# $p$-Adic Tate Conjectures and Abeloid Varieties

Oliver Gregory and Christian Liedtke

Received: May 7, 2019
Revised: September 26, 2019

Communicated by Thomas Geisser

Abstract. We explore Tate-type conjectures over $p$-adic fields, especially a conjecture of Raskind [Ra05] that predicts the surjectivity of

$$(\mathrm{NS}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_p)^{G_K} \longrightarrow H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K}$$

if $X$ is smooth and projective over a $p$-adic field $K$ and has totally degenerate reduction. Sometimes, this is related to $p$-adic uniformisation. For abelian varieties, Raskind's conjecture is equivalent to the question whether

$$\mathrm{Hom}(A,B) \otimes \mathbb{Q}_p \to \mathrm{Hom}_{G_K}(V_p(A), V_p(B))$$

is surjective if $A$ and $B$ are abeloid varieties over a $p$-adic field.

Using $p$-adic Hodge theory and Fontaine's functors, we reformulate both problems into questions about the interplay of $\mathbb{Q}$- versus $\mathbb{Q}_p$-structures inside filtered $(\varphi, N)$-modules. Finally, we disprove all of these conjectures and questions by showing that they can fail for algebraisable abeloid surfaces, that is, for abelian surfaces with totally degenerate reduction.

2010 Mathematics Subject Classification: 14F30, 11F80, 14C22, 14K02
Keywords and Phrases: Tate conjecture, abelian and abeloid varieties, $p$-adic fields and $p$-adic uniformisation, $p$-adic Hodge theory, filtered $(\varphi, N)$-module, totally degenerate reduction

# 1 Introduction

Let $F$ be a field and let $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ be the Galois group of a separable closure $F^{\mathrm{sep}}$ of $F$. If $X$ is a smooth and proper variety over $F$ and $\ell$ is a prime different from the characteristic $p$ of $F$, then the first Chern class map gives rise to an injective homomorphism of $\mathbb{Q}_\ell$-vector spaces

$$c_{1,\ell} : \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2_{\text{ét}}\left(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell(1)\right)^{G_F}. \qquad (\star)$$

This is far from being an isomorphism in general. For example, if one takes $F$ to be a separably closed field of characteristic $p = 0$, then the image of $c_{1,\ell}$ is a proper subspace for any smooth and proper variety $X$ with $h^2(\mathcal{O}_X) \neq 0$.

## 1.1 The classical Tate conjecture

However, the *Tate conjecture (for divisors)* predicts that ($\star$) is surjective if $F$ is finitely generated over its prime field (for example, if $F$ is a number field or a finite field), see [Ta65, Ta94]. This conjecture is known to be true if $X$ is an abelian variety [Fa83, Ta66, Zar75], if $X$ is a hyperkähler variety and $F$ is finitely generated over $\mathbb{Q}$ [An96, Ta89], as well as if $X$ is a K3 surface and $F$ is a finite field [Ch13, KM16, MP15, Ma14, Ny83, NO85]. In [Mo17], it has been established for surfaces $X$ with $h^2(\mathcal{O}_X) = 1$ if $F$ is finitely generated over $\mathbb{Q}$ and under the assumption that the Hodge structure on $H^2(X_{\mathbb{C}}, \mathbb{Q})$ varies sufficiently non-trivially in some family. We refer to [To17] for the current state of the Tate conjecture.

## 1.2 Raskind's $p$-adic Tate conjecture

Suppose now that $F$ is a number field, choose a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_F$ and let $F_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of $F$. Let $X$ be a smooth and proper variety over $F$. Then a standard argument (see Proposition 2.7) shows that if ($\star$) is surjective for the completion $X_{F_{\mathfrak{p}}}$ at $\mathfrak{p}$, then $X$ satisfies the Tate conjecture. It is therefore a natural idea to study the homomorphism ($\star$) for varieties defined over $p$-adic fields (by which we shall mean finite extensions of $\mathbb{Q}_p$). Such fields are *not* finitely generated over their prime field.

In light of the previous paragraph, let $X$ be a smooth and proper variety over a $p$-adic field $K$. Then it is well-known that one cannot expect ($\star$) to be surjective without imposing some further conditions on $X$ (see Appendix A for counter-examples). Nevertheless, Raskind [Ra05] has made a series of conjectures of Tate-type over such fields. In codimension one, that is, for divisors, they specialise to the following.

Conjecture 1.1 (Raskind). Let $K$ be a $p$-adic field, let $\ell = p$, and let $X$ be a smooth and proper variety over $K$ with totally degenerate reduction. Then, ($\star$) is surjective.

Of course, one has to specify what one means by *totally degenerate reduction*: roughly speaking, Raskind requires that $X$ has a strictly semi-stable model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$, he asks that the Chow groups of all intersections of all components of the special fibre $\mathcal{X}_0$ to be as trivial as possible, and he requires $\mathcal{X}_0$ to be ordinary, see [Ra05, §1], [RX07a, Definition 1], and Section 4.2 for details. In some sense, Conjecture 1.1 has a long history. We point out that the conjecture is true for varieties that are $p$-adically uniformisable by Drinfeld's upper half space (Proposition 2.5). This is an easy consequence of an observation of Rapoport and incorporated in work of Ito [It05, Appendix], which relies on a result of Schneider and Stuhler [SS91]. We also point out that it is a result of Tate that the conjecture is true when $X$ is the product of two Tate elliptic curves, as is explained in [Se68, Appendix A.1.4] (see also [RX07b, Corollary 19]). Moreover, a somewhat related conjecture has been formulated and established in several cases, such as abelian varieties, by Tankeev [Ta81, §1,2 and 3] for varieties over function fields over $\mathbb{C}$. Let us also remark that the restriction on $\ell$ being equal to $p$ and bad reduction is really necessary (see Appendix A).

### 1.3 TRANSLATION INTO A VARIATIONAL LOG-TATE CONJECTURE

Although Raskind's conjecture superficially looks to have a similar form as the Tate conjecture, we first show that it is in fact a variational conjecture. More precisely, using Fontaine's $\mathbb{D}_{st}$-functor we identify the conjectural image of ($\star$) with

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K} \;\cong\; H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0)^{\varphi=p, N=0} \cap \mathrm{Fil}^1 H^2_{\mathrm{dR}}(X/K),$$

where $K_0$ is the maximal unramified extension of $\mathbb{Q}_p$ contained in $K$, $\varphi$ and $N$ denote the Frobenius and monodromy operator on log-crystalline cohomology, respectively, and $\mathrm{Fil}^\bullet$ denotes the Hodge filtration. This translates Conjecture 1.1 into a "variational log-Tate conjecture" as follows:

1. By an appropriate log-version of Tate's conjecture for $\mathcal{X}_0$ over $k$, one might expect $H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0)^{\varphi=p,N=0}$ to be equal to the $\mathbb{Q}_p$-span of classes of invertible sheaves on $\mathcal{X}_0$, see Section 3.

2. Since $\mathcal{X}_0$ is totally degenerate, there exists a combinatorial description of $H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0)$, $\varphi$ and $N$. In fact, this cohomology group and its operators arise naturally from a $\mathbb{Q}$-vector space; a *rational structure* in the sense of Definition 3.4. In particular, one can be fairly explicit about the $\mathbb{Q}$-span and the $\mathbb{Q}_p$-span of classes of invertible sheaves.

3. The intersection with the Hodge filtration $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(X)$ as a necessary and sufficient condition to deform invertible sheaves from $\mathcal{X}_0$ to $X$ in the smooth case is a theorem of Berthelot and Ogus [BO83, Theorem 3.8] (based on ideas of Deligne and Illusie, see [De81, p. 124 b)]), which has been extended to the semi-stable situation by Yamashita [Ya11, Theorem 3.1].

Now, the crucial point is that the just-mentioned theorems of Berthelot, Ogus, and Yamashita deal with the deformation of the $\mathbb{Q}$-*span* of classes of invertible sheaves from the special to the generic fibre, whereas Raskind's conjecture predicts this to be true even for the $\mathbb{Q}_p$-*span* of classes of invertible sheaves when $\mathcal{X}_0$ is totally degenerate, see Remark 3.9. This translates Conjecture 1.1 into a question about the interplay and the intersection of certain $\mathbb{Q}$-vector spaces, certain $\mathbb{Q}_p$-vector spaces, and the filtration step $\mathrm{Fil}^1$ of a filtered $(\varphi, N)$-module. This leads to the notion of such a module being *Raskind-admissible* (Definition 3.10) and we obtain the following reformulation of Raskind's conjecture for divisors:

Theorem (Theorem 3.11). *Let $X$ be a smooth and proper variety over a $p$-adic field $K$ with totally degenerate reduction. Then, the following are equivalent:*

1. *The homomorphism $(\star)$ is surjective for $\ell = p$, that is, Conjecture 1.1 is true for $X$.*
2. *The filtered $(\varphi, N)$-module $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ is Raskind-admissible with respect to the rational structure arising from $\mathcal{X}_0$.*

One benefit of the reformulation is that it illuminates the known examples of varieties that satisfy Raskind's conjecture, and another is that Raskind-admissibility is a testable property in practice. For example, since (2) is a statement about filtered $(\varphi, N)$-modules, it is tempting to approach Conjecture 1.1 via this statement in semi-linear algebra. In fact, in Section 5.2 we use this approach to give a simple proof for the product of two Tate elliptic curves.

### 1.4 Abelian and abeloid varieties

After seeing that Conjecture 1.1 is actually a variational conjecture about $\mathbb{Q}_p$-classes of line bundles on totally degenerate varieties, we turn our attention to the study of Raskind's conjecture for abelian varieties. Let $A$ and $B$ be abelian varieties over a $p$-adic field $K$ and let $\ell$ be any prime number (possibly $\ell = p$). Functoriality gives a natural homomorphism of $\mathbb{Z}_\ell$-modules

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{G_K}(T_\ell(A), T_\ell(B)), \qquad (\star\star)$$

where the subscript $G_K$ on the right indicates homomorphisms that are $G_K$-equivariant. This homomorphism is injective and its cokernel is torsion free. Then a classical Künneth argument of Tate [Ta66] relates Conjecture 1.1 to the following question.

Question 1.2. Let $K$ be a $p$-adic field, let $\ell = p$, and let $A$ and $B$ be abelian varieties over $K$, both of which have totally degenerate reduction. Is it true that $(\star\star)$ is surjective?

Raskind and Xarles have established Question 1.2 if both $A$ and $B$ are Tate elliptic curves [RX07b, Theorem 18]. Moreover, Mumford's results on $p$-adic uniformisation of abelian varieties with totally degenerate reduction [Mum72b] make a positive answer plausible. Just as with Conjecture 1.1, all of the assumptions are necessary. For example, if $\ell \neq p$ and if $A$ and $B$ are elliptic curves with good reduction, then Lubin and Tate [LT66, §3.5] have given an example where surjectivity of $(\star\star)$ fails. For a comprehensive list of counterexamples where all combinations of assumptions are dropped, see Appendix A. We also point out in passing that using the Kuga-Satake correspondence [KS67], Conjecture 1.1 for abelian varieties implies a version of Conjecture 1.1 for projective hyperkähler varieties (by adapting the arguments of André [An96] or Tankeev [Ta89]).

There are several natural candidates for what it means for an abelian variety over a $p$-adic field $K$ to have totally degenerate reduction, but they turn out to be equivalent up to base change by a finite field extension, see Proposition 4.6. It turns out that the point of view of admitting $p$-adic uniformisation in the sense of Mumford [Mum72b] is a convenient framework for our studies and computations. In particular, from Section 4.3 onward, we will be working with lattices in $\mathbb{G}_{m,K}^g$ and *abeloid varieties*, which are rigid analytic varieties over $K$ that are not necessarily algebraic schemes. We study Question (1.2) in the enlarged context of abeloid varieties. Along the way prove and make use of the following results and computations, some of which may be of independent interest.

1. We describe the abelian groups

$$\operatorname{Hom}(A, B) \qquad \text{and} \qquad \operatorname{Hom}_{G_K}(T_\ell(A), T_\ell(B))$$

for abeloid varieties over a $p$-adic field $K$ in terms of their lattices (Theorem 4.7, Proposition 4.11). We note that the description of $\operatorname{Hom}(A, B)$ is essentially due to Gerritzen [Ge70, Ge71], see also [Kad07].

2. We explicitly compute the filtered $(\varphi, N)$-module $\mathbb{D}_{\mathrm{st}}(V_p(A))$ for an abeloid variety $A$ over a $p$-adic field in terms of a period matrix associated to a lattice (Theorem 4.12), where $V_p$ denotes the rational Tate module. This generalises work of Berger [Be04], Coleman [Co00], Coleman–Iovita [CI99], and Le Stum [LeS95]. As an application, we also describe

$$\operatorname{Hom}_{\mathrm{MF}_K^{\mathrm{wa},\varphi,N}}(\mathbb{D}_{\mathrm{st}}(V_p(A)), \mathbb{D}_{\mathrm{st}}(V_p(B)))$$

in terms of lattices (Proposition 4.16).

3. We introduce an $\mathcal{L}$-invariant for abeloid varieties that generalises the $\mathcal{L}$-invariant of a Tate elliptic curve. If the abeloid variety is the Jacobian $J$ of a Mumford curve $C$, then we show the Coleman-$\mathcal{L}$-invariant of $C$ introduced by Besser and de Shalit [BdS16] coincides with our $\mathcal{L}$-invariant for $J$ (Proposition 4.15).

### 1.5 COUNTER-EXAMPLES

Crucially, just as with Raskind's conjecture above, these descriptions reformulate Question 1.2 into a question about the interplay of certain $\mathbb{Q}$-*vector spaces* versus certain $\mathbb{Q}_\ell$-*vector spaces*, see Section 4.6. Using these computations we are able to give counter-examples to both Conjecture 1.1 and Question 1.2.

THEOREM (Theorem 6.1). *Let $p$ be a prime with $p \geq 5$ and $p \equiv 1 \mod 3$.*

1. *There exists a Tate elliptic curve $A$ and an algebraisable abeloid surface (that is, an abelian surface with totally degenerate reduction) $B$ over $\mathbb{Q}_p$ such that $(\star\star)$ is* not *surjective for $\ell = p$.*

2. *If $X = B$ with $B$ as in (1), then $(\star)$ is* not *surjective for $\ell = p$.*

On the other hand, our reformulation and following computations of Le Stum [LeS95] and Serre [Se68] allows us to confirm Conjecture 1.1 and Question 1.2 for abeloid varieties which are isogenous to arbitrary products of Tate elliptic curves.

PROPOSITION (Proposition 5.3). *Let $K$ be a $p$-adic field and let $A$ and $B$ be abelian varieties over $K$, both of which are isogenous to products of Tate elliptic curves.*

1. *Conjecture 1.1 is true for $A$, that is, $(\star)$ is surjective for $\ell = p$.*

2. *Question 1.2 is true for $A$ and $B$, that is, $(\star\star)$ is surjective for $\ell = p$.*

Finally, let us remark that in this article, we study Conjecture 1.1 and Question 1.2 over $p$-adic fields. Of course, they can also be formulated and studied over local fields of equicharacteristic $p > 0$. However, we expect that after replacing Yamashita's semi-stable Lefschetz theorem on $(1, 1)$ classes [Ya11] with results of Morrow, Lazda, and Pál [Mo14, LP17], one should be able to set up everything in equicharacteristic $p > 0$ and then, we expect that counter-examples similar to those of Theorem 6.1 and Appendix A should disprove them.

### 1.6 ORGANISATION

The article is organised as follows:
In Section 2, we establish general reduction steps for Raskind's conjecture, such as the behaviour under field extensions, dominant, or birational maps. These are familiar from the analogous results for the classical Tate conjectures. We also treat several simple cases and relate Raskind's conjecture to the classical Tate conjectures over number fields.
In Section 3 we translate Conjecture 1.1 into semi-linear algebra and filtered $(\varphi, N)$-modules, we introduce the notion of a rational structure, and we show that Conjecture 1.1 is in fact equivalent to a problem in semi-linear algebra.

In Section 4, we first establish reduction steps for Question 1.2 similar to those in Section 2. Then, we reformulate Raskind's notion of total degeneration for abelian varieties. As a result, we focus on abeloid varieties: we describe their homomorphisms, their $\ell$-adic Tate modules, and the filtered $(\varphi, N)$-modules associated to their $p$-adic Tate modules.
In Section 5, we do explicit computations with filtered $(\varphi, N)$-modules arising from the product of two Tate curves. This way, we prove Conjecture 1.1 for these varieties, but we also produce admissible $(\varphi, N)$-modules that do not satisfy a more general version of Conjecture 1.1.
In Section 6, we construct explicit examples that disprove Conjecture 1.1 and show that Question 1.2 has a negative answer.
In Appendix A, we collect examples which show that Conjecture 1.1 and Question 1.2 also have a negative answer if one allows $\ell \neq p$ or if one does not consider totally degenerate reduction. Here, we claim only little originality.

Acknowledgements. We thank Valentina Di Proietto, Thomas Geisser, Andreas Langer, Christopher Lazda, Bernard Le Stum, Johannes Nicaise, Wiesława Nizioł, Frans Oort, Otto Overkamp, Bjorn Poonen, Michael Rapoport, Wayne Raskind, and Burt Totaro for discussions and comments. We also thank the referee for comments and clarifications. Both authors are supported by the ERC Consolidator Grant 681838 "K3CRYSTAL".

## Notations and Conventions

Throughout the whole article, we fix the following notations

| | |
|---|---|
| $K$ | a $p$-adic field, that is, a finite field extension of $\mathbb{Q}_p$ |
| $\mathcal{O}_K$ | its ring of integers with maximal ideal $\mathfrak{m}_K$ |
| $k$ | its residue field, that is, $\mathcal{O}_K/\mathfrak{m}_K$ |
| $\pi_K$ | a uniformiser of $\mathcal{O}_K$ |
| $W(k)$ | the ring of Witt vectors, which we consider as subring of $\mathcal{O}_K$ |
| $K_0$ | the field of fractions of $W(k)$, which we consider as a subfield of $K$ |
| $\sigma$ | the Frobenius on $W(k)$ and $K_0$ |
| $\overline{K}, \overline{k}$ | algebraic closures of $K$ and $k$, respectively |
| $G_K, G_k$ | their absolute Galois groups |
| $\nu_p$ | the extension of the standard valuation from $\mathbb{Q}_p$ to $K$, that is, $\nu_p(p) = 1$ |
| $\log_p$ | the Iwasawa logarithm, normalised such that $\log_p(p) = 0$ |

By a *variety* over a field $F$, we mean a geometrically integral scheme of finite type over $\operatorname{Spec} F$. If $F'/F$ is a field extension and $X$ is a scheme over $F$, then we define $X_{F'} := X \times_{\operatorname{Spec} F} \operatorname{Spec} F'$.

## 2 Generalities

In this section, we recall some generalities concerning conjectures of Tate-type for divisors. These are well-known to the experts and we do not claim much, if any, originality.

## 2.1 Setup

Let $F$ be a field of characteristic $p \geq 0$, let $F^{\mathrm{sep}}$ be a separable closure, and let $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ be its absolute Galois group. If $X$ is a smooth and proper variety over $F$ and $\ell$ is a prime different from $p$, then the first Chern class map induces a $G_F$-equivariant and injective homomorphism of finite-dimensional $\mathbb{Q}_\ell$-vector spaces

$$c_{1,\ell} \,:\, \mathrm{NS}(X_{F^{\mathrm{sep}}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \,\longrightarrow\, H^2_{\text{ét}}\left(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell(1)\right).$$

Taking $G_F$-invariants, we obtain an inclusion of finite dimensional $\mathbb{Q}_\ell$-vector spaces $(\star)$. It is a natural question, whether this inclusion is in fact an isomorphism, that is, whether it is surjective.

## 2.2 Field extensions

Concerning this question, we have the following remarks, which are well-known in the context of the classical Tate conjecture (see, for example [Ta65, Ta94]), but perhaps not in this context:

Proposition 2.1. *Let $X$ be a smooth and proper variety over a field $F$, let $F \subset F'$ be a finite and separable field extension and let $\ell$ be a prime. If $(\star)$ is surjective with respect to $X_{F'}$, $G_{F'}$, and $\ell$, then $(\star)$ is surjective with respect to $X$, $G_F$, and $\ell$.*

Proof. This is well-known, but we give a proof here since we are not aware of a reference. We consider $G_{F'} := \mathrm{Gal}(F'^{\mathrm{sep}}/F') = \mathrm{Gal}(F^{\mathrm{sep}}/F')$ as subgroup of $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let $n$ be the degree of the extension $F'/F$. Suppose that $(\star)$ is an isomorphism for $X_{F'}$. Let $\alpha \in H^2_{\text{ét}}(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell(1))^{G_F}$. Then, $\alpha$ is fixed by the open subgroup $G_{F'} = \mathrm{Gal}(F^{\mathrm{sep}}/F') \subset G_F$ and hence, there is a $z \in \mathrm{NS}(X_{F'}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ with $c_{1,\ell}(z) = \alpha$. (Technical point: here, we are using that $(\mathrm{NS}(X_{F^{\mathrm{sep}}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{G_{F'}} \cong \mathrm{NS}(X_{F'}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. This is not always true integrally since the Brauer group of $F$ may be non-trivial. However, it is always true rationally since the Brauer group of a field is torsion.) Let $f : X_{F'} \to X$ be the finite morphism given by the base extension. Then, $f_*(z) \in \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ and $c_{1,\ell}(f_*(z)) = n c_{1,\ell}(z) = n\alpha$. Since we are working with rational coefficients, we see that $\alpha$ is the class of a cycle on $X$. □

## 2.3 Dominant and birational maps

Next, we study the question whether surjectivity of $(\star)$ is preserved under birational maps and dominant maps. To do so, we adapt Tate's arguments from [Ta94] to the $p$-adic case.

Proposition 2.2. *Let $K$ be a field that is finitely generated over its prime subfield or a $p$-adic field. Let $X$ and $Y$ be smooth and proper varieties over $K$ and assume that $(\star)$ is surjective for $X$.*

1. *If there exists a dominant and rational map $X \dashrightarrow Y$ of varieties over $K$, or*

2. *if $X$ and $Y$ are birationally equivalent varieties over $K$,*

*then $(\star)$ is surjective for $Y$.*

Proof. This is [Ta94, Theorem 5.2(b)] in the case that $K$ is finitely generated over its prime subfield. Tate shows that the $(\star)$ is an isomorphism for $X$ if and only if $(\star)$ is an isomorphism for an arbitrary dense open $U \subset X$ by using the Gysin sequence for $U \hookrightarrow X$. A weight argument then reduces showing that "the Tate conjecture for divisors on $X$ is equivalent to the Tate conjecture for divisors on $U$" to showing that numerical equivalence coincides with $\ell$-adic homological equivalence for divisors on $X_{\overline{K}}$, where $\overline{K}$ is an algebraic closure of $K$. The coincidence of numerical equivalence and homological equivalence (defined using any Weil cohomology theory) is known over algebraically closed fields, see [Ma57] or [An04, Proposition 3.4.6.1]. To prove the proposition for $p$-adic fields, the same proof works when $K$ since there is an appropriate theory of weights (see [Ja10] for a summary of both cases $\ell \neq p$ and $\ell = p$). □

## 2.4 A simple case

As an easy consequence of the Lefschetz theorem on $(1,1)$-classes and the Lefschetz principle, we obtain the following corollary of Proposition 2.1.

Proposition 2.3. *Let $F$ be a field of characteristic zero. Let $X$ be a smooth and proper variety over $K$ with $H^2(X, \mathcal{O}_X) = 0$. Then, $(\star)$ is surjective for all primes $\ell$.*

Proof. Being of finite type over $K$, there exists a subfield $F' \subseteq F$ that is finitely generated over $\mathbb{Q}$ such that $X$ can be defined over $F'$. Being finitely generated over $\mathbb{Q}$, we may choose an embedding $F' \hookrightarrow \mathbb{C}$. Let $X_{\mathbb{C}}$ be the base change of a model of $X$ over $F'$ to $\mathbb{C}$. Since $X_{\mathbb{C}}$ also satisfies $H^2(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}) = 0$, the Lefschetz theorem on $(1,1)$-cycles shows that the rank of $\mathrm{NS}(X_{\mathbb{C}})$ is equal to the second Betti number $b_2(X_{\mathbb{C}})$. Thus, already the rank of $\mathrm{NS}(X_{\overline{F'}})$ is equal to the $\mathbb{Q}_\ell$-dimension of $H^2_{\text{ét}}(X_{\overline{F'}}, \mathbb{Q}_\ell)$ for some algebraic closure $\overline{F'}$ of $F'$ inside $\mathbb{C}$. Since the Néron-Severi group is finitely generated, there exists a finite field extension $F' \subseteq F''$, such that the rank of $\mathrm{NS}(X_{F''})$ is equal to the $\mathbb{Q}_\ell$-dimension of $H^2_{\text{ét}}(X_{F''}, \mathbb{Q}_\ell)$. Thus, the $G_{F''}$-actions on $\mathrm{NS}(X_{\overline{F'}})$ and $H^2_{\text{ét}}(X_{F''}, \mathbb{Q}_\ell(1))$ are trivial and $(\star)$ is an isomorphism for $X_{F''}$. Thus, $(\star)$ is surjective for $X$ by Proposition 2.1. □

Remark 2.4. For example, this includes varieties that are birationally equivalent to smooth and proper varieties over $\overline{F}$ that are rationally connected (these satisfy $H^2(X, \mathcal{O}_X) = 0$), which includes rational and unirational varieties, and Fano varieties. It also includes geometrically ruled surfaces and Calabi-Yau varieties of dimension at least three (even in the liberal sense of varieties whose

canonical divisor class is numerically trivial and that satisfy $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$).
In particular, the conjectures of Tate and Raskind for divisor holds for these classes of varieties even without extra assumptions on finite generation of the field over its prime subfield or on total degeneration.

### 2.5 Drinfeld's upper half space

A second simple case is the following: let $X$ be a smooth and proper variety over a number field $F$. Assume that there exists a finite extension $F \subset F'$ and a finite place $w$ of $F'$ such that $X \times_F F'_w$ is isomorphic to $\Gamma\backslash\widehat{\Omega}^d_{F'_w}$, where $F'_w$ denotes the $w$-adic completion, where $\widehat{\Omega}^d_{F'_w}$ denotes the Drinfeld upper half space of dimension $d \geq 1$, and where $\Gamma \subset \mathrm{PGL}_{d+1}(F'_w)$ is a cocompact and torsion free discrete subgroup. In [It05, Theorem 7.1], Ito established the Tate conjecture for such varieties, which is based on ideas of Rapoport. Essentially the same proof shows the following observation, see also [It05, Remark 1.3].

Proposition 2.5. *Let $K$ be a $p$-adic field, let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact and torsion free discrete subgroup, and set $X_\Gamma := \Gamma\backslash\widehat{\Omega}^d_K$, Then, $(\star)$ is surjective for $X_\Gamma$ and all primes $\ell$.*

Proof. If $\ell \neq p$, then $H^2_{\text{ét}}(X_{\Gamma,\overline{K}}, \mathbb{Q}_\ell)$ is one-dimensional by [SS91, Theorem 4]. After choosing an embedding $K \hookrightarrow \mathbb{C}$ and using comparison theorems with singular cohomology, it follows that also $H^2_{\text{ét}}(X_{\Gamma,\overline{K}}, \mathbb{Q}_p)$ is one-dimensional. It follows from work of Kurihara and Mustafin [Ku80, Mus78] (see also the discussion in [It05, §6]) that $X_\Gamma$ is projective and thus, $\mathrm{NS}(X_\Gamma) \otimes \mathbb{Q}$ is non-zero. Thus, $(\star)$ is surjective for all primes $\ell$ for dimensional reasons. □

Remark 2.6. Such an $X_\Gamma$ admits a semi-stable model over $\mathcal{O}_K$, whose special fibre is totally degenerate in the sense of Raskind, see [It05, Remark 1.3] and [RX07a, Example 1.(iii)].

### 2.6 The Tate conjecture over number fields

Let us also relate the conjecture of Raskind over $p$-adic fields to the conjecture of Tate over number fields, which was already observed by Raskind [Ra05]. For a number field $F$, we let $\mathcal{O}_F$ be its ring of integers and for a prime ideal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_F$, we denote by $F_\mathfrak{p}$ the $\mathfrak{p}$-adic completion of $F$. The following is a slight generalisation of [Ra05, Proposition 1].

Proposition 2.7. *Let $X$ be a smooth and proper variety over a number field $F$ and assume that there exists a prime $\ell$ and a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ such that $(\star)$ is surjective for $X_{F_\mathfrak{p}}$ $G_{F_\mathfrak{p}}$, and $\ell$. Then, $(\star)$ is surjective for $X$, $G_F$, and $\ell$.*

Proof. Let $\alpha \in H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell(1))^{G_F}$. Then because $H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell(1)) \cong H^2_{\text{ét}}(X_{\overline{F_\mathfrak{p}}}, \mathbb{Q}_\ell(1))$ and $G_{F_\mathfrak{p}} \subset G_F$, we see that $\alpha \in H^2_{\text{ét}}(X_{\overline{F_\mathfrak{p}}}, \mathbb{Q}_\ell(1))^{G_{F_\mathfrak{p}}}$. Therefore there is a $z \in \mathrm{NS}(X_{F_\mathfrak{p}}) \otimes_\mathbb{Z} \mathbb{Q}_\ell$ with $c_{1,\ell}(z) = \alpha$ by assumption. Since

$\mathrm{NS}(X_{\overline{F}}) = \mathrm{NS}(X_{\overline{F_\mathfrak{p}}})$ (the Néron-Severi group is invariant under algebraically closed base extension), we see that $z \in \mathrm{NS}(X_{\overline{F}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$, and in particular $z \in \mathrm{NS}(X_{F'}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ for some finite extension $F'/F$. By making a finite extension if necessary, we may assume that $F'/F$ is Galois. Summing over the $\mathrm{Gal}(F'/F)$-conjugates of $z$, we obtain a $G_F$-invariant class $z'$, such that $c_{1,\ell}(z')$ is a non-zero multiple of $c_{1,\ell}(z)$. Thus, $c_{1,\ell}(z)$ lies in the image of ($\star$). □

## 3 A translation into semi-linear algebra

In this section, we use Fontaine's functor $\mathbb{D}_{\mathrm{st}}$ and Yamashita's $p$-adic semi-stable Lefschetz $(1,1)$-theorem to translate Raskind's conjecture for divisors (Conjecture 1.1) into a question about semi-linear algebra and filtered $(\varphi, N)$-modules.

More precisely, we define the notion of a *rational structure* on a $(\varphi, N)$-module and show how the special fibre of a model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ of some smooth and proper variety $X$ over $K$ with total degeneration gives rise to such a structure. Finally, we introduce the notion of *Raskind-admissibility*, which is the semi-linear algebra version of the Raskind conjecture for divisors on the level of filtered $(\varphi, N)$-modules with rational structure.

### 3.1 Translation into filtered $(\varphi, N)$-modules

Let $X$ be a smooth and proper variety over a $p$-adic field $K$ that admits a proper and semi-stable model

$$\pi \,:\, \mathcal{X} \,\to\, \operatorname{Spec} \mathcal{O}_K$$

over the ring of integers $\mathcal{O}_K$ of $K$, that is, $\mathcal{X}$ is a regular scheme, $\pi$ is a proper and flat morphism, the generic fibre of $\pi$ is isomorphic to $X$, and the special fibre $\mathcal{X}_0$ is a semi-stable scheme over the residue field $k$ of $\mathcal{O}_K$. Here, semi-stable means that $\mathcal{X}_0$ is a strict normal crossing divisor. In particular, the components of $\mathcal{X}_0$ are smooth and geometrically integral over $k$. Let $W(k)$ be the ring of Witt vectors, which we consider as subring of $\mathcal{O}_K$, and let $K_0$ be the field of fractions of $W(k)$. Then, $K_0$ is the maximal unramified extension of $\mathbb{Q}_p$ inside $K$. Endow $\mathcal{X}$ with the log structure induced by $\mathcal{X}_0$, and let $M$ denote the pullback of this log structure on $\mathcal{X}_0$. Then, $(\mathcal{X}_0, M)$ is a fine and log-smooth log scheme over $(\operatorname{Spec} k, \mathbb{N} \to k, 1 \mapsto 0)$, see [HK94, 2.13.2]. We shall write

$$H^n_{\mathrm{log-cris}}(\mathcal{X}_0/K_0) \,:=\, H^n(((\mathcal{X}_0, M)/(W(k),\mathbb{N}))_{\mathrm{cris}}, \mathcal{O}_{(\mathcal{X}_0,M)/(W(k),\mathbb{N})}) \otimes_{W(k)} K_0,$$

where $H^n(((\mathcal{X}_0, M)/(W(k),\mathbb{N}))_{\mathrm{cris}}, \mathcal{O}_{(\mathcal{X}_0,M)/(W(k),\mathbb{N})})$ is the log-crystalline cohomology of $(\mathcal{X}_0, M) \to (\operatorname{Spec} k, \mathbb{N})$. Then $H^n_{\mathrm{log-cris}}(\mathcal{X}_0/K_0)$ is equipped with a semi-linear endomorphism $\varphi$ (Frobenius) and a linear endomorphism $N$ (monodromy) satisfying the relation $N\varphi = p\varphi N$, making the triple

$$\big(H^n_{\mathrm{log-cris}}(\mathcal{X}_0/K_0), \varphi, N\big)$$

a $(\varphi, N)$-module. We refer to [HK94, §3] for the details.
Since $X$ has semi-stable reduction, the $G_K$-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable in the sense of Fontaine for every $n$ [Ts99, Theorem 0.2]. Next, let $B_{\text{st}}$ be Fontaine's period ring and if $V$ is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous $G_K$-action, that is, a $p$-adic Galois-representation, then we have a filtered $(\varphi, N)$-module over $K$

$$\mathbb{D}_{\text{st}}(V) := \left(V \otimes_{\mathbb{Q}_p} B_{\text{st}}\right)^{G_K},$$

that is, a $K_0$-vector space with a semi-linear operator $\varphi$, a linear operator $N$, and a filtration $\text{Fil}^\bullet$ on this vector space tensored with $K$. We recall that $V$ is said to be *semi-stable* if the inequality $\dim_{K_0} \mathbb{D}_{\text{st}}(V) \leq \dim_{\mathbb{Q}_p}(V)$ is an equality. We refer to [CF00] for details. Fontaine's functor $\mathbb{D}_{\text{st}}$ establishes an equivalence of categories between the category of semi-stable $G_K$-representations and the category $MF_K^{\text{wa},\varphi,N}$ of admissible filtered $(\varphi, N)$-modules over $K$ [CF00, Théorème A] .
By the semi-stable comparison theorem [Ts99, Theorem 0.2], the admissible filtered $(\varphi, N)$-module $\mathbb{D}_{\text{st}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ is equal to

$$D^n := \left(H^n_{\log-\text{cris}}(\mathcal{X}_0/K_0),\, \text{Fil}^\bullet H^n_{\text{dR}}(X/K),\, \varphi,\, N\right).$$

Using this translation, we have the following.

PROPOSITION 3.1. *Let $X$ be a smooth and proper variety over $K$ and assume that there exists a proper and semi-stable model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ of $X$. Let $\mathcal{X}_0$ be the special fibre. Then, there exists an isomorphism of $\mathbb{Q}_p$-vector spaces*

$$H^n_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Q}_p(m)\right)^{G_K} \cong H^n_{\log-\text{cris}}(\mathcal{X}_0/K_0)^{\varphi=p^m, N=0} \cap \text{Fil}^m H^n_{\text{dR}}(X/K) \quad (1)$$

*for all non-negative integers $m, n$.*

PROOF. This follows from the equalities and isomorphisms

$$\begin{aligned} H^n_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Q}_p(m)\right)^{G_K} &= \operatorname{Hom}_{G_K}\left(\mathbb{Q}_p,\, H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(m))\right) \\ &\cong \operatorname{Hom}_{\text{MF}_K^{\text{wa},\varphi,N}}\left(K,\, D^n(m)\right) \\ &\cong \left\{x \in H^n_{\log-\text{cris}}(\mathcal{X}_0/K_0) : \varphi(x) = p^m \cdot x,\, N(x) = 0\right\} \\ &\qquad \cap \text{Fil}^m H^n_{\text{dR}}(X/K), \end{aligned}$$

where $K$ denotes the trivial filtered $(\varphi, N)$-module. □

REMARK 3.2. If $\ell \neq p$, then we have

$$\dim_{\mathbb{Q}_\ell} H^n_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Q}_\ell(m)\right)^{G_K} = \dim_{\mathbb{Q}_p} H^n_{\log-\text{cris}}(\mathcal{X}_0/K_0)^{\varphi=p^m, N=0}$$

by [KM74]. This explains why the dimensions of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(m))^{G_K}$ behave differently for $\ell = p$ and $\ell \neq p$ and it also shows that for $\ell \neq p$, these vector spaces capture information about the special fibre $\mathcal{X}_0$ only. We refer to Consani's article [Co98] for background and some conjectures.

We end our discussion by presenting some probably well-known dimension estimates: since $K$ is of characteristic zero, the Frölicher spectral sequence

$$E_1^{r,s} = H^s(X, \Omega^r_{X/K}) \Rightarrow H^{r+s}_{\mathrm{dR}}(X/K)$$

degenerates at $E_1$. In particular, by Proposition 3.1 and Remark 3.2 we obtain the dimension estimates

$$\begin{aligned}\dim_{\mathbb{Q}_\ell} H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(m))^{G_K} - \sum_{i=m+1}^{n} h^{i,n-i}(X) \;&\le\; \dim_{\mathbb{Q}_p} H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(m))^{G_K} \\ &\le\; \dim_{\mathbb{Q}_\ell} H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(m))^{G_K}\,,\end{aligned}$$

where $h^{i,j}(X) := \dim_K H^j(X, \Omega^i_{X/K})$. In the case of interest to us, that is to say when $n = 2$ and $m = 1$, this gives

$$\begin{aligned}\dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K} - h^{2,0}(X) \;&\le\; \dim_{\mathbb{Q}_p} H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K} \\ &\le\; \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K}.\end{aligned}$$

### 3.2 Rational structures

Next, we deal with the log-crystalline cohomology of the special fibre $\mathcal{X}_0$ of a proper and semi-stable model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ of $X$. Let

$$Y \;=\; \bigcup_{i\in I} Y_i$$

be the decomposition of the special fibre $Y = \mathcal{X}_0$ into irreducible components. For a subset $J \subset I$, we denote by $Y_J$ the intersection of all $Y_j$ with $j \in J$. Since $Y$ is strict normal crossing, each $Y_J$ is a smooth, proper, and geometrically integral scheme over $k$. Moreover, we denote by $Y^{[i]}$ the disjoint union of all $Y_J$'s where $J$ has $(i+1)$ elements, that is, the subscript $i$ is equal to the codimension of $Y_J$ in $Y$. By Mokrane [Mo93, §3] and Nakkajima [Na05, §4], there exists a $p$-adic Steenbrink-Rapoport-Zink spectral sequence

$$E_1^{-k,h+k} \;=\; \bigoplus_{j\ge\max\{-k,0\}} H^{h-2j-k}_{\mathrm{cris}}(Y^{[2j+k]}/K_0)(-j-k) \;\Rightarrow\; H^h_{\log-\mathrm{cris}}(Y/K_0)\,, \tag{2}$$

which degenerates at $E_2$. This spectral sequence is compatible with the $F$-isocrystal structures on both sides and induces a monodromy operator $N$ on the right hand side.

Definition 3.3. A scheme $Y$ over a finite field $k$ is called *cohomologically totally degenerate* if it is strictly normal crossing, say, equal to $\bigcup_{i\in I} Y_i$, where the $Y_i$ are the irreducible components of $Y$, such that

1. for all $j$ and all odd integers $i$, the crystalline cohomology groups $H^i_{\mathrm{cris}}(Y^{[j]}/K_0)$ are zero and

2. for all $i$ and $j$, the cycle class maps

$$\mathrm{CH}^j(Y^{[i]}) \otimes K_0 \;\to\; H^{2j}_{\mathrm{cris}}(Y^{[i]}/K_0)(j)$$

are isomorphisms.

Put differently, all crystalline cohomology groups of all intersections of components of $Y$ are spanned by classes of algebraic cycles. In particular, since the Chow groups of a variety are $\mathbb{Q}$-vector spaces and since the action of Frobenius on Chow groups is trivial up to Tate twist, this implies that the log-crystalline cohomology of $\mathcal{X}_0$ is of a very simple form. For $H^2_{\mathrm{log-cris}}(\mathcal{X}_0/K_0)$, it leads to the following.

Definition 3.4. Let $k$ be a finite field, let $K_0 = \mathrm{Frac}(W(k))$, let $\sigma$ be the Frobenius on $K_0$, and let $H$ be a $(\varphi, N)$-module over $K_0$. A *rational structure* on a $(\varphi, N)$-module $H$ consists of a finite-dimensional $\mathbb{Q}$-vector $V$ space together with a direct sum decomposition

$$V \;=\; A \,\oplus\, B_0 \,\oplus\, B_1 \,\oplus\, C$$

and two $\mathbb{Q}$-linear endomorphisms $\varphi_V$ and $N_V$ such that

1. $N_V$ is zero on $B_1$ and $A$, and $N_V$ induces isomorphisms

$$C \xrightarrow{N} N(C) = B_0 \quad \text{ and } \quad B_0 \xrightarrow{N} N(B_0) = A\,.$$

2. $\varphi_V$ acts as identity on $A$, as multiplication by $p$ on $B_0 \oplus B_1$, and as multiplication by $p^2$ on $C$.

3. As $(\varphi, N)$-module, $H$ is isomorphic to $V \otimes_{\mathbb{Q}} K_0$ with $\varphi = \varphi_V \otimes \sigma$ and $N = N_V \otimes \mathrm{id}$.

If $V$ is a rational structure on a filtered $(\varphi, N)$-module $H$ over $K_0$, then we have an isomorphism of $\mathbb{Q}_p$-vector spaces

$$H^{\varphi=p, N=0} \;\cong\; B_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p. \tag{3}$$

For example, if $m = 1$ and $n = 2$ in Proposition 3.1 and if the $(\varphi, N)$-module $H^2_{\mathrm{log-cris}}(\mathcal{X}_0/K_0)$ there comes with a rational structure, then (3) makes the right hand side of (1) much easier to compute. Before establishing a natural rational structure on $H^2_{\mathrm{log-cris}}(\mathcal{X}_0/K_0)$, we need one more definition.

Definition 3.5. The *dual graph of* $Y$ is the simplicial complex $\Gamma$ that has one vertex $P_i$ for each component $Y_i$ of $Y$ and the simplex $\langle P_{i(0)}, ..., P_{i(k)}\rangle$ belongs to $\Gamma$ if and only if $Y_J$ for $J = \{i(0), ..., i(k)\}$ is non-empty. We define $H^*(\Gamma) := H^*_{\mathrm{sing}}(|\Gamma|, \mathbb{Q})$ to be the singular cohomology of the topological realisation $|\Gamma|$ of $\Gamma$.

PROPOSITION 3.6. *Let* $Y = \bigcup_{i\in I} Y_i$ *be a cohomologically totally degenerate scheme over a finite field* $k$ *and consider the* $(\varphi, N)$*-module* $H^2_{\log-\mathrm{cris}}(Y/K_0)$*. Then,*

1. *The cycle class maps*

$$\mathrm{CH}^j(Y^{[i]}) \otimes \mathbb{Q} \;\to\; H^{2j}_{\mathrm{cris}}(Y^{[i]}/K_0)(j)$$

*followed by the* $p$*-adic spectral sequence* (2) *induce a natural rational structure*

$$(V = A \oplus B_0 \oplus B_1 \oplus C, \varphi_V, N_V)$$

*on* $H^2_{\log-\mathrm{cris}}(Y/K_0)$*.*

2. *If* $\Gamma$ *denotes the dual complex of* $Y$ *and if* $H^2(\Gamma) \neq 0$*, then* $N$ *has maximally nilpotent monodromy, that is,* $N^2 \neq 0$*.*

3. *If* $Y^{\mathrm{sm}}$ *denotes the smooth locus of* $Y$ *and if* $Y$ *is equipped with its natural log-structure, then there exist homomorphisms*

$$\mathrm{Pic}(Y) \;\to\; \mathrm{Pic}(Y^{\mathrm{sm}}) \;\cong\; \mathrm{Pic}^{\log}(Y),$$

*where the first map is restriction and the second is an isomorphism.*

*The* $p$*-adic spectral sequence* (2) *gives rise to an isomorphism*

$$\mathrm{Pic}(Y) \otimes \mathbb{Q} \;\cong\; B_1.$$

*Moreover, the first Chern class maps give rise to a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(Y) \otimes F & & \\ \downarrow & \searrow & \\ \mathrm{Pic}^{\log}(Y) \otimes F & \longrightarrow & H^2_{\log-\mathrm{cris}}(Y/K_0)^{\varphi=p, N=0}, \end{array}$$

*where the images of both Chern class maps are equal to* $B_1$ *if* $F = \mathbb{Q}$ *and equal to* $H^2_{\log-\mathrm{cris}}(Y/K_0)^{\varphi=p,N=0}$ *if* $F = \mathbb{Q}_p$*.*

PROOF. Using the cycle class maps, the assumption on cohomological degeneracy, and the spectral sequence (2), we obtain a $\mathbb{Q}$-vector space $V$, such that $V \otimes K_0$ is naturally isomorphic to $H^2_{\log-\mathrm{cris}}(Y/K_0)$. Moreover, since Frobenius acts on cohomology classes of cycles by multiplication by some power of $p$, we obtain a direct sum decomposition $V = V_0 \oplus V_1 \oplus V_2$ of $\mathbb{Q}$-vector spaces together with a linear operator $\varphi_V$ that is multiplication by $p^i$ on $V_i$, such that the $F$-isocrystal structure on $H^2_{\log-\mathrm{cris}}(Y/K_0)$ is isomorphic to $(V \otimes K_0, \varphi_V \otimes \sigma)$. Also, the monodromy operator $N$ and the weight filtration arise from the spectral sequence and using the $\mathbb{Q}$-vector space structures, we obtain a monodromy operator on $V$. For details, we refer to the discussions in [BGS97, §1] or [Mo84].

The fact that $N^2 \neq 0$ is equivalent to $H^2(\Gamma) \neq 0$ is shown in [Mo84, §6] and although they are stated in the framework of complex geometry in loc.cit., the arguments carry over literally to our situation. This establishes claims (1) and (2).
Concerning claim (3): first, we have a restriction homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y^{\mathrm{sm}})$ and an isomorphism $\mathrm{Pic}(Y^{\mathrm{sm}}) \cong \mathrm{Pic}^{\log}(Y)$ by the discussion at the beginning of [Ya11, §2].
Next, let $\delta_i : Y^{[j]} \to Y^{[j-1]}$ be the morphisms induced by the obvious inclusions

$$Y_{\iota_1} \cap \ldots \cap Y_{\iota_{j+1}} \hookrightarrow Y_{\iota_1} \cap \ldots \cap Y_{\iota_{i-1}} \cap Y_{\iota_{i+1}} \cap \ldots \cap Y_{\iota_{j+1}} .$$

To give an invertible sheaf on $Y$ is equivalent to giving an invertible sheaf on $Y^{[0]}$ plus compatibilities under the restriction maps $\delta_1^*, \delta_2^* : Y^{[0]} \rightrightarrows Y^{[1]}$. In particular, we obtain an isomorphism

$$\mathrm{Pic}(Y) \otimes \mathbb{Q} \;\cong\; \ker\left( \mathrm{Pic}(Y^{[0]}) \otimes \mathbb{Q} \xrightarrow{\delta_2^* - \delta_1^*} \mathrm{Pic}(Y^{[1]}) \otimes \mathbb{Q} \right) .$$

Now, the boundary morphisms $\delta_i$ give an augmented simplicial scheme $Y^{[\bullet]} \to Y$, which is a proper smooth hypercovering of $Y$. Since each $Y^{[j]}$ is smooth, we have $H^*_{\mathrm{cris}}(Y^{[j]}/K_0) \cong H^*_{\mathrm{rig}}(Y^{[j]}/K_0)$ by [Be97, Proposition 1.9] and hence

$$H^*_{\mathrm{cris}}(Y^{[\bullet]}/K_0) \;\cong\; H^*_{\mathrm{rig}}(Y^{[\bullet]}/K_0) \cong H^*_{\mathrm{rig}}(Y/K_0),$$

where the second isomorphism is because rigid cohomology satisfies cohomological descent for proper hypercoverings [Ts03, Corollary 2.2.3]. Now, consider the spectral sequence

$$E_1^{s,t} \;=\; H^t_{\mathrm{rig}}(Y^{[s]}/K_0) \;\Rightarrow\; H^{s+t}_{\mathrm{rig}}(Y/K_0)$$

of the (hyper)covering (see, for example [Ts03, Theorem 4.5.1]). This degenerates at $E_2$ by a standard weight argument (our assumption that $Y$ is cohomologically totally degenerate makes this argument very easy, but see also [Ts03, Corollary 5.2.4] for the general statement). In particular, we find isomorphisms

$$\begin{aligned} H^2_{\mathrm{rig}}(Y/K_0) &\cong \ker\left( H^2_{\mathrm{rig}}(Y^{[0]}/K_0) \xrightarrow{\delta_2^* - \delta_1^*} H^2_{\mathrm{rig}}(Y^{[1]}/K_0) \right) \\ &\cong \ker\left( H^2_{\mathrm{cris}}(Y^{[0]}/K_0) \xrightarrow{\delta_2^* - \delta_1^*} H^2_{\mathrm{cris}}(Y^{[1]}/K_0) \right) \\ &\cong \ker\left( \mathrm{Pic}(Y^{[0]}) \otimes \mathbb{Q}_p \xrightarrow{\delta_2^* - \delta_1^*} \mathrm{Pic}(Y^{[1]}) \otimes \mathbb{Q}_p \right) \\ &\cong \mathrm{Pic}(Y) \otimes \mathbb{Q}_p, \end{aligned}$$

where the third isomorphism is induced by the crystalline Chern class map, and is an isomorphism by the assumption that $Y$ is cohomologically totally degenerate. The compatibility between rigid and crystalline Chern

classes [Pe03, Théorème 5.2.3] implies that the above isomorphism $\mathrm{Pic}(Y)\otimes\mathbb{Q}_p \xrightarrow{\sim} H^2_{\mathrm{rig}}(Y/K_0)$ is induced by the rigid Chern class map $c_1^{\mathrm{rig}} : \mathrm{Pic}(Y) \to H^2_{\mathrm{rig}}(Y/K_0)$.
Finally, one has a square

$$\begin{array}{ccc} \mathrm{Pic}(Y) & \longrightarrow & \mathrm{Pic}^{\log}(Y) \\ \downarrow{\scriptstyle c_1^{\mathrm{rig}}} & & \downarrow{\scriptstyle c_1} \\ H^2_{\mathrm{rig}}(Y/K_0) & \twoheadrightarrow & H^2_{\log-\mathrm{cris}}(Y/K_0)^{\varphi=p,N=0}, \end{array}$$

where the top arrow is induced by the inclusion $\mathcal{O}_Y^\times \hookrightarrow M^{\mathrm{gp}}$ (recall that $M$ is the log structure on $Y$). The surjection is the one given by the Clemens-Schmid exact sequence [CT14]. Recall from loc. cit. that this map is the composition

$$\begin{array}{rcl} H^2_{\mathrm{rig}}(Y/K_0) \;\cong\; H^2_{\mathrm{conv}}(Y/W(k)) & \to & H^2_{\log-\mathrm{conv}}((Y,M)/(W(k),\mathbb{N})) \\ & \cong & H^2_{\log-\mathrm{cris}}(Y/K_0), \end{array}$$

where the first isomorphism is because $Y$ is proper and the second isomorphism is because $(Y,M) \to (\mathrm{Spec}\ k, \mathbb{N})$ is log-smooth. Using the compatibility of the rigid and crystalline (resp. log-crystalline) Chern classes, one checks that the square commutes. Tensoring the square with $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) finishes the proof. □

Remark 3.7. One can give an elementary argument for the surjectivity of $c_1 : \mathrm{Pic}^{\log}(Y)\otimes\mathbb{Q}_p \to H^2_{\log-\mathrm{cris}}(Y/K_0)^{\varphi=p,N=0}$ using the Hyodo-Kato complex, but we have chosen to present the above proof because it nicely demonstrates the relationship between $\mathrm{Pic}(Y)$ and $\mathrm{Pic}^{\log}(Y)$.

### 3.3 Raskind-admissibility

Next, we recall the following $p$-adic Lefschetz $(1,1)$-theorem, due to Berthelot and Ogus [BO83, Theorem 3.8] in the smooth case and to Yamashita [Ya11, Theorem 3.1] in the semi-stable case.

Theorem 3.8 (Yamashita). *Let $X$ be a smooth and proper variety over $K$ and assume that there exists a proper and semi-stable model $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_K$ of $X$. Let $\mathcal{X}_0$ be the special fibre.*

1. *There exists a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Pic}(X) & \leftarrow & \mathrm{Pic}(\mathcal{X}) & \to & \mathrm{Pic}(\mathcal{X}_0) \\ \| & & \downarrow & & \downarrow \\ \mathrm{Pic}(X) & = & \mathrm{Pic}^{\log}(\mathcal{X}) & \to & \mathrm{Pic}^{\log}(\mathcal{X}_0). \end{array}$$

*where the vertical maps are restrictions and the horizontal maps are specialisations.*

2. *An invertible sheaf* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_0) \otimes \mathbb{Q}$ *(resp.* $\mathcal{L} \in \mathrm{Pic}^{\log}(\mathcal{X}_0) \otimes \mathbb{Q}$*) can be lifted to* $\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Q}$ *(resp.* $\mathrm{Pic}^{\log}(\mathcal{X}) \otimes \mathbb{Q}$ *) if and only if its first Chern class satisfies*

$$c_1(\mathcal{L}) \in \mathrm{Fil}^1 \cap \left(H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0) \otimes_{K_0} K\right),$$

*where* $\mathrm{Fil}^\bullet$ *denotes the Hodge filtration on* $H^2_{\mathrm{dR}}(X/K)$*.*

Proof. Claim (1) is in the discussion at the beginning of [Ya11, §2]. Claim (2) is [Ya11, Theorem 3.1]. □

After these preparations, we make the following key observation.

Remark 3.9. If $\mathcal{X}_0$ has cohomologically totally degenerate reduction, it looks at first glance as if the combination of Proposition 3.1, Proposition 3.6, and Theorem 3.8 might prove Conjecture 1.1. However, it is crucial to note that Theorem 3.8 deals with $\mathbb{Q}$-*classes* of invertible sheaves, whereas the other results deal with $\mathbb{Q}_p$-*classes*.

Definition 3.10. Let $K$ be a $p$-adic field and let $K_0$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $K$. Let $(H \otimes_{K_0} K, \mathrm{Fil}^\bullet, \varphi, N)$ be a filtered $(\varphi, N)$-module over $K$ and let $V$ be a rational structure on $H$. Then, $H$ is called *Raskind-admissible* if the natural inclusion of $\mathbb{Q}_p$-vector spaces

$$\left(\mathrm{Fil}^1 \cap V^{\varphi=p,N=0}\right) \otimes_{\mathbb{Q}} \mathbb{Q}_p \subset \mathrm{Fil}^1 \cap H^{\varphi=p,N=0}$$

is an equality.

We remark that $V^{\varphi=p,N=0} = B_1$ and $H^{\varphi=p,N=0} = B_1 \otimes \mathbb{Q}_p$ in the notation of Definition 3.4, see also Equation (3). After these preparations, we now reformulate Raskind's conjecture for divisors (Conjecture 1.1) into semi-linear algebra. In fact, the following equivalence holds under a slightly weaker assumptions than Raskind's requirement of total degeneracy.

Theorem 3.11. *Let $X$ be a smooth and proper variety over $K$ and assume that there exists a proper and semi-stable model $\mathcal{X} \to \mathrm{Spec}\, \mathcal{O}_K$ of $X$. Assume that the special fibre $\mathcal{X}_0$ is cohomologically totally degenerate. Then, the following are equivalent:*

1. *The homomorphism* $(\star)$ *is surjective for* $\ell = p$*.*
2. *The filtered* $(\varphi, N)$*-module* $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ *together with the rational structure arising from* $\mathcal{X}_0$ *is Raskind-admissible.*

Proof. By Proposition 3.6.(1), there exists a rational structure $(V = A \oplus B_0 \oplus B_1 \oplus C, \varphi_V, N_V)$ on $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ associated to $\mathcal{X}_0$. Next, by Proposition 3.6.(3), the first Chern class induces an isomorphism

$$\mathrm{Pic}(\mathcal{X}_0) \otimes_{\mathbb{Z}} \mathbb{Q} \cong B_1. \tag{4}$$

Using the first Chern class on $X$ we obtain a map

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Fil}^1 \cap H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0)^{\varphi=p, N=0}, \tag{5}$$

whose image lies inside the subspace $B_1$ of $H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0)^{\varphi=p,N=0}$. Using Yamashita's theorem (Theorem 3.8), it follows that the first map in

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Fil}^1 \cap B_1 \cong \mathrm{Fil}^1 \cap \mathrm{Pic}(\mathcal{X}_0) \otimes \mathbb{Q} \tag{6}$$

is surjective.

By Proposition 3.1 and (3), we have

$$\begin{aligned} H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(1))^{G_K} &\cong \mathrm{Fil}^1 \cap H^2_{\log-\mathrm{cris}}(\mathcal{X}_0/K_0)^{\varphi=p,N=0} \\ &= \mathrm{Fil}^1 \cap (B_1 \otimes \mathbb{Q}_p). \end{aligned}$$

Combining this with equation (5), we see that the homomorphism ($\star$) is surjective for $\ell = p$ if and only if

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to \mathrm{Fil}^1 \cap (B_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p)$$

is surjective. In view of (6), this is equivalent to asking whether

$$\left(\mathrm{Fil}^1 \cap B_1\right) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{Fil}^1 \cap (B_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p)$$

is surjective. But this is equivalent to the rational structure $V$ on the filtered $(\varphi, N)$-module $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ being Raskind-admissible. □

## 3.4 Ordinary representations

Using the Hyodo-Kato complex and work of Perrin-Riou [PR94] and Illusie [I94], we have the following description of the interplay between the $p$-adic Galois represenation of $G_K$ on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ and the special fibre of a semi-stable model.

Proposition 3.12. *Let $X$ be a smooth and proper variety over $K$ and assume that there exists a proper and semi-stable model $\mathcal{X} \to \mathrm{Spec}\, \mathcal{O}_K$ of $X$. Assume that the special fibre $\mathcal{X}_0$ is cohomologically totally degenerate and that every component of $\mathcal{X}_0^{[i]}$ for every $i$ is ordinary in the sense of Bloch-Kato-Illusie-Raynaud.*

1. *The $G_K$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is ordinary in the sense of Perrin-Riou [PR94, 1.2]. More precisely, if $\mathcal{F}^\bullet$ denotes the corresponding filtration, then there exist $G_K$-equivariant isomorphisms*

$$\mathrm{gr}^{2-i} H^2_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Q}_p\right) \cong H^{2-i}\left((\mathcal{X}_0)_{\overline{k}}, W\omega^i_{\log}\right) \otimes \mathbb{Q}_p(-i)\,.$$

2. *Let* $(V = A \oplus B_0 \oplus B_1 \oplus C, \varphi_V, N_V)$ *be the rational structure on* $\mathbb{D}_{\rm st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ *associated to* $\mathcal{X}_0$. *Then,*

$$\begin{array}{llcl} A & \otimes_{\mathbb{Q}} \mathbb{Q}_p & \cong & H^2(\mathcal{X}_0, W\omega^0_{\log}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ (B_0 \oplus B_1) & \otimes_{\mathbb{Q}} \mathbb{Q}_p & \cong & H^1(\mathcal{X}_0, W\omega^1_{\log}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ C & \otimes_{\mathbb{Q}} \mathbb{Q}_p & \cong & H^0(\mathcal{X}_0, W\omega^2_{\log}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \,. \end{array}$$

3. *We have the inequality*

$$\rho(X) \,\leq\, h^{1,1}(X) \,-\, h^{0,2}(X),$$

*where* $\rho(X)$ *denotes the Picard rank of* $X$.

Proof. Since every component of $\mathcal{X}_0^{[i]}$ is ordinary for every $i$, so is $\mathcal{X}_0$ itself [I94, Proposition 1.6]. Thus, claim (1) follows from [I94, Corollaire 2.7].
More precisely, we obtain a very explicit description of the filtered $(\varphi, N)$-module $\mathbb{D}_{\rm st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ via the rational structure $V$ and [I94, Corollaire 2.7] provides us with an equally explicit description of the $G_K$-action on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ via $H^{2-i}((\mathcal{X}_0)_{\overline{k}}, W\omega^i_{\log})$. Comparing these two descriptions, claim (2) follows.
Finally, the de Rham Chern class map $c_1 : \mathrm{Pic}(X) \to H^2_{\rm dR}(X/K)$ is compatible with the log-crystalline Chern class and its image lies inside $\mathrm{Fil}^1$ and $H^2_{\log-\rm cris}(\mathcal{X}_0/K_0)^{\varphi=p, N=0}$ (see, for example, [Ya11, §2]). In our situation, the latter is isomorphic to $B_1 \otimes \mathbb{Q}_p$, from which claim (3) immediately follows (noting that $h^{i,j}(X) = \dim_{K_0} H^j(\mathcal{X}_0, W\omega^i_{\mathcal{X}_0}) \otimes_{W(k)} K_0 = \dim_{\mathbb{Q}_p} H^j(\mathcal{X}_0, W\omega^i_{\log}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by ordinarity). □

Thus, when tensored with $\mathbb{Q}_p$, the rational structure on $\mathbb{D}_{\rm st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ arising from $\mathcal{X}_0$ has an interpretation via the logarithmic Hodge-Witt cohomology groups of the special fibre. Although this is not directly related to the conjectures discussed in this article, it might be of independent interest.

### 3.5 Concluding remarks

The definitions and notions of this section are a little bit ad hoc, since we only deal with Conjecture 1.1. A more conceptual approach, which would be needed when studying Raskind's conjectures for cycles of higher codimension, could proceed along the following lines:

1. One can directly construct $\mathbb{Q}$-structures on the groups $H^*_{\log-\rm cris}(Y/K_0)$ using the Chow complex of [BGS97]. Moreover, these would also come with $\mathbb{Q}$-linearisations of the Frobenius $\varphi$ and the monodromy $N$.

2. Concerning the definitions: one would have to define a *weight* $w$ of a filtered $(\varphi, N)$-module (in Definition 3.4, it would be $w = 2$), one would have to define such a module to be of *maximal nilpotent monodromy* if the monodromy operator $N$ satisfies $N^w \neq 0$ and then, a *rational structure* would be a $\mathbb{Q}$-vector space $V$ with a direct sum decomposition into subspaces upon which $\varphi_V$ acts as multiplication by $p^i$ for $i = 0, ..., w$, etc.

3. For an equivalence as in Theorem 3.11, one would also need a version of Yamashita's theorem (Theorem 3.8) for deforming cycles of higher codimension. This would be a semi-stable analogue of the $p$-adic variational Hodge conjecture (see for example [BEK14, Conjecture 1.2] for the good reduction case, where the conjecture is attributed to Fontaine-Messing). This conjecture is open in codimension $\geq 1$, even in the case of good reduction, but see [BEK14] for the state of the art.

To keep the discussion in this section shorter, we have decided not to develop the setup in general, but to stick to the case of divisors.

## 4 Abeloid varieties

From this section on, we study abelian varieties over $p$-adic fields with totally degenerate reduction. More precisely, we describe their morphisms, $\ell$-adic Tate modules, and the filtered $(\varphi, N)$-modules associated to the latter via $p$-adic uniformisation, that is, within the framework of abeloid varieties. The results of this section might be of independent interest and some of them might already be known to the experts.

### 4.1 Generalities

We start with the behaviour of Question 1.2 under field extension and under completion, similar to what we did for Raskind's conjecture for divisors (Conjecture 1.1) in Section 2.2 and Section 2.6.
Let $F$ be a field of characteristic $p \geq 0$ and let $\ell$ be a prime, possibly equal to $\ell$. If $A$ is an abelian variety of dimension $g$ over a field $F$, then the $\ell$-adic Tate module $T_\ell$ and the rational $\ell$-adic Tate module $V_\ell$ of $A$ are defined to be to be

$$T_\ell(A) := \varprojlim_n A(F^{\mathrm{sep}})[\ell^n] \quad \text{and} \quad V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

together with their $G_F$-actions. If $\ell \neq p$, then $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of rank $2g$.

Proposition 4.1. *Let $A$ and $B$ be abelian varieties over a field $F$, let $F \subset F'$ be a finite Galois extension and let $\ell$ be a prime. If the map $(\star\star)$ (please see §1.4) is surjective with r espect to $A_{F'}$, $B_{F'}$, $G_{F'}$, and $\ell$, then $(\star\star)$ is surjective with respect to $A$, $B$, $G_F$, and $\ell$.*

Proof. Suppose that $(\star\star)$ with respect to $A_{F'}$, $B_{F'}$, $G_{F'}$ is a surjection. Then, we get a surjection on the $\mathrm{Gal}(F'/F)$-invariants

$$(\mathrm{Hom}(A_{F'}, B_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)^{\mathrm{Gal}(F'/F)} \twoheadrightarrow \mathrm{Hom}_{G_{F'}}(T_\ell(A), T_\ell(B))^{\mathrm{Gal}(F'/F)} = \mathrm{Hom}_{G_F}(T_\ell(A), T_\ell(B)),$$

and the left-hand side is $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ by Galois descent for morphisms. □

REMARK 4.2. For abeloid varieties over $p$-adic fields, we will see a second proof of this result in Corollary 4.8 below.

PROPOSITION 4.3. *Let $A$ and $B$ be abelian varieties over a number field $F$ and assume that there exists a prime $\ell$ and a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ such that $(\star\star)$ is surjective for $A_{F_\mathfrak{p}}$, $B_{F_\mathfrak{p}}$, $G_{F_\mathfrak{p}}$, and $\ell$. Then, $(\star\star)$ is surjective for $A$, $B$, $G_F$, and $\ell$.*

PROOF. We view $G_{F_\mathfrak{p}}$ as a subgroup of $G_F$. Since we are allowed to make finite Galois extensions by Proposition 4.1, we may assume that

$$\mathrm{Hom}(A, B) = \mathrm{Hom}(A_{\overline{F}}, B_{\overline{F}})\,.$$

In particular, $G_{F_\mathfrak{p}}$ and $G_F$ act trivially on $\mathrm{Hom}(A_{\overline{F}}, B_{\overline{F}})$. Then we have the following commutative square

$$\begin{array}{ccc} \mathrm{Hom}(A_{\overline{F}}, B_{\overline{F}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \longrightarrow & \mathrm{Hom}(T_\ell(A), T_\ell(B)) \\ \downarrow \cong & & \downarrow \\ \mathrm{Hom}(A_{\overline{F_\mathfrak{p}}}, B_{\overline{F_\mathfrak{p}}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \longrightarrow & \mathrm{Hom}(T_\ell(A_{F_\mathfrak{p}}), T_\ell(B_{F_\mathfrak{p}})) \end{array}$$

Taking $G_{F_\mathfrak{p}}$-invariants gives

$$\begin{array}{ccc} \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \longrightarrow & \mathrm{Hom}_{G_{F_\mathfrak{p}}}(T_\ell(A), T_\ell(B)) \\ \downarrow \cong & & \downarrow \\ \mathrm{Hom}(A_{F_\mathfrak{p}}, B_{F_\mathfrak{p}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \twoheadrightarrow & \mathrm{Hom}_{G_{F_\mathfrak{p}}}(T_\ell(A_{F_\mathfrak{p}}), T_\ell(B_{F_\mathfrak{p}})) \end{array}$$

where the lower horizontal arrow is a surjection by assumption. We deduce therefore that the upper horizontal arrow is a surjection. This proves the proposition because of the inclusion $\mathrm{Hom}_{G_F}(T_\ell(A), T_\ell(B)) \subset \mathrm{Hom}_{G_{F_\mathfrak{p}}}(T_\ell(A), T_\ell(B))$. □

### 4.2 DEGENERATIONS OF ABELIAN VARIETIES

In the simple cases treated in Proposition 2.3, no assumption on the degeneration was needed and in Section 3, we worked with a weak form of total degeneration. In [Ra05, RX07a], Raskind suggested a degeneration assumption which is rather involved. For the purposes of this article, the following slight generalisation of [RX07a, Example 1.(i)] suffices.

LEMMA 4.4 (Raskind-Xarles + $\varepsilon$). *Let $Y = \bigcup_{i \in I} Y_i$ be a strict normal crossing scheme over a perfect field $F$. Assume that for every subset $J \subset I$ the intersection $Y_J$, if non-empty, is isomorphic to a successive blow-up of smooth, projective, and toric varieties along subvarieties that are also smooth and toric. Then, $Y$ is totally degenerate in the sense of Raskind.*

Proof. If every $Y_J$ is a smooth, projective, and toric variety, then this is [RX07a, Example 1.(i)]. Quite generally, if $\widetilde{X}$ is the blow-up of a smooth variety $X$ along a smooth subvariety $Y$, then one can express the $\ell$-adic cohomology groups, the crystalline cohomology groups, the Chow groups, and the cycle class maps for $\widetilde{X}$ in terms of $X$ and $Y$. From these formulas, it follows that the requirements (a)-(c) of [RX07a, Definition 1] also hold for our assumptions. Moreover, if $X$ and $Y$ are ordinary in the sense of Bloch-Kato-Illusie-Raynaud, then so is $\widetilde{X}$ [I90, Proposition 1.6], that is, also requirement (d) is satisfied. □

Remark 4.5. If $Y$ is a strict normal crossing scheme of dimension $N \leq 2$, such that all the $Y_J$ are smooth and rational varieties, then the assumptions of the lemma are satisfied. For example, this applies to the combinatorial degenerations of type III of the surfaces from [CL16, Definitions 5.4 to 5.7].

Next, we discuss the notion of totally degenerate reduction for abelian varieties. There are several obvious candidates, all of which are stable under finite field extensions and all of which are equivalent up to finite field extensions. The following is well-known, but maybe never explicitly stated in this way, which is why we include a short discussion with references.

Proposition 4.6. *Let $\mathcal{O}_K$ be a local and Henselian DVR with field of fractions $K$ and residue field $k$. Assume that $K$ is of characteristic zero and that $k$ is perfect of characteristic $p \geq 0$. Let $A$ be an abelian variety of dimension $g \geq 1$ over $K$. Consider the following properties:*

1. *$A$ admits uniformisation in the sense of Mumford [Mum72b].*
2. *The connected component of the special fibre of the Néron model of $A$ is a split torus.*
3. *The special fibre of the projective regular Künnemann-Mumford model [Mum72b, Kün98] of $A$ is a union of smooth and toric varieties.*
4. *$A$ has totally degenerate reduction in the sense of Raskind [Ra05].*
5. *For some (resp., all) $\ell \neq p$, the action of the inertia subgroup $I_K$ of $G_K$ on $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent with maximal number of $2 \times 2$-Jordan blocks, that is, its Jordan normal form has $g$ Jordan blocks of size $2 \times 2$ and generalised eigenvalue $1$.*
6. *For some (resp., all) $\ell \neq p$, the $I_K$-action on $H^g_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ is maximally unipotent, that is, its Jordan normal form has one Jordan block of size $g \times g$ and generalised eigenvalue $1$.*
7. *Assume $g \geq 2$. For some (resp., all) $\ell \neq p$, the $I_K$-action on $H^2_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent and its Jordan normal form has no Jordan block of size $2 \times 2$ and $\frac{1}{2}g(g-1)$ Jordan blocks of size $3 \times 3$.*

*If $K$ is a $p$-adic field, consider also the following properties*

8. *The $G_K$-representation $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable and the monodromy operator $N$ of the associated filtered $(\varphi, N)$-module is nilpotent with minimal kernel, that is,* $\ker(N)$ *is $g$-dimensional.*

9. *The $G_K$-representation $H^g_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable and the monodromy operator $N$ of the associated filtered $(\varphi, N)$-module is maximally nilpotent, that is, $N^g \neq 0$.*

10. *Assume $g \geq 2$. The $G_K$-representation $H^2_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable and the monodromy operator $N$ of the associated filtered $(\varphi, N)$-module has no Jordan block of size $2 \times 2$ and $\frac{1}{2}g(g-1)$ Jordan blocks of size $3 \times 3$.*

*Then,*

*(i) these properties are* stable under finite extension*, that is, if $A$ satisfies one of these properties, then $A_L$ satisfies the same property for every finite field extension $K \subset L$.*

*(ii) Moreover, these properties are* equivalent up to finite extension*, that is, if $A$ satisfies one of the above properties, then there exists a finite field extension $K \subset L$ such that $A_L$ satisfies all of these properties.*

PROOF. The assertion that all these properties are stable under finite extension are well-known or easy and we leave them to the reader.
The fact that (1) and (2) are equivalent up to finite extension is the main result of [Mum72b], see also the discussion in [Lüt16, §5.6].
The fact that (2) and (3) are equivalent up to finite extension is a special case of [Kün98, §3].
If an abelian variety satisfies (4), then the identity component of the Néron model is semi-abelian and the triviality of the conditions on the Chow group imply that it cannot have abelian parts, and thus, (4) is equivalent to (2) and (3) up to finite extension. Conversely, if an abelian variety satisfies (3), then Lemma 4.4 applies and thus, (3) is equivalent to (4) up to finite extension, see also [RX07a, Example 1.(ii)].
Assume that the Néron model of $A$ has semi-abelian reduction, let $t$ be the dimension of the toric part and $a$ be the dimension of the abelian part (and thus, $g = t + a$). Then, the $I_K$-action on $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent with index of unipotency at most $\leq 2$. More precisely, there are $2a$ Jordan blocks of size $1 \times 1$ and generalised eigenvalue 1 and there are $t$ Jordan blocks of size $2 \times 2$ and generalised eigenvalue 1. We refer to [BLR90, Theorem 6 in Chapter 7.4] or [Gro72, Exposé IX] for proofs and details. Similarly, the $G_K$-representation on $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is semi-stable in this case. The monodromy operator on the associated $(\varphi, N)$-module is nilpotent with nilpotency at most $\leq 2$. More precisely, there are $2a$ Jordan blocks of size $1 \times 1$ and eigenvalue 0 and there are $t$ Jordan blocks of size $2 \times 2$ and generalised eigenvalue 0.

From this discussion, it follows that (2) is equivalent to (5) up to finite extension and that (2) is equivalent to (8) up to finite extension.

In any case and for all $i$, there exist $G_K$-equivariant isomorphisms between $H^i_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ and $\wedge^i H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$. Moreover, for all $\ell \neq p$, the $I_K$-actions on $H^i_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ are quasi-unipotent and the $G_K$-representations $H^i_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ are potentially semi-stable. Replacing $K$ by a finite extension, we may assume that all these $I_K$-actions are unipotent for $\ell \neq p$ and all the $G_K$-representations are semi-stable for $\ell = p$.

In particular, if $\ell \neq p$, then the $I_K$-action on $\wedge^g H^g_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent with order of unipotency at most $g$. More precisely, there exists at most one Jordan block of size $g \times g$ and generalised eigenvalue 1 and there exists such a block if and only if $g = t$. We leave the straight forward exercise in linear algebra to the reader. This implies that (5) is equivalent to (6) up to finite extension. Similarly, (8) is equivalent to (9) up to finite extension.

If $g \geq 2$ and $\ell \neq p$, then the $I_K$-inertia on $\wedge^2 H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent with order of unipotency at most 3. More precisely, there are $r := \frac{1}{2}t(t-1)$ Jordan blocks of size $3 \times 3$ with generalised eigenvalue 1 and $s := 2at$ Jordan blocks of size $2 \times 2$ with generalised eigenvalue 1. Then, $t = \frac{r+4s}{2g-2}$ and thus, the $I_K$-action on $\wedge^2 H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ encodes $a$ and $t$. Again, we leave the details to the reader. These considerations show that (5) is equivalent to (7) up to finite extension. Similarly, (9) is equivalent to (11) up to finite extension. □

### 4.3 Abeloid varieties

From now on, we will adopt the point of view of $p$-adic uniformisation. In particular, this allows us to work with abeloid varieties, which are rigid analytic varieties over $p$-adic fields that are not necessarily algebraisable. We now establish and recall a couple of general facts about abeloid varieties.

Let $q_1, ..., q_g \in K^{\times g}$ be vectors, say $q_i = (q_{i,1}, ...q_{i,g})$, such that $\nu_p(q_{i,j}) > 1$ for all $i, j$. Let $Q = (q_{i,j}) \in M_{g \times g}(K)$ be the $g \times g$-matrix, whose rows are the $q_i$. Associated to $Q$, we have the matrix

$$\text{ord}_p(Q) \;:=\; (\nu_p(q_{i,j})) \;\in\; M_{g\times g}(\mathbb{Q}).$$

By definition, the abelian subgroup $\Lambda = q_1^{\mathbb{Z}} \cdots q_g^{\mathbb{Z}}$ of $K^{\times g}$ is called a *lattice*, if the columns of $\text{ord}_p(Q)$ span a lattice inside $\mathbb{R}^g$ or, equivalently, if the matrix $\text{ord}_p(Q)$ is invertible. (In the literature, this matrix is sometimes constructed with respect to a valuation $\nu_K$ on $K^\times$ with $\nu_K(\pi_K) = 1$ for some uniformiser $\pi_K \in \mathcal{O}_K$ and then, the matrix has integer entries rather then rational ones. We have decided to work with the valuation $\nu_p$ instead, which rescales the classical matrix by $\nu_p(\pi_K)$ and has the advantage of being stable under finite field extensions of $K$.)

Associated to a lattice $\Lambda \subset K^{\times g}$, there is a rigid analytic variety over $K$, the *abeloid variety* $\mathbb{G}^g_{m,K}/\Lambda$, see [Lüt16, Chapter 7]. The $g \times g$-matrix $Q$ associated to a choice of basis for $\Lambda$ is called a *period matrix*. The algebraisable

abeloid varieties are precisely the totally degenerating abelian varieties studied by Mumford [Mum72b]. Moreover, if $g = 1$, then a lattice $\Lambda \subset K^\times$ is generated by a single element $q \in K^\times$ with $\nu_p(q) > 1$, an abeloid variety of dimension one is always algebraisable, and these are precisely the *Tate elliptic curves*.
We introduce the following notation: if $R$ is a commutative ring and if $A$ is an $R$-module, then the set $\mathrm{Mat}_{m\times n}(A)$ of $m \times n$-matrices with values in $A$ is an Abelian group. Let $X \in \mathrm{Mat}_{m\times n}(A)$. Then, if $M \in \mathrm{Mat}_{s\times m}(R)$ and $N \in \mathrm{Mat}_{n\times t}(R)$ are matrices with entries in $R$ for some $s, t$, then the matrix products $M \odot X$ and $X \odot N$ are defined. In particular, $\mathrm{Mat}_{n\times n}(A)$ is a $\mathrm{Mat}_{n\times n}(R)$-bimodule. In the next theorem, we have $R = \mathbb{Z}$ and $A = K^\times$.

Theorem 4.7 (Gerritzen $+\varepsilon$). *Let $\Lambda_A \subset K^{\times g}$ and $\Lambda_B \subset K^{\times h}$ be lattices and let $A := \mathbb{G}^g_{m,K}/\Lambda_A$ and $B := \mathbb{G}^h_{m,K}/\Lambda_B$ be the associated abeloid varieties. Let $Q_A$ and $Q_B$ be period matrices for $\Lambda_A$ and $\Lambda_B$. Then, there exist isomorphisms*

$$\begin{array}{ll} & \mathrm{Hom}(A,B) \\ \cong & \{M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}) \,|\, \Lambda_A \odot M \subseteq \Lambda_B\} \\ \cong & \{M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}) \,|\, \exists N \in \mathrm{Mat}_{h\times g}(\mathbb{Z}) \,:\, Q_A \odot M = N \odot Q_B\} \\ \cong & \left\{M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}) \,|\, \left(\mathrm{ord}_p(Q_A)^{-1} \odot Q_A\right) \odot M = M \odot \left(\mathrm{ord}_p(Q_B)^{-1} \odot Q_B\right)\right\} . \end{array}$$

*In particular,*

1. *the natural map*
$$\mathrm{Hom}(A,B) \to \mathrm{Hom}(A_{\overline{K}}, B_{\overline{K}})$$
*is an isomorphism of abelian groups. In particular, the $G_K$-action on the right hand side is trivial.*
2. *$A$ and $B$ are $K$-isogenous if and only if they are $\overline{K}$-isogenous.*
3. *$A$ is $K$-simple if and only if it is $\overline{K}$-simple.*

Proof. If $M = (m_{ij}) \in M_{g\times h}(\mathbb{Z})$ is a $g \times h$-matrix, then it gives rise to a map $\psi_M : K^{\times g} \to K^{\times h}$ by sending

$$(x_1, ..., x_g) \mapsto \left( ..., \prod_{j=1}^{g} x_j^{m_{ji}}, ... \right).$$

If $\psi_M(\Lambda_A) = \Lambda_A \odot M$ is contained in $\Lambda_B$, then $\psi_M$ descends to a morphism $A \to B$ of abeloid varieties. Conversely, every morphism of abeloid varieties is of this form by the main theorem of [Ge70], see also the discussion in [Kad07, §3]. This establishes the first two isomorphisms describing of $\mathrm{Hom}(A,B)$. Taking valuations in the equality $Q_A \odot M = N \odot Q_B$, we find

$$N = \mathrm{ord}_p(Q_A) \cdot M \cdot \mathrm{ord}_p(Q_B)^{-1},$$

which implies the third isomorphism.

Now, since every morphism $A_{\overline{K}} \to B_{\overline{K}}$ can be defined over some finite field extension $L/K$ and since the above description of homomorphisms is also valid for homomorphisms over $L$, it follows from this description that every homomorphism $\mathrm{Hom}(A_L, B_L)$ can be defined over $K$. This establishes claim (1). In particular, $A$ and $B$ are isogenous over $K$ if and only if they are isogenous over $\overline{K}$, which establishes claim (2). Finally, $A$ is simple over $K$ if and only if there exists no non-trivial idempotent in $\mathrm{End}(A)$ if and only if there exists no non-trivial idempotent in $\mathrm{End}(A_{\overline{K}})$ (by the already established (1)) if and only if $A_{\overline{K}}$ is simple, which establishes claim (3). □

We will give another description of $\mathrm{Hom}(A, B) \otimes \mathbb{Q}$ in terms of $\mathcal{L}$-invariants in Proposition 4.14 below.

Corollary 4.8. *Let $A$ and $B$ be abeloid varieties over a $p$-adic field $K$, let $L/K$ be a finite field extension, and let $\ell$ be a prime. If $(\star\star)$ is surjective for $A_L$, $B_L$, and $\ell$, then $(\star\star)$ is surjective for $A$, $B$, and $\ell$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A, B) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{Hom}_{G_K}(T_\ell(A), T_\ell(B)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A_L, B_L) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{Hom}_{G_L}(T_\ell(A_L), T_\ell(B_L)) \end{array}$$

By Theorem 4.7.(1), the left vertical arrow is an isomorphism, from which the statement immediately follows. □

### 4.4 The Tate module of an abeloid variety

Let $A = \mathbb{G}_{m,K}^g/\Lambda$ be an abeloid variety over a $p$-adic field $K$ and let $\ell$ be a prime, possibly equal to $p$. It follows from the rigid analytic parametrisation $\overline{K}^{\times g}/\Lambda \cong A(\overline{K})$ that the Tate module $T_\ell(A)$ sits in a short exact sequence

$$0 \to \mathbb{Z}_\ell(1)^g \to T_\ell(A) \to \mathbb{Z}_\ell^g \to 0 \qquad (7)$$

that is compatible with the $G_K$-actions.

To describe its extension class, we follow and generalise some results due to Serre [Se68, Appendix A]. To state the result, we define $\mu_{\ell^\infty}(K)$ to be the group of those roots of unity of $K$ whose order is a power of $\ell$, we choose a uniformiser $\pi_K$, and we denote by $U^{(1)} := 1 + \mathfrak{m}_K = 1 + \pi_K \cdot \mathcal{O}_K$ the group of 1-units of $\mathcal{O}_K$. Then, there exists an isomorphism of abelian groups

$$K^\times \cong \mu(K) \times \pi_K^{\mathbb{Z}} \times U^{(1)}.$$

In the sequel, we consider the $\ell$-adic completion

$$\gamma_\ell : K^\times \to \widehat{K^\times}^\ell := \varprojlim_n K^\times/K^{\times \ell^n} \qquad (8)$$

of $K^\times$.

LEMMA 4.9 (Serre $+\varepsilon$). *Let $A := \mathbb{G}_{m,K}^g/\Lambda$ be an abeloid variety over $K$.*

1. *There exists an isomorphism*

$$\ker(\gamma_\ell) \;=\; \begin{cases} \mu_{\ell^\infty}(K) \times U^{(1)} & \text{if } \ell \neq p, \\ \mu_{p^\infty}(K) & \text{if } \ell = p. \end{cases}$$

   *In particular, $\ker(\gamma_\ell)$ is finite if and only if $\ell = p$.*

2. *Taking Galois invariants in (7), the boundary homomorphism in Galois cohomology gives rise to a homomorphism*

$$d_g \;:\; H^0(G_K, \mathbb{Z}_\ell^g) \;\to\; H^1(G_K, \mathbb{Z}_\ell(1)^g).$$

   *Let $e_i = (0, ..., 1, 0, ...)$, $i = 1, ...g$ be the standard basis of $\mathbb{Z}_\ell^g$. Then, the $\mathbb{Z}$-span $\Lambda'$ of $\{d_g(e_i)\}_{i=1,...,g}$ determines the extension class of (7).*

3. *Kummer theory induces an isomorphism*

$$(\widehat{K^\times}^{\ell})^g \;\cong\; H^1(G_K, \mathbb{Z}_\ell(1)^g).$$

   *Under this isomorphism, $\gamma_\ell(\Lambda)$ is equal to $\Lambda'$ from assertion (2).*

4. *The image $\gamma_\ell(\Lambda)$ is a lattice, that is, a free $\mathbb{Z}$-module of rank $g$. In particular, the sequence (7) does not split. In fact, there does not even exist a non-trivial and $G_K$-equivariant homomorphism $\mathbb{Z}_\ell \to T_\ell(A)$.*

PROOF. The description of the kernel in claim (1) for $\ell = p$ is shown in the proof of the implication (3) $\Rightarrow$ (2) of the theorem of [Se68, Appendix A.1.4]. If $\ell \neq p$, the valuation argument of loc.cit. still shows that $\pi_K^{\mathbb{Z}}$ has trivial intersection with $\ker(\gamma_\ell)$. This shows that $\ker(\gamma_\ell)$ is contained in $\tau(k^\times) \times U^{(1)}$, where $\tau$ denotes the Teichmüller lift from $k^\times$ to $K_0^\times \subset K^\times$. Hensel's lemma implies that $U^{(1)} \subset \ker(\gamma_\ell)$. The intersection of $\tau(k^\times)$ with $\ker(\gamma_\ell)$ is $\mu_{\ell^\infty}(K)$. Claims (2), (3), and (4) for $g = 1$ are the proposition and the corollary of [Se68, Appendix A.1.2]. The generalisations of claims (1) - (3) to arbitrary $g$ follow immediately by taking products and we leave them to the reader. For claim (4), we note that the valuation argument used in the proof of assertion (b) of the proposition in [Se68, Appendix A.1.2] still works, when being replaced by the valuation matrix $\mathrm{ord}_p(V)$ associated to a period matrix $V$ for $\Lambda$, which we introduced at the beginning of this section. □

REMARK 4.10. The $\ell$-adic completion is explicitly given by

$$\widehat{K^\times}^{\ell} \;\cong\; (\mu(K)/\mu_{\ell^\infty}(K)) \;\times\; \pi_K^{\mathbb{Z}_\ell} \;\times\; \begin{cases} \{1\} & \text{if } \ell \neq p \\ U^{(1)} & \text{if } \ell = p\,. \end{cases}$$

In this decomposition, the map $\gamma_\ell$ can be understood componentwise.

As a consequence, we now describe $G_K$-equivariant homomorphisms between $\ell$-adic Tate modules of abeloid varieties. In the one-dimensional case, this result is implicit in [Se68, Appendix A.1.4].

Proposition 4.11. *Let $\Lambda_A \subset K^{\times g}$ and $\Lambda_B \subset K^{\times h}$ be lattices and let $A := \mathbb{G}^g_{m,K}/\Lambda_A$ and $B := \mathbb{G}^h_{m,K}/\Lambda_B$ be the associated abeloid varieties. Let $Q_A$ and $Q_B$ be period matrices for $\Lambda_A$ and $\Lambda_B$. Then, there exist isomorphisms of $\mathbb{Z}_\ell$-modules*

$$\begin{aligned} &\mathrm{Hom}_{G_K}\left(T_\ell(A), T_\ell(B)\right) \\ &\cong \{M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell) \,|\, \gamma_\ell(\Lambda_A) \odot M \subseteq \gamma_\ell(\Lambda_B)\} \\ &\cong \{M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell) \,|\, \exists N \in \mathrm{Mat}_{h\times g}(\mathbb{Z}_\ell) : \gamma_\ell(Q_A) \odot M = N \odot \gamma_\ell(Q_B)\} \\ &\cong \left\{M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell) \,|\, \left(\mathrm{ord}_p(Q_A)^{-1} \odot \gamma_\ell(Q_A)\right) \odot M = M \odot \left(\mathrm{ord}_p(Q_B)^{-1} \odot \gamma_\ell(Q_B)\right)\right\} . \end{aligned}$$

*Here, $\gamma_\ell : K^\times \to \widehat{K^\times}^\ell$ denotes the $\ell$-adic completion from Lemma 4.9.*

Proof. Given a $G_K$-equivariant morphism $\varphi : T_\ell(A) \to T_\ell(B)$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_\ell(1)^g & \longrightarrow & T_\ell(A) & \longrightarrow & \mathbb{Z}_\ell^g & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \varphi & & \downarrow \sigma & & \\ 0 & \longrightarrow & \mathbb{Z}_\ell(1)^h & \longrightarrow & T_\ell(B) & \longrightarrow & \mathbb{Z}_\ell^h & \longrightarrow & 0 \end{array}$$

for some matrices $\rho, \sigma \in \mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell)$. Taking $G_K$-invariants and passing to cohomology, it follows that the diagram

$$\begin{array}{ccc} \mathbb{Z}_\ell^g & \xrightarrow{d_g} & H^1\left(G_K, \mathbb{Z}_\ell(1)^g\right) \\ \downarrow \sigma & & \downarrow \rho_* \\ \mathbb{Z}_\ell^h & \xrightarrow{d_h} & H^1\left(G_K, \mathbb{Z}_\ell(1)^h\right) \end{array} \tag{9}$$

commutes.

By Lemma 4.9, the images of $\Lambda'_A$ and $\Lambda'_B$ under $d_g$ and $d_h$ are lattices, and thus, the homomorphisms $d_g$ and $d_h$ are injective. In particular, $\rho$ determines $\rho_*$, which determines $\sigma$ uniquely and vice versa.

Using the results and identifications of Lemma 4.9, the commutativity of the above diagram implies

$$\gamma_\ell(Q_A) \odot \rho = \sigma \odot \gamma_\ell(Q_B)$$

with respect to the notation introduced above. Thus, $\gamma_\ell(\Lambda_A) \odot \rho$ lies in the $\mathbb{Z}_\ell$-span of $\gamma_\ell(\Lambda_B)$.

Conversely, let $M \in \mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell)$ be such that $\gamma_\ell(\Lambda_A) \odot M$ is contained in the $\mathbb{Z}_\ell$-span of $\gamma_\ell(\Lambda_B)$. Then, $M$ gives rise to map $\mathbb{Z}_\ell(1)^g \to \mathbb{Z}_\ell(1)^h$ and we can

find a unique matrix $N \in \mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell)$ defining a map $\mathbb{Z}_\ell^g \to \mathbb{Z}_\ell^h$ such that the diagram (9) commutes. This commutativity implies that $M$ and $N$ determine a unique $G_K$-equivariant map $\varphi : T_\ell(A) \to T_\ell(B)$.
The last isomorphism follows from taking valuations as in the proof of Theorem 4.7. □

### 4.5 The $p$-adic Galois representations

Let $A = \mathbb{G}^g_{m,K}/\Lambda_A$ be an abeloid variety over a $p$-adic field $K$ and let $Q_A = (q_{i,j})$ be a period matrix for $\Lambda_A$. As seen in (7), the $p$-adic Galois representation of $G_K$ on the rational Tate module $V_p(A)$ is an extension of $\mathbb{Q}_p(1)^g$ by $\mathbb{Q}_p^g$. We denote by $\log_p$ Iwasawa's $p$-adic logarithm, normalised such that $\log_p(p) = 0$. Associated to this data, we construct a filtered $(\varphi, N)$-module over $K$ as follows:

1. Let $V$ be the $2g$-dimensional vector space over $\mathbb{Q}$ with basis $x_1, ..., x_g, y_1, ..., y_g$ together with two linear operators $\varphi, N$:

$$\begin{array}{lllllll} \varphi(x_i) & = & p^{-1}\cdot x_i & \qquad & \varphi(y_i) & = & y_i \\ N(x_i) & = & 0 & & N(y_i) & = & \sum_{j=1}^g \nu_p(q_{i,j})\cdot x_j \end{array}$$

   that is, these operators are given by matrices

$$\begin{pmatrix} p^{-1}\cdot \mathrm{Id}_{g\times g} & 0 \\ 0 & \mathrm{Id}_{g\times g}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \mathrm{ord}_p(Q) \\ 0 & 0 \end{pmatrix}.$$

   We equip $V_{K_0} := V \otimes_{\mathbb{Q}} K_0$ with the $K_0$-linear extension $N \otimes \mathrm{id}_{K_0}$ of $N$ and with the $K_0$-semi-linear extension $\varphi \otimes \sigma$ of $\varphi$. Here, $\sigma$ denotes the lift of Frobenius on $K_0$ and by abuse of notation, we will denote these extensions again by $\varphi$ and $N$. This turns $(V_{K_0}, \varphi, N)$ into a $(\varphi, N)$-module.

2. A filtration on $V_K := V \otimes_{\mathbb{Q}} K$ defined by $\mathrm{Fil}^i = 0$ for $i \geq 1$, by $\mathrm{Fil}^i = V_K$ for $i < 0$, and $\mathrm{Fil}^0$ is the $g$-dimensional $K$-vector space spanned by

$$y_i + \sum_{j=1}^g \log_p(q_{i,j})\cdot x_j$$

   for $i = 1, ..., g$.

After these preparations, we obtain the following result, which was already known for Tate elliptic curves, that is, in the case where $g = 1$, see also Remark 4.13 below.

Theorem 4.12. *Let $A = \mathbb{G}^g_{m,K}/\Lambda_A$ be an abeloid variety over a $p$-adic field $K$. Then, the filtered $(\varphi, N)$-module $\mathbb{D}_{\mathrm{st}}(V_p(A))$ associated to the rational Tate module of $A$ is isomorphic to $(V_{K_0}, \varphi, N, \mathrm{Fil}^\bullet)$ constructed above.*

Proof. We use the notations of [Be04, §II.4] and generalise the computations there from $g = 1$ to arbitrary $g$. To obtain an explicit description of the rational Tate module $V_p(A)$, we fix a compatible system $\{\varepsilon^{(n)}\}_n$ of $p^n$-th roots of unity, as well as a compatible system $\{q_{i,j}^{(n)}\}_n$ of $p^n$-th roots of $q_{i,j}$. Via $p$-adic uniformisation, we obtain a $G_K$-equivariant parametrisation $\overline{K}^{\times g}/\Lambda_A \to A(\overline{K})$, which identifies the $p^n$-torsion subgroup of $A(\overline{K})$ with

$$\{x \in \overline{K}^{\times g} \,|\, x^{p^n} \in \Lambda_A\},$$

which shows that the $2g$ vectors

$$\begin{array}{rcl} e_i & := & \varprojlim_n (1, ..., \varepsilon^{(n)}, 1, ...) \\ f_i & := & \varprojlim_n (q_{i,1}^{(n)}, ..., q_{i,g}^{(n)}) \end{array}$$

with $i = 1, ..., g$ form a $\mathbb{Z}_p$-basis of the Tate module $T_p(A)$. Thus, if $g \in G_K$, we compute

$$g(e_i) \;=\; \varprojlim_n (1, ..., g(\varepsilon^{(n)}), 1, ...) \;=\; \chi(g) \cdot \varprojlim_n (1, ..., \varepsilon^{(n)}, 1, ...) \;=\; \chi(g) \cdot e_i,$$

where $\chi : G_K \to \mathbb{Z}_p^\times$ denotes the $p$-adic cyclotomic character. Moreover, we define a $g \times g$ matrix $C(g) = (c_{i,j}(g))$ with entries in $\mathbb{Z}_p$ via

$$\begin{array}{rcl} g(f_i) & = & \varprojlim_n \left( g(q_{i,1}^{(n)}), ..., g(q_{i,g}^{(n)}) \right) \\ & = & \varprojlim_n \left( q_{i,1}^{(n)} (\varepsilon^{(n)})^{c_{i,j}(g)}, ..., q_{i,g}^{(n)} (\varepsilon^{(n)})^{c_{i,j}(g)} \right) \\ & = & f_i + \displaystyle\sum_{j=1}^{g} c_{i,j}(g) \cdot e_j \,. \end{array}$$

Thus, the action of $g \in G_K$ on $T_p(A)$ is given by the matrix

$$\begin{pmatrix} \chi(g) \cdot \mathrm{Id}_{g\times g} & C(g) \\ 0 & \mathrm{Id}_{g\times g} \end{pmatrix} .$$

To determine the $p$-adic periods, we have $t = \log_p([\varepsilon]) \in B_{\mathrm{dR}}^+ \subset B_{\mathrm{dR}}$ and set

$$u_{i,j} \;:=\; \log_p(q_{i,j}) \;-\; \sum_{n=1}^{\infty} \frac{(1 - [\widetilde{q}_{i,j}])^n}{n}.$$

were

$$\widetilde{q}_{i,j} \;=\; (q_{i,j}^{(0)}, q_{i,j}^{(1)}, \ldots) \;\in\; \widetilde{\mathbb{E}}^+ \;=\; \varprojlim_{x \mapsto x^p} \mathcal{O}_{\widehat{\overline{K}}}$$

and $[\widetilde{q}_{i,j}] \in W(\widetilde{\mathbb{E}}^+)$ denotes its Teichmüller lift. This series converges in $B_{\mathrm{dR}}^+$ and as explained in [Be04, §II.4.3], one should think of $u_{i,j}$ as being equal to

$\log_p([\widetilde{q}_{i,j}])$ and one has $g(u_{i,j}) = u_{i,j} + c_{i,j}(g)t$. In this explicit description, it is easy to see that the $2g$ vectors

$$\begin{array}{lcl} x_i & := & \frac{1}{t} \otimes e_i \\ y_i & := & -\sum_{j=1}^{g} \frac{u_{i,j}}{t} \otimes e_j \;+\; 1 \otimes f_i \end{array}$$

with $i = 1, ..., g$ lie in $\mathbb{D}_{dR}(V_p(A)) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_p(A))^{G_K}$. These elements form a $2g$-dimensional vector space, which shows explicitly that the $G_K$-representation on $V_p(A)$ is de Rham. Now, $t$ and the $u_{i,j}$ lie in the subring $B^+_{\mathrm{cris}} \subset B^+_{\mathrm{dR}}$ and we have $\varphi(t) = p \cdot t$, $\varphi(u_{i,j}) = p \cdot u_{i,j}$, $\varphi(e_i) = e_i$, and $\varphi(f_i) = f_i$, see also [Be04, §II.4.3]. This implies

$$\varphi(x_i) \;=\; \frac{1}{pt} \otimes e_i \;=\; p^{-1} \cdot x_i \quad \text{ and } \quad \varphi(y_i) \;=\; -\sum_{j=1}^{g} \frac{p \cdot u_{i,j}}{p \cdot t} \otimes e_j + 1 \otimes f_i \;=\; y_i \,.$$

Then, we have $B_{\mathrm{st}} = B_{\mathrm{cris}}[Y]$ and the normalisation $\log_p(p) = 0$ determines an embedding of $B_{\mathrm{st}}$ into $B_{\mathrm{dR}}$, see [Be04, §II.3.3]. Then,

$$u_{i,j} \;=\; \log_p[\widetilde{q}_{i,j}] \;=\; \nu_p(q_{i,j}) \cdot Y \;+\; \log_p \left[ \frac{\widetilde{q}_{i,j}}{\nu_p(q_{i,j})} \right].$$

The monodromy operator is given by $N := -\frac{d}{dY}$ and we compute $N(x_i) = 0$. Rewriting $y_i$ as

$$y_i \;=\; -\sum_{j=1}^{g} \frac{\nu_p(q_{i,j})}{t} \cdot Y \otimes e_j \;-\; \sum_{j=1}^{g} \frac{1}{t} \cdot \log_p \left[ \frac{\widetilde{q}_{i,j}}{\nu_p(q_{i,j})} \right] \otimes e_j \;+\; 1 \otimes f_i \,,$$

we compute

$$N(y_i) \;=\; -\frac{d}{dY}\, y_i \;=\; \sum_{j=1}^{g} \frac{\nu_p(q_{i,j})}{t} \otimes e_j \;=\; \sum_{j=1}^{g} \nu_p(q_{i,j}) e_j.$$

By definition, the filtration on $B_{\mathrm{dR}}$ is defined by $\mathrm{Fil}^i(B_{\mathrm{dR}}) = t^i \cdot B^+_{\mathrm{dR}}$, from which it is easy to see that the induced filtration $\mathrm{Fil}^i$ on the $K$-span of the $x_i, y_i$, that is, the intersection with $t^i \cdot B^+_{\mathrm{dR}}$, is zero for $i \geq 1$ and it is equal to the whole space for $i < 0$. Moreover, the elements

$$y_i \;+\; \sum_{j=1}^{g} \log_p(q_{i,j}) \cdot x_j, \qquad i = 1, ..., g$$

lie in $\mathrm{Fil}^0$, see also [Be04, §II.4.3] and we leave it to the reader to show that these $g$ vectors actually span $\mathrm{Fil}^0$. This establishes the claim and we see from these explicit computations that the $G_K$-representation on $V_p(A)$ is semi-stable. We remark that the semi-stability of $V_p(A)$ is a special case of [CN17, Corollary 5.26]. □

Remark 4.13. For Tate elliptic curves, this result was established by Le Stum [LeS95, §9] and our computations extended the exposition in [Be04, §II.4]. For the description of $\varphi$ and $N$, we also refer to [Co00, CI99]. Since we followed the exposition for Tate elliptic curves in [Be04, §II.4], we have chosen to use the notation found therein. We note that $\widetilde{\mathbb{E}}^+ \cong \varprojlim_{x\mapsto x^p} \mathcal{O}_{\widehat{\overline{K}}}/p = \mathcal{O}^\flat_{\widehat{\overline{K}}}$ and also that $W(\widetilde{\mathbb{E}}^+)$ is commonly referred to as $\mathbb{A}_{\rm inf}$.

We end this discusssion by an analogue of Proposition 4.11 in the context of filtered $(\varphi, N)$-modules. Before stating the result, we extend the definition of the $\mathcal{L}$-invariant of a Tate elliptic curve [MTT86, Ch. II §1] to abeloid varieties of aribitrary dimension: if $Q_A$ is a period matrix for the abeloid variety $A := \mathbb{G}^g_{m,K}/\Lambda_A$, we set

$$\mathcal{L}(Q_A) := \mathrm{ord}_p(Q_A)^{-1}\cdot \log_p(Q_A) \in \mathrm{Mat}_{g\times g}(K^\times).$$

Here, $\log_p(Q_A)$ denotes the matrix obtained by applying $\log_p$ to every entry of $Q_A$. Note that by a definition of period matrices, $\mathrm{ord}_p(Q_A)\in \mathrm{Mat}_{g\times g}(\mathbb{Q})$ is an invertible matrix.

Proposition 4.14. *Let $A = \mathbb{G}^g_{m,K}/\Lambda_A$ and $B = \mathbb{G}^h_{m,K}/\Lambda_B$ be abeloid varieties and let $Q_A$ and $Q_B$ be period matrices for $A$ and $B$, respectively.*

1. *If $A$ is isomorphic to $B$, then there exists a $M \in \mathrm{GL}_{g\times g}(\mathbb{Z})$ such that*

$$\mathcal{L}(Q_B) = M^{-1}\cdot \mathcal{L}(Q_A)\cdot M\,.$$

   *In particular, this describes how $\mathcal{L}$ transforms under a change of period matrix of one abeloid variety.*

2. *There exists an isomorphism of $\mathbb{Q}$-vector spaces*

$$\begin{array}{ll} & \mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q} \\ \cong & \{M\in \mathrm{Mat}_{g\times h}(\mathbb{Q})\,|\,\mathcal{L}(Q_A)\cdot M = M\cdot \mathcal{L}(Q_B)\}\,. \end{array}$$

   *In particular, $A$ and $B$ are isogenous if and only if $g = h$ and there exists a $M \in \mathrm{GL}_g(\mathbb{Q})$ such that $\mathcal{L}(Q_A)\cdot M = M\cdot \mathcal{L}(Q_B)$.*

Proof. We start with the following computation: let $R$ be a subring of $\mathbb{Q}_p$, let $A = \mathbb{G}^g_{m,K}/\Lambda_A$ and $B = \mathbb{G}^h_{m,K}/\Lambda_B$ be abeloid varieties, and let $Q_A$ and $Q_B$ be period matrices. Moreover, let $M \in \mathrm{Mat}_{g\times h}(R)$ be a matrix such that there exists a $N \in \mathrm{Mat}_{g\times h}(R)$ with

$$Q_A \odot M = N \odot Q_B. \tag{10}$$

We have seen in the proof of Theorem 4.7 that we have $N = \mathrm{ord}_p(Q_A)\cdot M\cdot \mathrm{ord}_p(Q_B)^{-1}$ in this case, which then yields

$$(\mathrm{ord}_p(Q_A)^{-1}\odot Q_A)\odot M = M\odot(\mathrm{ord}_p(Q_B)^{-1}\odot Q_B).$$

Taking the Iwasawa logarithm on both sides, we obtain

$$\mathcal{L}(Q_A)\cdot M \;=\; M\cdot\mathcal{L}(Q_B).$$

This already implies claim (1): if $A\cong B$, then we can find $M,N\in\mathrm{GL}_g(\mathbb{Z})$ satisfying equation (10) and the assertion follows.

We now establish claim (2): given a homomorphism $\mathrm{Hom}(A,B)\otimes\mathbb{Q}$, there exist by Theorem 4.7 two matrices $M,N\in\mathrm{Mat}_{g\times h}(\mathbb{Q})$ that satisfy (10). By the above computations, we find that $\mathcal{L}(Q_A)\cdot M = M\cdot\mathcal{L}(Q_B)$.

Conversely, assume that we are given a matrix $M\in\mathrm{Mat}_{g\times h}(\mathbb{Q})$ such that $\mathcal{L}(Q_A)\cdot M = M\cdot\mathcal{L}(Q_B)$. We set $N:=\mathrm{ord}_p(Q_A)\cdot M\cdot\mathrm{ord}_p(Q_B)^{-1}$ and find

$$\log_p(Q_A)\cdot M \;=\; N\cdot\log_p(Q_B) \qquad\text{and}\qquad \mathrm{ord}_p(Q_A)\cdot M \;=\; N\cdot\mathrm{ord}_p(Q_B),$$

which shows that

$$\log_p(Q_A\odot M) \;=\; \log_p(N\odot Q_B) \qquad\text{and}\qquad \mathrm{ord}_p(Q_A\odot M) \;=\; \mathrm{ord}_p(N\odot Q_B).$$

Using properties of the Iwasawa logarithm, this shows that there exist roots of unity $\varepsilon_{i,j}\in K$ such that the $(i,j)$ entries of the matrices $Q_A\odot M$ and $N\odot Q_B$ differ by the factor $\varepsilon_{i,j}$, see also the proof of [LeS95, Proposition 6]. Thus, if $R$ is a positive integer such that $\varepsilon_{i,j}^R=1$ for all $i,j$, then $Q_A\odot(RM)=(RN)\odot Q_B$. In particular, $RM$ and $RN$ define an element of $\mathrm{Hom}(A,B)$ and claim (2) follows. □

Our definition of the $\mathcal{L}$-invariant of an abeloid variety generalises the $\mathcal{L}$-invariant of a Tate elliptic curve [MTT86, Ch. II §1]. We refer to [DT08] for a survey of various $\mathcal{L}$-invariants for varieties that are uniformised by Drinfeld's upper half plane $\widehat{\Omega}^1_K$. Before continuing with our discussion of Proposition 4.11 in the context of filtered $(\varphi,N)$-modules, we first show that the $\mathcal{L}$-invariant $\mathcal{L}^{\mathrm{Col}}(C)$ of a Mumford curve $C$, as defined by Besser-de Shalit [BdS16], coincides with our $\mathcal{L}$-invariant of the Jacobian $J$ of $C$.

Proposition 4.15. *Let $\Gamma\subset\mathrm{PGL}_2(K)$ be a Schottky group and let $C=\Gamma\backslash\widehat{\Omega}^1_K$ be the associated Mumford curve of genus $g$. Let $J=\mathbb{G}^g_{m,K}/\Lambda_J$ be the Jacobian of $C$. Then, there exists a choice of period matrix $Q_J$ for $J$ such that*

$$\mathcal{L}^{\mathrm{Col}}(C) \;=\; \mathcal{L}(Q_J)\,.$$

Proof. (We refer the reader to [BdS16, §2.1 and §2.2] for a summary of the de Rham cohomology of varieties that are uniformised by Drinfeld upper half spaces, and also for the appropriate references to the literature.)

Since we only consider de Rham cohomology in cohomological degree 1, we are in the situation $d=1$ and $i=0$ in the notation of [BdS16]. We have already fixed a uniformiser of $K$, so we shall drop this label from the notation of [BdS16]. In summary, we simplify the notation by setting $\mathcal{L}^{\mathrm{Col}}(C):=\mathcal{L}^{\mathrm{Col}}_{\pi,1}(C)$.

By definition, we have $\mathcal{L}^{\mathrm{Col}}(C) = \nu^{-1} \circ \lambda^{\mathrm{Col}}$ where

$$\nu := \mathrm{gr} N : \mathrm{gr}^0_\Gamma H^1_{\mathrm{dR}}(C/K) \to \mathrm{gr}^1_\Gamma H^1_{\mathrm{dR}}(C/K)$$

is the map induced by the monodromy operator on the graded pieces of the covering filtration, using the fact that the covering filtration coincides with the weight filtration up to a shift in index. It is an isomorphism by the monodromy-weight conjecture, which is a theorem in our situation. The covering filtration is opposite to the Hodge filtration, which gives the identifications

$$\mathrm{gr}^0_\Gamma H^1_{\mathrm{dR}}(C/K) \cong H^0(C, \Omega^1_{C/K})$$

and

$$\mathrm{gr}^1_\Gamma H^1_{\mathrm{dR}}(C/K) \cong H^1(C, \mathcal{O}_C)\,.$$

The monodromy operators of $C$ and $J$ coincide under the identification $H^1_{\mathrm{dR}}(C/K) = H^1_{\mathrm{dR}}(J/K)$, by [CI99, §3]. Choose a period matrix $Q_J$ of $J$. Then we have computed in Theorem 4.12 that $\nu$ is given by the matrix $\mathrm{ord}_p(Q_J)$. The map

$$\lambda^{\mathrm{Col}} : \mathrm{gr}^0_\Gamma H^1_{\mathrm{dR}}(C/K) \to \mathrm{gr}^1_\Gamma H^1_{\mathrm{dR}}(C/K)$$

is defined using harmonic cochains on the Bruhat-Tits building $\mathcal{T}$ of $\mathrm{PGL}_2(K)$ and the identifications

$$\mathrm{gr}^0_\Gamma H^1_{\mathrm{dR}}(C/K) \cong H^0(\Gamma, C^1_{\mathrm{har}}(\mathcal{T}))$$

and

$$\mathrm{gr}^1_\Gamma H^1_{\mathrm{dR}}(C/K) \cong H^1(\Gamma, C^0_{\mathrm{har}}(\mathcal{T}))\,.$$

Let $\mathbb{C}_p := \widehat{\overline{K}}$. It is shown in [Gro00, §3.1.2] that for Mumford curves, the map $\lambda^{\mathrm{Col}} \otimes_K \mathbb{C}_p$ coincides with Coleman integration.
Fix $z' \in \widehat{\Omega}^1_K(\mathbb{C}_p)$. For $\alpha \in \Gamma$, define

$$u_\alpha(z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(z')}{z - \gamma\alpha(z')}$$

(this is independent of the choice of $z'$), and for $\alpha, \beta \in \Gamma$, define

$$Q_{\alpha,\beta} := \frac{u_\alpha(z)}{u_\alpha(\beta z)} \in K^\times\,.$$

Then, $\Lambda_J$ is generated by elements of the form $Q_{\alpha,\beta}$, that is the $Q_{\alpha,\beta}$ form a period matrix $Q_J$ for $J$ [GvdP80, Ch. VI §2]. Recall as well that $H^0(C, \Omega^1_{C/K})$ is generated over $K$ by differential forms of the form

$$\omega_\alpha := \frac{du_\alpha}{u_\alpha},$$

where $\alpha \in \Gamma' := \Gamma/[\Gamma,\Gamma] \simeq \Lambda_J$. Using the notation of [Gro00], all paths $\gamma$ are linear combinations of the paths of the form $[z, \beta \cdot z]$ with $\beta \in \Gamma$, and hence to compute $\lambda^{\mathrm{Col}}$ we see that it suffices to calculate

$$\int_z^{\beta\cdot z} \frac{du_\alpha}{u_\alpha} \;=\; \log_p u_\alpha(\beta\cdot z) - \log_p u_\alpha(z) \;=\; \log_p Q_{\alpha,\beta}\,.$$

Altogether, we see that $\mathcal{L}^{\mathrm{Col}} = \nu^{-1}\circ\lambda^{\mathrm{Col}} = \mathrm{ord}_p(Q_J)^{-1}\cdot\log_p(Q_J) = \mathcal{L}(Q_J)$. □

We now return to our discussion of 4.11 in the context of filtered $(\varphi, N)$-modules.

Proposition 4.16. *Let $A := \mathbb{G}^g_{m,K}/\Lambda_A$ and $B := \mathbb{G}^h_{m,K}/\Lambda_B$ be abeloid varieties and let $Q_A$ and $Q_B$ be period matrices for $A$ and $B$, respectively. Then, there exists an isomorphism of $\mathbb{Q}_p$-vector spaces*

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{MF}_K^{\mathrm{wa},\varphi,N}}\left(\mathbb{D}_{\mathrm{st}}(V_p(A)), \mathbb{D}_{\mathrm{st}}(V_p(B))\right) \\ \cong\; &\{M \in \mathrm{Mat}_{g\times h}(\mathbb{Q}_p)\,|\,\mathcal{L}(Q_A)\cdot M \;=\; M\cdot\mathcal{L}(Q_B)\}\,. \end{aligned}$$

*In particular, $\mathbb{D}_{\mathrm{st}}(V_p(A))$ and $\mathbb{D}_{\mathrm{st}}(V_p(B))$ are isomorphic if and only if $g = h$ and there exists a $M \in \mathrm{GL}_g(\mathbb{Q}_p)$ such that $\mathcal{L}(Q_A)\cdot M = M\cdot\mathcal{L}(Q_B)$.*

Proof. Quite generally, let $D, D'$ be objects of $\mathrm{MF}_K^{\mathrm{wa},\varphi,N}$. Then, an element of $\mathrm{Hom}_{\mathrm{MF}_K^{\mathrm{wa},\varphi,N}}(D, D')$ is a $K_0$-linear map from the underlying $K_0$-vector space of $D$ to the underlying $K_0$-vector space of $D'$ that commutes with the Frobenius and monodromy operators, and such that the $K$-linear extension of this map sends the Hodge filtration of $D_K$ into the Hodge filtration of $D'_K$. After choosing bases, we represent this $K_0$-linear map by a matrix $M$ and let $F, F'$ be matrices representing the Frobenius operators on $D$ and $D'$, respectively. Then, compatibility with Frobenius is the condition that $FM = MF'$.
We claim that the matrix $M$ has coefficients in $\mathbb{Q}_p$: first, by considering

$$\left(\begin{smallmatrix} \mathrm{Id} & 0 \\ M & \mathrm{Id}\end{smallmatrix}\right) \;:\; D\oplus D' \;\to\; D\oplus D',$$

we see that it suffices to assume $D = D'$, $\varphi = \varphi'$, and $F = F'$. Recall that the Frobenius operator is semi-linear with respect to $\sigma$ by definition, that is, $FM = \sigma(M)F$. This together with condition $FM = MF$ shows that $\sigma(M) = M$, which proves the claim.
Now let us return to the explicit description of $\mathrm{Hom}_{\mathrm{MF}_K^{\mathrm{wa},\varphi,N}}(\mathbb{D}_{\mathrm{st}}(V_p(A)), \mathbb{D}_{\mathrm{st}}(V_p(B)))$. Using the bases from Section 4.5, we see that giving an element of this space is equivalent to giving a matrix $M \in \mathrm{Mat}_{2g\times 2h}(\mathbb{Q}_p)$ that satisfies the following three conditions

$$\begin{pmatrix} p^{-1}\cdot\mathrm{Id}_{g\times g} & 0 \\ 0 & \mathrm{Id}_{g\times g}\end{pmatrix} M \;=\; M \begin{pmatrix} p^{-1}\cdot\mathrm{Id}_{h\times h} & 0 \\ 0 & \mathrm{Id}_{h\times h}\end{pmatrix} \tag{a}$$

$$\begin{pmatrix} 0 & \mathrm{ord}_p(Q_A) \\ 0 & 0\end{pmatrix} M \;=\; M \begin{pmatrix} 0 & \mathrm{ord}_p(Q_B) \\ 0 & 0\end{pmatrix} \tag{b}$$

$$\left(\log_p(Q_A) \quad \mathrm{Id}_{g\times g}\right) M \;=\; \left(Z\cdot\log_p(Q_B) \quad Z\right) \tag{c}$$

for some $Z\in\mathrm{Mat}_{g\times h}(K)$. We see that (a) holds if and only if $M$ is of the form $\left(\begin{smallmatrix} X & 0 \\ 0 & Y\end{smallmatrix}\right)$ for some matrices $X, Y\in\mathrm{Mat}_{g\times h}(\mathbb{Q}_p)$. It then follows from (b) and (c) that to give an element of $\mathrm{Hom}_{\mathrm{MF}_K^{\mathrm{wa},\varphi,N}}(\mathbb{D}_{\mathrm{st}}(V_p(A)),\mathbb{D}_{\mathrm{st}}(V_p(B)))$ is the same as giving $X, Y\in\mathrm{Mat}_{g\times h}(\mathbb{Q}_p)$ that satisfy

$$\mathrm{ord}_p(Q_A)\cdot X \;=\; Y\cdot\mathrm{ord}_p(Q_B) \tag{d}$$

and

$$\log_p(Q_A)\cdot X \;=\; Y\cdot\log_p(Q_B)\,. \tag{e}$$

Since $Q_B$ is a period matrix, $\mathrm{ord}_p(Q_B)$ is invertible, and we find $Y=\mathrm{ord}_p(Q_A)\cdot X\cdot\mathrm{ord}_p(Q_B)^{-1}$. Plugging this into (e), the proposition follows. □

### 4.6 A translation of Question 1.2 into linear algebra

Let $A=\mathbb{G}_{m,K}^g/\Lambda_A$ and $B=\mathbb{G}_{m,K}^h/\Lambda_B$ be abeloid varieties over a $p$-adic field $K$ and let $\ell$ be a prime, possibly equal to $p$. We choose period matrices $Q_A$ and $Q_B$.

1. Under the identifications of Theorem 4.7 and Proposition 4.11 the homomorphism ($\star\star$)

$$\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \;\to\; \mathrm{Hom}_{G_K}\left(T_\ell(A),\, T_\ell(B)\right)$$

   is given by the homomorphism

$$\begin{array}{rl} & \{M\in\mathrm{Mat}_{g\times h}(\mathbb{Z})\,|\,\Lambda_A\odot M\subset\Lambda_B\}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \\ \to & \{M\in\mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell)\,|\,\gamma_\ell(\Lambda_A)\odot M\subset\gamma_\ell(\Lambda_B)\}, \end{array}$$

   or, equivalently,

$$\begin{array}{l} \left\{M\in\mathrm{Mat}_{g\times h}(\mathbb{Z})\,|\,\left(\mathrm{ord}_p(Q_A)^{-1}\odot Q_A\right)\odot M \;=\; M\odot\left(\mathrm{ord}_p(Q_B)^{-1}\odot Q_B\right)\right\}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \\ \to\left\{M\in\mathrm{Mat}_{g\times h}(\mathbb{Z}_\ell)\,|\,\left(\mathrm{ord}_p(Q_A)^{-1}\odot\gamma_\ell(Q_A)\right)\odot M \;=\; M\odot\left(\mathrm{ord}_p(Q_B)^{-1}\odot\gamma_\ell(Q_B)\right)\right\}\,. \end{array}$$

2. Moreover, if $\ell=p$, then under the identifications of Theorem 4.7 and Proposition 4.16, the analog of homomorphism ($\star\star$)

$$\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_p \;\to\; \mathrm{Hom}_{\mathrm{MF}_K^{\mathrm{wa},\varphi,N}}\left(\mathbb{D}_{\mathrm{st}}(V_p(A)),\mathbb{D}_{\mathrm{st}}(V_p(B))\right)$$

   is given by

$$\begin{array}{l} \left\{M\in\mathrm{Mat}_{g\times h}(\mathbb{Q})\,|\,\left(\mathrm{ord}_p(Q_A)^{-1}\odot Q_A\right)\odot M \;=\; M\odot\left(\mathrm{ord}_p(Q_B)^{-1}\odot Q_B\right)\right\}\otimes_{\mathbb{Q}}\mathbb{Q}_p \\ \to\{M\in\mathrm{Mat}_{g\times h}(\mathbb{Q}_p)\,|\,\mathcal{L}(Q_A)\cdot M \;=\; M\cdot\mathcal{L}(Q_B)\}\;, \end{array}$$

   or, equivalently,

$$\begin{array}{rl} & \{M\in\mathrm{Mat}_{g\times h}(\mathbb{Q})\,|\,\mathcal{L}(Q_A)\cdot M \;=\; M\cdot\mathcal{L}(Q_B)\}\otimes_{\mathbb{Q}}\mathbb{Q}_p \\ \to & \{M\in\mathrm{Mat}_{g\times h}(\mathbb{Q}_p)\,|\,\mathcal{L}(Q_A)\cdot M \;=\; M\cdot\mathcal{L}(Q_B)\}\;. \end{array}$$

This gives reformulations of Question 1.2 in terms of linear algebra. Note that it is particularly easy to see that ($\star\star$) is injective in some of these reformulations. We also see that the surjectivity of ($\star\star$) is equivalent to the interplay of $\mathbb{Q}$-structures versus $\mathbb{Q}_\ell$-structures, which is similar to Raskind-admissibility and the results of Section 3. In Lemma 4.9 and Remark 4.10, we have seen that $\gamma_\ell$ behaves very differently depending on whether $\ell \neq p$ or $\ell = p$:

1. In Proposition A.4, we will see that surjectivity of ($\star\star$) may fail if $A$ and $B$ are Tate elliptic curves and $\ell \neq p$.
2. Surjectivity of ($\star\star$) may look plausible if $\ell = p$: we will give a positive result in Proposition 5.3 below and disprove it in general in Theorem 6.1.

## 5 Products of Tate elliptic curves

In this section, we use the results of the previous section to study Raskind's conjecture for divisors (Conjecture 1.1) and Question 1.2 for abelian varieties that are isogenous to products of Tate elliptic curves. For the product $X$ of two Tate elliptic curves, we determine the rational structure on the filtered $(\varphi, N)$-module $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$, which leads to a direct verification of Conjecture 1.1. We also classify all filtered $(\varphi, N)$-modules that are ordinary in the sense of Perrin-Riou that have a fixed rational structure that is modeled on a surface with $p_g = 1$.

### 5.1 Products of Tate elliptic curves

We recall that a Tate elliptic curve over a $p$-adic field $K$ is the same as an abeloid variety of dimension one over $K$. In particular, they are of the form $E(q) := \mathbb{G}_{m,K}/q^{\mathbb{Z}}$ for some $q \in K^\times$ with $\nu_p(q) > 0$. As in the previous section, we set $\mathcal{L}(x) := \log_p(x)/\nu_p(x)$.

Theorem 5.1 (Le Stum, Serre). *Let $E(q_i)$, $i = 1, 2$ be two Tate elliptic curves over a $p$-adic field $K$ associated to $q_i \in K^\times$ with $\nu_p(q_i) > 0$. Then, the following are equivalent:*

1. *$E(q_1)$ and $E(q_2)$ are isogenous.*
2. *There exist positive integers $A_i$, $i = 1, 2$ such that $q_1^{A_1} = q_2^{A_2}$.*
3. *The rational Tate modules $V_p(E(q_i))$, $i = 1, 2$ are isomorphic as $p$-adic $G_K$-representations.*
4. *The $\mathbb{D}_{\mathrm{st}}(V(E(q_i)))$, $i = 1, 2$ are isomorphic as filtered $(\varphi, N)$-modules over $K$.*
5. *$\mathcal{L}(q_1) = \mathcal{L}(q_2)$.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are shown in [Se68, §A.1.4] and the equivalences (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are shown in [LeS95, Proposition 6]. □

Remark 5.2. In fact, using the results from Section 4, it is easy to deduce the equivalence (1) $\Leftrightarrow$ (2) from Theorem 4.7, to deduce the equivalence (2) $\Leftrightarrow$ (3) from Proposition 4.11, to prove the equivalence (1) $\Leftrightarrow$ (4) using Proposition 4.14, and to prove the equivalences (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) using Theorem 4.12.

Concerning Question 1.2, we have a positive answer in the following special case.

Proposition 5.3. *Let $K$ be a $p$-adic field and let $A$ and $B$ be abelian varieties over $K$, both of which are isogenous to products of Tate elliptic curves. Then, $(\star\star)$ (please see §1.4) is surjective for $A$, $B$, and $\ell = p$, that is, Question 1.2 has a positive answer for $A$ and $B$.*

Proof. First, the map $(\star\star)$ is injective [Mu70, IV.19.3] and since the cokernel is torsion-free [Ta66, Lemma 1], we are reduced to showing the surjectivity of the map

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow \mathrm{Hom}_{G_K}(V_p(A), V_p(B)) \ . \tag{11}$$

If $A$ is isogenous to a product $\prod_{i=1} E(q_i)$ of Tate elliptic curves, then an isogeny gives rise to an isomorphism of $p$-adic $G_K$-representations

$$V_p(A) \cong \bigoplus_{i=1}^{g} V_p(E(q_i)).$$

We are thus reduced to the case where both $A$ and $B$ are Tate elliptic curves. If they are not isogenous, then $\mathrm{Hom}(A, B) = 0$ and $\mathrm{Hom}(V_p(A), V_p(B)) = 0$ by Theorem 5.1. On the other hand, if they are isogenous, then we have $\mathrm{Hom}(A, B) = \mathbb{Z}$ and $\mathrm{Hom}(V_p(A), V_p(B)) = \mathbb{Q}_p$. This verifies the claim in both cases and the proposition follows. □

As a corollary, we establish Conjecture 1.1 in a special case.

Corollary 5.4. *Let $K$ be a $p$-adic field and let $A$ be an abelian variety over $K$ that is isogenous to a product of Tate elliptic curves. Then, $(\star)$ is surjective for $A$ and $\ell = p$, that is, Raskind's conjecture for divisors (Conjecture 1.1) is true for $A$.*

Proof. We may assume that $A$ is in fact isomorphic to a product of Tate elliptic curves. The argument to deduce surjectivity of $(\star)$ from the surjectivity of $(\star\star)$ is [Ta66, Theorem 3]. □

In Appendix A, we will see that both results fail to be true if $\ell \neq p$.

## 5.2 The product of two Tate curves

Let $E(q) := \mathbb{G}_{m,K}/q^{\mathbb{Z}}$ be the Tate elliptic curve associated to an element $q \in K^\times$ with $\nu_p(q) > 0$. Associated to $q$, we construct a filtered $(\varphi, N)$-module as follows

1. Let $V$ be a 2-dimensional vector space over $\mathbb{Q}$ with basis $e_1, e_2$ together with two linear operators $\varphi, N$ defined by
$$\begin{array}{rclcrcl} \varphi(e_1) & = & e_1 & \quad & \varphi(e_2) & = & p \cdot e_2 \\ N(e_1) & = & 0 & & N(e_2) & = & e_1 \end{array} .$$
We equip $V_{K_0} := V \otimes_{\mathbb{Q}} K_0$ with the $K_0$-linear extension of $N$ and the $K_0$-semi-linear extension of $\varphi$, which defines a $(\varphi, N)$-module over $K_0$.

2. Define a filtration on $V_K := V \otimes_{\mathbb{Q}} K$ defined by $\mathrm{Fil}^i = V_K$ for $i \geq 0$, by $\mathrm{Fil}^i = 0$ for $i \geq 2$, and $\mathrm{Fil}^1$ is the one-dimensional $K$-vector space spanned by
$$\mathcal{L}(q) \cdot e_1 + e_2.$$

Now, the $p$-adic $G_K$-representations $V_p(E(q))$ and $H^1_{\text{ét}}(E(q)_{\overline{K}}, \mathbb{Q}_p)$ are dual. Using the explicit description of $\mathbb{D}_{\text{st}}(V_p(E(q)))$ provided by Theorem 4.12 (although in the case of Tate elliptic curves, this was classically known, see [Be04, II.4.2]), the formulae from [Fo94, 4.2.4 and 4.3.4] to compute the dual, and after suitably rescaling, it is not difficult to see that the just constructed filtered $(\varphi, N)$-module $V$ is isomorphic to $\mathbb{D}_{\text{st}}(H^1_{\text{ét}}(E(q)_{\overline{K}}, \mathbb{Q}_p))$. Alternatively, one can also use Le Stum's computations [LeS95, §9].
Next, let $q_i \in K^\times$ with $\nu_p(q_i) > 0$ for $i = 1, 2$, let $E(q_i)$ be the assocated Tate elliptic curves over $K$, and set $X := E(q_1) \times E(q_2)$. Then, we have the following description of the filtered $(\varphi, N)$-module $\mathbb{D}_{\text{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$

1. Let $V^{(i)}$, $i = 1, 2$ be two 2-dimensional vector spaces with bases $\{e_1^{(i)}, e_2^{(i)}\}$, Frobenius, monodromy, and filtration on $V^{(i)} \otimes K$ associated to $q_i$ as above. Set $V := V^{(1)} \oplus V^{(2)}$.

2. Then, $W := \wedge^2(V) = \wedge^2(V^{(1)} \oplus V^{(2)})$ is a 6-dimensional $\mathbb{Q}$-vector space with basis
$$\begin{array}{l} a \,=\, e_1^{(1)} \wedge e_1^{(2)}, \\ b_0 \,=\, e_1^{(1)} \wedge e_2^{(2)} + e_2^{(1)} \wedge e_1^{(2)}, \\ b_1 \,=\, e_1^{(1)} \wedge e_2^{(2)} - e_2^{(1)} \wedge e_1^{(2)}, \qquad b_2 \,=\, e_1^{(1)} \wedge e_2^{(1)}, \qquad b_3 \,=\, e_1^{(2)} \wedge e_2^{(2)} \\ c \,=\, e_2^{(1)} \wedge e_2^{(2)}. \end{array}$$
We set
$$A := \langle a \rangle,\ B_0 := \langle b_0 \rangle,\ B_1 := \langle b_1, b_2, b_3 \rangle,\ B := B_0 \oplus B_1,\ \text{and}\ C := \langle c \rangle.$$

3. The $\varphi$'s on $V^{(1)}$ and $V^{(2)}$ induce a linear endomorphism $\varphi_W$ with
$$\varphi_W|_A \,=\, \mathrm{id}_A, \qquad \varphi_W|_B \,=\, p \cdot \mathrm{id}_A, \qquad \text{and} \qquad \varphi_W|_C \,=\, p^2 \cdot \mathrm{id}_A\,.$$
Similarly, we obtain a linear endomorphism $N_W$ with
$$N_W(c) \,=\, b_0, \qquad N_W(b_0) \,=\, 2a, \qquad \text{and} \qquad N_W|_{A \oplus B_1} \,=\, 0.$$
We extend $\varphi_W$ semi-linearly and $N_W$ linearly to $W \otimes K_0$ and thus obtain a $(\varphi, N)$-module.

4. Moreover, the decomposition $(W = A \oplus B_0 \oplus B_1 \oplus C, \varphi_W, N_W)$ defines a rational structure in the sense of Definition 3.4.

5. The filtrations on $V^{(i)} \otimes K$ give rise to a filtration on $W \otimes K$ with $\mathrm{Fil}^0 = W$, $\mathrm{Fil}^3 = 0$, and $\mathrm{Fil}^2$ is the one-dimensional $K$-span of

$$\underbrace{\mathcal{L}(q_1)\mathcal{L}(q_2)\cdot a}_{\in A} + \underbrace{\mathcal{L}(q_1)\cdot e_1^{(1)}\wedge e_2^{(2)} + \mathcal{L}(q_2)\cdot e_2^{(1)}\wedge e_1^{(2)}}_{\in B} + \underbrace{e_2^{(1)}\wedge e_2^{(2)}}_{=c\,\in C}. \tag{12}$$

To explain $\mathrm{Fil}^1$, we note that since $W = \wedge^2 V$, the wedge product induces a non-degenerate symmetric bilinear form $W \times W \to \wedge^4 V \cong \mathbb{Q}$. Then, $\mathrm{Fil}^2$ is isotropic with respect to this pairing and it is easy to see that

$$\mathrm{Fil}^1 \;=\; (\mathrm{Fil}^2)^{\perp}.$$

Thus, we obtain a filtered $(\varphi, N)$-module $(W \otimes K_0, \mathrm{Fil}^\bullet, \varphi_W \otimes \sigma, N_W \otimes \mathrm{id}_{K_0})$ and a rational structure $(W = A \oplus B_0 \oplus B_1 \oplus C, \varphi_W, N_W)$. We will see in Proposition 5.6 that it is isomorphic to $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$.

Lemma 5.5. *This just-constructed filtered $(\varphi, N)$-module together with its rational structure is Raskind-admissible. More precisely, we have*

$$\dim_{\mathbb{Q}}\left(\mathrm{Fil}^1 \cap B_1\right) \;=\; \begin{cases} 2 & \textit{if } \mathcal{L}(q_1) \neq \mathcal{L}(q_2) \textit{ and} \\ 3 & \textit{if } \mathcal{L}(q_1) = \mathcal{L}(q_2). \end{cases}$$

Proof. First, we note that an element lies in $\mathrm{Fil}^1$ if and only if it has zero intersection with the vector (12). This makes computations very easy.
If $\mathcal{L}(q_1) = \mathcal{L}(q_2)$, then $B_1 \subset \mathrm{Fil}^1$ from which the claim on the dimension and Raskind-admissibility easily follows.
If $\mathcal{L}(q_1) \neq \mathcal{L}(q_2)$, then $b_2, b_3 \in \mathrm{Fil}^1$ and thus, $\dim_{\mathbb{Q}}(\mathrm{Fil}^1 \cap B_1) \geq 2$. Moreover, we have that $\mathrm{Fil}^1 \cap (B_1 \otimes K)$ is strictly contained in $B_1 \otimes K$, which yields the chain of inequalities $\dim_{\mathbb{Q}}(\mathrm{Fil}^1 \cap B_1) \leq \dim_{\mathbb{Q}_p}(\mathrm{Fil}^1 \cap (B_1 \otimes \mathbb{Q}_p)) \leq \dim_K(\mathrm{Fil}^1 \cap (B_1 \otimes K)) \leq 2$. Together with the previous inequality, this implies that we have equality everywhere and establishes the claimed dimension, as well as Raskind-admissibility. ☐

Proposition 5.6. *Let $K$ be a $p$-adic field, let $q_i \in K^\times$ with $\nu_p(q_i) > 0$ for $i = 1, 2$, let $E(q_1), E(q_2)$ be the associated Tate elliptic curves over $K$, and set $X := E(q_1) \times E(q_2)$.*

1. *$X$ admits a proper and semi-stable model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$, whose special fibre $\mathcal{X}_0$ is cohomogically totally degenerate.*

2. *The filtered $(\varphi, N)$-module $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ together with the rational structure associated to $\mathcal{X}_0$ are isomorphic to the one constructed at the beginning of this subsection.*

*In particular, this filtered $(\varphi, N)$-module is Raskind-admissible and Conjecture 1.1 is true for $X$. More precisely, we have*

$$\rho(X) \;=\; \begin{cases} 2 & \textit{if } \mathcal{L}(q_1) \neq \mathcal{L}(q_2) \textit{ and} \\ 3 & \textit{if } \mathcal{L}(q_1) = \mathcal{L}(q_2). \end{cases}$$

*for the Picard rank of $X$.*

Proof. Using the decomposition

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \;\cong\; \bigwedge^2 H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \;\cong\; \bigwedge^2 \left(H^1_{\text{ét}}(E(q_1))_{\overline{K}}, \mathbb{Q}_p) \oplus H^1_{\text{ét}}(E(q_2)_{\overline{K}}, \mathbb{Q}_p)\right)$$

and the explicit description of $\mathbb{D}_{\text{st}}(H^1_{\text{ét}}(E(q_i)_{\overline{K}}, \mathbb{Q}_p))$ given at the beginning of Section 5.2, it is easy to see that the filtered $(\varphi, N)$-module constructed above is isomorphic to $\mathbb{D}_{\text{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$.

Let $\mathcal{E}_i \to \operatorname{Spec} \mathcal{O}_K$ the "standard" proper and semi-stable model of $E_i := E(q_i)$, whose special fibre $\mathcal{E}_{i,0}$ is a cycle of $\mathbb{P}^1$'s. It follows that $H^*_{\text{dR}}(E_i/K)$ and $H^*_{\log-\text{cris}}(\mathcal{E}_{i,0}/K_0)$ carry structures of $\mathbb{Q}$-vector spaces arising from classes of algebraic cycles of the special fibre. Tensoring with $K$, we obtain the filtered $(\varphi, N)$-module $V^{(i)} \otimes K$ constructed at the beginning of Section 5.2. Conversely, the vector $e_2^{(i)} \in V^{(i)}$ arises as an eigenvector of $\varphi$, which makes it canonical up to a factor $\lambda_i \in \mathbb{Q}_p^\times$. Since $e_1^{(i)} = N(e_2^{(i)})$, this also determines $e_1^{(i)}$. Thus, given $V^{(i)} \otimes K$, the just discussed rational structure must be the $\mathbb{Q}$-span $\langle \lambda_i e_1^{(i)}, \lambda_i e_2^{(i)} \rangle$ for some $\lambda_i \in \mathbb{Q}_p^\times$.

We obtain a proper and semi-stable model $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ of $X$ via a partial resolution of singularities of $\mathcal{E}_1 \times \mathcal{E}_2 \to \operatorname{Spec} \mathcal{O}_K$. Thus, the $\mathbb{Q}$-vector space structures on $H^*_{\text{dR}}(X/K)$ and $H^*_{\log-\text{cris}}(\mathcal{X}_0/K_0)$ arise (via Künneth) from the $\mathbb{Q}$-vector space structures on $H^*_{\text{dR}}(E_i/K)$ and $H^*_{\log-\text{cris}}(\mathcal{E}_{i,0}/K_0)$. Therefore, the rational structure on $H^*_{\log-\text{cris}}(\mathcal{X}_0/K_0)$ arising from $\mathcal{X}_0$ is of the form as discussed at the beginning of Section 5.2 and isomorphic to it, where the isomorphism is induced by multiplication by scalars $\lambda_i \in \mathbb{Q}_p^\times$ as just explained. We note that such a rescaling multiplies the vector (12) by $\lambda_1 \cdot \lambda_2$, that is, it still spans the same $K$-vector space $\text{Fil}^2$. Since $\text{Fil}^1 = (\text{Fil}^2)^\perp$, we see that rescaling also leaves $\text{Fil}^1$ invariant. Thus, the filtered $(\varphi, N)$-module together with its rational structure discussed at the beginning of Section 5.2 is isomorphic to $\mathbb{D}_{\text{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ with its rational structure arising from $\mathcal{X}_0$.

By Lemma 5.5, it is Raskind-admissible and thus, Conjecture 1.1 for $X$ follows from Theorem 3.11. The claim on the Picard ranks follows from the dimensions computed in Lemma 5.5. □

Remark 5.7. We note that Conjecture 1.1 for the product of two Tate curves was already established Tate, as is explained in [Se68, Appendix A.1.4] (see also [RX07b, §4.1, Corollary 19]), and it also follows from the more general Corollary 5.4.

The previous proposition raises the question, whether every admissible $(\varphi, N)$-module with rational structure is Raskind-admissible. This is *not* the case, as the following example shows.

Example 5.8. We keep the notations and assumptions of Proposition 5.6. Choose $\gamma \in \mathbb{Q}_p \backslash \mathbb{Q}$, choose $\lambda \in K \backslash \{0\}$, and define

$$v \;:=\; 2(\lambda^2 - \gamma)a \,+\, \lambda b_0 \,-\, \gamma b_2 \,+\, b_3 \,+\, c.$$

Then, $v$ is isotropic with respect to the pairing introduced at the beginning of Section 5.2, which allows us to define a filtration on $V \otimes K$ via

$$\mathrm{Fil}^2 \;:=\; K \cdot v \quad \text{ and } \quad \mathrm{Fil}^1 \;:=\; (\mathrm{Fil}^2)^{\perp}\,.$$

This filtration is admissible by Proposition 5.9 below. We leave it to the reader to check that

$$\begin{array}{lcl} \mathrm{Fil}^1 \cap (B_1 \otimes \mathbb{Q}_p) & = & \mathbb{Q}_p \langle b_1, \gamma b_2 + b_3 \rangle \\ \mathrm{Fil}^1 \cap B_1 & = & \mathbb{Q}\,\langle b_1 \rangle, \end{array}$$

which implies that $\mathrm{Fil}^\bullet$ is *not* Raskind-admissible. Varying $\gamma$ and $\lambda$, we obtain a whole family of such modules.

### 5.3 Admissibility

Given a $(\varphi, N)$-module and a rational structure in the sense of Definition 3.4 with $\dim A = \dim C = 1$, we now address the question when a filtration over $K$ is ordinary in the sense of Perrin-Riou [PR94, 1.2]. We have the following result, which should be the framework for rational structures on $\mathbb{D}_{\mathrm{st}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ where $X$ is a smooth and proper surface over $K$ with cohomological total degeneration and $h^{0,2} = h^{2,0} = 1$ (that is, $p_g = 1$ in classical terminology).

Proposition 5.9. *Let $(V = A \oplus B_0 \oplus B_1 \oplus C, \varphi_V, N_V)$ be a rational structure in the sense of Definition 3.4. Assume moreover that*

1. $\dim A = \dim C = 1$. *We also fix a non-zero element $c \in C$, that is, a basis of this vector space.*

2. *There exists a non-degenerate, symmetric, and bilinear pairing $Q : V \times V \to \mathbb{Q}$.*

*Let $K$ be a $p$-adic field and let $\mathrm{Fil}^\bullet$ be a filtration on $V \otimes K$ with $\mathrm{Fil}^0 = V \otimes K$ and $\mathrm{Fil}^3 = 0$.*

1. *If $\mathrm{Fil}^\bullet$ is ordinary, then $\mathrm{Fil}^2$ is 1-dimensional and spanned by a unique vector of the form*
$$v \;=\; v' \,+\, c \tag{13}$$
*with $v' \in (A \oplus B_0 \oplus B_1) \otimes K$.*

*2. If $v$ is of the form (13) with $Q(v,v)=0$ and $Q(v,N^2(c))\neq 0$, then*

$$\mathrm{Fil}^2 := K\cdot v \quad \text{and} \quad \mathrm{Fil}^1 := (\mathrm{Fil}^2)^\perp$$

*defines an ordinary filtration on $V\otimes K$.*

*In particular, there exists a bijection*

$$\left\{ v\in V\otimes K \;:\; \begin{array}{c} Q(v,v)=0,\ Q(v,N^2(c))\neq 0, \\ \text{and } v \text{ is of the form (13)} \end{array} \right\} \to \left\{ \begin{array}{c} \text{ordinary filtrations} \\ \text{of } V\otimes K \text{ with} \\ \mathrm{Fil}^1=(\mathrm{Fil}^2)^\perp \end{array} \right\}$$

*that is defined by sending $v$ to the filtration with $\mathrm{Fil}^1=(Kv)^\perp$ and $\mathrm{Fil}^2=Kv$.*

Proof. By [PR94, 2.6] it follows that $\mathrm{Fil}^\bullet$ is ordinary if and only if we have direct sum decompositions

$$V\otimes K \;=\; \mathrm{Fil}^1 \,\oplus\, (A\otimes K) \;=\; \mathrm{Fil}^2 \,\oplus\, ((A\oplus B)\otimes K)\,,$$

where $B=B_0\oplus B_1$,
In particular, $\mathrm{Fil}^2$ is one-dimensional and thus, generated by one element $v\in V\otimes K$. Being ordinary, we have $v\notin (A\oplus B)\otimes K$. Thus, after possibly rescaling, we may assume that $v$ is of the form $v'+c$ for some $v'\in(A\oplus B)\otimes K$ and this vector is unique. This establishes claim (1).
If $v\in V\otimes K$ satisfies $Q(v,v)=0$, then $\mathrm{Fil}^2:=Kv$ is contained in $\mathrm{Fil}^1:=(Kv)^\perp$, and we obtain a filtration. Since $Q$ is non-degenerate, it follows that $(Kv)^\perp$ is of codimension 1 in $V\otimes K$. If $v$ is moreover of the form (13), that is, $v=v'+c$ with $v'\in(A\oplus B_0\oplus B_1)\otimes K$, then $V\otimes K=\mathrm{Fil}^2\oplus(A\oplus B)\otimes K$. If $Q(v,N^2(c))\neq 0$, then $A$, which is spanned by $N^2(c)$, is not contained in $(Kv)^\perp=\mathrm{Fil}^1$, that is, we have $V\otimes K=\mathrm{Fil}^1\oplus A\otimes K$. This shows that $\mathrm{Fil}^\bullet$ is ordinary and establishes claim (2).
We leave the remaining assertion to the reader. □

Thus, ordinary filtrations on $V\otimes K$ with $\mathrm{Fil}^1=(\mathrm{Fil}^2)^\perp$ are parameterised by the $K$-rational points of a quasi-affine scheme over $\mathbb{Q}$. Explicitly: first, $b_0:=N(c)$ is a basis of $B_0$, then, $c:=N(b_0)=N^2(c)$ is a basis of $A$ and we choose a basis $b_1,...,b_s$ of $B_1$. Then, we define an affine quadric $\mathcal{Q}$ by the equation

$$Q\left(\lambda a+\sum_{i=0}^{s}\mu_i b_i+c,\ \lambda a+\sum_{i=0}^{s}\mu_i b_i+c\right) \;=\; 0$$

in the $(s+2)$-dimensional affine space with coordinates $(\lambda,\mu_0,...,\mu_s)$. We define the Zariski open subset $\mathcal{U}\subset\mathcal{Q}$ by the condition

$$Q\left(\lambda a+\sum_{i=0}^{s}\mu_i b_i+c,\ N^2(c)\right) \;\neq\; 0.$$

Then, $\mathcal{U}$ is a quasi-affine scheme of dimension $(s+1)$ over $\mathbb{Q}$, whose $K$-rational points are in bijection to ordinary filtrations $\mathrm{Fil}^\bullet$ on $V\otimes K$ with $\mathrm{Fil}^1=(\mathrm{Fil}^2)^\perp$.

## 6 A counter-example

For abelian varieties that are isogenous to products of Tate elliptic curves, we established Raskind's conjecture for divisors (Conjecture 1.1) and showed that Question 1.2 has a positive answer in the previous section. In this section, we will show that in general, Question 1.2 has a negative answer and that in general Raskind's conjecture for divisors is false.

### 6.1 Totally degenerate reduction and $\ell = p$

In view of Proposition 5.3, the first place to look for counter-examples are abeloid surfaces over $p$-adic fields.

Theorem 6.1. *Let $p$ be a prime with $p \geq 5$ and $p \equiv 1 \mod 3$. Then, there exists a Tate elliptic curve $A$ and an algebraisable abeloid surface $B$ over $\mathbb{Q}_p$, such that*

1. *the natural maps*

$$\begin{array}{rcl} \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & \mathrm{Hom}_{G_{\mathbb{Q}_p}}(T_p(A),\, T_p(B)) \\ \mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & \mathrm{End}_{G_{\mathbb{Q}_p}}(T_p(B)) \end{array}$$

*are not surjective. In particular,* ($\star\star$) *(please see §1.4) is not surjective and Question 1.2 has a negative answer for $\ell = p$.*

2. *The natural map induced by the first Chern class map*

$$\mathrm{Pic}(B) \otimes_{\mathbb{Z}} \mathbb{Q}_p \quad \to \quad H^2_{\text{ét}}(B_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p(1))^{G_{\mathbb{Q}_p}}$$

*is not surjective. In particular,* ($\star$) *is not surjective for $\ell = p$ and Raskind's conjecture for divisors (Conjecture 1.1) is false.*

Remark 6.2. As seen in Proposition 4.6, an algebraisable abeloid variety over $K$ is the same as an abelian variety over $K$ with totally degenerate reduction.

Proof. Let $\varepsilon \in 1 + p \cdot \mathbb{Z}_p$ be a non-trivial $p$-adic unit, set $q_1 := p$, and set $q_2 := \varepsilon \cdot p$. Let $A := E(q_1)$ be the Tate elliptic curve over $\mathbb{Q}_p$ with respect to the lattice $q_1^{\mathbb{Z}} \subset \mathbb{Q}_p^\times$. First, we set

$$V'_B := \begin{pmatrix} q_1 & 1 \\ 1 & q_2 \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p^\times)$$

and note that abeloid surface $B'$ over $\mathbb{Q}_p$ associated to the matrix $V'_B$ is isomorphic to the product of the two Tate elliptic curves $E(q_1) \times E(q_2)$.
For $v_1, v_2 \in \mathbb{Z}_p$, we define

$$S := \mathrm{Id}_{2\times 2} - 2 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 - 2v_1^2 & -2v_1v_2 \\ -2v_1v_2 & 1 - 2v_2^2 \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}_p).$$

Clearly, $S$ is symmetric and if $v_1^2+v_2^2=1$, which we will assume from now on, then $S^{-1}=S^t$ and thus, we find

$$S \;=\; \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \text{with} \qquad a := 1-2v_1^2, \qquad b := -2v_1v_2,$$

and note that we have $a^2+b^2=1$ (one should think of this matrix as the analog of an orthogonal matrix over $\mathbb{R}$ that describes the reflexion along the axis spanned by $(v_1\, v_2)$). We set

$$V_B \;:=\; S^{-1}\odot V_B'\odot S \;\in\; \mathrm{Mat}_{2\times 2}(\widehat{\mathbb{Q}_p^\times}^p),$$

(here $\widehat{\mathbb{Q}_p^\times}^p$ denotes the $p$-adic completion $\gamma_p$ from Lemma 4.9), which is equal to

$$V_B \;=\; \begin{pmatrix} q_1^{a^2}\cdot q_2^{b^2} & q_1^{ab}\cdot q_2^{-ab} \\ q_1^{ab}\cdot q_2^{-ab} & q_1^{b^2}\cdot q_2^{a^2} \end{pmatrix} \;=\; \begin{pmatrix} \varepsilon^{b^2}\cdot p & \varepsilon^{-ab} \\ \varepsilon^{-ab} & \varepsilon^{a^2}\cdot p \end{pmatrix}.$$

In particular, this matrix has actually coefficients in $\mathbb{Q}_p^\times$ rather than merely in $\widehat{\mathbb{Q}_p^\times}$, see also Remark 4.10. Moreover, the valuation $\nu_p:\mathbb{Q}_p^\times\to\mathbb{Q}$ sends $V_B$ to the identity matrix. Since $V_B$ is a symmetric matrix and since it is definite with respect to the $\mathbb{Q}$-linear functional $\nu_p$, it is a Riemann matrix in the sense of Gerritzen [Ge71]. We let $B$ be the abeloid surface over $\mathbb{Q}_p$ associated to $V_B$. By [Ge71, Theorem 11], this surface is algebraisable, that is, $B$ is an abelian surface over $\mathbb{Q}_p$.

In order to determine $\mathrm{Hom}(A,B)$ and $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(T_p(A),T_p(B))$, we are looking at the equation

$$\begin{pmatrix} q_1 \end{pmatrix}\odot\begin{pmatrix} x & y \end{pmatrix} \;=\; \begin{pmatrix} p^x & p^y \end{pmatrix} \;\overset{!}{=}\; \begin{pmatrix} x' & y' \end{pmatrix}\odot V_B \;=\; \begin{pmatrix} \varepsilon^{x'b^2-y'ab}\cdot p^{x'} & \varepsilon^{y'a^2-x'ab}\cdot p^{y'} \end{pmatrix}.$$

Taking valuations, we find $x=x'$ and $y=y'$. To avoid trivialities, we will also assume that $a\neq 0$ and $b\neq 0$, which leads to the general solution

$$y \;=\; \frac{b}{a}\cdot x\,.$$

Since there is always the $p$-adic solution $x=a$, $y=b$, Proposition 4.11 implies that $\mathrm{Hom}(V_p(A),V_p(B))$ is non-zero and in fact, isomorphic to $\mathbb{Q}_p$. On the other hand, Theorem 4.7 implies that $\mathrm{Hom}(A,B)\otimes\mathbb{Q}$ is non-zero if and only we can find a solution with $x\in\mathbb{Q}$ and $y\in\mathbb{Q}$. Thus, $\mathrm{Hom}(A,B)\otimes\mathbb{Q}$ is non-zero if and only if $a/b\in\mathbb{Q}$.

In order to establish the first claim, we have to show that we can find $v_1,v_2\in\mathbb{Z}_p$ that satisfy all the restrictions we made during the previous discussion and such that $a/b=(2v_1^2-1)/(2v_1v_2)$ does not lie in $\mathbb{Q}$. If $v_1=2$, then $v_2=\sqrt{-3}$ does not lie in $\mathbb{Q}$, but it is an element of $\mathbb{Z}_p$ if $p\geq 5$ and $p\equiv 1 \mod 3$ (the last statement easily follows from the law of quadratic reciprocity).

Similarly, to compute the endomorphism algebras $\mathrm{End}(B) \otimes \mathbb{Q}$ and $\mathrm{End}_{G_{\mathbb{Q}_p}}(V_p(B))$, we have to solve the equation

$$V_B \odot C = C' \odot V_B$$

for $2 \times 2$-matrices $C$ and $C'$ with entries in $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively. Taking valuations, we find $C = C'$. Moreover, if $C = (c_{ij})_{1 \leq i,j \leq 2}$, then we leave it to the reader to show that the general solution is given by

$$c_{12} = c_{21} \quad \text{and} \quad c_{22} - c_{11} = c_{12} \cdot \frac{b^2 - a^2}{ab}.$$

We thus always have the solution $c_{12} = c_{21} = 0$ and $c_{11} = c_{22}$, that is, multiplication by a scalar. If $(b^2 - a^2)/(ab)$ does not lie in $\mathbb{Q}$ (which is the case if $v_1 = 2$ and $v_2 = \sqrt{-3}$), then these are the only solutions in $\mathbb{Q}$, that is, $\mathrm{End}(B) \otimes \mathbb{Q} \cong \mathbb{Q}$. On the other hand, the above equation always has more solutions in $\mathbb{Q}_p$, that is, $\mathrm{End}(V_p(B))$ is strictly larger than $\mathbb{Q}_p$.
These computations establish the first claim. The second claim follows from the first claim by the same arguments as in the proof of Proposition 5.4. □

Remark 6.3. The restrictions on the prime $p$ in Theorem 6.1 are artificial in that they are only used to state a clean counterexample. The method of proof should give counterexamples to Conjecture 1.1 and Question 1.2 for any prime $p$.

## A Further (counter-)examples

So far, we have studied the surjectivity of the maps ($\star$) and ($\star\star$) (defined in §1) in the case where $\ell = p$ and where the varieties in question are smooth and proper over a $p$-adic field with totally degenerate reduction. If $\ell \neq p$ or if the variety has good reduction, then it was already more or less well-known to the experts that one cannot hope for such surjectivity results, but for the sake of completing the picture, we have decided to collect some examples. As a byproduct, we see that also "independence of $\ell$" fails. In this section, we claim only little originality.

### A.1 Good reduction and $\ell \neq p$

Let $X$ be a smooth and proper variety over a $p$-adic field $K$ that has good reduction, say, via a smooth and proper model $\mathcal{X} \to \mathrm{Spec}\, \mathcal{O}_K$ with special fibre $\mathcal{X}_0$ over the residue field $k$. By base-change in étale cohomology, the $G_K$-action on $H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified and factors through the $G_k$-action on $H^*_{\text{ét}}(\mathcal{X}_{0,\overline{k}}, \mathbb{Q}_\ell)$.

1. In particular, we have

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K} = H^2_{\text{ét}}(\mathcal{X}_{0,\overline{k}}, \mathbb{Q}_\ell(1))^{G_k}$$

and if $\mathcal{X}_0$ satisfies the Tate conjecture for divisors over the finite field $k$, then these spaces are isomorphic to $\mathrm{Pic}(\mathcal{X}_0) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$.

2. Let $A$ and $B$ be abelian varieties with good reduction over $K$ and let let $\mathcal{A}_0$ and $\mathcal{B}_0$ be the special fibres of their Néron models. In this case, base-change implies that

$$\mathrm{Hom}_{G_K}(T_\ell(A), T_\ell(B)) = \mathrm{Hom}_{G_k}(T_\ell(\mathcal{A}_0), T_\ell(\mathcal{B}_0)),$$

which is isomorphic to $\mathrm{Hom}(\mathcal{A}_0, \mathcal{B}_0) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ by Tate's theorem [Ta66].

In particular, the right hand sides of ($\star$) and ($\star\star$) compute invariants of the special fibre, see also Remark 3.2. After these preparations, it is easy to give the desired counter-examples - in fact, "almost all" elliptic curves provide counter-examples.

PROPOSITION A.1. *Let $K$ be a $p$-adic field, let $E$ be an elliptic curve over $K$ with good reduction, and let $\mathcal{E}_0$ be the special fibre of its Néron model. If $\mathcal{E}_0$ is supersingular or if $\mathcal{E}_0$ is ordinary and $E$ does not have CM, then ($\star$) is not surjective for $X = E \times E$ and $\ell \neq p$ and ($\star\star$) is not surjective for $A = B = E$ and $\ell \neq p$.*

PROOF. The first claim follows from the second claim by the same arguments as in the proof of Proposition 5.4. Therefore, it suffices to show that the natural map $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{End}_{G_K}(V_\ell(E))$ is not surjective.
First, assume that $\mathcal{E}_0$ is supersingular. Then, $\mathrm{End}(\mathcal{E}_0)$ is an order in a quaternion algebra and thus, $\mathrm{End}(\mathcal{E}_0) \otimes \mathbb{Q}_\ell$ is 4-dimensional. Hence, $\mathrm{End}_{G_K}(V_\ell(E))$, which is isomorphic to $\mathrm{End}_{G_k}(V_\ell(\mathcal{E}_0))$, is 4-dimensional by Tate's theorem [Ta66, Main Theorem]. On the other hand, $\mathrm{End}(E)$ is isomorphic to $\mathbb{Z}$ or to an order in a quadratic imaginary field, which implies that $\mathrm{End}(E) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{G_K}(V_\ell(E))$ cannot be surjective.
Similarly, if $E$ does not have CM, then $\mathrm{End}(E) \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell$. Moreover, if $\mathcal{E}_0$ is ordinary, then $\mathrm{End}_{G_k}(\mathcal{E}_0)$ is an order in a quadratic imaginary field and $\mathrm{End}_{G_k}(\mathcal{E}_0) \otimes \mathbb{Q}_\ell$ is 2-dimensional. □

### A.2 GOOD REDUCTION AND $\ell = p$

Next, we show that Conjecture 1.1 and Question 1.2 have a negative answer if $\ell = p$ and in the case of good reduction.

PROPOSITION A.2. *Let $K$ be a $p$-adic field and let $\mathcal{E}_0$ be an ordinary elliptic curve over $k$.*

1. *Let $A$ be a lift of $\mathcal{E}_0$ over $K$ with CM, for example, the canonical lift, and*
2. *let $B$ be a lift of $\mathcal{E}_0$ over $K$ without CM.*

*Then, ($\star$) is not surjective for $X = A \times B$ and $\ell = p$ and ($\star\star$) is not surjective for $A$, $B$, and $\ell = p$.*

PROOF. The first claim follows from the second claim by the same arguments as in the proof of Proposition 5.4. Therefore, it suffices to show that the natural map $\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_p \to \mathrm{Hom}_{G_K}(V_p(A),V_p(B))$ is not surjective.
Since $A$ cannot be isogenous to $B$, the source $\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_p$ is trivial. Next, for $E\in\{A,B\}$ there exists a short exact sequence of $p$-adic $G_K$-representations

$$0 \to X \to V_p(E) \to Y \to 0. \tag{14}$$

More precisely, $X$ corresponds to the Tate module associated to the connected component of the $p$-divisible group $\mathcal{E}_0[p^\infty]$ and $Y$ corresponds to the Tate module associated to the étale quotient. In particular, the $G_K$-representations $X$ and $Y$ only depend on $\mathcal{E}_0$ and not on the choice of lift $E$. Moreover, the sequence of $G_K$-representations (14) splits if and only if the lift of $\mathcal{E}_0$ has CM, that is, it splits for $A$ but not for $B$. We refer to [Se68, Appendix A.2.4] for details and proof. But this implies that the target $\mathrm{Hom}_{G_K}(V_p(A),V_p(B))$ is non-trivial: taking the monomorphism $X\to V_p(B)$ from (14) and the zero map $Y\to V_p(B)$, we obtain a non-trivial and $G_K$-equivariant map

$$V_p(A) \;=\; X\oplus Y \;\to\; V_p(B).$$

This establishes the second claim. □

REMARK A.3. In [LT66, §3.5], Lubin and Tate constructed elliptic curves $E$ over $p$-adic fields having good and supersingular reduction such that the the monomorphism

$$\mathrm{End}(E)\otimes_{\mathbb{Z}}\mathbb{Z}_p \hookrightarrow \mathrm{End}_{G_K}(T_p(E))$$

is not surjective. Therefore, $(\star)$ is not surjective for $X=E\times E$ and $\ell=p$ and $(\star\star)$ is not surjective for $A=E$, $B=A$, and $\ell=p$.

### A.3 TOTALLY DEGENERATE REDUCTION AND $\ell\neq p$

Now, we show that Proposition 5.3 and Corollary 5.4 fail if $\ell\neq p$.

PROPOSITION A.4. *For every prime $p$, there exist Tate elliptic curves $A$ and $B$ over $\mathbb{Q}_p$, such that $(\star)$ is not surjective for $X=A\times B$ and $\ell\neq p$ and $(\star\star)$ is not surjective for $A$, $B$, and $\ell\neq p$.*

PROOF. Let $\varepsilon\in 1+p\cdot\mathbb{Z}_p$ be a non-trivial $p$-adic unit and let $A:=E(p)$ and $B:=E(\varepsilon\cdot p)$ be the Tate elliptic curves associated to $p\in\mathbb{Q}_p^\times$ and $\varepsilon\cdot p\in\mathbb{Q}_p^\times$. Then, $A$ and $B$ are not isogenous by Serre's criterion (Theorem 5.1.(2)) and thus, $\mathrm{Hom}(A,B)=0$.
We let $\gamma_\ell$ be the $\ell$-adic completion from Lemma 4.9. By Proposition 4.11, we have

$$\mathrm{Hom}_{G_K}(T_\ell(A),T_\ell(B)) \;\cong\; \{m\in\mathbb{Z}_\ell \,|\, \exists n\in\mathbb{Z}_\ell \,:\, \gamma_\ell(p)^m=\gamma_\ell(\varepsilon\cdot p)^n\}\;.$$

By Lemma 4.9 and Remark 4.10, we have $\gamma_\ell(\varepsilon)=1$, that is, we have $\gamma_\ell(p)=\gamma_\ell(\varepsilon\cdot p)$, which implies that $\mathrm{Hom}_{G_K}(T_\ell(A),T_\ell(B))$ is non-zero (in fact, isomorphic to $\mathbb{Z}_\ell$). □

### A.4 Étale fundamental groups and all primes at once

If $A$ is an abelian variety of dimension $g$ over a field characteristic zero field $F$, then there exists an isomorphism of $G_F$-representations

$$\pi_1^{\text{ét}}(A_{\overline{F}}) \cong \prod_\ell T_\ell(A),$$

where the product is taken over all primes. As an abelian group, this is a free $\widehat{\mathbb{Z}}$-module of rank $2g$. Now, if $A$ and $B$ are abelian varieties over $F$, then one may ask whether the natural map

$$\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}} \to \mathrm{Hom}_{G_F}\left(\pi_1^{\text{ét}}(A_{\overline{F}}),\, \pi_1^{\text{ét}}(B_{\overline{F}})\right) \qquad (\star\star\star)$$

is an isomorphism.

Proposition A.5. *For every prime $p$, there exist Tate elliptic curves $A$ and $B$ over $\mathbb{Q}_p$, such that $(\star\star\star)$ is not surjective.*

Proof. Fix a prime $\ell_0 \neq p$ and let $A$ and $B$ be counterexamples as provided by Proposition A.4, that is, $(\star\star)$ is not surjective for $A$, $B$, and $\ell_0$. Since $(\star\star\star)$ factors through

$$\prod_\ell \mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to \prod_\ell \mathrm{Hom}_{G_F}(T_\ell(A),T_\ell(B))$$

and since this map is not surjective at the factor corresponding to $\ell_0$, the claim follows. □

### A.5 Independence of $\ell$

From the previous computations, we conclude that also "independence of $\ell$" fails in the $p$-adic world, see also Remark 3.2 and the subsequent discussion.

Proposition A.6. *For every prime $p$, there exist Tate elliptic curves $A$ and $B$ over $\mathbb{Q}_p$, such that*

$$\dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}((A\times B)_{\overline{\mathbb{Q}}_p},\mathbb{Q}_\ell(1))^{G_{\mathbb{Q}_p}} = \begin{cases} 2 & \text{if } \ell = p \\ 3 & \text{if } \ell \neq p \end{cases}.$$

*In particular, this dimension depends on the prime $\ell$.*

Proof. Let $A$ and $B$ the Tate elliptic curves from the proof of Proposition A.4. There, we have seen that $A$ and $B$ are not isogenous and that $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_\ell(A),V_\ell(B))$ is one-dimensional if $\ell\neq p$. On the other hand, since $A$ and $B$ are not isogenous, $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_p(A),V_p(B))$ is zero by Proposition 5.3. The arguments from the proof of Proposition 5.4 show that the sought $\mathbb{Q}_\ell$-dimensions are equal to $2+\dim_{\mathbb{Q}_\ell}\mathrm{Hom}_{G_K}(V_\ell(A),V_\ell(B))$ and the claim follows. □

References

[An96] Y. André, *On the Shafarevich and Tate conjectures for hyper-Kähler varieties*, Math. Ann. 305 (1996), no. 2, 205–248.

[An04] Y. André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses 17, Société Mathématique de France, 2004.

[Be04] L. Berger, *An introduction to the theory of p-adic representations*, Geometric aspects of Dwork theory, Vol. I, II, 255–292, Walter de Gruyter (2004).

[BO83] P. Berthelot, A. Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. 72 (1983), no. 2, 159–199.

[Be97] P. Berthelot, *Finitude et pureté cohomologique en cohomologie rigide*, Invent. Math. 128 (1997), no. 2, 329–377.

[BdS16] A. Besser, E. de Shalit, *$\mathcal{L}$-invariants of p-adically uniformized varieties*, Ann. Math. Qué. 40 (2016), no. 1, 29–54.

[BEK14] S. Bloch, H. Esnault, M. Kerz, *p-adic deformation of algebraic cycle classes*, Invent. Math. 195 (2014), no. 3, 673– 722.

[BGS97] S. Bloch, H. Gillet, C. Soulé, *Algebraic cycles on degenerate fibers*, Arithmetic geometry (Cortona, 1994), 45–69, Sympos. Math., XXXVII, Cambridge University Press (1997).

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer (1990).

[Ch13] F. Charles, *The Tate conjecture for K3 surfaces over finite fields*, Invent. Math. 194 (2013), no. 1, 119–145.

[CL16] B. Chiarellotto, C. Lazda, *Combinatorial degenerations of surfaces and Calabi-Yau threefolds*, Algebra Number Theory 10 (2016), no. 10, 2235–2266.

[CT14] B. Chiarellotto, N. Tsuzuki, *Clemens-Schmid exact sequence in characteristic p*, Math. Ann. 358 (2014), no. 3-4, 971–1004.

[Co00] R. Coleman, *The monodromy pairing*, Asian J. Math. 4 (2000), no. 2, 315–330.

[CI99] R. Coleman, A. Iovita, *The Frobenius and monodromy operators for curves and abelian varieties*, Duke Math. J. 97 (1999), no. 1, 171–215.

[CF00] P. Colmez, J.-M. Fontaine, *Construction des represéntations p-adiques semi-stables*, Invent. Math. 140 (2000), no. 1, 1–43.

[CN17] P. Colmez, W. Nizioł, *Syntomic complexes and p-adic nearby cycles*, Invent. Math. 208 (2017), no. 1, 1–108.

[Co98] C. Consani, *Double complexes and Euler L-factors*, Compositio Math. 111 (1998), no. 3, 323–358.

[DT08] S. Dasgupta, J. Teitelbaum, *The p-adic upper half plane*, p-adic geometry, 65–121, Univ. Lecture Series 45, Amer. Math. Soc. 2008.

[De81] P. Deligne, *Cristaux ordinaires et coordonnées canoniques*, With the collaboration of L. Illusie. With an appendix by Nicholas M. Katz. Lecture Notes in Math., 868, Algebraic surfaces (Orsay, 1976-78), 80–137, Springer, Berlin-New York, 1981.

[Fa83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73 (1983), no. 3, 349–366.

[Fo94] J.-M. Fontaine, *Représentations p-adiques semi-stables*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque no. 223 (1994), 113–184.

[Ge70] L. Gerritzen, *Über Endomorphismen nichtarchimedischer holomorpher Tori*, Invent. Math. 11 (1970), 27–36.

[Ge71] L. Gerritzen, *On multiplication algebras of Riemann matrices*, Math. Ann. 194 (1971), 109–122.

[GvdP80] L. Gerritzen, M. van der Put, *Schottky Groups and Mumford Curves*, Springer (1980).

[Gro00] M. Gros, *Sur les $(K_0, \phi, N)$-structures attachées aux courbes de Mumford*, Rend. Sem. Mat. Univ. Padova 103 (2000), 233–260.

[Gro72] A. Grothendieck, *Modèles de Néron et monodromie*, Groupes de monodromie en géometrie algébrique, SGA 7 I, Lecture Notes in Math. 288, 313–523, Springer (1972).

[HK94] O. Hyodo, K. Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque No. 223 (1994), 221–268.

[I90] L. Illusie, *Ordinarité des intersections complètes générales*, The Grothendieck Festschrift, Vol. II, 376–405, Progr. Math., 87, Birkhäuser Boston 1990.

[I94] L. Illusie, *Réduction semi-stable ordinaire, cohomologie étale p-adique et cohomologie de de Rham d'après Bloch-Kato et Hyodo*, appendix to [PR94].

[It05] T. Ito, *Weight-monodromy conjecture for $p$-adically uniformized varieties*, Invent. Math. 159 (2005), no. 3, 607–656.

[Ja10] U. Jannsen, *Weights in arithmetic geometry*, Jpn. J. Math. 5 (2010), no. 1, 73–102.

[Kad07] S. Kadziela, *Rigid analytic uniformization of curves and the study of isogenies*, Acta Appl. Math. 99 (2007), no. 2, 185–204.

[KM74] N. Katz, W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. 23 (1974), 73–77.

[KM16] W. Kim, K. Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma 4 (2016), e28, 34 pp.

[KS67] M. Kuga, I. Satake, *Abelian varieties attached to polarized $K_3$-surfaces*, Math. Ann. 169 (1967), 239–242.

[Kün98] K. Künnemann, *Projective regular models for abelian varieties, semistable reduction, and the height pairing*, Duke Math. J. 95 (1998), no. 1, 161–212.

[Ku80] A. Kurihara, *Construction of $p$-adic unit balls and the Hirzebruch proportionality*, Am. J. Math. 102, 565–648 (1980).

[LP17] C. Lazda, A. Pál, *Cycle classes in overconvergent rigid cohomology and a semistable Lefschetz* $(1,1)$ *theorem*, Compositio Math. 155 (2019), 1025–1045.

[LeS95] B. Le Stum, *La structure de Hyodo-Kato pour les courbes*, Rend. Sem. Mat. Univ. Padova 94 (1995), 279–301.

[LT66] J. Lubin, J. Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France 94 (1966), 49–59.

[Lüt16] W. Lütkebohmert, *Rigid geometry of curves and their Jacobians*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 61. Springer (2016).

[MP15] K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. 201 (2015), no. 2, 625–668.

[Ma57] T. Matsusaka, *The criteria for algebraic equivalence and the torsion group*, Amer. J. Math. 79 (1957), 53–66.

[Ma14] D. Maulik, *Supersingular K3 surfaces for large primes*, With an appendix by Andrew Snowden, Duke Math. J. 163 (2014), no. 13, 2357–2425.

[MTT86] B. Mazur, J. Tate, J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. 84 (1986), no. 1, 1–48.

[Mo93] F. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. 72 (1993), no. 2, 301–337.

[Mo17] B. Moonen, *On the Tate and Mumford-Tate conjectures in codimension* 1 *for varieties with* $h^{2,0} = 1$, Duke Math. J. 166 (2017), no. 4, 739–799.

[Mo84] D. Morrison, *The Clemens-Schmid exact sequence and applications*, Topics in transcendental algebraic geometry, 101–119, Ann. of Math. Stud. 106, Princeton Univ. Press (1984).

[Mo14] M. Morrow, *A Variational Tate Conjecture in crystalline cohomology*, arXiv:1408.6783 (2014).

[Mu70] D. Mumford, *Abelian varieties*, Oxford University Press (1970).

[Mum72b] D. Mumford, *An analytic construction of degenerating abelian varieties over complete local rings*, Compositio Math. 24 (1972), 239–272.

[Mus78] G. A. Mustafin, *Non-Archimedean uniformization*, Mat. Sb. (N.S.) 105 (147) (1978), no. 2, 207–237, 287.

[Na05] Y. Nakkajima, *p-adic weight spectral sequences of log varieties*, J. Math. Sci. Univ. Tokyo 12 (2005), no. 4, 513–661.

[Ny83] N. O. Nygaard, *The Tate conjecture for ordinary K3 surfaces over finite fields*, Invent. Math. 74 (1983), 213–237.

[NO85] N. O. Nygaard, A. Ogus, *Tate's conjecture for K3 surfaces of finite height*, Ann. Math. 122 (1985), 461–507.

[PR94] B. Perrin-Riou, *Représentations p-adiques ordinaires*. With an appendix by Luc Illusie. Périodes $p$-adiques (Bures-sur-Yvette, 1988), Astérisque no. 223 (1994), 185–220.

[Pe77] U. Persson, *On degenerations of algebraic surfaces*, Mem. Amer. Math. Soc. 11, no. 189 (1977).

[Pe03] D. Petrequin, *Classes de Chern et classes de cycles en cohomologie rigide*, Bull. Soc. Math. France 131 (2003), no. 1, 59–121.

[Ra05] W. Raskind, *A generalized Hodge-Tate conjecture for algebraic varieties with totally degenerate reduction over p-adic fields*, Algebra and number theory, 99–115, Hindustan Book Agency, Delhi, 2005.

[RX07a] W. Raskind, X. Xarles, *On the étale cohomology of algebraic varieties with totally degenerate reduction over p-adic fields*, J. Math. Sci. Univ. Tokyo 14 (2007), no. 2, 261–291.

[RX07b] W. Raskind, X. Xarles, *On p-adic intermediate Jacobians*, Trans. Amer. Math. Soc. 359 (2007), no. 12, 6057–6077.

[SS91] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces*, Invent. Math. 105 (1991), no. 1, 47–122.

[Se68] J.-P. Serre, *Abelian ℓ-Adic Representations and Elliptic Curves*, W. A. Benjamin, Inc., New York–Amsterdam (1968).

[Ta81] S. G. Tankeev, *Algebraic cycles on surfaces and abelian varieties*, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 2, 398–434, 463–464.

[Ta89] S. G. Tankeev, *Surfaces of type K3 over number fields and ℓ-adic representations*, Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), no. 6, 1252–1271, 1328; translation in Math. USSR-Izv. 33 (1989), no. 3, 575–595

[Ta65] J. T. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), 93–110, Harper and Row (1965).

[Ta66] J. T. Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. 2 (1966), 134–144.

[Ta94] J. T. Tate, *Conjectures on algebraic cycles in ℓ-adic cohomology*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. 55, 71–83, Amer. Math. Soc. (1994).

[To17] B. Totaro, *Recent progress on the Tate conjecture*, Bull. Amer. Math. Soc. (N.S.) 54 (2017), no. 4, 575–590.

[Ts99] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. 137 (1999), no. 2, 233–411.

[Ts03] N. Tsuzuki, *Cohomological descent of rigid cohomology for proper coverings*, Invent. Math. 151 (2003), no. 1, 101–133.

[Ya11] G. Yamashita, *The p-adic Lefschetz* $(1,1)$ *theorem in the semistable case, and the Picard number jumping locus*, Math. Res. Lett. 18 (2011), no. 1, 107–124.

[Zar75] Ju. Zarhin, *Endomorphisms of Abelian varieties over fields of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 2, 272–277, 471.

Oliver Gregory and Christian Liedtke
TU München
Zentrum Mathematik - M11
Boltzmannstr. 3
85748 Garching bei München
Germany
gregory@ma.tum.de
liedtke@ma.tum.de