Constructible 1-Motives and Exactness of Realisation Functors

Simon Pepin Lehalleur

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Abstract. The triangulated category of cohomological 1-motives with rational coefficients over a base scheme admits a motivic t-structure. We prove that this t-structure restricts to the subcategory of compact objects, and that pullbacks along arbitrary morphisms, as well as Betti and étale realisation functors, are t-exact relative to this t-structure. These exactness properties follow from a structural result: compact objects in the heart behave like a constructible sheaf of Deligne 1-motives.

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Introduction

This paper takes place in the context of triangulated categories of mixed motivic sheaves in the sense of Morel-Voevodsky, and is a follow-up to [12]. Let S be a finite dimensional noetherian excellent scheme. Write $\mathbf{DA}(S)$ for the triangulated category of mixed motives over S with rational coefficients. Let $\mathbf{DA}^1(S) \subset \mathbf{DA}(S)$ be the localizing subcategory generated by compactly supported cohomological motives of relative curves over S. This category is a natural environment to study cohomology of families of curves and their degenerations. The category $\mathbf{DA}(S)$ is conjectured to admit a motivic t-structure compatible with standard t-structures on derived categories of sheaves via realisation functors. In [12], we constructed a candidate for the motivic t-structure on the subcategory $\mathbf{DA}^1(S)$. Its heart is an abelian category $\mathbf{MM}^1(S)$. In this paper, we solve some of the main questions left open in [12] about the motivic t-structure and the category $\mathbf{MM}^1(S)$.

We prove (under a mild hypothesis on S) that pullbacks along arbitrary morphisms are t-exact on compact objects (Theorem 2.1 (iii)), and that Betti and étale realisation functors are t-exact if the target categories of realisation functors are equipped with their standard t-structure (Theorem 2.1 (iii) (iv) (v)). In particular, this justifies our claim that the motivic t-structure on $\mathbf{DA}^1_c(S)$ is the restriction to the conjectural motivic t-structure on $\mathbf{DA}_c(S)$. To make sense of these statements, we first have to prove that the motivic t-structure on $\mathbf{DA}^1(S)$ restricts to the category $\mathbf{DA}^1_c(S)$ of compact cohomological 1-motives (Theorem 2.1 (i)).

Let $\mathbf{MM}^1_c(S)$ be the category of constructible cohomological 1-motives, that is, the heart of the restricted t-structure on $\mathbf{DA}^1_c(S)$. We show that, for any $M \in \mathbf{MM}^1_c(S)$, there exists a stratification of the base S by regular locally closed subschemes such that the restrictions of M to strata are Deligne 1-motives (Theorem 2.1 (ii)). This structural result easily implies the other statements of Theorem 2.1 and its proof occupies most of the paper. The result is true generically since $\mathbf{MM}^1_c(k)$ is equivalent to the category of Deligne 1-motives over a field k (Proposition 1.7, and by a continuity argument holds over a dense open subset of S. Using localisation triangles and easy homological algebra, we are reduced to a statement about 1-motivic degeneration of Deligne 1-motives, with the key case being the degeneration of the 1-motive associated to the Jacobian of a smooth projective pointed curve (Lemma 2.2). We show using results of De Jong that we can assume that the curve extends to a semi-stable curve with regular total space. In this geometric situation, we can conclude with an explicit computation.

Related work

The fact that the t-structure restricts to $\mathbf{DA}^1_c(S)$ has been obtained previously by V. Vaish in [14]. Vaish's approach relies on an elegant combination of the gluing procedure for t-structures of [4] and the "weight truncation" t-structures of [9]. He first gives an alternative construction of the functor $\omega^1 : \mathbf{DA}^{\mathrm{coh}}_c(S) \to \mathbf{DA}^1_c(S)$ (see Definition 1.2 and Theorem 1.3) by gluing the analoguous functors $\omega^1 : \mathbf{DA}^{\mathrm{coh}}_c(k(s)) \to \mathbf{DA}^1_c(k(s))$ for all points $s \in S$ (which exist by [3]), and then uses gluing data of the form

$$(j_! \dashv j^* \dashv \omega^1 j_*, i^* \dashv i_* \dashv \omega^1 i^!)$$

for $j : U \to S \leftarrow Z : i$ complementary open and closed immersions to glue together the t-structures on the $\mathbf{DA}^1_c(k(s))$ (which exist by [11]). It is not clear to us how to prove the other results in Theorem 2.1 using the approach of [14]; we plan to come back to this point and to combine the strengths of our approaches in future work.

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Conventions

All schemes are assumed to be finite dimensional, noetherian and excellent. Unless specified, smooth morphisms are assumed to be separated of finite type. The notation Sm/S denotes the category of all smooth S-schemes considered as a site with the étale topology.
In this paper, a semi-abelian scheme over a scheme S is an S-group scheme which is an extension of an abelian scheme by a torus; in other words, a semi-abelian scheme in the more general sense which is of constant toric rank.

Definition 0.1. We say that a scheme S allows resolution of singularities by alterations if for any separated S-scheme X of finite type and any nowhere dense closed subset $Z \subset X$, there is a projective alteration $g : X' \to X$ with X' regular and such that $g^{-1}(Z)$ is a strict normal crossing divisor.

The best result available in this direction is due to Temkin [13, Theorem 1.2.4]: any S which is of finite type over a quasi-excellent scheme of dimension ≤ 3 allows resolution of singularities by alterations.

1 Background on relative 1-motives

For the comfort of the reader, we review some definitions and results from [12]. Let S be a scheme. The category $\mathbf{DA}(S) := \mathbf{DA}^{\text{ét}}(S, \mathbb{Q})$ is the triangulated category of rational étale motives coming from the stable homotopical 2-functor $\mathbf{DA}^{\text{ét}}(-, \mathbb{Q})$ considered in [2, §3].

Definition 1.1. The category $\mathbf{DA}^{\text{coh}}(S)$ of *cohomological motives* is the localising subcategory of $\mathbf{DA}(S)$ generated by

$$\{f_*\mathbb{Q}_X \mid f : X \to S \text{ proper morphism}\}.$$

The category $\mathbf{DA}^1(S)$ of *cohomological* 1-*motives* is the localising subcategory of $\mathbf{DA}(S)$ generated by

$$\{f_*\mathbb{Q}_X \mid f : X \to S \text{ proper morphism of relative dimension} \leq 1\}.$$

Definition 1.2. The full embedding $\mathbf{DA}^1(S) \hookrightarrow \mathbf{DA}^{\mathrm{coh}}(S)$ preserves small sums, thus by Neeman's version of Brown representability for compactly generated triangulated categories (see e.g. [10, Theorem 8.3.3]), admits a right adjoint $\omega^1 : \mathbf{DA}^{\mathrm{coh}}(S) \to \mathbf{DA}^1(S)$.

One of the main results of [12] (which was reproved later by Vaish in [14] with a different method) is the following.

Theorem 1.3. *Let S be a noetherian finite-dimensional excellent scheme. Assume that S allows resolution of singularities by alterations. Then the functor $\omega^1 : \mathbf{DA}^{\mathrm{coh}}(S) \to \mathbf{DA}^1(S)$ preserves compact objects.*

We recall the definition of Deligne 1-motives.

Definition 1.4. Let S be a scheme. A 2-term complex of commutative S-group schemes:

$$M = [\overset{0}{L} \longrightarrow \overset{-1}{G}]$$

is called a *Deligne* 1-*motive* over S if L is a lattice (i.e. an S-group scheme which is étale locally isomorphic to a free abelian group of finite rank) and G is a semi-abelian scheme. We denote by $\mathcal{M}_1(S)$ the category of Deligne 1-motives with rational coefficients (i.e., the idempotent completion of the category whose morphisms groups are morphism groups of Deligne 1-motives tensored with $\mathbb{Q}$).

Recall that sets of isomorphism classes of compact objects generate t-structures in compactly generated triangulated categories [1, Lemme 2.1.69, Proposition 2.1.70].

Definition 1.5. The *motivic t-structure* $t^1_{\mathbf{MM}}(S)$ on $\mathbf{DA}^1(S)$ is the t-structure generated by the family

$$\mathcal{DG}_S = \{e_\sharp \Sigma^\infty(\mathbb{M}) |\ e : U \to S \text{ étale },\ \mathbb{M} \in \mathcal{M}_1(U)\}.$$

of compact objects.

Let $(\mathcal{T}, \mathcal{T}_{\geq 0}, \mathcal{T}_{<0})$ be a triangulated category with a t-structure, written with the homological convention. In [12] we used the terminology "t-positive" for objects in $\mathcal{T}_{\geq 0}$ and "t-negative" for objects in $\mathcal{T}_{\leq 0}$; we adopt here the more correct english usage of "t-non-negative" for objects in $\mathcal{T}_{\geq 0}$ and "t-non-positive" for objects in $\mathcal{T}_{\leq 0}$.
The main properties of $t^1_{\mathbf{MM}}(S)$ from [12, §4] which we will use are the following.

- Elementary exactness properties [12, Proposition 4.14].
- Compact objects are bounded for $t^1_{\mathbf{MM}}(S)$ [12, Corollary 4.29].
- There is a functor $\Sigma^\infty(-)(-1) : \mathcal{M}_1(S) \to \mathbf{MM}^1(S)$ which is fully faithful when S is regular [12, Theorem 4.22, Theorem 4.31]. Furthermore, if $e : U \to S$ is any étale morphism and $\mathbb{M} \in \mathcal{M}_1(U)$, then $e_! \Sigma^\infty \mathbb{M}(-1) \in \mathbf{MM}^1(S)$.

For the theory over an imperfect field, we have the following results which complements the treatment in [12].

Proposition 1.6. *Let k be a field and l/k be a purely inseparable field extension. Then the base change functor*

$$\mathcal{M}_1(k) \to \mathcal{M}_1(l)$$

is an equivalence of categories.

Proof. Let us first prove that this functor is fully faithful. This can be deduced from the embedding into $\mathbf{MM}^1$ recalled above, but we give a direct proof. The idempotent completion of a fully faithful functor is fully faithful, so that we only have to study morphism groups between Deligne 1-motives. Let $\mathbb{M} = [L \xrightarrow{u} G]$ and $\mathbb{M}' = [L' \xrightarrow{u'} G']$ in $\mathcal{M}_1(k)$.
Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, it is enough to show faithfulness for the functor $\mathcal{M}_1(k, \mathbb{Z}) \to \mathcal{M}_1(l, \mathbb{Z})$. The base change functor from group schemes over k to group schemes over l is faithful, and this implies the result.
Since the category of lattices only depends on the small étale site $k_{\text{ét}}$ and $k_{\text{ét}} \simeq l_{\text{ét}}$, we have $\mathcal{M}_1(k)(L, L') \simeq \mathcal{M}_1(l)(L_l, L'_l)$. By [5, Theorem 3.11], $\mathrm{Hom}_{\underline{\mathcal{C}}_k}(G, G') \simeq \mathrm{Hom}_{\underline{\mathcal{C}}_l}(G_l, G'_l)$ with $\underline{\mathcal{C}}_k$ the category of smooth commutative k-groups up to isogeny in the sense of loc.cit. Since semi-abelian varieties over any field are divisible, this implies by [5, Proposition 3.6] that

$$\begin{aligned}\mathcal{M}_1(k)(G, G') &:= \mathrm{Hom}(G, G') \otimes \mathbb{Q} \\ &\simeq \mathrm{Hom}_{\underline{\mathcal{C}}_k}(G, G') \\ &\simeq \mathrm{Hom}_{\underline{\mathcal{C}}_l}(G_l, G'_l) \\ &\simeq \mathcal{M}_1(l)(G, G').\end{aligned}$$

We can now prove fullness. Let $g = f \otimes \frac{1}{n} \in \mathcal{M}_1(l)(\mathbb{M}_l, \mathbb{M}'_l)$ with $f = (f^L, f^G) \in \mathcal{M}_1(l, \mathbb{Z})(\mathbb{M}_l, \mathbb{M}'_{l'})$. By the previous paragraph, there exist preimages $f_0^L : L \to L'$ and $f_0^G : G \to G'$ of f^L, f^G. The pair (f_0^L, f_0^G) is a morphism of complexes if and only if $u' \circ f_0^L = f_0^G \circ u : L \to G'$. Because the base change for group schemes from k to l is faithful, we can check this over l, where it follows from the fact that f is a morphism. This concludes the proof of fullness.
We prove essential surjectivity. The idempotent completion of an equivalence of categories is an equivalence of categories, so we have to show that Deligne 1-motives lie in the essential image. Let $\mathbb{M} = [L \xrightarrow{u} G] \in \mathcal{M}_1(l)$. By étale descent and semi-simplicity of lattices up to isogeny, we can assume furthermore that $L \simeq \mathbb{Z}^r$ is split. By standard spreading-out arguments, we see that $\mathbb{M}$ is defined over a finitely generated (hence finite since it is purely inseparable) subextension of l. Hence we assume that l is finite over k, which implies that $l^q \subset k$ with $q = p^N$ is a large enough power of p. Since l/k is purely inseparable, there is a lattice L_0 over k such that $L \simeq (L_0)_l$ as group schemes. By [5, Theorem 3.11] (again combined with [5, Proposition 3.6] and divisibility

of semi-abelian varieties), there exists a semi-abelian variety G_0 over k and an isogeny $\lambda : G \to (G_0)_l$. We thus get a morphism $\lambda u : \mathbb{Z}^r \to (G_0)_l$. By [7, Exp. VIIA §4.3], we see that $[q]\lambda u$ factors through a morphism $L_0 \to G_0$, which makes $[L_0 \to G_0] \in \mathcal{M}_1(k)$ into a pre-image of $\mathbb{M}$. This concludes the proof. ◻

PROPOSITION 1.7. *Over a field k, the t-structure restricts to compact objects and the functor $\Sigma^\infty(-)(-1) : D^b(\mathcal{M}_1(k)) \to \mathbf{DA}^1_c(k)$ is an equivalence of t-categories, so that $\Sigma^\infty(-)(-1) : \mathcal{M}_1(k) \simeq \mathbf{MM}^1_c(k)$.*

Proof. This follows from [12, Proposition 4.21] combined with Proposition 1.6. ◻

In the same vein, here is a result implicit in [12] which we make explicit for later reference.

LEMMA 1.8. *Let $f : T \to S$ be a finite surjective radicial morphism. Then $f^* : \mathbf{DA}^1(S) \to \mathbf{DA}^1(T)$ is an equivalence of t-categories.*

Proof. For such a morphism, $f^* \simeq f^! : \mathbf{DA}_{(c)}(S) \to \mathbf{DA}_{(c)}(T)$ is an equivalence by [1, Corollaire 2.1.164]. Since f is finite, the functor f_* sends $\mathbf{DA}^1(S)$ to $\mathbf{DA}^1(T)$, so that f^* induces an equivalence between $\mathbf{DA}^1(S)$ and $\mathbf{DA}^1(T)$. Finally, the t-exactness follows from $f^* \simeq f^!$ and [12, Proposition 4.14]. ◻

2 EXACTNESS PROPERTIES OF CONSTRUCTIBLE 1-MOTIVES

Here is the main theorem of this paper.

THEOREM 2.1. *Let S be an scheme allowing resolution of singularities by alterations. Then*

(i) *The t-structure $t^1_{\mathbf{MM}}(S)$ restricts to the subcategory $\mathbf{DA}^1_c(S)$ of constructible 1-motives. Denote its heart by $\mathbf{MM}^1_c(S)$.*

(ii) *Let M be in $\mathbf{DA}^1_c(S)$. Then M is in $\mathbf{MM}^1(S)$ if and only if there exists a locally closed stratification $(i_\alpha : S_\alpha \to S)$ of S_{red} such that for all α, we have*

$$i_\alpha^* M \simeq \Sigma^\infty \mathbb{M}_\alpha(-1)$$

with $\mathbb{M}_\alpha$ a Deligne 1-motive on S_α. Moreover, we can assume the S_α to be regular.

(iii) *Let $f : T \to S$ be a morphism. Then the functor $f^* : \mathbf{DA}^1_c(S) \to \mathbf{DA}^1_c(T)$ is t-exact (with respect to the restricted t-structures from (i)).*

(iv) *Let ℓ be a prime number invertible on S. Then the functor $R_\ell : \mathbf{DA}^1_c(S) \to D^b_c(S_{\text{ét}}, \mathbb{Q}_\ell)$ obtained by restricting the rational ℓ-adic realisation functor from [2, Definition 9.6] is t-exact for the motivic t-structure of (i) on the source and the standard t-structure on the target.*

(v) *Assume that S is a finite type k-scheme with k a field of characteristic 0 admitting an embedding $\sigma : k \to \mathbb{C}$. Then the Betti realisation functor $R_{B,\sigma} : \mathbf{DA}^1_c(S) \to D^b_c(S_\sigma(\mathbb{C}), \mathbb{Q})$ is t-exact for the motivic t-structure of (i) on the source and the standard t-structure on the target.*

(vi) *Statements (i)-(v) also hold for homological 1-motives $\mathbf{DA}_1(-)$ (resp. for 0-motives $\mathbf{DA}^0(-)$) provided one replaces $\mathbb{M}(-1)$ by $\mathbb{M}$ (resp. by $\mathcal{F}$ with $\mathcal{F}$ a locally free sheaf of $\mathbb{Q}$-vector spaces) in (ii) (cf. [12] for the relevant definitions for homological 1-motives and 0-motives).*

Proof. First, a word of caution about notation. Since we do not know yet that $t^1_{\mathbf{MM}}(S)$ restricts to compact objects, we refrain from using the notation $\mathbf{MM}^1_c(S)$ and always write $\mathbf{MM}^1(S) \cap \mathbf{DA}^1_c(S)$ for the compact objects in the heart.

By Lemma 1.8, we can assume in every statement (i)-(v) that the schemes involved are reduced, and we will do so in the rest of the proof.

For every $d \in \mathbb{N}$ and every statement (i)-(v), write $(\text{i})_d$-$(\text{v})_d$ for the corresponding statements where the dimension of the schemes involved is less or equal to d (in particular, for $(\text{iii})_d$, we consider morphisms between schemes of dimension $\leq d$. We are going to prove all the statements by induction on d. More precisely, we show that

- $(\text{i})_0$ and $(\text{ii})_0$ hold.
- $((\text{i})_d$ and $(\text{ii})_d)$ imply $((\text{iii})_d$, $(\text{iv})_d$ and $(\text{v})_d)$ for all $d \in \mathbb{N}$.
- $((\text{i})_{d-1}$ and $(\text{ii})_{d-1})$ imply $((\text{i})_d$ and $(\text{ii})_d)$ for all $d \geq 1$.

Let us prove $(\text{i})_0$ and $(\text{ii})_0$. Let S be reduced of dimension 0. Then S is a finite disjoint union of spectra of fields, and the result follows from Proposition 1.7.

Let us show that $(\text{i})_d$ and $(\text{ii})_d$ imply $(\text{iii})_d$. Let $f : T \to S$ be a morphism of schemes with $\dim(S) \leq d$ and $\dim(T) \leq d$. By $(\text{i})_d$, $t^1_{\mathbf{MM}}(S)$ and $t^1_{\mathbf{MM}}(T)$ restrict to the subcategories of compact objects. We want to show that f^* is t-exact. By [12, Proposition 4.14], it is enough to show that $f^* : \mathbf{DA}^1_c(S) \to \mathbf{DA}^1_c(T)$ is t-non-positive. Let $M \in \mathbf{DA}^1_c(S)_{\leq 0}$. We have to show that f^*M is t-non-positive. By [12, Corollary 4.29], the motive M has finitely many non-zero homology objects, and thus can be obtained by finitely many extensions starting with non-positive shifts of objects in $\mathbf{MM}^1(S) \cap \mathbf{DA}^1_c(S)$. So it is enough to prove that $M \in \mathbf{MM}^1(S) \cap \mathbf{DA}^1_c(S) \Rightarrow f^*M$ is t-non-positive.

By $(\text{ii})_d$, there exists a stratification $\{S_\alpha\}$ of S so that we have

$$i_\alpha^* M \simeq \Sigma^\infty \mathbb{M}_\alpha(-1)$$

with $\mathbb{M}_\alpha$ a Deligne 1-motive on S_α. Consider the induced stratification $T_\alpha := f^{-1}(S_\alpha)$ of T, with $i'_\alpha : T_\alpha \to T$ and $f_\alpha : T_\alpha \to S_\alpha$. The T_α are not necessarily regular, but we can refine the stratification and assume they are. By [12, Corollary 2.21], we have $i'^*_\alpha f^* M \simeq f^*_\alpha \Sigma^\infty \mathbb{M}_\alpha(-1)$ is a Deligne 1-motive. By the other direction of (ii)$_d$, this shows that f^*M is in $\mathbf{MM}^1(T)\cap\mathbf{DA}^1_c(S)$ and in particular is t-non-positive.

We show (iv)$_d$ and (v)$_d$ assuming (i)$_d$ and (ii)$_d$. The argument is the same in both cases, so we only present the ℓ-adic case. By [12, Corollary 4.29], the motivic t-structure on $\mathbf{DA}^1_c(S)$ is bounded, so that to prove t-exactness it is enough to show that an object in the heart $\mathbf{MM}^1(S)\cap\mathbf{DA}^1_c(S)$ is sent to a constructible ℓ-adic sheaf. Let $M\in\mathbf{MM}^1(S)\cap\mathbf{DA}^1_c(S)$. By (ii)$_d$, there exists a locally closed stratification (S_α) of S_{red} such that for all α, if $i_\alpha : S_\alpha\to S$ is the natural immersion, we have

$$i_\alpha^* M\simeq \Sigma^\infty\mathbb{M}_\alpha(-1)$$

with $\mathbb{M}_\alpha$ a Deligne 1-motive on S_α. By gluing and exactness of pullbacks for ℓ-adic sheaves, it is enough to show that for any α, the object $i_\alpha^*R_\ell M\simeq R_\ell(\Sigma^\infty\mathbb{M}_\alpha)(-1)$ is a constructible ℓ-adic sheaf. This fact is established in the proof of [12, Proposition 4.15].

So it remains to show that ((i)$_{d-1}$ and (ii)$_{d-1}$) imply ((i)$_d$ and (ii)$_d$) for all $d\geq 1$. We assume (i)$_{d-1}$ and (ii)$_{d-1}$ for the rest of the proof. Let S be a scheme of dimension $\leq d$.

Let us show the "if" direction of Statement (ii)$_d$. Let $M\in\mathbf{DA}^1_c(S)$. Assume that there exists a stratification $\{S_\alpha\}$ of S so that we have

$$i_\alpha^* M\simeq \Sigma^\infty\mathbb{M}_\alpha(-1)$$

with $\mathbb{M}_\alpha$ a Deligne 1-motive on S_α. Write U for the union of the open strata, and Z for the complement, equipped with the reduced scheme structure (i.e. the union of all the other strata). Write $j : U\to S$ for the open immersion and $i : Z\to S$ for the complementary reduced closed immersion. Then Z is of dimension $< d$. We see that i^*M satisfies the same hypothesis, with the restricted stratification (since the pullback of a Deligne 1-motive is a Deligne 1-motive, [12, Corollary 2.21]). By (ii)$_{d-1}$, the motive i^*M is in $\mathbf{MM}^1(Z)$; by [12, Proposition 4.14], we get $i_*i^*M\in\mathbf{MM}^1(S)$. Moreover, j^*M is a Deligne 1-motive. By [12, Theorem 4.22], this implies that $j_!j^*M$ is in $\mathbf{MM}^1(S)$. By localisation, we have a distinguished triangle

$$j_!j^*M\to M\to i_*i^*M\xrightarrow{+}$$

which shows that $M\in\mathbf{MM}^1(S)$ as required.

In the rest of the proof, we establish the second part of (ii)$_d$ and (i)$_d$. Both statements will be established modulo the key geometric Lemma 2.2 below.

We first prove the rest of (ii)$_d$. Let $A\in\mathbf{MM}^1(S)\cap\mathbf{DA}^1_c(S)$. Let η be the scheme of generic points of S which is a disjoint union of spectra of fields. Let us show that η^*A lies in $\mathbf{MM}^1(\eta)\cap\mathbf{DA}^1_c(\eta)$. The functor η^* is t-non-negative by Proposition [12, Proposition 4.14] (where no finite type hypothesis is required). Let us show that η^*A is t-non-positive. We have to show that for any P in a compact generating family of $t^1_{\mathbf{MM}}(\eta)$ and $n>0$, we have

$\mathbf{DA}(\eta)(P[n], \eta^*A) \simeq 0$. By [12, Proposition 4.21], we can assume for instance that P is of the form $\Sigma^\infty \mathbb{M}'_\eta(-1)$ with $\mathbb{M}'_\eta \in \mathcal{M}_1(\eta)$.
By continuity results for Deligne 1-motives [12, Proposition A.10] and $\mathbf{DA}_c(-)$, we can find an open set $\eta \in V \xrightarrow{j} S$ such that there exist $Q = \Sigma^\infty \widetilde{\mathbb{M}}''(-1)$ for $\widetilde{\mathbb{M}}'' \in \mathcal{M}_1(V)$ and $P \simeq (\eta/V)^*Q$. In particular, in both cases, Q is t-non-negative. We can then use continuity for $\mathbf{DA}_c(-)$ to write

$$\begin{aligned} \mathbf{DA}(\eta)(P[n], \eta^*A) &\simeq \mathbf{DA}(\eta)((\eta/V)^*Q[n], (\eta/V)^*j^*A) \\ &\simeq \operatorname{Colim}_{\eta\in W\subset V} \mathbf{DA}(W)((W/V)^*Q[n], (W/V)^*j^*A). \end{aligned}$$

For such an intermediate open W, we see from [12, Proposition 4.14] that $(W/V)^*Q$ is t-non-negative while $(W/V)^*j^*A$ is t-non-positive. This implies that every morphism group in the colimit vanishes, and completes the proof that η^*A is in $\mathbf{MM}^1(\eta) \cap \mathbf{DA}^1_c(\eta)$.
Over η, which is a finite disjoint union of spectra of fields, thanks to Proposition 1.7, we understand completely the structure of compact objects in $\mathbf{MM}^1(\eta)$. Namely, there exists a Deligne 1-motive $\mathbb{M}_\eta \in \mathcal{M}_1(\eta)$ such that

$$\eta^*A \simeq \Sigma^\infty \mathbb{M}_\eta(-1)$$

The motive $\mathbb{M}_\eta$ has three components $\mathrm{Gr}^W_i \mathbb{M}_\eta$ for $i = -2, -1, 0$. By [8, Theorem 11], we can write $\mathrm{Gr}^W_{-1}\mathbb{M}_\eta$ as a direct factor of the Jacobian of a smooth projective geometrically connected curve C_η of genus ≥ 2 with a rational point σ_η; that is, $\mathrm{Gr}^W_{-1}\mathbb{M}_\eta = \mathrm{Im}(\pi_\eta : \mathrm{Jac}(C_\eta) \otimes \mathbb{Q} \to \mathrm{Jac}(C_\eta) \otimes \mathbb{Q})$ with π_η an idempotent.
By continuity for Deligne 1-motives and $\mathbf{DA}(-)$, we can find an open set $\eta \in U \xrightarrow{j} S$ such that $j^*A \simeq \Sigma^\infty \widetilde{\mathbb{M}}(-1)$ for $\widetilde{\mathbb{M}} \in \mathcal{M}_1(U)$. We can also assume that U is regular (since S is excellent and reduced), and that $\mathrm{Gr}^W_{-1}\mathbb{M}$ is a direct factor of the Jacobian of a smooth projective curve $f : C \to U$ with geometrically connected fibers which comes together with a section $\sigma : U \to C$. Similarly, by restricting U, we can assume that the lattice $\mathrm{Gr}^W_0\mathbb{M}$ and the character lattice of the torus $\mathrm{Gr}^W_{-2}\mathbb{M}$ are direct factors of permutation lattices over U, that is, lattices of the form $e_*\mathbb{Z}$ with $e : V \to U$ a finite étale morphism, since this holds over a field.
Write $i : Z \to S$ for the reduced closed immersion complementary to U. We prove that i^*A lies in $\mathbf{MM}^1(Z)$. By [12, Proposition 4.14], since A is t-non-negative, the motive i^*A is t-non-negative. It remains to show it is t-non-positive. Applying $\omega^1 i^*$ to a localisation triangle, we get

$$\omega^1 i^! A \to i^*A \to \omega^1 i^* j_* j^* A \xrightarrow{+} .$$

The functor $\omega^1 i^!$ is t-non-positive by [12, Proposition 4.14], hence it is enough to show that $\omega^1 i^* j_* j^* A$ is t-non-positive. We have seen that j^*A is of the form $\Sigma^\infty \widetilde{\mathbb{M}}(-1)$. By Lemma 2.2 below, $\omega^1 i^* j_* j^* A$ is t-non-positive, and we conclude that i^*A lies in the heart as claimed.

By (ii)$_{d-1}$, there exists a locally closed stratification $(i_\alpha : Z_\alpha \to Z)$ such that all the Z_α are regular and such that $i_\alpha^* i^* A$ is a Deligne 1-motive. This stratification combines with $j : U \to S$ to yield a stratification of S with regular strata and such that A restricted to each stratum is a Deligne 1-motive. This proves (ii)$_d$.
Let us show (i)$_d$. We have to show that for any $M \in \mathbf{DA}^1_c(S)$, we have $\tau_{\geq 0} M \in \mathbf{DA}^1_c(S)$. By [12, Corollary 4.29], M is bounded for $t^1_{\mathbf{MM}}(S)$. By induction on the $t^1_{\mathbf{MM}}(S)$-amplitude of M, we see that it is enough to show that for a morphism $f : A \to B$ with $A, B \in \mathbf{MM}^1(S) \cap \mathbf{DA}_c(S)$, the motive $\tau_{\geq 1}\mathrm{Cone}(f) \simeq \mathrm{Ker}_{\mathbf{MM}^1(S)}(f)$ is compact (or equivalently, that $\tau_{\leq 0}\mathrm{Cone}(f) \simeq \mathrm{Coker}_{\mathbf{MM}^1(S)}(f)$ is compact).
The idea is to describe f over a dense open of S in terms of Deligne 1-motives and to try to degenerate it to the boundary and apply the induction hypothesis.
Let us first describe A and B generically. By the same arguments in the proof of (ii)$_d$ above, we can find a regular open subscheme $\eta \in U \stackrel{j}{\to} S$ such that $j^* A \simeq \Sigma^\infty \widetilde{\mathbb{M}}(-1)$ for $\widetilde{\mathbb{M}} \in \mathcal{M}_1(U)$ and that $j^* B \simeq \Sigma^\infty \widetilde{\mathbb{N}}(-1)$ for $\widetilde{\mathbb{N}} \in \mathcal{M}_1(U)$. We can also assume that with $\mathrm{Gr}^W_{-1}\mathbb{M}$ (resp. $\mathrm{Gr}^W_{-1}\widetilde{\mathbb{N}}$) is a direct factor of the Jacobian of a smooth projective curve $f : C \to U$ (resp. $g : P \to U$) with geometrically connected fibers which comes together with a section $\sigma : U \to C$ (resp. $\theta : U \to P$). Similarly, we can assume that the lattices (resp. tori) of A and B are direct factors of permutation lattices (resp. that their character lattices are direct factors of permutation lattices).
The functor $\Sigma^\infty(-)(-1) : \mathcal{M}_1(U) \to \mathbf{MM}^1(U)$ is fully faithful by [12, Theorem 4.31], so that we can identify the morphism $j^* f : j^* A \to j^* B$ modulo the isomorphisms above with a morphism $F : \widetilde{\mathbb{M}} \to \widetilde{\mathbb{N}}$. Restricting U, we can also assume that the kernel K and cokernel Q of F in the abelian category $\mathcal{M}_1(\eta)$ extend to Deligne 1-motives $\widetilde{K}$, $\widetilde{Q}$ over U. Consider the morphism $\Sigma^\infty F(-1)$. The cone of $\mathrm{Cone}(\eta^* \Sigma^\infty F(-1)) \simeq \mathrm{Cone}(\eta^* f)$ fits into a distinguished triangle

$$\eta^* \Sigma^\infty \widetilde{K}(-1)[1] \to \eta^* \mathrm{Cone}(\Sigma^\infty \widetilde{F}(-1)) \to \eta * \Sigma^\infty \widetilde{Q}(-1) \stackrel{+}{\to} .$$

By continuity for $\mathbf{DA}(-)$, again by restricting U, we can assume there is a distiguished triangle

$$\Sigma^\infty \widetilde{K}(-1)[1] \to \mathrm{Cone}(\Sigma^\infty \widetilde{F}(-1)) \to \Sigma^\infty \widetilde{Q}(-1) \stackrel{+}{\to} .$$

By [12, Theorem 4.22], we have $j_! \Sigma^\infty \widetilde{K}(-1) \in \mathbf{MM}^1(S)$ and $j_! \Sigma^\infty \widetilde{Q}(-1) \in \mathbf{MM}^1(S)$. This implies that the distinguished triangle

$$j_! \Sigma^\infty \widetilde{K}(-1)[1] \to j_! j^* \mathrm{Cone}(f) \to j_! \Sigma^\infty \widetilde{Q}(-1) \stackrel{+}{\to}$$

is the truncation triangle of $j_! j^* \mathrm{Cone}(f)$ for t^1_{MM}, i.e., $H_1(j_! j^* \mathrm{Cone}(f)) \simeq j_! \Sigma^\infty \widetilde{K}(-1)$ and $H_0(j_! j^* \mathrm{Cone}(f)) \simeq j_! \Sigma^\infty \widetilde{Q}(-1)$.
As in the proof of (ii)$_d$ above, we see that $i^* A$ and $i^* B$ are in $\mathbf{MM}^1(Z)$. By [12, Proposition 4.14], the motives $i_* i^* A$ and $i_* i^* B$ are in $\mathbf{MM}^1(S)$, and this

implies that the morphism of localisation triangles

$$\begin{array}{ccccccc}
j_!\Sigma^\infty\widetilde{\mathbb{M}}(-1) & \longrightarrow & A & \longrightarrow & i_*i^*A & \xrightarrow{+} & \\
\Big\downarrow {\scriptstyle j_!\Sigma^\infty\widetilde{F}(-1)} & & \Big\downarrow {\scriptstyle f} & & \Big\downarrow {\scriptstyle i_*i^*f} & & \\
j_!\Sigma^\infty\widetilde{\mathbb{M}}'(-1) & \longrightarrow & B & \longrightarrow & i_*i^*B & \xrightarrow{+} &
\end{array}$$

is in fact a morphism of short exact sequences in $\mathbf{MM}^1(S)$, to which we can apply the Snake lemma and get a six term exact sequence

$$\begin{aligned}0 \to j_!\Sigma^\infty\widetilde{K}(-1) \to H_1(\mathrm{Cone}(f)) \to H_1(\mathrm{Cone}(i_*i^*f))\\ \to j_!\Sigma^\infty\widetilde{Q}(-1) \to H_0(\mathrm{Cone}(f)) \to H_0(\mathrm{Cone}(i_*i^*f)) \to 0.\end{aligned}$$

Again by Proposition [12, Proposition 4.14], we have $H_1(\mathrm{Cone}(i_*i^*f)) \simeq i_*H_1(\mathrm{Cone}(i^*f))$ and $H_0(\mathrm{Cone}(i_*i^*f)) \simeq i_*H_0(\mathrm{Cone}(i^*f))$.
By (i)$_{d-1}$ applied to the morphism $i^*f : i^*A \to i^*B$ on the proper closed subset Z, we deduce that $H_1(\mathrm{Cone}(i^*f))$ and $H_0(\mathrm{Cone}(i^*f))$ are compact. Since i_* preserves compact objects, $H_1(\mathrm{Cone}(i_*i^*f))$ and $H_0(\mathrm{Cone}(i_*i^*f))$ are compact. By adjunction and localisation, we have a sequence of isomorphisms

$$\begin{aligned}
& \mathbf{DA}(S)(i_*H_1(\mathrm{Cone}(i^*f)), j_!\Sigma^\infty\widetilde{Q}(-1))\\
\simeq\ & \mathbf{DA}(Z)(H_1(\mathrm{Cone}(i^*f)), i^!j_!\Sigma^\infty\widetilde{Q}(-1))\\
\simeq\ & \mathbf{DA}(Z)(H_1(\mathrm{Cone}(i^*f)), i^*j_*\Sigma^\infty\widetilde{Q}(-1)[-1])\\
\simeq\ & \mathbf{DA}(Z)(H_1(\mathrm{Cone}(i^*f)), \omega^1 i^*j_*\Sigma^\infty\widetilde{Q}(-1)[-1]).
\end{aligned}$$

Again by Lemma 2.2 below, the motive $\omega^1 i^*j_*\Sigma^\infty\widetilde{Q}(-1)$ is t-non-positive. Since the motive $H_1(\mathrm{Cone}(i^*f))$ is t-non-negative, we deduce that the morphism group above vanishes. This shows that the six term exact sequence above splits into two short exact sequences,

$$0 \to j_!\Sigma^\infty\widetilde{K}(-1) \to H_1(\mathrm{Cone}(f)) \to H_1(\mathrm{Cone}(i_*i^*f)) \to 0$$

and

$$0 \to j_!\Sigma^\infty\widetilde{Q}(-1) \to H_0(\mathrm{Cone}(f)) \to H_0(\mathrm{Cone}(i_*i^*f)) \to 0.$$

We have proved that the outer terms in both those sequences are compact, and we deduce that $H_1(\mathrm{Cone}(f))$ and $H_0(\mathrm{Cone}(f))$ are compact.
The theorem is now established modulo the following lemma, whose proof occupies the end of this section.

Lemma 2.2. *Assume* (i)$_{d-1}$ *and* (ii)$_{d-1}$. *Let S be a scheme of dimension $\leq d$ allowing resolution of singularities by alterations. Let $j : U \to S$ be an open immersion with U_{red} regular and $i : Z \to S$ the complementary reduced closed immersion. Let $\mathbb{M} \in \mathcal{M}_1(U)$ and $M = (\Sigma^\infty\mathbb{M})(-1)$.*

Assume moreover that the abelian scheme part of $\mathbb{M}$ is a direct factor of the Jacobian scheme of a smooth projective curve C with geometrically connected fibres and a section σ, that the lattice part of $\mathbb{M}$ is a direct factor of a permutation lattice, and that the toric part has a character lattice which is a direct factor of a permutation lattice.
Then the motive $\omega^1 i^* j_* M$ is $t^1_{\mathbf{MM}}(Z)$-non-positive.

Using Lemma 1.8, we can assume that S is reduced. Let us show that we can in fact assume S to be integral. Let $q : \widetilde{S} \to S$ be the normalisation morphism. Since U is assumed to be regular, q is an isomorphism above U. Consider the diagram of schemes with cartesian squares

$$\begin{array}{ccccc} U & \xrightarrow[\widetilde{j}]{} & \widetilde{S} & \xleftarrow[\widetilde{\imath}]{} & \widetilde{Z} \\ \| & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_Z} \\ U & \xrightarrow[j]{} & S & \xleftarrow[i]{} & Z. \end{array}$$

By proper base change and [12, Proposition 3.3 (iii)], we have

$$\begin{aligned} \omega^1 i^* j_* j^* M &\simeq \omega^1 i^* q_* \widetilde{j}_* j^* M \\ &\simeq \omega^1 q_{Z*} \omega^1 \widetilde{\imath}^* \widetilde{j}_* j^* M. \end{aligned}$$

The functor $\omega^1 \pi_{Z*}$ is t-non-positive [12, Proposition 4.14]. So it is enough to prove Lemma 2.2 in the case S is integral.
We improve the geometric situation using alterations.

Lemma 2.3. *There exists a projective alteration $\pi : S' \to S$ such that*

- *S' is regular, and*
- *$C \times_S S'$ extends to a projective semi-stable curve $\bar{f}' : \bar{C}' \to S'$ with geometrically connected fibers such that $\bar{C}'$ is regular and $\sigma' = \sigma \times_S S'$ extends to a section $\bar{\sigma}'$ of $\bar{C}'/S'$, which lands in the smooth locus.*

Proof. The pair (C, σ) determines a U-point of the stack $\overline{\mathcal{M}}_{g,1}$ of genus g stable curves with a section. By a standard argument using the existence and properness of the moduli stack $\overline{\mathcal{M}}_{g,1}$, there exists a projective alteration $\pi_1 : S_1 \to S$ (with S_1 again integral) such that, if we write $U_1 = U \times_S S_1$, $C_1 = C \times_S S_1$ and so on, the pair (C_1, σ_1) extends to a point $(\bar{C}_1, \bar{\sigma}_1) \in \overline{\mathcal{M}}_{g,1}$. Note that, by definition of $\overline{\mathcal{M}}_{g,1}$, such a curve still has geometrically connected fibers. In particular, $\bar{\sigma}_1$ factors through the S_1-smooth locus $\overline{C}_1^{\mathrm{sm}}$ of $\overline{C}_1$. By [6, Lemma 5.7], by replacing S_1 by a further projective alteration, we can also assume that $\overline{C}_1$ is quasi-split over S_1 in the sense of loc. cit., i.e., that on every fiber the singular points and the tangents to the singular points are rational.
The closed subset $f_1(\mathrm{Sing}(\overline{C}_1))$ is a proper closed subset since $\overline{C}_1$ is generically smooth over a generically regular scheme S_1. By resolution of singularities by

alterations applied to the pair $(S_1, f_1(\mathrm{Sing}(\overline{C}_1)))$, which is possible by hypothesis on S, there exists a projective alteration $\pi_2 : S_2 \to S_1$ with S_2 integral and regular, and with $D := \pi_2^{-1}(f_1(\mathrm{Sing}(\overline{C}_1)))$ a strict normal crossings divisor. Put $U_2 = U \times_S S_2$ and so on. Then $(\overline{C}_2, \bar{\sigma}_2) \in \overline{\mathcal{M}}_{g,1}(S_2)$ is still a quasi-split stable curve, which is moreover smooth outside of the strict normal crossings divisor D.

By [6, Proposition 5.11], there exists a projective modification $\phi_3 : \overline{C}_3 \to \overline{C}_2$ which is an isomorphism outside of $\mathrm{Sing}(\overline{C}_2)$ (so in particular over $\overline{C}_2^{\mathrm{sm}}$, and over $S_2 \setminus D$ via $\bar{f}_2$), and such that $\overline{C}_3$ is regular and the composite $\bar{f}_3 = \bar{f}_2 \circ \phi_3 : \overline{C}_3 \to S_2$ is a projective semi-stable curve. Since ϕ_3 is an isomorphism on $\overline{C}_2^{\mathrm{sm}}$, the section $\bar{\sigma}_2$ lifts to a section $\bar{\sigma}_3 : S_2 \to \overline{C}_3$ of $\bar{f}_3$.

We now put $S' = S_2$, $\bar{C}' = \bar{C}_3$, $\bar{f}' = \bar{f}_3$, and $\bar{\sigma}' = \bar{\sigma}_3$. These satisfy all the requirements of the conclusion of the lemma. □

Let us fix an alteration π as in Lemma 2.3. Consider the following diagram of schemes with cartesian squares.

$$\begin{array}{ccccc} V' & \xrightarrow{j'} & S' & \xleftarrow{i'} & Z' \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Z} \\ V & \xrightarrow{j} & S & \xleftarrow{i} & Z \end{array}$$

By construction, the morphism π_V is the composite of a finite étale morphism followed by a finite purely inseparable morphism. By [1, Lemme 2.1.165], we have that j^*M is a direct factor of $\pi_{V*}\pi_V^*j^*M$. We are thus reduced to show that $\omega^1 i^* j_* \pi_{V*}\pi_V^* j^* M$ is t-non-positive. By proper base change and [12, Proposition 3.3 (iii)], we have

$$\begin{aligned} \omega^1 i^* j_* \pi_{V*}\pi_V^* j^* M &\simeq \omega^1 i^* \pi_* j'_* \pi_V^* \Sigma^\infty \mathbb{M}(-1) \\ &\simeq \omega^1 \pi_{Z*} \omega^1 i'^* j'_* \Sigma^\infty \pi_V^* \mathbb{M}(-1). \end{aligned}$$

The functor $\omega^1 \pi_{Z*}$ is t-non-positive [12, Proposition 4.14], and $\pi_V^*\mathbb{M}(-1)$ is a Deligne 1-motive. Hence we can assume that $S' = S$ and that C itself has an extension $\bar{f} : \bar{C} \to S$ satisfying the conclusions of Lemma 2.3.

Since the lattice $\mathrm{Gr}_0^W \mathbb{M}$ (resp. the character lattice of the torus $\mathrm{Gr}_{-2}^W \mathbb{M}$) is by assumption a direct factor of a permutation lattice, the same argument shows that we can assume these lattices to be direct factors of trivial lattices.

By the distinguished triangles associated to the weight filtration of $\mathbb{M}$, we can treat each piece separately and assume that $\mathbb{M}$ is pure.

Let us quickly take care of the cases where $\mathbb{M}$ is either a lattice or a torus. By the reductions above, we are immediately reduced to the case where those are trivial lattices or split tori, and from there to the case $\mathbb{M} = [\mathbb{Z} \to 0]$ or $\mathbb{M} = [0 \to \mathbb{G}_{\mathrm{m}}]$. So we have to prove that $\omega^1(i^* j_* \mathbb{Q})$ and $\omega^1(i^* j_* \mathbb{Q}(-1))$ are t-non-positive. By localisation, we have distinguished triangles

$$\mathbb{Q} \to \omega^1 i^* j_* \mathbb{Q} \to \omega^1 i^! \mathbb{Q}[+1]$$

and

$$\mathbb{Q}(-1) \to \omega^1(i^* j_* \mathbb{Q}(-1)) \to \omega^1(i^! \mathbb{Q}(-1))[+1]$$

The motives $\mathbb{Q}$ and $\mathbb{Q}(-1)$ are associated to Deligne 1-motives over Z, so they lie in $\mathbf{MM}^1(Z)$. By Lemma 2.4 below and [12, Corollary 3.9 (iv)], we have that $\omega^1 i^! \mathbb{Q}[+1]$ is in $\mathbf{DA}^1(Z)_{\leq -1}$ (in particular t-non-positive), and that $\omega^1 i^! \mathbb{Q}(-1) \simeq 0$. This concludes the proof in the lattice and torus case.
It remains to treat the case where $\mathbb{M}$ is a direct factor of $\mathrm{Jac}(C/V)$, and it is clearly enough to study the case $\mathbb{M} = \mathrm{Jac}(C/V)$. Consider the diagram with cartesian squares

$$\begin{array}{ccccc} C & \xrightarrow{\tilde{j}} & \bar{C} & \xleftarrow{\tilde{i}} & \partial C \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \partial f} \\ U & \xrightarrow{j} & S & \xleftarrow{i} & Z. \end{array}$$

By [12, Corollary 3.20], the motive $\Sigma^\infty \mathrm{Jac}(C/U)(-1)[-1]$ is a direct factor of $f_* \mathbb{Q}_C[+1]$, via the section σ (we use the fact that C/U has geometrically connected fibers). More precisely, using the adjunction (σ^*, σ_*) and the fact that $f\sigma = \mathrm{id}$, we get a map

$$\pi_0 : f_* \mathbb{Q}_C \to f_* \sigma_* \sigma^* \mathbb{Q}_C \simeq \mathbb{Q}_U.$$

From [12, Corollary 3.20], we deduce that $\Sigma^\infty \mathrm{Jac}(C/U)(-1)[-2]$ is a direct factor of $\mathrm{Fib}(\pi_0)$, hence that $\Sigma^\infty \mathrm{Jac}(C/U)(-1)[-1]$ is a direct factor of $\mathrm{Fib}(\pi_0[+1])$. So it is enough to show that that $\omega^1 i^* j_* \mathrm{Fib}(\pi_0[+1]) \simeq \mathrm{Fib}(\omega^1 i^* j_* \pi_0)[+1]$ is t-non-positive.
By base change and [12, Proposition 3.3 (iii)], we have $\omega^1 i^* j_* f_* \mathbb{Q}_C \simeq \omega^1 (\partial f)_* \omega^1 \tilde{i}^* \tilde{j}_* \mathbb{Q}_C$. Moreover, if we write $\partial \pi_0 : \partial f_* \mathbb{Q}_{\partial C} \to \mathbb{Q}_Z$ obtained from $\partial \sigma$ in the same fashion as π_0 was obtained from σ, the square

$$\begin{array}{ccc} \omega^1(\partial f)_* \mathbb{Q}[+1] & \longrightarrow & \omega^1 i^* j_* f_* \mathbb{Q}[+1] \\ {\scriptstyle \omega^1 \partial \pi_0}\downarrow & & \downarrow{\scriptstyle \omega^1 i^* j_* \pi_0} \\ \mathbb{Q}[+1] & \longrightarrow & \omega^1 i^* j_* \mathbb{Q}[+1] \end{array}$$

obtained by applying ω^1 to a commutative square of unit of pullback-pushforwards adjunctions is commutative. Applying localisation, we can complete it into the following diagram with distinguished rows.

$$\begin{array}{ccccccc} \omega^1(\partial f)_* \mathbb{Q}[+1] & \longrightarrow & \omega^1 i^* j_* f_* \mathbb{Q}[+1] & \longrightarrow & \omega^1 \partial f_* \omega^1 \tilde{i}^! \mathbb{Q}[+2] & \xrightarrow{+} & \\ {\scriptstyle \omega^1 \partial \pi_0}\downarrow & & {\scriptstyle \omega^1 i^* j_* \pi_0}\downarrow & & \downarrow & & \\ \mathbb{Q}[+1] & \longrightarrow & \omega^1 i^* j_* \mathbb{Q}[+1] & \longrightarrow & \omega^1 i^! \mathbb{Q}[+2] & \xrightarrow{+} & . \end{array}$$

By Lemma 2.4 below and the fact that $\omega^1(\delta f)_*$ is t-non-positive [12, Proposition 4.14] , the rightmost terms of both triangles are both t-non-positive. We deduce that the central terms lie in $\mathbf{DA}^1(Z)_{\leq 1}$. In particular, we also have $\omega^1 i^* j_* \mathrm{Fib}(\pi_0) \in \mathbf{DA}^1(Z)_{\leq 1}$ and there is an isomorphism

$$H_1(\omega^1 i^* j_* \mathrm{Fib}(\pi_0)) \simeq \mathrm{Ker}(H_1(\omega^1 i^* j_* f_* \mathbb{Q}[+1]) \to H_1(\omega^1 i^* j_* \mathbb{Q}[+1]))$$

and it remains to show that this last morphism, or equivalently $H_0(\omega^1 i^* j_* f_* \mathbb{Q}) \to H_0(\omega^1 i^* j_* \mathbb{Q})$ is injective. From the same diagram with distinguished rows and Lemma 2.4, we deduce a commutative square with horizontal isomorphisms

$$\begin{array}{ccc} H_0(\omega^1(\partial f)_* \mathbb{Q})) & \xrightarrow{\sim} & H_0(\omega^1 i^* j_* f_* \mathbb{Q}) \\ \downarrow{\scriptstyle \omega^1 \partial \pi_0} & & \downarrow{\scriptstyle \omega^1 i^* j_* \pi_0} \\ H_0 \mathbb{Q} \simeq \mathbb{Q} & \xrightarrow{\sim} & H_0(\omega^1 i^* j_* \mathbb{Q}). \end{array}$$

So we have to prove that the morphism $H_0(\omega^1 \partial f_* \mathbb{Q}) \to \mathbb{Q}$ induced by $\partial \sigma$ is injective. We prove that it is in fact an isomorphism. First of all, since ∂f is a proper curve, we have $\partial f_* \mathbb{Q} \in \mathbf{DA}^1_c(Z)$, so that $\omega^1 \partial f_* \mathbb{Q} \simeq \partial f_* \mathbb{Q}$. By [2, Proposition 3.24], it is enough to show that, for all points $z \in Z$, the morphism $z^* H_0(\partial f_* \mathbb{Q}_{\partial C}) \to \mathbb{Q}_z$ is an isomorphism. We have $\dim(Z) < \dim(S)$, so (iii)$_{d-1}$ (which as we have seen earlier in the proof follows from (i)$_{d-1}$ and (ii)$_{d-1}$), the pullback functor z^* is t-exact for the 1-motivic t-structures. We thus have $z^* H_0(\partial f_* \mathbb{Q}) \simeq H_0(z^* \partial f_* \mathbb{Q}) \simeq H_0(\partial (f_z)_* \mathbb{Q})$, with $\partial f_z : \bar{C}_z \to z$. The curve $\bar{C}_z$ is geometrically connected, hence the morphism $H_0(\partial f_{z*} \mathbb{Q}) \to \mathbb{Q}$ induced by the point $\bar{\sigma}_z \in \bar{C}_z(z)$ is an isomorphism. This concludes the proof. □

The following lemma was used in the proof of Theorem 2.1.

Lemma 2.4. *Let S be a regular scheme, and $i : Z \to S$ be a closed immersion with Z reduced and nowhere dense. Let $k_0 : Z_0 \to Z$ be an open immersion with Z_0 regular of pure codimension 1 in S, containing all points of Z which are of codimension 1 in S (i.e., all generic points of irreducible components of Z which are divisors); note that Z_0 can be empty. Let $M \in \mathbf{MM}^{1,\mathrm{sm}}_c(S)$ be a smooth constructible 1-motive. Then*

$$\omega^1 i^! M \simeq \omega^1(k_{0*} k_0^* i^* M(-1))[-2] \simeq \omega^0(k_{0*} k_0^* i^* M)(-1)[-2].$$

In particular, $\omega^1 i^! \mathbb{Q}[+2]$ is $t^1_{\mathbf{MM}}$-non-positive.

Proof. Extend Z_0 into a stratification $\emptyset = \bar{Z}_{m+1} \subset \bar{Z}_m \subset \ldots \subset \bar{Z}_0 = Z$ by closed subsets with $Z_i := \bar{Z}_i \setminus \bar{Z}_{i-1}$ regular and equidimensional. Write c_i for the codimension of Z_i in S; by hypothesis, we can arrange that $c_i \geq 2$ for $i > 0$. Write $Z_i \overset{k_i}{\to} \bar{Z}_i \overset{l_i}{\leftarrow} \bar{Z}_{i+1}$. By localisation, we have distinguished triangles

$$l_{0*} l_0^! i^! M \to i^! M \to k_{0*} k_0^! i^! M \xrightarrow{+}$$

$$l_{1*}l_1^!l_0^!i^!M \to l_0^!i^! \to k_{1*}k_1^!l_0^!M \xrightarrow{+}$$

$$\dots$$

$$l_{m*}l_m^!l_{m-1}^! \dots i^!Ml_{d-1}^! \dots i^!M \to k_{m*}k_m^!l_{d-1}^! \dots i^!M$$

For all i, since S and Z_i are regular, the immersion $k_i = i \circ l_1 \circ \dots l_{i-1} : Z_i \to S$ is a regular immersion. By absolute purity in the form of [12, Proposition 1.7] for the smooth motive M, we deduce that $k_i^!M \simeq k_i^*M(-c_i)[-2c_i]$. Combining this formula with the triangles above and the fact that $\omega^1 M(-2) = 0$ for any $M \in \mathbf{DA}^{\mathrm{coh}}$ implies that $\omega^1 i^! M \simeq \omega^1(k_{0*}k_0^*i^*M(-1)[-2])$.
We have $\omega^1(k_{0*}k_0^*i^*M(-1)) \simeq \omega^0(k_{0*}k_0^*i^*M)(-1)$ by [12, Corollary 3.9 (iv)]. This concludes the proof of the main statement.
We specialise to the case $M = \mathbb{Q}[+2]$. We have $\mathbb{Q}(-1) \in \mathbf{MM}^1(Z_0)$. By [12, Proposition 4.14 (v)], the motive $\omega^1 i^!\mathbb{Q}[+2] \simeq \omega^1 k_{0*}(\mathbb{Q}(-1))$ is then t^1-negative, as claimed. □

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Simon Pepin Lehalleur
Freie Universität Berlin
Arnimallee 3
14195 Berlin
Germany
simon.pepin.lehalleur@gmail.com