactness of the rows follows from [Wit13b, Cor. 3.3]. To extend $\circledast$, it suffices to show that for any strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$-modules which is also perfect as complex of $\Lambda[[H]]$-modules, the complex $(P^\bullet)^\circledast$ is perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules. By [Wit14, Prop. 4.8] we may check this after tensoring with $(\Lambda/\mathrm{Jac}(\Lambda))^{\mathrm{op}}[[G/V]]$ with $V \subset G$ a closed normal pro-p-subgroup which is open in H. Using Lemma 5.7, we may therefore assume that Λ and H are finite.

By Lemma 3.6, $S \coloneqq S_{\Lambda[[G]]} \subset \Lambda[[G]]$ is a left and right denominator set and $\sharp$ maps $S_{\Lambda[[G]]}$ to the set $S_{\Lambda^{\mathrm{op}}[[G]]} \subset \Lambda^{\mathrm{op}}[[G]]$. Moreover $(P^\bullet)^\circledast$ is perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules if and only if its cohomology is $S_{\Lambda^{\mathrm{op}}[[G]]}$-torsion. As $P^\bullet$ has $S_{\Lambda[[G]]}$-torsion cohomology and as

$$(\Lambda[[G]]_S)^{\mathrm{op}} \otimes_{\Lambda[[G]]^{\mathrm{op}}} (P^\bullet)^{*_{\Lambda[[G]]}} \cong (\Lambda[[G]]_S \otimes_{\Lambda[[G]]} P^\bullet)^{*_{\Lambda[[G]]_S}},$$

we conclude that $(P^\bullet)^\circledast$ is indeed perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules. $\qquad\square$

We may also extend $\circledast$ to the Waldhausen category $\mathbf{SP}(\Lambda[[H]], G)$ from Definition 3.10.

DEFINITION 5.9. For any $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)^{\mathrm{op}}$ we set

$$(P^\bullet)^\circledast \coloneqq \Lambda^{\mathrm{op}}[[H]]^\sharp \otimes_{\Lambda[[H]]^{\mathrm{op}}} (P^\bullet)^{*_{\Lambda[[H]]}}[-1].$$

and define a left action of $\gamma \in \Gamma$ on $(P^\bullet)^\circledast$ by

$$\gamma\colon (P^\bullet)^\circledast \to (P^\bullet)^\circledast, \qquad \lambda \otimes f \mapsto \gamma\lambda\gamma^{-1} \otimes \gamma f(\gamma^{-1}\cdot)\gamma^{-1}.$$

One checks that $(P^\bullet)^\circledast$ is indeed an object of $\mathbf{SP}(\Lambda^{\mathrm{op}}[[H]], G)$, such that we obtain a Waldhausen exact equivalence of categories

$$\circledast\colon \mathbf{SP}(\Lambda[[H]], G)^{\mathrm{op}} \to \mathbf{SP}(\Lambda^{\mathrm{op}}[[H]], G).$$

PROPOSITION 5.10. *Let* $\gamma \in \Gamma$ *be a topological generator. Then for any* $P^\bullet$ *in* $\mathbf{SP}(\Lambda[[H]], G)^{\mathrm{op}}$ *we have a commutative diagram*

$$
\begin{array}{ccc}
\Lambda^{\mathrm{op}}[[G]] \otimes_{\Lambda^{\mathrm{op}}[[H]]} (P^\bullet)^\circledast[1] & \xrightarrow{\mathrm{id} - \cdot\gamma^{-1}\otimes\gamma} & \Lambda^{\mathrm{op}}[[G]] \otimes_{\Lambda^{\mathrm{op}}[[H]]} (P^\bullet)^\circledast[1] \\
\cong \downarrow \alpha & & \cong \downarrow \alpha \\
(\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet)^\circledast & \xrightarrow{\mathrm{id} - (\cdot\gamma\otimes\gamma^{-1}\cdot)^\circledast} & (\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet)^\circledast
\end{array}
$$

in $\mathbf{SP}(\Lambda^{op}[[G]])$ *inducing a canonical isomorphism*

$$C_{\gamma^{-1}}(P^\bullet)^\circledast \cong C_\gamma((P^\bullet)^\circledast).$$

In particular, we have

$$[P^\bullet]^\circledast = [(P^\bullet)^\circledast]$$

in $\mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]], S)$.

Proof. For any degree n and any $f \in (P^n)^{*\Lambda[[H]]}$, we write

$$\tilde{f}: \Lambda[[G]] \otimes_{\Lambda[[H]]} P^n \to \Lambda[[G]], \qquad \lambda \otimes p \mapsto \lambda f(p).$$

We then set

$$
\begin{array}{ccc}
\Lambda^{\mathrm{op}}[[G]] \otimes_{\Lambda^{\mathrm{op}}[[H]]} \Lambda^{\mathrm{op}}[[H]]^\sharp \otimes_{\Lambda[[H]]^{\mathrm{op}}} (P^n)^{*\Lambda[[H]]} & & \lambda \otimes \mu \otimes f \\
\alpha \downarrow \cong & & \downarrow \\
\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda[[G]]^{\mathrm{op}}} (\Lambda[[G]] \otimes_{\Lambda[[H]]} P^n)^{*\Lambda[[G]]} & & \lambda \otimes \tilde{f}\mu^\sharp
\end{array}
$$

It is then straightforward to check that the above diagram commutes. The cone of the first row is $C_\gamma((P^\bullet)^{\circledast})[1]$, the cone of the second row is the $\circledast$-dual of the cocone of

$$\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\mathrm{id}-(\cdot\gamma \otimes \gamma^{-1}\cdot)} \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet$$

in $\mathbf{SP}^{w_H}(\Lambda[[G]])$, which is the same as $C_{\gamma^{-1}}(P^\bullet)[-1]$. Finally, recall from Proposition 3.12 that the class of $C_\gamma(P^\bullet)$ in $\mathrm{K}_0(\Lambda[[G]], S)$ is independent of the choice of the topological generator γ. Hence,

$$[P^\bullet]^{\circledast} = [C_\gamma(P^\bullet)^{\circledast}] = [C_{\gamma^{-1}}(P^\bullet)^{\circledast}] = [C_\gamma((P^\bullet)^{\circledast})] = [(P^\bullet)^{\circledast}].$$

$\square$

6 NON-COMMUTATIVE ALGEBRAIC L-FUNCTIONS

Let $G = H \rtimes \Gamma$ as before. Recall the split exact sequence

$$0 \to \mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda[[G]]_S) \xrightarrow{d} \mathrm{K}_0(\Lambda[[G]], S) \to 0.$$

[Wit13b, Cor. 3.4], which is central for the formulation of the non-commutative main conjecture: The map $\mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda[[G]]_S)$ is the obvious one; the boundary map

$$d: \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_0(\Lambda[[G]], S)$$

on the class $[f]$ of an endomorphism f which is a weak equivalence in the Waldhausen category $w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ is given by

$$d[f] = -[\mathrm{Cone}(f)^\bullet]$$

where $\mathrm{Cone}(f)^\bullet$ denotes the cone of f [Wit14, Thm. A.5]. (Note that other authors use $-d$ instead.) For a fixed choice of a topological generator $\gamma \in \Gamma$, a splitting s_γ of d is given by

$$s_\gamma([P^\bullet]) := [\Lambda[[G]] \hat{\otimes}_{\Lambda[[H]]} P^\bullet \xrightarrow{x\hat{\otimes}y \mapsto x\hat{\otimes}y - x\gamma^{-1}\hat{\otimes}\gamma y} \Lambda[[G]] \hat{\otimes}_{\Lambda[[H]]} P^\bullet]^{-1} \quad (6.1)$$

for any $P^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$, where the precise definition of $\Lambda[[G]] \hat{\otimes}_{\Lambda[[H]]} P^\bullet$ as an object of the Waldhausen category $w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ is

$$\Lambda[[G]] \hat{\otimes}_{\Lambda[[H]]} P^\bullet = (\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} \Lambda[[G]]/I \otimes_{\Lambda[[H]]} P_J^\bullet)_{I \in \mathfrak{I}_{\Lambda[[G]]}}$$

[Wit13b, Def. 2.12]. A short inspection of the definition shows that s_γ only depends on the image of γ in G/H. Following [Bur09], we may call $s_\gamma(-A)$ the *non-commutative algebraic L-function* of $A \in \mathrm{K}_0(\Lambda[[G]], S)$.

PROPOSITION 6.1. *Consider an element* $A \in \mathrm{K}_0(\Lambda[[G]], S)$.

1. *Let* Λ' *be another adic* $\mathbb{Z}_p$-*algebra. For any complex* $P^\bullet$ *of* Λ'-$\Lambda[[G]]$-*bimodules which is strictly perfect as complex of* Λ'-*modules we have*

$$\Psi_{P[[G]]^{\delta \bullet}}(s_\gamma(A)) = s_\gamma(\Psi_{P[[G]]^{\delta \bullet}}(A))$$

in $\mathrm{K}_1(\Lambda'[[G]]_S)$.

2. *Let* $G' = H' \rtimes \Gamma'$ *such that* H' *has an open, topologically finitely generated pro-p-subgroup and* $\Gamma' \cong \mathbb{Z}_p$. *Assume that* $\alpha \colon G \to G'$ *is a continuous homomorphism such that* $\alpha(G) \not\subseteq H'$. *Set* $r \coloneqq [G' : \alpha(G)H']$. *Let* $\gamma' \in \Gamma'$ *be a topological generator such that* $\alpha(\gamma) = (\gamma')^r$ *in* G'/H'. *Then*

$$\Psi_{\Lambda[[G']]}(s_\gamma(A)) = s_{\gamma'}(\Psi_{\Lambda[[G']]}(A))$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$.

3. *Assume that* G' *is an open subgroup of* G *and set* $H' \coloneqq H \cap G'$, $r \coloneqq [G : G'H]$. *Consider* $\Lambda[[G]]$ *as a* $\Lambda[[G']]$-$\Lambda[[G]]$-*bimodule. Then* γ^r *generates* $G'/H' \subset G/H$ *and*

$$\Psi_{\Lambda[[G]]}(s_\gamma(A)) = s_{\gamma^r}(\Psi_{\Lambda[[G]]}(A))$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$.

Proof. For (1), we first note that by applying the Waldhausen additivity theorem [Wal85, Prop. 1.3.2] to the short exact sequences resulting from stupid truncation, we have

$$\Psi_{P[[G]]^{\delta \bullet}} = \sum_{i \in \mathbb{Z}} (-1)^i \Psi_{P^i[[G]]^\delta}$$

as homomorphisms between the K-groups. Hence we may assume that $P = P^\bullet$ is concentrated in degree 0. We now apply [Wit13b, Prop 2.14.1] to the $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodule $M \coloneqq P[[G]]^\delta$ and its $\Lambda'[[H]]$-$\Lambda[[H]]$-sub-bimodule

$$N \coloneqq \Lambda'[[H]] \otimes_{\Lambda'} P$$

(with the diagonal right action of H) and $t_1 \coloneqq t_2 \coloneqq \gamma - 1$, $\gamma_1 \coloneqq \gamma_2 \coloneqq \gamma$.

For (2), we first assume that α induces an isomorphism $G/H \cong G/H'$ and that $\gamma' = \alpha(\gamma)$. We then apply [Wit13b, Prop 2.14.1] to $M \coloneqq \Lambda[[G']]$, $N \coloneqq \Lambda[[H']]$, and $t_1 \coloneqq \gamma - 1$, $t_2 \coloneqq \alpha(\gamma) - 1$, $\gamma_1 \coloneqq \gamma$, $\gamma_2 \coloneqq \alpha(\gamma)$.

Next, we assume that $G \subset G'$, $H = H'$, and $\gamma = (\gamma')^r$. This case is not covered by [Wit13b, Prop 2.14] and therefore, we will give more details. Consider the isomorphism of $\Lambda[[G']]$-$\Lambda[[G]]$-bimodules

$$\kappa \colon \Lambda[[G']] \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]]^r \to \Lambda[[G']] \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G']],$$

$$\mu \hat{\otimes} \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{r-1} \end{pmatrix} \mapsto \sum_{i=0}^{r-1} \mu(\gamma')^{-i} \hat{\otimes} (\gamma')^i (\lambda_i).$$

Then the map $\mu \hat{\otimes} \lambda \mapsto \mu \hat{\otimes} \lambda - \mu(\gamma')^{-1} \hat{\otimes} \gamma' \lambda$ on the righthand side corresponds to left multiplication with the matrix

$$A \coloneqq \begin{pmatrix} \mathrm{id} & 0 & \cdots & 0 & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ -\mathrm{id} & \mathrm{id} & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \mathrm{id} & 0 \\ 0 & \cdots & 0 & -\mathrm{id} & \mathrm{id} \end{pmatrix}$$

on the left-hand side. Let $P^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$. Then κ induces an isomorphism

$$\kappa \colon \Psi_{\Lambda[[G']]}(\Lambda[[G]] \hat{\otimes}_{\Lambda[[H]]} (P^\bullet)^r) \to \Lambda[[G']] \hat{\otimes}_{\Lambda[[H]]} \Psi_{\Lambda[[G']]}(P^\bullet)$$

in $w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G']])$ while $A \circlearrowright \Psi_{\Lambda[[G']]}(\Lambda[[G]] \hat{\otimes}_{\Lambda[[H]]} (P^\bullet)^r)$ is a weak equivalence. Hence,

$$[A]^{-1} = s_{\gamma'}([\Psi_{\Lambda[[G']]}(P^\bullet)])$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$. Moreover,

$$\begin{pmatrix} \mathrm{id} & 0 & \cdots & \cdots & 0 \\ \mathrm{id} & \mathrm{id} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathrm{id} & 0 \\ \mathrm{id} & \cdots & \cdots & \mathrm{id} & \mathrm{id} \end{pmatrix} A = \begin{pmatrix} \mathrm{id} & 0 & \cdots & 0 & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ 0 & \mathrm{id} & \ddots & \vdots & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \mathrm{id} & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ 0 & \cdots & \cdots & 0 & \mathrm{id} - (\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \end{pmatrix}.$$

The relations (R1)–(R7) in the definition of $\mathcal{D}_\bullet(\mathbf{W})$ imply that the class of a triangular matrix is the product of the classes of its diagonal entries in $\mathrm{K}_1(\Lambda[[G']]_S)$. Hence, $[A]^{-1} = \Psi_{\Lambda[[G']]}(s_\gamma[P^\bullet])$, as desired.

In the general case, we note that the image of α is contained in the subgroup G'' of G' topologically generated by $(\gamma')^r$ and H' and recall that s_γ only depends on the image of γ in G/H. We are then reduced to the two cases already treated above.

For (3), we first treat the case $r = 1$, i.e. $G' \to G/H$ is a surjection. Hence, we may assume $\gamma \in G'$. We then apply [Wit13b, Prop 2.14.1] to $M \coloneqq \Lambda[[G]]$,

$N := \Lambda[[H]]$, and $t_1 := t_2 := \gamma - 1$, $\gamma_1 := \gamma_2 := \gamma$ as above. If $r > 1$ we can thus reduce to the case that G' is topologically generated by H and γ^r and apply [Wit13b, Prop 2.14.2].

In [Wit13b], we use a slightly different Waldhausen category for the construction of the K-theory of $\Lambda[[G]]$, but the proof of [Wit13b, Prop 2.14] goes through without changes. $\qquad\square$

Example 6.2.

1. Assume that M is a $\Lambda[[G]]$-module which is finitely generated and projective as a $\Lambda[[H]]$-module. Then the complex

$$C_\gamma(M)\colon \qquad \underbrace{\Lambda[[G]] \otimes_{\Lambda[[H]]} M}_{\text{degree } -1} \xrightarrow{\mathrm{id}-(\cdot\gamma^{-1}\otimes\gamma\cdot)} \underbrace{\Lambda[[G]] \otimes_{\Lambda[[H]]} M}_{\text{degree } 0}$$

    is an object of $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ whose cohomology is M in degree 0 and zero otherwise. Moreover,

$$s_\gamma([M]) = s_\gamma([C_\gamma(M)]) = [\mathrm{id} - (\cdot\gamma^{-1}\otimes\gamma\cdot) \circlearrowleft \Lambda[[G]] \otimes_{\Lambda[[H]]} M]^{-1}$$

    in $\mathrm{K}_1(\Lambda[[G]]_S)$. If $\Lambda[[G]]$ is commutative, then the image of the element $s_\gamma([M])^{-1}$ under

$$\det\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \Lambda[[G]]_S^\times$$

    is precisely the reverse characteristic polynomial

$$\det\nolimits_{\Lambda[[H]][t]}(\mathrm{id} - (\cdot t \otimes \gamma\cdot) \circlearrowleft \Lambda[[H]][t] \otimes_{\Lambda[[H]]} M)$$

    evaluated at $t = \gamma^{-1} \in \Gamma$. In fact, one may extend this to non-commutative $\Lambda[[H]]$ and $G = H \times \Gamma$ as well, using the results of the appendix.

2. If $M = \Lambda[[G]]/\Lambda[[G]]f$ with

$$f := t^n + \sum_{i=0}^{n-1} \lambda_i t^i \in \Lambda[[G]]$$

    a polynomial of degree n in $t := \gamma - 1$ with $\lambda_i \in \mathrm{Jac}(\Lambda[[H]])$, then M is finitely generated and free as $\Lambda[[H]]$-module. A $\Lambda[[H]]$-basis is given by the residue classes of $1, t, \dots, t^{n-1} \in \Lambda[[G]]$. If we use this basis to identify $\Lambda[[G]] \otimes_{\Lambda[[H]]} M$ with $\Lambda[[G]]^n$, then the $\Lambda[[G]]$-linear endomorphism $\mathrm{id} - (\cdot\gamma^{-1}\otimes\gamma\cdot)$ is given by right multiplication with the matrix

$$A := \begin{pmatrix} \gamma^{-1}t & -\gamma^{-1} & 0 & \cdots & 0 \\ 0 & \gamma^{-1}t & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma^{-1}t & -\gamma^{-1} \\ \gamma^{-1}\lambda_0 & \gamma^{-1}\lambda_1 & \cdots & \gamma^{-1}\lambda_{n-2} & \gamma^{-1}(t+\lambda_{n-1}) \end{pmatrix}.$$

By right multiplication with

$$
E \coloneqq \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
t & 1 & \ddots & & \vdots \\
t^2 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
t^{n-1} & \cdots & t^2 & t & 1
\end{pmatrix}
$$

one can transform A into

$A' \coloneqq$

$$
\begin{pmatrix}
0 & -\gamma^{-1} & 0 & \cdots & & 0 \\
\vdots & 0 & \ddots & & \ddots & \vdots \\
\vdots & \vdots & \ddots & & \ddots & 0 \\
0 & 0 & \cdots & & 0 & -\gamma^{-1} \\
\gamma^{-1}f & \gamma^{-1}\left(\frac{f-\lambda_0}{t}\right) & \cdots & \gamma^{-1}(t^2+\lambda_{n-1}t+\lambda_{n-2}) & \gamma^{-1}(t+\lambda_{n-1})
\end{pmatrix}.
$$

By left multiplication with

$$
P \coloneqq \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 1 & \ddots & \vdots & 0 \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

one can exchange the first and last row of A' to obtain a triangular matrix. In $\mathrm{K}_1(\Lambda[[G]])$, we have

$$
[\,\cdot E \circlearrowright \Lambda[[G]]^n\,] = 1,
$$
$$
[\,\cdot P \circlearrowright \Lambda[[G]]^n\,] = [\,-1 \circlearrowright \Lambda[[G]]\,]^{n-1}.
$$

We conclude

$$
s_\gamma([M])^{-1} = [\,\cdot A \circlearrowright \Lambda[[G]]^n\,] = [\,\cdot A' \circlearrowright \Lambda[[G]]^n\,]
$$
$$
= [\,\cdot P \circlearrowright \Lambda[[G]]^n\,]^{-1}[\,\cdot(-\gamma^{-1}) \circlearrowright \Lambda[[G]]\,]^{n-1}[\,\cdot\gamma^{-1}f \circlearrowright \Lambda[[G]]\,]
$$
$$
= [\,\cdot\gamma^{-n}f \circlearrowright \Lambda[[G]]\,].
$$

The section $s_\gamma\colon \mathrm{K}_0(\Lambda[[G]],S) \to \mathrm{K}_1(\Lambda[[G]]_S)$ also commutes with the homomorphisms $\circledast\colon \mathrm{K}_0(\Lambda[[G]]) \to \mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]])$, $\circledast\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]]_S)$ from Definition 5.5 in the following sense.

PROPOSITION 6.3. *For any element $A \in \mathrm{K}_0(\Lambda[[G]],S)$,*

$$
s_{\gamma^{-1}}(A)^\circledast = s_\gamma(A^\circledast)
$$

in $\mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S)$.

Proof. Since $\mathrm{K}_0(\mathbf{SP}(\Lambda[[H]],G))$ surjects onto $\mathrm{K}_0(\Lambda[[G]],S)$ by Proposition 3.12, it suffices to prove the formula for $C_\gamma(M)$ with M a $\Lambda[[G]]$-module that is finitely generated and projective over $\Lambda[[H]]$. The equality is then a direct consequence of the diagram in Proposition 5.10. $\quad\square$

Remark 6.4. Note that

$$s_\gamma([C_\gamma(M)]) = s_{\gamma^{-1}}([C_\gamma(M)])[-(\cdot\gamma^{-1}\otimes\gamma\cdot)\circlearrowright\Lambda[[G]]\otimes_{\Lambda[[H]]}M]$$

for any topological generator γ of Γ and any $\Lambda[[G]]$-module M that is finitely generated and projective over $\Lambda[[H]]$.

7 Perfect complexes of adic sheaves

We will use étale cohomology instead of Galois cohomology to formulate the main conjecture. The main advantage is that we have a little bit more flexibility in choosing our coefficient systems. Instead of being restricted to locally constant sheaves corresponding to Galois modules, we can work with constructible sheaves. An alternative would be the use of cohomology for Galois modules with local conditions.

As Waldhausen models for the derived categories of complexes of constructible sheaves, we will use the Waldhausen categories introduced in [Wit08, § 5.4–5.5] for separated schemes of finite type over a finite field. The same constructions still work with some minor changes if we consider subschemes of the spectrum of a number ring.

Fix an odd prime p. Let F be a number field with ring of integers $\mathcal{O}_F$ and assume that U is an open or closed subscheme of $X := \mathrm{Spec}\,\mathcal{O}_F$. Recall that for a finite ring R, a complex $\mathcal{F}^\bullet$ of étale sheaves of left R-modules on U is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. It is *perfect* if it is quasi-isomorphic to a strictly perfect complex. We call it *DG-flat* if for each geometric point of U, the complex of stalks is *DG*-flat.

Let Λ be an adic $\mathbb{Z}_p$-algebra.

DEFINITION 7.1. The *category* $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ *of perfect complexes of adic sheaves on U* is the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ are inverse systems $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ such that:

1. for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}_I^\bullet$ is *DG*-flat perfect complex of étale sheaves of Λ/I-modules on U,

2. for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism

$$\varphi_{IJ}: \mathcal{F}_I^\bullet \to \mathcal{F}_J^\bullet$$

of the system induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I^\bullet \xrightarrow{\sim} \mathcal{F}_J^\bullet.$$

Weak equivalences and cofibrations are defined as in Definition 3.2.

Definition 7.2. Any system $\mathcal{F} = (\mathcal{F}_I)_{I \in \mathfrak{J}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ consisting of flat, constructible sheaves $\mathcal{F}_I$ of Λ/I-modules on U, regarded as complexes concentrated in degree 0, will be called a Λ-*adic sheaf* on U. If in addition, the $\mathcal{F}_I$ are locally constant, we call $\mathcal{F}$ a *smooth Λ-adic sheaf*. We write $\mathbf{S}(U, \Lambda)$ and $\mathbf{S}^{\mathrm{sm}}(U, \Lambda)$ for the full Waldhausen categories of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ consisting of Λ-adic sheaves and smooth Λ-adic sheaves, respectively.

Definition 7.3. If U is an open dense subscheme of $\mathrm{Spec}\,\mathcal{O}_F$, we will call a complex $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{J}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ to be *smooth at ∞* if for each $I \in \mathfrak{J}_\Lambda$, the stalk of $\mathcal{F}_I^\bullet$ in $\mathrm{Spec}\,\overline{F}$ is quasi-isomorphic to a strictly perfect complex of Λ/I-modules with trivial action of any complex conjugation $\sigma \in \mathrm{Gal}_F$. The full subcategory of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ of complexes smooth at ∞ will be denoted by

$$\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$$

Since we assume $p \neq 2$, it is immediate that if in an exact sequence

$$0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$, the complexes $\mathcal{F}^\bullet$ and $\mathcal{H}^\bullet$ are smooth at ∞, then so is $\mathcal{G}^\bullet$. It then follows from [Wit08, Prop. 3.1.1] that $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$ is a Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.

We will write Λ_U for the smooth Λ-adic sheaf on U given by the system of constant sheaves $(\Lambda/I)_{I \in \mathfrak{J}_\Lambda}$ on U. Further, if p is invertible on U, we will write μ_{p^n} for the sheaf of p^n-th roots of unity on U, and

$$(\mathcal{F}_I^\bullet)_{I \in \mathfrak{J}_\Lambda}(1) := (\varprojlim_n \mu_{p^n} \otimes_{\mathbb{Z}_p} \mathcal{F}_I^\bullet)_{I \in \mathfrak{J}_\Lambda}$$

for the Tate twist of a complex in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.

We will consider Godement resolutions of the complexes in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$. To be explicit, we will fix an algebraic closure $\overline{F}$ of F and for each place x of F an embedding $\overline{F} \subset \overline{F}_x$ into a fixed algebraic closure of the local field F_x in x. In particular, we also obtain an embedding of the residue field $k(x)$ of x into the algebraically closed residue field $\overline{k(x)}$ of $\overline{F}_x$ for each closed point x of U. We write $\hat{x}$ for the corresponding geometric point $\hat{x}: \mathrm{Spec}\,\overline{k(x)} \to U$ over x and let U^0 denote the set of closed points of U.

For each étale sheaf $\mathcal{F}$ on U we set

$$(\mathrm{G}_U\,\mathcal{F})^n := \underbrace{\prod_{u \in U^0} \hat{u}_* \hat{u}^* \cdots \prod_{u \in U^0} \hat{u}_* \hat{u}^*}_{n+1} \mathcal{F}$$

and turn $(\mathrm{G}_U\,\mathcal{F})^\bullet$ into a complex by taking as differentials

$$\partial^n: (\mathrm{G}_U\,\mathcal{F})^n \to (\mathrm{G}_U\,\mathcal{F})^{n+1}$$

the alternating sums of the maps induced by the natural transformation

$$\mathcal{F} \to \prod_{u \in U^0} \hat{u}_* \hat{u}^* \mathcal{F}.$$

The Godement resolution of a complex of étale sheaves is given by the total complex of the corresponding double complex as in [Wit08, Def. 4.2.1]. The Godement resolution of a complex $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ is given by applying the Godement resolution to each of the complexes $\mathcal{F}_I^\bullet$ individually. We may define the total derived section functor

$$\mathrm{R}\,\Gamma(U, \cdot) : \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

by the formula

$$\mathrm{R}\,\Gamma(U, (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) \coloneqq (\Gamma(U, \mathrm{G}_U\, \mathcal{F}_I^\bullet))_{I \in \mathfrak{I}_\Lambda}.$$

This agrees with the usual construction if we consider $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ as an object of the 'derived' category of adic sheaves, e.g. as defined in [KW01] for $\Lambda = \mathbb{Z}_p$. In addition, however, we see that $\mathrm{R}\,\Gamma(U, \cdot)$ is a Waldhausen exact functor and hence, induces homomorphisms

$$\mathrm{R}\,\Gamma(U, \cdot) : \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)) \to \mathrm{K}_n(\Lambda)$$

for all n [Wit08, Prop. 4.6.6, Def. 5.4.13]. Here, we use the finiteness and the vanishing in large degrees of the étale cohomology groups $\mathrm{H}^n(U, \mathcal{G})$ for constructible sheaves $\mathcal{G}$ of abelian groups in order to assure that $\mathrm{R}\,\Gamma(U, (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda})$ is indeed an object of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. In particular, for each $I \in \mathfrak{I}_\Lambda$, $\mathrm{R}\,\Gamma(U, \mathcal{F}_I^\bullet)$ is a perfect complex of Λ/I-modules. Note that we do not need to assume that p is invertible on U (see the remark after [Mil06, Thm. II.3.1]).

If $j : U \to V$ is an open immersion, we set

$$j_!(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \coloneqq (j_! \mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda},$$
$$\mathrm{R}\,j_*(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \coloneqq (j_*\, \mathrm{G}_U\, \mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}.$$

for any $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$. While the extension by zero $j_!$ always gives us a Waldhausen exact functor

$$j_! : \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(V, \Lambda),$$

the total direct image

$$\mathrm{R}\,j_* : \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(V, \Lambda)$$

is only a well-defined Waldhausen exact functor if p is invertible on $V - U$. If $V - U$ contains places above p, then $\mathrm{R}\,j_*(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ is still a system of DG-flat complexes compatible in the sense of Definition 7.1.(2), but for $I \in \mathfrak{I}_\Lambda$ the cohomology of the complex of stalks of the complexes $\mathrm{R}\,j_* \mathcal{F}_I^\bullet$ in the geometric points over places above p is in general not finite, such that $\mathrm{R}\,j_* \mathcal{F}_I^\bullet$ fails to be

a perfect complex. In any case, we may consider $\mathrm{R}\,j_*$ as a Waldhausen exact functor from $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ to the Waldhausen category of complexes over the abelian category of inverse systems of étale sheaves of Λ-modules, indexed by $\mathfrak{I}_\Lambda$.

The pullback f^* along a morphism of schemes f and the pushforward f_* along a finite morphism of schemes are also defined as Waldhausen exact functors by degreewise application. No Godement resolution is needed, since these functors are exact on all étale sheaves.

As a shorthand, we set

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U,(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) \coloneqq \mathrm{R}(X,j_!(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda})$$

for $j\colon U \to X$ the open immersion into $X = \operatorname{Spec}\mathcal{O}_F$. Under our assumption that $p \neq 2$, this agrees with the definition of cohomology with proper support in [Mil06, §II.2]. If F is a totally real number field and $(\mathcal{F}_I^\bullet(-1))_{I\in\mathfrak{I}_\Lambda}$ is smooth at ∞, then it also agrees with the definition in [FK06, §1.6.3], but in general, the two definitions differ by a contribution coming from the complex places.

For any closed point x of X and any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(x,\Lambda)$, we set

$$\mathrm{R}\,\Gamma(\hat{x}, \mathcal{F}^\bullet) \coloneqq \Gamma(\operatorname{Spec}\overline{k(x)}, \hat{x}^*\,\mathrm{G}_x\,\mathcal{F}^\bullet)$$

and let $\mathfrak{F}_x \in \operatorname{Gal}(\overline{k(x)}/k(x))$ denote the geometric Frobenius of $k(x)$. We obtain an exact sequence

$$0 \to \mathrm{R}\,\Gamma(x,\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ [Wit08, Prop. 6.1.2]. Note that if $\hat{x}'$ is the geometric point corresponding to another choice of an embedding $\overline{F} \subset \overline{F}_x$ and if $\mathfrak{F}_x'$ denotes the associated geometric Frobenius, then there is a canonical isomorphism

$$\sigma\colon \mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma(\hat{x}',\mathcal{F}^\bullet)$$

such that

$$\sigma \circ (\mathrm{id} - \mathfrak{F}_x) = (\mathrm{id} - \mathfrak{F}_x') \circ \sigma. \tag{7.1}$$

At some point, we will also make use of the categories $\mathbf{PDG}^{\mathrm{cont}}(\operatorname{Spec}F_x,\Lambda)$ for the local fields F_x together with the associated total derived section functors. In this case, one can directly appeal to the constructions in [Wit08, Ch. 5]. We write F_x^{nr} for the maximal unramified extension field of F_x in $\overline{F}_x$ and note that we have a canonical identification $\operatorname{Gal}(F_x^{\mathrm{nr}}/F_x) \cong \operatorname{Gal}(\overline{k(x)}/k(x))$.

LEMMA 7.4. *Let $j\colon U \to V$ denote the open immersion of two open dense sub-schemes of X and assume that $i\colon x \to V$ is a closed point in the complement of U not lying over p. Write $\eta_x\colon \operatorname{Spec}F_x \to U$ for the map to the generic point of U. Then there exists a canonical chain of weak equivalences*

$$\mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,j_*\,\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec}F_x^{\mathrm{nr}}, \eta_x^*\,\mathrm{G}_U\,\mathcal{F}^\bullet) \xleftarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec}F_x^{\mathrm{nr}}, \eta_x^*\mathcal{F}^\bullet) \tag{7.2}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ *compatible with the operation of the Frobenius on each complex and hence, a canonical chain of weak equivalences*

$$\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\, j_* \,\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^* \,\mathrm{G}_U \,\mathcal{F}^\bullet) \xleftarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^* \mathcal{F}^\bullet) \quad (7.3)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

Proof. From [Mil80, Thm. III.1.15] we conclude that for each $I \in \mathfrak{I}_\Lambda$, the complex $\eta_x^* \,\mathrm{G}_U \,\mathcal{F}_I^\bullet$ is a complex of flabby sheaves on $\operatorname{Spec} F_x$ and that

$$\mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\, j_* \,\mathcal{F}_I^\bullet) \to \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^* \,\mathrm{G}_U \,\mathcal{F}_I^\bullet)$$

is an isomorphism. Write G_{F_x} for the Godement resolution on $\operatorname{Spec} F_x$ with respect to $\operatorname{Spec} \overline{F}_x \to \operatorname{Spec} F_x$. Then

$$\eta_x^* \,\mathrm{G}_U \,\mathcal{F}_I^\bullet \to \mathrm{G}_{F_x} \,\eta_x^* \,\mathrm{G}_U \,\mathcal{F}_I^\bullet \leftarrow \mathrm{G}_{F_x} \,\eta_x^* \mathcal{F}_I^\bullet$$

are quasi-isomorphisms of complexes of flabby sheaves on $\operatorname{Spec} F_x$. Hence, they remain quasi-isomorphisms if we apply the section functor $\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -)$ in each degree. Since the Frobenius acts compatibly on F_x^{nr} and $\overline{k(x)}$, the induced operation on the complexes is also compatible. The canonical exact sequence

$$0 \to \Gamma(\operatorname{Spec} F_x, -) \to \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -) \xrightarrow{\mathrm{id} - \mathfrak{F}_x} \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -) \to 0$$

on flabby sheaves on $\operatorname{Spec} F_x$ implies that the morphisms in the chain (7.3) are also quasi-isomorphisms. $\square$

Remark 7.5. Note that for x lying over p, the proof of the lemma remains still valid, except that the complexes in the chain (7.2) do not lie in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

It will be useful to introduce an explicit strictly perfect complex weakly equivalent to $\mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^* \mathcal{F})$ in the case that $\mathcal{F}$ is a Λ-adic sheaf on U. Assume that $x \in X$ does not lie over p. Let N be the compact $\operatorname{Gal}(\overline{F}_x/F_x)$-module corresponding to $\eta_x^* \mathcal{F}$ and write $F_x^{\mathrm{nr},(p)}$ for the maximal pro-p extension of F_x^{nr} inside $\overline{F}_x$, such that $\operatorname{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}}) \cong \mathbb{Z}_p$.

We set $N' := N^{\operatorname{Gal}(\overline{F}_x/F_x^{\mathrm{nr},(p)})}$. Note that N' is a direct summand of the finitely generated, projective Λ-module N, because the p-Sylow subgroups of the Galois group $\operatorname{Gal}(\overline{F}_x/F_x^{\mathrm{nr},(p)})$ are trivial by our assumption that p is different from the characteristic of $k(x)$. In particular, N' is itself finitely generated and projective over Λ.

Fix a topological generator τ of $\operatorname{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})$ and a lift $\varphi \in \operatorname{Gal}(F_x^{\mathrm{nr},(p)}/F_x)$ of the geometric Frobenius $\mathfrak{F}_x$. Then τ and φ are topological generators of the profinite group $\operatorname{Gal}(F_x^{\mathrm{nr},(p)}/F_x)$ and

$$\varphi \tau \varphi^{-1} = \tau^{q^{-1}}$$

with $q = q_x$ the number of elements of $k(x)$ [NSW00, Thm. 7.5.3].

DEFINITION 7.6. We define a strictly perfect complex $D_{\hat{x}}^{\bullet}(\mathcal{F})$ of Λ-modules with an action of $\mathfrak{F}_x$ as follows: For $k \neq 0, 1$ we set $D_{\hat{x}}^{k}(\mathcal{F}) \coloneqq 0$. As Λ-modules we have $D_{\hat{x}}^{0}(\mathcal{F}) = D_{\hat{x}}^{1}(\mathcal{F}) = N'$ and the differential is given by $\mathrm{id} - \tau$. The geometric Frobenius $\mathfrak{F}_x$ acts on $D_{\hat{x}}^{0}(\mathcal{F})$ via φ and on $D_{\hat{x}}^{1}(\mathcal{F})$ via

$$\varphi\left(\frac{\tau^q - 1}{\tau - 1}\right) \in \Lambda[[\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x)]]^{\times}.$$

LEMMA 7.7. *There exists a weak equivalence*

$$D_{\hat{x}}^{\bullet}(\mathcal{F}) \xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Spec}\, F_x^{\mathrm{nr}}, \eta_x^* \mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ *that is compatible with the operation of the geometric Frobenius* $\mathfrak{F}_x$ *on both sides.*

Proof. Clearly, we have

$$\Lambda/I \otimes_{\Lambda} D_{\hat{x}}^{\bullet}(\mathcal{F}) \cong D_{\hat{x}}^{\bullet}(\mathcal{F}_I)$$

for all $I \in \mathfrak{I}_{\Lambda}$. We may therefore reduce to the case that Λ is a finite $\mathbb{Z}_p$-algebra. By construction, the perfect complex of Λ-modules $\mathrm{R}\,\Gamma(F_x^{\mathrm{nr}}, \eta_x^* \mathcal{F})$ may be canonically identified with the homogenous cochain complex

$$X^{\bullet}(\mathrm{Gal}(\overline{F}_x/F_x), N)^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}$$

(in the notation of [NSW00, Ch. I, §2]) of the finite $\mathrm{Gal}(\overline{F}_x/F_x)$-module N. Recall that the elements of $X^n(\mathrm{Gal}(\overline{F}_x/F_x), N)$ are continuous maps

$$f \colon \mathrm{Gal}(\overline{F}_x/F_x)^{n+1} \to N$$

and the operation of $\sigma \in \mathrm{Gal}(\overline{F}_x/F_x)$ on $f \in X^n(\mathrm{Gal}(\overline{F}_x/F_x), N)$ is defined by

$$\sigma f \colon \mathrm{Gal}(\overline{F}_x/F_x)^{n+1} \to N, \qquad (\sigma_0, \ldots, \sigma_n) \mapsto \sigma f(\sigma^{-1}\sigma_0, \ldots, \sigma^{-1}\sigma_n).$$

The inflation map provides a quasi-isomorphism

$$X^{\bullet}(\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x), N')^{\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})} \xrightarrow{\sim} X^{\bullet}(\mathrm{Gal}(\overline{F}_x/F_x), N)^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})},$$

which is compatible with the operation of $\mathfrak{F}_x$ by a lift to $\mathrm{Gal}(\overline{F}_x/F_x)$ on both sides.
We define a quasi-isomorphism

$$\alpha \colon D_{\hat{x}}^{\bullet}(\mathcal{F}) \xrightarrow{\sim} X^{\bullet}(\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x), N')^{\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})}$$

compatible with the $\mathfrak{F}_x$-operation by

$$\alpha(n)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x) \to N', \qquad \tau^a \varphi^b \mapsto \tau^a n \qquad\qquad \text{for } n \in D_{\hat{x}}^{0}(\mathcal{F}),$$

$$\alpha(n)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x)^2 \to N', \qquad (\tau^a \varphi^b, \tau^c \varphi^d) \mapsto \frac{\tau^c - \tau^a}{1 - \tau} n \quad \text{for } n \in D_{\hat{x}}^{1}(\mathcal{F}),$$

with $a, c \in \mathbb{Z}_p$, $b, d \in \hat{\mathbb{Z}}$. Note that

$$\frac{\tau^c - \tau^a}{1 - \tau} = \tau^c \sum_{n=1}^{\infty} \binom{a-c}{n} (\tau - 1)^{n-1}$$

is a well-defined element of $\Lambda[[\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x)]]$ for any $a, c \in \mathbb{Z}_p$. $\square$

Assume again that $U \subset X$ is an open or closed subscheme. If Λ' is another adic $\mathbb{Z}_p$-algebra and $M^\bullet$ a complex of Λ'-Λ-bimodules which is strictly perfect as complex of Λ'-modules, we may extend $\Psi_{M^\bullet}$ to a Waldhausen exact functor

$$\Psi_{M^\bullet} \colon \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda'),$$
$$(\mathcal{P}_J^\bullet)_{J \in \mathfrak{I}_\Lambda} \mapsto \big(\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_\Lambda M^\bullet \otimes_\Lambda \mathcal{P}_J^\bullet \big)_{I \in \mathfrak{I}_{\Lambda'}}$$

such that

$$\Psi_{M^\bullet} \mathrm{R}\,\Gamma(U, \mathcal{P}^\bullet) \to \mathrm{R}\,\Gamma(U, \Psi_{M^\bullet}(\mathcal{P}^\bullet))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda')$ [Wit08, Prop. 5.5.7].

If $i \colon \Sigma \to V$ is the embedding of a closed subscheme Σ of X into an open subscheme V of X with complement $j \colon U \to V$ and $\mathcal{F}$ is an étale sheaf of abelian groups on V, then we may consider the sheaf

$$i^! \mathcal{F} \coloneqq \ker(i^* \mathcal{F} \to i^* j_* j^* \mathcal{F})$$

on Σ. Its global sections $i^! \mathcal{F}(\Sigma)$ are the global sections of $\mathcal{F}$ on V with support on Σ. The right derived functor $\mathrm{R}\,i^!$ can also be defined via Godement resolution:

Lemma 7.8. *Assume that p is invertible on Σ.*

$$\mathrm{R}\,i^! \colon \mathbf{PDG}^{\mathrm{cont}}(V, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Sigma, \Lambda), \qquad (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \mapsto (i^! \mathrm{G}_V(\mathcal{F}_I^\bullet))_{I \in \mathfrak{I}_\Lambda}$$

is a Waldhausen exact functor and for every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(V, \Lambda)$ there is an exact sequence

$$0 \to i_* \mathrm{R}\,i^! \mathcal{F}^\bullet \to \mathrm{G}_V(\mathcal{F}^\bullet) \to \mathrm{R}\,j_* j^* \mathcal{F}^\bullet \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(V, \Lambda)$. In particular, if $i^* \mathcal{F}^\bullet$ is weakly equivalent to 0, then there exists a chain of weak equivalences

$$i^* \mathrm{R}\,j_* j^* \mathcal{F}^\bullet \sim \mathrm{R}\,i^! \mathcal{F}^\bullet[1].$$

Proof. Note that for any abelian étale sheaf $\mathcal{F}$ on V, we habe $j^* \mathrm{G}_V(\mathcal{F}) = \mathrm{G}_U(\mathcal{F})$. Moreover, by [AGV72b, Exp. XVII, Prop. 4.2.3], $\mathrm{G}_U(\mathcal{F})$ is a complex of flasque sheaves in the sense of [AGV72b, Exp. V, Def. 4.1]. In particular, $\mathrm{G}_V(\mathcal{F}) \to j_* j^* \mathrm{G}_V(\mathcal{F})$ is surjective in the category of presheaves by [AGV72a, Exp. V, Prop. 4.7]. If $\mathcal{F}^\bullet$ is a complex of abelian sheaves, $\mathrm{G}_V(\mathcal{F}^\bullet)$

is constructed as the total complex of the double complex obtained by taking the Godement resolution of each individual sheaf. In particular, $G_V(\mathcal{F}^\bullet)$ is a complex of possibly infinite sums of flasque sheaves. Note that infinite sums of flasque sheaves are not necessarily flasque. Still, as étale cohomology of noetherian schemes commutes with filtered direct limits, $G_V(\mathcal{F}^\bullet) \to j_* j^* G_V(\mathcal{F}^\bullet)$ is always surjective in the category of presheaves. This proves the exactness of the above sequence. Moreover, it implies that $G_V(\mathcal{F}^\bullet)$ is a $i^!$-acyclic resolution of $\mathcal{F}^\bullet$ such that $i^! G_V$ preserves quasi-isomorphisms and injections. If $\mathcal{F}^\bullet$ is a perfect complex of sheaves of Λ-modules on V for any finite ring Λ, then $i^! G_V(\mathcal{F}^\bullet)$ is perfect since this is true for $i^* G_V(\mathcal{F}^\bullet)$ and $i^* j_* j^* G_V(\mathcal{F}^\bullet)$. Similarly, we see that $i^! G_V$ commutes with tensor products with finitely generated right Λ-modules. In particular, $R\, i^!$ does indeed take values in $\mathbf{PDG}^{\mathrm{cont}}(\Sigma, \Lambda)$ for any adic ring Λ. Finally, if $i^* \mathcal{F}^\bullet$ is weakly equivalent to 0, then we obtain the chain of weak equivalences

$$i^* R\, j_* j^* \mathcal{F}^\bullet \xleftarrow{\sim} \mathrm{Cone}\left(R\, i^! \mathcal{F}^\bullet \to i^* G_V(\mathcal{F}^\bullet)\right) \xrightarrow{\sim} R\, i^! \mathcal{F}^\bullet[1].$$

$\square$

8 Duality for adic sheaves

For any scheme Z, any ring R and any two étale sheaves of R-modules $\mathcal{F}$, $\mathcal{G}$ on Z, let

$$\mathcal{H}om_{R,Z}(\mathcal{F}, \mathcal{G})$$

denote the sheaf of R-linear morphisms $\mathcal{F} \to \mathcal{G}$ on Z. As before, we fix an adic $\mathbb{Z}_p$-algebra Λ. Let $U \subset X = \mathrm{Spec}\, \mathcal{O}_F$ be an open or closed subscheme. Unfortunately, we cannot present a construction of a Waldhausen exact functor

$$*\colon \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)^{\mathrm{op}} \to \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda^{\mathrm{op}})$$

that would give rise to the usual total derived $\mathcal{H}om$-functor $\mathcal{F} \mapsto R\,\mathcal{H}om_{\Lambda,U}(\mathcal{F}, \Lambda_U)$ on the 'derived' category of Λ-adic sheaves. Instead, we will construct a Waldhausen exact duality functor on the Waldhausen subcategory $\mathbf{S}^{\mathrm{sm}}(U, \Lambda)$ of smooth Λ-adic sheaves.
For any smooth Λ-adic sheaf $\mathcal{F}$,

$$\begin{aligned}
\mathcal{F}^{*_\Lambda} &:= \left(\mathcal{H}om_{\Lambda/I, U}(\mathcal{F}_I, (\Lambda/I)_U)\right)_{I \in \mathfrak{I}_\Lambda} \\
&= \left(\mathcal{H}om_{\mathbb{Z}, U}\left(\mathcal{H}om_{\mathbb{Z}, U}((\Lambda/I)_U, (\mathbb{Q}_p/\mathbb{Z}_p)_U) \otimes_{\Lambda/I} \mathcal{F}_I, (\mathbb{Q}_p/\mathbb{Z}_p)_U\right)\right)_{I \in \mathfrak{I}_\Lambda}
\end{aligned}$$

is a smooth Λ^{op}-adic sheaf on U. In this way, we obtain a Waldhausen exact equivalence

$$*\colon \mathbf{S}^{\mathrm{sm}}(U, \Lambda)^{\mathrm{op}} \to \mathbf{S}^{\mathrm{sm}}(U, \Lambda^{\mathrm{op}})$$

and, by composing with $I\colon \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U, \Lambda)) \xrightarrow{\cong} \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U, \Lambda)^{\mathrm{op}})$, isomorphisms

$$*\colon \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U, \Lambda)) \to \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U, \Lambda^{\mathrm{op}}))$$

for each $n \geq 0$.

Assume that U is an open subscheme of X such that p is invertible on U. If $\mathcal{F}$ is a smooth Λ-adic sheaf on U, we can find a strictly perfect complex of Λ-modules $P^\bullet$ together with a weak equivalence

$$P^\bullet \xrightarrow{\sim} \mathrm{R}\,\Gamma_{\mathrm{c}}(U, \mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. As a consequence of Artin-Verdier duality [Mil06, Thm. II.3.1], we then also have a weak equivalence

$$(P^\bullet)^* \xrightarrow{\sim} \mathrm{R}\,\Gamma(U, \mathcal{F}^*(1))[-3]. \tag{8.1}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$.

We could proceed in the same way for local duality and duality over finite fields, but instead, we prove the following finer results.

LEMMA 8.1. *Assume that U is an open subscheme of X such that p is invertible on U and that $i \colon x \to X$ is a closed point not lying over p. For any smooth Λ-adic sheaf $\mathcal{F}$ on U, there exists a weak equivalence*

$$D_{\hat{x}}^\bullet(\mathcal{F})^* \xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Spec}\, F_x^{\mathrm{nr}}, \eta_x \mathcal{F}^*(1))[-1]$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$, compatible with the operation of $\mathfrak{F}_x^*$ on the left and of $\mathfrak{F}_x^{-1}$ on the right.

Proof. As in the proof of Lemma 7.7, we can replace $\mathrm{R}\,\Gamma(\mathrm{Spec}\, F_x^{\mathrm{nr}}, \eta_x \mathcal{F}^*(1))$ by the homogenous cochain complex

$$X^\bullet\big(\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x), (N')^*(1)\big)^{\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})}.$$

By choosing a basis of the free $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$, i.e. a compatible system of p^n-th roots of unity, we may identify the underlying Λ-modules of $(N')^*$ and $(N')^*(1)$. The operation of $\sigma \in \mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})$ on $f \in (N')^*$ is given by

$$\sigma f := f \circ (\sigma^*)^{-1}.$$

The operation of $\mathfrak{F}_x^*$ on $f \in D_{\hat{x}}^1(\mathcal{F})^* = (N')^*$ is then given by

$$\mathfrak{F}_x^*(f) := \left(\frac{\tau^{-q} - 1}{\tau^{-1} - 1}\right)\varphi^{-1} f$$

and on $g \in D_{\hat{x}}^0(\mathcal{F})^* = (N')^*$ by

$$\mathfrak{F}_x^*(g) := \varphi^{-1} g,$$

with $\varphi, \tau \in \mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x)$ denoting our fixed topological generators and $q \in \Lambda^\times$ denoting the order of the residue field $k(x)$. Set for $b \in \hat{\mathbb{Z}}$

$$s(b) := q^{-b}\left(\frac{\tau^{-1} - 1}{\tau^{-q^{-b}} - 1}\right) \in \Lambda[[\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})]]^\times$$

and note that s satisfies the cocycle relation

$$s(b+1) = q^{-1}\varphi s(b)\left(\frac{\tau^{-q}-1}{\tau^{-1}-1}\right)\varphi^{-1} = q^{-1}\varphi s(b)\varphi^{-1}s(1).$$

We define a weak equivalence

$$\beta\colon D_{\hat{x}}^{\bullet}(\mathcal{F})^* \xrightarrow{\sim} X^{\bullet}(\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x),(N')^*(1))^{\mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x^{\mathrm{nr}})}[-1]$$

by

$$\beta(f)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x) \to (N')^*(1), \qquad \tau^a\varphi^b \mapsto \tau^a s(b)f$$

for $f \in D_{\hat{x}}^1(\mathcal{F})^*$ and by

$$\beta(g)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(p)}/F_x)^2 \to (N')^*(1), \qquad (\tau^a\varphi^b, \tau^c\varphi^d) \mapsto \left(\frac{\tau^a s(b) - \tau^c s(d)}{1-\tau^{-1}}\right)g$$

for $g \in D_{\hat{x}}^0(\mathcal{F})^*$, $a,c \in \mathbb{Z}_p$, $b,d \in \hat{\mathbb{Z}}$.

Using the cocycle relation for s, it is easily checked that

$$\beta \circ \mathfrak{F}_x^* = \mathfrak{F}_x^{-1} \circ \beta,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In particular, if $Q^{\bullet}$ denotes the cocone of

$$D_{\hat{x}}^{\bullet}(\mathcal{F}) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} D_{\hat{x}}^{\bullet}(\mathcal{F}),$$

then $Q^{\bullet}$ is a strictly perfect complex of Λ-modules and there exist weak equivalences

$$\begin{aligned}Q^{\bullet} &\xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Spec}\,F_x, \eta_x^*\mathcal{F}), \\ (Q^{\bullet})^* &\xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Spec}\,F_x, \eta_x^*\mathcal{F}^*(1))[-2].\end{aligned} \tag{8.2}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$.

Let now $\mathcal{G}$ be a complex in $\mathbf{S}(x,\Lambda) = \mathbf{S}^{\mathrm{sm}}(x,\Lambda)$ and let

$$\mathcal{G}_{\hat{x}} := \varprojlim_{I \in \mathfrak{I}_{\Lambda}} (\mathcal{G}_I)_{\hat{x}}$$

be the stalk of $\mathcal{G}$ in the geometric point $\hat{x}$ over x. Then $\mathcal{G}_{\hat{x}}$ is a finitely generated, projective Λ-module, equipped with a natural operation of $\mathfrak{F}_x$. Clearly, the natural morphism

$$\mathcal{G}_{\hat{x}} \xrightarrow{\sim} \mathrm{R}\,\Gamma(\hat{x}, \mathcal{G}) \tag{8.3}$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ that is compatible with the operation of $\mathfrak{F}_x$ on both sides. In particular, the cocone $C^{\bullet}$ of

$$\mathcal{G}_{\hat{x}} \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathcal{G}_{\hat{x}}$$

is weakly equivalent to $\mathrm{R}\,\Gamma(x, \mathcal{G})$.

Lemma 8.2. *With $\mathcal{G}$ as above, there exists an isomorphism*

$$(\mathcal{G}_{\hat{x}})^* \xrightarrow{\cong} (\mathcal{G}^*)_{\hat{x}}$$

of finitely generated, projective Λ-modules, compatible with the operation of $\mathfrak{F}_x^*$ on the left and of $\mathfrak{F}_x^{-1}$ on the right.

Proof. Let R be any finite ring. Under the equivalence between the categories of étale sheaves of R-modules on x and of discrete $R[[\mathrm{Gal}(\overline{k(x)}/k(x))]]$-modules, given by $\mathcal{F} \mapsto \mathcal{F}_{\hat{x}}$, the dual sheaf $\mathcal{F}^{*R}$ corresponds to the R^{op}-module $(\mathcal{F}_{\hat{x}})^{*R}$ with $\sigma \in \mathrm{Gal}(\overline{k(x)}/k(x))$ acting on $f\colon \mathcal{F}_{\hat{x}} \to R$ by $f \circ (\sigma^{*R})^{-1}$. □

Consequently, we obtain a weak equivalence

$$(C^\bullet)^* \xrightarrow{\sim} \mathrm{R}\Gamma(x, \mathcal{G}^*)[-1] \tag{8.4}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$. If $\mathcal{G} = i^*\mathcal{F}$ with $\mathcal{F}$ a smooth Λ-adic sheaf on U as above, then by the exchange formula [Fu11, Thm. 8.4.7], there exists a chain of weak equivalences

$$
\begin{aligned}
(i^*\mathcal{F})^* &= \left(\mathrm{R}\,\mathcal{H}om_{\Lambda/I,x}(i^*\mathcal{F}_I, (\Lambda/I)_x)\right)_{I \in \mathfrak{I}_\Lambda} \\
&\sim \left(\mathrm{R}\,\mathcal{H}om_{\Lambda/I,x}(i^*\mathcal{F}_I, \mathrm{R}\,i^!(\Lambda/I)_U(1)[-2])\right)_{I \in \mathfrak{I}_\Lambda} \\
&\sim \left(\mathrm{R}\,i^!\,\mathcal{H}om_{\Lambda/I,U}(\mathcal{F}_I, (\Lambda/I)_U(1))[-2]\right)_{I \in \mathfrak{I}_\Lambda} \\
&= \mathrm{R}\,i^!\mathcal{F}^*(1)[-2]
\end{aligned}
\tag{8.5}
$$

in $\mathbf{PDG}^{\mathrm{cont}}(x, \Lambda^{\mathrm{op}})$.

9   Admissible extensions

As before, we fix an odd prime p and a number field F. Assume that F_∞/F is a possibly infinite Galois extension unramified over an open or closed subscheme $U = U_F$ of $X = \mathrm{Spec}\,\mathcal{O}_F$. Let $G \coloneqq \mathrm{Gal}(F_\infty/F)$ be its Galois group. We also assume that G has a topologically finitely generated, open pro-p-subgroup, such that for any adic $\mathbb{Z}_p$-algebra Λ, the profinite group ring $\Lambda[[G]]$ is again an adic ring [Wit14, Prop. 3.2]. For any intermediate number field K of F_∞/F, we will write U_K for the base change with $X_K \coloneqq \mathrm{Spec}\,\mathcal{O}_K$ and $f_K\colon U_K \to U$ for the corresponding Galois covering of U, such that we obtain a system of Galois coverings $(f_K\colon U_K \to U)_{F \subset K \subset F_\infty}$, which we denote by

$$f\colon U_{F_\infty} \to U.$$

As in [Wit14, Def. 6.1] we make the following construction.

Definition 9.1. Let Λ be any adic $\mathbb{Z}_p$-algebra. For $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ we set

$$f_! f^* \mathcal{F}^\bullet \coloneqq \left(\varprojlim_{I \in \mathfrak{I}_\Lambda} \varprojlim_{F \subset K \subset F_\infty} \Lambda[[G]]/J \otimes_{\Lambda[[G]]} f_{K!} f_K^* \mathcal{F}_I^\bullet\right)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$$

Remark 9.2. The functor $f_!f^*$ corresponds to the composition of the restriction and compact induction functors on the level of compact Galois modules. On finite level, the extension by zero $f_{K!}$ agrees with the direct image f_{K*}. We use the notation of the extension by zero to emphasise its role as a left adjoint of f_K^*.

As in [Wit14, Prop. 6.2] one verifies that we thus obtain a Waldhausen exact functor

$$f_!f^*\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda[[G]]).$$

In particular, if U is open and dense in X and if $k\colon U \to W$ denotes the open immersion into another open dense subscheme W of X, we obtain for each complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ a complex

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.

Remark 9.3. For U open and dense in X, set $V \coloneqq U \cup (X - W)$ and let $j\colon U \to V$ denote the corresponding open immersion. Write $j'\colon V \to X$ and $k'\colon W \to X$ for the open immersions into X. For any étale sheaf $\mathcal{G}$ on U, the canonical morphism

$$k'_! k_* \,\mathrm{G}_U\,\mathcal{G} \cong j'_* j_!\,\mathrm{G}_U\,\mathcal{G} \to j'_*\,\mathrm{G}_V\, j_!\mathcal{G}$$

is seen to be a quasi-isomorphism by checking on the stalks. Hence, for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$, there is a weak equivalence

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(V, j_! f_! f^* \mathcal{F}^\bullet).$$

We recall that the righthand complex is always in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$. Hence, the same is true for the left-hand complex without any condition on U and W, even if $\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet$ fails to be a perfect complex. In particular, we may use the two complexes interchangeably in our results.

We recall how the functor $f_!f^*$ transforms under the change of the extension F_∞/F and under changes of the coefficient ring Λ.

PROPOSITION 9.4. *Let $f\colon U_{F_\infty} \to U$ be the system of Galois coverings of the open or closed subscheme U of X associated to the extension F_∞/F with Galois group G which is unramified over U. Let further Λ be an adic $\mathbb{Z}_p$-algebra and $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.*

1. *Let Λ' be another adic $\mathbb{Z}_p$-algebra and let $P^\bullet$ be a complex of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules. Then there exists a natural isomorphism*

$$\Psi_{P[[G]]^{\delta\bullet}} f_! f^* \mathcal{F}^\bullet \cong f_! f^* \Psi_{P^\bullet} f_! f^* \mathcal{F}^\bullet$$

2. *Let $F'_\infty \subset F_\infty$ be a subfield such that F'_∞/F is a Galois extension with Galois group G' and let $f':U_{F'_\infty} \to U$ denote the corresponding system of Galois coverings. Then there exists a natural isomorphism*

$$\Psi_{\Lambda[[G']]}f_! f^* \mathcal{F}^\bullet \cong (f')_!(f')^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G']])$.

3. *Let F'/F be a finite extension inside F_∞/F, let $f_{F'}:U_{F'} \to U$ denote the associated étale covering of U and let $g:U_{F_\infty} \to U'_F$ be the restriction of the system of coverings f to $U_{F'}$. Write $G' \subset G$ for the corresponding open subgroup and view $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then there exists a natural isomorphism*

$$\Psi_{\Lambda[[G]]}f_! f^* \mathcal{F}^\bullet \cong f_{F'*}(g_! g^*) f_{F'}^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G']])$.

4. *With the notation of (3), let $\mathcal{G}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda)$ and view $\Lambda[[G]]$ as a $\Lambda[[G]]$-$\Lambda[[G']]$-bimodule. Then there exists a natural isomorphism*

$$\Psi_{\Lambda[[G]]}f_{F'*} g_! g^* \mathcal{G}^\bullet \cong f_! f^* (f_{F'*} \mathcal{G}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]])$.

Proof. Part (1) – (3) are proved in [Wit14, Prop. 6.5, 6.7]. We prove (4). First, note that for any finite Galois extension F''/F with $F' \subset F'' \subset F_\infty$ and any $I \in \mathfrak{I}_\Lambda$ the canonical map

$$g_{F''!} g_{F''}^* (\Lambda/I)_{U_{F'}} \to f_{F'}^* f_{F''!} f_{F''}^* (\Lambda/I)_U$$

induces an isomorphism

$$\Lambda/I[\mathrm{Gal}(F''/F)] \otimes_{\Lambda/I[\mathrm{Gal}(F''/F')]} g_{F''!} g_{F''}^* (\Lambda/I)_{U_{F'}} \cong f_{F'}^* f_{F''!} f_{F''}^* (\Lambda/I)_U.$$

Hence,

$$\Psi_{\Lambda[[G]]}(g_! g^* \Lambda_{U_{F'}}) \cong f_{F'}^* f_! f^* \Lambda_U$$

in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda[[G]])$. We further recall that in the notation of [Wit14, Prop. 6.3], there exists an isomorphism

$$f_! f^* f_{F'*} \mathcal{G}^\bullet \cong \Psi_{f_! f^* \Lambda_U} f_{F'*} \mathcal{G}^\bullet.$$

The projection formula then implies

$$\begin{aligned}
\Psi_{f_! f^* \Lambda_U} f_{F'*} \mathcal{G}^\bullet &\cong f_{F'*}(\Psi_{f_{F'}^* f_! f^* \Lambda_U}(\mathcal{G}^\bullet)) \\
&\cong f_{F'*}(\Psi_{\Psi_{\Lambda[[G]]}(g_! g^* \Lambda_{U_{F'}})}(\mathcal{G}^\bullet)) \\
&\cong f_{F'*}(\Psi_{\Lambda[[G]]}(g_! g^* \mathcal{G}^\bullet))
\end{aligned}$$

as desired. $\square$

To understand Part (1) of this proposition, note that if ρ is a representation of G on a finitely generated and projective Λ-module and $\rho^\sharp$ is the corresponding Λ-$\mathbb{Z}_p[[G]]$-bimodule as in Example 3.17, then

$$\mathcal{M}(\rho) := \Psi_{\rho^\sharp} f_! f^*(\mathbb{Z}_p)_U \tag{9.1}$$

is simply the smooth Λ-adic sheaf on U associated to ρ [Wit14, Prop. 6.8]. In general,

$$\Psi_{\tilde{P}^\bullet} \mathcal{F}^\bullet := \Psi_{P^\bullet} f_! f^* \mathcal{F}^\bullet \tag{9.2}$$

should be understood as the derived tensor product over Λ of the complex of sheaves associated to $P^\bullet$ and the complex $\mathcal{F}^\bullet$.

Assume that $\mathcal{F}$ is a smooth Λ-adic sheaf on U. As before, we write

$$\mathcal{F}^{*\Lambda} := \left(\mathcal{H}om_{\Lambda/I,U}(\mathcal{F}_I, \Lambda/I)\right)_{I \in \mathfrak{I}_{\Lambda/I}} \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda^{\mathrm{op}})$$

for the Λ-dual of $\mathcal{F}$ and $\Lambda^{\mathrm{op}}[[G]]^\sharp$ for the $\Lambda^{\mathrm{op}}[[G]]$-$\Lambda[[G]]^{\mathrm{op}}$-bimodule with $g \in G$ acting by g^{-1} from the right. We then have a natural isomorphism

$$f_! f^* \mathcal{F}^{*\Lambda} \cong \Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp} (f_! f^* \mathcal{F})^{*\Lambda[[G]]}. \tag{9.3}$$

This can then be combined with the duality assertions (8.1), (8.2), and (8.4). For example, we may find a strictly perfect complex of $\Lambda^{\mathrm{op}}[[G]]$-modules $P^\bullet$ and weak equivalences

$$\begin{aligned} P^\bullet &\xrightarrow{\sim} \mathrm{R}\Gamma_\mathrm{c}(U, f_! f^* \mathcal{F}^{*\Lambda}(1)), \\ (P^\bullet)^\circledast &\xrightarrow{\sim} \mathrm{R}\Gamma(U, f_! f^* \mathcal{F})[-3] \end{aligned} \tag{9.4}$$

if ℓ is invertible on U.

Let F_{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of F and let M be the maximal abelian p-extension of F_{cyc} unramified outside the places over p. Assume that F is a totally real field. By the validity of the weak Leopoldt conjecture for F_{cyc}, the Galois group $\mathrm{Gal}(M/F_{\mathrm{cyc}})$ is a finitely generated torsion module of projective dimension less or equal 1 over the classical Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(F_{\mathrm{cyc}}/F)]]$ [NSW00, Thm. 11.3.2]. Like in [Kak13], we will assume the vanishing of its Iwasawa μ-invariant in the following sense:

CONJECTURE 9.5. *For every totally real field F, the Galois group over F_{cyc} of the maximal abelian p-extension of F_{cyc} unramified outside the places over p is a finitely generated $\mathbb{Z}_p$-module.*

In particular, for any totally real field F and any finite set Σ of places of F containing the places over p, the Galois group over F_{cyc} of the maximal abelian p-extension of F_{cyc} unramified outside Σ is also a finitely generated $\mathbb{Z}_p$-module, noting that no finite place is completely decomposed in F_{cyc}/F [NSW00, Cor. 11.3.6]. We also observe that the Galois group $\mathrm{Gal}(F_\Sigma^{(p)}/F_{\mathrm{cyc}})$ of the maximal p-extension of F unramified outside Σ is then a free pro-p-group topologically generated by finitely many elements [NSW00, Thm. 11.3.7].

DEFINITION 9.6. Let F be a number field. An extension F_∞/F inside $\overline{F}$ is called *admissible* if

(1) F_∞/F is Galois and unramified outside a finite set of places,

(2) F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc},

(3) $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ contains an open pro-p subgroup that is topologically finitely generated.

An admissible extension F_∞/F is called *really admissible* if F_∞ and F are totally real.

The notion of really admissible extensions is slightly weaker than the notion of admissible extension used in [Kak13, Def. 2.1]: We do not need to require $\mathrm{Gal}(F_\infty/F)$ to be a p-adic Lie group. For example, as a result of the preceding discussion, we see that we could choose $F_\infty = F_\Sigma^{(p)}$ for some finite set of places Σ of F containing the places above p, provided that Conjecture 9.5 is valid.

If F_∞/F is an admissible extension, we let $G := \mathrm{Gal}(F_\infty/F)$ denote its Galois group and set $H := \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$, $\Gamma := \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. We may then choose a continuous splitting $\Gamma \to G$ to identify G with the corresponding semi-direct product $G = H \rtimes \Gamma$.

If a really admissible extension F_∞/F is unramified over the open dense subscheme $U = W$ of X, $\Lambda = \mathbb{Z}_p$ and $\mathcal{F}^\bullet = (\mathbb{Z}_p)_U(1)$, then

$$\varprojlim_{I \in \mathfrak{I}_{\mathbb{Z}_p[[G]]}} \mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))[-3]$$

is by Artin-Verdier duality and comparison of étale and Galois cohomology quasi-isomorphic to the complex $C(F_\infty/F)$ featuring in the main conjecture [Kak13, Thm. 2.11]. In particular,

$$\mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))$$

is in fact an object of $\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_p[[G]])$ under Conjecture 9.5. We will generalise this statement in the next section.

10 THE S-TORSION PROPERTY

Assume that F_∞/F is an admissible extension that is unramified over the open dense subscheme U of $X = \mathrm{Spec}\,\mathcal{O}_F$ and that $k\colon U \to W$ is the open immersion into another open dense subscheme of X. Note that p must be invertible on U, because the cyclotomic extension F_{cyc}/F is ramified in all places over p. We also fix an adic $\mathbb{Z}_p$-algebra Λ. Our purpose is to prove:

THEOREM 10.1. *Assume that F_∞/F is really admissible and that p is invertible on W. Let $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$ be a complex of Λ-adic sheaves smooth at ∞. If Conjecture 9.5 is valid, then the complexes*

$$\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet(1)), \quad \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

In the course of the proof, we will also need to consider the following local variant, whose validity is independent of Conjecture 9.5.

Theorem 10.2. *Assume that F_∞/F is an admissible extension with $k\colon U \to W$ as above. Let $i\colon \Sigma \to W$ denote a closed subscheme of W and assume that p is invertible on Σ. For any complex of Λ-adic sheaves $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$, the complexes*

$$\mathrm{R}\,\Gamma(\Sigma, i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet), \mathrm{R}\,\Gamma(\Sigma, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

Using [Wit14, Prop. 4.8] we may at once reduce to the case that Λ is a finite semi-simple $\mathbb{Z}_p$-algebra and that $F_\infty/F_{\mathrm{cyc}}$ is a finite extension. It then suffices to show that the complexes appearing in the above theorems have finite cohomology groups. We may then replace $\mathcal{F}^\bullet$ by a quasi-isomorphic strictly perfect complex. Using stupid truncation and induction on the length of the strictly perfect complex we may assume that $\mathcal{F}$ is in fact a flat and constructible sheaf (unramified in ∞). Note further that the cohomology groups

$$\mathrm{H}_c^n(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) = \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}_c^n(W_K, \mathrm{R}\, k_* f_K^* \mathcal{F}(1)),$$

$$\mathrm{H}^n(W, k_! f_! f^* \mathcal{F}) = \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^n(W_K, k_! f_K^* \mathcal{F}),$$

$$\mathrm{H}^n(\Sigma, i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}) = \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^n(\Sigma_K, i^* \,\mathrm{R}\, k_* f_K^* \mathcal{F})$$

$$\mathrm{H}^n(\Sigma, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}) = \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^n(\Sigma_K, \mathrm{R}\, i^! k_! f_K^* \mathcal{F})$$

do not change if we replace F by a finite extension of F inside F_∞. So, we may assume that $F_\infty = F_{\mathrm{cyc}}$ and that no place in Σ splits in F_∞/F. Further, we may reduce to the case that Σ consists of a single place x. In particular, x does not split or ramify in F_∞/F and x does not lie above p.

We consider Theorem 10.2 in the case that $x \in U$ and write $i'\colon x \to U$ for the inclusion map. Under the above assumptions on x, there exists a chain of weak equivalences

$$\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}) \xleftarrow{\sim} \mathrm{R}\,\Gamma(x, i'^* f_! f^* \mathcal{F}) \xrightarrow{\sim} \mathrm{R}\,\Gamma(x, g_! g^* i'^* \mathcal{F})$$

where $g\colon x_\infty \to x$ is the unique $\mathbb{Z}_p$-extension of x. We can now refer directly to [Wit14, Thm. 8.1] or identify

$$\mathrm{H}^n(x, g_! g^* i'^* \mathcal{F}) = \mathrm{H}^n(\mathrm{Gal}_{k(x)}, \mathbb{F}_p[[\Gamma]]^\sharp \otimes_{\mathbb{F}_p} M)$$

with $\mathrm{Gal}_{k(x)}$ the absolute Galois group of the residue field $k(x)$ of x, M the stalk of $\mathcal{F}$ in a geometric point over x and $\mathbb{F}_p[[\Gamma]]^\sharp$ being the $\mathrm{Gal}_{k(x)}$-module $\mathbb{F}_p[[\Gamma]]$ with $\sigma \in \mathrm{Gal}_{k(x)}$ acting by right multiplication with the image of

σ^{-1} in Γ. It is then clear that the only non-vanishing cohomology group is $H^1(\mathrm{Gal}_{k(x)}, \mathbb{F}_p[[\Gamma]]^\sharp \otimes_{\mathbb{F}_p} M)$, of order bounded by the order of M.

Write $U' \coloneqq U - \{x\}$ and let $\ell \colon U' \to U$ denote the inclusion morphism. Then there is an exact sequence

$$0 \to \mathrm{R}\, i^! k_! \ell_! \ell^* f_! f^* \mathcal{F} \to \mathrm{R}\, i^! k_! f_! f^* \mathcal{F} \to \mathrm{R}\, i^! i_* i^* k_! \mathcal{F} \to 0.$$

Moreover, there exists a chain of weak equivalences

$$i'^* f_! f^* \mathcal{F} \xrightarrow{\cong} i^* k_! f_! f^* \mathcal{F} \xrightarrow{\sim} \mathrm{R}\, i^! i_* i^* k_! f_! f^* \mathcal{F}.$$

Since we already know that the groups $\mathrm{H}^n(x, i'^* f_! f^* \mathcal{F})$ are finite, it is sufficient to prove that $\mathrm{H}^n(x, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F})$ is finite in the case that $x \in W - U$.

Now we prove Theorem 10.2 in the case that $x \in W - U$. First, note that the complex $\mathrm{R}\, i^! \mathrm{R}\, k_* f_! f^* \mathcal{F}$ is quasi-isomorphic to 0. Hence, there is a chain of weak equivalences

$$i^* \mathrm{R}\, k_* f_! f^* \mathcal{F} \sim \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}[1]$$

by Lemma 7.8. So, it suffices to consider the left-hand complex. By Lemma 7.4 and the smooth base change theorem there exists a chain of weak equivalences

$$\mathrm{R}\,\Gamma(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}) \sim \mathrm{R}\,\Gamma(\mathrm{Spec}\, F_x, h_! h^* \eta_x^* \mathcal{F}),$$

where F_x is the local field in x with valuation ring $\mathcal{O}_{F_x}$, $\eta_x \colon \mathrm{Spec}\, F_x \to U$ is the map to the generic point of U, and $h \colon \mathrm{Spec}(F_x)_{\mathrm{cyc}} \to \mathrm{Spec}\, F_x$ is the unique $\mathbb{Z}_p$-extension of F_x inside $\overline{F}_x$. We may now identify

$$\mathrm{H}^n(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}) = \mathrm{H}^n(\mathrm{Gal}_{F_x}, \mathbb{F}_p[[\Gamma]]^\sharp \otimes_{\mathbb{F}_p} M)$$

with Gal_{F_x} the absolute Galois group of the local field F_x in x, M the finite Gal_{F_x}-module corresponding to $\eta_x^* \mathcal{F}$ and $\mathbb{F}_p[[\Gamma]]^\sharp$ being the Gal_{F_x}-module $\mathbb{F}_p[[\Gamma]]$ with $\sigma \in \mathrm{Gal}_{F_x}$ acting by right multiplication with the image of σ^{-1} in Γ. The finiteness of the cohomology group on the righthand side is well-known: We can use local duality to identify it with the Pontryagin dual of

$$\mathrm{H}^{2-n}(\mathrm{Gal}_{(F_x)_{\mathrm{cyc}}}, M^\vee(1))$$

where $M^\vee(1)$ is the first Tate twist of the Pontryagin dual of M.

Finally, we prove Theorem 10.1. Assume that F_∞/F is really admissible, that $\mathcal{F}$ is smooth at ∞, and that p is invertible on W. We begin with the case of étale cohomology with proper support. Letting $i \colon \Sigma \to W$ denote the complement of U in W, we have the exact excision sequence

$$0 \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, k_! k^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}(1))$$
$$\to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, i_* i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \to 0$$

and chains of weak equivalences

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, k_! k^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \sim \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}(1)),$$
$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, i_* i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \sim \mathrm{R}\,\Gamma(\Sigma, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)).$$

By Theorem 10.2, we may thus reduce to the case $W = U$. Furthermore, we may shrink U ad libitum. Hence, we may assume that $\mathcal{F}$ is locally constant on U and smooth at ∞. Consequently, there exists a finite Galois extension F'/F such that F' is totally real, $g_{F'}\colon U_{F'} \to U$ is étale and $g_{F'}^*\mathcal{F}$ is constant. Then F'_{cyc}/F is an admissible extension and

$$\rho \coloneqq g_{F'}^*\mathcal{F}(U_{F'})$$

may be viewed as a continuous representation of $G = \mathrm{Gal}(F'_{\mathrm{cyc}}/F)$ on a finitely generated, projective Λ-module. Write $g\colon U_{F'_{\mathrm{cyc}}} \to U$ for the corresponding system of coverings of U and observe that there exists a weak equivalence

$$\Phi_\rho(\mathrm{R}\,\Gamma_{\mathrm{c}}(U, g_! g^*(\mathbb{Z}_p)_U(1))) \xrightarrow{\sim} \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}(1))$$

with Φ_ρ being defined by (3.5) [Wit14, Prop. 5.9, 6.3, 6.5, 6.7]. Since Φ_ρ takes complexes in $\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_p[[G]])$ to complexes in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[\Gamma]])$, it remains to show that the cohomology groups $\mathrm{H}_c^n(U, g_! g^*(\mathbb{Z}_p)_U(1))$ are finitely generated as $\mathbb{Z}_p$-modules. Now

$$\mathrm{H}_c^n(U, g_! g^*(\mathbb{Z}_p)_U(1)) = \begin{cases} 0 & \text{if } n \neq 2, 3, \\ \mathrm{Gal}(M/F'_{\mathrm{cyc}}) & \text{if } n = 2, \\ \mathbb{Z}_p & \text{if } n = 3, \end{cases}$$

with M denoting the maximal abelian p-extension of F'_{cyc} unramified over U [Kak13, p. 548]. At this point, we make use of Conjecture 9.5 on the vanishing of the μ-invariant to finish the proof for the first complex.

We now turn to the second complex. We still assume that Λ is a finite ring. Write $\Sigma \coloneqq W - U$, $V \coloneqq U \cup (X - W)$ and $j\colon U \to V$, $\ell\colon V \to X$, $i\colon \Sigma \to X$ for the natural immersions. As mentioned in Remark 9.3, the exists a chain of weak equivalences

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(V, \mathrm{R}\,j_* f_! f^* \mathcal{F}) \sim \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}).$$

Moreover, there is an exact sequence

$$0 \to \ell_! \mathrm{R}\,j_* f_! f^* \mathcal{F} \to \mathrm{R}(\ell \circ j)_* f_! f^* \mathcal{F} \to i_* i^* \mathrm{R}\,j_* f_! f^* \mathcal{F} \to 0$$

Using Theorem 10.2 we may thus reduce to the case that $V = X$, $W = U$ and $\mathcal{F}$ locally constant on U and smooth at ∞. Let $P^\bullet$ be a strictly perfect complex of $\Lambda^{\mathrm{op}}[[G]]$-modules quasi-isomorphic to $\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^{*_\Lambda}(1))$. By what we have proved above, $P^\bullet$ is also perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules. By (9.4) we obtain a weak equivalence

$$(P^\bullet)^\circledast \xrightarrow{\sim} \mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F}).$$

From Prop. 5.8 we conclude that $\mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F})$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(U, \Lambda[[G]])$, as claimed. This finishes the proof of Theorem 10.1.

11   Non-commutative Euler factors

Assume as before that F_∞/F is an admissible extension of a number field F which is unramified over a dense open subscheme U of X and write $f\colon U_{F_\infty} \to U$ for the system of Galois coverings of U corresponding to F_∞/F. Let W be another dense open subscheme of X containing U, but no place over p and let $k\colon U \to W$ denote the corresponding open immersion. We consider a complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. As the complexes

$$\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ for $i\colon x \to W$ a closed point, we conclude that the endomorphism

$$\mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$$

is in fact a weak equivalence in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$. Hence, it gives rise to an element in $\mathrm{K}_1(\Lambda[[G]])_S$.

DEFINITION 11.1.   The non-commutative Euler factor $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)$ of $\mathrm{R}\,k_* \mathcal{F}^\bullet$ at x is the inverse of the class of the above weak equivalence in $\mathrm{K}_1(\Lambda[[G]])_S$:

$$\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet) \coloneqq [\mathrm{id} - \mathfrak{F}_x \,\circlearrowright\, \mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)]^{-1}$$

Note that $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)$ is independent of our specific choice of a geometric point above x. Indeed, by (7.1) and relation (R5) in the definition of $\mathcal{D}_\bullet(\mathbf{W})$, we conclude that the classes $[\mathrm{id} - \mathfrak{F}_x]$ and $[\mathrm{id} - \mathfrak{F}'_x]$ agree in $\mathrm{K}_1(\Lambda[[G]])_S$. Moreover, $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)$ does not change if we enlarge W by adding points not lying over p or shrink U by removing a finite set of points different from x.

PROPOSITION 11.2.   *The non-commutative Euler factor is a characteristic element for* $\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$:

$$d\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet) = -[\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)]$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

Proof. The complex $\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$ is weakly equivalent to the cone of the endomorphism

$$\mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)$$

shifted by one. Hence, the result follows from the explicit description of d given in (5.2).   □

DEFINITION 11.3.   For a topological generator $\gamma \in \Gamma$, we define the local modification factor at x to be the element

$$\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet) \coloneqq \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet) s_\gamma([\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)]).$$

in $\mathrm{K}_1(\Lambda[[G]])$.

We obtain the following transformation properties.

Proposition 11.4. *With $k\colon U \to W$ as above, let Λ be any adic $\mathbb{Z}_p$-algebra and let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.*

1. *Let Λ' be another adic $\mathbb{Z}_p$-algebra. For any complex $P^\bullet$ of Λ'-$\Lambda[[G]]$-bimodules which is strictly perfect as complex of Λ'-modules we have*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)) = \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda'[[G]]_S)$ and

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)) = \mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_* \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda'[[G]])$.

2. *Let F'_∞/F be an admissible subextension of F_∞/F with Galois group G'. Then*

$$\Psi_{\Lambda[[G']]}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)) = \mathcal{L}_{F'_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ and

$$\Psi_{\Lambda[[G']]}(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)) = \mathcal{M}_{F'_\infty/F,\gamma}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

3. *Let F'/F be a finite extension inside F_∞/F. Set $r \coloneqq [F' \cap F_{\mathrm{cyc}} : F]$. Write $f_{F'}\colon U_{F'} \to U$ for the corresponding étale covering and $x_{F'}$ for the fibre in $\operatorname{Spec}\mathcal{O}_{F'}$ above x. Let $G' \subset G$ be the Galois group of the admissible extension F_∞/F' and consider $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then*

$$\Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_* f_{F'}^* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ and

$$\Psi_{\Lambda[[G]]}\big(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{M}_{F_\infty/F',\gamma^r}(y, \mathrm{R}\,k_* f_{F'}^* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

4. *With the notation of (3), assume that $\mathcal{G}^\bullet$ is a complex in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'},\Lambda)$ and consider $\Lambda[[G]]$ as a $\Lambda[[G]]$-$\Lambda[[G']]$-bimodule. Then*

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_* \mathcal{G}^\bullet)\big) = \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* f_{F'*} \mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$ and

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{M}_{F_\infty/F',\gamma^r}(y, \mathrm{R}\,k_* \mathcal{G}^\bullet)\big) = \mathcal{M}_{F_\infty/F',\gamma}(x, \mathrm{R}\,k_* f_{F'*} \mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]])$.

Proof. Note that the functor Ψ commutes up to weak equivalences with $\mathrm{R}\,\Gamma$, i^*, and $\mathrm{R}\,k_*$ [Wit08, 5.5.7] and apply Proposition 9.4 and Proposition 6.1. Part (1) and (2) are direct consequences.

For Part (3), we additionally need the same reasoning as in the proof of [Wit14, Thm. 8.4.(3)] to verify that for any $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda)$

$$[\mathrm{id} - \mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(y \times_x \hat{x}, \mathrm{R}\,k_* g_! g^* \mathcal{G}^\bullet)] = [\mathrm{id} - \mathfrak{F}_y \circlearrowright \mathrm{R}\,\Gamma(\hat{y}, \mathrm{R}\,k_* g_! g^* f_{F'}^* \mathcal{G}^\bullet)] \quad (11.1)$$

in $\mathrm{K}_1(\Lambda[[G']])_S$. Here, $g\colon U_{F_\infty} \to U_{F'}$ denotes the system of coverings induced by f. This implies the formula for $\Psi_{\Lambda[[G]]}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet))$. Moreover, we have a weak equivalence

$$\Psi_{\Lambda[[G]]} \mathrm{R}\,\Gamma(x, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(x_{F'}, \mathrm{R}\,k_* g_! g^* f_{F'}^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G']])$. In particular,

$$s_{\gamma^r}([\Psi_{\Lambda[[G]]} \mathrm{R}\,\Gamma(x, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet)]) = \prod_{y \in x_{F'}} s_{\gamma^r}([\mathrm{R}\,\Gamma(y, \mathrm{R}\,k_* g_! g^* f_{F'}^* \mathcal{F}^\bullet)])$$

from which the formula for $\Psi_{\Lambda[[G]]}(\mathcal{M}_{F_\infty/F, \gamma}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet))$ follows.

For Part (4) we use (11.1) to show

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}(\mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_* \mathcal{G}^\bullet)) =$$

$$= \Psi_{\Lambda[[G]]}([\mathrm{id} - \mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(x_{F'} \times_x \hat{x}, \mathrm{R}\,k_* g_! g^* \mathcal{G}^\bullet)]^{-1})$$

$$= [\mathrm{id} - \mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,k_* f_! f^* f_{F'*} \mathcal{G}^\bullet)]^{-1}$$

$$= \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* f_{F'*} \mathcal{G}^\bullet).$$

On the other hand, we also have a weak equivalence

$$\Psi_{\Lambda[[G]]} \mathrm{R}\,\Gamma(x_{F'}, \mathrm{R}\,k_* g_! g^* \mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(x, \mathrm{R}\,k_* f_! f^* f_{F'*} \mathcal{G}^\bullet),$$

thence the formula for the local modification factors. $\square$

If $\mathcal{G}$ is a smooth Λ-adic sheaf on U and x is a point in U, it makes sense to consider the element

$$\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{G}^{*\Lambda}(1))^\circledast \in \mathrm{K}_1(\Lambda[[G]], S)$$

as an alternative Euler factor, which does not agree with $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{G})$ in general. We shall show below that

$$\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_* \mathcal{G}^{*\Lambda}(1))^\circledast = [\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^! k_! f_! f^* \mathcal{G})]$$

and take this as a definition for arbitrary complexes $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.

DEFINITION 11.5. The dual non-commutative Euler factor of $k_! \mathcal{F}^\bullet$ at $x \in W$ is the element

$$\mathcal{L}_{F_\infty/F}^\circledast(x, k_! \mathcal{F}^\bullet) := [\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^! k_! f_! f^* \mathcal{F}^\bullet)]$$

in $\mathrm{K}_1(\Lambda[[G]])_S$.

Proposition 11.6. *The inverse of the dual non-commutative Euler factor is a characteristic element for* $\mathrm{R}\,\Gamma(x,\mathrm{R}\,i^{!}k_{!}f_{!}f^{*}\mathcal{F}^{\bullet})$:

$$d\mathcal{L}^{\circledast}_{F_{\infty}/F}(x,k_{!}\mathcal{F}^{\bullet}) = [\mathrm{R}\,\Gamma(x,\mathrm{R}\,i^{!}k_{!}f_{!}f^{*}\mathcal{F}^{\bullet})]$$

in $\mathrm{K}_{0}(\Lambda[[G]],S)$.

Proof. The complex $\mathrm{R}\,\Gamma(x,i^{*}\,\mathrm{R}\,k_{*}f_{!}f^{*}\mathcal{F}^{\bullet})$ is weakly equivalent to the cone of the endomorphism

$$\mathrm{R}\,\Gamma(\hat{x},\mathrm{R}\,i^{!}k_{!}f_{!}f^{*}\mathcal{F}^{\bullet}) \xrightarrow{\mathrm{id}-\mathfrak{F}_{x}^{-1}} \mathrm{R}\,\Gamma(\hat{x},\mathrm{R}\,i^{!}k_{!}f_{!}f^{*}\mathcal{F}^{\bullet})$$

shifted by one. Hence, the result follows from the explicit description of d given in [Wit14, Thm. A.5]. $\square$

Definition 11.7. *For a topological generator* $\gamma \in \Gamma$, *the dual local modification factor* $k_{!}\mathcal{F}^{\bullet}$ *at* x *is the element*

$$\mathcal{M}^{\circledast}_{F_{\infty}/F,\gamma}(x,k_{!}\mathcal{F}^{\bullet}) \coloneqq \mathcal{L}^{\circledast}_{F_{\infty}/F}(x,k_{!}\mathcal{F}^{\bullet})s_{\gamma^{-1}}([\mathrm{R}\,\Gamma(x,\mathrm{R}\,i^{!}k_{!}f_{!}f^{*}\mathcal{F}^{\bullet})])^{-1}.$$

We obtain the following transformation properties.

Proposition 11.8. *With* $k\colon U \to W$ *as above, let* Λ *be any adic* $\mathbb{Z}_{p}$-*algebra and let* $\mathcal{F}^{\bullet}$ *be a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.

1. *Let* Λ' *be another adic* $\mathbb{Z}_{p}$-*algebra. For any complex* $P^{\bullet}$ *of* Λ'-$\Lambda[[G]]$-*bimodules which is strictly perfect as complex of* Λ'-*modules we have*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{L}^{\circledast}_{F_{\infty}/F}(x,k_{!}\mathcal{F}^{\bullet})) = \mathcal{L}^{\circledast}_{F_{\infty}/F}(x,k_{!}\Psi_{\tilde{P}\bullet}(\mathcal{F}^{\bullet}))$$

   in $\mathrm{K}_{1}(\Lambda'[[G]]_{S})$ *and*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{M}^{\circledast}_{F_{\infty}/F,\gamma}(x,k_{!}\mathcal{F}^{\bullet})) = \mathcal{M}^{\circledast}_{F_{\infty}/F,\gamma}(x,k_{!}\Psi_{\tilde{P}\bullet}(\mathcal{F}^{\bullet}))$$

   in $\mathrm{K}_{1}(\Lambda'[[G]])$.

2. *Let* F'_{∞}/F *be an admissible subextension of* F_{∞}/F *with Galois group* G'. *Then*

$$\Psi_{\Lambda[[G']]}(\mathcal{L}^{\circledast}_{F_{\infty}/F}(x,k_{!}\mathcal{F}^{\bullet})) = \mathcal{L}^{\circledast}_{F'_{\infty}/F}(x,k_{!}\mathcal{F}^{\bullet})$$

   in $\mathrm{K}_{1}(\Lambda[[G']]_{S})$ *and*

$$\Psi_{\Lambda[[G']]}(\mathcal{M}^{\circledast}_{F_{\infty}/F,\gamma}(x,k_{!}\mathcal{F}^{\bullet})) = \mathcal{M}^{\circledast}_{F'_{\infty}/F,\gamma}(x,k_{!}\mathcal{F}^{\bullet})$$

   in $\mathrm{K}_{1}(\Lambda[[G']])$.

3. Let F'/F be a finite extension inside F_∞/F. Set $r \coloneqq [F' \cap F_{\mathrm{cyc}} : F]$. Write $f_{F'}\colon U_{F'} \to U$ for the corresponding étale covering and $x_{F'}$ for the fibre in $\operatorname{Spec}\mathcal{O}_{F'}$ above x. Let $G' \subset G$ be the Galois group of the admissible extension F_∞/F' and consider $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then

$$\Psi_{\Lambda[[G]]}\big(\mathcal{L}^{\oplus}_{F_\infty/F}(x, k_! \mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{L}^{\oplus}_{F_\infty/F'}(y, k_! f_{F'}^* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ and

$$\Psi_{\Lambda[[G]]}\big(\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(x, k_! \mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{M}^{\oplus}_{F_\infty/F',\gamma^r}(y, k_! f_{F'}^* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

4. With the notation of (3), assume that $\mathcal{G}^\bullet$ is a complex in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda)$ and consider $\Lambda[[G]]$ as a $\Lambda[[G]]$-$\Lambda[[G']]$-bimodule. Then

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}^{\oplus}_{F_\infty/F'}(y, k_! \mathcal{G}^\bullet)\big) = \mathcal{L}^{\oplus}_{F_\infty/F}(x, k_! f_{F'*} \mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$ and

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{M}^{\oplus}_{F_\infty/F',\gamma^r}(y, k_! \mathcal{G}^\bullet)\big) = \mathcal{M}^{\oplus}_{F_\infty/F',\gamma}(x, k_! f_{F'*} \mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]])$.

Proof. The arguments are the same as in the proof of Prop. 11.4. $\qquad\square$

PROPOSITION 11.9.

1. Let $\mathcal{G}$ be a smooth Λ-adic sheaf on U. Then

$$(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, k_* \mathcal{G}^{*\Lambda}(1)))^\oplus = \mathcal{L}^{\oplus}_{F_\infty/F}(x, k_! \mathcal{G}) =$$

$$= \begin{cases} [-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x}, i^* f_! f^* \mathcal{G}(-1))]^{-1} \mathcal{L}_{F_\infty/F}(x, \mathcal{G}(-1))^{-1} & \text{if } x \in U \\ [-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x}, i^* \mathrm{R}\, k_* f_! f^* \mathcal{G})] \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, k_* \mathcal{G}) & \text{if } x \in W - U \end{cases}$$

in $\mathrm{K}_1(\Lambda[[G]], S)$ and

$$(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\, k_* \mathcal{G}^{*\Lambda}(1)))^\oplus = \mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(x, k_! \mathcal{G})$$

in $\mathrm{K}_1(\Lambda[[G]])$.

2. Let $\mathcal{G}$ be a Λ-adic sheaf on $x \in U$. Then

$$(\mathcal{L}_{F_\infty/F}(x, i_* \mathcal{G}^{*\Lambda}))^\oplus = \mathcal{L}^{\oplus}_{F_\infty/F}(x, i_* \mathcal{G})$$

$$= [-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x}, i^* f_! f^* i_* \mathcal{G})]^{-1} \mathcal{L}_{F_\infty/F}(x, i_* \mathcal{G})^{-1}$$

in $\mathrm{K}_1(\Lambda[[G]], S)$ and

$$(\mathcal{M}_{F_\infty/F,\gamma}(x, i_* \mathcal{G}^{*\Lambda}))^\oplus = \mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(x, i_* \mathcal{G})$$

in $\mathrm{K}_1(\Lambda[[G]])$.

Proof. We only need to prove the formulas for the non-commutative Euler factors, the formulas for the local modification factors then follow from Proposition 6.3.

We begin by proving (1) in the case that $x \in W - U$. By (9.3), combined with Lemma 7.7 and Lemma 7.4, we have

$$
\mathcal{L}_{F_\infty/F}(x, i_* \mathrm{R}\, k_* \mathcal{G}^{*\wedge}(1)) =
$$
$$
= \Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp}\left([\mathrm{id} - \mathfrak{F}_x \circlearrowright \mathrm{R}\Gamma(\hat{x}, (i^* \mathrm{R}\, k_* f_! f^* \mathcal{G})^{*\Lambda[[G]]}(1))]^{-1}\right)
$$
$$
= \Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp}\left([\mathrm{id} - \mathfrak{F}_x \circlearrowright D_{\hat{x}}^\bullet((\mathrm{R}\, k_* f_! f^* \mathcal{G})^{*\Lambda[[G]]}(1))]^{-1}\right).
$$

From the Definition 5.5 of $\circledast$, Lemma 8.1, and again Lemma 7.4, we conclude

$$
\left(\Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp}\left([\mathrm{id} - \mathfrak{F}_x \circlearrowright D_{\hat{x}}^\bullet((\mathrm{R}\, k_* f_! f^* \mathcal{G})^{*\Lambda[[G]]}(1))]^{-1}\right)\right)^{\circledast} =
$$
$$
= [\mathrm{id} - \mathfrak{F}_x^{*\Lambda[[G]]^{\mathrm{op}}} \circlearrowright D_{\hat{x}}^\bullet((\mathrm{R}\, k_* f_! f^* \mathcal{G})^{*\Lambda[[G]]}(1))^{*\Lambda[[G]]^{\mathrm{op}}}]
$$
$$
= [\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\Gamma(\hat{x}, i^* \mathrm{R}\, k_* f_! f^* \mathcal{G})]^{-1}
$$
$$
= [-\mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\Gamma(\hat{x}, i^* \mathrm{R}\, k_* f_! f^* \mathcal{G})]^{-1} \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, k_* \mathcal{G}).
$$

Finally,
$$
[\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\Gamma(\hat{x}, i^* \mathrm{R}\, k_* f_! f^* \mathcal{G})]^{-1} = \mathcal{L}_{F_\infty/F}^{\circledast}(x, k_! \mathcal{G})
$$

by Lemma 7.8.

The validity of the first equality in (2) follows similarly from Lemma 8.2 and the exchange formula (8.5):

$$
(\mathcal{L}_{F_\infty/F}(x, i_* \mathcal{G}^{*\wedge}))^{\circledast} = ([\mathrm{id} - \mathfrak{F}_x \circlearrowright (i^* f_! f^* i_* \mathcal{G}^{*\wedge})_{\hat{x}}]^{-1})^{\circledast}
$$
$$
= \Psi_{\Lambda[[G]]^\sharp}\left([\mathrm{id} - \mathfrak{F}_x^{*\Lambda[[G]]^{\mathrm{op}}} \circlearrowright ((i^* f_! f^* i_* \mathcal{G}^{*\wedge})_{\hat{x}})^{*\Lambda[[G]]^{\mathrm{op}}}]\right)
$$
$$
= \Psi_{\Lambda[[G]]^\sharp}\left([\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\Gamma(\hat{x}, (i^* f_! f^* i_* \mathcal{G}^{*\wedge}))^{*\Lambda[[G]]^{\mathrm{op}}}]\right)
$$
$$
= [\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\Gamma(\hat{x}, \mathrm{R}\, i^! f_! f^*(i_* \mathcal{G}^{*\wedge}))^{*\Lambda^{\mathrm{op}}}]
$$
$$
= \mathcal{L}_{F_\infty/F}^{\circledast}(x, i_* \mathcal{G})
$$

Further, write $\ell: U' \to U$ for the complement of x in U. Then

$$
\mathrm{R}\,\ell_* \ell^* f_! f^* i_* \mathcal{G} \cong \mathrm{R}\,\ell_* f_! f^* \ell^* i_* \mathcal{G} = 0
$$

and hence,

$$
\mathrm{R}\, i^! f_! f^* i_* \mathcal{G} \cong i^* f_! f^* i_* \mathcal{G},
$$

from which the second equality in (2) follows.

For the proof of (1) in the case that $x \in U$, we observe that

$$
(\mathcal{L}_{F_\infty/F}(x, \mathcal{G}^{*\wedge}(1)))^{\circledast} = (\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,\ell_* \ell^* \mathcal{G}^{*\wedge}(1)))^{\circledast}(\mathcal{L}_{F_\infty/F}(x, i_* \mathrm{R}\, i^! \mathcal{G}^{*\wedge}(1)))^{\circledast}
$$
$$
= (\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,\ell_* \ell^* \mathcal{G}^{*\wedge}(1)))^{\circledast}(\mathcal{L}_{F_\infty/F}(x, i_* (i^* \mathcal{G})^{*\wedge}))^{\circledast}
$$
$$
= \mathcal{L}_{F_\infty/F}^{\circledast}(x, \ell_! \ell^* \mathcal{G}) \mathcal{L}_{F_\infty/F}^{\circledast}(x, i^* \mathcal{G})
$$
$$
= \mathcal{L}_{F_\infty/F}^{\circledast}(x, \mathcal{G})
$$

by what we have proved above. For the second equality, we use that by absolute purity [Mil06, Ch. II, Cor. 1.6], there exists chain of weak equivalences

$$i^* f_! f^* \mathcal{G}(-1) \sim \mathrm{R}\, i^! f_! f^* \mathcal{G}[2].$$

$\square$

12 Euler factors for the cyclotomic extension

In the case $F_\infty = F_{\mathrm{cyc}}$, we can give a different description of $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, k_* \mathcal{F}^\bullet)$. We will undergo the effort to allow arbitrary adic $\mathbb{Z}_p$-algebras Λ as coefficient rings, but in the end, we will use the results only in the case that Λ is the valuation ring in a finite extension of $\mathbb{Q}_p$. If one restricts to this case, some of the technical constructions that follow may be skipped.

Let $\Lambda[t]$ be the polynomial ring over Λ in the indeterminate t that is assumed to commute with the elements of Λ. In the appendix we define a Waldhausen category $w_t \mathbf{P}(\Lambda[t])$: The objects are perfect complexes of $\Lambda[t]$-modules and cofibrations are injective morphism of complexes such that the cokernel is again perfect. A weak equivalence is a morphism $f \colon P^\bullet \to Q^\bullet$ of perfect complexes of $\Lambda[t]$-modules such that $\Lambda \otimes^{\mathbb{L}}_{\Lambda[t]} f$ is a quasi-isomorphism of complexes of Λ-modules. Here, Λ is considered as a Λ-$\Lambda[t]$-bimodule via the augmentation map and $\Lambda \otimes^{\mathbb{L}}_{\Lambda[t]} \cdot$ denotes the total derived tensor product as functor between the derived categories.

If Λ is noetherian, then the subset

$$S_t \coloneqq \{ f(t) \in \Lambda[t] \mid f(0) \in \Lambda^\times \} \subset \Lambda[t]$$

is a left and right denominator set, the localisation $\Lambda[t]_{S_t}$ is semi-local and $\Lambda[t] \to \Lambda[t]_{S_t}$ induces an isomorphism

$$\mathrm{K}_1(w_t \mathbf{P}(\Lambda[t])) \cong \mathrm{K}_1(\Lambda[t]_{S_t})$$

(Proposition A.1). For commutative adic rings, which are always noetherian [War93, Cor. 36.35], we may further identify

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \cong \Lambda[t]_{S_t}^\times$$

via the determinant map. In general, S_t is not a left or right denominator set. We then take

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \coloneqq \mathrm{K}_1(w_t \mathbf{P}(\Lambda[t]))$$

as a definition.

For any adic $\mathbb{Z}_p$-algebra Λ and any $\gamma \in \Gamma \cong \mathbb{Z}_p$, the ring homomorphism

$$\mathrm{ev}_\gamma \colon \Lambda[t] \mapsto \Lambda[[\Gamma]], \quad f(t) \mapsto f(\gamma).$$

induces a homomorphism

$$\mathrm{ev}_\gamma \colon \mathrm{K}_1(\Lambda[t]_{S_t}) \to \mathrm{K}_1(\Lambda[[\Gamma]]_S)$$

with $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$ as defined in (3.2) (Proposition A.2). In the noetherian case, the proof boils down to a verification that $\mathrm{ev}_\gamma(S_t) \subset S$ with S denoting Venjakob's canonical Ore set (3.1).

DEFINITION 12.1. For $\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ we define

$$L(x, \mathrm{R}\,k_* \mathcal{F}^\bullet, t) \coloneqq [\mathrm{id} - t\mathfrak{F}_x \circlearrowright P^\bullet]^{-1} \in \mathrm{K}_1(\Lambda[t]_{S_t})$$
$$L^\circledast(x, k_! \mathcal{F}^\bullet, t) \coloneqq [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowright Q^\bullet] \in \mathrm{K}_1(\Lambda[t]_{S_t})$$

where

$$P^\bullet \coloneqq \Lambda[t] \otimes_\Lambda \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathrm{R}\,k_* \mathcal{F}^\bullet),$$
$$Q^\bullet \coloneqq \Lambda[t] \otimes_\Lambda \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^! k_! \mathcal{F}^\bullet)$$

For any $1 \neq \gamma \in \Gamma$, we write $L(x, \mathrm{R}\,k_* \mathcal{F}^\bullet, \gamma)$ and $L^\circledast(x, k_! \mathcal{F}^\bullet, \gamma)$ for the images of $L(x, \mathrm{R}\,k_* \mathcal{F}^\bullet, t)$ and $L^\circledast(x, k_! \mathcal{F}^\bullet, t)$ under

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow{\mathrm{ev}_\gamma} \mathrm{K}_1(\Lambda[[\Gamma]]_S).$$

Since the endomorphism $\mathrm{id} - t\mathfrak{F}_x$ is canonical, it follows easily from the relations in the definition of $\mathcal{D}_\bullet(\mathbf{W})$ that $L(x, \mathcal{F}^\bullet, t)$ does only depend on the weak equivalence class of $\mathcal{F}^\bullet$ and is multiplicative on exact sequences. So, it defines a homomorphism

$$L(x, \mathrm{R}\,k_*(-), t) \colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)) \to \mathrm{K}_1(\Lambda[t]_{S_t}).$$

The same is also true for $L^\circledast(x, k_! \mathcal{F}^\bullet, t)$.

PROPOSITION 12.2. Let $\gamma_x \in \Gamma$ be the image of $\mathfrak{F}_x$ in Γ. Then

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_* \mathcal{F}^\bullet) = L(x, \mathrm{R}\,k_* \mathcal{F}^\bullet, \gamma_x^{-1}),$$
$$\mathcal{L}^\circledast_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}^\bullet) = L^\circledast(x, k_! \mathcal{F}^\bullet, \gamma_x).$$

Proof. Since p is invertible on W, the extension F_{cyc}/F is unramified over W. By the smooth base change theorem applied to the étale morphism $f_K \colon W_K \to W$ for each finite subextension K/F of F_{cyc}/F and the quasi-compact morphism $k \colon U \to W$ there exists a weak equivalence

$$f_! f^* \mathrm{R}\,k_* \mathcal{F}^\bullet \xrightarrow{\sim} \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(W, \Lambda)$. By the proper base change theorem, there exists also an isomorphism

$$f_! f^* k_! \mathcal{F}^\bullet \cong k_! f_! f^* \mathcal{F}^\bullet$$

Hence, we may assume $x \in U = W$.

For any finite subextension K/F in F_{cyc}/F write x_K for the set of places of K lying over x and $g\colon x_{F_{\mathrm{cyc}}} \to x$ for the corresponding system of Galois covers. (We note that this system might not be admissible in the sense of [Wit14, Def. 2.6] for any base field $\mathbb{F} \subset k(x)$: for example if $F = \mathbb{Q}$ and $x = (\ell)$ with $\ell \neq p$ splitting in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.) By the proper base change theorem there exists an isomorphism

$$i^* f_! f^* \mathcal{F}^\bullet \cong g_! g^* i^* \mathcal{F}^\bullet.$$

From Lemma 7.8 we can then also infer the existence of a weak equivalence

$$\mathrm{R}\, i^! f_! f^* \mathcal{F}^\bullet \xrightarrow{\sim} g_! g^* \mathrm{R}\, i^! \mathcal{F}^\bullet.$$

We will now concentrate on the proof of the equality

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathcal{F}^\bullet) = L(x, \mathcal{F}^\bullet, \gamma_x^{-1}).$$

The proof of the equality for the dual Euler factors follows along the same lines, with $\mathfrak{F}_x^{-1}$ replaced by $\mathfrak{F}_x^{-1}$ and γ_x^{-1} replaced by γ_x.

By our choice of the embedding $\overline{F} \subset \overline{F}_x$, we have a compatible system of morphisms $\operatorname{Spec} \overline{k(x)} \to x_K$ for each $K \subset F_{\mathrm{cyc}}$ and hence, distinguished isomorphisms

$$\alpha\colon \mathbb{Z}[\mathrm{Gal}(K/F)] \otimes_{\mathbb{Z}} \mathcal{M}_{\hat{x}} \to (g_{K!} g_K^* \mathcal{M})_{\hat{x}}$$

for the stalk $\mathcal{M}_{\hat{x}}$ in $\hat{x}$ of any étale sheaf $\mathcal{M}$ on x. The action of the Frobenius $\mathfrak{F}_x$ on the righthand side corresponds to the operation of $\cdot \gamma_x^{-1} \otimes \mathfrak{F}_x$ on the left-hand side. By compatibility, we may extend α to an isomorphism

$$\alpha\colon \Psi_{\Lambda[[\Gamma]]} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathcal{F}^\bullet) \cong \mathrm{R}\,\Gamma(\hat{x}, g_! g^* i^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma]])$. Hence,

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathcal{F}^\bullet) = [\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \; \circlearrowright \; \Psi_{\Lambda[[\Gamma]]} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathcal{F}^\bullet)]^{-1}$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. Furthermore, we may choose a strictly perfect complex of Λ-modules $P^\bullet$ with an endomorphism f and a quasi-isomorphism

$$\beta\colon P^\bullet \to \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, \mathcal{G}^\bullet)$$

under which f and $\mathfrak{F}_x$ are compatible up to chain homotopy [Wit08, Lemma 3.3.2]. The endomorphism

$$\mathrm{id} - t f\colon \Lambda[t] \otimes_\Lambda P^\bullet \to \Lambda[t] \otimes_\Lambda P^\bullet$$

is clearly a weak equivalence in $w_t \mathbf{P}(\Lambda[t])$. By [Wit08, Lemma 3.1.6], homotopic weak auto-equivalences have the same class in the first K-group. Hence, we may conclude

$$[\mathrm{id} - t f \; \circlearrowright \; \Lambda[t] \otimes_\Lambda P^\bullet]^{-1} = L(x, \mathrm{R}\, k_* \mathcal{F}^\bullet, t)$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$L(x, \mathrm{R}\,k_*\,\mathcal{F}^\bullet, \gamma_x^{-1}) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_*\,\mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We will make this construction a little more explicit in the case that $\mathcal{F}$ is a Λ-adic sheaf on U. If $x \in U$, recall from (8.3) that there is a weak equivalence

$$\mathcal{F}_{\hat{x}} \xrightarrow{\sim} \mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_*\,\mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ compatible with the operation of the Frobenius $\mathfrak{F}_x$ on both sides.

Hence, we have

$$L(x, \mathrm{R}\,k_*\,\mathcal{F}, t) = \left[\Lambda[t] \otimes_\Lambda \mathcal{F}_{\hat{x}} \xrightarrow{\mathrm{id}-t\mathfrak{F}_x} \Lambda[t] \otimes_\Lambda \mathcal{F}_{\hat{x}}\right]^{-1} \qquad (12.1)$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_*\,\mathcal{F}) = \left[\Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}_{\hat{x}} \xrightarrow{\mathrm{id}-\gamma_x^{-1}\otimes\mathfrak{F}_x} \Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}_{\hat{x}}\right]^{-1}$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. In particular, if Λ is commutative, then the isomorphism

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow{\det} \Lambda[t]_{S_t}^\times$$

sends $L(x, \mathrm{R}\,k_*\,\mathcal{F}, t)$ to the inverse of the reverse characteristic polynomial of the geometric Frobenius operation on $\mathcal{F}_{\hat{x}}$.

If $\mathcal{F}$ is smooth in $x \in U$, then by absolute purity [Mil06, Ch. II, Cor. 1.6], there exists chain of weak equivalences

$$\mathcal{F}(-1)_{\hat{x}} \sim \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^! k_!\,\mathcal{F})[2].$$

Hence,

$$L^\circledast(x, k_!\,\mathcal{F}, t) = \left[\Lambda[t] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}} \xrightarrow{\mathrm{id}-t\mathfrak{F}_x^{-1}} \Lambda[t] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}}\right] \qquad (12.2)$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}^\circledast_{F_{\mathrm{cyc}}/F}(x, k_!\,\mathcal{F}) = \left[\Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}} \xrightarrow{\mathrm{id}-\gamma_x\otimes\mathfrak{F}_x^{-1}} \Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}}\right]$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. If $\mathcal{F} = i_*\mathcal{G}$ for some $\mathcal{G}$ in $\mathbf{PDG}^{\mathrm{cont}}(x, \Lambda)$, then there exists a weak equivalence

$$\mathcal{G}_{\hat{x}} \xrightarrow{\sim} \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^! k_!\,\mathcal{F}).$$

Hence,

$$L^\circledast(x, k_!\,\mathcal{F}, t) = \left[\Lambda[t] \otimes_\Lambda \mathcal{G}_{\hat{x}} \xrightarrow{\mathrm{id}-t\mathfrak{F}_x^{-1}} \Lambda[t] \otimes_\Lambda \mathcal{G}_{\hat{x}}\right] \qquad (12.3)$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}) = [\Lambda[[\Gamma]] \otimes_\Lambda \mathcal{G}_{\hat{x}} \xrightarrow{\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1}} \Lambda[[\Gamma]] \otimes_\Lambda \mathcal{G}_{\hat{x}}]$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$.

If $x \in W - U$, there exists by Lemma 7.4, Lemma 7.7 and Lemma 7.8 a chain of weak equivalences

$$D^\bullet_{\hat{x}}(\mathcal{F}) \sim \mathrm{R}\,\Gamma(\hat{x}, i^* \,\mathrm{R}\,k_* f_! f^* \mathcal{F}) \sim \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^! k_! f_! f^* \mathcal{F})[1]$$

compatible with the Frobenius operation.

We conclude that for $x \in W - U$,

$$L(x, \mathrm{R}\,k_* \mathcal{F}, t) = [\mathrm{id} - t\mathfrak{F}_x \circlearrowleft \Lambda[t] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1}[\mathrm{id} - t\mathfrak{F}_x \circlearrowleft \Lambda[t] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})],$$

$$L^{\circledast}(x, k_! \mathcal{F}, t) = [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowleft \Lambda[t] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1}[\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowleft \Lambda[t] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})]$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_* \mathcal{F}) = [\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1}$$
$$[\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})],$$

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}) = [\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1} \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1}$$
$$[\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1} \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})].$$

Let N be the stalk of $\mathcal{F}$ in the geometric point $\mathrm{Spec}\,\overline{F}$, viewed as $\mathrm{Gal}(\overline{F}_x/F_x)$-module. If the image of $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$ in the automorphism group of N has trivial p-Sylow subgroups, then $N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})} = D^0_{\hat{x}}(\mathcal{F})$ and the differential of $D^\bullet_{\hat{x}}(\mathcal{F})$ is trivial. Our formula then simplifies to

$$L(x, \mathrm{R}\,k_* \mathcal{F}, t) = [\mathrm{id} - t\mathfrak{F}_x \circlearrowleft \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}]^{-1}$$
$$[\mathrm{id} - t q_x \mathfrak{F}_x \circlearrowleft \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}],$$

$$L^{\circledast}(x, k_! \mathcal{F}, t) = [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowleft \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}]^{-1}$$
$$[\mathrm{id} - t q_x^{-1} \mathfrak{F}_x^{-1} \circlearrowleft \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}],$$

where q_x is the order of the residue field $k(x)$. In particular, this is the case if N is unramified in x.

Conversely, assume that the differential of $D^\bullet_{\hat{x}}(\mathcal{F})$ is an isomorphism. Since the operation of $\mathfrak{F}_x$ commutes with the differential, we then have

$$L(x, \mathrm{R}\,k_* \mathcal{F}, t) = L^{\circledast}(x, k_! \mathcal{F}, t) = 1.$$

If $\Lambda = \mathcal{O}_C$ is the valuation ring of a finite field extension C of $\mathbb{Q}_p$, then we may replace $C \otimes_{\mathcal{O}_C} N$ by its semi-simplification $(C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}}$ as a $\mathrm{Gal}(\overline{F}_x/F_x)$-module and obtain a corresponding decomposition $(C \otimes_{\mathcal{O}_C} D^\bullet_{\hat{x}}(\mathcal{F}))^{\mathrm{ss}}$ of the complex $D^\bullet_{\hat{x}}(\mathcal{F})$. Note that

$$(C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}} = ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})} \oplus V$$

with $V^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})} = 0$. In particular, on each simple part of $(C \otimes_{\mathcal{O}_C} D_{\hat{x}}^\bullet(\mathcal{F}))^{\mathrm{ss}}$, the differential is either trivial or an isomorphism. We conclude

$$
\begin{aligned}
\det L(x, \mathrm{R}\, k_* \mathcal{F}, t) &= \det(\mathrm{id} - t\mathfrak{F}_x \mathrel{\circlearrowright} ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})})^{-1} \\
&\qquad \det(\mathrm{id} - tq_x\mathfrak{F}_x \mathrel{\circlearrowright} ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}), \\
\det L^{\circledast}(x, k_! \mathcal{F}, t) &= \det(\mathrm{id} - t\mathfrak{F}_x^{-1} \mathrel{\circlearrowright} ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})})^{-1} \\
&\qquad \det(\mathrm{id} - tq_x^{-1}\mathfrak{F}_x^{-1} \mathrel{\circlearrowright} ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}),
\end{aligned}
\tag{12.4}
$$

in the units $(C[t]_{(t)})^\times$ of the localisation of $C[t]$ at the prime ideal (t).
We can also give more explicit formulas for the local modification factors. We will not make use of the following calculations in any other part of the text. Let M be one of the finitely generated and projective Λ-modules $\mathcal{F}_{\hat{x}}$, $\mathcal{F}(-1)_{\hat{x}}$, $D_{\hat{x}}^0(\mathcal{F})$, $D_{\hat{x}}^1(\mathcal{F})$. Then M comes equipped with a continuous action of the Galois group $\mathrm{Gal}(\overline{k(x)}/k(x))$. Let $k(x)_{\mathrm{cyc}}$ denote the unique $\mathbb{Z}_p$-extension of $k(x)$ and let r be the index of the image of $\Gamma' := \mathrm{Gal}(k(x)_{\mathrm{cyc}}/k(x))$ in $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. Fix a topological generator $\gamma \in \Gamma$.
Clearly,

$$
[\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \mathrel{\circlearrowright} \Lambda[[\Gamma]] \otimes_\Lambda M] = \Psi_{\Lambda[[\Gamma]]}([\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \mathrel{\circlearrowright} \Lambda[[\Gamma']] \otimes_\Lambda M])
$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$, while

$$
\begin{aligned}
\Psi_{\Lambda[[\Gamma]]}(s_{\gamma^r}([\mathrm{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \mathrel{\circlearrowright} \Lambda[[\Gamma']] \otimes_\Lambda M)])) &= \\
s_\gamma(\Psi_{\Lambda[[\Gamma]]}([\mathrm{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \mathrel{\circlearrowright} \Lambda[[\Gamma']] \otimes_\Lambda M)])) &= \\
= s_\gamma([\mathrm{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \mathrel{\circlearrowright} \Lambda[[\Gamma]] \otimes_\Lambda M)])
\end{aligned}
$$

by Proposition 6.1. Hence, it suffices to consider the case that x does not split in F_{cyc}/F. So, we assume from now on that $r = 1$.
The image of $\mathrm{Gal}(\overline{k(x)}/k(x)_{\mathrm{cyc}})$ in the automorphism group of M is a finite commutative group Δ of order d prime to p. Write

$$
e_\Delta := \frac{1}{d} \sum_{\delta \in \Delta} \delta
$$

for the corresponding idempotent in the endomorphism ring. We thus obtain a canonical decomposition

$$
M \cong M' \oplus M''
$$

of M with $M' := e_\Delta M$ and $M'' := (\mathrm{id} - e_\Delta)M$.
Since

$$
(\mathrm{id} - \delta)(\mathrm{id} - e_\Delta)m = (\mathrm{id} - \delta)m
$$

for every $\delta \in \Delta$ and every $m \in M$, the action of Δ on M'' is faithful. Since d is prime to p, the action of Δ on $M''/\mathrm{Jac}(\Lambda)M''$ is still faithful. Indeed, the kernel K of $\mathrm{id} - \delta \mathrel{\circlearrowright} M''$ is a direct summand of M'' with $K \neq M''$.

The Nakayama lemma then implies $K/\operatorname{Jac}(\Lambda)K \neq M''/\operatorname{Jac}(\Lambda)M''$ such that $\delta \circlearrowleft M''/\operatorname{Jac}(\Lambda)M''$ cannot be the identity.

Note that K is trivial if δ is a generator of Δ. In this case, $\operatorname{id} - \delta \circlearrowleft M''/\operatorname{Jac}(\Lambda)M''$ must be an automorphism. We may apply this to a suitable p^n-th power of $\mathfrak{F}_x$ to infer that $\operatorname{id} - \mathfrak{F}_x \circlearrowleft M''/\operatorname{Jac}(\Lambda)M''$ is an automorphism. By the Nakayama lemma for $\Lambda[[\Gamma]]$ we conclude that the endomorphism $\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x$ of $\Lambda[[\Gamma]] \otimes_\Lambda M''$ is also an automorphism.

The Λ-module M' can be seen as $\Lambda[[\Gamma]]$-module with γ_x acting as $\mathfrak{F}_x$ and by Example 6.2 we have

$$[\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M''] = s_{\gamma_x}([\operatorname{coker}(\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M')])^{-1}$$

We conclude that

$$[\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M]s_\gamma([\operatorname{coker}(\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M)]) =$$
$$[\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M'']\frac{s_\gamma}{s_{\gamma_x}}([\operatorname{coker}(\operatorname{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M')]).$$

In particular, if $x \in U$ and $\operatorname{H}^0(\operatorname{Spec} k(x)_{\mathrm{cyc}}, i^* \mathcal{F}) = 0$, then

$$\mathcal{M}_{F_{\mathrm{cyc}}/F, \gamma}(x, \operatorname{R} k_* \mathcal{F}) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \operatorname{R} k_* \mathcal{F}).$$

If $x \in U$, $\operatorname{H}^0(\operatorname{Spec} k(x)_{\mathrm{cyc}}, i^* \mathcal{F}) = \mathcal{F}_{\hat{x}}$ and $\gamma = \gamma_x$, then

$$\mathcal{M}_{F_{\mathrm{cyc}}/F, \gamma}(x, \operatorname{R} k_* \mathcal{F}) = 1.$$

The same considerations apply to the dual local modification factors.

Remark 12.3. Let V^0 denote the set of closed points of V. Note that the infinite product

$$\prod_{x \in V^0} \mathcal{M}_{F_{\mathrm{cyc}}/F, \gamma}(x, \operatorname{R} k_* \mathcal{F})$$

does not converge in the compact topology of $\operatorname{K}_1(\Lambda[[\Gamma]])$. Indeed, by the Chebotarev density theorem, we may find for each non-trivial finite subextensions F'/F of F_{cyc}/F an infinite subset $S \subset U$ of closed points such that the elements

$$\mathcal{M}_{F_{\mathrm{cyc}}/F, \gamma}(x, \operatorname{R} k_* \mathcal{F}) \in \operatorname{K}_1(\Lambda[[\Gamma]])$$

for $x \in S$ have a common non-trivial image in $\operatorname{K}_1(\Lambda/\operatorname{Jac}(\Lambda)[[\operatorname{Gal}(F'/F)]])$.

13 Artin representations

Let $\mathcal{O}_C$ be the valuation ring of a finite extension field C of $\mathbb{Q}_p$ inside the fixed algebraic closure $\overline{\mathbb{Q}}_p$ and assume as before that $\Gamma \cong \mathbb{Z}_p$. The augmentation map $\varphi : \mathcal{O}_C[[\Gamma]] \to \mathcal{O}_C$ extends to a map

$$\varphi : \operatorname{K}_1(\mathcal{O}_C[[\Gamma]]_S) \xrightarrow[\cong]{\det} \mathcal{O}_C[[\Gamma]]_S^\times \to \mathbb{P}^1(C).$$

Indeed, let $\frac{a}{s} \in \mathcal{O}_C[[\Gamma]]_S^\times$. Since $\mathcal{O}_C[[\Gamma]]$ is a unique factorisation domain and the augmentation ideal is a principal prime ideal, we may assume that not both a and s are contained in the augmentation ideal. Hence, we obtain a well-defined element

$$\varphi\left(\frac{a}{s}\right) := [\varphi(a) : \varphi(s)] \in \mathbb{P}^1(C) = C \cup \{\infty\},$$

with $[x : y]$ denoting the standard projective coordinates of $\mathbb{P}^1(C)$. Note that this map agrees with φ' in [Kak13, §2.4]. We further note that $\det a^\circledast = ((\det a)^\sharp)^{-1}$ for any $a \in \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S)$. Since $\sharp \colon \mathcal{O}_C[[\Gamma]]_S \to \mathcal{O}_C[[\Gamma]]_S$ maps $\gamma \in \Gamma$ to γ^{-1} and is given by the identity on $\mathcal{O}_C$, we conclude that

$$\varphi(a^\circledast) = \varphi(a)^{-1}. \tag{13.1}$$

Finally, note that the diagram

$$
\begin{array}{ccc}
\mathrm{K}_1(\mathcal{O}_C[t]_{S_t}) & \xrightarrow{\ \det\ } & (C[t]_{(t)})^\times \\
\downarrow{\scriptstyle t \mapsto \gamma} & & \downarrow{\scriptstyle \frac{f}{g} \mapsto [f(1):g(1)]} \\
\mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S) & \xrightarrow{\ \varphi\ } & \mathbb{P}^1(C)
\end{array}
$$

commutes for any choice of $\gamma \in \Gamma$ with $\gamma \neq 1$. On the right downward pointing map, $\frac{f}{g}$ denotes a reduced fraction.

From now on, we let F denote a totally real number field. Consider an Artin representation $\rho \colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ (i.e. with open kernel) over $\mathcal{O}_C$. We will write

$$\check{\rho} \colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C), \qquad g \mapsto \rho(g^{-1})^t$$

for the dual representation. Assume for simplicity that ρ is unramified over U: For each $x \in U$, the restriction $\rho \restriction_{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}$ of ρ to $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$ is trivial. Then ρ corresponds to the smooth $\mathcal{O}_C$-adic sheaf $\mathcal{M}(\rho)$ on $U \subset W$ defined by (9.1) and therefore, to an object in $\mathbf{PDG}^{\mathrm{cont}}(U, \mathcal{O}_C)$. Analogously, $\check{\rho}$ corresponds to $\mathcal{M}(\check{\rho}) = \mathcal{M}(\rho)^{*\mathcal{O}_C}$.

We set $\Sigma := \mathrm{Spec}(\mathcal{O}_F) - W$, $\mathrm{T} := W - U$. Since the image of $\mathrm{Gal}(\overline{F}_x/F_x)$ in $\mathrm{Gl}_d(\mathcal{O}_C)$ is finite, the base change of $\rho \restriction_{\mathrm{Gal}(\overline{F}_x/F_x)}$ to C is automatically a semi-simple representation of $\mathrm{Gal}(\overline{F}_x/F_x)$ for any $x \in X$. We let ρ_x denote the representation of $\mathrm{Gal}(F_x^{\mathrm{nr}}/F_x)$ obtained from $\rho \restriction_{\mathrm{Gal}(\overline{F}_x/F_x)}$ by taking invariants under $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$.

From (12.1), (12.2), and (12.4) we conclude

$$
\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{M}(\rho)(n)))
$$
$$
= \begin{cases} [1 : \det(1 - \rho_x(\mathfrak{F}_x) q_x^{-n})] & \text{if } x \in U, \\ [\det(1 - \rho_x(\mathfrak{F}_x) q_x^{1-n}) : \det(1 - \rho_x(\mathfrak{F}_x) q_x^{-n})] & \text{if } x \in \mathrm{T}, \end{cases} \tag{13.2}
$$
$$
\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}^{\circledast}(x, k_! \mathcal{M}(\rho)(n)))
$$
$$
= \begin{cases} [\det(1 - \check{\rho}_x(\mathfrak{F}_x) q_x^{n-1}) : 1] & \text{if } x \in U, \\ [\det(1 - \check{\rho}_x(\mathfrak{F}_x) q_x^{n-1}) : \det(1 - \check{\rho}_x(\mathfrak{F}_x) q_x^{n})] & \text{if } x \in \mathrm{T}, \end{cases}
$$

where q_x denotes the number of elements of the residue field $k(x)$. Classically, one often demands that ρ is also unramified over T. Therefore, we stress that the above considerations hold even if this is not the case. Note that $\det(1 - \rho_x(\mathfrak{F}_x) q_x^{-n}) = 0$ if and only if $n = 0$ and ρ_x contains the trivial representation as a subrepresentation.

For any open dense subscheme V of X, we write V^0 for the set of closed points of V. Let $\alpha \colon \overline{\mathbb{Q}}_p \to \mathbb{C}$ be an embedding of $\overline{\mathbb{Q}}_p$ into the complex numbers. We can then associate to the complex Artin representation $\alpha \circ \rho$ the classical Σ-truncated T-modified Artin L-function with the product formula

$$
L_{\Sigma,\mathrm{T}}(\alpha \circ \rho, s) := \prod_{x \in U^0} \det(1 - \alpha \circ \rho_x(\mathfrak{F}_x) q_x^{-s})^{-1} \prod_{x \in \mathrm{T}} \frac{\det(1 - \alpha \circ \rho_x(\mathfrak{F}_x) q_x^{1-s})}{\det(1 - \alpha \circ \rho_x(\mathfrak{F}_x) q_x^{-s})}
$$

for $\operatorname{Re} s > 1$. Note that we follow the geometric convention of using the geometric Frobenius in the definition of the Artin L-function as in [CL73]. With this convention, we have

$$
L_{\Sigma,\mathrm{T}}(\alpha \circ \rho, n) = \prod_{x \in W^0} \alpha(\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{M}(\rho)(n))))
$$
$$
= \prod_{x \in W^0} \alpha(\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}^{\circledast}(x, k_! \mathcal{M}(\check{\rho})(1-n))))^{-1}
$$

for all $n \in \mathbb{Z}$, $n > 1$.

By [CL73, Cor 1.4] there exists for each $n \in \mathbb{Z}$, $n < 0$ a well defined number $L_{\Sigma,\mathrm{T}}(\rho, n) \in C$ such that

$$
\alpha(L_{\Sigma,\mathrm{T}}(\rho, n)) = L_{\Sigma,\mathrm{T}}(\alpha \circ \rho, n) \in \mathbb{C}
$$

Consequently,

$$
L_{\Sigma',\mathrm{T}'}(\rho, n) = L_{\Sigma,\mathrm{T}}(\rho, n) \prod_{x \in \Sigma' \cup \mathrm{T}' - \Sigma \cup \mathrm{T}} \varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{M}(\rho)(n)))^{-1}
$$

if $\Sigma \subset \Sigma'$ and $\mathrm{T} \subset \mathrm{T}'$ with disjoint subsets Σ' and T' of $X = \operatorname{Spec} \mathcal{O}_F$ such that ρ is unramified over $X - \Sigma' - \mathrm{T}'$ and all the primes over p are contained in Σ'. Let $\kappa_F \colon \operatorname{Gal}_F \to \mathbb{Z}_p^{\times}$ denote the cyclotomic character such that

$$
\sigma(\zeta) = \zeta^{\kappa_F(\sigma)}
$$

for every $\sigma \in \mathrm{Gal}_F$ and $\zeta \in \mu_{p^\infty}$. Further, we write $\omega_F \colon \mathrm{Gal}_F \to \mu_{p-1}$ for the Teichmüller character, i. e. the composition of κ_F with the projection $\mathbb{Z}_p^\times \to \mu_{p-1}$. Finally, we set $\epsilon_F \coloneqq \kappa_F \omega^{-1}$ and note that ϵ_F factors through $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. Assume that ρ factors through the Galois group of a totally real field extension of F. Then $\mathcal{M}(\rho)$ is smooth at ∞. Assume further that $U = W$, such that T is empty. Under Conjecture 9.5 it follows from [Gre83] and from the validity of the classical main conjecture that for every integer n there exist unique elements

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho\omega_F^n)(1-n)) \in \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S)$$

such that

$$\begin{aligned}
\Phi_{\epsilon_F^n}(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho\omega_F^n)(1-n))) &= \mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1)), \\
\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho\omega_F^n)(1-n))) &= L_{\Sigma,\varnothing}(\rho\omega_F^n, 1-n) \quad \text{if } n > 1.
\end{aligned} \tag{13.3}$$

with $\Phi_{\epsilon_F^n}$ as defined in Example 3.17. Beware that Greenberg uses the arithmetic convention for L-functions.

DEFINITION 13.1. Let $\gamma \in \Gamma$ be a topological generator. We define the *global modification factor* for $\mathcal{M}(\rho\omega_F^n)(1)$ and $f \colon U_{F_{\mathrm{cyc}}} \to U$ to be the element

$$\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \mathcal{M}(\rho)(1)) \coloneqq \mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1)) s_\gamma([\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{M}(\rho)(1))])$$

in $\mathrm{K}_1(\mathcal{O}_C[[\Gamma]])$.

14   NON-COMMUTATIVE L-FUNCTIONS FOR Λ-ADIC SHEAVES

Throughout this section, we assume the validity of Conjecture 9.5. We recall the main theorem of [Kak13].

THEOREM 14.1. *Let $U \subset X$ be a dense open subscheme with complement Σ and assume that p is invertible on U. Assume that F_∞/F is a really admissible extension which is unramified over U and that $G \coloneqq \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group. Then there exists unique elements $\tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_p[[G]]_S)/\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[G]])$ such that*

1.

$$d\tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_p)_U(1))],$$

2. For any Artin representation ρ factoring through G

$$\Phi_\rho(\tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1))$$

Proof. This is [Kak13, Thm. 2.11] translated into our notations. Recall that our $\Phi_{\rho\kappa_F^{-n}}$ corresponds to $\Phi_{\bar\rho\kappa_F^n}$ in the notation of the cited article. Moreover, Kakde uses the arithmetic convention in the definition of L-values. Further, note that the p-adic L-function $\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1))$ is uniquely determined by

the values $\varphi(\Phi_{\epsilon_F^n}(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1))))$ for $n < 0$ and $n \equiv 0 \mod p-1$. Finally, Kakde's complex $\mathcal{C}(F_\infty/F)$ corresponds to $\mathrm{R}\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))$ shifted by 3 and therefore, the images of the two complexes under d differ by a sign, but at the same time, his definition of d differs by a sign from ours. $\qquad\square$

We will improve this theorem as follows. Let $\Xi := \Xi_F$ be the set of pairs (U, F_∞) such that $U \subset X$ is a dense open subscheme with p invertible on U and F_∞/F is a really admissible extension unramified over U.

THEOREM 14.2. *Let $\gamma \in \Gamma := \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a topological generator. There exists a unique family of elements*

$$\big(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))\big)_{(U,F_\infty)\in\Xi}$$

such that

1. $\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$,

2. *if $U \subset U'$ with complement Σ and $(U, F_\infty), (U', F_\infty) \in \Xi$, then*

$$\mathcal{M}_{F_\infty/F,\gamma}(U', (\mathbb{Z}_p)_{U'}(1)) = $$
$$\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)) \prod_{x \in \Sigma} \mathcal{M}_{F_\infty/F,\gamma}(x, (\mathbb{Z}_p)_{U'}(1)),$$

3. *if $(U, F_\infty), (U, F'_\infty) \in \Xi$ such that $F'_\infty \subset F_\infty$ is a subfield, then*

$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{M}_{F'_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)),$$

4. *if $(U, F_\infty) \in \Xi$ and $\rho: \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_n(\mathcal{O}_C)$ is an Artin representation, then*
$$\Phi_\rho(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \mathcal{M}(\rho)(1))$$

   with $\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \mathcal{M}(\rho)(1))$ as in Definition 13.1.

Proof. Uniqueness: Assume that $m_k(U, F_\infty)$, $k = 1, 2$ are two families with the listed properties. Then

$$d(F_\infty) := m_2(U, F_\infty)^{-1} m_1(U, F_\infty)$$

does not depend on U as a consequence of (2).
Let $(U, F_\infty) \in \Xi$ be any pair such that $F_\infty/F_{\mathrm{cyc}}$ is finite and write $f: U_{F_\infty} \to U$ for the system of coverings of U associated to F_∞/F. Then (4) implies that the elements
$$m_i(U, F_\infty) s_\gamma(-[\mathrm{R}\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))])$$
both agree with $\tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))$ modulo $\widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$. Hence,

$$d(F_\infty) \in \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]).$$

By Corollary 2.4, we may find a pair $(U', F'_\infty) \in \Xi$ such that F'_∞/F_∞ is finite, $U' \subset U$, and

$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]} : \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]) \quad (14.1)$$

is the zero map. We conclude from (3) that $d(F_\infty) = 1$ for all (U, F_∞) with $F_\infty/F_{\mathrm{cyc}}$ finite. Now for any really admissible extension F_∞/F,

$$\mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]) = \varprojlim_{F'_\infty} \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$$

where F'_∞ runs through the really admissible subextensions of F_∞/F with $F'_\infty/F_{\mathrm{cyc}}$ finite [FK06, Prop. 1.5.3]. We conclude that $d(F_\infty) = 1$ in general.
Existence: It suffices to construct the elements for $(U, F_\infty) \in \Xi$ with $F_\infty/F_{\mathrm{cyc}}$ finite. Choose (U', F'_∞) as above such that the map in (14.1) becomes trivial. Pick any $m \in \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$ such that

$$ms_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))]) \equiv$$
$$\tilde{\mathcal{L}}_{F'_\infty/F}(U', (\mathbb{Z}_p)_{U'}(1)) \mod \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]])$$

Define

$$\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)) := \Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]}(m) \prod_{x \in U - U'} \mathcal{M}_{F_\infty/F,\gamma}(x, (\mathbb{Z}_p)_U(1)).$$

By Proposition 6.1 and Proposition 11.4 we conclude that

$$\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)) s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1))]) \equiv \tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))$$
$$\mod \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]])$$

and that $\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))$ satisfies

$$\Phi_\rho(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \mathcal{M}(\rho)(1))$$

for any Artin representation $\rho : \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_n(\mathcal{O}_C)$. In particular, the system

$$(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)))_{(F_\infty, U) \in \Xi}$$

satisfies (1) and (4). By construction and again by Proposition 11.4, it is independent of the choices of (U', F'_∞) and m and satisfies (2) and (3). $\qquad \square$

Corollary 14.3. *There exists a unique family of elements*

$$\left(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) \right)_{(U, F_\infty) \in \Xi}$$

such that

1. $\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]_S)$,

2. if $(U, F_\infty) \in \Xi$ and $f: U_{F_\infty} \to U$ denotes the associated system of coverings, then
$$d\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) = -[\mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))]$$

3. if $U' \subset U$ with complement Σ and $(U', F_\infty), (U, F_\infty) \in \Xi$, then
$$\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) = \mathcal{L}_{F_\infty/F}(U', (\mathbb{Z}_p)_{U'}(1)) \prod_{x \in \Sigma} \mathcal{L}_{F_\infty/F}(x, (\mathbb{Z}_p)_U(1)),$$

4. if $(U, F_\infty), (U, F'_\infty) \in \Xi$ such that $F'_\infty \subset F_\infty$ is a subfield, then
$$\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F'_\infty/F}(U, (\mathbb{Z}_p)_U(1)),$$

5. if $(U, F_\infty) \in \Xi$ and $\rho: \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation, then
$$\Phi_\rho(\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \mathcal{M}(\rho)(1)).$$

Proof. Fix a topological generator $\gamma \in \Gamma$ and set
$$\mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1)) := \mathcal{M}_{F_\infty/F, \gamma}(U, (\mathbb{Z}_p)_U(1)) s_\gamma(-[\mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))]).$$

If $(\ell(U, F_\infty))_{(U, F_\infty) \in \Xi}$ is a second family with the listed properties, then
$$\ell(U, F_\infty) s_\gamma([\mathrm{R}\,\Gamma_c(U, f_! f^*(\mathbb{Z}_p)_U(1))]) = \mathcal{M}_{F_\infty/F, \gamma}(U, (\mathbb{Z}_p)_U(1))$$

by the uniqueness of $\mathcal{M}_{F_\infty/F, \gamma}(U, (\mathbb{Z}_p)_U(1))$. $\qquad\square$

Let $\Theta := \Theta_F$ be the set of triples (U, F_∞, Λ) such that $U \subset X$ is a dense open subscheme with p invertible on U, F_∞/F is a really admissible extension unramified over U and Λ is an adic $\mathbb{Z}_p$-algebra.

THEOREM 14.4. *Let* $\gamma \in \Gamma := \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ *be a topological generator. There exists a unique family of homomorphisms*

$$\left(\begin{matrix} \mathcal{M}_{F_\infty/F, \gamma}(U, (-)(1)): \\ \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}, \infty}(U, \Lambda)) \to \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]) \end{matrix} \right)_{(U, F_\infty, \Lambda) \in \Theta}$$

such that

1. *for any* $(U, F_\infty, \mathbb{Z}_p) \in \Theta$, $\mathcal{M}_{F_\infty/F, \gamma}(U, (\mathbb{Z}_p)_U(1))$ *is the element constructed in Theorem 14.2,*

2. *if* $j: U' \to U$ *is an open immersion and* $(U', F_\infty, \Lambda), (U, F_\infty, \Lambda) \in \Theta$, *then*
$$\mathcal{M}_{F_\infty/F, \gamma}(U, \mathcal{F}^\bullet(1)) = \mathcal{M}_{F_\infty/F, \gamma}(U', j^* \mathcal{F}^\bullet(1)) \prod_{x \in U - U'} \mathcal{M}_{F_\infty/F, \gamma}(x, \mathcal{F}^\bullet(1)),$$

    for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont}, \infty}(U, \Lambda)$.

3. *if $(U, F_\infty, \Lambda), (U, F'_\infty, \Lambda) \in \Theta$ such that $F'_\infty \subset F_\infty$ is a subfield, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))) = \mathcal{M}_{F'_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)),$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

4. *if $(U, F_\infty, \Lambda), (U, F_\infty, \Lambda') \in \Theta$ and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then*

$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))) = \mathcal{M}_{F_\infty/F,\gamma}(U, \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)(1))$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

Proof. Applying (4) to the Λ/I-$\Lambda[[G]]$-bimodule $\Lambda/I[[G]]$ for any open two-sided ideal I of Λ and using

$$\mathrm{K}_1(\Lambda[[G]]) = \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{K}_1(\Lambda/I[[G]]),$$

we conclude that it is sufficient to consider triples $(U, F_\infty, \Lambda) \in \Theta$ with Λ a finite ring. So, let Λ be finite. Since $\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))$ depends only on the class of $\mathcal{F}^\bullet$ in $\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda))$, we may assume that $\mathcal{F}^\bullet$ is a bounded complex of flat constructible étale sheaves of Λ-modules. Using (2) we may shrink U until $\mathcal{F}^\bullet$ is a complex of locally constant étale sheaves. Hence, there exists a $(U, F'_\infty, \Lambda) \in \Theta$ such that F_∞/F is a subextension of F'_∞/F and such that the restriction of $\mathcal{F}^\bullet$ to U_K for some finite subextension K/F of F'_∞/F is a complex of constant sheaves. By (3), we may replace F_∞ by F'_∞. We may then find a complex of Λ-$\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$-bimodules $P^\bullet$, strictly perfect as complex of Λ modules and a weak equivalence

$$\Psi_{P^\bullet} f_! f^*(\mathbb{Z}_p)_U(1) \xrightarrow{\sim} \mathcal{F}^\bullet(1)$$

[Wit14, Prop. 6.8]. By (4), the only possible definition of $\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))$ is

$$\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) \coloneqq \Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))).$$

It is then clear that this construction satisfies the given properties. $\qquad\square$

Theorem 14.5. *Let $\gamma \in \Gamma \coloneqq \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a topological generator. There exists a unique family of homomorphisms*

$$\left(\mathcal{M}^\circledast_{F_\infty/F,\gamma}(U, -) \colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)) \to \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]) \right)_{(U,F_\infty,\Lambda) \in \Theta}$$

such that

1. *for any $(U, F_\infty, \mathbb{Z}_p) \in \Theta$,*

$$\mathcal{M}^\circledast_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U) = (\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1)))^\circledast$$

2. if $j : U' \to U$ is an open immersion and $(U', F_\infty, \Lambda), (U, F_\infty, \Lambda) \in \Theta$, then

$$\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet) = \mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U', j^* \mathcal{F}^\bullet) \prod_{x \in U - U'} \mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(x, \mathcal{F}^\bullet),$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

3. if $(U, F_\infty, \Lambda), (U, F'_\infty, \Lambda) \in \Theta$ such that $F'_\infty \subset F_\infty$ is a subfield, then

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))) = \mathcal{M}^{\oplus}_{F'_\infty/F,\gamma}(U, \mathcal{F}^\bullet),$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

4. if $(U, F_\infty, \Lambda), (U, F_\infty, \Lambda') \in \Theta$ and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then

$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet)) = \mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U, \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

Moreover, for any $(U, F_\infty, \Lambda) \in \Theta$ and any smooth Λ-adic sheaf $\mathcal{F}$ on U, we have

$$\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U, \mathcal{F}) = (\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^{*\Lambda}(1)))^{\oplus}.$$

Proof. We proceed as in Theorem 14.4 and use Lemma 5.7. $\qquad\square$

PROPOSITION 14.6. *Assume that γ, γ' are two topological generators of Γ. Then*

$$\frac{\mathcal{M}_{F_\infty/F,\gamma}}{\mathcal{M}_{F_\infty/F,\gamma'}}(U, \mathcal{F}^\bullet(1)) = \frac{s_\gamma}{s_{\gamma'}}([\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\bullet(1))])$$

$$\frac{\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}}{\mathcal{M}^{\oplus}_{F_\infty/F,\gamma'}}(U, \mathcal{F}^\bullet) = \frac{s_{(\gamma')^{-1}}}{s_{\gamma^{-1}}}([\mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F}^\bullet)])$$

for any $(U, F_\infty, \Lambda) \in \Theta$ and any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

Proof. By definition, these identities hold for the local modification factors and by Corollary 14.3 and Proposition 6.3 they hold for $\mathcal{F}^\bullet = (\mathbb{Z}_p)_U$. Hence,

$$\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) = \frac{s_\gamma}{s_{\gamma'}}([\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\bullet(1))])\mathcal{M}_{F_\infty/F,\gamma'}(U, \mathcal{F}^\bullet(1))$$

$$\mathcal{M}^{\oplus}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet) = \frac{s_{(\gamma')^{-1}}}{s_{\gamma^{-1}}}([\mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F}^\bullet)])\mathcal{M}^{\oplus}_{F_\infty/F,\gamma'}(U, \mathcal{F}^\bullet)$$

by the uniqueness assertion in Theorem 14.4 and Theorem 14.5. $\qquad\square$

THEOREM 14.7. *Let F'/F be a finite extension of totally real fields. Set $r := [F' \cap F_{\mathrm{cyc}} : F]$ and let $\gamma \in \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a topological generator. Assume that $(U, F_\infty, \Lambda) \in \Theta_F$ with $F' \subset F_\infty$ and write $f_{F'} : U_{F'} \to U$ for the associated covering. Then*

1. *for every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$,*

$$\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'}, f_{F'}^*\mathcal{F}^\bullet(1)) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F,\gamma}(U,\mathcal{F}^\bullet(1)),$$
$$\mathcal{M}_{F_\infty/F',\gamma^r}^{\circledast}(U_{F'}, f_{F'}^*\mathcal{F}^\bullet) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F,\gamma}^{\circledast}(U,\mathcal{F}^\bullet);$$

2. *for every $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'},\Lambda)$,*

$$\mathcal{M}_{F_\infty/F,\gamma}(U, f_{F'*}\mathcal{G}^\bullet(1)) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'},\mathcal{G}^\bullet(1)),$$
$$\mathcal{M}_{F_\infty/F,\gamma}^{\circledast}(U, f_{F'*}\mathcal{G}^\bullet) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F',\gamma^r}^{\circledast}(U_{F'},\mathcal{G}^\bullet).$$

Proof. We prove the identities for the global modification factors; the proof for the global dual modification factors is the same.

We first note that for any complex $P^\bullet$ of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, there exists an obvious isomorphism of complexes of $\Lambda'[[\mathrm{Gal}(F_\infty/F')]]$-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules

$$\Lambda'[[\mathrm{Gal}(F_\infty/F)]] \otimes_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} P[[\mathrm{Gal}(F_\infty/F)]]^{\delta^\bullet} \cong$$
$$P[[\mathrm{Gal}(F_\infty/F')]]^{\delta^\bullet} \otimes_{\Lambda[[\mathrm{Gal}(F_\infty/F')]]} \Lambda[[\mathrm{Gal}(F_\infty/F)]].$$

Hence,

$$\Psi_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} \circ \Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta^\bullet}} = \Psi_{P[[\mathrm{Gal}(F_\infty/F')]]^{\delta^\bullet}} \circ \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]} \quad (14.2)$$

as homomorphisms from $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])$ to $\mathrm{K}_1(\Lambda'[[\mathrm{Gal}(F_\infty/F')]])$. Likewise, for a complex $Q^\bullet$ of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F')]]$-bimodules, strictly perfect as complex of Λ'-modules, we have an equality

$$\Psi_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} \circ \Psi_{Q[[\mathrm{Gal}(F_\infty/F')]]^{\delta^\bullet}} = \Psi_{Q[[\mathrm{Gal}(F_\infty/F)]]^{\delta^\bullet}} \circ \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}.$$
$$(14.3)$$

in $\mathrm{Hom}(\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F')]]), \mathrm{K}_1(\Lambda'[[\mathrm{Gal}(F_\infty/F)]]))$.

In particular, we may reduce to the case of finite $\mathbb{Z}_p$-algebras Λ by choosing $P^\bullet = \Lambda = Q^\bullet$ with the trivial action of $\mathrm{Gal}(F_\infty/F)$ and $\mathrm{Gal}(F_\infty/F')$, respectively. By Proposition 11.4.(4) we may then shrink U until $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ may be assumed to be strictly perfect complexes of locally constant étale sheaves. Using the identities (14.2) and (14.3) again, we may reduce to the case $\Lambda = \mathbb{Z}_p$ and $\mathcal{F}^\bullet = (\mathbb{Z}_p)_U$, $\mathcal{G}^\bullet = (\mathbb{Z}_p)_{U_{F'}}$. We may then further reduce to the case that $F_\infty/F'_{\mathrm{cyc}}$ is a finite extension.
Setting

$$q \coloneqq \frac{\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'}, f_{F'}^*(\mathbb{Z}_p)_{U_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_p)_U(1))} \in \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F')]]),$$

$$q' \coloneqq \frac{\mathcal{M}_{F_\infty/F,\gamma}(U, f_{F'*}(\mathbb{Z}_p)_{U_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'},(\mathbb{Z}_p)_U(1))} \in \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]),$$

it suffices to show that $q = 1$ and $q' = 1$.

Let $g \colon U_{F_\infty} \to U_{F'}$ denote the restriction of $f \colon U_{F_\infty} \to U$. Write

$$M := \mathbb{Z}_p[\operatorname{Gal}(F_\infty/F') \backslash \operatorname{Gal}(F_\infty/F)]$$

for the $\mathbb{Z}_p$-$\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]$-bimodule freely generated as $\mathbb{Z}_p$-module by the right cosets $\operatorname{Gal}(F_\infty/F)\sigma$ for $\sigma \in \operatorname{Gal}(F_\infty/F)$ and on which $\tau \in \operatorname{Gal}(F_\infty/F)$ operates by right multiplication. From Proposition 9.4 we conclude

$$\Psi_{\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]} \operatorname{R}\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1)) \sim \operatorname{R}\Gamma_{\mathrm{c}}(U, f_{F'*} g_! g^* f^*_{F'}(\mathbb{Z}_p)_U(1))$$
$$\sim \operatorname{R}\Gamma_{\mathrm{c}}(U_{F'}, g_! g^*(\mathbb{Z}_p)_{U_{F'}}(1)),$$

$$\Psi_{\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]} \operatorname{R}\Gamma_{\mathrm{c}}(U_{F'}, g_! g^*(\mathbb{Z}_p)_{U_{F'}}(1))$$
$$\sim \Psi_{\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]} \operatorname{R}\Gamma_{\mathrm{c}}(U, f_{F'*} g_! g^*(\mathbb{Z}_p)_{U_{F'}}(1))$$
$$\sim \operatorname{R}\Gamma_{\mathrm{c}}(U, f_! f^* f_{F'*}(\mathbb{Z}_p)_{U_{F'}}(1))$$
$$\sim \Psi_{M[[\operatorname{Gal}(F_\infty/F)]]^\delta} \operatorname{R}\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_p)_U(1)).$$

Additionally, we note that

$$\mathcal{M}_{F_\infty/F,\gamma}(U, f_{F'*}(\mathbb{Z}_p)_U(1)) = \Psi_{M[[\operatorname{Gal}(F_\infty/F)]]^\delta} \mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_p)_U(1))$$

by Theorem 14.4.

From this and from Proposition 6.1, we conclude

$$q = \frac{\mathcal{L}_{F_\infty/F'}(U_{F'}, (\mathbb{Z}_p)_{U_{F'}}(1))}{\Psi_{\Lambda[[\operatorname{Gal}(F_\infty/F)]]} \mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))},$$

$$q' = \frac{\Psi_{M[[\operatorname{Gal}(F_\infty/F)]]^\delta} \mathcal{L}_{F_\infty/F}(U, (\mathbb{Z}_p)_U(1))}{\Psi_{\Lambda[[\operatorname{Gal}(F_\infty/F)]]} \mathcal{L}_{F_\infty/F'}(U_{F'}, (\mathbb{Z}_p)_U(1))}.$$

Let $C/\mathbb{Q}_p$ be a finite field extension and

$$\rho' \colon \operatorname{Gal}(F_\infty/F') \to \operatorname{Gl}_d(\mathcal{O}_C)$$
$$\rho \colon \operatorname{Gal}(F_\infty/F) \to \operatorname{Gl}_d(\mathcal{O}_C)$$

be Artin representations. Write

$$\varphi_F \colon \mathcal{O}_C[[\operatorname{Gal}(F_{\mathrm{cyc}}/F)]] \to \mathcal{O}_C$$
$$\varphi_{F'} \colon \mathcal{O}_C[[\operatorname{Gal}(F'_{\mathrm{cyc}}/F')]] \to \mathcal{O}_C$$

for the augmentation maps. We denote by $\operatorname{Ind}_{F'}^F \rho'$ and $\operatorname{Res}_F^{F'} \rho$ the induced an restricted representations, respectively.

Then for every $n \in \mathbb{Z}$

$$\varphi_{F'} \circ \Phi_{\rho' \epsilon_{F'}^n} \circ \Psi_{\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]} = \varphi_F \circ \Phi_{\operatorname{Ind}_{F'}^F \rho' \epsilon_{F'}^n} = \varphi_F \circ \Phi_{\epsilon_F^n \operatorname{Ind}_{F'}^F \rho'}$$

as maps from $\operatorname{K}_1(\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]_S)$ to $\mathbb{P}^1(C)$ and

$$\varphi_F \circ \Phi_{\rho \epsilon_F^n} \circ \Psi_{\mathbb{Z}_p[[\operatorname{Gal}(F_\infty/F)]]} = \varphi_{F'} \circ \Phi_{\operatorname{Res}_F^{F'} \rho \epsilon_F^n} = \varphi_{F'} \circ \Phi_{\epsilon_{F'}^n \operatorname{Res}_F^{F'} \rho}$$

as maps from $K_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]]_S)$ to $\mathbb{P}^1(C)$. From (13.3) and the transformation properties of the complex Artin L-functions with respect to inflation and restriction we conclude that for $n < -1$ and $\Sigma \coloneqq X - U$

$$
\begin{aligned}
\varphi_{F'} \circ \Phi_{\rho'\epsilon_{F'}^n}(\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1))) &= \\
&= L_{\Sigma,\varnothing}(\omega_F^{-n}\mathrm{Ind}_{F'}^F \rho', 1+n) \\
&= L_{\Sigma_{F'},\varnothing}(\rho'\omega_{F'}^{-n}, 1+n) \\
&= \varphi_{F'} \circ \Phi_{\rho'\epsilon_{F'}^n}(\mathcal{L}_{F_\infty/F'}(U_{F'},(\mathbb{Z}_p)_{U_{F'}}(1))), \\
\varphi_F \circ \Phi_{\rho\epsilon_F^n}(\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F'}(U_{F'},(\mathbb{Z}_p)_{U_{F'}}(1)) &= \\
&= L_{\Sigma_{F'},\varnothing}(\omega_{F'}^{-n}\mathrm{Res}_F^{F'}\rho, 1+n) \\
&= L_{\Sigma,\varnothing}(\omega_F^{-n}\mathrm{Ind}_F^{F'}\mathrm{Res}_F^{F'}\rho, 1+n) \\
&= \varphi_F \circ \Phi_{\epsilon_F^n\mathrm{Ind}_F^{F'}\mathrm{Res}_{F'}^{F'}\rho}(\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1)))) \\
&= \varphi_F \circ \Phi_{\rho\epsilon_F^n}(\Psi_{M[[\mathrm{Gal}(F_\infty/F)]]}^\delta\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_p)_U(1))).
\end{aligned}
$$

From [Bur15, Lemma 3.4] we conclude that $\Phi_{\rho'}(q) = 1$ in $K_1(\mathcal{O}_C[[\Gamma]])$ and thus $\varphi_{F'}(\Phi_{\rho'}(q)) = 1$ in C for every Artin representation ρ' of $\mathrm{Gal}(F_\infty/F')$. In particular, with K running through the finite Galois extension fields of F in F_∞, the images of q in the groups $K_1(\mathbb{Q}_p[\mathrm{Gal}(K/F)])$ are trivial. This implies

$$
q \in \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]]).
$$

Using Corollary 2.4 we find a suitable admissible extension L_∞/F unramified over $U' \subset U$ such that

$$
\Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]]}\colon \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(L_\infty/F')]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]])
$$

is the zero map. As

$$
q = \Psi_{\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F')]]}\left(\frac{\mathcal{M}_{L_\infty/F',\gamma^r}(U'_{F'}, f_{F'}^*(\mathbb{Z}_p)_{U'_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(L_\infty/F)]]}\mathcal{M}_{L_\infty/F,\gamma}(U',(\mathbb{Z}_p)_{U'}(1))}\right),
$$

we conclude $q = 1$. The proof that $q' = 1$ follows the same pattern. $\qquad\square$

DEFINITION 14.8. Let F be a totally real field, $k\colon U \to W$ be an open immersion of open dense subschemes of $X = \mathrm{Spec}\,\mathcal{O}_F$ such that p is invertible on W, and Λ be an adic $\mathbb{Z}_p$-algebra. Fix a topological generator $\gamma \in \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, and any really admissible extension F_∞/F unramified over U, we set

$$
\begin{aligned}
\mathcal{M}_{F_\infty/F,\gamma}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1)) &\coloneqq \mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) \\
&\qquad \prod_{x \in W-U} \mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1)), \\
\mathcal{M}^\circledast_{F_\infty/F,\gamma}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet) &\coloneqq \mathcal{M}^\circledast_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet) \prod_{x \in W-U} \mathcal{M}^\circledast_{F_\infty/F,\gamma}(x, k_!f_!f^*\mathcal{F}^\bullet)
\end{aligned}
$$

in $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])$ and

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1)) \coloneqq \mathcal{M}_{F_\infty/F,\gamma}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1))$$
$$s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1))]),$$
$$\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}^\bullet) \coloneqq \mathcal{M}^\oplus_{F_\infty/F,\gamma}(W, k_!\mathcal{F}^\bullet)s_{\gamma^{-1}}([\mathrm{R}\,\Gamma(W, k_!f_!f^*\mathcal{F}^\bullet)])$$

in $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]_S)$.

Note that we do not assume that F_∞/F is unramified over W. If it is unramified over W, then

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1)) = \mathrm{R}\,\Gamma_{\mathrm{c}}(W, f_!f^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)),$$
$$\mathrm{R}\,\Gamma(W, k_!f_!f^*\mathcal{F}^\bullet) = \mathrm{R}\,\Gamma(W, f_!f^*k_!\mathcal{F}^\bullet)$$

and the two possible definitions of the elements $\mathcal{M}_{F_\infty/F,\gamma}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ and $\mathcal{M}^\oplus_{F_\infty/F,\gamma}(W, k_!\mathcal{F}^\bullet)$ agree. Moreover, by Proposition 14.6, $\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ and $\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}^\bullet)$ do not depend on the choice of γ.

In the following corollary, we compile a list of the transformation properties of $\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ and $\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}^\bullet)$.

COROLLARY 14.9. *Let F be a totally real field, $k\colon U \to W$ be an open immersion of open dense subschemes of $X = \mathrm{Spec}\,\mathcal{O}_F$ such that p is invertible on W, and Λ be an adic $\mathbb{Z}_p$-algebra. Fix a $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, and a really admissible extension F_∞/F unramified over U.*

1. *Write $f\colon U_{F_\infty} \to U$ for the system of coverings associated to F_∞/F. Then*

$$d\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1))],$$
$$d\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}^\bullet) = [\mathrm{R}\,\Gamma(W, k_!f_!f^*\mathcal{F}^\bullet)]$$

2. *If $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ are weakly equivalent in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, then*

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{G}^\bullet(1)),$$
$$\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}^\bullet) = \mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{G}^\bullet).$$

3. *If $0 \to \mathcal{F}'^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}''^\bullet \to 0$ is an exact sequence in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, then*

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}'^\bullet(1))\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}''^\bullet(1)),$$
$$\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}^\bullet) = \mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}'^\bullet)\mathcal{L}^\oplus_{F_\infty/F}(W, k_!\mathcal{F}''^\bullet).$$

4. *If W' is an open dense subscheme of X on which p is invertible and $k':W \to W'$ is an open immersion, then*

$$\mathcal{L}_{F_\infty/F}(W', \mathrm{R}(k'k)_* \mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))$$
$$\prod_{x \in W'-W} \mathcal{L}_{F_\infty/F}(x, \mathrm{R}(k'k)_* \mathcal{F}^\bullet(1)),$$
$$\mathcal{L}^\oplus_{F_\infty/F}(W', (k'k)_! \mathcal{F}^\bullet) = \mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)$$
$$\prod_{x \in W'-W} \mathcal{L}^\oplus_{F_\infty/F}(x, (k'k)_! \mathcal{F}^\bullet).$$

5. *If $i:x \to U$ is a closed point, then*

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* i_* i^* \mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(x, \mathcal{F}^\bullet(1)),$$
$$\mathcal{L}^\oplus_{F_\infty/F}(W, k_! i_* \mathrm{R}\, i^! \mathcal{F}^\bullet) = \mathcal{L}^\oplus_{F_\infty/F}(x, \mathcal{F}^\bullet).$$

6. *If F'_∞/F is a really admissible subextension of F_∞/F, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))) = \mathcal{L}_{F'_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)) = \mathcal{L}^\oplus_{F'_\infty/F}(W, k_! \mathcal{F}^\bullet).$$

7. *If Λ' is another adic $\mathbb{Z}_p$-algebra and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then*

$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)(1)),$$
$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)) = \mathcal{L}^\oplus_{F_\infty/F}(W, k_! \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)).$$

8. *If F'/F is a finite extension inside F_∞ and $f_{F'}:U_{F'} \to U$ the associated covering, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1))) = \mathcal{L}_{F_\infty/F'}(W_{F'}, \mathrm{R}\, k_* f_{F'}^* \mathcal{F}^\bullet(1)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)) = \mathcal{L}^\oplus_{F_\infty/F'}(W_{F'}, k_! f_{F'}^* \mathcal{F}^\bullet).$$

9. *With the notation of (8), if $\mathcal{G}^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'}, \Lambda)$, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}_{F_\infty/F'}(W_{F'}, \mathrm{R}\, k_* \mathcal{G}^\bullet(1))) = \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* f_{F'*} \mathcal{G}^\bullet(1)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^\oplus_{F_\infty/F'}(W_{F'}, k_! \mathcal{G}^\bullet)) = \mathcal{L}^\oplus_{F_\infty/F}(W, k_! f_{F'*} \mathcal{G}^\bullet).$$

10. *If $\mathcal{F}$ is a smooth Λ-adic sheaf on U which is smooth at ∞, then*

$$\mathcal{L}^\oplus_{F_\infty/F}(W, k_! \mathcal{F}) = (\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^{*\Lambda}(1)))^\circledast.$$

11. *If $C/\mathbb{Q}_p$ is a finite field extension and $\rho\colon\mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation, then*

$$\Phi_{\rho\epsilon_F^{-n}}(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*(\mathbb{Z}_p)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\,k_*\mathcal{M}(\rho\omega_F^n)(1-n)),$$
$$\Phi_{\rho\epsilon_F^n}(\mathcal{L}_{F_\infty/F}^{\oplus}(W, \mathrm{R}\,k_*(\mathbb{Z}_p)_U)) = \mathcal{L}_{F_{\mathrm{cyc}}/F}^{\oplus}(W, \mathrm{R}\,k_*\mathcal{M}(\rho\omega_F^{-n})(n)),$$

for any integer n.

12. *If $C/\mathbb{Q}_p$ is a finite field extension and $\rho\colon\mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation which factors through a totally real field and which is unramified over U, then*

$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\,k_*\mathcal{M}(\rho\omega_F^n)(1-n))) = L_{\Sigma,\mathrm{T}}(\rho\omega_F^n, 1-n),$$
$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}^{\oplus}(W, k_!\mathcal{M}(\rho\omega_F^{-n})(n))) = L_{\Sigma,\mathrm{T}}(\check{\rho}\omega_F^n, 1-n)^{-1}$$

with $\Sigma \coloneqq X - W$, $\mathrm{T} \coloneqq W - U$ and any integer $n > 1$.

Proof. Properties (1)–(4) are clear by definition. For Property (5) we notice that for $y \in W$

$$\mathcal{L}_{F_\infty/F}(y, \mathrm{R}\,k_* i_* i^* \mathcal{F}^{\bullet}(1)) = \begin{cases} \mathcal{L}_{F_\infty/F}(x, \mathcal{F}^{\bullet}(1)) & \text{if } y = x, \\ 1 & \text{else.} \end{cases}$$

Hence,

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_* i_* i^* \mathcal{F}^{\bullet}(1)) = \mathcal{L}_{F_\infty/F}(U, i_* i^* \mathcal{F}(1)) = \mathcal{L}_{F_\infty/F}(x, \mathcal{F}^{\bullet}(1))$$

by (4) and by Theorem 14.4.(2). The proof for the dual L-function is analogous. Properties (6) and (7) follow from Theorem 14.4 or Theorem 14.5 combined with Proposition 6.1 and either Proposition 11.4 or Proposition 11.8. For Properties (8) and (9) one applies Theorem 14.7. Property (10) follows from the last part of Theorem 14.5 combined with Proposition 6.3 and Proposition 11.9. Property (11) is just a special case of (7) in a different notation.
It remains to prove (12). The first identity is simply the combination of (13.3) and (13.2). The second identity follows from (13.1), Property (10) and the first identity. $\qquad\square$

15 CM-admissible extensions

DEFINITION 15.1. Let F be a totally real number field and F_∞/F an admissible extension. We call F_∞/F *CM-admissible* if F_∞ is a CM-field, i.e. it is totally imaginary and there exists a (unique) involution $\iota \in \mathrm{Gal}(F_\infty/F)$ such that the fixed field F_∞^+ of ι is totally real.

Note that for a CM-admissible extension F_∞/F with Galois group

$$G \coloneqq \mathrm{Gal}(F_\infty/F),$$

the automorphism ι is uniquely determined and commutes with every other field automorphism of F_∞. As usual, we write

$$e_- \coloneqq \frac{1-\iota}{2}, \quad e_+ \coloneqq \frac{1+\iota}{2} \in \Lambda[[G]].$$

for the corresponding central idempotents.

The extension F_∞^+/F is Galois and hence, a really admissible extension. We set $G^+ \coloneqq \mathrm{Gal}(F_\infty^+/F)$. Moreover, we fix as before an immersion $k\colon U \to W$ of open dense subschemes of $X = \mathrm{Spec}\, F$ such that F_∞/F is unramified over U and $p \neq 2$ is invertible on W. Let

$$f^+\colon U_{F_\infty^+} \to U$$

denote the restriction of the family of coverings $f\colon U_{F_\infty} \to U$ to $U_{F_\infty^+}$.

If F_∞ contains the p-th roots of unity and hence, the p^n-th roots of unity for all $n \geq 1$, the cyclotomic character

$$\kappa_F\colon \mathrm{Gal}_F \to \mathbb{Z}_p^\times, \quad g\zeta = \zeta^{\kappa_F(g)}, \quad g \in \mathrm{Gal}_F, \zeta \in \mu_{p^\infty}$$

factors through $G \coloneqq \mathrm{Gal}(F_\infty/F)$. We then obtain for every odd $n \in \mathbb{Z}$ a ring isomorphism

$$\Lambda[[G]] \to \Lambda[[G_+]] \times \Lambda[[G_+]], \qquad G \ni g \mapsto (g^+, \kappa_F(g)^n g^+),$$

where g^+ denotes the image of $g \in G$ in G^+. The projections onto the two components corresponds to the decomposition of $\Lambda[[G]]$ with respect to e_+ and e_-.

We will construct the corresponding decomposition of $\mathbf{A}(\Lambda[[G]])$, where

$$\mathbf{A} \in \{\mathbf{PDG}^{\mathrm{cont}}, w_H\mathbf{PDG}^{\mathrm{cont}}, \mathbf{PDG}^{\mathrm{cont}, w_H}\}.$$

Write $\Lambda(\kappa_F^n)^\sharp$ for the Λ-$\Lambda[[G]]$-bimodule Λ with $g \in G$ acting by $\kappa_F^n(g^{-1})$ from the right and $\Lambda(\kappa_F^n)^\sharp[[G]]^\delta$ for the $\Lambda[[G]]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]] \otimes_\Lambda \Lambda(\kappa_F^n)^\sharp$ with the diagonal right action of G. According to Example 3.15, we obtain Waldhausen exact functors

$$\Psi_{\Lambda(\kappa_F^n)^\sharp[[G]]^\delta}\colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]]).$$

Moreover, considering $\Lambda[[G^+]]$ as a $\Lambda[[G^+]]$-$\Lambda[[G]]$-bimodule or as a $\Lambda[[G]]$-$\Lambda[[G^+]]$-bimodule, we obtain Waldhausen exact functors

$$\Psi_{\Lambda[[G^+]]}\colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G^+]]), \qquad \Psi_{\Lambda[[G^+]]}\colon \mathbf{A}(\Lambda[[G^+]]) \to \mathbf{A}(\Lambda[[G]]).$$

Note there exists isomorphisms of $\Lambda[[G]]$-$\Lambda[[G]]$-bimodules

$$e_+\Lambda[[G]] \cong \Lambda[[G^+]] \otimes_{\Lambda[[G^+]]} \Lambda[[G^+]]$$
$$e_-\Lambda[[G]] \cong \Lambda(\kappa_F^n)^\sharp[[G]]^\delta \otimes_{\Lambda[[G]]} e_+\Lambda[[G]] \otimes_{\Lambda[[G]]} \Lambda(\kappa_F^{-n})^\sharp[[G]]^\delta$$

for every odd $n \in \mathbb{Z}$ such that the composition

$$\Psi_{\Lambda[[G^+]]} \circ \Psi_{\Lambda[[G^+]]} \colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]])$$

is just the projection onto the e_+-component, whereas the projection onto the e_--component may be written as

$$\Psi_{\Lambda(\kappa_F^n)^\sharp[[G]]^\delta \otimes_{\Lambda[[G]]} \Lambda[[G^+]]} \circ \Psi_{\Lambda[[G^+]] \otimes_{\Lambda[[G]]} \Lambda(\kappa_F^{-n})^\sharp[[G]]^\delta} \colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]]).$$

We further note that

$$\Psi_{\Lambda(\kappa_F^n)^\sharp[[G]]^\delta}\big(f_! f^* \mathcal{F}^\bullet\big) \cong f_! f^* \mathcal{F}^\bullet(n).$$

If Λ' is another adic $\mathbb{Z}_p$-algebra and $P^\bullet$ is a complex of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, we set

$$P_+^\bullet \coloneqq P^\bullet e_+, \qquad P_-^\bullet \coloneqq P^\bullet e_- \tag{15.1}$$

such that ι acts trivially on $P_+^\bullet$ and by -1 on $P_-^\bullet$. Both are again complex of Λ'-$\Lambda[[G]]$-bimodules and strictly perfect as complex of Λ'-modules. In particular, we have an isomorphism of complexes of $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodules

$$P[[G]]^{\delta \bullet} \cong P_+[[G]]^{\delta \bullet} \oplus P_-[[G]]^{\delta \bullet}.$$

Beware that $P_+[[G]]^{\delta \bullet}$ differs from $P[[G]]^{\delta \bullet} e_+$. The element ι acts as $\iota \otimes \mathrm{id}$ on the first complex and trivially on the second. In fact, we have

$$P_+[[G]]^{\delta \bullet} e_+ = e_+ P_+[[G]]^{\delta \bullet}, \qquad P_+[[G]]^{\delta \bullet} e_- = e_- P_+[[G]]^{\delta \bullet},$$
$$P_-[[G]]^{\delta \bullet} e_+ = e_- P_-[[G]]^{\delta \bullet}, \qquad P_-[[G]]^{\delta \bullet} e_- = e_+ P_-[[G]]^{\delta \bullet}.$$

Moreover, the Waldhausen exact functors

$$\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda'), \qquad \mathcal{F}^\bullet \mapsto \Psi_{\widetilde{P_+}^\bullet}(\mathcal{F}^\bullet),$$
$$\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda'), \qquad \mathcal{F}^\bullet \mapsto \Psi_{\widetilde{P_-}^\bullet}(\mathcal{F}^\bullet)(1)$$

map complexes in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$ to complexes in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda')$. Throughout the rest of this section, we assume the validity of Conjecture 9.5.

COROLLARY 15.2. *Assume that F_∞/F is any CM-admissible extension unramified over U. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, the complexes*

$$e_+ \, \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet(1)), \qquad e_- \, \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet),$$
$$e_+ \, \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet), \qquad e_- \, \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet(1))$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

Proof. Without loss of generality, we may enlarge F_∞ by adjoining the p-th roots of unity. The claim of the corollary is then an immediate consequence of Theorem 10.1 applied to

$$\Psi_{\Lambda[[G^+]]}(\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*\,f_!f^*\mathcal{F}^\bullet(1))) \sim \mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*(f^+)_!(f^+)^*\mathcal{F}^\bullet(1)),$$
$$\Psi_{\Lambda[[G^+]]}(\mathrm{R}\,\Gamma(W,k_!\,f_!f^*\mathcal{F}^\bullet)) \sim \mathrm{R}\,\Gamma(W,k_!(f^+)_!(f^+)^*\mathcal{F}^\bullet),$$
$$\Psi_{\Lambda[[G^+]]\otimes_{\Lambda[[G]]}\Lambda(\kappa_F)^\sharp[[G]]^\delta}(\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*\,f_!f^*\mathcal{F}^\bullet)) \sim$$
$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*(f^+)_!(f^+)^*\mathcal{F}^\bullet(1)),$$
$$\Psi_{\Lambda[[G^+]]\otimes_{\Lambda[[G]]}\Lambda(\kappa_F^{-1})^\sharp[[G]]^\delta}(\mathrm{R}\,\Gamma(W,k_!\,f_!f^*\mathcal{F}^\bullet(1))) \sim \mathrm{R}\,\Gamma(W,k_!(f^+)_!(f^+)^*\mathcal{F}^\bullet)$$

$\square$

Assume that F_∞/F is CM-admissible and that F_∞ contains the p-th roots of unity. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, we set

$$\mathcal{L}^+_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)) \coloneqq \Psi_{\Lambda[[G^+]]}(\mathcal{L}_{F^+_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))),$$
$$\mathcal{L}^{\otimes,+}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet) \coloneqq \Psi_{\Lambda[[G^+]]}(\mathcal{L}^\otimes_{F^+_\infty/F}(W,k_!\mathcal{F}^\bullet)),$$
$$\mathcal{L}^-_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet) \coloneqq \Psi_{\Lambda(\kappa_F^{-1})^\sharp[[G]]^\delta\otimes_{\Lambda[[G]]}\Lambda[[G^+]]}(\mathcal{L}_{F^+_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))),$$
$$\mathcal{L}^{\otimes,-}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet(1)) \coloneqq \Psi_{\Lambda(\kappa_F)^\sharp[[G]]^\delta\otimes_{\Lambda[[G]]}\Lambda[[G^+]]}(\mathcal{L}^\otimes_{F^+_\infty/F}(W,k_!\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$. We extend this definition to CM-admissible subextensions F'_∞/F with F'_∞ not containing the p-th roots of unity by taking the image of the elements under

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F'_\infty/F)]]_S).$$

Furthermore, for $\varepsilon \in \{+,-\}$, $x \in W$ and $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ we set

$$\mathcal{L}^\varepsilon_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet) \coloneqq \Psi_{e_\varepsilon\Lambda[[G]]}(\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet)),$$
$$\mathcal{L}^{\otimes,\varepsilon}_{F_\infty/F}(x,k_!\mathcal{F}^\bullet) \coloneqq \Psi_{e_\varepsilon\Lambda[[G]]}(\mathcal{L}^\otimes_{F_\infty/F}(x,k_!\mathcal{F}^\bullet)).$$

We will write $-\varepsilon \in \{+,-\}$ for the opposite sign.

Assume that $C/\mathbb{Q}_p$ is a finite field extension and $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation unramified over U. If $\rho(\sigma) = -\mathrm{id}$ for every complex conjugation $\sigma \in \mathrm{Gal}_F$, then ρ factors through a CM-extension of F. In particular, $\mathcal{M}(\rho\omega_F^{-1})$ is smooth on U and at ∞ and we may define elements

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\mathcal{M}(\rho\omega_F^n)(-n)), \quad \mathcal{L}^\otimes_{F_{\mathrm{cyc}}/F}(W,k_!\mathcal{M}(\rho\omega_F^n)(1-n)) \qquad (15.2)$$

by identifying $\mathcal{M}(\rho\omega_F^n)(-n)$ with $\mathcal{M}(\rho\omega_F^{-1}\omega_F^{n+1})(1-(n+1))$. In particular,

$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\mathcal{M}(\rho\omega_F^n)(-n))) = L_{\Sigma,\mathrm{T}}(\rho\omega_F^n,-n),$$
$$\varphi(\mathcal{L}^\otimes_{F_{\mathrm{cyc}}/F}(W,k_!\mathcal{M}(\rho\omega_F^{-n})(1-n))) = L_{\Sigma,\mathrm{T}}(\check{\rho}\omega_F^n,-n)^{-1}$$

with $\Sigma \coloneqq X - W$, $\mathrm{T} \coloneqq W - U$ and any integer $n > 0$. If ρ is any Artin representation that factors through a CM-extension, then we can decompose it as in (15.1) into two subrepresentations ρ_+ and ρ_- such that

$$\rho_+(\sigma) = \mathrm{id}, \qquad \rho_-(\sigma) = -\mathrm{id}$$

for all complex conjugations $\sigma \in \mathrm{Gal}_F$.

COROLLARY 15.3. *Let F be a totally real field, $k : U \to W$ be an open immersion of open dense subschemes of $X = \mathrm{Spec}\, \mathcal{O}_F$ such that p is invertible on W, and Λ be an adic $\mathbb{Z}_p$-algebra. Fix a $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, and a CM-admissible extension F_∞ / F unramified over U. If $\varepsilon = +$, we choose n to be an even integer. We choose n to be odd if $\varepsilon = -$.*

1. *Write $f : U_{F_\infty} \to U$ for the system of coverings associated to F_∞ / F. Then*

$$d\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1+n)) = -[e_\varepsilon\, \mathrm{R}\, \Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet(1+n))],$$
$$d\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n)) = [e_\varepsilon\, \mathrm{R}\, \Gamma(W, k_! f_! f^* \mathcal{F}^\bullet(n))]$$

2. *If $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ are weakly equivalent in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, then*

$$\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1+n)) = \mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{G}^\bullet(1+n)),$$
$$\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n)) = \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{G}^\bullet(n)).$$

3. *If $0 \to \mathcal{F}'^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}''^\bullet \to 0$ is an exact sequence in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, then*

$$\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1+n)) = \mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}'^\bullet(1+n))$$
$$\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}''^\bullet(1+n)),$$
$$\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n)) = \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}'^\bullet(n)) \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}''^\bullet(n)).$$

4. *If W' is an open dense subscheme of X on which p is invertible and $k' : W \to W'$ is an open immersion, then*

$$\mathcal{L}^\varepsilon_{F_\infty/F}(W', \mathrm{R}(k'k)_* \mathcal{F}^\bullet(1+n)) = \mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet(1+n))$$
$$\prod_{x \in W'-W} \mathcal{L}^\varepsilon_{F_\infty/F}(x, \mathrm{R}(k'k)_* \mathcal{F}^\bullet(1+n)),$$
$$\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W', (k'k)_! \mathcal{F}^\bullet(n)) = \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n))$$
$$\prod_{x \in W'-W} \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(x, (k'k)_! \mathcal{F}^\bullet(n)).$$

5. *If $i : x \to U$ is a closed point, then*

$$\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\, k_* i_* i^* \mathcal{F}^\bullet(1+n)) = \mathcal{L}_{F_\infty/F}(x, \mathcal{F}^\bullet(1+n)),$$
$$\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! i_* \mathrm{R}\, i^! \mathcal{F}^\bullet(n)) = \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(x, \mathcal{F}^\bullet(n)).$$

6. If F'_∞/F is a CM-admissible subextension of F_∞/F, then

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n))) = \mathcal{L}^\varepsilon_{F'_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n))) = \mathcal{L}^{\oplus,\varepsilon}_{F'_\infty/F}(W, k_! \mathcal{F}^\bullet(n)).$$

If F'_∞/F is a really admissible subextension of F_∞/F, then

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^+_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n))) = \mathcal{L}_{F'_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\oplus,+}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n))) = \mathcal{L}^\oplus_{F'_\infty/F}(W, k_! \mathcal{F}^\bullet(n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^-_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n))) = 1$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\oplus,-}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n))) = 1.$$

7. If Λ' is another adic $\mathbb{Z}_p$-algebra and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then

$$\Psi_{P_+[[\mathrm{Gal}(F_\infty/F)]]}{}^{\delta\bullet}(\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n))) =$$
$$\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* \Psi_{\widetilde{P_+}^\bullet}(\mathcal{F}^\bullet)(1+n)),$$
$$\Psi_{P_-[[\mathrm{Gal}(F_\infty/F)]]}{}^{\delta\bullet}(\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n))) =$$
$$\mathcal{L}^{-\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_* \Psi_{\widetilde{P_-}^\bullet}(\mathcal{F}^\bullet)(1+n)),$$
$$\Psi_{P_+[[\mathrm{Gal}(F_\infty/F)]]}{}^{\delta\bullet}(\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n))) =$$
$$\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \Psi_{\widetilde{P_+}^\bullet}(\mathcal{F}^\bullet(n))),$$
$$\Psi_{P_-[[\mathrm{Gal}(F_\infty/F)]]}{}^{\delta\bullet}(\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet(n))) =$$
$$\mathcal{L}^{\oplus,-\varepsilon}_{F_\infty/F}(W, k_! \Psi_{\widetilde{P_-}^\bullet}(\mathcal{F}^\bullet(n))).$$

8. If F'/F is a finite extension inside F_∞ such that F' is totally real and $f_{F'}: U_{F'} \to U$ is the associated covering, then

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^\bullet(1+n))) = \mathcal{L}^\varepsilon_{F_\infty/F'}(W_{F'}, \mathrm{R}\,k_* f^*_{F'} \mathcal{F}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)(n)) = \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F'}(W_{F'}, k_! f^*_{F'} \mathcal{F}^\bullet(n)).$$

9. With the notation of (8), if $\mathcal{G}^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'}, \Lambda)$, then

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^\varepsilon_{F_\infty/F'}(W_{F'}, \mathrm{R}\,k_* \mathcal{G}^\bullet(1+n))) = \mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* f_{F'*} \mathcal{G}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F'}(W_{F'}, k_! \mathcal{G}^\bullet(n))) = \mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! f_{F'*} \mathcal{G}^\bullet(n)).$$

10. If $\mathcal{F}$ is a smooth Λ-adic sheaf on U which is smooth at ∞, then

$$\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, k_! \mathcal{F}(n)) = (\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_* \mathcal{F}^{*\Lambda}(1-n)))^\oplus.$$

11. *If $C/\mathbb{Q}_p$ is a finite field extension and $\rho\colon\mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation, then*

$$\Phi_\rho(\mathcal{L}^\varepsilon_{F_\infty/F}(W, \mathrm{R}\,k_*(\mathbb{Z}_p)_U(1+n))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\,k_*\mathcal{M}(\rho_\varepsilon)(1+n)),$$
$$\Phi_\rho(\mathcal{L}^{\oplus,\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*(\mathbb{Z}_p)_U(n))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\,k_*\mathcal{M}(\rho_\varepsilon)(n)),$$

Proof. This is an easy consequence of the preceding remarks and Corollary 14.9.
□

16   Calculation of the cohomology

We retain the notation from the previous section. Our objective is to investigate the cohomology of the complexes

$$\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*f_!f^*\mathcal{F}(1)), \quad \mathrm{R}\,\Gamma(W, k_!f_!f^*\mathcal{F}), \quad \mathrm{R}\,\Gamma(\Sigma, i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{F})$$

for a Λ-adic sheaf $\mathcal{F}$ on U in order to tie the connection to classical objects in Iwasawa theory.

The following two propositions are direct consequences of Proposition 4.1, Theorem 10.2, and Corollary 15.2.

PROPOSITION 16.1. *Let F_∞/F be any admissible extension unramified over U. Assume that $i\colon x \to W$ is a closed point not lying over p. Then*

$$\mathrm{H}^s(x, i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{F}) \cong \varprojlim_{F'} \mathrm{H}^{s-1}(x_{F'_{\mathrm{cyc}}}, i^*\,\mathrm{R}\,k_*\mathcal{F})$$

where F' runs through the finite subextensions of F_∞/F and $x_{F'_{\mathrm{cyc}}} = x \times_W W_{F'_{\mathrm{cyc}}}$ contains the places of F'_{cyc} lying over x. In particular,

$$\mathrm{H}^s(x, i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{F}) = 0$$

for $s \neq 1$ if $x \in U$ and for $s \neq 1, 2$ if $x \in W - U$.

PROPOSITION 16.2. *Let F be totally real and F_∞/F be a CM-admissible extension unramified over U. Assume that p is invertible on W and that $\mathcal{F}$ is smooth at ∞. If Conjecture 9.5 is valid, then*

$$e_+\,\mathrm{H}^s_c(W, \mathrm{R}\,k_*f_!f^*\mathcal{F}(1)) \cong \varprojlim_{F'} e_+\,\mathrm{H}^{s-1}_c(W_{F'_{\mathrm{cyc}}}, \mathrm{R}\,k_*\mathcal{F}(1))$$

$$e_-\,\mathrm{H}^s_c(W, \mathrm{R}\,k_*f_!f^*\mathcal{F}) \cong \varprojlim_{F'} e_-\,\mathrm{H}^{s-1}_c(W_{F'_{\mathrm{cyc}}}, \mathrm{R}\,k_*\mathcal{F})$$

$$e_+\,\mathrm{H}^s(W, k_!f_!f^*\mathcal{F}) \cong \varprojlim_{F'} e_+\,\mathrm{H}^{s-1}(W_{F'_{\mathrm{cyc}}}, k_!\mathcal{F})$$

$$e_-\,\mathrm{H}^s(W, k_!f_!f^*\mathcal{F}(1)) \cong \varprojlim_{F'} e_-\,\mathrm{H}^{s-1}(W_{F'_{\mathrm{cyc}}}, k_!\mathcal{F}(1))$$

where F' runs through the finite subextensions of F_∞/F. In particular, if $\mathcal{F}$ is smooth over U,

1. $e_+ \operatorname{H}_c^s(W, \operatorname{R} k_* f_! f^* \mathcal{F}(1)) = e_- \operatorname{H}_c^s(W, \operatorname{R} k_* f_! f^* \mathcal{F}) = 0$ *for* $s \neq 2$ *if* $U \neq W$ *and for* $s \neq 2, 3$ *if* $W = U$.

2. $e_+ \operatorname{H}^s(W, k_! f_! f^* \mathcal{F}) = e_- \operatorname{H}^s(W, k_! f_! f^* \mathcal{F}(1)) = 0$ *for* $s \neq 2$ *if* $U \neq W$ *or if* $U = W$ *and* $F_\infty / F_{\mathrm{cyc}}$ *is infinite and for* $s \neq 1, 2$ *if* $U = W$ *and* F_∞ / F *is finite.*

In particular, we obtain the following corollary. We will explain in the next section in what sense this is a generalisation of [GP15, Thm. 4.6].

COROLLARY 16.3. *Let* F *be totally real and* F_∞ / F *be a CM-admissible extension unramified over* U. *Assume that* p *is invertible on* $W \neq U$ *and that* $\mathcal{F}$ *is smooth over* U *and at* ∞. *If Conjecture 9.5 is valid, then*

$$
\begin{aligned}
e_+ \operatorname{H}_c^2(W, \operatorname{R} k_* f_! f^* \mathcal{F}(1)), & \qquad e_- \operatorname{H}_c^2(W, \operatorname{R} k_* f_! f^* \mathcal{F}), \\
e_+ \operatorname{H}^2(W, k_! f_! f^* \mathcal{F}), & \qquad e_- \operatorname{H}^2(W, k_! f_! f^* \mathcal{F}(1))
\end{aligned}
$$

are finitely generated and projective as $\Lambda[[H]]$-*modules. In particular, they have strictly perfect resolutions of length equal to* 1 *as* $\Lambda[[G]]$-*modules. Hence, we may consider their classes in* $\operatorname{K}_0(\Lambda[[G]], S)$ *and obtain*

$$
\begin{aligned}
[e_+ \operatorname{H}_c^2(W, \operatorname{R} k_* f_! f^* \mathcal{F}(1))] &= [e_+ \operatorname{R} \Gamma_{\mathrm{c}}(W, \operatorname{R} k_* f_! f^* \mathcal{F}(1))], \\
[e_- \operatorname{H}_c^2(W, \operatorname{R} k_* f_! f^* \mathcal{F})] &= [e_- \operatorname{R} \Gamma_{\mathrm{c}}(W, \operatorname{R} k_* f_! f^* \mathcal{F})], \\
[e_+ \operatorname{H}^2(W, k_! f_! f^* \mathcal{F})] &= [e_+ \operatorname{R} \Gamma(W, k_! f_! f^* \mathcal{F})], \\
[e_- \operatorname{H}^2(W, k_! f_! f^* \mathcal{F}(1))] &= [e_- \operatorname{R} \Gamma(W, k_! f_! f^* \mathcal{F}(1))].
\end{aligned}
$$

Proof. We give the argument for $X \coloneqq e_+ \operatorname{H}^2(W, k_! f_! f^* \mathcal{F})$; the proof of the other cases is essentially the same. The $\Lambda[[G]]$-module X is the only non-vanishing cohomology group of the perfect complex of $\Lambda[[G]]$-modules

$$
P^\bullet \coloneqq \varprojlim_{I \in \mathfrak{J}_{\Lambda[[G]]}} e_+ \operatorname{R} \Gamma(W, k_!(f_! f^* \mathcal{F})_I).
$$

Since for any simple $\Lambda[[G]]$-module M,

$$
M \otimes_{\Lambda[[G]]}^{\mathbb{L}} P^\bullet \sim e_+ \operatorname{R} \Gamma(W, k_! M \otimes_{\Lambda[[G]] / \operatorname{Jac}(\Lambda[[G]])} (f_! f^* \mathcal{F})_{\operatorname{Jac}(\Lambda[[G]])})
$$

has no cohomology except in degrees 1 and 2, we conclude that there exists a strictly perfect complex $Q^\bullet$ of $\Lambda[[G]]$-modules concentrated in degrees -1 and 0 and quasi-isomorphic to X. By Corollary 15.2, we know that $Q^\bullet$ is also perfect as complex of $\Lambda[[H]]$-modules. By Lemma 3.4, we conclude that X is finitely generated and projective as $\Lambda[[H]]$-module.
We then have

$$
[X] = [Q^\bullet] = [e_+ \operatorname{R} \Gamma(W, k_! f_! f^* \mathcal{F})]
$$

in $\operatorname{K}_0(\Lambda[[G]], S)$. $\qquad\square$

17 Realisations of abstract 1-motives

As a central result of this section, we want to establish the link with the theory of abstract 1-motives considered in [GP15].
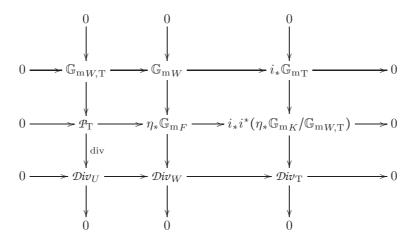
Assume that F is any number field and let $U \subset W$ be two open dense subschemes of $X = \operatorname{Spec} \mathcal{O}_F$. Write $k \colon U \to W$ for the corresponding open immersion. Fix a closed subscheme structure on the complement T of U in W and write $i \colon T \to W$ for the closed immersion. For any scheme S we write $\mathbb{G}_{\mathrm{m}S}$ for the étale sheaf of units on S. Recalling that the stalk of $\mathbb{G}_{\mathrm{m}W}$ in a geometric point over a closed point is given by the units of the strict henselisation of the local ring in this closed point, we see that

$$\mathbb{G}_{\mathrm{m}W} \to i_* \mathbb{G}_{\mathrm{m}T}.$$

is a surjection. We define $\mathbb{G}_{\mathrm{m}W,\mathrm{T}}$ to be its kernel. Write $\eta \colon \operatorname{Spec} F \to W$ for the generic point and set

$$\mathcal{P}_\mathrm{T} \coloneqq \ker \eta_* \mathbb{G}_{\mathrm{m}F} \to i_* i^* (\eta_* \mathbb{G}_{\mathrm{m}F}/\mathbb{G}_{\mathrm{m}W,\mathrm{T}}).$$

In other words, $\mathcal{P}_\mathrm{T}$ is the subsheaf of $\eta_* \mathbb{G}_{\mathrm{m}F}$ of elements congruent to 1 modulo T. For any subscheme Z of W we let $\mathcal{D}iv_Z$ denote the sheaf of divisors on W with support on Z. Hence, we obtain an exact 9-diagram

The third row is clearly also exact in the category of presheaves on W. Moreover,

$$i_* i^* (\eta_* \mathbb{G}_{\mathrm{m}F}/\mathbb{G}_{\mathrm{m}W,\mathrm{T}})(W) = \bigoplus_{v \in \mathrm{T}} F_v^\times / \mathcal{U}_{v,n_v}$$

where n_v is the multiplicity of v in T, F_v is the completion of F at v and $\mathcal{U}_{n,n_v} \subset F_v$ is the group of units f such that the valuation of $1 - f$ is larger or equal to n_v. From the weak approximation theorem we conclude that the second row of the 9-diagram is also exact in the category of presheaves. The

same is also true for the third column. Hilbert 90 and the Leray spectral sequence then imply that

$$\mathrm{H}^1(W, \mathcal{P}_\mathrm{T}) = \mathrm{H}^1(W, \eta_* \mathbb{G}_{\mathrm{m}F}) = 0.$$

We conclude that

$$\mathrm{H}^1(W, \mathbb{G}_{\mathrm{m}W,\mathrm{T}}) = \operatorname{coker} \mathcal{P}_\mathrm{T}(W) \xrightarrow{\operatorname{div}} \mathcal{D}iv_U(W)$$
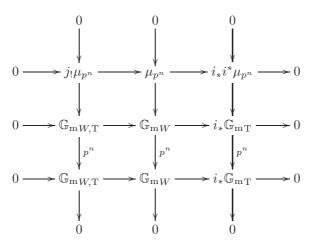
is the ray class group of W with respect to the modulus T. If K/F is a possibly infinite algebraic extension of F, it follows from [AGV72a, VII, Cor. 5.8] that

$$\mathrm{H}^1(W_K, \mathbb{G}_{\mathrm{m}W_K,\mathrm{T}_K}) = \operatorname{coker}\left(\mathcal{P}_{\mathrm{T}_K}(W_K) \to \varinjlim_{K' \subset K} \mathcal{D}iv_{U_{K'}}(W_{K'}) \right)$$

with

$$\varinjlim_{K' \subset K} \mathcal{D}iv_{U_{K'}}(W_{K'}) = \bigoplus_v \Gamma_v,$$

where v ranges over the places of K lying over the closed points of U and Γ_v denotes the value group of the associated, possibly non-discrete valuation. Assume now that p is invertible on W. We then obtain an exact 9-diagram

and hence, an exact sequence

$$0 \to j_! \mu_{p^n} \to \mathcal{P}_\mathrm{T} \xrightarrow{(p^n, \operatorname{div})} \mathcal{P}_\mathrm{T} \oplus \mathcal{D}iv_U \xrightarrow{\left(\begin{smallmatrix} \operatorname{div} \\ -p^n \end{smallmatrix}\right)} \mathcal{D}iv_U \to 0.$$

We take global sections on W. Since $\mathrm{H}^1(W, \mathcal{P}_\mathrm{T}) = 0$ and since multiplication by p^n is injective on $\mathcal{D}iv_U(W)$ we obtain

$$\mathrm{H}^1(W, j_! \mu_{p^n}) = \left\{ f \in \mathcal{P}_\mathrm{T}(W) \left| \begin{array}{l} \operatorname{div}(f) = p^n D, \\ D \in \mathcal{D}iv_U(W) \end{array} \right\} \middle/ \{ g^{p^n} \mid g \in \mathcal{P}_\mathrm{T}(W) \}.\right.$$

Note that this group does not depend on the subscheme structure of T. So, we might as well consider it with the reduced scheme structure.

We now assume in addition that F is totally real and fix a CM-admissible extension F_∞/F such that $F_\infty/F_{\mathrm{cyc}}$ is finite. Passing to the direct limit over all finite subextensions F'/F of F_∞/F, we obtain

$$\mathrm{H}^1(W_{F_\infty}, j_!\mu_{p^n}) =$$

$$\left\{ f \in \mathscr{P}_{\mathrm{T}_{F_\infty}}(W_{F_\infty}) \;\middle|\; \begin{matrix} \mathrm{div}(f) = p^n D, \\ D \in \mathscr{D}iv_{U_{F_\infty}}(W_{F_\infty}) \end{matrix} \right\} \Big/ \{ g^{p^n} \mid g \in \mathscr{P}_{\mathrm{T}_{F_\infty}}(W_{F_\infty}) \}. \tag{17.1}$$

Write Σ for the complement of W in $\mathrm{Spec}\,\mathcal{O}_F[\frac{1}{p}]$. The Iwasawa-theoretic 1-motive associated to $(F_\infty, \Sigma_{F_\infty}, \mathrm{T}_{F_\infty})$ is the complex of abelian groups

$$\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}: \qquad \mathscr{D}iv_{\Sigma_{F_\infty}}(X_{F_\infty}) \xrightarrow{\delta} \mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

sitting in degrees 0 and 1 [GP15, §3.1]. Its group of p^n-torsion points is defined to be

$$\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[p^n] := \left\{ (D, c) \;\middle|\; \begin{matrix} D \in \mathscr{D}iv_{\Sigma_{F_\infty}}(X_{F_\infty}), \\ c \in \mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p, \\ \delta(D) = p^n c \end{matrix} \right\} \otimes_{\mathbb{Z}} \mathbb{Z}/(p^n)$$

$$= \mathrm{H}^0(\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^n))$$

and its p-adic Tate module is given by

$$\mathrm{T}_p\,\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}} := \varprojlim_n \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[p^n]$$

[GP15, Def. 2.2, Def. 2.3].

Remark 17.1. The complex of abelian groups $\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ is an abstract 1-motive in the sense of [GP15, Def. 2.1] only if $\mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is divisible of finite corank. The proof of [GP15, Lemma 2.8] shows that this is true if and only if $\mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is divisible of finite corank. By [NSW00, Thm. 11.1.8] this is equivalent to the Galois group $\mathcal{X}_{\mathrm{nr}}(F_\infty)$ of the maximal abelian unramified pro-p-extension of F_∞ being a finitely generated $\mathbb{Z}_p$-module. This is true if $\mathcal{X}_{\mathrm{nr}}(F_\infty(\mu_p))$ is a finitely generated $\mathbb{Z}_p$-module. By [Was97, Thm 13.24] the latter statement is equivalent to $e_-\mathcal{X}_{\mathrm{nr}}(F_\infty(\mu_p))$ being finitely generated over $\mathbb{Z}_p$, which is in turn equivalent to the Galois group of the maximal abelian pro-p-extension unramified outside the primes over p of the maximal totally real subfield $F_\infty(\mu_p)^+$ being finitely generated over $\mathbb{Z}_p$ [NSW00, Cor. 11.4.4]. Hence, $\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ is an abstract 1-motive under Conjecture 9.5.

PROPOSITION 17.2. *There is a short exact sequence*

$$0 \to \mathrm{H}^0(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}/(p^n) \to \mathrm{H}^1(W_{F_\infty}, k_!\mu_{p^n}) \to \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[p^n] \to 0.$$

In particular, there are isomorphisms

$$e_- \, \mathrm{H}^1(W_{F_\infty}, k_! \mu_{p^n}) \cong e_- \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[p^n],$$

$$e_- \, \mathrm{H}^1(W_{F_\infty}, k_!(\mathbb{Z}_p)_{W_{F_\infty}}(1)) \cong e_- \, \mathrm{T}_p \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}.$$

Proof. This follows from (17.1) and [GP15, Prop. 3.2, Cor. 3.4]. Note that the proofs of these statements do not make use of the divisibility of the group $\mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_\mathbb{Z} \mathbb{Z}_p$. □

Remark 17.3. We refer to [GP17] for related work of Greither and Popescu.

Writing again $f \colon U_{F_\infty} \to U$ for the system of coverings associated to F_∞/F we conclude from Proposition 16.2:

COROLLARY 17.4. *Assume that F_∞/F is unramified over U. Under Conjecture 9.5, there are isomorphisms*

$$e_- \, \mathrm{H}^2(W, k_! f_! f^* \mu_{p^n}) \cong e_- \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[p^n],$$

$$e_- \, \mathrm{H}^2(W, k_! f_!(\mathbb{Z}_p)_U(1)) \cong e_- \, \mathrm{T}_p \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}.$$

In particular,

$$d\mathcal{L}^{\oplus,-}_{F_\infty/F}(W, k_!(\mathbb{Z}_p)_U(1)) = \left[e_- \, \mathrm{T}_p \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}} \right]$$

in $\mathrm{K}_0(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]], S)$.

Proof. Combine Proposition 17.2 with Proposition 16.2 and use Corollary 15.3. □

In particular, [GP15, Thm. 4.6] reduces to the special case $\mathcal{F} = (\mathbb{Z}_p)_U$ of Corollary 16.3. Moreover, if $\mathrm{Gal}(F_\infty/F)$ is commutative, we may identify the Fitting ideal and the characteristic ideal of $e_- \, \mathrm{T}_p \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ over $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$. The characteristic ideal may then be viewed as an element of

$$(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]_S)^\times / \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]^\times \cong \mathrm{K}_0(\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]], S).$$

Under this identification, it corresponds to the class $\left[e_- \, \mathrm{T}_p \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}} \right]^{-1}$. Furthermore, the interpolation property (11) in Corollary 15.3 shows that the element $\mathcal{L}^{\oplus,-}_{F_\infty/F}(W, k_!(\mathbb{Z}_p)_U(1))^{-1}$ agrees with the element $e_+ + \theta^{(\infty)}_{\Sigma, \mathrm{T}}$ with $\Sigma \coloneqq X - W$, $\mathrm{T} \coloneqq W - U$ in the notation of [GP15, Def. 5.16]. In particular, we recover the version of the equivariant main conjecture formulated in [GP15, Thm. 5.6] as a special case of Corollary 17.4.

In the same way, one can also recover its non-commutative generalisation in [Nic13, Thm. 3.3], which states that Nickel's non-commutative Fitting invariant of $e_- \, \mathrm{T}_\ell \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ is generated by the reduced norm of $\mathcal{L}^{\oplus,-}_{F_\infty/F, \Sigma, \mathrm{T}}(\mathbb{Z}_\ell(1))$. By the argument before [Nic13, Conj. 2.1], this is in fact equivalent to Cor. 17.4. However, Nickel only considers the case that F_∞/F is unramified over W.

With some technical effort, one can further extend Prop. 17.2 and Cor. 17.4 by allowing U to contain a finite number of points x such that F_∞/F is ramified over x, but with a ramification index prime to ℓ. The interested reader may consult [Wit17, Ch. 5, Ch. 6] for a detailed exposition, which parallels the discussion in [Wit13c] on the same phenomenon in the function field case.

Appendix: Localisation in polynomial rings

Let R be any associative ring with 1 and let $R[t]$ be the polynomial ring over R in one indeterminate t that commutes with the elements of R. Write $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$ for the Waldhausen categories of strictly perfect and perfect complexes of $R[t]$-modules. Consider R as a R-$R[t]$-bimodule via the augmentation map

$$R[t] \to R, \qquad t \mapsto 0.$$

We then define full subcategories

$$\mathbf{SP}^{w_t}(R[t]) \coloneqq \{P^\bullet \in \mathbf{SP}(R[t]) \mid R \otimes_{R[t]} P^\bullet \text{ is acyclic}\},$$
$$\mathbf{P}^{w_t}(R[t]) \coloneqq$$
$$\{P^\bullet \in \mathbf{P}(R[t]) \mid P^\bullet \text{ is quasi-isomorphic to a complex in } \mathbf{SP}^{w_t}(R[t])\}.$$

These categories are in fact Waldhausen subcategories of $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$, respectively, since they are closed under shifts and extensions [Wit08, 3.1.1]. We can then construct new Waldhausen categories $w_t\mathbf{SP}(R[t])$ and $w_t\mathbf{P}(R[t])$ with the same objects, morphisms, and cofibrations as $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$, but with weak equivalences being those morphisms with cone in $\mathbf{SP}^{w_t}(R[t])$ and $\mathbf{P}^{w_t}(R[t])$, respectively. By the approximation theorem [TT90, 1.9.1], the inclusion functor $w_t\mathbf{SP}(R[t]) \to w_t\mathbf{P}(R[t])$ induces isomorphisms

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \mathrm{K}_n(w_t\mathbf{P}(R[t]))$$

for all $n \geq 0$.

It might be reassuring to know that, if R is noetherian, we can identify these K-groups for $n \geq 1$ with the K-groups of a localisation of $R[t]$: Set

$$S_t \coloneqq \{f(t) \in R[t] \mid f(0) \in R^\times\}$$

PROPOSITION A.1. *Assume that R is noetherian. Then S_t is a left (and right) denominator set in the sense of [GW04, Ch. 10] such that the localisation $R[t]_{S_t}$ exists and is noetherian. Its Jacobson radical $\mathrm{Jac}(R[t]_{S_t})$ is generated by the Jacobson radical $\mathrm{Jac}(R)$ of R and t. In particular, if R is semi-local, then so is $R[t]_{S_t}$.*
Moreover, the category $\mathbf{SP}^{w_t}(R[t])$ consists precisely of those complexes $P^\bullet$ in $\mathbf{SP}(R[t])$ with S_t-torsion cohomology. In particular,

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \mathrm{K}_n(R[t]_{S_t})$$

for $n \geq 1$.

Proof. Clearly, the set S_t consists of non-zero divisors, such that we only need to check the Ore condition:

$$\forall s \in S_t \colon \forall a \in R[t] \colon \exists x \in R[t] \colon \exists y \in S_t \colon xs = ya.$$

Moreover, we may assume that $s(0) = y(0) = 1$. Write

$$s = 1 - \sum_{i=1}^{\infty} s_i t^i, \quad a = \sum_{i=0}^{\infty} a_i t^i, \quad x = \sum_{i=0}^{\infty} x_i t^i, \quad y = 1 + \sum_{i=1}^{\infty} y_i t^i.$$

and assume that $s_i = a_{i-1} = 0$ for $i > n$. Comparing coefficients, we obtain the recurrence equation

$$(\ast) \qquad x_i = \sum_{j=0}^{i-1} x_j s_{i-j} + \sum_{j=1}^{i} y_j a_{i-j} + a_i = \sum_{j=1}^{i} y_j b_{i-j} + b_i$$

with

$$b_i \coloneqq \sum_{j=0}^{i-1} b_j s_{i-j} + a_i.$$

Write $B_i \coloneqq (b_{i-n+1}, \dots, b_i) \in R^n$ with the convention that $b_i = 0$ for $i < 0$. Then for $i \geq n$

$$B_i = B_{i-1} S = B_{n-1} S^{i-n+1}$$

with

$$S \coloneqq \begin{pmatrix} 0 & \cdots & \cdots & s_n \\ 1 & \ddots & & s_{n-1} \\ 0 & \ddots & 0 & \vdots \\ \vdots & & 1 & s_1 \end{pmatrix}.$$

Since R was assumed to be noetherian, there exists a $m \geq n$ and $y_n, \dots, y_m \in R$ such that

$$0 = \sum_{j=n}^{m} y_j B_{m-j} + B_m = \sum_{j=n}^{m} y_j B_{i-j} + B_i$$

for all $i \geq m$. Hence, we can find a solution $(x_i, y_i)_{i=0,1,2\dots}$ of equation $(\ast)$ with $x_i = y_i = 0$ for $i > m$ and $y_i = 0$ for $i < n$. This shows that S_t is indeed a left denominator set such that $R[t]_{S_t}$ exists and is noetherian [GW04, Thm. 10.3, Cor. 10.16].

Let $N \subset R[t]$ be the semi-prime ideal of $R[t]$ generated by t and the Jacobson ideal $\mathrm{Jac}(R)$ of R. Then S_t is precisely the set of elements of $\Lambda[t]$ which are units modulo N. In particular, the localisation N_{S_t} is a semi-prime ideal of $R[t]_{S_t}$ such that

$$R[t]_{S_t}/N_{S_t} = R[t]/N = R/\mathrm{Jac}(R)$$

[GW04, Thm. 10.15, 10.18]. We conclude $\mathrm{Jac}(R[t]_{S_t}) \subset N_{S_t}$. For the other inclusion it suffices to note that for every $s \in S_t$ and every $n \in N$, the element $s + n$ is a unit modulo N.

The Nakayama lemma implies that for any noetherian ring R with Jacobson radical $\mathrm{Jac}(R)$, a strictly perfect complex of R-modules $P^\bullet$ is acyclic if and only if $R/\mathrm{Jac}(R) \otimes_R P^\bullet$ is acyclic. Hence, if $P^\bullet$ is a strictly perfect complex of $R[t]$-modules, then $R \otimes_{R[t]} P^\bullet$ is acyclic if and only if $R[t]_{S_t} \otimes_R P^\bullet$ is acyclic. This shows that $\mathbf{SP}^{w_t}(R[t])$ consists precisely of those complexes $P^\bullet$ in $\mathbf{SP}(R[t])$ with S_t-torsion cohomology. From the localisation theorem in [WY92] we conclude that the Waldhausen exact functor

$$w_t\mathbf{SP}(R[t]) \to \mathbf{SP}(R[t]_{S_t}), \qquad P^\bullet \mapsto R[t]_{S_t} \otimes_{R[t]} P^\bullet$$

induces isomorphisms

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \begin{cases} \mathrm{K}_n(R[t]_{S_t}) & \text{if } n > 0, \\ \mathrm{im}\,(\mathrm{K}_0(R[t]) \to \mathrm{K}_0(R[t]_{S_t})) & \text{if } n = 0. \end{cases}$$

$\square$

The set S_t fails to be a left denominator set if $R = \mathbb{F}_p\langle\langle x, y\rangle\rangle$ is the power series ring in two non-commuting indeterminates: $a(1 - xt) = by$ has no solution with $a \in R[t]$, $b \in S_t$. Note also that a commutative adic ring is always noetherian [War93, Cor. 36.35]. In this case, S_t is the union of the complements of all maximal ideals of $\Lambda[t]$ containing t and the determinant provides an isomorphism

$$\mathrm{K}_1(w_t\mathbf{SP}(\Lambda[t])) \cong \mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow[\cong]{\det} \Lambda[t]_{S_t}^\times.$$

For any adic $\mathbb{Z}_p$-algebra Λ and any $\gamma \in \Gamma \cong \mathbb{Z}_p$, we have a ring homomorphism

$$\mathrm{ev}_\gamma\colon \Lambda[t] \mapsto \Lambda[[\Gamma]], \quad f(t) \mapsto f(\gamma).$$

inducing homomorphisms $\mathrm{K}_n(\Lambda[t]) \to \mathrm{K}_n(\Lambda[[\Gamma]])$.

PROPOSITION A.2. *Assume that $\gamma \neq 1$. Then the ring homomorphism ev_γ induces homomorphisms*

$$\mathrm{ev}_\gamma\colon \mathrm{K}_n(w_t\mathbf{P}(\Lambda[[t]])) \cong \mathrm{K}_n(w_t\mathbf{SP}(\Lambda[t])) \to \mathrm{K}_n(\Lambda[[\Gamma]]_S)$$

for all $n \geq 0$.

Proof. It suffices to show that for any complex $P^\bullet$ in $\mathbf{SP}^{w_t}(\Lambda[t])$, the complex

$$Q^\bullet := \Lambda[[\Gamma]] \otimes_{\Lambda[t]} P^\bullet$$

is perfect as complex of Λ-modules. We can check this after factoring out the Jacobson radical of Λ [Wit14, Prop. 4.8]. Hence, we may assume that Λ is semi-simple, i. e.

$$\Lambda = \prod_{i=1}^m M_{n_i}(k_i)$$

where $M_{n_i}(k_i)$ is the algebra of $n_i \times n_i$-matrices over a finite field k_i of characteristic p. By the Morita theorem, the tensor product over Λ with the $\prod_i k_i$-Λ-bimodule

$$\prod_{i=1}^{m} k_i^{n_i}$$

induces equivalences of categories

$$\mathbf{SP}^{w_t}(\Lambda[t]) \to \mathbf{SP}^{w_t}\left(\prod_{i=1}^{n} k_i[t]\right),$$

$$\mathbf{SP}^{w_H}(\Lambda[[\Gamma]]) \to \mathbf{SP}^{w_H}(\prod_{i=1}^{n} k_i[[\Gamma]]),$$

with $H \subset \Gamma$ being the trivial subgroup. Hence, we are reduced to the case

$$\Lambda = \prod_{i=1}^{m} k_i.$$

In this case, the set $S \subset \Lambda[[\Gamma]]$ defined in (3.1) consists of all non-zero divisors of $\Lambda[[\Gamma]]$, i. e. all elements with non-trivial image in each component $k_i[[\Gamma]]$. Since $\Lambda[[\Gamma]]$ is commutative, this is trivially a left denominator set. Moreover, the complex $Q^\bullet$ is perfect as complex of Λ-modules precisely if its cohomology groups are S-torsion. On the other hand, as a trivial case of Proposition A.1, we know that S_t is a left denominator set and that the cohomology groups of $P^\bullet$ are S_t-torsion. Since $f(0)$ is a unit in Λ for each $f \in S_t$, the element $f(\gamma)$ has clearly non-trivial image in each component $k_i[[\Gamma]]$. Hence, ev_γ maps S_t to S and $Q^\bullet$ is indeed perfect as complex of Λ-modules. $\square$

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Malte Witte
Ruprecht-Karls-Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
D-69120 Heidelberg
Germany
witte@mathi.uni-heidelberg.de

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