Boardman's Whole-Plane Obstruction Group
for Cartan–Eilenberg Systems

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ABSTRACT. Each extended Cartan–Eilenberg system (H, ∂) gives rise to two exact couples and one spectral sequence. We show that the canonical colim-lim interchange morphism associated to H is a surjection, and that its kernel is isomorphic to Boardman's whole-plane obstruction group W, for each of the two exact couples.

1 Introduction

In a classic study, Boardman [3] analyzed the convergence of the spectral sequence $E^r \implies G$ arising from an exact couple (A, E) in the sense of Massey [14]. He identified a condition on the exact couple, called conditional convergence, which in the case of half-plane spectral sequences with exiting differentials is sufficient to ensure strong convergence. In the case of half-plane spectral sequences with entering differentials, conditional convergence and the vanishing of an obstruction group RE^∞ guarantee strong convergence. Finally, in the case of whole-plane spectral sequences, Boardman identified another obstruction group

$$W = \operatorname*{colim}_s \operatorname*{Rlim}_r K_\infty \operatorname{Im}^r A_s \,,$$

such that conditional convergence and the vanishing of both RE^∞ and W imply strong convergence.

It is an insight of the first author that for spectral sequences arising from Cartan–Eilenberg systems (H, ∂), as defined in [6], there is a natural isomorphism

$$W \cong \operatorname{Ker}(\kappa) \,,$$

where κ is the canonical colim-lim interchange morphism

$$\kappa \colon \operatorname*{colim}_i \lim_j H(i,j) \longrightarrow \lim_j \operatorname*{colim}_i H(i,j)\,.$$

We prove this in Theorems 6.7 and 7.5. Furthermore, κ is always a surjection, cf. Theorem 5.2. We find that this direct relationship to the interchange morphism clarifies the role of Boardman's whole-plane obstruction group W. For example, we obtain a straightforward proof of his criterion for the vanishing of W, see Proposition 5.3.

In order to introduce notations we review spectral sequences, exact couples and Cartan–Eilenberg systems in Sections 2, 3 and 4, respectively. Our treatment of convergence is very close to that of [3], but we obtain slightly more general conclusions in particular cases.

Our novel work begins in Section 5, where we introduce the canonical interchange morphism κ and show that it is always surjective for Cartan–Eilenberg systems. In Sections 6 and 7 we consider right and left Cartan–Eilenberg systems and their associated exact couples (A'', E^1) and (A', E^1), and identify the kernel of κ with the respective whole-plane obstruction groups. In Section 8 we give some examples of spectral sequences that arise from biinfinite sequences of spectra, illustrating that the whole-plane obstruction W can be highly nontrivial, and giving topological interpretations of its meaning.

2   Spectral sequences

Let R be a ring and let $\mathcal{A}$ be the graded abelian category of graded R-modules $M = (M_t)_t$, where $t \in \mathbb{Z}$ is the *internal degree*. (Greater abstraction is possible, but beware the counterexample of Neeman–Deligne [15] to Roos' theorem [19], as repaired in [20].)

Definition 2.1 (Leray [11], Koszul [10]). A *spectral sequence* is a sequence $(E^r, d^r)_r$ of differential graded objects in $\mathcal{A}$, for $r \geq 1$, equipped with isomorphisms $E^{r+1} \cong H(E^r, d^r)$. Each term $E^r = (E_s^r)_s$ is a graded object in $\mathcal{A}$, where $s \in \mathbb{Z}$ is the *filtration degree*. We assume that each differential $d^r = (d_s^r)_s$, with $d_s^r \colon E_s^r \to E_{s-r}^r$, has filtration degree $-r$ and internal degree -1. It satisfies $d^r \circ d^r = 0$, and its homology $H(E^r, d^r)$ is the graded object with $H_s(E^r, d^r) = \operatorname{Ker}(d_s^r)/\operatorname{Im}(d_{s+r}^r)$ in $\mathcal{A}$.

Lemma 2.2. *For any spectral sequence $(E^r, d^r)_r$ there are subobjects*

$$0 = B_s^1 \subset \cdots \subset B_s^r \subset \cdots \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s^1$$

in $\mathcal{A}$, with $E_s^r \cong Z_s^r/B_s^r$ for all integers $r \geq 1$ and s.

Proof. This is clear for $r = 1$. By induction on r we may assume that $E_s^r \cong Z_s^r/B_s^r$ for all $s \in \mathbb{Z}$. Then $\operatorname{Im}(d_s^r) \subset \operatorname{Ker}(d_s^r) \subset E_s^r$ correspond to $B_s^{r+1}/B_s^r \subset Z_s^{r+1}/B_s^r \subset Z_s^r/B_s^r$ for well-defined subobjects B_s^{r+1} and Z_s^{r+1} of Z_s^r, with $B_s^r \subset B_s^{r+1} \subset Z_s^{r+1} \subset Z_s^r$. This completes the inductive step. $\square$

DEFINITION 2.3. For each integer s, let

$$Z_s^\infty = \lim_r Z_s^r$$
$$B_s^\infty = \operatorname*{colim}_r B_s^r$$
$$E_s^\infty = Z_s^\infty / B_s^\infty$$

denote the infinite cycles, the infinite boundaries, and the E^∞-term, respectively. Following Boardman [3, (5.1)] let

$$RE_s^\infty = \operatorname*{Rlim}_r Z_s^r \,.$$

Remark 2.4. Fix a filtration degree s. If $d_s^r = 0$ for all sufficiently large r, then the descending sequence $(Z_s^r)_r$ is eventually constant and $RE_s^\infty = 0$. This argument may be applied in one internal degree t at a time.

DEFINITION 2.5. A *filtration* of an object G in $\mathcal{A}$ is a diagram of subobjects

$$\cdots \subset F_{s-1}G \subset F_s G \subset \cdots \subset G \,,$$

where $s \in \mathbb{Z}$. Let $F_\infty G = \operatorname{colim}_s F_s G$, $F_{-\infty}G = \lim_s F_s G$ and $RF_{-\infty}G = \operatorname{Rlim}_s F_s G$. The filtration is *exhaustive* if $F_\infty G \to G$ is an isomorphism, it is *Hausdorff* if $F_{-\infty}G = 0$, and *complete* if $RF_{-\infty}G = 0$.

LEMMA 2.6. *For an exhaustive complete Hausdorff filtration we can recover G from the filtration subquotients $F_j G/F_i G$ for $i \le j$ in $\mathbb{Z}$, by either one of the isomorphisms*

$$\operatorname*{colim}_j \lim_i F_j G/F_i G \;\cong\; G \;\cong\; \lim_i \operatorname*{colim}_j F_j G/F_i G \,.$$

Proof. This follows from the isomorphisms $F_j G \cong \lim_i F_j G/F_i G$ and $G/F_i G \cong \operatorname{colim}_j F_j G/F_i G$, respectively. $\qquad\square$

DEFINITION 2.7 ([6, XV.2], [3, 5.2]). A spectral sequence $(E^r, d^r)_r$ *converges weakly* to a filtered object G if the filtration is exhaustive and there are isomorphisms

$$E_s^\infty \cong F_s G/F_{s-1} G$$

for each integer s. The spectral sequence *converges* if it converges weakly and the filtration is Hausdorff. It *converges strongly* if it converges and the filtration is complete.
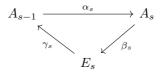
Remark 2.8. For a strongly convergent spectral sequence we can recover the target G from the E^∞-term if we are able to resolve the extension problems, i.e., to determine the diagram of filtration subquotients $F_j G/F_i G$ from knowledge of the minimal subquotients $F_s G/F_{s-1} G \cong E_s^\infty$ for $i < s \le j$ and other available information.
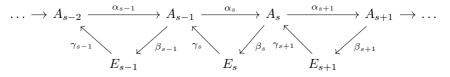
3 EXACT COUPLES

DEFINITION 3.1 ([14, §4]). An *exact couple* $(A, E, \alpha, \beta, \gamma)$ in $\mathcal{A}$ is a pair of graded objects $A = (A_s)_s$ and $E = (E_s)_s$, and three morphisms $\alpha = (\alpha_s \colon A_{s-1} \to A_s)_s$, $\beta = (\beta_s \colon A_s \to E_s)_s$, $\gamma = (\gamma_s \colon E_s \to A_{s-1})_s$ of filtration degree $+1$, 0 and -1, respectively, such that the triangle

$$
\begin{array}{ccc}
A_{s-1} & \xrightarrow{\ \alpha_s\ } & A_s \\
& \underset{\gamma_s}{\nwarrow} \quad \underset{\beta_s}{\swarrow} & \\
& E_s &
\end{array}
$$

is exact for each $s \in \mathbb{Z}$. We assume that the internal degree of α is 0, and that the internal degrees of β and γ are either 0 and -1, or -1 and 0, respectively.

Remark 3.2. Equivalently we require that each triangle in the diagram

$$
\begin{array}{ccccccccc}
\cdots \to A_{s-2} & \xrightarrow{\alpha_{s-1}} & A_{s-1} & \xrightarrow{\ \alpha_s\ } & A_s & \xrightarrow{\alpha_{s+1}} & A_{s+1} & \to \cdots \\
\underset{\gamma_{s-1}}{\searrow} & \underset{\beta_{s-1}}{\nearrow} \ \underset{\gamma_s}{\searrow} & & \underset{\beta_s}{\nearrow} \ \underset{\gamma_{s+1}}{\searrow} & & \underset{\beta_{s+1}}{\nearrow} & & \\
E_{s-1} & & E_s & & E_{s+1} & & &
\end{array}
$$

is exact. This object is called an *unrolled*, or unraveled, exact couple [3, §0].

DEFINITION 3.3. Let $(A, E, \alpha, \beta, \gamma)$ be an exact couple. For integers $r \geq 1$ and s let

$$
\begin{aligned}
Z_s^r &= \gamma^{-1} \operatorname{Im}(\alpha^{r-1} \colon A_{s-r} \to A_{s-1}) \\
B_s^r &= \beta \operatorname{Ker}(\alpha^{r-1} \colon A_s \to A_{s+r-1}) \\
E_s^r &= Z_s^r / B_s^r \,.
\end{aligned}
$$

Let $d_s^r \colon E_s^r \to E_{s-r}^r$ be given by $d_s^r([x]) = [\beta(y)]$ where $\gamma(x) = \alpha^{r-1}(y)$.

LEMMA 3.4. $\operatorname{Ker}(d_s^r) = Z_s^{r+1}/B_s^r$ *and* $\operatorname{Im}(d_{s+r}^r) = B_s^{r+1}/B_s^r$, *so* $H_s(E^r, d^r) \cong E_s^{r+1}$ *and* $(E^r, d^r)_r$ *is a spectral sequence.* $\qquad\square$

Remark 3.5. The objects Z_s^r and B_s^r of Definition 3.3 agree with those associated in Lemma 2.2 to the spectral sequence $(E^r, d^r)_r$.

DEFINITION 3.6. Given a sequence $\cdots \to A_{s-1} \xrightarrow{\alpha_s} A_s \to \ldots$, consider the colimit $A_\infty = \operatorname{colim}_s A_s$, the limit $A_{-\infty} = \lim_s A_s$, and the derived limit $RA_{-\infty} = \operatorname{Rlim}_s A_s$. Let $\iota_s \colon A_s \to A_\infty$ and $\pi_s \colon A_{-\infty} \to A_s$ be the colimit and limit structure maps, respectively. The colimit A_∞ is filtered, for $s \in \mathbb{Z}$, by

$$
F_s A_\infty = \operatorname{Im}(\iota_s \colon A_s \to A_\infty) \,.
$$

The limit $A_{-\infty}$ is filtered, also for $s \in \mathbb{Z}$, by

$$
F_s A_{-\infty} = \operatorname{Ker}(\pi_s \colon A_{-\infty} \to A_s) \,.
$$

DEFINITION 3.7 ([3, 5.10]). An exact couple $(A, E, \alpha, \beta, \gamma)$ *converges conditionally to the colimit* if $A_{-\infty} = 0$ and $RA_{-\infty} = 0$. It *converges conditionally to the limit* if $A_{\infty} = 0$.

Remark 3.8. One often says that a spectral sequence is conditionally convergent, but conditional convergence is, strictly speaking, a property of an exact couple.

The following two variants of Boardman's results show that conditional convergence, when combined with the vanishing of RE^{∞}, suffices to give quite good convergence results. In the first case the target is correct, but we may not get strong convergence because the filtration (with limit $F_{-\infty}A_{\infty}$) might not be Hausdorff. In the second case we get strong convergence, but the target is sometimes only a quotient (by $RA_{-\infty}$) of the most desirable target object. In both cases the error term is given by the whole-plane obstruction object W for the exact couple, whose definition is reviewed directly after the two theorems.

THEOREM 3.9 (cf. [3, 8.10]). *Let* $(A, E, \alpha, \beta, \gamma)$ *be an exact couple that converges conditionally to the colimit, and assume that* $RE^{\infty} = 0$. *Then the associated spectral sequence converges weakly to* A_{∞}, *and the filtration of* A_{∞} *is complete. Moreover,* $F_{-\infty}A_{\infty} \cong W$.

THEOREM 3.10 (cf. [3, 8.13]). *Let* $(A, E, \alpha, \beta, \gamma)$ *be an exact couple that converges conditionally to the limit, and assume that* $RE^{\infty} = 0$. *Then the associated spectral sequence converges strongly to* $A_{-\infty}$. *Moreover,* $RA_{-\infty} \cong W$.

DEFINITION 3.11 ([3, §3, §8]). Let $(A, E, \alpha, \beta, \gamma)$ be any exact couple. For integers $r \geq 1$ and s let

$$\mathrm{Im}^r A_s = \mathrm{Im}(\alpha^r \colon A_{s-r} \to A_s)$$
$$Q_s = \lim_r \mathrm{Im}^r A_s$$
$$RQ_s = \mathrm{Rlim}_r \mathrm{Im}^r A_s$$
$$K_{\infty} \mathrm{Im}^r A_s = \mathrm{Ker}(\iota_s \colon A_s \to A_{\infty}) \cap \mathrm{Im}^r A_s$$
$$W = \mathrm{colim}_s \mathrm{Rlim}_r K_{\infty} \mathrm{Im}^r A_s .$$

Remark 3.12. Boardman employs transfinite induction to define image subsequences $\mathrm{Im}^{\sigma} A_s$ for arbitrary ordinals σ, and uses these to prove that the sufficient conditions he gives for strong convergence are also necessary. We will only establish the sufficiency of our conditions, and for these results there is no need to invoke transfinite induction.

LEMMA 3.13 (cf. [3, 5.4(b)]). *The filtration of* $A_{-\infty} = \lim_s \mathrm{Im}(\pi_s) = \lim_s Q_s$ *is complete and Hausdorff.*

Proof. The colimit structure map $\pi_s \colon A_{-\infty} \to A_s$ factors through $\mathrm{Im}(\pi_s) \subset \mathrm{Im}^r A_s \subset A_s$ for each r, hence also through $\mathrm{Im}(\pi_s) \subset Q_s \subset A_s$. Therefore the

identity map of $A_{-\infty}$ factors through $\lim_s \mathrm{Im}(\pi_s) \subset \lim_s Q_s \subset \lim_s A_s$, which implies that $\lim_s \mathrm{Im}(\pi_s) = \lim_s Q_s = \lim_s A_s$. The short exact sequences $0 \to F_s A_{-\infty} \to A_{-\infty} \to \mathrm{Im}(\pi_s) \to 0$ give an exact sequence

$$0 \to F_{-\infty} A_{-\infty} \longrightarrow A_{-\infty} \xrightarrow{\;\cong\;} \lim_s \mathrm{Im}(\pi_s) \longrightarrow RF_{-\infty} A_{-\infty} \longrightarrow 0$$

upon passage to limits. Hence $F_{-\infty} A_{-\infty} = 0$ and $RF_{-\infty} A_{-\infty} = 0$. ☐

LEMMA 3.14. *Consider any two morphisms* $A \xleftarrow{\;f\;} C \xrightarrow{\;g\;} B$ *in* $\mathcal{A}$. *There is an isomorphism*

$$\mathrm{Im}(f)/f(\mathrm{Ker}(g)) \xrightarrow{\;\cong\;} \mathrm{Im}(g)/(g(\mathrm{Ker}(f))$$

given by $[a] \mapsto [g(c)]$, *where* $f(c) = a$.

Proof. The isomorphism factors through $C/(\mathrm{Ker}(f) + \mathrm{Ker}(g))$. ☐

LEMMA 3.15 (cf. [3, 5.6]). *(a) There is a natural short exact sequence*

$$0 \to F_s A_\infty / F_{s-1} A_\infty \longrightarrow E_s^\infty \longrightarrow Z_s^\infty / \mathrm{Ker}(\gamma) \to 0 \,.$$

(b) There is a natural six term exact sequence

$$0 \to Z_s^\infty / \mathrm{Ker}(\gamma) \xrightarrow{\;\gamma\;} Q_{s-1} \xrightarrow{\;\alpha\;} Q_s \to RE_s^\infty \xrightarrow{\;\gamma\;} RQ_{s-1} \xrightarrow{\;\alpha\;} RQ_s \to 0 \,.$$

(c) If $RE^\infty = 0$ *then* $\lim_s Q_s \to Q_s$ *is surjective.*

Proof. (a) The inclusions $B_s^\infty \subset \mathrm{Im}(\beta) = \mathrm{Ker}(\gamma) \subset Z_s^\infty$ lead to a short exact sequence

$$0 \to \mathrm{Im}(\beta)/B_s^\infty \to E_s^\infty \to Z_s^\infty / \mathrm{Ker}(\gamma) \to 0 \,.$$

By Lemma 3.14 applied to the two morphisms $E_s^1 \xleftarrow{\;\beta\;} A_s \xrightarrow{\;\iota_s\;} A_\infty$ there is an isomorphism

$$\mathrm{Im}(\beta)/B_s^\infty \cong F_s A_\infty / F_{s-1} A_\infty \,.$$

(b) The short exact sequences

$$0 \to Z_s^r / \mathrm{Ker}(\gamma) \xrightarrow{\;\gamma\;} \mathrm{Im}^{r-1} A_{s-1} \xrightarrow{\;\alpha\;} \mathrm{Im}^r A_s \to 0$$

give the stated six term exact sequence upon passage to limits.
(c) If $RE_s^\infty = 0$ for all s, then $\alpha\colon Q_{s-1} \to Q_s$ is surjective for each s. This implies that $\lim_s Q_s \to Q_s$ is surjective for each s. ☐

Proof of Theorem 3.9. We are assuming that $A_{-\infty} = 0$, $RA_{-\infty} = 0$ and $RE^\infty = 0$. The filtration of A_∞ is exhaustive because

$$F_\infty A_\infty = \operatorname*{colim}_s \mathrm{Im}(A_s \to A_\infty) \cong \mathrm{Im}(\operatorname*{colim}_s A_s \to A_\infty) = A_\infty \,.$$

It is also complete, because the derived limit of the surjections $A_s \twoheadrightarrow F_s A_\infty$ is a surjection $0 = RA_{-\infty} \twoheadrightarrow RF_{-\infty} A_\infty$. By Lemma 3.13, $\lim_s Q_s = A_{-\infty} = 0$. Lemma 3.15 then implies that $Q_s = 0$ for all $s \in \mathbb{Z}$. Furthermore, $RQ_{s-1} \cong RQ_s$, $Z_s^\infty / \operatorname{Ker}(\gamma) = 0$ and $F_s A_\infty / F_{s-1} A_\infty \cong E_s^\infty$, for all $s \in \mathbb{Z}$.

The short exact sequences

$$0 \to K_\infty \operatorname{Im}^r A_s \to \operatorname{Im}^r A_s \xrightarrow{\iota_s|} F_{s-r} A_\infty \to 0$$

give the exact sequence

$$Q_s \to F_{-\infty} A_\infty \to \operatorname*{Rlim}_r K_\infty \operatorname{Im}^r A_s \to RQ_s$$

upon passage to limits over r. The Mittag–Leffler short exact sequence

$$0 \to \operatorname*{Rlim}_s Q_s \longrightarrow RA_{-\infty} \longrightarrow \lim_s RQ_s \to 0 \tag{3.1}$$

of [3, 3.4(b)] simplifies to $0 = RA_{-\infty} \cong \lim_s RQ_s \cong RQ_s$. Hence $F_{-\infty} A_\infty \cong \operatorname{Rlim}_r K_\infty \operatorname{Im}^r A_s$ for all s, and $F_{-\infty} A_\infty \cong W$. □

Proof of Theorem 3.10. We are assuming that $A_\infty = 0$ and $RE^\infty = 0$. The filtration of $A_{-\infty}$ is complete and Hausdorff by Lemma 3.13. It is exhaustive because

$$F_\infty A_{-\infty} = \operatorname*{colim}_s \operatorname{Ker}(A_{-\infty} \to A_s) \cong \operatorname{Ker}(A_{-\infty} \to \operatorname*{colim}_s A_s) = A_{-\infty} ,$$

since $\operatorname{colim}_s A_s = A_\infty = 0$.

By Lemma 3.15 we have short exact sequences

$$0 \to E_s^\infty \xrightarrow{\gamma} Q_{s-1} \xrightarrow{\alpha} Q_s \to 0$$

and isomorphisms $\alpha \colon RQ_{s-1} \cong RQ_s$, for all $s \in \mathbb{Z}$. Furthermore, $\lim_s Q_s \to Q_s$ is surjective. Hence, by Lemma 3.13 the image of $\pi_s \colon A_{-\infty} \to A_s$ is equal to Q_s. The surjections $A_{-\infty} \twoheadrightarrow Q_{s-1} \xrightarrow{\alpha} Q_s$ lead to the short exact sequence

$$0 \to F_{s-1} A_{-\infty} \longrightarrow F_s A_{-\infty} \longrightarrow \operatorname{Ker}(\alpha) \to 0 .$$

Thus $F_s A_{-\infty} / F_{s-1} A_{-\infty} \cong \operatorname{Ker}(\alpha) \cong E_s^\infty$.

The Mittag–Leffler sequence (3.1) simplifies to an isomorphism $RA_{-\infty} \cong \lim_s RQ_s \cong \operatorname{colim}_s RQ_s$. Furthermore, $K_\infty \operatorname{Im}^r A_s = \operatorname{Im}^r A_s$ since $A_\infty = 0$, so $\operatorname{Rlim}_r K_\infty \operatorname{Im}^r A_s = RQ_s$ and $W = \operatorname{colim}_s RQ_s$. □
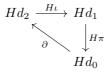
4 Cartan–Eilenberg systems

DEFINITION 4.1. Let $\mathcal{I}$ be a linearly ordered set, and let $\mathcal{I}^{[1]}$ be its *arrow category*, with one object (i, j) for each pair in $\mathcal{I}$ with $i \le j$, and a single morphism from (i, j) to (i', j'), where $i' \le j'$, precisely when $i \le i'$ and $j \le j'$.

Let $\mathcal{I}^{[2]}$ be the category with one object (i,j,k) for each triple in $\mathcal{I}$ with $i \leq j \leq k$, and a single morphism $(i,j,k) \to (i',j',k')$, where $i' \leq j' \leq k'$, precisely when $i \leq i'$, $j \leq j'$ and $k \leq k'$.

Let d_0, d_1 and $d_2 \colon \mathcal{I}^{[2]} \to \mathcal{I}^{[1]}$ be the functors mapping (i,j,k) to (j,k), (i,k) and (i,j), respectively. There are natural transformations $\iota \colon d_2 \to d_1$ and $\pi \colon d_1 \to d_0$, with components $\iota \colon (i,j) \to (i,k)$ and $\pi \colon (i,k) \to (j,k)$, respectively.

View the set $\mathbb{Z}$ of integers as linearly ordered, with the usual ordering.

DEFINITION 4.2 ([6, XV.7]). An $\mathcal{I}$-*system* (H, ∂) in $\mathcal{A}$ is a functor $H \colon \mathcal{I}^{[1]} \to \mathcal{A}$ and a natural transformation $\partial \colon Hd_0 \to Hd_2$ of functors $\mathcal{I}^{[2]} \to \mathcal{A}$, such that the triangle

$$
\begin{array}{ccc}
Hd_2 & \xrightarrow{\ H\iota\ } & Hd_1 \\
 & \nwarrow{\scriptstyle \partial} \quad \searrow & \downarrow{\scriptstyle H\pi} \\
 & & Hd_0
\end{array}
$$

is exact. We assume that the internal degrees of $H\iota$ and $H\pi$ are 0, and that ∂ has internal degree -1. A $\mathbb{Z}$-system is called a (homological) *Cartan–Eilenberg system*.

Remark 4.3. The functor H assigns an object $H(i,j)$ to each pair (i,j) with $i \leq j$ in $\mathcal{I}$, and a morphism

$$\eta \colon H(i,j) \longrightarrow H(i',j')$$

to each morphism $(i,j) \to (i',j')$. By functoriality, $\eta \colon H(i,j) \to H(i,j)$ is the identity, and the composite $\eta \circ \eta \colon H(i,j) \to H(i',j') \to H(i'',j'')$ is equal to $\eta \colon H(i,j) \to H(i'',j'')$. In particular, $\eta = H\pi \circ H\iota \colon H(i,j) \to H(i',j')$ for $\iota \colon (i,j) \to (i,j')$ and $\pi \colon (i,j') \to (i',j')$. The natural transformation ∂ has components

$$\partial_{(i,j,k)} \colon H(j,k) \longrightarrow H(i,j)$$

for each triple (i,j,k) with $i \leq j \leq k$ in $\mathcal{I}$, and

$$\eta \circ \partial_{(i,j,k)} = \partial_{(i',j',k')} \circ \eta \colon H(j,k) \to H(i',j')$$

whenever there is a morphism $(i,j,k) \to (i',j',k')$. The triangle

$$
\begin{array}{ccc}
H(i,j) & \xrightarrow{\ \eta\ } & H(i,k) \\
 & \nwarrow{\scriptstyle \partial} \quad \searrow & \downarrow{\scriptstyle \eta} \\
 & & H(j,k)
\end{array}
\tag{4.1}
$$

is exact, where $\partial = \partial_{(i,j,k)}$. In particular, $H(i,i) = 0$ for each i in $\mathcal{I}$.

Example 4.4. In the language of [12, 1.2.2]: If $\mathcal{C}$ is a stable ∞-category equipped with a t-structure, and if $X \in \mathrm{Gap}(\mathcal{I}, \mathcal{C})$ is an $\mathcal{I}$-complex in $\mathcal{C}$, then $H = \pi_*(X)$ with $H(i, j) = (\pi_t X(i, j))_t$ is an $\mathcal{I}$-system. In particular, if X is a $\mathbb{Z}$-complex then $H = \pi_*(X)$ is a Cartan–Eilenberg system.

DEFINITION 4.5. Let (H, ∂) be a Cartan–Eilenberg system. For each integer s let $E_s^1 = H(s - 1, s)$, and for each integer $r \geq 1$ let

$$Z_s^r = \mathrm{Ker}(\partial \colon E_s^1 \to H(s - r, s - 1))$$
$$B_s^r = \mathrm{Im}(\partial \colon H(s, s + r - 1) \to E_s^1)$$
$$E_s^r = Z_s^r / B_s^r$$

define the r-cycles, r-boundaries and E^r-term, respectively. These form sequences

$$0 = B_s^1 \subset \cdots \subset B_s^r \subset \cdots \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s^1 \,,$$

so each E^r-term is a subquotient of the E^1-term.

Remark 4.6. We note that

$$Z_s^r = \mathrm{Im}(\eta \colon H(s - r, s) \to E_s^1)$$
$$B_s^r = \mathrm{Ker}(\eta \colon E_s^1 \to H(s - 1, s + r - 1)) \,,$$

by exactness. Furthermore, $\partial \colon E_s^1 \to H(s - r, s - 1)$ factors through $\eta \colon E_s^1 \to H(s - 1, s + r - 1)$ by naturality, so $B_s^r \subset Z_s^r$ by exactness.

LEMMA 4.7. *For integers $r \geq 1$ and s there is an isomorphism*

$$Z_s^r / Z_s^{r+1} \xrightarrow{\cong} B_{s-r}^{r+1} / B_{s-r}^r$$

given by $[x] \mapsto [\partial(\tilde{x})]$ for $\eta(\tilde{x}) = x$, where $\eta \colon H(s - r, s) \to E_s^1$ and $\partial \colon H(s - r, s) \to E_{s-r}^1$.

Proof. Apply Lemma 3.14 to the two morphisms

$$E_s^1 \xleftarrow{\ \eta\ } H(s - r, s) \xrightarrow{\ \partial\ } E_{s-r}^1 \,,$$

noting that $\mathrm{Im}(\eta) = Z_s^r$, $\eta(\mathrm{Ker}(\partial)) = Z_s^{r+1}$, $\mathrm{Im}(\partial) = B_{s-r}^{r+1}$ and $\partial(\mathrm{Ker}(\eta)) = B_{s-r}^r$. $\qquad\square$

DEFINITION 4.8. For integers $r \geq 1$ and s let $d_s^r \colon E_s^r \longrightarrow E_{s-r}^r$ be the composite morphism

$$E_s^r = Z_s^r/B_s^r \longrightarrow\!\!\!\!\!\rightarrow Z_s^r/Z_s^{r+1} \xrightarrow{\cong} B_{s-r}^{r+1}/B_{s-r}^r \rightarrowtail Z_{s-r}^r/B_{s-r}^r = E_{s-r}^r \,.$$

More explicitly, it is given by $[x] \mapsto [\partial(\tilde{x})]$, where $x \in Z_s^r \subset E_s^1$ and $\tilde{x} \in H(s-r,s)$ satisfy $\eta(\tilde{x}) = x$, and $\partial(\tilde{x}) \in Z_{s-r}^r \subset E_{s-r}^1$.

PROPOSITION 4.9. *For each Cartan–Eilenberg system (H, ∂), the associated sequence $(E^r, d^r)_{r \geq 1}$ is a spectral sequence. More precisely, $\mathrm{Ker}(d_s^r) = Z_s^{r+1}/B_s^r$ contains $\mathrm{Im}(d_{s+r}^r) = B_s^{r+1}/B_s^r$, so $d_s^r \circ d_{s+r}^r = 0$, and there is an isomorphism*

$$E_s^{r+1} \cong H_s(E^r, d^r)$$

given by $[x] \mapsto [\bar{x}]$, where $x \in Z_s^{r+1}$ maps to $\bar{x} \in Z_s^{r+1}/B_s^r$.

Proof. The calculation of $\mathrm{Ker}(d_s^r)$ and $\mathrm{Im}(d_{s-r}^r)$ is evident from the definition of d_s^r. The isomorphism in question is the Noether isomorphism $Z_s^{r+1}/B_s^{r+1} \cong (Z_s^{r+1}/B_s^r)/(B_s^{r+1}/B_s^r)$. $\qquad\square$
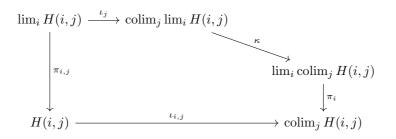
Remark 4.10. The objects Z_s^r and B_s^r of Definition 4.5 agree with those associated in Lemma 2.2 to the spectral sequence $(E^r, d^r)_r$.

5   The canonical interchange morphism

DEFINITION 5.1. Given any functor $H \colon \mathbb{Z}^{[1]} \to \mathcal{A}$, let κ be the canonical *interchange morphism*

$$\kappa \colon \operatornamewithlimits{colim}_{j} \lim_{i} H(i,j) \longrightarrow \lim_{i} \operatornamewithlimits{colim}_{j} H(i,j) \,,$$

as defined in [13, IX.2(3)]. The restriction $\kappa\iota_j$ of κ to $\lim_i H(i,j)$ is the limit over i of the colimit structure maps $\iota_{i,j} \colon H(i,j) \to \operatornamewithlimits{colim}_j H(i,j)$. The projection $\pi_i \kappa$ of κ to $\operatornamewithlimits{colim}_j H(i,j)$ is the colimit over j of the limit structure maps $\pi_{i,j} \colon \lim_i H(i,j) \to H(i,j)$.

$$
\begin{array}{ccc}
\lim_i H(i,j) & \xrightarrow{\;\iota_j\;} & \operatornamewithlimits{colim}_j \lim_i H(i,j) \\[4pt]
{\scriptstyle\pi_{i,j}}\big\downarrow & {}^{\kappa}\searrow & \\[4pt]
 & & \lim_i \operatornamewithlimits{colim}_j H(i,j) \\[4pt]
 & & \big\downarrow{\scriptstyle\pi_i} \\[4pt]
H(i,j) & \xrightarrow[\;\iota_{i,j}\;]{} & \operatornamewithlimits{colim}_j H(i,j)
\end{array}
$$

THEOREM 5.2. *Let (H, ∂) be a Cartan–Eilenberg system. There is a natural exact sequence*

$$0 \to \operatorname*{colim}_{j} \operatorname*{Rlim}_{i} H(i,j) \xrightarrow{\lambda} \operatorname*{Rlim}_{i} \operatorname*{colim}_{j} H(i,j)$$

$$\longrightarrow \operatorname*{colim}_{j} \lim_{i} H(i,j) \xrightarrow{\kappa} \lim_{i} \operatorname*{colim}_{j} H(i,j) \to 0 \,,$$

where $\operatorname{Cok}(\lambda) = \operatorname{colim}_{j} \operatorname{Rlim}_{i} \operatorname{Cok}(\iota_{i,j}) \cong \operatorname{colim}_{j} \lim_{i} \operatorname{Ker}(\iota_{i,j}) = \operatorname{Ker}(\kappa)$ and the middle morphism has internal degree -1. In particular, the interchange morphism κ is surjective.

Proof. To simplify the notation, let

$$D_i = \operatorname*{colim}_{k} H(i,k)$$

for each integer i. The colimit over k of (4.1) is the exact triangle

$$
\begin{array}{ccc}
H(i,j) & \xrightarrow{\iota_{i,j}} & D_i \\
& \nwarrow_{\partial} \quad \searrow & \downarrow \eta \\
& & D_j \,.
\end{array}
$$

Let $K(i,j) = \operatorname{Ker}(\iota_{i,j})$, $I(i,j) = \operatorname{Im}(\iota_{i,j})$ and $C(i,j) = \operatorname{Cok}(\iota_{i,j})$. We then have short exact sequences

$$0 \to K(i,j) \longrightarrow H(i,j) \longrightarrow I(i,j) \to 0 \,,$$
$$0 \to I(i,j) \longrightarrow D_i \longrightarrow C(i,j) \to 0 \,,$$
$$0 \to C(i,j) \longrightarrow D_j \xrightarrow{\partial} K(i,j) \to 0 \,,$$

for each $i \le j$. Passing to limits over i we obtain the exact sequences

$$0 \to \lim_{i} K(i,j) \longrightarrow \lim_{i} H(i,j) \longrightarrow \lim_{i} I(i,j)$$

$$\xrightarrow{\delta} \operatorname*{Rlim}_{i} K(i,j) \longrightarrow \operatorname*{Rlim}_{i} H(i,j) \longrightarrow \operatorname*{Rlim}_{i} I(i,j) \to 0 \,,$$

$$0 \to \lim_{i} I(i,j) \longrightarrow \lim_{i} D_i \longrightarrow \lim_{i} C(i,j)$$

$$\xrightarrow{\delta} \operatorname*{Rlim}_{i} I(i,j) \longrightarrow \operatorname*{Rlim}_{i} D_i \longrightarrow \operatorname*{Rlim}_{i} C(i,j) \to 0 \,,$$

$$0 \to \lim_{i} C(i,j) \longrightarrow D_j \longrightarrow \lim_{i} K(i,j)$$

$$\xrightarrow{\delta} \operatorname*{Rlim}_{i} C(i,j) \longrightarrow 0 \longrightarrow \operatorname*{Rlim}_{i} K(i,j) \to 0 \,.$$

Here $\lim_i D_j = D_j$ and $\operatorname{Rlim}_i D_j = 0$, so $\operatorname{Rlim}_i K(i,j) = 0$. Passing to colimits

over j we get the exact sequences

$$0 \to \operatorname*{colim}_{j} \operatorname*{lim}_{i} K(i,j) \longrightarrow \operatorname*{colim}_{j} \operatorname*{lim}_{i} H(i,j) \longrightarrow \operatorname*{colim}_{j} \operatorname*{lim}_{i} I(i,j) \to 0 \,,$$

$$0 \to \operatorname*{colim}_{j} \operatorname{Rlim}_{i} H(i,j) \xrightarrow{\cong} \operatorname*{colim}_{j} \operatorname{Rlim}_{i} I(i,j) \to 0 \,,$$

$$0 \to \operatorname*{colim}_{j} \operatorname*{lim}_{i} I(i,j) \longrightarrow \operatorname*{lim}_{i} D_i \longrightarrow \operatorname*{colim}_{j} \operatorname*{lim}_{i} C(i,j)$$

$$\xrightarrow{\delta} \operatorname*{colim}_{j} \operatorname{Rlim}_{i} I(i,j) \longrightarrow \operatorname{Rlim}_{i} D_i \longrightarrow \operatorname*{colim}_{j} \operatorname{Rlim}_{i} C(i,j) \to 0 \,,$$

$$0 \to \operatorname*{colim}_{j} \operatorname*{lim}_{i} C(i,j) \longrightarrow \operatorname*{colim}_{j} D_j \longrightarrow \operatorname*{colim}_{j} \operatorname*{lim}_{i} K(i,j)$$

$$\xrightarrow{\delta} \operatorname*{colim}_{j} \operatorname{Rlim}_{i} C(i,j) \to 0 \,.$$

Here $\operatorname{colim}_j D_j = 0$, since $H(j,j) = 0$ for each j. This gives us a natural isomorphism

$$\delta \colon \operatorname*{colim}_{j} \operatorname*{lim}_{i} K(i,j) \xrightarrow{\cong} \operatorname*{colim}_{j} \operatorname{Rlim}_{i} C(i,j)$$

of internal degree $+1$. Furthermore, $\operatorname{colim}_j \operatorname{lim}_i C(i,j) = 0$, which implies that $\operatorname{colim}_j \operatorname{lim}_i I(i,j) \cong \operatorname{lim}_i D_i$. We therefore have natural short exact sequences

$$0 \to \operatorname*{colim}_{j} \operatorname*{lim}_{i} K(i,j) \longrightarrow \operatorname*{colim}_{j} \operatorname*{lim}_{i} H(i,j) \xrightarrow{\kappa} \operatorname*{lim}_{i} D_i \to 0 \,,$$

$$0 \to \operatorname*{colim}_{j} \operatorname{Rlim}_{i} H(i,j) \xrightarrow{\lambda} \operatorname{Rlim}_{i} D_i \longrightarrow \operatorname*{colim}_{j} \operatorname{Rlim}_{i} C(i,j) \to 0 \,.$$

By using δ^{-1} to splice these together, we obtain the asserted four-term exact sequence. □

The whole-plane obstruction W and the interchange kernel $\operatorname{Ker}(\kappa)$ depend on the underlying exact couple and Cartan–Eilenberg system, respectively, not just the associated spectral sequence. Boardman gave a useful sufficient criterion for the vanishing of W, which only depends on data internal to the spectral sequence. The analogous statement for $\operatorname{Ker}(\kappa)$ admits the following direct proof.
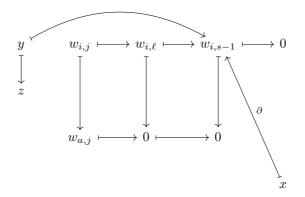
PROPOSITION 5.3 (cf. [3, 8.1]). *Let (H, ∂) be a Cartan–Eilenberg system. Suppose that a and b are such that $d_s^r \colon E_s^r \to E_{s-r}^r$ is zero whenever $s - r \leq a$ and $s > b$. Then $\operatorname{Ker}(\kappa) = 0$.*

Proof. Let $w \in \operatorname{Ker}(\kappa) \subset \operatorname{colim}_j \operatorname{lim}_i H(i,j)$ be the image of $w_j = (w_{i,j})_i \in \operatorname{lim}_i H(i,j)$, for a fixed $j \geq a$. Then $w_{i,j} \mapsto 0$ under $H(i,j) \to \operatorname{colim}_j H(i,j)$, for each $i \leq j$. Let $w_{i,k}$ denote the image of $w_{i,j}$ under $\eta \colon H(i,j) \to H(i,k)$, for each $k \geq j$. Choose $\ell \geq b$ so large that $w_{a,\ell} = 0$. Then $w_\ell = (w_{i,\ell})_i \in \operatorname{lim}_i H(i,\ell)$ maps to w. Consider any $i \leq a$. Suppose, for a contradiction, that $w_{i,\ell} \neq 0$. Then there is a minimal $s > \ell$ such that $w_{i,s} = 0$. By exactness at $H(i,s-1)$ we can choose an $x \in H(s-1,s) = E_s^1$ with $\partial(x) = w_{i,s-1}$. There is

also a minimal $s - r > i$ such that $w_{s-r,s-1} = 0$. We know that $s - r \leq a$, and by exactness at $H(i, s-1)$ we can choose a $y \in H(i, s-r)$ with $\eta(y) = w_{i,s-1}$. Let $z \in H(s-r-1, s-r) = E_{s-r}^1$ be the image of y.

$$
\begin{array}{ccccccc}
y & & w_{i,j} & \longmapsto & w_{i,\ell} & \longmapsto & w_{i,s-1} & \longmapsto & 0 \\
\downarrow & & \downarrow & & \downarrow & & \uparrow \\
z & & & & & & \quad \searrow \partial \\
& & \downarrow & & \downarrow & & \downarrow \\
& & w_{a,j} & \longmapsto & 0 & \longmapsto & 0 \\
& & & & & & & \searrow \\
& & & & & & & & x
\end{array}
$$

By construction there is now a nonzero differential $d_s^r([x]) = [z]$. This contradicts the hypothesis that these differentials are zero. Hence $w_{i,\ell} = 0$ for all i, so $w_\ell = 0$ and $w = 0$. Since $w \in \mathrm{Ker}(\kappa)$ was arbitrary, this proves that $\mathrm{Ker}(\kappa) = 0$. □

Remark 5.4. This argument may be applied in one internal degree t at a time: If there are integers $a(t)$ and $b(t)$ such that $d_s^r \colon (E_s^r)_{t+1} \to (E_{s-r}^r)_t$ is zero whenever $s - r \leq a(t)$ and $s > b(t)$, then $\mathrm{Ker}(\kappa)_t = 0$.
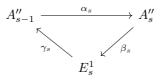
6   Right Cartan–Eilenberg systems

Let $\mathbb{Z}_{+\infty} = \mathbb{Z} \cup \{+\infty\}$, $\mathbb{Z}_{-\infty} = \mathbb{Z} \cup \{-\infty\}$ and $\mathbb{Z}_{\pm\infty} = \mathbb{Z} \cup \{\pm\infty\}$, and extend the linear ordering on $\mathbb{Z}$ to these sets by letting $+\infty$ and $-\infty$ be the greatest and least elements, respectively. Recall Definition 4.2.

DEFINITION 6.1. A $\mathbb{Z}_{+\infty}$-system is called a *right Cartan–Eilenberg system*, a $\mathbb{Z}_{-\infty}$-system is called a *left Cartan–Eilenberg system*, and a $\mathbb{Z}_{\pm\infty}$-system is called an *extended Cartan–Eilenberg system*. By restriction to $\mathbb{Z}$, each such system has an *underlying* Cartan–Eilenberg system.

We discuss right Cartan–Eilenberg systems in this section, and turn to left Cartan–Eilenberg systems in the next section.

DEFINITION 6.2. Let (H, ∂) be a right Cartan–Eilenberg system. Let $A_s'' = H(s, \infty)$ and $E_s^1 = H(s-1, s)$ for each $s \in \mathbb{Z}$, and let

$$
\begin{array}{ccc}
A_{s-1}'' & \xrightarrow{\ \alpha_s\ } & A_s'' \\
& \nwarrow{\scriptstyle \gamma_s} \qquad \swarrow{\scriptstyle \beta_s} & \\
& E_s^1 &
\end{array}
$$

be given by

$$\alpha_s = \eta \colon H(s-1, \infty) \longrightarrow H(s, \infty)$$
$$\beta_s = \partial \colon H(s, \infty) \longrightarrow H(s-1, s)$$
$$\gamma_s = \eta \colon H(s-1, s) \longrightarrow H(s-1, \infty) \,.$$

We call $(A'', E^1, \alpha, \beta, \gamma)$ the *right couple* associated to (H, ∂).

LEMMA 6.3. *The right couple $(A'', E^1, \alpha, \beta, \gamma)$ is an exact couple, and the associated spectral sequence is equal to the one associated to the underlying Cartan–Eilenberg system of (H, ∂).*
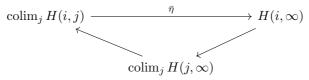
Proof. The exact triangles for $(s-1, s, \infty)$ of the right Cartan–Eilenberg system form the right (exact) couple. Diagram chases show that $\gamma^{-1} \operatorname{Im}(\alpha^{r-1}) = Z_s^r = \operatorname{Ker}(\partial \colon E_s^1 \to H(s-r, s-1))$, that $\beta \operatorname{Ker}(\alpha^{r-1}) = B_s^r = \operatorname{Im}(\partial \colon H(s, s+r-1) \to E_s^1)$, and that the definitions of the d^r-differentials agree, for all $r \geq 1$ and s. $\square$

PROPOSITION 6.4. *Let (H, ∂) be a right (resp. extended) Cartan–Eilenberg system. The right couple (A'', E^1) is conditionally convergent to the limit if and only if*

$$\bar{\eta} \colon \operatorname*{colim}_j H(i, j) \to H(i, \infty)$$

is an isomorphism for some $i \in \mathbb{Z}$ (resp. $i \in \mathbb{Z}_{-\infty}$), in which case it is an isomorphism for every $i \in \mathbb{Z}$ (resp. $i \in \mathbb{Z}_{-\infty}$).
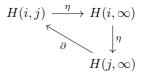
Proof. The colimit over j of the exact triangles for (i, j, ∞) gives an exact triangle

$$\operatorname*{colim}_j H(i, j) \xrightarrow{\quad\bar{\eta}\quad} H(i, \infty)$$
$$\operatorname*{colim}_j H(j, \infty)$$

for each i, so $A''_\infty = \operatorname{colim}_j H(j, \infty)$ is zero if and only if $\bar{\eta} \colon \operatorname{colim}_j H(i, j) \to H(i, \infty)$ is an isomorphism for some i, and this implies that $\bar{\eta}$ is an isomorphism for each i. $\square$

LEMMA 6.5. *Each underlying (resp. left) Cartan–Eilenberg system (H, ∂) can be prolonged, in an essentially unique way, to a right (resp. extended) Cartan–Eilenberg system whose right couple (A'', E^1) is conditionally convergent to its limit.*

Proof. Let $H(i, \infty) = \operatorname{colim}_j H(i, j)$ for each i. The exact triangle

$$H(i, j) \xrightarrow{\quad\eta\quad} H(i, \infty)$$
$$\downarrow{\scriptstyle\eta}$$
$$\partial \qquad H(j, \infty)$$

is the colimit over k of the exact triangles for (i, j, k). $\square$

Remark 6.6. Cartan and Eilenberg assume in [6, XV.7] that $H(i,j)$ is defined for all $-\infty \leq i \leq j \leq \infty$, with $\mathrm{colim}_j H(i,j) \cong H(i,\infty)$ for all $i \in \mathbb{Z}_{-\infty}$. In our terminology this means that they only consider extended Cartan–Eilenberg systems with right couples that are conditionally convergent to their limits. We emphasize the underlying Cartan–Eilenberg systems, with $H(i,j)$ defined for finite i and j, since this structure suffices to define the interchange morphism κ.
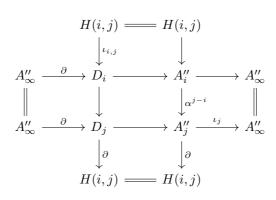
THEOREM 6.7. *Let (H,∂) be a right Cartan–Eilenberg system. There is a natural isomorphism*

$$W'' \xrightarrow{\cong} \mathrm{Ker}(\kappa)$$

of internal degree -1, where $W'' = \mathrm{colim}_s \mathrm{Rlim}_r K_\infty \mathrm{Im}^r A_s''$ is Boardman's whole-plane obstruction group for the right couple (A'', E^1), and κ is the interchange morphism.

Proof. For each $i \leq j \leq k < \infty$ we have a commutative diagram

$$
\begin{array}{ccc}
H(i,j) & \!\!=\!\!\!=\!\!\!=\!\! & H(i,j) \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccc}
H(k,\infty) & \xrightarrow{\ \partial\ } & H(i,k) & \longrightarrow & H(i,\infty) & \longrightarrow & H(k,\infty) \\
\| & & \downarrow & & \downarrow & & \| \\
H(k,\infty) & \xrightarrow{\ \partial\ } & H(j,k) & \longrightarrow & H(j,\infty) & \longrightarrow & H(k,\infty) \\
& & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
& & H(i,j) & \!\!=\!\!\!=\!\!\!=\!\! & H(i,j) & &
\end{array}
$$

with exact rows and columns. Passing to colimits over k we get the commutative diagram

$$
\begin{array}{ccc}
H(i,j) & \!\!=\!\!\!=\!\!\!=\!\! & H(i,j) \\
\downarrow{\scriptstyle \iota_{i,j}} & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccc}
A_\infty'' & \xrightarrow{\ \partial\ } & D_i & \longrightarrow & A_i'' & \longrightarrow & A_\infty'' \\
\| & & \downarrow & & \downarrow{\scriptstyle \alpha^{j-i}} & & \| \\
A_\infty'' & \xrightarrow{\ \partial\ } & D_j & \longrightarrow & A_j'' & \xrightarrow{\ \iota_j\ } & A_\infty'' \\
& & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
& & H(i,j) & \!\!=\!\!\!=\!\!\!=\!\! & H(i,j) & &
\end{array}
$$

with exact rows and columns. Here $D_i = \mathrm{colim}_k H(i,k)$ and $A_i'' = H(i,\infty)$, while $A_\infty'' = \mathrm{colim}_k A_k''$. The homomorphism $D_j \to A_j''$ maps

$$\mathrm{Ker}(\partial\colon D_j \to H(i,j)) = \mathrm{Im}(D_i \to D_j) \cong \mathrm{Cok}(\iota_{i,j}) = C(i,j)$$

onto

$$\mathrm{Im}(D_j \to A_j'') \cap \mathrm{Ker}(\partial\colon A_j'' \to H(i,j))$$
$$= \mathrm{Ker}(\iota_j\colon A_j'' \to A_\infty'') \cap \mathrm{Im}(\alpha^{j-i}\colon A_i'' \to A_j'')$$
$$= K_\infty \mathrm{Im}^{j-i} A_j''\,,$$

with kernel $\mathrm{Im}(\partial\colon A_\infty'' \to D_j)$. Hence there is an exact sequence

$$A_\infty'' \xrightarrow{\ \partial\ } C(i,j) \longrightarrow K_\infty \mathrm{Im}^{j-i} A_j'' \to 0\,.$$

By right exactness of Rlim_i, we obtain an exact sequence

$$\mathrm{Rlim}_i A_\infty'' \xrightarrow{\ \partial\ } \mathrm{Rlim}_i C(i,j) \longrightarrow \mathrm{Rlim}_i K_\infty \mathrm{Im}^{j-i} A_j'' \to 0$$
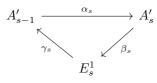
where $\mathrm{Rlim}_i A_\infty'' = 0$. Hence the right hand map is an isomorphism. Passing to colimits over j we obtain the isomorphism

$$\mathrm{colim}_j \mathrm{Rlim}_i C(i,j) \cong \mathrm{colim}_j \mathrm{Rlim}_i K_\infty \mathrm{Im}^{j-i} A_j'' = W''\,.$$

By Theorem 5.2, the left hand side is isomorphic to $\mathrm{Ker}(\kappa)$. $\qquad\square$

7   LEFT CARTAN–EILENBERG SYSTEMS

DEFINITION 7.1. Let (H, ∂) be a left Cartan–Eilenberg system. Let $A_s' = H(-\infty, s)$ and $E_s^1 = H(s-1, s)$ for each $s \in \mathbb{Z}$, and let

be given by

$$\alpha_s = \eta\colon H(-\infty, s-1) \longrightarrow H(-\infty, s)$$
$$\beta_s = \eta\colon H(-\infty, s) \longrightarrow H(s-1, s)$$
$$\gamma_s = \partial\colon H(s-1, s) \longrightarrow H(-\infty, s-1)\,.$$

We call $(A', E^1, \alpha, \beta, \gamma)$ the *left couple* associated to (H, ∂).

LEMMA 7.2. *The left couple $(A', E^1, \alpha, \beta, \gamma)$ is an exact couple, and the associated spectral sequence is equal to the one associated to the underlying Cartan–Eilenberg system of (H, ∂).*

Proof. The exact triangles for $(-\infty, s-1, s)$ of the left Cartan–Eilenberg system form the left (exact) couple. The map of exact couples $(A'', E^1) \to (A', E^1)$ given by $\partial\colon A_s'' = H(s, \infty) \to H(-\infty, s) = A_s'$ and $\mathrm{id}\colon E_s^1 \to E_s^1$, for each $s \in \mathbb{Z}$, induces a map of spectral sequences. It is the identity at the E^1-term, hence is also the identity map at all later E^r-terms. $\qquad\square$
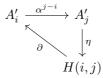
Proposition 7.3. *Let (H, ∂) be a left (resp. extended) Cartan–Eilenberg system whose left couple (A', E^1) is conditionally convergent to its colimit. Then for each $j \in \mathbb{Z}$ (resp. $j \in \mathbb{Z}_{+\infty}$) there is a short exact sequence*

$$0 \to \mathrm{Rlim}_i H(i,j) \xrightarrow{\partial} H(-\infty, j) \xrightarrow{\tilde{\eta}} \lim_i H(i,j) \to 0\,,$$

where ∂ has internal degree -1.

Proof. Fix j, and consider the exact triangles

$$
\begin{array}{ccc}
A'_i & \xrightarrow{\ \alpha^{j-i}\ } & A'_j \\
& {}_{\partial}\nwarrow\ \ \searrow & \downarrow{\scriptstyle \eta} \\
& & H(i,j)
\end{array}
$$

for integers i with $i \le j$. (Some notational modifications are appropriate for $j = \infty$, but otherwise the argument is the same.) Let

$$
\begin{aligned}
K'(i,j) &= \mathrm{Ker}(\alpha^{j-i} \colon A'_i \to A'_j)\,, \\
I'(i,j) &= \mathrm{Im}(\alpha^{j-i} \colon A'_i \to A'_j)\,, \\
C'(i,j) &= \mathrm{Cok}(\alpha^{j-i} \colon A'_i \to A'_j)\,,
\end{aligned}
$$

leading to the short exact sequences

$$
\begin{aligned}
0 &\to K'(i,j) \longrightarrow A'_i \longrightarrow I'(i,j) \to 0\,, \\
0 &\to I'(i,j) \longrightarrow A'_j \longrightarrow C'(i,j) \to 0\,, \\
0 &\to C'(i,j) \longrightarrow H(i,j) \xrightarrow{\partial} K'(i,j) \to 0\,.
\end{aligned}
$$

Here $I'(i,j) = \mathrm{Im}^{j-i} A'_j$, so $\lim_i I'(i,j) = Q_j$ and $\mathrm{Rlim}_i I'(i,j) = RQ_j$ in Boardman's notation (Definition 3.11). Passing to limits over i, we obtain the exact sequences

$$0 \to \lim_i K'(i,j) \to A'_{-\infty} \to Q_j \to \mathrm{Rlim}_i K'(i,j) \to RA'_{-\infty} \to RQ_j \to 0\,,$$

$$0 \to Q_j \to A'_j \xrightarrow{\tilde{\eta}} \lim_i C'(i,j) \to RQ_j \to 0 \to \mathrm{Rlim}_i C'(i,j) \to 0\,,$$

$$0 \to \lim_i C'(i,j) \to \lim_i H(i,j) \xrightarrow{\partial} \lim_i K'(i,j)$$

$$\to \mathrm{Rlim}_i C'(i,j) \to \mathrm{Rlim}_i H(i,j) \xrightarrow{\partial} \mathrm{Rlim}_i K'(i,j) \to 0\,.$$

By the assumption of conditional convergence, $A'_{-\infty} = \lim_j A'_j = 0$ and $RA'_{-\infty} = \mathrm{Rlim}_j A'_j = 0$, so $\lim_i K'(i,j) = 0$, $Q_j \cong \mathrm{Rlim}_i K'(i,j)$, $RQ_j = 0$, $\mathrm{Rlim}_i C'(i,j) = 0$, $\lim_i C'(i,j) \cong \lim_i H(i,j)$ and $\partial \colon \mathrm{Rlim}_i H(i,j) \cong$

$\mathrm{Rlim}_i\, K'(i,j)$, so the short exact sequence $0 \to Q_j \to A'_j \to \lim_i C'(i,j) \to 0$ can be rewritten in the form

$$0 \to \underset{i}{\mathrm{Rlim}}\, H(i,j) \xrightarrow{\partial} A'_j \xrightarrow{\tilde{\eta}} \lim_i H(i,j) \to 0\,,$$

as claimed. $\qquad\square$

Remark 7.4. There is probably no canonical prolongation of a Cartan–Eilenberg system (H,∂) to a left Cartan–Eilenberg system, even if we require that the left couple $(A', E^1, \alpha, \beta, \gamma)$ should be conditionally convergent to its colimit. The objects $H(i,j)$ for integers $i \leq j$ determine $\lim_i H(i,j)$ and $\mathrm{Rlim}_i\, H(i,j)$, but these only determine $A'_j = H(-\infty, j)$ up to an extension.

Theorem 7.5. *Let (H,∂) be a left Cartan–Eilenberg system. There is a natural isomorphism*

$$W' \xrightarrow{\cong} \mathrm{Ker}(\kappa)$$

of internal degree 0, where $W' = \mathrm{colim}_s\, \mathrm{Rlim}_r\, K_\infty\, \mathrm{Im}^r A'_s$ is Boardman's whole-plane obstruction group for the left couple (A', E^1), and κ is the interchange morphism.

Proof. The proof is very similar to that of Theorem 6.7. For each $-\infty < i \leq j \leq k$ we have a commutative diagram

$$
\begin{array}{ccc}
H(i,j) & =\!\!=\!\!= & H(i,j) \\
\downarrow & & \downarrow{\scriptstyle\partial} \\
H(-\infty,k) \longrightarrow H(i,k) \xrightarrow{\partial} H(-\infty,i) \longrightarrow H(-\infty,k) \\
\| \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \| \\
H(-\infty,k) \longrightarrow H(j,k) \xrightarrow{\partial} H(-\infty,j) \longrightarrow H(-\infty,k) \\
\downarrow{\scriptstyle\partial} \qquad\qquad \downarrow \\
H(i,j) & =\!\!=\!\!= & H(i,j)
\end{array}
$$

with exact rows and columns. Passing to colimits over k we get the commutative diagram

$$
\begin{array}{ccc}
H(i,j) & =\!\!=\!\!= & H(i,j) \\
\downarrow{\scriptstyle \iota_{i,j}} & & \downarrow{\scriptstyle\partial} \\
A'_\infty \longrightarrow D_i \xrightarrow{\partial} A'_i \longrightarrow A'_\infty \\
\| \qquad\qquad \downarrow \qquad\qquad \downarrow{\scriptstyle \alpha^{j-i}} \qquad\qquad \| \\
A'_\infty \longrightarrow D_j \xrightarrow{\partial} A'_j \xrightarrow{\iota_j} A'_\infty \\
\downarrow{\scriptstyle\partial} \qquad\qquad \downarrow \\
H(i,j) & =\!\!=\!\!= & H(i,j)
\end{array}
$$

with exact rows and columns. Here $D_i = \operatorname{colim}_k H(i,k)$, $A'_i = H(-\infty, i)$ and $A'_\infty = \operatorname{colim}_k A'_k$. The homomorphism $\partial \colon D_j \to A'_j$ maps

$$\operatorname{Ker}(\partial \colon D_j \to H(i,j)) = \operatorname{Im}(D_i \to D_j) \cong \operatorname{Cok}(\iota_{i,j}) = C(i,j)$$

onto

$$
\begin{aligned}
\operatorname{Im}(\partial \colon D_j \to A'_j) &\cap \operatorname{Ker}(A'_j \to H(i,j)) \\
&= \operatorname{Ker}(\iota_j \colon A'_j \to A'_\infty) \cap \operatorname{Im}(\alpha^{j-i} \colon A'_i \to A'_j) \\
&= K_\infty \operatorname{Im}^{j-i} A'_j \,,
\end{aligned}
$$

with kernel $\operatorname{Im}(A'_\infty \to D_j)$. Hence there is an exact sequence

$$A'_\infty \longrightarrow C(i,j) \xrightarrow{\ \partial\ } K_\infty \operatorname{Im}^{j-i} A'_j \to 0 \,.$$

By right exactness of Rlim_i, we obtain an exact sequence

$$\operatorname{Rlim}_i A'_\infty \longrightarrow \operatorname{Rlim}_i C(i,j) \xrightarrow{\ \partial\ } \operatorname{Rlim}_i K_\infty \operatorname{Im}^{j-i} A'_j \to 0 \,,$$
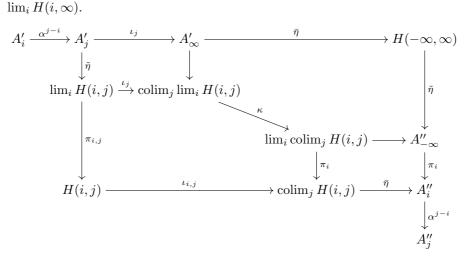
where $\operatorname{Rlim}_i A'_\infty = 0$. Hence the right hand map is an isomorphism. Passing to colimits over j we obtain the isomorphism

$$\partial \colon \operatorname{colim}_j \operatorname{Rlim}_i C(i,j) \xrightarrow{\ \cong\ } \operatorname{colim}_j \operatorname{Rlim}_i K_\infty \operatorname{Im}^{j-i} A'_j = W' \,.$$

By Theorem 5.2, the left hand side is isomorphic to $\operatorname{Ker}(\kappa)$. $\square$

DEFINITION 7.6. An extended Cartan–Eilenberg system (H, ∂) is *conditionally convergent* if the left couple (A', E^1) is conditionally convergent to the colimit and the right couple (A'', E^1) is conditionally convergent to the limit.

Remark 7.7. We summarize the discussion. The spectral sequence $(E^r, d^r)_r$ associated to a extended Cartan–Eilenberg system (H, ∂) has three plausible target groups, namely $A'_\infty = \operatorname{colim}_j H(-\infty, j)$, $H(-\infty, \infty)$ and $A''_{-\infty} = \lim_i H(i, \infty)$.

When (H, ∂) is conditionally convergent, Theorems 3.9 and 3.10 apply to the left couple (A', E^1) and to the right couple (A'', E^1), respectively. Hence there are isomorphisms $\operatorname{colim}_j H(i, j) \cong H(i, \infty)$ for all $i \in \mathbb{Z}_{-\infty}$, and short exact sequences $0 \to \operatorname{Rlim}_i H(i, j) \to H(-\infty, j) \to \lim_i H(i, j) \to 0$ for all $j \in \mathbb{Z}_{+\infty}$. In particular,

$$\bar{\eta} \colon A'_\infty \xrightarrow{\cong} H(-\infty, \infty)$$

in an isomorphism and

$$0 \to RA''_{-\infty} \xrightarrow{\partial} H(-\infty, \infty) \xrightarrow{\tilde{\eta}} A''_{-\infty} \to 0$$

is exact. Under the additional hypothesis $RE^\infty = 0$, we know that $(E^r, d^r)_r$ converges weakly to a complete filtration of A'_∞, with $F_{-\infty} A'_\infty \cong W$. We also know that (E^r, d^r) converges strongly to $A''_{-\infty}$, with $RA''_{-\infty} \cong W$. Hence, if $W = 0$ then (E^r, d^r) converges strongly to $A'_\infty \cong H(-\infty, \infty) \cong A''_{-\infty}$. When $W \neq 0$, the spectral sequence is not strongly convergent to A'_∞, because the filtration $\{F_s A'_\infty\}_s$ fails to be Hausdorff, with limit $F_{-\infty} A'_\infty \cong W$. The spectral sequence is strongly convergent to $A''_{-\infty}$, which is the quotient of $H(-\infty, \infty)$ by $RA''_{-\infty} \cong W$. Hence $H(-\infty, \infty)$ plays the role of the ideal target group, and $W \cong \operatorname{Ker}(\kappa)$ precisely measures the failure of the spectral sequence to converge strongly to this target.

8 SEQUENCES OF SPECTRA

Cartan–Eilenberg systems naturally arise from filtered objects, and are well suited for the construction of multiplicative spectral sequences. Consider a biinfinite sequence of spectra

$$X_{-\infty} \to \cdots \to X_{s-1} \to X_s \to \cdots \to X_\infty \,,$$

with $X_{-\infty} = \operatorname{holim}_s X_s$ and $X_\infty = \operatorname{hocolim}_s X_s$. We obtain an extended Cartan–Eilenberg system (H, ∂), with

$$H(i, j) = \pi_*(\operatorname{cone}(X_i \to X_j))$$

for $-\infty \leq i \leq j \leq \infty$, and two exact couples (A', E^1) and (A'', E^1), with

$$A'_s = \pi_*(\operatorname{cone}(X_{-\infty} \to X_s))$$
$$A''_s = \pi_*(\operatorname{cone}(X_s \to X_\infty))$$

for $s \in \mathbb{Z}$. The three associated spectral sequences are all equal, and begin with

$$E^1_s = \pi_*(\operatorname{cone}(X_{s-1} \to X_s)) \,.$$

A typical aim is to calculate $G = \pi_*(X_\infty)$ under the assumption that $X_{-\infty} \simeq *$. Under these hypotheses the extended Cartan–Eilenberg system (H, ∂) is conditionally convergent, because of the short exact sequence

$$0 \to \operatorname*{Rlim}_s \pi_{*+1}(X_s) \longrightarrow \pi_*(X_{-\infty}) \longrightarrow \lim_s \pi_*(X_s) \to 0$$

and the isomorphism

$$\operatorname*{colim}_{s} \pi_*(X_s) \xrightarrow{\cong} \pi_*(X_\infty)\,.$$

The first (left) exact couple (A', E^1) is conditionally convergent to the colimit

$$A'_\infty = \operatorname*{colim}_{s} \pi_*(\operatorname{cone}(X_{-\infty} \to X_s)) \cong G\,,$$

which is the target of interest, cf. Theorem 3.9, which is a variant of [3, 8.10]. When $RE^\infty = 0$, which can often be verified from the differential structure in the spectral sequence, the spectral sequence is weakly convergent to a complete filtration $\{F_s A'_\infty\}_s$ of G, but the filtration may fail to be Hausdorff. We can therefore only hope to recover the quotient $G/F_{-\infty}A'_\infty$, where $F_{-\infty}A'_\infty = \lim_s F_s A'_\infty$ is the limit of the filtration. In this case there is an isomorphism $W' \cong F_{-\infty}A'_\infty$, where W' is Boardman's group for the exact couple (A', E^1), and our Theorem 7.5 identifies this error term with $\operatorname{Ker}(\kappa)$.

The second (right) exact couple (A'', E^1) is conditionally convergent to the limit

$$A''_{-\infty} = \lim_s \pi_*(\operatorname{cone}(X_s \to X_\infty)) \cong G/RA''_{-\infty}\,,$$

where $RA''_{-\infty} = \operatorname{Rlim}_s \pi_{*+1}(\operatorname{cone}(X_s \to X_\infty))$, cf. Theorem 3.10, which is a variant of [3, 8.13]. When $RE^\infty = 0$ the spectral sequence is strongly convergent to this limit. Since G is the group we are principally interested in, we also need to understand the subgroup $RA''_{-\infty}$. In this case there is an isomorphism $W'' \cong RA''_{-\infty}$, where W'' is Boardman's group for the exact couple (A'', E^1), and our Theorem 6.7 identifies this error term with $\operatorname{Ker}(\kappa)$.

Example 8.1. Let Hk be the Eilenberg–MacLane spectrum of a field k, and let

$$X_s = \prod_{i \geq |s|} Hk$$

for $s, i \in \mathbb{Z}$. Let $X_{s-1} \to X_s$ be given by the identity map on the i-th factor, except when $s \leq 0$ and $i = |s|$, when it is given by $* \to Hk$, and when $s > 0$ and $i = |s| - 1$, when it is given by $Hk \to *$. Then

$$X_\infty \simeq \operatorname{cone}(\bigvee_{i \geq 0} Hk \to \prod_{i \geq 0} Hk)$$

and $X_{-\infty} \simeq *$. We obtain an exact couple (A', E^1) converging conditionally to its colimit, with $A'_s = \prod_{i \geq |s|} k$, $A'_\infty = \prod_{i \geq 0} k / \bigoplus_{i \geq 0} k$, $A'_{-\infty} = 0$ and $RA'_{-\infty} = 0$. Here $E^1_s = k$ for $s \leq 0$ and $E^1_s = \Sigma k$ for $s \geq 1$, with $d_r^{2r-1} \colon E_r^{2r-1} \to E_{1-r}^{2r-1}$ an isomorphism for each $r \geq 1$. Hence $E^\infty = 0$ and $RE^\infty = 0$. Also $\iota_s \colon A'_s \to A'_\infty$ is surjective for each s, so $F_s A'_\infty = A'_\infty$. Hence

$$W' = F_{-\infty}A'_\infty = F_\infty A'_\infty = \prod_{i \geq 0} k / \bigoplus_{i \geq 0} k$$

and $RF_{-\infty}A'_\infty = 0$. Note that this is not a half-plane spectral sequence with exiting or entering differentials, in the sense of [3, §6, §7], even if the E^1-term is concentrated in the intersection of the two half-planes $t \geq 0$ and $t \leq 1$, where t is the internal degree.

Example 8.2. Let $J_p = L_{K(1)}S$ be the image-of-J spectrum, completed at an odd prime p. Its homotopy groups are

$$\pi_n(J_p) \cong \begin{cases} \mathbb{Z}_p & \text{for } n \in \{-1, 0\}, \\ \mathbb{Z}/p^{m+1} & \text{for } n = (2p-2)p^m q - 1,\ p \nmid q, \\ 0 & \text{otherwise,} \end{cases}$$

and its mod p homotopy groups form the graded ring

$$\pi_*(J/p) = \mathbb{Z}/p[\alpha, v^{\pm 1}]/(\alpha^2)\,,$$

with $|\alpha| = 2p-3$ and $|v| = 2p-2$. Let $J_p^{tS^1} = [\tilde{E}S^1 \wedge F(ES_+^1, J_p)]^{S^1}$ denote the Tate construction [7] for the trivial S^1-action on J_p. The biinfinite Greenlees filtration of $\tilde{E}S^1$ leads to a sequence of spectra with associated whole-plane Tate spectral sequence

$$\hat{E}_{s,n}^2(S^1, J_p) \Longrightarrow_s \pi_{s+n}(J_p^{tS^1})\,.$$

Any filtration of $\mathbb{Z}_p$ is complete, so $RE^\infty = 0$. For bidegree reasons the only nonzero differentials are of the form $d_{s,n}^r$ with $s \neq 0$ even and $n = 0$, so $W = 0$ by Boardman's criterion. Hence this spectral sequence is strongly convergent, to the abutment calculated by Hesselholt and Madsen in [8, 0.2]. On the other hand, the S^1-Tate spectral sequence for J/p is

$$\hat{E}_{s,n}^2(S^1, J/p) = \mathbb{Z}/p[t^{\pm 1}] \otimes \mathbb{Z}/p[\alpha, v^{\pm 1}]/(\alpha^2)$$
$$\Longrightarrow_s \pi_{s+n}(J^{tS^1}/p)\,.$$

Bökstedt and Madsen [4] showed that it has nonzero differentials

$$d^{2(p^{k+1}-1)}(t^{p^k - p^{k+1}}) \doteq v^{p(p^k-1)/(p-1)} \cdot t^{p^k - 1}\alpha$$

(up to units in $\mathbb{Z}/p$) for each $k \geq 0$, and the classes α and v are infinite cycles. Hence

$$\hat{E}_{*,*}^\infty(S^1, J/p) = \mathbb{Z}/p[v^{\pm 1}]\{1, t^{-1}\alpha\}$$

and $RE^\infty = 0$. The spectral sequence is thus weakly convergent to $G = \mathbb{Z}/p[v^{\pm 1}]\{1, t^{-1}\alpha\}$, for a complete filtration that might not be Hausdorff. Boardman's criterion for $W = 0$ applies for the differentials landing in even total degrees, but not for the differentials landing in odd total degrees. Indeed, the surjection $\pi_*(J^{tS^1}/p) \twoheadrightarrow G$ with kernel $W \cong F_{-\infty}A_\infty$ is an isomorphism in even degrees, but has the large kernel $(\prod_{i \geq 0} \mathbb{Z}/p)/(\bigoplus_{i \geq 0} \mathbb{Z}/p)$ in each odd degree [8, 4.4].

Example 8.3. Similar patterns are found in the Tate spectral sequences for $\hat{T}(\mathbb{Z})^{tS^1}$ and $\hat{T}(\mathbb{Z})^{tS^1}/p$ for odd primes p, where $T(\mathbb{Z}) = THH(\mathbb{Z})$ denotes the topological Hochschild homology spectrum of the integers and $\varphi_p \colon T(\mathbb{Z}) \to \hat{T}(\mathbb{Z}) = T(\mathbb{Z})^{tC_p}$ is its p-cyclotomic structure map, in the sense of [16, II.1.1]. The S^1-Tate construction for $T(\mathbb{Z})$ is a refinement of Connes' periodic cyclic homology, and is denoted $TP(\mathbb{Z}) = T(\mathbb{Z})^{tS^1}$ in [9]. It is expected to have deep arithmetic significance, by analogy with the results for THH and TP of smooth and proper schemes over finite fields in [9], and over perfectoid bases in [1] and [2]. However, the structure of the Tate spectral sequence for $T(\mathbb{Z})^{tS^1}$ is not fully known, cf. [17, §3] and [18, §1]. The S^1-Tate spectral sequence

$$\hat{E}^2_{*,*}(S^1, \hat{T}(\mathbb{Z})) \Longrightarrow \pi_*(\hat{T}(\mathbb{Z})^{tS^1})$$

is strongly convergent. The mod p spectral sequence

$$\hat{E}^2_{*,*}(S^1, \hat{T}(\mathbb{Z})/p) = \mathbb{Z}/p[t^{\pm 1}] \otimes \mathbb{Z}/p[e, f^{\pm 1}]/(e^2)$$
$$\Longrightarrow \pi_*(\hat{T}(\mathbb{Z})^{tS^1}/p) \,,$$

where $|e| = 2p - 1$ and $|f| = 2p$, is only known to be weakly convergent for a complete filtration. Bökstedt and Madsen [5] showed that the latter spectral sequence has nonzero differentials

$$d^{2p(p^{k+1}-1)/(p-1)}(t^{p^k - p^{k+1}}) \doteq (tf)^{p(p^k-1)/(p-1)} \cdot t^{p^k} e$$

(up to units in $\mathbb{Z}/p$), for each $k \geq 0$. Hence

$$\hat{E}^\infty_{*,*}(S^1, \hat{T}(\mathbb{Z})/p) = \mathbb{Z}/p[(tf)^{\pm 1}]\{1, e\} \,,$$

and there is a surjection $\pi_*(\hat{T}(\mathbb{Z})^{tS^1}/p) \twoheadrightarrow G = \mathbb{Z}/p[(tf)^{\pm 1}]\{1, e\}$, with kernel $W \cong F_{-\infty} A_\infty$. By Boardman's criterion, W is again zero in even total degrees, but may, very well, be nonzero in odd total degrees.

REFERENCES

[1] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397.

[2] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. 129 (2019), 199–310.

[3] J. Michael Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.

[4] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Astérisque 226 (1994), 7–8, 57–143. K-theory (Strasbourg, 1992).

[5] M. Bökstedt and I. Madsen, *Algebraic K-theory of local number fields: the unramified case*, Prospects in topology (Princeton, NJ, 1994), Ann. of Math. Stud., vol. 138, Princeton Univ. Press, Princeton, NJ, 1995, pp. 28–57.

[6] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.

[7] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 113 (1995), no. 543, viii+178.

[8] Lars Hesselholt and Ib Madsen, *The S^1-Tate spectrum for J*, Bol. Soc. Mat. Mexicana (2) 37 (1992), no. 1-2, 215–240. Papers in honor of José Adem (Spanish).

[9] Lars Hesselholt, *Topological Hochschild homology and the Hasse-Weil zeta function*, An alpine bouquet of algebraic topology, Contemp. Math., vol. 708, Amer. Math. Soc., Providence, RI, 2018, pp. 157–180.

[10] Jean-Louis Koszul, *Sur les opérateurs de dérivation dans un anneau*, C. R. Acad. Sci. Paris 225 (1947), 217–219 (French).

[11] Jean Leray, *Structure de l'anneau d'homologie d'une représentation*, C. R. Acad. Sci. Paris 222 (1946), 1419–1422 (French).

[12] Jacob Lurie, *Higher Algebra*. `www.math.harvard.edu/~lurie/papers/HA.pdf`.

[13] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[14] W. S. Massey, *Exact couples in algebraic topology. I, II*, Ann. of Math. (2) 56 (1952), 363–396.

[15] Amnon Neeman, *A counterexample to a 1961 "theorem" in homological algebra*, Invent. Math. 148 (2002), no. 2, 397–420. With an appendix by P. Deligne.

[16] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, Acta Math. 221 (2018), no. 2, 203–409.

[17] John Rognes, *Trace maps from the algebraic K-theory of the integers (after Marcel Bökstedt)*, J. Pure Appl. Algebra 125 (1998), no. 1-3, 277–286.

[18] John Rognes, *Topological cyclic homology of the integers at two*, J. Pure Appl. Algebra 134 (1999), no. 3, 219–286.

[19] Jan-Erik Roos, *Sur les foncteurs dérivés de $\varprojlim$. Applications*, C. R. Acad. Sci. Paris 252 (1961), 3702–3704 (French).

[20] Jan-Erik Roos, *Derived functors of inverse limits revisited*, J. London Math. Soc. (2) 73 (2006), no. 1, 65–83.

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