Spinors and the Tangent Groupoid

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Abstract. The purpose of this article is to study Ezra Getzler's approach to the Atiyah-Singer index theorem from the perspective of Alain Connes' tangent groupoid. We shall construct a "rescaled" spinor bundle on the tangent groupoid, define a convolution operation on its smooth, compactly supported sections, and explain how the algebra so-obtained incorporates Getzler's symbol calculus.

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1 Introduction

In this paper we shall investigate the relationship between Alain Connes' tangent groupoid [Con94, Sec. II.5] and Ezra Getzler's approach to the Atiyah-Singer index theorem for the Dirac operator on a spin manifold [Get83, BGV92]. We shall construct a variation of Connes' convolution algebra for the tangent groupoid that incorporates Getzler's rescaling of Clifford variables. The new algebra carries a family of supertraces that smoothly vary between operator traces and an integral of differential forms (and so in index-theoretic contexts, where the operator traces are integer-valued, the integral of differential forms actually computes the operator trace).

Connes introduced the tangent groupoid in order to conceptualize the construction by Atiyah and Singer [AS68] of the K-theoretic analytic index map,

$$\mathrm{Ind}_a\colon K(T^*M) \longrightarrow \mathbb{Z},$$

and thereby streamline the K-theory proof of the Atiyah-Singer index theorem. In contrast, Getzler's approach to the index theorem was purely local in character, and on the surface at least, quite far removed from global, K-theoretic considerations. So it is an interesting problem to try to harmonize the two approaches. The issue has certainly been considered by others, but as far as we are aware little has been published on this topic. It is the purpose of this paper to help fill this gap.

We should say at the outset that virtually everything that follows is implicit either in Getzler's original work or in Connes' definition of the tangent groupoid. But a fresh looks seems to be worthwhile, especially in view of the new role that these ideas are finding in Bismut's work on the hypoelliptic Laplacian; see for example [Bis11]. The latter was, in fact, an important motivation for us.

The tangent groupoid of a smooth manifold M is, among other things, a smooth manifold $\mathbb{T}M$ equipped with a submersion onto $M\times\mathbb{R}$ via the *source map* that is part of the groupoid structure. The fibers of the source map have the form

$$\mathbb{T}M_{(m,\lambda)} \cong \begin{cases} M & \lambda \neq 0 \\ T_m M & \lambda = 0. \end{cases} \tag{1.0.1}$$

So the tangent groupoid smoothly interpolates between the curved manifold M and its linear tangent spaces.

Now let D be a linear partial differential operator on M. For $m \in M$, denote by D_m the constant-coefficient, linear partial differential *model operator* on the tangent space $T_m M$ that is obtained by freezing the coefficient functions for D in local coordinates at m, and then dropping lower-order terms; the resulting operator is invariantly defined on the tangent space and carries the same information as the principal symbol of D at m. The foundation of the relationship between the tangent groupoid and partial differential operators, and eventually between the tangent groupoid and index theory, is the following result:

Theorem. *If D has order q, then the operators*

$$D_{(m,\lambda)} = \begin{cases} \lambda^q D & \lambda \neq 0 \\ D_m & \lambda = 0 \end{cases}$$

constitute, under the identifications (1.0.1)*, a smooth family of differential operators on the source fibers of the tangent groupoid.*

The theorem is easy to prove, as we shall recall in Section 2. In fact it is more or less incorporated into the *definition* of the tangent groupoid.

Suppose now that M is a Riemannian spin manifold with spinor bundle S. In Section 3 we shall construct from S a smooth vector bundle $\mathbb{S}$ on $\mathbb{T}M$ whose restrictions to the fibers in (1.0.1) are as follows:

$$\mathbb{S}|_{\mathbb{T}M_{(m,\lambda)}} \cong \begin{cases} S \otimes S^*_m & \lambda \neq 0 \\ \wedge^* T_m M & \lambda = 0. \end{cases} \tag{1.0.2}$$

The most important, and indeed defining, feature of $\mathbb{S}$, is its relation to Getzler's filtration of the algebra of linear partial differential operators acting on the sections of the spinor bundle [Get83]. The filtration associates to any operator D a new family of model operators $D_{\langle m\rangle}$ on the tangent spaces $T_m M$. These are typically *not* constant-coefficient operators, and moreover they reflect the Riemannian geometry of M in a rather subtle way. We shall prove the following result.

THEOREM. *If D is a linear partial differential operator on M, acting on the sections of S, and if D has Getzler-order no more than q, then the operators*

$$D_{(m,\lambda)} = \begin{cases} \lambda^q D & \lambda \neq 0 \\ D_{\langle m\rangle} & \lambda = 0 \end{cases}$$

constitute a smooth family of operators on the source-fibers of $\mathbb{T}M$, acting on the sections of the smooth vector bundle $\mathbb{S}$.

Let us return to Connes' work. He constructs a *convolution algebra* $C_c^\infty(\mathbb{T}M)$ of smooth, compactly supported, complex-valued functions on the tangent groupoid that brings the geometric object $\mathbb{T}M$ into close contact with operator theory. The tangent groupoid decomposes into a family of smooth, closed subgroupoids parametrized by $\lambda \in \mathbb{R}$, namely the fibers of the composite submersion

$$\mathbb{T}M \longrightarrow M\times\mathbb{R} \longrightarrow \mathbb{R}.$$

These subgroupoids are

$$\mathbb{T}M_\lambda \cong \begin{cases} M\times M & \lambda \neq 0 \\ TM & \lambda = 0, \end{cases} \tag{1.0.3}$$

where $M\times M$ carries the pair groupoid structure and the tangent bundle TM is a made into a groupoid using the vector group structures on its fibers. By restricting functions on $\mathbb{T}M$ to these subgroupoids, Connes obtains algebra homomorphisms

$$\varepsilon_\lambda \colon C_c^\infty(\mathbb{T}M) \longrightarrow \mathfrak{K}^\infty(L^2(M)) \tag{1.0.4}$$

for $\lambda \neq 0$ and

$$\varepsilon_0 \colon C_c^\infty(\mathbb{T}M) \longrightarrow C_c^\infty(TM), \tag{1.0.5}$$

where $\mathfrak{K}^\infty(L^2(M))$ is the algebra of smoothing operators on $L^2(M)$, and $C_c^\infty(TM)$ is the fiberwise convolution algebra of smooth, compactly supported functions on the tangent bundle. The study of these is the next step in Connes' K-theoretic approach to index theory—not surprisingly so since at the level of K-theory, the morphisms (1.0.4) are related to the analytic index of elliptic operators, whereas (1.0.5) is related to the symbol class in K-theory, and the index theorem is all about relating these two quantities. See [Con94, Sec. II.5] or [Hig93], for further details.

We shall explore similar constructions in the spinorial context, although we shall do so here at the level of supertraces rather than K-theory. The first step is to construct a suitable convolution algebra. In Section 4 we shall prove that the bundle $\mathbb{S}$ carries a natural *multiplicative structure*, which is to say a smoothly varying and associative family of complex-linear maps

$$\mathbb{S}_\gamma \otimes \mathbb{S}_\eta \longrightarrow \mathbb{S}_{\gamma\circ\eta}, \tag{1.0.6}$$

among the fibers of $\mathbb{S}$, where $(\gamma, \eta) \mapsto \gamma \circ \eta$ is the tangent groupoid composition law. Using the multiplicative structure, the space $C_c^\infty(\mathbb{T}M, \mathbb{S})$ of smooth, compactly supported sections of $\mathbb{S}$ may be given a convolution product and becomes a complex associative algebra.

Of special interest are the multiplication maps (1.0.6) in the case where $\lambda = 0$, so that γ and η correspond to tangent vectors X_m and Y_m, and the spaces $\mathbb{S}_\gamma$ and $\mathbb{S}_\eta$ are copies of $\wedge^* T_m M$. We shall compute the multiplication maps in this case, as follows:

THEOREM. *When $\lambda = 0$ the morphism (1.0.6) is given by the formula*

$$\alpha \otimes \beta \longmapsto \alpha \wedge \beta \wedge \exp\big(-\tfrac{1}{2}\kappa(X_m, Y_m)\big), \tag{1.0.7}$$

where $\kappa(X_m, Y_m)$ is the Riemannian curvature $R(X_m, Y_m)$ viewed as an element of $\wedge^2 T_m M$.

Continuing, and following Connes' work, we can construct algebra homomorphisms

$$\varepsilon_\lambda \colon C_c^\infty(\mathbb{T}M, \mathbb{S}) \longrightarrow \mathfrak{K}^\infty(L^2(M, S)) \tag{1.0.8}$$

for $\lambda \neq 0$ and

$$\varepsilon_0 \colon C_c^\infty(\mathbb{T}M, \mathbb{S}) \longrightarrow C_c^\infty(TM, \wedge^* TM)), \tag{1.0.9}$$

by restricting sections of $\mathbb{S}$ to the subgroupoids (1.0.3). Here $\mathfrak{K}^\infty(L^2(M, S))$ is the algebra of smoothing operators acting on the sections of the spinor bundle, but the algebra $C_c^\infty(TM, \wedge^* TM))$ is more interesting. It is the algebra of smooth, compactly supported sections of the pullback of the exterior algebra bundle of M to TM but with a *twisted* convolution multiplication related to (1.0.7).

In Section 5 we shall construct and analyze our family of supertraces

$$\operatorname{STr}_\lambda \colon C_c^\infty(\mathbb{T}M, \mathbb{S}) \longrightarrow \mathbb{C},$$

parametrized by $\lambda \in \mathbb{R}$. When $\lambda \neq 0$ the supertrace is defined by the diagram

$$\begin{array}{ccc} C_c^\infty(\mathbb{T}M, \mathbb{S}) & \xrightarrow{\ \varepsilon_\lambda\ } & \mathfrak{K}^\infty(L^2(M, S)) \\ & \searrow^{\operatorname{STr}_\lambda} & \downarrow{\scriptstyle \operatorname{STr}} \\ & & \mathbb{C}. \end{array} \tag{1.0.10}$$

Here ε_λ is the evaluation morphism in (1.0.8) and STr is the standard operator supertrace. The supertrace STr_0 uses the morphism

$$\int \colon C_c^\infty(TM, \wedge^* TM)) \longrightarrow \mathbb{C}$$

that is given by restricting a form to the zero section $M \subseteq TM$, then integrating its top-degree component over M (we use the Riemmanian metric to identify the top degree component with a top-degree differential form). We define STr_0 by means of the diagram

$$\begin{array}{ccc} C_c^\infty(\mathbb{T}M, \mathbb{S}) & \xrightarrow{\varepsilon_0} & C_c^\infty(TM, \wedge^* TM)) \\ & \searrow^{\mathrm{STr}_0} & \downarrow{\scriptstyle (2/i)^{\dim(M)/2}\cdot\int} \\ & & \mathbb{C}, \end{array} \tag{1.0.11}$$

where ε_0 is the evaluation morphism in (1.0.9).

Theorem. *If $\sigma \in C_c^\infty(\mathbb{T}M, \mathbb{S})$, then the supertraces $\mathrm{STr}_\lambda(\sigma) \in \mathbb{C}$ vary smoothly with $\lambda \in \mathbb{R}$.*

As we have already hinted, this is a sort of "index theorem without a Dirac operator," which relates the operator supertrace to differential forms, and we shall conclude the paper with a brief reminder, following Getzler, of how the actual index theorem for the Dirac operator quickly follows from it.

In future work we aim to study Fréchet and Banach algebra completions of the convolution algebra $C_c^\infty(\mathbb{T}M, \mathbb{S})$ from a K-theoretic perspective, and also consider variations appropriate to other occurences of Getzler's rescaling method, for instance in the work on the hypoelliptic Laplacian that we already mentioned.

2 The Tangent Groupoid

In this section we shall review the construction of the tangent groupoid and discuss its relations with linear partial differential operators. The tangent groupoid is a special case of the deformation to the normal cone construction from algebraic geometry, and generally speaking we shall follow the algebraic geometric approach towards its definition. The particular adaptations needed to handle smooth manifolds as opposed to algebraic varieties are taken from [HSSH18]. (A more direct account would be possible, see for instance [Hig10], but it would be a bit less convenient for our later purposes.)

2.1 Deformation to the Normal Cone

Throughout this section V will be a smooth manifold, and M will be a smoothly embedded submanifold of V. We shall denote by $C^\infty(V)$ the $\mathbb{R}$-algebra of real-valued smooth functions on V.

Recall that if p is a positive integer, then a smooth, real-valued function on V is said to *vanish to order $\geq p$ on M* if it is locally a sum of products of p or more smooth, real-valued functions on V, all of which vanish on M. It will be convenient to extend this concept to nonpositive p: let us agree that if p is nonpositive, then *every* smooth, real-valued function vanishes to order $\geq p$. We shall define the *deformation space* (or *deformation to the normal cone*) $\mathbb{N}_V M$ using the filtration on the smooth, real-valued functions by order of vanishing on M, as encoded in the following Rees [Ree56] construction:

2.1.1 DEFINITION. Denote by $A(V,M) \subseteq C^\infty(V)[t^{-1},t]$ the $\mathbb{R}$-algebra of those Laurent polynomials

$$\sum_{p\in\mathbb{Z}} f_p t^{-p}$$

for which each coefficient f_p is a smooth, real-valued function on V that vanishes to order $\geq p$ on M (and all but finitely many f_p are zero).

2.1.2 DEFINITION. A *character* of an associative algebra A over $\mathbb{R}$ is a nonzero algebra homomorphism from A to $\mathbb{R}$. The *character spectrum* of A, which we shall denote by $\operatorname{Char}\operatorname{Spec}(A)$, is the set of all characters of A. We equip $\operatorname{Char}\operatorname{Spec}(A)$ with the weak topology, that is, the topology with the fewest open sets for which the evaluation morphisms $\varphi \mapsto \varphi(a)$ are continuous.

In the case of $A(V,M)$, two kinds of characters present themselves:

(i) Simple evaluations at points in V and a nonzero values in $\mathbb{R}$, given by the formula

$$\varepsilon_{(v,\lambda)} \colon \sum_{p\in\mathbb{Z}} f_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} f_p(v)\lambda^{-p}. \tag{2.1.1}$$

(ii) Evaluations at normal vectors $X_m \in T_mV/T_mM$, given by the formula

$$\varepsilon_{X_m} \colon \sum_{p\in\mathbb{Z}} f_p t^{-p} \mapsto \sum_{p\geq 0} \frac{1}{p!} X_m^p(f_p). \tag{2.1.2}$$

Here, in order to evaluate the right-hand side, the normal vector $X_m \in T_mV/T_mM$ is first lifted to a tangent vector on V, and then extended to a vector field X on V, so that the p'th iterated derivative $X^p(f_p)$ can be formed; the value of $X^p(f_p)$ at m depends only on the normal vector X_p.

2.1.3 REMARK. If M has a Riemannian structure, then the evaluation (2.1.2) at a normal vector can be written alternatively as

$$\sum_{p\in\mathbb{Z}} f_p t^{-p} \mapsto \lim_{\lambda\to 0} \sum_{p\in\mathbb{Z}} f_p(\exp_m(\lambda X_m))\lambda^{-p}, \tag{2.1.3}$$

which should help explain the phrase "evaluation at a normal vector." Compare also Proposition 2.2.5 below.

2.1.4 THEOREM (See for example [HSSH18, Sec. 3]). *The character spectrum of the algebra $A(V,M)$ consists precisely of the characters of the form (2.1.1) and (2.1.2). All of them are distinct, and so the character spectrum may be identified with the disjoint union*

$$NM\times\{0\} \ \sqcup \ V\times\mathbb{R}^\times.$$

where $NM = TV|_M/TM$ is the normal bundle of M in V, and $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. □

2.1.5 Definition. We shall denote by $\mathbb{N}_V M$ the above character spectrum of $A(V, M)$.

The topological space $\mathbb{N}_V M$ may be equipped with a smooth manifold structure, as follows:

2.1.6 Definition. We shall denote by $\boldsymbol{A}_{\mathbb{N}_V M}$ the sheaf of those real-valued continuous functions on (open subsets of) $\mathbb{N}_V M$ that are locally of the form

$$\operatorname{Char}\operatorname{Spec}\bigl(A(V, M)\bigr) \ni \varphi \longmapsto h(\varphi(f_1), \dots, \varphi(f_k)) \in \mathbb{R},$$

where $k \in \mathbb{N}$, $f_1, \dots, f_k \in A(V, M)$, and where h is a smooth function on $\mathbb{R}^k$.

This is a fine sheaf (that is, there are partitions of unity; see [God58, Sec. 3.7] or [Wel08, Def. 3.3]), and so it is determined by its space of global sections, which is the algebra of smooth, real-valued functions on $\mathbb{N}_V M$. Nevertheless the sheaf point of view will be useful later, when we examine sheaves of modules over $\mathbb{N}_V M$.

2.1.7 Theorem (See for example [HSSH18, Sec. 3] again.). *The deformation space $\mathbb{N}_V M$ carries a unique smooth manifold structure for which $\boldsymbol{A}_{\mathbb{N}_V M}$ is the sheaf of smooth, real-valued functions.* □

Of course every element of $A(V, M)$ can be viewed as a smooth function on $\mathbb{N}_V M$, and from here onwards we shall occasionally refer to $A(V, M)$ as the *coordinate algebra* of the deformation space, and use the more suggestive notation

$$A(\mathbb{N}_V M) = A(V, M).$$

Not every smooth function on the deformation space belongs to the coordinate algebra, since for instance every function in the coordinate algebra is polynomial in each fiber of the normal bundle. Hence the need to consider the sheaf $\boldsymbol{A}_{\mathbb{N}_V M}$ above.

2.2 Vector Fields on the Deformation Space

In this subsection we shall present a new view of the characters of $A(\mathbb{N}_V M)$ that correspond to tangent vectors. It is an algebraic counterpart of the description involving the Riemannian exponential map that was mentioned in Remark 2.1.3, and it will be quite useful to us later on.
We begin with the following very simple fact:

2.2.1 Lemma. *Let M be a smooth submanifold of a smooth manifold V and let X be a vector field on V. If a smooth function f on V vanishes to order $\geq p$ on M, then $X(f)$ vanishes to order $\geq p-1$.* □

If follows from Lemma 2.2.1 that the formula

$$\sum f_p t^{-p} \longmapsto \sum X(f_p) t^{-(p-1)} \tag{2.2.1}$$

defines a derivation of the coordinate algebra $A(\mathbb{N}_V M)$. We shall study the action of this derivation on the following quotient of $A(\mathbb{N}_V M)$:

2.2.2 Definition. We shall denote by $A_0(\mathbb{N}_V M)$ the quotient of $A(\mathbb{N}_V M)$ by the ideal generated by $t \in A(\mathbb{N}_V M)$. We shall view $A_0(\mathbb{N}_V M)$ as a graded algebra, with components in degrees $p{\geq}0$ generated by the images of the elements $f_p t^{-p}$ (note that the elements $f_p t^{-p}$ with $p{<}0$ map to zero in the quotient).

2.2.3 Remark. Thanks to the way that $A(\mathbb{N}_V M)$ was defined, the quotient $A_0(\mathbb{N}_V M)$ identifies with the associated graded algebra for the decreasing filtration on $C^\infty(V)$ given by order of vanishing on M.

The derivation (2.2.1) annihilates t, and hence it descends to the quotient algebra $A_0(\mathbb{N}_V M)$. We shall use the following notation:

2.2.4 Definition. If X is a vector field on V, then we shall denote by

$$\boldsymbol{X} \colon A_0(\mathbb{N}_V M) \longrightarrow A_0(\mathbb{N}_V M)$$

the derivation on $A_0(\mathbb{N}_V M)$ induced from the derivation of $A(\mathbb{N}_V M)$ in (2.2.1).

The action of $\boldsymbol{X}$ lowers the grading degree in $A_0(\mathbb{N}_V M)$, and therefore the operator $\boldsymbol{X}$ is locally nilpotent. Because of this, we can form the exponential

$$\exp(\boldsymbol{X}) \colon A_0(\mathbb{N}_V M) \longrightarrow A_0(\mathbb{N}_V M)$$

using the power series. Now, let us denote by

$$\varepsilon_m \colon A_0(\mathbb{N}_V M) \longrightarrow \mathbb{R}$$

the character (2.1.2) obtained by evaluation at the zero normal vector $0_m \in N_{V,m}M$. Algebraically it is given by the formula

$$\varepsilon_m \colon \sum f_p t^{-p} \longmapsto f_0(m).$$

The following proposition is then merely a reformulation of the general formula (2.1.2). As we noted at the beginning of this subsection, it should be compared with the formula in Remark 2.1.3.

2.2.5 Proposition. *Let M be a smooth submanifold of a smooth manifold V, let X be a vector field on V. If $f \in A_0(\mathbb{N}_V M)$ and $m \in M$, then*

$$\varepsilon_{X_m}(f) = \varepsilon_m\big(\exp(\boldsymbol{X})f\big). \qquad \square$$

2.2.6 Remark. Since $\boldsymbol{X}$ is a derivation of $A_0(\mathbb{N}_V M)$, the map $\exp(\boldsymbol{X})$ is an algebra automorphism. This makes it clear that the formula for ε_{X_m} in (2.1.2) does indeed define a character.

We shall use algebraic exponentials similar to $\exp(\boldsymbol{X})$ quite extensively later on, and the following proposition gives an indication of how we shall do so (but using local coordinates, it is also very easy to prove this particular result by a direct computation).

2.2.7 Proposition. *Let X be a vector field on V. The smooth family of vector fields*

$$\lambda X : C^\infty(V\times\{\lambda\}) \longrightarrow C^\infty(V\times\{\lambda\})$$

defined on the fibers of $\mathbb{N}_V M$ over $\lambda \neq 0$, extends to a smooth family of vector fields on the fibers of $\mathbb{N}_V M$ over all $\lambda \in \mathbb{R}$ (here by a smooth family of vector fields on the fibers of a submersion we mean a vector field on the total space that is tangent to each fiber). The extension to λ=0 acts by differentiation in each normal space $N_{V,m}M$ in the direction of the normal vector determined by the tangent vector X_m

Proof. According to [HSSH18, §2] there is a vector field on the deformation space that implements the derivation (2.2.1). The vector field is tangent to each λ-fiber since the derivation annihilates $t \in A(\mathbb{N}_V M)$, and clearly it restricts to λX on $V\times\{\lambda\}$. To compute its action on the fiber over λ=0 we can proceed as follows.

The derivations of $A_0(\mathbb{N}_V M)$ that are associated to any pair of vector fields always commute with one another. This is a consequence of Remark 2.2.3 and the fact that the commutator of vector fields, being itself a vector field, only lowers the order of vanishing on M by one, not two. Using this, we compute that

$$\begin{aligned}
\varepsilon_{Y_m}(\boldsymbol{X} f) &= \frac{d}{dt}\Big|_{t=0} \varepsilon_{Y_m}(\exp(t\boldsymbol{X})f) \\
&= \frac{d}{dt}\Big|_{t=0} \varepsilon_m(\exp(\boldsymbol{Y})\exp(t\boldsymbol{X})f) \\
&= \frac{d}{dt}\Big|_{t=0} \varepsilon_m(\exp(\boldsymbol{Y}+t\boldsymbol{X})f) = \frac{d}{dt}\Big|_{t=0} \varepsilon_{Y_m+tX_m}(f).
\end{aligned}$$

The extension therefore acts on the fiber of the normal bundle over $m \in M$ as directional differentiation in the direction X_m, as required. □

2.2.8 Remark. The characters (2.1.2) determine an isomorphism from $A_0(\mathbb{N}_V M)$ to the algebra of smooth functions on the normal bundle that are polynomial in each fiber (and are of uniformly bounded degree). The proposition asserts that $\boldsymbol{X}$ acts by differentiation in each $N_{V,m}M$ in the direction of the normal vector determined by X_m.

2.3 Functoriality Properties

The deformation space is functorial in the following sense. Given a commuting diagram

$$\begin{array}{ccc} V & \longrightarrow & V' \\ \uparrow & & \uparrow \\ M & \longrightarrow & M' \end{array} \tag{2.3.1}$$

in which the vertical maps are inclusions of submanifolds and the horizontal maps are arbitrary smooth maps, there is an induced map

$$\mathbb{N}_V M \longrightarrow \mathbb{N}_{V'} M'. \tag{2.3.2}$$

Indeed the algebra map from $C^\infty(V')$ to $C^\infty(V)$ determined by (2.3.1) determines, in turn, an algebra map

$$A(\mathbb{N}_{V'} M') \longrightarrow A(\mathbb{N}_V M),$$

and so a map on character spectra in the reverse direction. In terms of the determination of the spectrum in Theorem 2.1.4, the formula for the induced map on deformation spaces is the obvious one determined by (2.3.1).
Here are some properties related to functoriality that we shall use later on:

(i) If the horizontal maps in (2.3.1) are open inclusions, then so is the induced map on deformation spaces.

(ii) Given a diagram

$$\begin{array}{ccccc} V_1 & \xrightarrow{\Phi_1} & W & \xleftarrow{\Phi_2} & V_2 \\ \uparrow & & \uparrow & & \uparrow \\ M_1 & \xrightarrow[\Psi_1]{} & P & \xleftarrow[\Psi_2]{} & M_2 \end{array}$$

of submanifolds in which the horizontal maps are submersions, if we form the fiber product manifolds

$$V = V_1 \underset{W}{\times} V_2 = \{\, (v_1, v_2) \in V_1 \times V_2 : \Phi_1(v_1) = \Phi_2(v_2) \,\}$$

and

$$M = M_1 \underset{P}{\times} M_2 = \{\, (m_1, m_2) \in M_1 \times M_2 : \Psi_1(m_1) = \Psi_2(m_2) \,\},$$

then the natural map

$$\mathbb{N}_V M \longrightarrow \mathbb{N}_{V_1} M_1 \underset{\mathbb{N}_W P}{\times} \mathbb{N}_{V_2} M_2$$

coming from functoriality is a diffeomorphism.

(iii) If $M = V$, that is, if the submanifold M is the entire manifold V, then there are of course no nonzero normal vectors, and the deformation space identifies with the product $M{\times}\mathbb{R}$ via Theorem 2.1.4:

$$\mathbb{N}_M M \cong M{\times}\mathbb{R} \tag{2.3.3}$$

2.4 Tangent Groupoid

The *tangent groupoid* of a smooth manifold M, denoted $\mathbb{T}M$, is the deformation space associated to the diagonal embedding of M into its square. That is,

$$\mathbb{T}M = \mathbb{N}_{M^2} M,$$

where $M^2 = M{\times}M$ (it will be helpful to use this compressed notation for the powers of M in this subsection). Throughout the paper we shall identify the normal bundle for the diagonal embedding with the tangent bundle of M via the projection onto the first coordinate. So as a set

$$\mathbb{T}M = TM{\times}\{0\} \sqcup M^2{\times}\mathbb{R}^{\times}. \tag{2.4.1}$$

We shall use the following notation for the coordinate algebra of the tangent groupoid:

$$A(\mathbb{T}M) = A(\mathbb{N}_{M^2} M).$$

The tangent groupoid inherits a Lie groupoid structure from the pair groupoid structure on $M{\times}M$ using the functoriality of the deformation space construction, as follows. Consider the commuting diagram

$$\begin{array}{ccc} M^2 & \rightrightarrows & M \\ \uparrow & & \uparrow \\ M & \xrightarrow[=]{} & M \end{array}$$

in which the top maps are the first and second coordinate projections—these are the target and source maps, respectively, for the pair groupoid—while the upwards maps are diagonal maps, viewed as inclusions of submanifolds. By functoriality of the deformation space construction, the diagram gives rise to maps

$$\mathbb{N}_{M^2} M \rightrightarrows \mathbb{N}_M M,$$

and therefore to maps

$$t, s\colon \mathbb{T}M \rightrightarrows M{\times}\mathbb{R}.$$

These are the target and source maps for the tangent groupoid.

The composition law in the tangent groupoid is obtained in the same fashion. The space of composable pairs of elements in the tangent groupoid is

$$\begin{aligned} \mathbb{T}M^{(2)} &= \{\, (\gamma,\eta) \in \mathbb{T}M{\times}\mathbb{T}M : s(\gamma) = t(\eta) \,\} \\ &= \mathbb{T}M \underset{M\times\mathbb{R}}{\times} \mathbb{T}M \end{aligned}$$

According to the previous subsection, the diagram

$$\begin{array}{ccccc} M^2 & \xrightarrow{p_2} & M & \xleftarrow{p_1} & M^2 \\ \uparrow & & \uparrow & & \uparrow \\ M & = & M & = & M \end{array} \tag{2.4.2}$$

in which the top maps are the projections onto the second and factors respectively, gives rise to a diffeomorphism

$$\mathbb{N}_{M^3} M \xrightarrow{\cong} \mathbb{T}M \underset{M\times\mathbb{R}}{\times} \mathbb{T}M. \tag{2.4.3}$$

On the other hand the diagram

$$\begin{array}{ccc} M^3 & \longrightarrow & M^2 \\ \uparrow & & \uparrow \\ M & \underset{=}{\longrightarrow} & M, \end{array}$$

in which the top map is projection onto the first and third factors induces a map from $\mathbb{N}_{M^3} M$ to $\mathbb{N}_{M^2} M$, and hence a map

$$c\colon \mathbb{T}M^{(2)} \longrightarrow \mathbb{T}M. \tag{2.4.4}$$

This is the composition law for the tangent groupoid. Clearly

$$(m_1, m_2, \lambda) \circ (m_2, m_3, \lambda) = (m_1, m_3, \lambda) \tag{2.4.5}$$

when $\lambda{\neq}0$. The formula when $\lambda{=}0$ is only a little harder to derive. We shall do the calculation in a somewhat roundabout way that will be helpful later (although for the tangent groupoid itself, an easy direct computation in local coordinates is possible).

Start with the following commutative diagram, in which the various evaluation characters ε are labeled by the indicated tangent vectors on the diagonal (which of course project to normal vectors):

$$\begin{array}{ccccc} A(\mathbb{N}_{M^2} M) & \xrightarrow{p_{12}^*} & A(\mathbb{N}_{M^3} M) & \xleftarrow{p_{23}^*} & A(\mathbb{N}_{M^2} M) \\ {\scriptstyle \varepsilon_{(X_m,0_m)}}\downarrow & & {\scriptstyle \varepsilon_{(X_m,0_m,-Y_m)}}\downarrow & & {\scriptstyle \varepsilon_{(Y_m,0_m)}}\downarrow {\scriptstyle = \varepsilon_{(0_m,-Y_m)}} \\ \mathbb{R} & = & \mathbb{R} & = & \mathbb{R}. \end{array} \tag{2.4.6}$$

Commutativity follows by direct computation from the definitions. It follows from the diagram that the isomorphism (2.4.3) maps the point of $\mathbb{N}_{M^3} M$ associated to the character $\varepsilon_{(X_m,0_m,-Y_m)}$ to $\big((X_m, 0), (Y_m, 0)\big) \in \mathbb{T}M^{(2)}$.

2.4.1 Lemma. $(X_m, 0) \circ (Y_m, 0) = (X_m + Y_m, 0)$.

Proof. Consider the diagram

$$\begin{array}{ccccc} A_0(\mathbb{T}M) & \xrightarrow{\quad c^* \quad} & A_0(\mathbb{TM}^{(2)}) & \xrightarrow{\varepsilon_{(X_m,0_m,-Y_m)}} & \mathbb{R} \\ & & \uparrow & & \uparrow \cong \\ & & A_0(\mathbb{T}M) \otimes_{\mathbb{R}} A_0(\mathbb{T}M) & \xrightarrow[\varepsilon_{(Y_m,0_m)} \otimes \varepsilon_{(X_m,0)}]{} & \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \end{array}$$

in which the leftmost map is induced from the composition law (2.4.4), while the left vertical map is induced from

$$f_1 \otimes f_2 \longmapsto \Big[(m_1, m_2, m_3) \mapsto f_1(m_1, m_2) f_2(m_2, m_3)\Big],$$

or equivalently from the inclusion of $\mathbb{T}M^{(2)}$ into $\mathbb{T}M \times \mathbb{T}M$. It follows from (2.4.6) that the square in the diagram is commutative. The composition of morphisms along the top is the character of $A_0(\mathbb{T}M)$ associated to $(X_m, 0) \circ (Y_m, 0)$. But this composition is

$$\varepsilon_{(X_m,-Y_m)} \colon A_0(\mathbb{T}M) \longrightarrow \mathbb{R}$$

and $\varepsilon_{(X_m,-Y_m)} = \varepsilon_{(X_m+Y_m,0)}$ since the tangent vectors $(X_m, -Y_m)$ and $(X_m + Y_m, 0)$ at the diagonal point (m, m) determine the same normal vector for the diagonal embedding. This completes the proof. □

2.5 FAMILIES OF DIFFERENTIAL OPERATORS

Let D be a linear partial differential operator on M. The source fibers of the pair groupoid $M \times M$ have the form $M \times \{m\}$, and if we place a copy of D on each one, then we obtain a smooth, equivariant family of linear partial differential operators on the source fibers.

A small extension of the above produces a family of linear partial differential operators on the source fibers of the tangent groupoid. First, if $m \in M$, then denote by D_m the translation-invariant *model operator* on $T_m M$ that is obtained from D by freezing coefficients in a local coordinate expression for D and dropping lower order terms.

2.5.1 THEOREM. *Let M be a smooth manifold and let D be a linear partial differential operator on M of order q. The formula*

$$D_{(m,\lambda)} = \begin{cases} \lambda^q D & \lambda \neq 0 \\ D_m & \lambda = 0 \end{cases}$$

defines a smooth and equivariant family of differential operators on the source fibers of $\mathbb{T}M$.

Proof. We need to show that if f is a smooth function on $\mathbb{T}M$, and if we apply the above family of differential operators to f fiberwise, then the result is again a smooth function on $\mathbb{T}M$.
In fact it suffices to prove this when D is a vector field, since $\lambda^q D$ is a sum of products

$$\lambda^{q-d} \cdot h \cdot \lambda D_1 \cdot \ldots \cdot \lambda D_d$$

where h is a smooth function, the operators D_i are vector fields, and where $d \leq q$, and since the model operator D_m is the sum of those products

$$h(m) \cdot D_{1,m} \cdot \ldots \cdot D_{d,m}$$

where $q = d$. The vector field case is handled by Proposition 2.2.7. □

3 A Rescaled Spinor Bundle

In this section we shall explain how to construct a "rescaled" spinor bundle on the tangent groupoid of a Riemannian spin manifold.

3.1 Clifford Algebras

We begin with a very quick review of some points in Clifford algebra theory to fix notation and terminology. Let E be a finite-dimensional euclidean vector space. We shall denote by $\mathrm{Cliff}_{\mathbb{R}}(E)$ the real algebra generated by a copy of E subject to the relations

$$ef + fe = -2\langle e, f\rangle 1$$

for all $e, f \in E$, and we shall denote by $\mathrm{Cliff}_{\mathbb{C}}(E)$, or simply $\mathrm{Cliff}(E)$, its complexification. We shall generally follow the conventions in the monograph [Mei13], and in terms of that book, $\mathrm{Cliff}(E)$ is the complex Clifford algebra associated to the bilinear form B given by the *negative* of the inner product on E.
There is a real-linear *quantization isomorphism*

$$q\colon \wedge^* E \longrightarrow \mathrm{Cliff}_{\mathbb{R}}(E) \tag{3.1.1}$$

as in [Mei13, Sec. 2.2.5]. If $\{e_1, \ldots, e_n\}$ is any orthonormal basis for E, then

$$q(e_{i_1} \wedge \cdots \wedge e_{i_d}) = e_{i_1} \cdot \ldots \cdot e_{i_d}$$

for all indices $i_1 < \cdots < i_d$. The quantization isomorphism equips the Clifford algebra with a vector space grading. This grading is not compatible with the multiplication operation in the Clifford algebra, but the underlying increasing filtration *is* compatible with multiplication. We shall write it as

$$\mathbb{C} \cdot I = \mathrm{Cliff}_0(E) \subseteq \mathrm{Cliff}_1(E) \subseteq \cdots \subseteq \mathrm{Cliff}_{\dim(E)}(E) = \mathrm{Cliff}(E), \tag{3.1.2}$$

where $\mathrm{Cliff}_d(E)$ is the sum of all $q(\wedge^a E)$ with $a \leq d$.

3.1.1 Remark. For later purposes it will be convenient to extend this filtration to all $d \in \mathbb{Z}$ so that

$$\operatorname{Cliff}_d(E) = \begin{cases} 0 & d < 0 \\ \operatorname{Cliff}(E) & d > \dim(E). \end{cases} \tag{3.1.3}$$

The associated "Clifford order" of $0 \in \operatorname{Cliff}(E)$ will be $-\infty$. Observe that the quantization map gives rise to an isomorphism

$$\wedge^d E \xrightarrow{\cong} \operatorname{Cliff}_d(E)/\operatorname{Cliff}_{d-1}(E) \tag{3.1.4}$$

for all d.

The subspace $q(\wedge^2 E) \subseteq \operatorname{Cliff}_{\mathbb{R}}(E)$ is closed under the ordinary commutator bracket in the Clifford algebra, and so acquires a Lie algebra structure. Moreover

$$\big[q(\wedge^2 E), q(\wedge^1 E)\big] \subseteq q(\wedge^1 E),$$

so that the Lie algebra $q(\wedge^2 E)$ acts on $E \cong q(\wedge^1 E)$ by commutator bracket in the Clifford algebra. This action determines a Lie algebra homomorphism

$$q(\wedge^2 E) \longrightarrow \mathfrak{gl}(E),$$

and indeed Lie algebra *isomorphism*

$$q(\wedge^2 E) \xrightarrow{\cong} \mathfrak{so}(E). \tag{3.1.5}$$

Now define a vector space isomorphism $\gamma\colon \mathfrak{so}(E) \to \wedge^2 E$ by means of the following commuting diagram:

$$\begin{array}{ccccc} & & q(\wedge^2 E) & & \\ & \overset{q}{\nearrow} & & \overset{(3.1.5)}{\searrow} & \\ \wedge^2 E & & \xleftarrow{\quad\gamma\quad} & & \mathfrak{so}(E). \end{array} \tag{3.1.6}$$

We shall not use it, but γ is given by the beautiful explicit formula

$$\gamma(T) = \tfrac{1}{4} \sum T(e_i) \wedge e_i.$$

See [Mei13, Section 2.2.10].

3.2 Spinor Bundles

From now on M will be an even-dimensional, Riemannian spin manifold. We shall review some facts concerning spinors on M.

Let $S \to M$ be a complex irreducible spinor vector bundle, equipped with the canonical Riemannian connection ∇ (also known as the Levi-Civita connection), as in [LM89, Sec. II.4] or [Roe98, Ch. 4]. The bundle S and connection ∇ have the following properties:

(i) S is a smooth, $\mathbb{Z}/2$-graded Hermitian vector bundle over M.

(ii) There is a morphism of smooth real vector bundles

$$c\colon TM \longrightarrow \operatorname{End}(S)^{\text{odd}}_{\text{skew-adjoint}}$$

with

$$c(X)^2 = -\|X\|^2 \cdot I$$

for every vector field X.

(iii) The morphism c induces an irreducible representation of $\operatorname{Cliff}_{\mathbb{C}}(T_mM)$ on S_m for every $m \in M$, and indeed a $\mathbb{Z}/2$-graded algebra isomorphism

$$c\colon \operatorname{Cliff}_{\mathbb{C}}(T_mM) \xrightarrow{\cong} \operatorname{End}(S_m). \tag{3.2.1}$$

(iv) If X and Y are vector fields on M, and if s is a smooth section of S, then

$$\nabla_Y(c(X)s) = c(\nabla^{\text{LC}}_Y(X))s + c(X)\nabla_Y s, \tag{3.2.2}$$

where ∇^{LC} is the Levi-Civita connection on TM.

We shall also use in a crucial way a simple formula that relates the curvature operator

$$K(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

of the Riemannian connection on S to the Riemann curvature tensor

$$R(X,Y) = \nabla^{\text{LC}}_X\nabla^{\text{LC}}_Y - \nabla^{\text{LC}}_Y\nabla^{\text{LC}}_X - \nabla^{\text{LC}}_{[X,Y]}$$

on TM. For any pair of tangent vectors $X_m, Y_m \in T_mM$ we have, of course

$$R(X_m,Y_m) \in \mathfrak{so}(T_mM) \quad \text{and} \quad K(X_m,Y_m) \in \operatorname{End}(S_m).$$

Moreover

$$K(X,Y) = c \circ q \circ \gamma\big(R(X,Y)\big) \tag{3.2.3}$$

where γ is the morphism (3.1.6).

3.3 The Scaling Filtration

Denote by $S \boxtimes S^*$ the bundle over $M \times M$ whose fiber over (m_1, m_2) is $S_{m_1} \otimes S^*_{m_2}$. In this subsection we shall construct a decreasing filtration of the space of smooth sections of $S \boxtimes S^*$ that is based on the vanishing behavior of sections near the diagonal in $M \times M$. The construction uses the following *Getzler filtration* of the algebra of linear partial differential operators acting on the smooth sections of S over M, the definition of which takes advantage of the fact that the algebra is generated by Clifford multiplications, covariant derivatives and multiplications by scalar functions.

3.3.1 Definition. We say such a differential operator D has *Getzler order* $\leq p$ if in a neighborhood of any point in M it can be expressed as a finite sum of operators of the form

$$f \cdot D_1 \cdots D_p,$$

where f is a smooth function and each D_j is either a covariant derivative ∇_X, or a Clifford multiplication operator $c(X)$, or the identity operator.

3.3.2 Examples. If X is any vector field on M, then Getzler-order$(\nabla_X) \leq 1$, and the order is equal to 1 unless $X = 0$. In addition Getzler-order$(c(X)) \leq 1$, and again the order is equal to 1 unless $X = 0$.

The construction also uses the following increasing filtration of the fibers of $S \boxtimes S^*$ over the diagonal in $M \times M$. Using (3.2.1), these fibers admit canonical identifications

$$S_m \otimes S_m^* \cong \mathrm{End}(S_m) \cong \mathrm{Cliff}(T_m M), \tag{3.3.1}$$

and we equip them with the canonical increasing Clifford algebra filtration from (3.1.2).

3.3.3 Definition. Let $q \in \mathbb{Z}$. We shall say that a smooth section of $S \boxtimes S^*$ has *Clifford order* $\leq q$ if its value at each diagonal point (m, m) lies in the order q subspace $\mathrm{Cliff}_q(T_m M) \subseteq \mathrm{Cliff}(T_m M)$.

3.3.4 Examples. If q is negative, then a section with Clifford order $\leq q$ at $m \in M$ must vanish. At the other extreme, every section has Clifford order $\leq \dim(M)$.

In the following definition we shall consider linear partial differential operators D that act on the smooth sections of the spinor bundle S over M. We shall consider D as also acting on the smooth sections of $S \boxtimes S^*$ over $M \times M$ by differentiation in the first factor of $M \times M$ alone.

3.3.5 Definition. Let $p \in \mathbb{Z}$. We shall say that a section σ of $S \boxtimes S^*$ over $M \times M$ has *scaling order* $\geq p$ if

$$\text{Clifford-order}(D\sigma) \leq q - p$$

for every differential operator D of Getzler order $\leq q$. If $m \in M$, then we shall say that σ has *scaling order* p *near* m if the above condition holds in a neighborhood of (m, m).

3.3.6 Remark. The definition can be compared as follows to the ordinary notion of vanishing to order p along the diagonal of a real-valued function on $M \times M$ that we used to construct the tangent groupoid. Suppose we write $\mathrm{val}(f) = -\infty$ if f vanishes on the diagonal, while $\mathrm{val}(f) = 0$ otherwise. Then f vanishes to order p on the diagonal in $M \times M$ if and only if

$$\mathrm{val}(Df) \leq \mathrm{order}(D) - p$$

for every linear partial differential operator D on M acting on functions on $M \times M$ through the first factor.

3.3.7 Example. Since the Clifford order of a section of $S \boxtimes S^*$ is never more than $\dim(M)$, every section has scaling order $\geq -\dim(M)$.

Let us record two easy consequences of the definition. The first is immediate:

3.3.8 Lemma. *If a smooth section σ of $S \boxtimes S^*$ has scaling order $\geq p$, and if D has Getzler order $\leq q$, then $D\sigma$ has scaling order $\geq p-q$.* □

3.3.9 Lemma. *If a smooth section σ of $S \boxtimes S^*$ has scaling order $\geq p_1$, and if a smooth function f on $M \times M$ vanishes to order $\geq p_2$ on the diagonal of $M \times M$, then the section $f \cdot \sigma$ has scaling order $\geq p_1 + p_2$.*

Proof. If D has Getzler order $\leq q$, then

$$D(f \cdot \sigma) = \sum (E_j f)(D_j \sigma)$$

for suitable operators D_j acting on sections of on S, and suitable partial differential operators E_j acting on scalar-valued functions, with

$$\text{order}(E_j) + \text{Getzler-order}(D_j) \leq q.$$

The proof follows from this. □

In particular, the sections of $S \boxtimes S^*$ of scaling order $\geq p$ form a $C^\infty(M \times M)$-module. A deeper result concerning scaling order is the following fact, whose proof we shall give in an appendix; see Section 6.

3.3.10 Proposition. *Let $m \in M$ and let $d \geq 0$. Every smooth section of the bundle $\text{Cliff}(TM)$ over M that has Clifford order $\leq d$ near m is the restriction to the diagonal in $M \times M$ of a smooth section of $S \boxtimes S^*$ of scaling order $\geq -d$ near m.*

3.4 The Rescaled Spinor Module

In this subsection we shall define a module $S(\mathbb{T}M)$ over the coordinate algebra $A(\mathbb{T}M)$ using the scaling filtration from the previous subsection and the Rees construction. As we shall soon see, it may be viewed as the module of "regular" sections of a bundle $\mathbb{S}$ over the tangent groupoid, just as $A(\mathbb{T}M)$ may be viewed as the algebra of "regular" functions on the tangent groupoid. Here we shall compute the fibers of the module $S(\mathbb{T}M)$, which will be the fibers of the bundle $\mathbb{S}$.

3.4.1 Definition. Denote by $S(\mathbb{T}M)$ the complex vector space of Laurent polynomials

$$\sum_{p \in \mathbb{Z}} \sigma_p t^{-p}$$

where each σ_p is a smooth section of $S \boxtimes S^*$ of scaling order at least p. It follows from Lemma 3.3.9 that $S(\mathbb{T}M)$ is a module over $A(\mathbb{T}M)$ by ordinary multiplication of Laurent polynomials. For each point $\gamma \in \mathbb{T}M$ let $I_\gamma \subseteq A(\mathbb{T}M)$ be the corresponding vanishing ideal. The *fiber* of $S(\mathbb{T}M)$ over γ is

$$S(\mathbb{T}M)|_\gamma = S(\mathbb{T}M)\big/ I_\gamma \cdot S(\mathbb{T}M)$$

The most interesting fibers are those for which the morphism $\gamma \in \mathbb{T}M$ has the form $\gamma = (X_m, 0)$, where X_m is a tangent vector on M, and most of this subsection will be devoted to studying them.

3.4.2 Definition. We shall denote by $S_0(\mathbb{T}M)$ the vector space quotient

$$S_0(\mathbb{T}M) = S(\mathbb{T}M)\big/ t \cdot S(\mathbb{T}M).$$

Note that the $A(\mathbb{T}M)$-module structure on $S(\mathbb{T}M)$ descends to an $A_0(\mathbb{T}M)$-module structure on $S_0(\mathbb{T}M)$.

The quotient space $S_0(\mathbb{T}M)$ is a graded vector space with nonzero components in integer degrees $-\dim(M)$ and up. Indeed it is the associated graded space for the decreasing filtration of the smooth sections of $S \boxtimes S^*$ by scaling order (to be clear, we place in degree p the images of the elements $\sigma_p t^{-p}$, or in other words the sections of scaling order at least p, modulo the sections of scaling order $\geq p{+}1$).
If the morphism $\gamma \in \mathbb{T}M$ has the form $\gamma = (X_m, 0)$, where X_m is a tangent vector on M, then the quotient map from $S(\mathbb{T}M)$ to $S_0(\mathbb{T}M)$ induces an isomorphism

$$S(\mathbb{T}M)|_\gamma \cong S_0(\mathbb{T}M)\big/ I_{X_m} \cdot S_0(\mathbb{T}M),$$

where $I_{X_m} \lhd A_0(\mathbb{T}M)$ is the vanishing ideal for $X_m \in TM$. We shall use this to compute $S(\mathbb{T}M)|_\gamma$.

3.4.3 Definition. Let $m \in M$. We shall denote by

$$\varepsilon_m \colon S_0(\mathbb{T}M) \longrightarrow \wedge^* T_m M$$

the *evaluation map* at $m \in M$ defined by the formula

$$\varepsilon_m \colon \sum_p \sigma_p t^{-p} \longmapsto \sum_d [\sigma_{-d}(m,m)]_d,$$

where $[\_]_d$ denotes the image in the quotient $\operatorname{Cliff}_d(T_mM)\big/ \operatorname{Cliff}_{d-1}(T_mM)$ of an element in $\operatorname{Cliff}_d(T_mM)$, and we identify the quotient with $\wedge^d T_m M$ via the quantization map.

3.4.4 Lemma. *The evaluation map has the property that*

$$\varepsilon_m(f\sigma) = \varepsilon_m(f)\varepsilon_m(\sigma)$$

for all $f \in A_0(\mathbb{T}M)$ and all $\sigma \in S_0(\mathbb{T}M)$. □

3.4.5 DEFINITION. Let X be a vector field on M. Denote by

$$\boldsymbol{\nabla}_{\boldsymbol{X}} \colon S_0(\mathbb{T}M) \longrightarrow S_0(\mathbb{T}M)$$

the linear operator determined by the formula

$$\boldsymbol{\nabla}_{\boldsymbol{X}} \colon \sum \sigma_p t^{-p} \longmapsto \sum \nabla_X \sigma_p t^{-(p-1)}.$$

3.4.6 LEMMA. *The operator $\boldsymbol{\nabla}_{\boldsymbol{X}}$ is compatible with the derivation $\boldsymbol{X}$ of the coordinate algebra $A_0(\mathbb{T}M)$ in the sense that*

$$\boldsymbol{\nabla}_{\boldsymbol{X}}(f\sigma) = \boldsymbol{X}(f)\sigma + f\boldsymbol{\nabla}_{\boldsymbol{X}}(\sigma)$$

for every $f \in A_0(\mathbb{T}M)$ and every $\sigma \in S_0(\mathbb{T}M)$. □

The operator $\boldsymbol{\nabla}_{\boldsymbol{X}}$ has grading degree minus one, and is therefore locally nilpotent. So we can form the exponential

$$\exp(\boldsymbol{\nabla}_{\boldsymbol{X}}) \colon S_0(\mathbb{T}M) \longrightarrow S_0(\mathbb{T}M)$$

using the power series. Inspired by the discussion in Subsection 2.2, let us now make the following definition:

3.4.7 DEFINITION. Let X be a vector field on M and let $m \in M$. Denote by

$$\varepsilon_{X_m} \colon S_0(\mathbb{T}M) \longrightarrow \wedge^* T_m M$$

the map defined by the commuting diagram

$$\begin{array}{ccc} S_0(\mathbb{T}M) & \xrightarrow{\varepsilon_{X_m}} & \wedge^* T_m M \\ {\scriptstyle \exp(\boldsymbol{\nabla}_{\boldsymbol{X}})}\downarrow & & \Vert \\ S_0(\mathbb{T}M) & \xrightarrow[\varepsilon_m]{} & \wedge^* T_m M. \end{array}$$

3.4.8 LEMMA. *The morphism ε_{X_m} depends only on the tangent vector X_m, and not on the values of the vector field X at other points in M. Moreover $\varepsilon_{X_m}(f\sigma) = \varepsilon_{X_m}(f)\varepsilon_{X_m}(\sigma)$.*

Proof. This follows from Lemma 3.4.6 and the definitions. □

3.4.9 PROPOSITION. *Let $m \in M$ and let $X_m \in T_m M$. The morphism*

$$\varepsilon_{X_m} \colon S_0(\mathbb{T}M) \longrightarrow \wedge^* T_m M$$

induces an isomorphism

$$S(\mathbb{T}M)|_{(X_m,0)} \xrightarrow{\cong} \wedge^* T_m M.$$

The proof will use the following local form for sections of $S\boxtimes S^*$ of scaling order p. Let $n = \dim(M)$, choose a local orthonormal frame $\{ e_1, \dots, e_n \}$ for TM near $m \in M$, and for each index

$$I = (i_1 < \cdots < i_d)$$

of length $\ell(I) = d$, form the local section

$$e_I = e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_d}$$

of the bundle $\mathrm{Cliff}_{\ell(I)}(TM)$ over M. View this as a local section of the restriction of $S\boxtimes S^*$ to the diagonal, and use Proposition 3.3.10 to extend it to a section over $M\times M$ with scaling order $-\ell(I)$ near m. We shall use the same notation e_I for the extension. The smooth sections e_I constitute a local frame for $S\boxtimes S^*$ near $(m,m) \in M\times M$. So any smooth section σ may be expanded in the form

$$\sigma = \sum_I h_I \cdot e_I \tag{3.4.1}$$

near (m,m), where the h_I are smooth, complex-valued functions on $M\times M$.

3.4.10 Lemma. *Let $p \in \mathbb{Z}$. The section σ in (3.4.1) has scaling order $\geq p$ near m if and only if each scalar function h_I vanishes on the diagonal of $M\times M$ near (m,m) to order $\geq p + \ell(I)$.*

Proof. The sufficiency of the condition follows from Lemma 3.3.9. As for necessity, given a section $\sigma = \sum h_I e_I$ of scaling order p, let p' be the minimum of all the integers $\mathrm{order}(h_I) - \ell(I)$, where $\mathrm{order}(h_I)$ means here the order of vanishing of h_I on the diagonal of $M\times M$ near (m,m). We need to prove that $p' \geq p$.
Let $\mathcal{I}_{\min}$ be the set of all those indices I for which the minimum p' is achieved, let q be the least value of $\mathrm{order}(h_I)$ among all $I \in \mathcal{I}_{\min}$. If D has Getzler order q, then for any $I \in \mathcal{I}_{\min}$, the section $D(h_I e_I)$ restricts to a function times e_I on the diagonal. Moreover for any fixed $I_0 \in \mathcal{I}_{\min}$ we can find some such D so that $D(h_{I_0} e_{I_0})$ is not identically zero in a neighborhood of (m,m) on the diagonal. So for this D,

$$\text{Clifford-order}\big(\textstyle\sum_{I\in\mathcal{I}_{\min}} D(h_I e_I)\big) \geq \ell(I_0)$$

near m. As a result, since $\ell(I_0) = q - p'$,

$$\text{Scaling-order}\big(\textstyle\sum_{I\in\mathcal{I}_{\min}} D(h_I e_I)\big) \leq p'$$

near m. On the other hand, it follows from Lemma 3.3.9 that

$$\text{Scaling-order}\big(\textstyle\sum_{I\notin\mathcal{I}_{\min}} D(h_I e_I)\big) \geq p' + 1$$

near m. Hence the scaling order of σ is $\leq p'$, and so $p' \geq p$ as required. ∎

Proof of Proposition 3.4.9. Lemma 3.4.8 shows that ε_{X_m} does at least induce a vector space morphism

$$S(\mathbb{T}M)|_{(X_m,0)} \longrightarrow \wedge^* T_m M.$$

In addition, if $I = (i_1 < \cdots < i_d)$, then under this morphism an element $e_I t^{\ell(I)} \in S(\mathbb{T}M)$ is mapped to an element of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_d} + \text{higher-degree terms in } \wedge^* T_m M. \tag{3.4.2}$$

It follows from this that the morphism is surjective.

Now suppose that $\sum \sigma_p t^{-p} \in S(\mathbb{T}M)$ is mapped to zero by ε_{X_m}. We need to prove that it lies in $I_\gamma \cdot S(\mathbb{T}M)$. In doing so, we can assume that each σ_p is supported near (m,m). Indeed, if φ is any smooth function on $M \times M$ that is equal to 1 near (m,m), then we can write

$$\sum \sigma_p t^{-p} = (1-\varphi) \cdot \sum \sigma_p t^{-p} + \sum \varphi \sigma_p t^{-p},$$

and the first term on the right-hand side belongs to $I_{(X_m,0)} \cdot S(\mathbb{T}M)$ and so is mapped to zero by ε_{X_m}. So we can replace each σ_p by $\varphi\sigma_p$.

Assuming then that each σ_p is supported near (m,m), we can write

$$\sigma_p = \sum_I h_{p,I} e_I,$$

as in (3.4.1). According to Lemma 3.4.10, each $h_{p,I}$ vanishes to order $p+\ell(I)$ or higher on the diagonal in $M \times M$. Hence we may write

$$\sum_p \sigma_p t^{-p} = \sum_I \Big(\sum_p h_{p,I} t^{-(p+\ell(I))} \Big) \cdot \big(e_I t^{\ell(I)} \big) \tag{3.4.3}$$

where each $\sum_p h_{p,I} t^{-(p+\ell(I))}$ is an element of (the complexification of) $A(\mathbb{T}M)$. To prove the proposition it suffices to show that if $\sum \sigma_p t^{-p}$ maps to zero under the morphism ε_{X_m} in the statement of the proposition, then each function

$$\sum_p h_{p,I} t^{-(p+\ell(I))} \in A(\mathbb{T}M)$$

evaluates to zero at X_m. But according to (3.4.2), the elements $e_I t^{\ell(I)} \in S(\mathbb{T}M)$ map to linearly independent elements under ε_{X_m}. So the required vanishing follows from Lemma 3.4.8. ☐

3.4.11 Proposition. *Let $m_1, m_2 \in M$ and $\lambda \in \mathbb{R}^\times$. The morphism*

$$\varepsilon_{(m_1,m_2,\lambda)} \colon S_0(\mathbb{T}M) \longrightarrow S_{m_1} \otimes S^*_{m_2}$$

defined by the formula

$$\varepsilon_{(m_1,m_2,\lambda)} \colon \sum \sigma_p t^{-p} \longmapsto \sum \lambda^{-p} \sigma_p(m_1,m_2)$$

induces an isomorphism

$$S(\mathbb{T}M)|_{(m_1,m_2,\lambda)} \xrightarrow{\cong} S_{m_1} \otimes S^*_{m_2}.$$

Proof. The morphism $\varepsilon_{(m_1,m_2,\lambda)}$ above is obviously surjective and factors through the fiber, so it remains to prove injectivity of the induced map on the fiber.

Suppose first that $m_1 = m = m_2$. Using the argument and notation of the previous proof, if $\lambda \neq 0$, and if

$$\sum_p \sum_I \lambda^{-p} h_{p,I}(m,m) e_I(m,m) = 0,$$

then for each I

$$\sum_p \lambda^{-(p+\ell(I))} h_{p,I}(m,m) = 0,$$

since the vectors $\lambda^{-\ell(I)} e_I(m,m)$ are linearly independent. So the formula (3.4.3) expresses any element of $S(\mathbb{T}M)$ that maps to zero in $S_m \otimes S_m^*$ as a combination of elements in $A(\mathbb{T}M)$ that vanish at (m,m,λ), times elements in $S(\mathbb{T}M)$, as required.

If $m_1 \neq m_2$, then we need only replace the local frame $\{e_I\}$ near (m,m) by any local frame of $S \boxtimes S^*$ near (m_1,m_2) (away from the diagonal there is no need to invoke Proposition 3.3.10). Then we may proceed as above. □

3.5 THE RESCALED SPINOR BUNDLE

We are now ready to construct the *rescaled spinor bundle* $\mathbb{S}$ over $\mathbb{T}M$.

3.5.1 DEFINITION. Define a family of vector spaces $\mathbb{S}_\gamma$ parametrized by $\gamma \in \mathbb{T}M$ as follows:

$$\mathbb{S}_\gamma = \begin{cases} S_{m_1} \otimes S_{m_2}^* & \gamma = (m_1,m_2,\lambda) \\ \wedge^* T_m M & \gamma = (X_m, 0). \end{cases} \tag{3.5.1}$$

Denote by $\sigma \mapsto \widehat{\sigma}$ the morphism of $A(\mathbb{T}M)$-modules

$$S(\mathbb{T}M) \longrightarrow \prod_{\gamma \in \mathbb{T}M} \mathbb{S}_\gamma,$$

that associates to each $\sigma \in S(\mathbb{T}M)$ its value in each fiber $S(\mathbb{T}M)|_\gamma$ under the identifications in Propositions 3.4.9 and 3.4.11.

3.5.2 LEMMA. *The above morphism is injective.*

Proof. Let $(m_1,m_2) \in M \times M$. If an element $\sum_p \sigma_p t^{-p}$ maps to zero, then it follows from the formula for the morphisms $\varepsilon_{(m_1,m_2,\lambda)}$ that

$$\sum_p \lambda^{-p} \sigma_p(m_1,m_2) = 0$$

for all $\lambda \neq 0$. But this implies that $\sigma_p(m_1,m_2) = 0$ for all p. Hence $\sigma_p = 0$ for all p, and so $\sum_p \sigma_p t^{-p} = 0$. □

3.5.3 Definition. We shall denote by $\boldsymbol{S}_{\mathbb{T}M}$ the sheaf on $\mathbb{T}M$ consisting of sections

$$\mathbb{T}M \ni \gamma \longmapsto \tau(\gamma) \in \mathbb{S}_\gamma$$

that are locally of the form

$$\tau(\gamma) = \sum_{j=1}^{N} f_j(\gamma) \cdot \widehat{\sigma}_j(\gamma)$$

for some $N \in \mathbb{N}$, where $f_1, \dots, f_N$ are smooth, complex-valued functions on $\mathbb{T}M$ and $\sigma_1, \dots, \sigma_N$ belong to $S(\mathbb{T}M)$.

3.5.4 Theorem. *The sheaf $\boldsymbol{S}_{\mathbb{T}M}$ is locally free, of rank $2^{\dim(M)}$, as a sheaf of modules over $\boldsymbol{A}_{\mathbb{T}M}$.*

Proof. Let us prove that the sheaf is free in a neighborhood of $\gamma = (X_m, 0)$; the other $\gamma \in \mathbb{T}M$ are handled in the same way. Consider a section τ of $\boldsymbol{S}_{\mathbb{T}M}$ as in Definition 3.5.3 above. Locally we may write each $\sigma_j \in S(\mathbb{T}M)$ as

$$\sigma_j = \sum_I \Big(\sum_p h_{j,p,I} t^{-(p+\ell(I))} \Big) \cdot \big(e_I t^{\ell(I)} \big)$$

as in (3.4.3). So near γ, the section τ is a linear combination the sections $\widehat{e_I t^{\ell(I)}}$. But these spanning sections are also linearly independent in each fiber $\mathbb{S}_\eta$, for η near γ. So they are linearly independent over the smooth functions on $\mathbb{T}M$, near γ, as required. □

3.5.5 Definition. We shall denote by $\mathbb{S}$ the unique smooth vector bundle over $\mathbb{T}M$ whose fibers are the spaces $\mathbb{S}_\gamma$ in (3.5.1) and whose smooth sections are the sections of the sheaf $\boldsymbol{S}_{\mathbb{T}M}$.

3.6 Families of Differential Operators and the Getzler Symbol

We wish to prove the following spinorial counterpart of Theorem 2.5.1:

3.6.1 Theorem. *Let D be a linear partial differential operator on M, acting on the sections of the spinor bundle, of Getzler order q. The family of linear partial differential operators*

$$D_{(m,\lambda)} = \lambda^q D,$$

defined on those source fibers of $\mathbb{T}M$ with $\lambda \neq 0$, extends to a smooth family of linear partial differential operators on all the source fibers of the tangent groupoid, acting on sections of $\mathbb{S}$.

As with scalar case, it suffices to consider generators of the algebra of linear partial differential operators, in this case covariant derivatives ∇_X and Clifford multiplication operators $c(X)$. Let us begin with the latter, which are easier.

3.6.2 Lemma. *Let X be a vector field on M. The family of Clifford multiplication operators*

$$D_{(m,\lambda)} = \lambda c(X),$$

defined on the source fibers of $\mathbb{T}M$ with $\lambda \neq 0$, and acting on sections of $\mathbb{S}$, extends to a smooth family on all the source fibers of $\mathbb{T}M$. The operator on the source fiber $\mathbb{T}M_{(m,0)} \cong T_m M$ is the exterior multiplication operator

$$\wedge^* T_m M \ni \omega \longmapsto X_m \wedge \omega \in \wedge^* T_m M.$$

Proof. The formula

$$\sum \sigma_p t^{-p} \longmapsto \sum c(X)\sigma_p t^{-(p-1)}$$

defines an $A(\mathbb{T}M)$-linear operator on $S(\mathbb{T}M)$ and hence a endomorphism of the bundle $\mathbb{S}$. Its restriction to $\lambda{\neq}0$ is the operator of left multiplication by $\lambda c(X)$ on $S \boxtimes S^*$. We need to compute its restriction to $\lambda{=}0$.
For this, we need to show that if $\boldsymbol{c(X)}$ is the induced operator on $S_0(\mathbb{T}M)$, then for any tangent vector Y_m,

$$\varepsilon_{Y_m}\big(\boldsymbol{c(X)}\sigma\big) = X_m \wedge \varepsilon_{Y_m}(\sigma).$$

Note first that

$$\varepsilon_m\big(\boldsymbol{c(X)}\tau\big) = X_m \wedge \varepsilon_m(\tau\big)$$

for any $\tau \in S_0(\mathbb{T}M)$, which is clear from the definitions. Next, the formula

$$\nabla_Y c(X) - c(X)\nabla_Y = c(\nabla_Y X)\colon C^\infty(M{\times}M, S\boxtimes S^*) \longrightarrow C^\infty(M{\times}M, S\boxtimes S^*)$$

shows that the Getzler order-one operators $c(X)$ and ∇_Y commute up to an operator of Getzler order one, not two. As a result,

$$\boldsymbol{\nabla_Y c(X)} = \boldsymbol{c(X)\nabla_Y}\colon S_0(\mathbb{T}M) \longrightarrow S_0(\mathbb{T}M),$$

and it therefore follows that

$$\begin{aligned} \varepsilon_{Y_m}\big(\boldsymbol{c(X)}\sigma\big) &= \varepsilon_m\big(\exp(\boldsymbol{\nabla_Y})\boldsymbol{c(X)}\sigma\big) \\ &= \varepsilon_m\big(\boldsymbol{c(X)}\exp(\boldsymbol{\nabla_Y})\sigma\big) \\ &= X_m \wedge \varepsilon_{Y_m}(\sigma), \end{aligned}$$

as required. □

3.6.3 Lemma. *Let X be a vector field on M. The family of covariant derivatives*

$$D_{(m,\lambda)} = \lambda\nabla_X,$$

defined on those source fibers of $\mathbb{T}M$ with $\lambda \neq 0$, and acting on sections of $\mathbb{S}$, extends to a smooth family on all the source fibers of $\mathbb{T}M$. The operator on *the source fiber* $\mathbb{T}M_{(m,0)} \cong T_mM$ *is the sum of directional differentiation in the direction* X_m *and exterior multiplication by the linear function*

$$T_mM \ni Y_m \longmapsto \tfrac{1}{2}\kappa(Y_m, X_m) \in \wedge^* T_mM,$$

where, as in (3.2.3)*, the section* $\kappa(Y,X)$ *of* $\wedge^2 TM$ *is related to the Riemannian curvature and the curvature of* S *by*

$$\gamma(R(Y,X)) = \kappa(Y,X) \quad \textit{and} \quad c(q(\kappa(Y,X))) = K(Y,X).$$

The proof will use following simple algebraic fact:

3.6.4 LEMMA. *If* A*,* B *and* $[A,B]$ *are locally nilpotent linear operators on a rational vector space, and if* $[A,B]$ *commutes with both* A *and* B*, then* $A+B$ *is locally nilpotent and*

$$\exp(A)\exp(B) = \exp(\tfrac{1}{2}[A,B])\exp(A+B). \qquad \square$$

Proof of Lemma 3.6.3. It follows from the definition of the curvature operator that

$$[\nabla_Y, \nabla_X] - \nabla_{[Y,X]} = K(Y,X)$$

as operators on smooth sections of $S \boxtimes S^*$ over $M \times M$. So if we define

$$\boldsymbol{K(Y,X)} \colon S_0(\mathbb{T}M) \longrightarrow S_0(\mathbb{T}M)$$

by

$$\sum \sigma_p t^{-p} \longmapsto \sum K(Y,X)\sigma_p t^{-(p-2)},$$

then, since $\nabla_{[Y,X]}$ has Getzler order one, not two, we obtain

$$[\boldsymbol{\nabla_Y}, \boldsymbol{\nabla_X}] = \boldsymbol{K(Y,X)} \colon S_0(\mathbb{T}M) \longrightarrow S_0(\mathbb{T}M).$$

Moreover, as in the proof of Lemma 3.6.2, each of $\boldsymbol{\nabla_Y}$ and $\boldsymbol{\nabla_X}$ commutes with $\boldsymbol{K(Y,X)}$, and so by Lemma 3.6.4,

$$\exp\big(\boldsymbol{\nabla_Y}\big)\exp\big(\boldsymbol{\nabla_X}\big) = \exp\big(\boldsymbol{\nabla_{Y+X}}\big)\exp\big(\tfrac{1}{2}\boldsymbol{K(Y,X)}\big).$$

We can now compute that

$$\begin{aligned}
\varepsilon_Y\big(\boldsymbol{\nabla_X} s\big) &= \frac{d}{dt}\Big|_{t=0} \varepsilon_Y\big(\exp(\boldsymbol{\nabla_{tX}})s\big) \\
&= \frac{d}{dt}\Big|_{t=0} \varepsilon_0\big(\exp(\boldsymbol{\nabla_Y})\exp(\boldsymbol{\nabla_{tX}})s\big) \\
&= \frac{d}{dt}\Big|_{t=0} \varepsilon_0\big(\exp(\boldsymbol{\nabla_{Y+tX}})\exp(\tfrac{1}{2}\boldsymbol{K(Y,tX)})s\big) \\
&= \frac{d}{dt}\Big|_{t=0} \varepsilon_{Y+tX}\big(\exp(\tfrac{1}{2}\boldsymbol{K(Y,tX)})s\big) \\
&= \frac{d}{dt}\Big|_{t=0} \varepsilon_{Y+tX}\big(s\big) + \tfrac{1}{2}\kappa(Y_m, X_m) \wedge \varepsilon_Y(s).
\end{aligned}$$

In the last line we used the Leibniz rule and Lemma 3.6.2. We have now computed the action of the family $\{\lambda\nabla_X\}$ in the statement of the lemma on "algebraic" sections of $\mathbb{S}$ (associated to elements of $S(\mathbb{T}M)$). The lemma follows from this. $\square$

3.7 Tangent Vectors Versus Normal Vectors

So far, when discussing the tangent groupoid we have been identifying TM with the normal bundle for the diagonal in $M\times M$ by associating to a tangent vector X_m at $m \in M$ the tangent vector $(X_m, 0_m)$ at the diagonal point $(m,m) \in M\times M$. In this subsection we shall examine the effect of using other identifications.

Let X be a vector field on M. Instead of writing ∇_X for the covariant derivative on $M\times M$ associated to the action on the left copy of M, let us temporarily write $\nabla_{(X,0)}$. Let us similarly write $c(X,0)$ for left Clifford multiplication. There are also obvious right operators $\nabla_{(0,X)}$ and $c(0,X)$, and let us begin by noting that all the right operators commute with the all the left operators.

It follows from this commutativity that each right operator decreases the scaling order of a section σ of $S\boxtimes S^*$ by at most one. Consider for example the covariant derivative $\nabla_{(0,X)}$. If the scaling order of σ is at least p, and if D is a differential operator of Getlzer order $\leq q$ on M, acting on the left factor of $M\times M$, then we need to show that

$$\text{Clifford-order}(D\nabla_{(0,X)}\sigma) \leq q - p + 1.$$

Write $\tau = D\sigma$, which is a section of scaling order $\geq p{-}q$. Since $D\nabla_{(0,X)}\sigma = \nabla_{(0,X)}D\sigma$, we need to show that

$$\text{Clifford-order}(\nabla_{(0,X)}\tau) \leq q - p + 1.$$

Next write

$$\nabla_{(0,X)} = \nabla_{(X,X)} - \nabla_{(X,0)}.$$

The operator $\nabla_{(X,X)}$ preserves Clifford order since along the diagonal of $M\times M$ the Riemannian connection is the standard connection on $\mathrm{Cliff}(TM)$, while of course $\nabla_{(X,0)}$ increases the Clifford order of τ by at most one, by definition of the scaling filtration. The proof is complete. The proof for Clifford multiplications is similar, but simpler since the last step above is not needed.

It follows from these computations that we can define the scaling order using either left operators, or the right operators, or both.

Now let X and Y be vector fields on M. Since $\nabla_{(X,Y)}$ decreases scaling order by at most one, there is an induced, degree minus one operator

$$\boldsymbol{\nabla}_{\boldsymbol{(X,Y)}} \colon S_0(\mathbb{T}M) \longrightarrow S_0(\mathbb{T}M)$$

given by the now-usual formula

$$\sum \sigma_p t^{-p} \longmapsto \sum \nabla_{(X,Y)}\sigma_p t^{-(p-1)}$$

on $S(\mathbb{T}M)$. Define the evaluation morphism

$$\varepsilon_{(X_m,Y_m)} \colon S_0(\mathbb{T}M) \longrightarrow \wedge^* T_m M$$

via the commuting diagram

$$\begin{array}{ccc} S_0(\mathbb{T}M) & \xrightarrow{\varepsilon_{(X_m,Y_m)}} & \wedge^* T_m M \\ {\scriptstyle \exp(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{Y})})}\downarrow & & \Vert \\ S_0(\mathbb{T}M) & \xrightarrow[\varepsilon_m]{} & \wedge^* T_m M. \end{array}$$

In the context of $A_0(\mathbb{T}M)$, the map $\varepsilon_{(X_m,Y_m)}$ only depends on the normal vector determined by (X_m, Y_m), but the following computation shows that this is not the case for $S_0(\mathbb{T}M)$. Write $\kappa(X,Y) = \gamma(R(X,Y))$, as in Lemma 3.6.3. View $\kappa(X,Y)$ as an operator on the exterior algebra bundle by exterior multiplication.

3.7.1 Proposition. *The diagram*

$$\begin{array}{ccc} S_0(\mathbb{T}M) & \xrightarrow{\varepsilon_{(X_m,Y_m)}} & \wedge^* T_m M \\ \Vert & & \downarrow {\scriptstyle \exp(\frac{1}{2}\kappa(X_m,Y_m))} \\ S_0(\mathbb{T}M) & \xrightarrow[\varepsilon_{(X_m-Y_m,0)}]{} & \wedge^* T_m M \end{array}$$

is commutative.

Proof. Let us first prove the following special case: if Y is any vector field on M, then

$$\varepsilon_{(0_m,Y_m)} = \varepsilon_{(-Y_m,0_m)} \colon S_0(\mathbb{T}M) \longrightarrow \wedge^* T_m M. \tag{3.7.1}$$

To do so, use the fact that $\nabla_{(0,Y)}$ and $\nabla_{(-Y,0)}$ commute as operators on smooth sections of $S \boxtimes S^*$ to write

$$\begin{aligned} &\nabla^n_{(0,Y)} - \nabla^n_{(-Y,0)} \\ &\qquad = \nabla_{(Y,Y)} \Big(\nabla^{n-1}_{(0,Y)} + \nabla^{n-2}_{(0,Y)} \nabla_{(-Y,0)} + \cdots + \nabla_{(0,Y)} \nabla^{n-2}_{(-Y,0)} + \nabla^{n-1}_{(-Y,0)} \Big). \end{aligned}$$

The operator $\nabla_{(Y,Y)}$ does not increase the Clifford order of sections, so if σ is a section of scaling order p, then the section

$$\nabla^n_{(0,Y)}\sigma - \nabla^n_{(-Y,0)}\sigma$$

has Clifford order $\leq p+n-1$. It follows now from the definitions that

$$\varepsilon_m\big(\boldsymbol{\nabla}_{(\mathbf{0},\boldsymbol{Y})}\sigma\big) = \varepsilon_m\big(\boldsymbol{\nabla}_{(-\boldsymbol{Y},\mathbf{0})}\sigma\big),$$

and (3.7.1) follows.

For the general case, it follows from Lemma 3.6.4 that

$$\exp(\boldsymbol{\nabla}_{(-\boldsymbol{Y},\mathbf{0})}) \exp(\boldsymbol{\nabla}_{(\boldsymbol{X},\mathbf{0})}) = \exp(\tfrac{1}{2}\boldsymbol{K}(\boldsymbol{X},\boldsymbol{Y})) \exp(\boldsymbol{\nabla}_{(\boldsymbol{X}-\boldsymbol{Y},\mathbf{0})}) \tag{3.7.2}$$

as operators on $S_0(\mathbb{T}M)$. Therefore

$$\begin{aligned}\varepsilon_{(X_m,Y_m)}(\sigma) &= \varepsilon_m\big(\exp(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{Y})})\sigma\big)\\ &= \varepsilon_m\big(\exp(\boldsymbol{\nabla}_{(\boldsymbol{0},\boldsymbol{Y})})\exp(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{0})})\sigma\big)\\ &= \varepsilon_m\big(\exp(\boldsymbol{\nabla}_{(-\boldsymbol{Y},\boldsymbol{0})})\exp(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{0})})\sigma\big)\\ &= \varepsilon_m\big(\exp(\tfrac{1}{2}\boldsymbol{K}(\boldsymbol{X},\boldsymbol{Y}))\exp(\boldsymbol{\nabla}_{(\boldsymbol{X}-\boldsymbol{Y},\boldsymbol{0})})\sigma\big)\\ &= \exp(\tfrac{1}{2}\kappa(X_m,Y_m))\varepsilon_{X_m-Y_m}(\sigma)\end{aligned}$$

as required. □

4 Multiplicative Structure

In this section we shall equip the space $C_c^\infty(\mathbb{T}M,\mathbb{S})$ of smooth, compactly supported sections of the rescaled spinor bundle over $\mathbb{T}M$ with a convolution product.

4.1 Convolution Algebras of Smooth Groupoids

We being by reviewing some basic facts about the convolution algebras of smooth groupoids [Con94, Section 2.5]. Let $s,t\colon \mathbb{G}\rightrightarrows M$ be a Lie groupoid. In order to build a convolution algebra of functions on $\mathbb{G}$ we shall fix a suitable family of measures on the target fibers of $\mathbb{G}$ (an alternative approach uses half-densities, but in our tangent groupoid example the family of measures has an extremely simple form).

4.1.1 Definition (Compare [Ren80, Section 1.2]). A *smooth left Haar system* on $\mathbb{G}$ is a family of smooth measures μ^m on the target fibers

$$\mathbb{G}^m = \{\, \gamma\in\mathbb{G} : t(\gamma)=m \,\}$$

of $\mathbb{G}$ having the following two properties:

(i) For any compactly supported smooth function f on $\mathbb{G}$, the assignment

$$m\mapsto \int_{\mathbb{G}^m} f(\gamma)\,d\mu^m(\gamma)$$

defines a smooth function on M.

(ii) For any morphism $\gamma_1 : m\to p$ and any compactly supported smooth function f on $\mathbb{G}$ we have

$$\int_{\mathbb{G}^m} f(\gamma_1\circ\gamma)\,d\mu^m(\gamma) = \int_{\mathbb{G}^p} f(\gamma)\,d\mu^p(\gamma)$$

Given a smooth left Haar system on $\mathbb{G}$, the formula

$$f_1\star f_2(\eta) = \int_{\mathbb{G}^{t(\eta)}} f_1(\gamma)f_2(\gamma^{-1}\circ\eta)\,d\mu^{t(\eta)}(\gamma),$$

defines an associative product on $C_c^\infty(\mathbb{G})$. This is the convolution algebra of the Lie groupoid $\mathbb{G}$.

4.2 Multiplicative Structures on Bundles Over Groupoids

Let $s, t\colon \mathbb{G} \rightrightarrows M$ be a Lie groupoid, once again. Form the space of composable pairs

$$\mathbb{G}^{(2)} = \{\, (\gamma, \eta) \in \mathbb{G}\times\mathbb{G} : s(\gamma) = t(\eta)\,\},$$

and denote by

$$c\colon \mathbb{G}^{(2)} \longrightarrow \mathbb{G} \quad \text{and} \quad p_1, p_2\colon \mathbb{G}^{(2)} \longrightarrow \mathbb{G}$$

the composition map $c(\gamma, \eta) = \gamma \circ \eta$ and the two coordinate projections.

4.2.1 Definition. Let $\mathbb{V}$ be a smooth vector bundle over $\mathbb{G}$. A *multiplicative structure* on $\mathbb{V}$ is a morphism of vector bundles

$$p_1^* \mathbb{V} \otimes p_2^* \mathbb{V} \xrightarrow{\ \circ\ } c^* \mathbb{V},$$

or in other words a smoothly varying family of vector space morphisms

$$\mathbb{V}_\eta \otimes \mathbb{V}_\gamma \xrightarrow{\ \circ\ } \mathbb{V}_{\eta\circ\gamma},$$

that is associative in the natural sense that

$$v_\alpha \in \mathbb{V}_\alpha, \quad v_\beta \in \mathbb{V}_\beta, \quad v_\gamma \in \mathbb{V}_\gamma \quad \Rightarrow \quad v_\alpha \circ (v_\beta \circ v_\gamma) = (v_\alpha \circ v_\beta) \circ v_\gamma \in \mathbb{V}_{\alpha\circ\beta\circ\gamma}$$

for all composable α, β and γ.

4.2.2 Example. If $\mathbb{G} \rightrightarrows M$ is any smooth groupoid, and if V is a vector bundle on M then the bundle on $\mathbb{G}$ with fibers

$$\mathbb{V}_\gamma = V_{t(\gamma)} \otimes V^*_{s(\gamma)} = \operatorname{Hom}\big(V_{s(\gamma)}, V_{t(\gamma)}\big)$$

has an obvious multiplicative structure given by contraction/composition that we shall call the *standard multiplicative structure*.

4.2.3 Lemma. *Let $\mathbb{G}$ be a Lie groupoid equipped with a smooth left Haar system. If $\mathbb{V}$ is a vector bundle on $\mathbb{G}$ with multiplicative structure, then the formula*

$$f_1 \star f_2(\eta) = \int_{\mathbb{G}^{t(\eta)}} f_1(\gamma) \circ f_2(\gamma^{-1} \circ \eta)\, d\mu^{t(\eta)}(\gamma),$$

defines an associative product on the smooth, compactly supported sections of $\mathbb{V}$. □

4.3 Multiplicative Structure on the Rescaled Spinor Bundle

The rescaled spinor bundle $\mathbb{S}$ over the tangent groupoid that we constructed in Section 3 carries the standard multiplicative structure away from $\lambda = 0$ since

$$\mathbb{S}|_{\lambda\neq 0} = S \boxtimes S^*.$$

The purpose of this section is to prove the following result:

4.3.1 Theorem. *There is a unique multiplicative structure on the rescaled spinor bundle* $\mathbb{S}$ *over* $\mathbb{TM}$ *whose restriction away from* $\lambda = 0$ *is the standard multiplicative structure. On fibers at* $\lambda = 0$ *the multiplication map*

$$\mathbb{S}_{(X_m,0)} \otimes \mathbb{S}_{(Y_m,0)} \longrightarrow \mathbb{S}_{(X_m+Y_m,0)}$$

is given by the formula

$$\alpha \otimes \beta \longmapsto \alpha \wedge \beta \wedge \exp\bigl(-\tfrac{1}{2}\kappa(X_m, Y_m)\bigr).$$

The uniqueness statement in the theorem is clear since $\mathbb{T}M \setminus TM \times \{0\}$ is dense in $\mathbb{T}M$. To prove the existence statement we shall show that if $\rho, \tau \in S(\mathbb{T}M)$, and if the associated sections of $\mathbb{S}$ are pulled back to $\mathbb{T}M^{(2)}$ via p_1 and p_2, and then multiplied according to the formula in the statement of the theorem, then the result, namely

$$\begin{cases}(m_1, m_2, m_3, \lambda) \mapsto \varepsilon_{(m_1,m_2,\lambda)}(\rho) \circ \varepsilon_{(m_2,m_3,\lambda)}(\tau) & (m_1, m_2, m_3{\in}M, \lambda{\neq}0)\\ (X_m, Y_m, 0) \mapsto \varepsilon_{X_m}(\rho){\wedge}\varepsilon_{Y_m}(\tau){\wedge}\exp\bigl(\tfrac{1}{2}\kappa(Y_m, X_m)\bigr) & (X_m, Y_m \in T_mM)\end{cases} \tag{4.3.1}$$

is a smooth section of the pullback bundle $c^*\mathbb{S}$ over $\mathbb{T}M^{(2)}$. This will suffice. By linearity it further suffices to consider elements $\rho, \tau \in S(\mathbb{T}M)$ of the form

$$\rho = \rho_{p_1} t^{-p_1} \quad \text{and} \quad \tau = \tau_{p_2} t^{-p_2},$$

where ρ_{p_1} and τ_{p_2} have scaling orders at least p_1 and p_2, respectively. Form the pointwise composition

$$M{\times}M{\times}M \ni (m_1, m_2, m_3) \longmapsto \rho_{p_1}(m_1, m_2) \circ \tau_{p_2}(m_2, m_3) \in S_{m_1} \otimes S^*_{m_3},$$

which is a smooth section of the pullback to $M{\times}M{\times}M$ of $S{\boxtimes}S^*$ along the projection onto the first and third factors (which is the composition map for the pair groupoid). As we did in Subsection 3.4, choose a local frame $\{e_I\}$ of $S{\boxtimes}S^*$ consisting of sections whose scaling orders are at least the negatives of their Clifford orders. We can of course write

$$\rho_{p_1}(m_1, m_2) \circ \tau_{p_2}(m_2, m_3) = \sum_I f_I(m_1, m_2, m_3) \cdot e_I(m_1, m_3) \tag{4.3.2}$$

where each f_I is a smooth function on (an open subset of) $M{\times}M{\times}M$.

4.3.2 Lemma. *Each function* f_I *defined above vanishes to order* $\geq p_1{+}p_2{+}\ell(I)$ *on the diagonal* $M \subseteq M{\times}M{\times}M$.

Proof. Let us call an index I *regular* if f_I vanishes to order $\geq p_1{+}p_2{+}\ell(I)$ on the diagonal, and *deficient* otherwise. Write

$$\rho_{p_1}(m_1, m_2) \circ \tau_{p_2}(m_2, m_3) - \sum_{I\text{ regular}} f_I(m_1, m_2, m_3) \cdot e_I(m_1, m_3)$$
$$= \sum_{I\text{ deficient}} f_I(m_1, m_2, m_3) \cdot e_I(m_1, m_3). \tag{4.3.3}$$

The left-hand side has scaling order $\geq p_1 + p_2$ in the sense Definition 3.3.5, except using covariant derivatives and Clifford multiplications in both the first and third factors in $M \times M \times M$. If there were any deficient indices at all, then we could choose a deficient $I_{\min}$ for which the vanishing order of f_I was minimal. Call the vanishing order q; of course

$$q < p_1 + p_2 + \ell(I_{\min}) \tag{4.3.4}$$

by definition of deficiency. We could then find a differential operator D of order q so that $D(f_I e_I)$ is a smooth function multiple of e_I along the diagonal for all deficient I, and a nonzero function multiple for $I_{\min}$. But the the Clifford order of the right-hand side of (4.3.3) after applying D would be at least $\ell(I_{\min})$, whereas the Clifford order of the left-hand side after applying D would be at most $q - p_1 - p_2$. This contradicts (4.3.4). □

Now write

$$F_I = f_I t^{-(p_1+p_2+\ell(I))} \in A(\mathbb{T}M^{(2)}) \quad \text{and} \quad \sigma_I = e_I t^{\ell(I)} \in S(\mathbb{T}M).$$

We should like to prove that the section (4.3.1) is given by the formula

$$\begin{cases} (m_1, m_2, m_3, \lambda) \longmapsto \sum_I \varepsilon_{(m_1,m_2,m_3,\lambda)}(F_I)\varepsilon_{(m_1,m_3,\lambda)}(\sigma) \\ (X_m, Y_m, 0) \longmapsto \sum_I \varepsilon_{(X_m,0_m,-Y_m)}(F_I)\varepsilon_{X_m+Y_m}(\sigma_I). \end{cases} \tag{4.3.5}$$

See (2.4.6) for the notation. Since (4.3.5) is a combination of smooth functions on $\mathbb{T}M^{(2)}$, times pullbacks to $\mathbb{T}M^{(2)}$ of smooth sections of $\mathbb{S}$, this will suffice. The identity of (4.3.1) and (4.3.5) away from $\lambda = 0$ is clear, and we have seen in Proposition 3.7.1 that

$$\varepsilon_{(X_m+Y_m,0_m)}(\sigma_I) = \exp\bigl(\tfrac{1}{2}\kappa(Y_m, X_m)\bigr) \wedge \varepsilon_{(X_m,-Y_m)}(\sigma_I).$$

So in fact it suffices to prove that

$$\varepsilon_{X_m}(\rho) \wedge \varepsilon_{Y_m}(\tau) = \sum_I \varepsilon_{(X_m,0_m,-Y_m)}(F_I)\varepsilon_{(X_m,-Y_m)}(\sigma_I),$$

or, using (3.7.1), that

$$\varepsilon_{(X_m,0_m)}(\rho) \wedge \varepsilon_{(0_m,-Y_m)}(\tau) = \sum_I \varepsilon_{(X_m,0_m,-Y_m)}(F_I)\varepsilon_{(X_m,-Y_m)}(\sigma_I). \tag{4.3.6}$$

A systematic way to check this formula is to introduce the space $S(\mathbb{T}M^{(2)})$ of Laurent polynomials $\sum \sigma_p t^{-p}$ in which σ_p is a smooth section of $S \boxtimes \mathbb{C} \boxtimes S^*$ over $M \times M \times M$ that has scaling order p, as in the proof of Lemma 4.3.2. This is a module over $A(\mathbb{T}M^{(2)})$, and we have the obvious identity

$$\rho \circ \tau = \sum_I F_I \cdot \sigma_I$$

in $S(\mathbb{T}M^{(2)})$, and hence in the quotient

$$S_0(\mathbb{T}M^{(2)}) = S(\mathbb{T}M^{(2)})/t \cdot S(\mathbb{T}M^{(2)}).$$

The morphism

$$\varepsilon_m \colon S_0(\mathbb{T}M^{(2)}) \longrightarrow \wedge^* T_m M$$

defined, following Definition 3.4.3, by

$$\varepsilon_m \colon \sum \sigma_p t^{-p} \longmapsto \sum [\sigma_{-d}(m,m,m)]_d$$

has the properties that

$$\varepsilon_m(\rho \circ \tau) = \varepsilon_m(\rho) \wedge \varepsilon_m(\tau)$$

and that

$$\varepsilon_m(F \cdot \sigma) = \varepsilon_m(F) \cdot \varepsilon_m(\sigma),$$

and these settle (4.3.6) in the special case where $X_m = Y_m = 0$. The general case is settled by applying the special case to the elements

$$\overline{\rho} = \exp\big(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{0})}\big)\rho, \quad \overline{\tau} = \exp\big(\boldsymbol{\nabla}_{(\boldsymbol{0},\boldsymbol{-Y})}\big)\tau,$$
$$\overline{F_I} = \exp\big(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{0},\boldsymbol{-Y})}\big)F_I, \quad \text{and} \quad \overline{\sigma_I} = \exp\big(\boldsymbol{\nabla}_{(\boldsymbol{X},\boldsymbol{-Y})}\big)\sigma_I,$$

for which $\overline{\rho} \circ \overline{\tau} = \sum_I \overline{F_I} \cdot \overline{\sigma_I}$.

5 Convolution Algebra and Traces

The multiplicative structure on $\mathbb{S}$ provides us with a convolution algebra $C_c^\infty(\mathbb{T}M, \mathbb{S})$. In this section we shall construct our family of supertraces on this algebra.

5.1 A Haar System for the Tangent Groupoid

Let M be a smooth manifold. The target fibers of $\mathbb{T}M$ are of course

$$\mathbb{T}M^{(m,\lambda)} = \{m\} \times M \times \{\lambda\}$$

and

$$\mathbb{T}M^{(X_m,0)} = T_m M \times \{0\}$$

If we fix a smooth measure μ on M, and if we denote by μ^m the associated translation-invariant measures on the tangent spaces $T_m M$, then the formulas

$$\begin{cases} \mu^{(m,\lambda)} = |\lambda|^{-n}\mu \\ \mu^{(X_m,0)} = \mu^m \end{cases} \tag{5.1.1}$$

define a smooth left Haar system for $\mathbb{T}M$. So we can now form the associated *tangent groupoid algebra* $C_c^\infty(\mathbb{T}M)$. The definition is due to Connes, and we summarize the basic facts from [Con94, Sec. II.5]:

5.1.1 PROPOSITION. *For $\lambda \neq 0$ the linear map*

$$\varepsilon_\lambda \colon C_c^\infty(\mathbb{T}M) \longrightarrow \mathfrak{K}^\infty(L^2(M))$$

given by the formula

$$\varepsilon_\lambda(f) \colon (m_1, m_2) \longmapsto \lambda^{-n} f(m_1, m_2, \lambda)$$

is a homomorphism of algebras. In addition the linear map

$$\varepsilon_0 \colon C_c^\infty(\mathbb{T}M) \longrightarrow C_c^\infty(TM)$$

given by the formula

$$\varepsilon_0(f) \colon X_m \longmapsto f(X_m, 0)$$

is a homomorphism of algebras, too, if the target $C_c^\infty(TM)$ is equipped with the fiberwise convolution product. □

The factor λ^{-n} that appears above comes from the fact that we are using a fixed measure on M to define $L^2(M)$, whereas the chosen Haar system is comprised of measures on copies of M that vary with λ.

5.2 Twisted Convolution on the Tangent Bundle

The first statement in Proposition 5.1.1 has an obvious spinorial counterpart, whose proof requires no new ideas:

5.2.1 PROPOSITION. *For $\lambda \neq 0$ the morphism*

$$\varepsilon_\lambda \colon C_c^\infty(\mathbb{T}M, \mathbb{S}) \longrightarrow \mathfrak{K}^\infty(L^2(M, S))$$

given by the formula

$$\varepsilon_\lambda(\sigma) \colon (m_1, m_2) \longmapsto \lambda^{-n} \sigma(m_1, m_2, \lambda)$$

is a homomorphism of algebras. □

We can also define

$$\varepsilon_0 \colon C_c^\infty(\mathbb{T}M, \mathbb{S}) \longrightarrow C_c^\infty(TM, \wedge^* TM)$$

by restriction, so that

$$\varepsilon_0(\sigma) \colon X_m \longmapsto \sigma(X_m, 0).$$

But in order to make this a homomorphism of algebras we need to adjust the convolution operation on $C_c^\infty(TM, \wedge^* TM)$, in accordance with Theorem 4.3.1. Other than this adjustment, the proof of the following theorem is identical to the case of the ordinary tangent groupoid algebra considered by Connes.

5.2.2 Proposition. *If the space $C_c^\infty(TM, \wedge^* TM)$ is equipped with the twisted convolution product*

$$(\varphi_1 \star \varphi_2)(X_m) = \int_{T_m M} \varphi_1(X_m - Y_m) \wedge \varphi_2(Y_m) \wedge \exp(\tfrac{1}{2}\kappa(Y_m, X_m))\, d\mu^m(Y_m),$$

then the restriction map

$$\varepsilon_0 \colon C_c^\infty(\mathbb{T}M, \mathbb{S}) \longrightarrow C_c^\infty(TM, \wedge^* TM)$$

is a homomorphism of algebras. □

5.2.3 Remark. This is an appropriate time to note that our approach shares much with a manuscript of Siegel [Sie10]. The above proposition is, however at variance with the corresponding formula there [Sie10, p.16].

5.3 Supertraces on the Clifford Algebra

Let E be an even-dimensional and oriented Euclidean vector space. Let $e_1, \dots, e_n$ be an oriented orthonormal basis for E, and for $I = (i_1 < i_2 < \dots < i_d)$ let

$$e_I = e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_d} \in \operatorname{Cliff}(E).$$

The linear functional

$$\operatorname{str} \colon \operatorname{Cliff}(E) \longrightarrow \mathbb{C}$$

defined by

$$\operatorname{str}(e_I) = \begin{cases} 1 & I = (1, 2, \dots, n) \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.1}$$

is independent of the choice of oriented, orthonormal basis, and is a supertrace on the Clifford algebra. See [Mei13, Sec. 2.2.8]. Note that

$$\operatorname{str}|_{\operatorname{Cliff}_{n-1}(E)} = 0. \tag{5.3.2}$$

The supertrace can be calculated using the irreducible representation

$$c \colon \operatorname{Cliff}(E) \xrightarrow{\cong} \operatorname{End}(S)$$

as follows. The element

$$s = i^{\frac{n}{2}} e_1 \cdot \ldots \cdot e_n \in \operatorname{Cliff}(E) \tag{5.3.3}$$

is independent of the choice of oriented orthonormal basis and satisfies $s^2 = 1$. The self-adjoint operator $c(s)$ determines a $\mathbb{Z}/2$-grading of the vector space S, and

$$\operatorname{str}(x) = \left(\tfrac{i}{2}\right)^{\frac{n}{2}} \operatorname{Tr}(c(s)c(x)) \tag{5.3.4}$$

for all $x \in \operatorname{Cliff}(E)$.

5.4 Supertraces on the Convolution Algebra

The Hilbert space $L^2(M,S)$ carries a $\mathbb{Z}/2$-grading that is defined as follows. If $e_1,\dots,e_n$ is any local oriented orthonormal frame for the tangent bundle of M, then the product

$$c(s) = i^{\frac{n}{2}} c(e_1)c(e_2)\dots c(e_n)$$

defines locally an endomorphism of S whose square is the identity. It is in fact independent of the choice of local oriented orthonormal frame, and so the formula above defines a canonical global endomorphism of S. It is self-adjoint and squares to the identity, and is by definition the grading operator for the $\mathbb{Z}/2$-grading on $L^2(M,S)$.
We shall denote by

$$\mathrm{STr}\colon \mathfrak{K}^\infty(L^2(M,S)) \longrightarrow \mathbb{C}$$

the associated supertrace, and we shall use this to define a family of supertraces on $C_c^\infty(\mathbb{T}M,\mathbb{S})$, as follows:

5.4.1 Definition. For $\lambda \in \mathbb{R}\setminus\{0\}$ we shall denote by

$$\mathrm{STr}_\lambda\colon C_c^\infty(\mathbb{TM},\mathbb{S}) \longrightarrow \mathbb{C}$$

the composition

$$C_c^\infty(\mathbb{TM},\mathbb{S}) \xrightarrow{\varepsilon_\lambda} \mathfrak{K}^\infty(L^2(M,S)) \xrightarrow{\mathrm{STr}} \mathbb{C},$$

as in (1.0.10). In addition, we define the supertrace STr_0 using (1.0.11).

5.4.2 Theorem. *If $\tau \in C_c^\infty(\mathbb{T}M,\mathbb{S})$, then $\lambda \mapsto \mathrm{STr}_\lambda(\tau)$ is a smooth function of $\lambda \in \mathbb{R}$.*

5.4.3 Remark. The ordinary tangent groupoid algebra carries a family of traces, parametrized by $\lambda \neq 0$, that are obtained by composing the homomorphisms (1.0.4) with the usual operator trace on smoothing operators:

$$C_c^\infty(\mathbb{T}M) \xrightarrow{\varepsilon_\lambda} \mathfrak{K}^\infty(L^2(M)) \xrightarrow{\mathrm{Tr}} \mathbb{C}.$$

Roughly speaking, local, or algebraic, index theory is the study of these traces as $\lambda \to 0$. The traces do not converge as $\lambda \to 0$, and instead more elaborate strategies must be developed, for instance replacing the traces with equivalent cyclic cocycles. See for example [NT95] or [Per13] for two perspectives on this. It is a remarkable fact, discovered of course by Getzler, that in the supersymmetric context the traces *do* converge.

Proof of Theorem 5.4.2. The supertrace on $\mathcal{K}^\infty(L^2(M,S))$ can be written

$$\mathrm{STr}(k) = \int_M \mathrm{str}\big(k(m,m)\big)\, d\mu(m),$$

where str is the pointwise supertrace on $\mathrm{End}(S_m)$. So according to the definitions, if $\tau \in C_c^\infty(\mathbb{T}M, \mathbb{S})$ and $\lambda \neq 0$, then

$$\mathrm{STr}_\lambda(\tau) = \lambda^{-n} \int_M \mathrm{str}\big(\tau(m, m, \lambda)\big)\, d\mu(m).$$

We shall show that for any smooth section τ the map

$$(m, \lambda) \longmapsto \lambda^{-n}\, \mathrm{str}\big(\tau(m, m, \lambda)\big)$$

extends to a smooth function on $M \times \mathbb{R}$, and then calculate the value of the extension at $0 \in \mathbb{R}$ to be

$$(m, 0) \longmapsto \mathrm{str}(\tau(0_m, 0)), \tag{5.4.1}$$

where $0_m \in T_m M$ is the zero tangent vector and the supertrace is the coefficient of $e_1 \wedge \cdots \wedge e_n \in \wedge^* T_m M$, with $e_1, \ldots, e_n$ as above. This will suffice.

Any smooth section of $\mathbb{S}$ over $\mathbb{T}M$ is locally a finite sum of products $f \cdot \widehat{\sigma}$, where f is a smooth function on $\mathbb{T}M$ and $\sigma \in S(\mathbb{T}M)$; see Definition 3.5.3. Since

$$\lambda^{-n}\, \mathrm{str}\big((f \cdot \widehat{\sigma})(m, m, \lambda)\big) = f(m, m, \lambda) \cdot \lambda^{-n}\, \mathrm{str}\big(\widehat{\sigma}(m, m, \lambda)\big)$$

it suffices to show that $\lambda^{-n}\, \mathrm{str}(\widehat{\sigma}(m, m, \lambda))$ extends to a smooth function on $M \times \mathbb{R}$, and calculate that the value of the extension at $\lambda = 0$ agrees with (5.4.1).

If $\sigma = \sum \sigma_p t^{-p}$, then

$$\lambda^{-n}\, \mathrm{str}\big(\widehat{\sigma}(m, m, \lambda)\big) = \sum \lambda^{-p-n} \sigma_p(m, m)$$

Now if $p > -n$, then the restriction of σ_p to the diagonal point (m, m) lies in

$$c\big(\mathrm{Cliff}_{n-1}(T_m M)\big) \subseteq \mathrm{End}(S_m),$$

and hence by (5.3.2) it has supertrace zero. So after writing $q = -p$ we find that

$$\lambda^{-n}\, \mathrm{str}\big(\widehat{\sigma}(m, m, \lambda)\big) = \sum_{q \geq n} \lambda^{q-n}\, \mathrm{str}\big(\widehat{\sigma}_{-q}(m, m)\big),$$

which is clearly a smooth function of $m \in M$ and $\lambda \in \mathbb{R}$. The value at $\lambda = 0$ is $\mathrm{str}(\widehat{\sigma}_{-n}(m, m))$, and if we write

$$\sigma_{-n} = \sum_I h_I e_I$$

as in (3.4.1), then from (5.3.1) we find that

$$\mathrm{str}\big(\widehat{\sigma}_{-n}(m, m)\big) = h_{I_n}(m, m)$$

where $I_n = (1, 2, \ldots, n)$. This is the coefficient of $e_1 \wedge \cdots \wedge e_n$ in the fiber $\wedge^* T_m M$, as required. $\square$

5.5 Final Comments on Index Theory

In this concluding subsection we shall comment on the roles that the tangent groupoid and rescaling play in index theory, and suggest future developments, which we aim to pursue elsewhere.

Let us return to Theorem 2.5.1. We noted there that the family of operators $\{D_{(m,\lambda)}\}$ on the source fibers of the tangent groupoid that is associated to a single linear partial differential operator on M is *equivariant* for the (right) action of the groupoid $\mathbb{T}M$ on itself. This has the following consequence:

5.5.1 Lemma. *The family of operators $\{D_{(m,\lambda)}\}$ on the source fibers of $\mathbb{T}M$ acts on the function space $C_c^\infty(\mathbb{T}M)$ as a right $C_c^\infty(\mathbb{T}M)$-module endomorphism.*

If D is in addition elliptic, then we can say more. To make the cleanest statement it is convenient to introduce the quotient algebra $C_c^\infty(\mathbb{T}M)_{[0,1]}$ of $C_c^\infty(\mathbb{T}M)$ by the ideal of all smooth, compactly supported functions on $\mathbb{T}M$ that vanish for all $\lambda \in [0,1]$. Of course the family $\{D_{(m,\lambda)}\}$ acts on this algebra by right module endomorphisms, too. Ellipticity implies that this action is almost invertible:

5.5.2 Theorem. *If M is closed, and if D is elliptic, then the associated right-module endomorphism of $C_c^\infty(\mathbb{T}M)_{[0,1]}$ is invertible modulo left multiplications by elements of $C_c^\infty(\mathbb{T}M)_{[0,1]}$.*

To be explicit, the theorem asserts that there are right module maps

$$\mathbb{D}\colon C_c^\infty(\mathbb{T}M)_{[0,1]} \longrightarrow C_c^\infty(\mathbb{T}M)_{[0,1]} \quad \text{and} \quad \mathbb{Q}\colon C_c^\infty(\mathbb{T}M)_{[0,1]} \longrightarrow C_c^\infty(\mathbb{T}M)_{[0,1]},$$

the first associated to $\{D_{(m,\lambda)}\}$, for which the operators

$$\mathbb{I} - \mathbb{D}\mathbb{Q},\ \mathbb{I} - \mathbb{Q}\mathbb{D}\colon C_c^\infty(\mathbb{T}M)_{[0,1]} \longrightarrow C_c^\infty(\mathbb{T}M)_{[0,1]}$$

are left multiplications by elements of $C_c^\infty(\mathbb{T}M)_{[0,1]}$. The theorem may be proved using pseudodifferential operator theory (and see [EY17] for an account of the theory of pseudodifferential operators that is particularly well suited to the present context).

The theorem implies that $\mathbb{D}$ defines a class in $K_0(C_c^\infty(\mathbb{T}M)_{[0,1]})$; see for example [Mil71, Sec. 2]. This is an essential step in Connes' approach to index theory via K-theory and the tangent groupoid.

5.5.3 Remarks. Actually when considering K-theory it is preferable to pass to a Fréchet algebra completion of $C_c^\infty(\mathbb{T}M)_{[0,1]}$, as in [CR08], or, even better, the C^*-algebra completion considered by Connes in [Con94, Sec. II.5]. In addition, in order to get a sufficiently rich class of examples, one should introduce operators acting on sections of bundles, and use the associated modified convolution algebras, as in Example 4.2.2 and Lemma 4.2.3 above.

It is an interesting challenge to fit the rescaled bundle and the algebra $C_c^\infty(\mathbb{T}M, \mathbb{S})$ into this type of K-theory picture. The main issue is that the Dirac operator $\not{D}$ gives rise to a family of operators for which the analogue of Lemma 5.5.1 holds, but *not* the analogue of Theorem 5.5.2, the latter because the model operators $\not{D}_{(m,0)}$ are not elliptic, as they are in the standard case (as is well known they are in fact the de Rham differentials on the tangent fibers). Perhaps Kasparov's Dirac operator d_M from [Kas88, Def. 4.2] has a role to play here.

There are other interesting challenges, too. For instance although the convolution algebra $C_c^\infty(\mathbb{T}M, \mathbb{S})$ admits natural Fréchet and Banach algebra completions [Yi19], there is no C^*-algebra completion.

Getzler took a different approach that focussed not on $\not{D}$ but on the Laplace-type operator $\Delta = \not{D}^2$, for which the model operators $\Delta_{(m,0)}$ are variants of the quantum harmonic oscillator (and are elliptic). Supersymmetry relates the supertraces considered in Subsection 5.4 to the index of the Dirac operator:

5.5.4 Lemma. *The supertrace* $\mathrm{STr}\big(\exp(-\lambda^2\Delta)\big)$ *is the index of the Dirac operator* $\not{D}$*, and is in particular a constant, integer-valued function of* $\lambda \neq 0$.

As Getzler pointed out, the smoothness of the family of supertraces STr_λ from Subsection 5.4 now allows one to compute the index from the value at λ=0, which involves only the operators $\Delta_{(m,0)}$, which depend only on the Riemannian curvature of M. See [Get83, BGV92, Roe98]. It will be interesting to explore this more thoroughly from the point of view of the cyclic cohomology of the algebra $C_c^\infty(\mathbb{T}M, \mathbb{S})$, and also discover what lessons can be learned in K-theory and K-homology about the use of $\not{D}^2$ rather than $\not{D}$ here.

6 Appendix. Taylor Expansions

The purpose of this appendix is to prove Proposition 3.3.10. We shall use the exponential map

$$TM \ni X_m \longmapsto (\exp_m(X_m), m) \in M \times M,$$

which is a diffeomorphism from a neighborhood of the zero section in the tangent bundle onto a neighborhood of the diagonal in $M\times M$, and the associated Euler vector field E, defined on a neighborhood of the diagonal in $M\times M$, by

$$E_{(\exp(X_m),m)} = \frac{d}{ds}\Big|_{s=1} (\exp_m(sX_m), m).$$

The Euler vector field is tangent to each source fiber $M\times\{m\}$ of the pair groupoid, and if $(x_1, \ldots, x_n)$ are geodesic local coordinates on M that are centered at m, then

$$E = \sum_{i=1}^n x_i \partial_i$$

on $M\times\{m\}$. We shall also use the concept of Taylor series that is explained in the following two definitions.

6.0.1 Definition. We shall say that a smooth section σ of $S \boxtimes S^*$ is *synchronous near* $m \in M$, if $\nabla_E \sigma = 0$ in a neighborhood of $(m, m) \in M \times M$.

By parallel translation, every smooth section of $S \boxtimes S^*$ on the diagonal extends to a smooth section that is synchronous near the diagonal.

6.0.2 Definition. Let $m \in M$ and let $(x_1, \ldots, x_n)$ be smooth functions defined in a neighborhood of $(m, m) \in M \times M$ that restrict to geodesic local coordinates at (m', m') on each $M \times \{m'\}$. Let σ be a smooth section $S \boxtimes S^*$. A *Taylor expansion* of the section σ at $m \in M$ is a formal series

$$\sum_{\alpha \geq 0} x^\alpha \sigma_\alpha, \tag{6.0.1}$$

where

(i) the sum is over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, with each α_k a nonnegative integer, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$;

(ii) each σ_α is a smooth section of $S \boxtimes S^*$ that is synchronous near m (note that since it is synchronous near m, σ_α is determined by its values along the diagonal near m)

(iii) the series is asymptotic to σ near the diagonal and near $(m, m) \in M \times M$ in the sense that for every $N \in \mathbb{N}$ the difference

$$\sigma - \sum_{|\alpha| < N} x^\alpha \sigma_\alpha$$

vanishes to order N on the diagonal near (m, m) (here $|\alpha| = \alpha_1 + \cdots + \alpha_n$).

Every smooth section has a unique Taylor expansion. Proposition 3.3.10 is a consequence of the following result:

6.0.3 Proposition. *Let σ be a smooth section of the $S \boxtimes S^*$, and let $m \in M$. If*

$$\sigma \sim \sum_{\alpha \geq 0} x^\alpha \sigma_\alpha$$

is the Taylor series of σ near m, then

$$\text{Scaling-order}(\sigma) \geq \min_\alpha \{ \, |\alpha| - \text{Clifford-order}(\sigma_\alpha) \, \} \tag{6.0.2}$$

near m. In particular, if σ is synchronous near m, then

$$\text{Scaling-order}(\sigma) \geq -\text{Clifford-order}(\sigma).$$

near m.

The proposition is proved as follows. Let us temporarily call the quantity on the right hand side of (6.0.2) the *Taylor order* of σ. Obviously

$$\text{Taylor-order}(\sigma) \leq -\text{Clifford-order}(\sigma)$$

If we can prove that applying an operator D to σ decreases the Taylor order by at most the Getzler order of D, then we shall get

$$\begin{aligned}\text{Taylor-order}(\sigma) - \text{Getzler-order}(D) &\leq \text{Taylor-order}(D\sigma)\\ &\leq -\text{Clifford-order}(D\sigma)\end{aligned}$$

and hence

$$\text{Clifford-order}(D\sigma) \leq \text{Getzler-order}(D) - \text{Taylor-order}(\sigma)$$

In view of Definition 3.3.5, the proposition follows immediately from this. As for the effect on the Talyor order of applying D, it is clear that a Clifford multiplication $c(X)$ increases it by at most one; the other case to consider, that of a covariant derivative ∇_X, is handled by the following lemma:

6.0.4 Lemma. *Let σ be a smooth section of the $S\boxtimes S^*$ that is synchronous near $m \in M$, and let X be a vector field on M. The Taylor series at m of the section $\nabla_X \sigma$ has the form*

$$\nabla_X \sigma \sim \sum_{|\alpha|\geq 1} x^\alpha \, c(q(\omega_\alpha))\, \sigma,$$

where each ω_α is the germ near $m \in M$ of a smooth section of $\wedge^2 TM$. Here we regard $c(q(\omega_\alpha))$ as a section of $S\boxtimes S^*$ defined on the diagonal near m, and extend it to a section over $M\times M$ that is synchronous near m.

Proof. (Compare [Roe98, Prop. 12.22].) A general vector field X on M can be written as a combination $\sum_i f_i \partial_i$, and by expanding the smooth coefficient functions f_i in Taylor series we see that it suffices to prove the lemma for the coordinate vector fields $X = \partial_i$.
According to the definition of curvature,

$$\nabla_E \nabla_X \sigma - \nabla_X \nabla_E \sigma - \nabla_{[E,X]} \sigma = K(E,X)\, \sigma. \tag{6.0.3}$$

Since the section σ is synchronous near m,

$$\nabla_E \sigma = 0 \tag{6.0.4}$$

in a neighborhood of $(m,m) \in M\times M$. Moreover, since X is a coordinate vector field,

$$[E,X] = -X. \tag{6.0.5}$$

Inserting (6.0.4) and (6.0.5) into (6.0.3) we find that

$$\nabla_E \nabla_X \sigma + \nabla_X \sigma = K(E,X)\, \sigma. \tag{6.0.6}$$

Now expand $\nabla_X \sigma$ as a Taylor series at $m \in$M,

$$\nabla_X \sigma \sim \sum_{|\alpha|\geq 1} x^\alpha \sigma_\alpha \tag{6.0.7}$$

(there is no order zero term because the section σ is synchronous). Using the formula

$$\nabla_E \, x^\alpha \, \sigma_\alpha = |\alpha| \, x^\alpha \, \sigma_\alpha$$

for the Euler vector field we find that the Taylor series for $\nabla_E \nabla_X \sigma$ is

$$\nabla_E \nabla_X \, \sigma \sim \sum_\alpha |\alpha| \, x^\alpha \sigma_\alpha. \tag{6.0.8}$$

Next, recall that the curvature operator $K(E, X)$ may be written as

$$K(E, X) = c\big(\gamma(R(E, X))\big).$$

See (3.2.3). Write the section $\gamma(R(E, X))$ of $\wedge^2 TM$ as a Taylor series

$$\gamma(R(E, X)) \sim \sum_{|\alpha|\geq 1} x^\alpha \eta_\alpha, \tag{6.0.9}$$

where each $\eta_\alpha \in \wedge^2 TM$ is synchronous at $m \in M$ for the Levi-Civita connection (there is no order zero term in this Taylor expansion either, this time because the vector field E vanishes at $m \in M$). Inserting (6.0.7), (6.0.8) and (6.0.9) into (6.0.6) we obtain an identity of Taylor expansions

$$\sum_{|\alpha|\geq 1} (1 + |\alpha|) \, x^\alpha \sigma_\alpha = \sum_{|\alpha|\geq 1} x^\alpha c(q(\eta_\alpha))\sigma.$$

The lemma follows from this. □

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