

COISOTROPIC TRIPLES, REDUCTION AND CLASSICAL LIMIT

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ABSTRACT. Coisotropic reduction from Poisson geometry and deformation quantization is cast into a general and unifying algebraic framework: we introduce the notion of coisotropic triples of algebras for which a reduction can be defined. This allows to construct also a notion of bimodules for such triples leading to bicategories of bimodules for which we have a reduction functor as well. Morita equivalence of coisotropic triples of algebras is defined as isomorphism in the ambient bicategory and characterized explicitly. Finally, we investigate the classical limit of coisotropic triples of algebras and their bimodules and show that classical limit commutes with reduction in the bicategory sense.

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1 INTRODUCTION

Coisotropic reduction is one of the standard constructions in Poisson geometry leading to a new reduced Poisson manifold obtained out of the given data of a Poisson manifold with a coisotropic submanifold. Of course, geometrically certain circumstances have to be met in order to obtain a smooth reduced Poisson manifold. Ignoring these geometric assumptions, an algebraic formulation of coisotropic reduction is possible and works in general, yielding a reduced Poisson algebra out of a given Poisson algebra with a coisotropic ideal: here an associative ideal in a Poisson algebra is called coisotropic if it is a Poisson subalgebra, though not necessarily a Poisson ideal.

The original motivation to consider coisotropic submanifolds and the corresponding reduction comes from Dirac's program [22, 23] to handle constraint mechanical systems: the notion of a coisotropic submanifold corresponds to the first-class constraints. Dirac's intention ultimately was of course to pass

to a quantum theory. This leads to the task to find a quantized version of coisotropic reduction as well.

Among many approaches one can favour deformation quantization [1] as starting point. Here various versions of phase space reduction are available, starting with a BRST approach in [7] and more general coisotropic reduction schemes found in e.g. [18, 16, 17, 6, 5, 4]. The general idea is that the functions vanishing on the coisotropic submanifold should be deformed into a *left ideal* of the ambient algebra of all functions. The reduced algebra is then the quotient of the normalizer of this left ideal modulo the left ideal. Also, it is worthwhile to mention that the quantization of coisotropic subgroups has been considered in the context of quantum groups, see e.g [19, 21].

Since both versions, the classical reduction as well as the quantum reduction, can be formulated entirely in an algebraic fashion it is reasonable to explore the algebraic features further, independent of the possible geometric origin. Then one important question is how the relations between different algebras with coisotropic ideals behave after reduction. A standard question beyond the behaviour of isomorphisms is then the behaviour of Morita equivalence.

Thus the first main question we want to address is how Morita equivalences between reduced algebras can be encoded in the data before reduction.

The main idea to approach this is to put Morita theory in a slightly larger context of an appropriate bicategory: for algebras (or rings) it is a well-known procedure that the bicategory of all bimodules \mathbf{Bimod} encodes Morita equivalence as the notion of isomorphism in the bicategorical sense. However, now one has much more structure as also bimodules between algebras enter the game which are not necessarily equivalence bimodules: they can carry important information themselves.

Thus our first step is to construct a bicategory for the situation before reduction which allows for a functorial reduction. It turns out that the first idea of algebras \mathcal{A}_{tot} with a specified left ideal \mathcal{A}_0 are not yet the suitable notion of objects in this bicategory. One simply can not define a reasonable notion of bimodules over such pairs that is compatible with reduction. Thus our approach consists in taking triples of an ambient algebra: the total algebra \mathcal{A}_{tot} , a subalgebra $\mathcal{A}_{\mathcal{N}}$ of weakly observables in Dirac's original sense, and a two-sided ideal \mathcal{A}_0 in this subalgebra, corresponding to the left ideal from before. The idea in mind is that starting with an algebra with left ideal one has to add the normalizer of this left ideal as algebra in the middle. Nevertheless, we give classes of interesting examples where one needs additional freedom to choose this algebra in the middle, thus justifying to consider what we call *coisotropic triples* of algebras $\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_{\mathcal{N}}, \mathcal{A}_0)$ in the following. We then define a notion of bimodules over such triples allowing for a good tensor product: this ultimately leads us to the *bicategory of coisotropic triples* $\mathbf{C}_3\mathbf{Bimod}$ where the 1-morphisms are bimodules over coisotropic triples of algebras as objects with an appropriate notion of bimodule morphisms as 2-morphisms.

Having this bicategory, it allows now to speak of Morita equivalence of coisotropic triples of algebras which, by definition, is isomorphism in the sense

of $\mathbf{C}_3\mathbf{Bimod}$. As a first result we give an explicit characterization of Morita equivalent triples in Theorem 5.5: it implies Morita equivalence of the corresponding tot- and N-components together with a compatibility condition between the three components of the triples.

The second step consists now in extending the reduction of algebras to bimodules. We show in Theorem 6.1 that this is indeed possible and leads to a reduction functor

$$\text{red}: \mathbf{C}_3\mathbf{Bimod} \longrightarrow \mathbf{Bimod}$$

in the bicategory sense. Since we have an honest functor between bicategories, the reduction maps isomorphic objects to isomorphic objects and hence preserves Morita equivalence being the notion of isomorphism in $\mathbf{C}_3\mathbf{Bimod}$ and \mathbf{Bimod} , respectively. However, being a functor we get much more detailed information from reduction, say a bigroupoid morphism of the corresponding Picard bigroupoids, i.e. the bigroupoids of isomorphisms in these bicategories. For technical reasons it is convenient to consider only the components $(\mathcal{A}_N, \mathcal{A}_0)$ of a coisotropic triple of algebras leading to the notion of a coisotropic pair of algebras: the reduction uses only this information. Now the construction of $\mathbf{C}_3\mathbf{Bimod}$ can be adjusted to yield also a bicategory $\mathbf{C}_2\mathbf{Bimod}$ of bimodules over such pairs together with the corresponding reduction functor

$$\text{red}: \mathbf{C}_2\mathbf{Bimod} \longrightarrow \mathbf{Bimod}.$$

As we have seen, one of the main motivations to consider coisotropic reduction is to pass from a classical to a quantum system and use the classical data to investigate the reduced quantum system. Thus, in a last step, we consider general deformations of coisotropic triples of algebras and their bimodules. While a *quantization* of bimodules is typically obstructed, not unique, and fairly difficult to understand in general, the *classical limit* is always rather easy to study and unobstructed. We define a classical limit of coisotropic triples of algebras over a ring $\mathbb{R}[[\lambda]]$ of formal power series in a formal parameter λ with coefficients in a ring \mathbb{R} as the quotient by the ideals of multiples of λ . The idea is that the algebras over $\mathbb{R}[[\lambda]]$ are interpreted as deformations of algebras over \mathbb{R} . While the classical limit of the algebras is rather straightforward, we then are able to extend the classical limit also to bimodules leading to a functor

$$\text{cl}: \mathbf{C}_3\mathbf{Bimod}_{\mathbb{R}[[\lambda]]} \longrightarrow \mathbf{C}_3\mathbf{Bimod}_{\mathbb{R}}$$

of bicategories where now we explicitly indicate the underlying ring of scalars. As before, we know that the classical limit preserves Morita equivalence and yields a bigroupoid morphism between the Picard bigroupoids. In e.g. [12, 13] it was demonstrated that a similar classical limit can be successfully used to determine the Picard groups of deformed algebras and thus their Morita theory. The final result is now that the two functors *red* and *cl* commute in the sense of functors between bicategories: we explicitly construct the relevant natural

transformations and modifications in Theorem 7.13 to obtain a commutative diagram

$$\begin{array}{ccc}
 \mathbf{C}_2\mathbf{Bimod}_R[[\lambda]] & \xrightarrow{\text{cl}} & \mathbf{C}_2\mathbf{Bimod}_R \\
 \text{red} \downarrow & & \downarrow \text{red} \\
 \mathbf{Bimod}_R[[\lambda]] & \xrightarrow{\text{cl}} & \mathbf{Bimod}_R
 \end{array}$$

of functors between bicategories. Here it suffices to restrict to coisotropic pairs instead of triples since the reduction only uses the information of pairs anyway. In particular, the functors restrict to commuting bigroupoid morphisms for the corresponding Picard bigroupoids thus encoding the behaviour of Morita equivalence under classical limit and reduction completely.

After arriving at this conceptually clear and fairly general picture of how coisotropic reduction extends to bimodules and relates to the classical limit several questions arise. We do not address their solutions in this work but come back to them later on.

1. The question of *quantization* of coisotropic algebras and their modules now becomes more urgent, once having understood their classical limit. Here a question of particular interest is to understand the quantization of equivalence bimodules provided the quantization of the algebras is given. One can then use commutativity of reduction and classical limit to actually find a good classification of coisotropic triples of e.g. star product algebras in geometric terms like the characteristic classes of the underlying star products. This should eventually lead to a comparison of the Morita classification of equivariant star products initiated in [31, 30], see also [38, 39, 25], extending the Morita classification from [15, 11, 12]. On the classical level, coisotropic relations provide particular coisotropic triples which can then also be taken as starting point for quantization [20].
2. The geometric nature of the description of the equivalence bimodules from Theorem 5.5 has to be clarified further. Here the case of star products is again the guiding example and raises the questions what the *semi-classical* limit is: the first order structures of equivalence bimodules should give analogs of covariant (or contravariant) derivatives, now adapted to the coisotropic triple point of view analogously to the ordinary case [12, 8].
3. Already on the classical level one can try to incorporate the first order structures, i.e. the Poisson structures, into the game at a more fundamental level. Then a more geometric approach to Morita theory in this context could ultimately lead to a notion of Morita equivalence of coisotropic triples in Poisson geometry yielding the usual Morita equivalence for the reduced Poisson manifolds, see [41]. Then the question of the behaviour of Picard groups as studied in [9, 14] under reduction would be one of the first tasks.

4. We know that the reduction functor maps equivalence bimodules between the triples to usual equivalence bimodules between the reduced algebras. Which other classes of (bi-)modules behave well under reduction? Here we want to find suitable criteria to obtain e.g. projective modules etc. making contact to the geometric framework of reducing vector bundles.
5. A final longer term project is to incorporate $*$ -involutions into the definition of coisotropic triples. For applications in mathematical physics this is of course crucial but requires some severe changes: the main obstacle is that there is no reasonable compatibility to require between a left ideal and a $*$ -involution. The naive compatibility that the left ideal is closed under the involution yields immediately a two-sided ideal which in the examples of deformation quantization is known to be not relevant at all. One idea would be to require the existence of a positive functional having the left ideal as Gel'fand ideal as this was done in [29] and induce a $*$ -involution for the reduced algebra this way. Nevertheless, at the moment it seems to be quite unclear how to incorporate the corresponding structures like algebra-valued inner products on the modules as in [13, 10]. Ultimately, one would like to have a definition for *strong Morita equivalence* of coisotropic triples of algebras.

The paper is organized as follows: in Section 2 we recall the basic constructions of coisotropic reduction together with some principal examples. Section 3 contains the definition of coisotropic triples and pairs of algebras together with some first functorial properties. The bicategories of bimodules over coisotropic triples and pairs are constructed in Section 4 while Section 5 contains the characterization of Morita equivalence bimodules. The reduction functor for bimodules is constructed in Section 6. Finally, Section 7 contains the definition of the classical limit functor together with the proof of our main result that classical limit commutes with reduction. In a small appendix we collect the basic definitions of bicategories, functors, natural transformations, and modifications as unfortunately there are several competing versions in the literature: we want to make clear which definitions we are actually using.

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2 PRELIMINARIES

In this section we recall some well-known reduction constructions in the settings of Poisson geometry and deformation quantization to fix our notation.

Let (M, π) be a Poisson manifold, which models the phase space of a classical mechanical system, and assume that $\iota: C \rightarrow M$ is a closed coisotropic submanifold of M , called the *constraint surface*. We denote the vanishing ideal of C by

$$\mathcal{I}_C = \{f \in \mathcal{C}^\infty(M) \mid \iota^* f = 0\}, \quad (1)$$

which is clearly an ideal in the associative commutative algebra $\mathcal{C}^\infty(M)$ of smooth functions on M .

LEMMA 2.1 *Let (M, π) be a Poisson manifold and $C \subseteq M$ be a submanifold. Then the following statements are equivalent:*

1. *The submanifold C is coisotropic.*
2. *For a function $f \in \mathcal{I}_C$ the Hamiltonian vector field $X_f = (df)^\sharp \in \Gamma^\infty(TM)$ is tangent to C , i.e. $X_f(p) \in T_p C \subseteq T_p M$ for all $p \in C$.*
3. *Its vanishing ideal \mathcal{I}_C is a Poisson subalgebra of $\mathcal{C}^\infty(M)$.*

Note that in most interesting situations, \mathcal{I}_C will not be a Poisson ideal: this is equivalent to the statement that C is even a Poisson submanifold, a situation which is rarely of interest in the context of reduction.

The distribution on C spanned by the Hamiltonian vector fields X_f of functions $f \in \mathcal{I}_C$ is called the *characteristic distribution* of C . It turns out that this distribution is integrable and, under suitable circumstances, has a nice leaf space C/\sim . For simplicity, we assume that the leaf space is a smooth manifold and the projection onto the leaf space

$$\text{pr}: C \longrightarrow C/\sim =: M_{\text{red}} \quad (2)$$

is a surjective submersion. In this case, M_{red} is itself a Poisson manifold with Poisson structure determined as follows: one can characterize the functions on C which are constant along the leaves as restrictions of functions $f \in \mathcal{C}^\infty(M)$ with the property that $\{f, g\} \in \mathcal{I}_C$ for all $g \in \mathcal{I}_C$, i.e. as the Lie normalizer (or Lie idealizer) of \mathcal{I}_C inside $\mathcal{C}^\infty(M)$. We denote this normalizer as

$$\mathcal{B}_C = \{f \in \mathcal{C}^\infty(M) \mid \iota^*(X_g f) = 0 \text{ for all } g \in \mathcal{I}_C\}. \quad (3)$$

It is now an easy verification that \mathcal{B}_C is a Poisson subalgebra of all functions and $\mathcal{I}_C \subseteq \mathcal{B}_C$ is a Poisson ideal in its normalizer. Thus, as an immediate consequence we obtain the following claim.

LEMMA 2.2 *Let (M, π) be a Poisson manifold and $C \subseteq M$ be a coisotropic submanifold. Then the quotient $\mathcal{B}_C/\mathcal{I}_C$ is a Poisson algebra.*

Finally, we can observe that the pull-back with the projection yields an isomorphism

$$\text{pr}^*: \mathcal{C}^\infty(M_{\text{red}}) \longrightarrow \mathcal{B}_C/\mathcal{I}_C \quad (4)$$

of associative algebras. Since the right hand side is a Poisson algebra in a natural way, this induces the Poisson structure π_{red} on the reduced space M_{red} whenever M_{red} is a manifold at all with pr being a surjective submersion. In this case, the isomorphism turns it into a Poisson manifold as claimed. But even if this geometric assumption is not satisfied, we can take $\mathcal{B}_C/\mathcal{I}_C$ as a valid replacement for M_{red} .

EXAMPLE 2.3 (MARSDEN-WEINSTEIN REDUCTION, CLASSICAL) *A particular but important case of the above procedure is the Marsden-Weinstein reduction. Here one assumes to have a smooth action $\Phi: G \times M \rightarrow M$ of a connected Lie group such that the Poisson structure π is preserved. Moreover, one requires an ad^* -equivariant momentum map $J: M \rightarrow \mathfrak{g}^*$ where \mathfrak{g}^* is the dual of the Lie algebra \mathfrak{g} of G , i.e. for all $\xi \in \mathfrak{g}$ the fundamental vector field $\xi_M \in \Gamma^\infty(TM)$ is given by $\xi_M = -X_{J_\xi} = \{J_\xi, \cdot\}$ where we define the scalar function $J_\xi \in \mathcal{C}^\infty(M)$ as the function obtained by pairing the result of J with ξ . The equivariance then reads as $\{J_\xi, J_\eta\} = J_{[\xi, \eta]}$ for all $\xi, \eta \in \mathfrak{g}$. Equivalently, J is a Poisson map with respect to the linear Poisson structure on \mathfrak{g}^* . Now one considers the zero level set $C = J^{-1}(\{0\})$ of J , provided 0 is a regular value and $C \neq \emptyset$. Then C is indeed coisotropic and the foliation of C is just the foliation by orbits of G . Hence in this case*

$$\mathcal{B}_C = \{f \in \mathcal{C}^\infty(M) \mid \iota^* f \text{ is } G\text{-invariant}\}. \tag{5}$$

Moreover, $M_{\text{red}} = C/G$, provided the action of G on C is sufficiently nice: here we assume that G acts freely and properly on C so that $\text{pr}: C \rightarrow M_{\text{red}}$ becomes a G -principal fiber bundle. There are of course many generalizations of this particularly simple situation allowing for less restrictive assumptions, see e.g. the textbooks [37, 35] for further information.

EXAMPLE 2.4 *Another example comes from the setting of actions of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π) . One assumes to have a smooth action $\Phi: G \times M \rightarrow M$ of a Poisson Lie group that sends the Poisson structure $\pi_G \oplus \pi$ into π . In this case a momentum map, if it exists, is a map $J: M \rightarrow G^*$ where G^* is the dual of the Poisson Lie group (G, π_G) . Its definition has been introduced in [33]. Assuming that J is a Poisson map, one can easily see that for any dressing orbit \mathcal{O}_μ its preimage $C := J^{-1}(\{\mathcal{O}_\mu\})$ by J is a coisotropic submanifold of M . Thus in a similar way as the case discussed above, we can obtain a reduced Poisson manifold. For further details see [24]. Furthermore, the relation between coisotropic submanifolds and left ideals in this setting has been proposed in [34].*

As a next step we want to incorporate the quantum picture as well. Here we choose the approach of deformation quantization [1], see e.g. [40] for an introduction. Thus we assume to have a formal star product \star given on (M, π) , i.e. a $\mathbb{C}[[\lambda]]$ -bilinear associative product for $\mathcal{C}^\infty(M)[[\lambda]]$ written as

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g) \tag{6}$$

with bilinear operators $C_r: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ extended $\mathbb{C}[[\lambda]]$ -bilinearly as usual, such that

$$C_0(f, g) = fg \quad \text{and} \quad C_1(f, g) - C_1(g, f) = i\{f, g\} \tag{7}$$

for all $f, g \in \mathcal{C}^\infty(M)$, the constant function 1 is the unit of \star , and C_r is bidifferential for all $r \in \mathbb{N}_0$. The resulting algebra is denoted by $\mathcal{A} = (\mathcal{C}^\infty(M)[[\lambda]], \star)$. The formal parameter λ corresponds in convergent situations to the physical Planck constant \hbar .

To formulate a quantum version of reduction, we first need to introduce a quantum analog of the ideal and Poisson subalgebra \mathcal{I}_C . One way consists in requiring the existence of a deformation of ι^* into a quantum restriction map

$$\iota^* = \iota^* \circ S \quad \text{with} \quad S = \text{id} + \sum_{r=1}^{\infty} \lambda^r S_r, \tag{8}$$

where $S_r: \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ are differential operators to be found in such a way that

$$\mathcal{I}_C = \ker \iota^* = \{f \in \mathcal{C}^\infty(M)[[\lambda]] \mid \iota^* f = 0\} \tag{9}$$

becomes a left ideal with respect to \star . As before, we can then consider the functions on the constraint surface C and find that

$$\iota^*: \mathcal{C}^\infty(M)[[\lambda]]/\mathcal{I}_C \rightarrow \mathcal{C}^\infty(C)[[\lambda]] \tag{10}$$

becomes an isomorphism thanks to the above assumption that ι^* starts with ι^* in zeroth order of λ . Now we can proceed as in the classical case by considering the *normalizer*, now in the associative sense, i.e.

$$\mathcal{B}_C = N(\mathcal{I}_C) = \{f \in \mathcal{C}^\infty(M)[[\lambda]] \mid g \star f \in \mathcal{I}_C \text{ for all } g \in \mathcal{I}_C\}. \tag{11}$$

Note that this is equivalent to the condition $[f, g]_\star \in \mathcal{I}_C$ for all $g \in \mathcal{I}_C$ since \mathcal{I}_C is already a left ideal. A simple check shows that \mathcal{B}_C is a subalgebra of $\mathcal{C}^\infty(M)[[\lambda]]$ and \mathcal{I}_C is a two-sided ideal in \mathcal{B}_C . Thus it is tempting to define the reduction on the quantum side as the quotient algebra $\mathcal{A}_{\text{red}} = \mathcal{B}_C/\mathcal{I}_C$ in complete analogy to the above classical case.

We point out now an alternative but equivalent definition of this reduced algebra:

PROPOSITION 2.5 *The functions on the constraint surface become a left module of the algebra \mathcal{A} by (10) in a canonical way. Moreover, the module endomorphisms of this left module are isomorphic to the opposite algebra of \mathcal{A}_{red} via*

$$\mathcal{B}_C/\mathcal{I}_C \ni [f] \mapsto ([g] \mapsto [g \star f]) \in \text{End}_{\mathcal{A}}(\mathcal{C}^\infty(C)[[\lambda]])^{\text{opp}}, \tag{12}$$

where $[g]$ denotes an equivalence class in the quotient (10).

This idea from [5] puts the role of the constraint surface in a much clearer light: it carries a bimodule structure for the original big algebra \mathcal{A} acting from the left and the reduced algebra acting from the right such that the reduced algebra coincides with (the opposite of) the module endomorphisms. Note, however, that the \mathcal{A}_{red} -endomorphisms contain \mathcal{A} but are typically strictly larger, see [29]. In particular, this bimodule is typically *not* a Morita equivalence bimodule.

It is now a final check necessary to show that $\mathfrak{A}_{\text{red}}$ defined as above actually gives a deformation quantization of the reduced Poisson manifold $(M_{\text{red}}, \pi_{\text{red}})$. This is by far not trivial and in fact not true in general. Simple examples of ill-adjusted star products on M where this fails are discussed e.g. in [7]. More profound obstructions are discussed in [5, 4] in the symplectic case and in [18, 17, 16] for the Poisson case.

However, in many reasonably nice situations the program can be carried through and yields a star product \star_{red} for M_{red} , see again [5, 4]: whenever the reduced space M_{red} exists in the case of a symplectic manifold M , then one can find a suitable star product \star on M for which the construction yields a reduced star product \star_{red} . A more specific situation is the analog of the Marsden-Weinstein reduction, the construction relying on BRST cohomological arguments [7]:

EXAMPLE 2.6 (MARS DEN-WEINSTEIN REDUCTION, QUANTUM) *Suppose that we are in the same situation as in Example 2.3. Suppose moreover, that \star is a star product on M , invariant under the group action Φ which allows for a quantum momentum map $\mathbf{J} = J + \sum_{r=1}^{\infty} \lambda^r J_r$, i.e. one has $[\mathbf{J}_\xi, f]_\star = i\lambda \xi_M f$ for all $f \in \mathcal{C}^\infty(M)[[\lambda]]$ and $[\mathbf{J}_\xi, \mathbf{J}_\eta]_\star = i\lambda \mathbf{J}_{[\xi, \eta]}$. Then one can construct a deformation \mathfrak{A}^* as needed and $\mathfrak{A}_{\text{red}}$ turns out to be isomorphic to $\mathcal{C}^\infty(M_{\text{red}})[[\lambda]]$ as $\mathbb{C}[[\lambda]]$ -module inducing thereby a star product \star_{red} . Moreover, explicit formulas for the bimodule structure on $\mathcal{C}^\infty(C)[[\lambda]]$ can be given, see e.g. [29]. The existence of invariant star products, quantum momentum maps, and the corresponding reduction is discussed in detail in [36, 28, 26] culminating in classification results [38, 39] in the symplectic case. Finally, the existence and classification of equivariant star products in the Poisson case has been recently proved in [25].*

We will come back to this construction at several instances. It will serve as the main motivation in the following: based on these observations we shall put the reduction process into a purely algebraic framework.

3 COISOTROPIC TRIPLES AND PAIRS OF ALGEBRAS

In the following we fix a commutative unital ring \mathbb{k} of scalars which in most situations will be even a field. It will sometimes be convenient to assume $\mathbb{Q} \subseteq \mathbb{k}$. All occurring algebras and modules will be over \mathbb{k} and linearity always will include linearity over \mathbb{k} .

As we want to discuss reduction with respect to some coisotropic data, we start with some unital *total algebra* \mathcal{A}_{tot} in which we suppose to have a left ideal $\mathcal{A}_0 \subseteq \mathcal{A}_{\text{tot}}$. The correspondence with the above geometric situation is that the total algebra stands for the functions on the total phase space while the left ideal corresponds to the functions vanishing on the constraint surface. To ensure maximal flexibility, we need to specify an additional algebra, which we call the *weakly observables* according to Dirac's original discussion of constraint systems in [22, 23]. We thus have to specify a unital subalgebra $\mathcal{A}_N \subseteq \mathcal{A}_{\text{tot}}$

containing the left ideal \mathcal{A}_0 as a two-sided ideal $\mathcal{A}_0 \subseteq \mathcal{A}_N$. With other words, \mathcal{A}_N will be a unital subalgebra of the normalizer of the left ideal, i.e.

$$\mathcal{A}_0 \subseteq \mathcal{A}_N \subseteq N(\mathcal{A}_0). \quad (13)$$

However, note that we explicitly allow \mathcal{A}_N to be strictly smaller than the normalizer of \mathcal{A}_0 : in the commutative situation above we took the Lie normalizer with respect to the additional Poisson bracket, a structure we do not want to introduce at this level. Summarizing, this leads now to the following definition of a coisotropic triple of algebras:

DEFINITION 3.1 (COISOTROPIC TRIPLE OF ALGEBRAS) *A coisotropic triple of algebras over \mathbb{k} is a triple $\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_N, \mathcal{A}_0)$ consisting of a unital algebra \mathcal{A}_{tot} , a unital subalgebra $\mathcal{A}_N \subseteq \mathcal{A}_{\text{tot}}$, and a left ideal $\mathcal{A}_0 \subseteq \mathcal{A}_{\text{tot}}$ such that $\mathcal{A}_0 \subseteq \mathcal{A}_N$ is a two-sided ideal.*

With this definition, our geometric situation provides us some first and guiding examples:

EXAMPLE 3.2 (COISOTROPIC TRIPLES OF ALGEBRAS FROM GEOMETRY) *Let M be a Poisson manifold with a coisotropic submanifold $\iota: C \rightarrow M$.*

1. *Setting $\mathcal{A}_{\text{tot}} = \mathcal{C}^\infty(M)$ and $\mathcal{A}_0 = \mathcal{J}_C = \ker \iota^*$ gives a total algebra and a (left) ideal inside. However, since \mathcal{A}_{tot} is commutative, taking the normalizer of \mathcal{A}_0 would reproduce \mathcal{A}_{tot} , a too simple choice to be interesting. However, taking $\mathcal{A}_N = \mathcal{B}_C$ gives an interesting coisotropic triple of algebras over \mathbb{C} .*
2. *Suppose now in addition that we can find a star product \star and a deformation ι^* of the restriction map. Then $\mathcal{A}_{\text{tot}} = \mathcal{C}^\infty(M)[[\lambda]]$, equipped with the star product \star serves as total algebra, $\mathcal{A}_0 = \ker \iota^*$ will be the left ideal and the normalizer $\mathcal{A}_N = N(\mathcal{A}_0)$ can be taken as weakly observables. We obtain a truly noncommutative coisotropic triple in this case, over the ring $\mathbb{C}[[\lambda]]$ as underlying scalars.*

EXAMPLE 3.3 *We recall a concrete example of coisotropic triples already discussed in [19] following [20]. Let $\mathcal{E}_q(2)$ be the \ast -algebra generated by the following relations:*

$$\begin{aligned} vv^{-1} &= 1 = v^{-1}v \\ vn^* &= q^{-1}nv \\ vn &= qnv \\ nn^* &= n^*n, \end{aligned}$$

where q is a real parameter. Denote by \mathcal{J}_λ the right ideal generated by $\lambda(v-1)+n$ and $\bar{\lambda}(v^{-1}-1)+n^$, with $\lambda \in \mathbb{C}$. It is easy to see that $\mathcal{E}_q(2)$, \mathcal{J}_λ and the corresponding normalizer form a coisotropic triple.*

Before investigating more examples and constructions we introduce the following notion of morphisms between coisotropic triples of algebras: we define a *morphism* from the coisotropic triple $\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_N, \mathcal{A}_0)$ to the coisotropic triple $\mathcal{B} = (\mathcal{B}_{\text{tot}}, \mathcal{B}_N, \mathcal{B}_0)$ to be a unital algebra morphism

$$\Phi: \mathcal{A}_{\text{tot}} \longrightarrow \mathcal{B}_{\text{tot}}, \tag{14}$$

such that

$$\Phi(\mathcal{A}_N) \subseteq \mathcal{B}_N \quad \text{and} \quad \Phi(\mathcal{A}_0) \subseteq \mathcal{B}_0. \tag{15}$$

It is clear that the composition of morphisms is again a morphism and hence we ultimately obtain a category of coisotropic triples of algebras which we denote by C_3Alg or C_3Alg_k if we want to emphasize the underlying ring k of scalars.

REMARK 3.4 *If one would only focus on the pair $(\mathcal{A}_{\text{tot}}, \mathcal{A}_0)$ then this notion of morphisms becomes less obvious: in that case a natural candidate for a morphism from one such pair to another would be a unital algebra morphism $\Phi: \mathcal{A}_{\text{tot}} \longrightarrow \mathcal{B}_{\text{tot}}$ with $\Phi(\mathcal{A}_0) \subseteq \mathcal{B}_0$. However, simple examples show that then the normalizer $N(\mathcal{A}_0)$ needs not to be mapped into the normalizer $N(\mathcal{B}_0)$. As we will base many constructions on the choice of \mathcal{A}_N , we need to take care of this part of the triple by hand.*

We denote the category of unital algebras (with unital algebra morphisms) by Alg while the not necessarily unital algebras are then denoted by alg . Then we have several obvious functors. First, projecting on one of the three components of a triple is of course functorial leading to functors

$$\text{C}_3\text{Alg} \ni \mathcal{A} \longmapsto \mathcal{A}_{\text{tot}} \in \text{Alg} \quad \text{and} \quad \text{C}_3\text{Alg} \ni \mathcal{A} \longmapsto \mathcal{A}_N \in \text{Alg} \tag{16}$$

as well as

$$\text{C}_3\text{Alg} \ni \mathcal{A} \longmapsto \mathcal{A}_0 \in \text{alg}, \tag{17}$$

each with the obvious restriction of morphisms. But we can also go the other way and build coisotropic triples out of single algebras. Here we have several options. The first is the *trivial triple*

$$\mathcal{A}_{\text{trivial}} = (\mathcal{A}, \mathcal{A}, \mathcal{A}) \tag{18}$$

for a unital algebra \mathcal{A} . Alternatively, we can construct the *un-reduce triple*

$$\mathcal{A}_{\text{unred}} = (\mathcal{A}, \mathcal{A}, \{0\}) \tag{19}$$

for a unital algebra \mathcal{A} . Both versions yield functors $\text{Alg} \longrightarrow \text{C}_3\text{Alg}$. Finally, more important for our original motivation, is the *Dirac triple* we can build out of a pair of a unital algebra \mathcal{A} and a left ideal $\mathcal{J} \subseteq \mathcal{A}$. Here we set

$$\mathcal{A}_{\text{Dirac}} = (\mathcal{A}, N(\mathcal{J}), \mathcal{J}). \tag{20}$$

In view of Remark 3.4 this becomes again functorial if we consider the category LeftIdealAlg of pairs of unital algebras with left ideals together with unital

algebra morphisms mapping one left ideal into the other *and* mapping the normalizer of the first left ideal into the normalizer of the second. Then the canonical triple becomes a functor

$$\text{Dirac: LeftIdealAlg} \longrightarrow \text{C}_3\text{Alg}. \quad (21)$$

After having established the notion of coisotropic triples one can define the reduction of them in the following way, mimicking the situation of star products in the geometric situation: Let $\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_N, \mathcal{A}_0)$ be a coisotropic triple of algebras. Then the reduction of \mathcal{A} is defined to be the unital algebra

$$\mathcal{A}_{\text{red}} = \mathcal{A}_N / \mathcal{A}_0. \quad (22)$$

For a morphism $\Phi: \mathcal{A} \longrightarrow \mathcal{B}$ between coisotropic triples of algebras we see that the restriction of Φ to the weak observables passes to the quotient and thus defines a unital algebra morphism

$$\Phi_{\text{red}}: \mathcal{A}_{\text{red}} \longrightarrow \mathcal{B}_{\text{red}}. \quad (23)$$

Clearly, this yields a functorial reduction:

PROPOSITION 3.5 *Reduction of coisotropic triples of algebras yields a functor*

$$\text{red: C}_3\text{Alg} \longrightarrow \text{Alg}. \quad (24)$$

COROLLARY 3.6 *The reduction of the un-reduce triple is naturally isomorphic to the identity functor on Alg. The reduction of the trivial triple is naturally isomorphic to the zero-functor on Alg sending an algebra to the zero algebra $\{0\}$.*

Surprisingly, the reduction uses only the information of the pair $(\mathcal{A}_N, \mathcal{A}_0)$ instead of the full triple. The ambient total algebra does not play a role here. This raises of course the question whether one can not just start with a category of *coisotropic pairs* consisting of a unital algebra together with a two-sided ideal inside. To some extent this is true and many of the following constructions will only use the pair instead of the triple. Thus we also state the definition of a coisotropic pair as follows:

DEFINITION 3.7 (COISOTROPIC PAIR) *A coisotropic pair of algebras is a pair $\mathcal{A} = (\mathcal{A}_N, \mathcal{A}_0)$ of a unital associative algebra \mathcal{A}_N over \mathbb{k} together with a two-sided ideal $\mathcal{A}_0 \subseteq \mathcal{A}_N$. A morphism between two coisotropic pairs \mathcal{A} and \mathcal{B} is a unital algebra morphism $\Phi: \mathcal{A}_N \longrightarrow \mathcal{B}_N$ with $\Phi(\mathcal{A}_0) \subseteq \mathcal{B}_0$.*

Clearly, this gives again a categorical framework for coisotropic pairs of algebras. We denote the resulting category by C_2Alg or $\text{C}_2\text{Alg}_{\mathbb{k}}$ whenever we need to emphasize the underlying ring of scalars.

Forgetting the total algebra yields then a functor

$$\text{C}_3\text{Alg} \longrightarrow \text{C}_2\text{Alg}. \quad (25)$$

This also results in a functor

$$\text{Dirac}: \text{LeftIdealAlg} \longrightarrow \text{C}_2\text{Alg}, \tag{26}$$

sending an algebra \mathcal{A}_{tot} with left ideal $\mathcal{A}_0 \subseteq \mathcal{A}_{\text{tot}}$ to the coisotropic pair $(N(\mathcal{A}_0), \mathcal{A}_0)$. Note that here the correct definition of morphisms in LeftIdealAlg is crucial to make this functorial. Conversely, we can extend a coisotropic pair $(\mathcal{A}_N, \mathcal{A}_0)$ always to a coisotropic triple in a trivial way by mapping it to $(\mathcal{A}_N, \mathcal{A}_N, \mathcal{A}_0)$. Forgetting the triple gives back the pair we started with. In fact this is a left adjoint to the forgetful functor forgetting \mathcal{A}_{tot} . Moreover, this allows us to view C_2Alg as a subcategory of C_3Alg . We will often give definitions only for coisotropic triples, and the appropriate definitions for coisotropic pairs will then be given by restricting to this subcategory. However, the interesting triples are those where $\mathcal{A}_0 \subseteq \mathcal{A}_{\text{tot}}$ is *only* a left ideal. Hence these will not show up as images of this inclusion functor $\text{C}_2\text{Alg} \longrightarrow \text{C}_3\text{Alg}$.

Note also that we have a *trivial* coisotropic pair for every unital algebra \mathcal{A} by setting

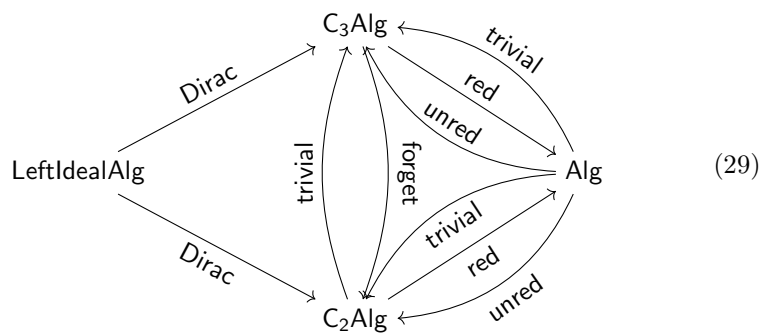
$$\mathcal{A}_{\text{trivial}} = (\mathcal{A}, \mathcal{A}) \tag{27}$$

as well as the *un-reduce pair*

$$\mathcal{A}_{\text{unred}} = (\mathcal{A}, \{0\}). \tag{28}$$

Both notions are of course compatible with the trivial and the un-reduce triples and the functor (25).

As mentioned before, the reduction functor only uses the information of a pair and thus gives a reduction $\text{red}: \text{C}_2\text{Alg} \longrightarrow \text{Alg}$. Ultimately, we arrive at the following diagram



of functors. Thus, in particular, the reduction of the un-reduce pair reproduces the algebra one started with and the reduction of the trivial triple is the zero algebra.

REMARK 3.8 While the pair point of view simplifies the reduction picture drastically, the original motivation is to generalize the geometric situation of phase

space reduction: there the algebra in the middle \mathcal{A}_N is typically the most difficult one to get, both in the classical and the quantum situation. Instead, one starts with the ambient algebra \mathcal{A}_{tot} . Then in the quantum version it is already difficult enough (and sometimes obstructed) to deform the classical vanishing ideal into a left ideal \mathcal{A}_0 . Only in the last step one can then find \mathcal{A}_N . Thus we will discuss triples and pairs in parallel to keep in mind that a serious application will always require to actually find the triples out of more simple data. Ultimately, we will be interested in starting with $(\mathcal{A}_{\text{tot}}, \mathcal{A}_0)$ in `LeftIdealAlg` and construct the relevant data out of this. While for the algebras there is a functorial way by using the normalizers, in the case of (bi-)modules we will see that one typically has no obvious functorial way to accomplish this.

Before moving to the categories of (bi-)modules over coisotropic triples and pairs, we mention the following canonical bimodule relating the total algebra and the reduced one:

PROPOSITION 3.9 *Let $\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_N, \mathcal{A}_0)$ be a coisotropic triple of algebras over \mathbb{k} .*

1. Then

$$\mathbf{C}(\mathcal{A}) = \mathcal{A}_{\text{tot}}/\mathcal{A}_0 \tag{30}$$

is a $(\mathcal{A}_{\text{tot}}, \mathcal{A}_{\text{red}})$ -bimodule, cyclic with respect to \mathcal{A}_{tot} , and one has

$$\text{End}_{\mathcal{A}_{\text{tot}}}(\mathbf{C}(\mathcal{A}))^{\text{opp}} = \mathbf{N}(\mathcal{A}_0)/\mathcal{A}_0. \tag{31}$$

2. For a morphism $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ of coisotropic triples of algebras the map

$$\mathbf{C}(\Phi): \mathbf{C}(\mathcal{A}) \ni [a] \mapsto [\Phi(a)] \in \mathbf{C}(\mathcal{B}) \tag{32}$$

is a bimodule morphism along the two algebra morphisms $\Phi_{\text{tot}}: \mathcal{A}_{\text{tot}} \rightarrow \mathcal{B}_{\text{tot}}$ and $\Phi_{\text{red}}: \mathcal{A}_{\text{red}} \rightarrow \mathcal{B}_{\text{red}}$.

3. Mapping \mathcal{A} to $\mathbf{C}(\mathcal{A})$ and morphisms $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ to $\mathbf{C}(\Phi)$ gives a functor

$$\mathbf{C}: \mathbf{C}_3\text{Alg} \rightarrow \text{Bimodule} \tag{33}$$

into the category `Bimodule` of bimodules with morphisms being bimodule morphisms along algebra morphisms of the involved algebras.

PROOF: Indeed, since \mathcal{A}_0 is a left ideal in \mathcal{A}_{tot} , the quotient $\mathbf{C}(\mathcal{A})$ is a left \mathcal{A} -module. Since \mathcal{A}_{tot} is unital, $\mathbf{C}(\mathcal{A})$ is cyclic with cyclic element $[1] \in \mathcal{C}(\mathcal{A})$. As already indicated in the case of star products in (12), the opposite of the module endomorphisms of this left \mathcal{A} -module is given by $\mathbf{N}(\mathcal{A}_0)/\mathcal{A}_0$ using the right multiplications. Since $\mathcal{A}_N \subseteq \mathbf{N}(\mathcal{A}_0)$ by assumption, this gives the canonical right module structure, showing the first claim. For the second, we note that $[\Phi(a)] \in \mathbf{C}(\mathcal{B})$ only depends on the class $[a]$ since $\Phi(\mathcal{A}_0) \subseteq \mathcal{B}_0$. Then it is clear that $\mathbf{C}(\Phi)(a \cdot [x]) = \Phi_{\text{tot}}(a) \cdot \mathbf{C}(\Phi)([x])$ and $\mathbf{C}(\Phi)([x] \cdot [a']) = \mathbf{C}(\Phi)([x]) \cdot \Phi_{\text{red}}([a'])$ for $a, x \in \mathcal{A}_{\text{tot}}$ and $a' \in \mathcal{A}_N$ since we can check these relations on representatives. From this the second part follows. But then the claimed functoriality is clear. \square

Geometrically, $C(\mathcal{A})$ corresponds to the functions on the constraint surface. Even though in the classical (commutative) case this is an algebra itself, we will consider it only as a $(\mathcal{A}_{\text{tot}}, \mathcal{A}_{\text{red}})$ -bimodule, since this is the only structure remaining in the noncommutative situation. Note that here we need the coisotropic triples instead of mere coisotropic pairs of algebras in order to define the bimodule $C(\mathcal{A})$.

REMARK 3.10 *We should remark that this observation stands at the beginning of the reduction idea of Bordemann in [5, 4] where the geometric situation is analyzed in detail, including a description of obstructions for the deformation quantization of coisotropic submanifolds in the symplectic framework.*

4 TRIPLES AND PAIRS OF BIMODULES

Let us come back to the geometric picture of Section 2, where $\iota: C \rightarrow M$ is a coisotropic submanifold of a Poisson manifold (M, π) and we assume to have a surjective submersion $\text{pr}: C \rightarrow M_{\text{red}}$. Here M_{red} is again the leaf space C/\sim of the characteristic distribution of C . Now let in addition $p: E \rightarrow M$ be a vector bundle over M . Then we know that $\mathcal{E}_{\text{tot}} = \Gamma^\infty(E)$ is a $\mathcal{C}^\infty(M)$ -module and we can define a submodule

$$\mathcal{E}_0 = \{s \in \Gamma^\infty(E) \mid s|_C = 0\} \tag{34}$$

of all sections vanishing on C . In order to define a reduced vector bundle $p_{\text{red}}: E_{\text{red}} \rightarrow M_{\text{red}}$ we would like to use the sections of E that are constant along the characteristic distribution of C . Of course, there is no canonical way to make sense out of such a statement. Instead, we need to use some additional data. For this, let ∇ be a covariant derivative for the vector bundle E and consider those sections of E whose covariant derivative in the direction of Hamiltonian vector fields of functions in the vanishing ideal \mathcal{I}_C vanish on C . We denote this subset by

$$\mathcal{E}_N = \{s \in \Gamma^\infty(E) \mid (\nabla_{X_f} s)|_C = 0 \text{ for all } f \in \mathcal{I}_C\}. \tag{35}$$

Note that for the definition of \mathcal{E}_N we used the additional information of a covariant derivative on E while \mathcal{E}_0 was still canonical. This is different from coisotropic algebras, where we could define \mathcal{A}_N as the Poisson normalizer \mathcal{B}_C . We use this geometric situation as motivation for the definition of bimodules over coisotropic triples.

DEFINITION 4.1 (BIMODULES OVER COISOTROPIC TRIPLES) *Let \mathcal{A} and \mathcal{B} be coisotropic triples of algebras over \mathbb{k} .*

1. *A triple $\mathcal{E} = (\mathcal{E}_{\text{tot}}, \mathcal{E}_N, \mathcal{E}_0)$ consisting of a $(\mathcal{B}_{\text{tot}}, \mathcal{A}_{\text{tot}})$ -bimodule \mathcal{E}_{tot} and $(\mathcal{B}_N, \mathcal{A}_N)$ -bimodules \mathcal{E}_N and \mathcal{E}_0 together with a bimodule morphism $\iota_{\mathcal{E}}: \mathcal{E}_N \rightarrow \mathcal{E}_{\text{tot}}$ along the inclusions $\mathcal{B}_N \subseteq \mathcal{B}_{\text{tot}}$ and $\mathcal{A}_N \subseteq \mathcal{A}_{\text{tot}}$ is called a $(\mathcal{B}, \mathcal{A})$ -bimodule if $\mathcal{E}_0 \subseteq \mathcal{E}_N$ is a sub-bimodule such that*

$$\mathcal{B}_0 \cdot \mathcal{E}_N \subseteq \mathcal{E}_0 \quad \text{and} \quad \mathcal{E}_N \cdot \mathcal{A}_0 \subseteq \mathcal{E}_0. \tag{36}$$

2. A morphism $\Phi: \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ between $(\mathcal{B}, \mathcal{A})$ -bimodules is a pair $(\Phi_{\text{tot}}, \Phi_N)$ of a $(\mathcal{B}_{\text{tot}}, \mathcal{A}_{\text{tot}})$ -bimodule morphism $\Phi_{\text{tot}}: \mathcal{E}_{\text{tot}} \rightarrow \tilde{\mathcal{E}}_{\text{tot}}$ and $(\mathcal{B}_N, \mathcal{A}_N)$ -bimodule morphism $\Phi: \mathcal{E}_N \rightarrow \tilde{\mathcal{E}}_N$ such that $\Phi_{\text{tot}} \circ \iota_{\mathcal{E}} = \iota_{\tilde{\mathcal{E}}} \circ \Phi_N$ and $\Phi_N(\mathcal{E}_0) \subseteq \tilde{\mathcal{E}}_0$.
3. The category of $(\mathcal{B}, \mathcal{A})$ -bimodules is denoted by $\mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$.

The motivation is to mimic the two-sided ideal property of the 0-component also on the level of (bi-)modules. It is of course a routine check that the obvious composition of morphisms is again a morphism and we thus get a category. The reason we did not choose to require an inclusion $\mathcal{E}_N \subseteq \mathcal{E}_{\text{tot}}$ is that with this broader notion tensor products turn out to become easier as we will not have to insist on flatness in order to guarantee the injectivity of the tensor product of the inclusions. Nevertheless, viewing a coisotropic triple \mathcal{A} as bimodule over itself $\iota_{\mathcal{A}}: \mathcal{A}_N \rightarrow \mathcal{A}_{\text{tot}}$ is still the inclusion map. Similar to the case of coisotropic triples of algebras we can also define modules over coisotropic pairs by simply ignoring the tot-component.

DEFINITION 4.2 (BIMODULES OVER COISOTROPIC PAIRS) *Let \mathcal{A} and \mathcal{B} be coisotropic pairs of algebras over k .*

1. A pair $\mathcal{E} = (\mathcal{E}_N, \mathcal{E}_0)$ of $(\mathcal{B}_N, \mathcal{A}_N)$ -bimodules is called a $(\mathcal{B}, \mathcal{A})$ -bimodule if $\mathcal{E}_0 \subseteq \mathcal{E}_N$ is a sub-bimodule such that

$$\mathcal{B}_0 \cdot \mathcal{E}_N \subseteq \mathcal{E}_0 \quad \text{and} \quad \mathcal{E}_N \cdot \mathcal{A}_0 \subseteq \mathcal{E}_0. \tag{37}$$

2. A morphism $\Phi: \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ between $(\mathcal{B}, \mathcal{A})$ -bimodules is a $(\mathcal{B}_N, \mathcal{A}_N)$ -bimodule morphism $\Phi: \mathcal{E}_N \rightarrow \tilde{\mathcal{E}}_N$ such that $\Phi(\mathcal{E}_0) \subseteq \tilde{\mathcal{E}}_0$.
3. The category of $(\mathcal{B}, \mathcal{A})$ -bimodules is denoted by $\mathbf{C}_2\text{Bimod}(\mathcal{B}, \mathcal{A})$.

By forgetting the total bimodule we get a forgetful functor

$$\mathbf{C}_3\text{Bimod} \longrightarrow \mathbf{C}_2\text{Bimod}. \tag{38}$$

Conversely, we can go the other way by mapping a $(\mathcal{B}, \mathcal{A})$ -bimodule \mathcal{E} over coisotropic pairs \mathcal{A} and \mathcal{B} to the bimodule $(\mathcal{E}_N, \mathcal{E}_N, \mathcal{E}_0)$ over the coisotropic triples $(\mathcal{A}_N, \mathcal{A}_N, \mathcal{A}_0)$ and $(\mathcal{B}_N, \mathcal{B}_N, \mathcal{B}_0)$. Similar to the case of coisotropic algebras this is a left adjoint to the forgetful functor forgetting the tot-component. The bicategory Bimod of bimodules over algebras with the tensor product as composition functors is one of the most basic examples of a bicategory. The goal of this section is to prove that we can construct bicategories $\mathbf{C}_3\text{Bimod}$ and $\mathbf{C}_2\text{Bimod}$ building on the above categories as well. Thus we can realize $\mathbf{C}_2\text{Bimod}$ as a sub-bicategory of $\mathbf{C}_3\text{Bimod}$. To show this we need to define a tensor product of bimodules over coisotropic triples and pairs and to check that there exist natural transformations of associativity as well as left and right

identities. This is done in the following lemmas. As a first step we construct a tensor product functor

$$\otimes_{\mathcal{B}} : \mathbf{C}_3\text{Bimod}(\mathcal{C}, \mathcal{B}) \times \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathbf{C}_3\text{Bimod}(\mathcal{C}, \mathcal{A}) \quad (39)$$

by tensoring the components of the triple as follows:

LEMMA 4.3 *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be coisotropic triples of algebras and let $\mathcal{F} \in \mathbf{C}_3\text{Bimod}(\mathcal{C}, \mathcal{B})$, $\mathcal{E} \in \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be corresponding bimodules. Then ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ given by the components*

$$({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})_{\text{tot}} = \mathcal{F}_{\text{tot}} \otimes_{\mathcal{B}_{\text{tot}}} \mathcal{E}_{\text{tot}}, \quad (40)$$

$$({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})_{\text{N}} = \mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}}, \quad (41)$$

$$({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})_0 = \mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_0 + \mathcal{F}_0 \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}} \quad (42)$$

is a $(\mathcal{C}, \mathcal{A})$ -bimodule, where we use the tensor product $\iota_{\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}} = \iota_{\mathcal{F}} \otimes \iota_{\mathcal{E}}$ to map the N-component into the tot-component.

PROOF: Note that the tensor product $\mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_0$ is *not* a submodule of $\mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}}$ directly, thus (42) has to be understood in one of the two following equivalent ways: either view $\mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_0$ as the submodule of $\mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}}$ generated by all elements of the form $y \otimes x$, with $y \in \mathcal{F}_{\text{N}}$, $x \in \mathcal{E}_0$, or as the image of $\text{id}_{\mathcal{F}_{\text{N}}} \otimes \iota_{\mathcal{E}_0}$ where $\iota_{\mathcal{E}_0}$ is the inclusion of \mathcal{E}_0 into \mathcal{E}_{N} . Similarly for $\mathcal{F}_0 \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}}$. Now observe that $(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{tot}}$ is a $(\mathcal{C}_{\text{tot}}, \mathcal{A}_{\text{tot}})$ -bimodule and $(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{N}}$ and $(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_0$ are clearly $(\mathcal{C}_{\text{N}}, \mathcal{A}_{\text{N}})$ -bimodules. Moreover, the map $\iota_{\mathcal{F}} \otimes \iota_{\mathcal{E}} : (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{N}} \longrightarrow (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{tot}}$ is a bimodule morphism and

$$\mathcal{B}_0 \cdot (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{N}} = (\mathcal{B}_0 \cdot \mathcal{F}_{\text{N}}) \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}} \subseteq \mathcal{F}_0 \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_{\text{N}} \subseteq (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_0$$

and

$$(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{N}} \cdot \mathcal{A}_0 = \mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} (\mathcal{E}_{\text{N}} \cdot \mathcal{A}_0) \subseteq \mathcal{F}_{\text{N}} \otimes_{\mathcal{B}_{\text{N}}} \mathcal{E}_0 \subseteq (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_0$$

hold. Hence $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$ is a $(\mathcal{B}, \mathcal{A})$ -bimodule. □

LEMMA 4.4 *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be coisotropic triples of algebras. Moreover, let $\Psi : \mathcal{F} \longrightarrow \mathcal{F}'$ and $\Phi : \mathcal{E} \longrightarrow \mathcal{E}'$ be morphisms of $(\mathcal{C}, \mathcal{B})$ -bimodules $\mathcal{F}, \mathcal{F}'$, and $(\mathcal{B}, \mathcal{A})$ -bimodules $\mathcal{E}, \mathcal{E}'$, respectively. Then $\Psi \otimes \Phi$ given by*

$$(\Psi \otimes \Phi)_{\text{tot}} = \Psi_{\text{tot}} \otimes \Phi_{\text{tot}} \quad (43)$$

and

$$(\Psi \otimes \Phi)_{\text{N}} = \Psi_{\text{N}} \otimes \Phi_{\text{N}} \quad (44)$$

is a bimodule morphism from $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$ to $\mathcal{F}' \otimes_{\mathcal{B}} \mathcal{E}'$.

PROOF: It is clear that $(\Psi \otimes \Phi)_{\text{tot}}$ and $(\Psi \otimes \Phi)_N$ are a morphism of $(\mathcal{C}_{\text{tot}}, \mathcal{A}_{\text{tot}})$ - and $(\mathcal{C}_N, \mathcal{A}_N)$ -bimodules, respectively, fulfilling $(\Psi \otimes \Phi)_{\text{tot}} \circ (\iota_{\mathcal{F}} \otimes \iota_{\mathcal{E}}) = (\iota_{\mathcal{F}'} \otimes \iota_{\mathcal{E}'}) \circ (\Psi_N \otimes \Phi_N)$. Moreover.

$$\begin{aligned} (\Psi \otimes \Phi)_N (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_0 &= (\Psi \otimes \Phi)_N (\mathcal{F}_N \otimes_{\mathcal{B}_N} \mathcal{E}_0 + \mathcal{F}_0 \otimes_{\mathcal{B}_N} \mathcal{E}_N) \\ &= \Psi_N(\mathcal{F}_N) \otimes_{\mathcal{B}_N} \Phi_N(\mathcal{E}_0) + \Psi_N(\mathcal{F}_0) \otimes_{\mathcal{B}_N} \Phi_N(\mathcal{E}_N) \\ &\subseteq \mathcal{F}'_N \otimes_{\mathcal{B}_N} \mathcal{E}'_0 + \mathcal{F}'_0 \otimes_{\mathcal{B}_N} \mathcal{E}'_N \\ &= (\mathcal{F}' \otimes_{\mathcal{B}} \mathcal{E}')_0 \end{aligned}$$

holds. Hence $\Psi \otimes \Phi$ is a bimodule morphism from $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$ to $\mathcal{F}' \otimes_{\mathcal{B}} \mathcal{E}'$. \square

Note that by embedding $\text{C}_2\text{Bimod}(\mathcal{B}, \mathcal{A})$ into $\text{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ we can also define a bimodule $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$ for bimodules over coisotropic pairs. Putting these two lemmas together we obtain functors

$$\otimes_{\mathcal{B}} : \text{C}_3\text{Bimod}(\mathcal{C}, \mathcal{B}) \times \text{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A}) \longrightarrow \text{C}_3\text{Bimod}(\mathcal{C}, \mathcal{A}). \tag{45}$$

and

$$\otimes_{\mathcal{B}} : \text{C}_2\text{Bimod}(\mathcal{C}, \mathcal{B}) \times \text{C}_2\text{Bimod}(\mathcal{B}, \mathcal{A}) \longrightarrow \text{C}_2\text{Bimod}(\mathcal{C}, \mathcal{A}). \tag{46}$$

as wanted.

As a second step we need to show that the tensor product fulfills the associativity and identity properties of a bicategory.

LEMMA 4.5 *For coisotropic triples $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ of algebras over \mathbb{k} there is a natural isomorphism*

$$\text{asso} : \otimes_{\mathcal{B}} \circ (\otimes_{\mathcal{C}} \times \text{id}) \implies \otimes_{\mathcal{C}} \circ (\text{id} \times \otimes_{\mathcal{B}}), \tag{47}$$

given by the natural isomorphisms of associativity for usual bimodules

$$\begin{aligned} \text{asso}_{\text{tot}} : (\mathcal{G}_{\text{tot}} \otimes_{\mathcal{C}_{\text{tot}}} \mathcal{F}_{\text{tot}}) \otimes_{\mathcal{B}_{\text{tot}}} \mathcal{E}_{\text{tot}} &\longrightarrow \mathcal{G}_{\text{tot}} \otimes_{\mathcal{C}_{\text{tot}}} (\mathcal{F}_{\text{tot}} \otimes_{\mathcal{B}_{\text{tot}}} \mathcal{E}_{\text{tot}}) \\ (z \otimes y) \otimes x &\longmapsto z \otimes (y \otimes x) \end{aligned} \tag{48}$$

and

$$\begin{aligned} \text{asso}_N : (\mathcal{G}_N \otimes_{\mathcal{C}_N} \mathcal{F}_N) \otimes_{\mathcal{B}_N} \mathcal{E}_N &\longrightarrow \mathcal{G}_N \otimes_{\mathcal{C}_N} (\mathcal{F}_N \otimes_{\mathcal{B}_N} \mathcal{E}_N) \\ (z \otimes y) \otimes x &\longmapsto z \otimes (y \otimes x) \end{aligned} \tag{49}$$

for $\mathcal{G} \in \text{C}_3\text{Bimod}(\mathcal{D}, \mathcal{C})$, $\mathcal{F} \in \text{C}_3\text{Bimod}(\mathcal{C}, \mathcal{B})$, and $\mathcal{E} \in \text{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$.

PROOF: First it is clear that the above definitions have the necessary multilinearity to extend to the tensor products at all. We then need to check that $\text{asso}_{\text{tot}} \circ ((\iota_{\mathcal{G}} \otimes \iota_{\mathcal{F}}) \otimes \iota_{\mathcal{E}}) = (\iota_{\mathcal{G}} \otimes (\iota_{\mathcal{F}} \otimes \iota_{\mathcal{E}})) \circ \text{asso}_N$ holds and that the morphisms asso_N preserve the submodules. This is an easy computation done on factorizing tensors. \square

LEMMA 4.6 *For coisotropic triples of algebras \mathcal{A} and \mathcal{B} over \mathbb{k} there are natural isomorphisms*

$$\text{left} : \otimes_{\mathcal{B}} \circ (\text{Id}_{\mathcal{B}} \times \text{id}) \implies \text{id} \quad \text{and} \quad \text{right} : \otimes_{\mathcal{A}} \circ (\text{id} \times \text{Id}_{\mathcal{A}}) \implies \text{id}, \quad (50)$$

given by the left and right identities of the tensor product of usual bimodules

$$\text{left}_{\text{tot}} : \mathcal{B}_{\text{tot}} \otimes_{\mathcal{B}_{\text{tot}}} \mathcal{E}_{\text{tot}} \ni b \otimes x \longmapsto bx \in \mathcal{E}_{\text{tot}} \quad (51)$$

and

$$\text{left}_{\mathbb{N}} : \mathcal{B}_{\mathbb{N}} \otimes_{\mathcal{B}_{\mathbb{N}}} \mathcal{E}_{\mathbb{N}} \ni b \otimes x \longmapsto bx \in \mathcal{E}_{\mathbb{N}} \quad (52)$$

as well as

$$\text{right}_{\text{tot}} : \mathcal{E}_{\text{tot}} \otimes_{\mathcal{A}_{\text{tot}}} \mathcal{A}_{\text{tot}} \ni x \otimes a \longmapsto xa \in \mathcal{E}_{\text{tot}} \quad (53)$$

and

$$\text{right}_{\mathbb{N}} : \mathcal{E}_{\mathbb{N}} \otimes_{\mathcal{A}_{\mathbb{N}}} \mathcal{A}_{\mathbb{N}} \ni x \otimes a \longmapsto xa \in \mathcal{E}_{\mathbb{N}}. \quad (54)$$

PROOF: Since left_{tot} and $\text{left}_{\mathbb{N}}$ are natural isomorphisms we only need to show that they form morphisms of bimodules over triples when put together. For this let $b \in \mathcal{B}_{\mathbb{N}}$ and $x \in \mathcal{E}_{\mathbb{N}}$, then

$$(\text{left}_{\text{tot}} \circ (\iota_{\mathcal{B}} \otimes \iota_{\mathcal{E}}))(b \otimes x) = b \cdot \iota_{\mathcal{E}}(x) = \iota_{\mathcal{E}}(bx) = (\iota_{\mathcal{E}} \circ \text{left}_{\mathbb{N}})(b \otimes x)$$

holds. Additionally, observe that

$$\text{left}_{\mathbb{N}}((\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E})_0) = \text{left}_{\mathbb{N}}(\mathcal{B}_{\mathbb{N}} \otimes_{\mathcal{B}_{\mathbb{N}}} \mathcal{E}_{\mathbb{N}} + \mathcal{B}_0 \otimes_{\mathcal{B}_{\mathbb{N}}} \mathcal{E}_{\mathbb{N}}) = \mathcal{B}_{\mathbb{N}} \cdot \mathcal{E}_0 + \mathcal{B}_0 \cdot \mathcal{E}_{\mathbb{N}} = \mathcal{E}_0,$$

since $\mathcal{B}_{\mathbb{N}}$ is unital and $\mathcal{B}_0 \cdot \mathcal{E}_{\mathbb{N}} \subseteq \mathcal{E}_0$. Hence left is a natural isomorphism as claimed. An analogous computation shows that also right is a natural isomorphism. \square

Finally, Lemmas 4.3, 4.4, 4.5, and 4.6 imply that C_3Bimod and C_2Bimod in fact form bicategories.

THEOREM 4.7 (BICATEGORY OF COISOTROPIC MODULES) *Taking coisotropic triples of algebras as 0-morphisms, bimodules over coisotropic triples of algebras as 1-morphisms and morphisms between such bimodules as 2-morphisms, together with the tensor product, associativity and identities as constructed in Lemmas 4.3, 4.4, 4.5, and 4.6 we obtain a bicategory.*

PROOF: For any two coisotropic triples of algebras \mathcal{A} and \mathcal{B} over \mathbb{k} we have a category $\text{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ by Definition 4.2. The tensor product as introduced in Lemma 4.3 is functorial due to Lemma 4.4. Moreover, Lemma 4.5 and Lemma 4.6 ensure the existence of natural transformations of associativity and identity. Finally, the coherences have to be checked, but since the associativity and identity natural transformations are for every component those of usual bimodules it is clear that the coherence diagrams from Definition A.1 commute. \square

Again, by forgetting the tot-component we obtain also a bicategory of coisotropic pairs, modules over coisotropic pairs and morphisms between them:

DEFINITION 4.8 (BICATEGORY OF COISOTROPIC MODULES) *The bicategory with coisotropic triples of algebras as 0-morphisms, bimodules over coisotropic triples of algebras as 1-morphisms and morphisms between such bimodules as 2-morphisms from Theorem 4.7 is called the bicategory of coisotropic triples and will be denoted by $C_3\text{Bimod}$. Similarly, the bicategory of coisotropic pairs of algebras, bimodules over coisotropic pairs, and bimodule morphisms is called the bicategory of coisotropic pairs and will be denoted by $C_2\text{Bimod}$.*

We can embed the category Alg of unital algebras into the bicategory Bimod of unital Algebras with bimodules by turning an algebra morphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ into a bimodule ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\phi}$, with right multiplication twisted by ϕ , and adding as 2-morphisms only identities. Similarly, we can view the category $C_3\text{Alg}$ of coisotropic triples of algebras as a bicategory, by simply adding as 2-morphisms only the identities. This allows us to embed $C_3\text{Alg}$ into the bicategory $C_3\text{Bimod}$.

PROPOSITION 4.9 *The following data defines a functor of bicategories $L: C_3\text{Alg} \rightarrow C_3\text{Bimod}$:*

1. For $\mathcal{A} \in C_3\text{Alg}$ define $L(\mathcal{A}) = \mathcal{A}$.
2. For $\mathcal{A}, \mathcal{B} \in C_3\text{Alg}$ a functor $L: C_3\text{Alg}(\mathcal{B}, \mathcal{A}) \rightarrow C_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ defined by

$$L_{\mathcal{B}\mathcal{A}}(\phi) = \left({}_{\mathcal{B}_{\text{tot}}}(\mathcal{B}_{\text{tot}})_{\mathcal{A}_{\text{tot}}}^{\phi}, {}_{\mathcal{B}_N}(\mathcal{B}_N)_{\mathcal{A}_N}^{\phi}, {}_{\mathcal{B}_0}(\mathcal{B}_0)_{\mathcal{A}_0}^{\phi} \right) \tag{55}$$

for $\phi \in C_3\text{Alg}(\mathcal{B}, \mathcal{A})$. Here ϕ in superscript means that the right module structure is twisted with ϕ .

3. For $\mathcal{A}, \mathcal{B}, \mathcal{C} \in C_3\text{Alg}$ a natural isomorphism $m: \otimes_{\mathcal{B}} \circ (L_{\mathcal{C}\mathcal{B}} \times L_{\mathcal{B}\mathcal{A}}) \rightarrow L_{\mathcal{C}\mathcal{A}} \circ (\circ_{C_2\text{Alg}})$ given by

$$m(\psi, \phi): {}_{\mathcal{C}}\mathcal{C}_{\mathcal{B}}^{\psi} \otimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\phi} \ni c \otimes b \mapsto c\psi(b) \in {}_{\mathcal{C}}\mathcal{C}_{\mathcal{A}}^{\psi \circ \phi} \tag{56}$$

for $\phi \in C_3\text{Alg}(\mathcal{B}, \mathcal{A})$ and $\psi \in C_3\text{Alg}(\mathcal{C}, \mathcal{B})$.

This functor is even an embedding of bicategories.

PROOF: The map (55) defines a functor by the usual extension to the discrete category. Since for modules of the form (55) we have $\mathcal{B}_N^{\phi} \subseteq \mathcal{B}_{\text{tot}}^{\phi}$, we only need to check that the natural transformation (56) preserves the N- and 0-components: for $c \otimes b \in {}_{\mathcal{C}}\mathcal{C}_{\mathcal{B}}^{\psi} \otimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\phi}$ we have $m(\psi, \phi)(c \otimes b) = c\psi(b) \in {}_{\mathcal{C}}\mathcal{C}_N$ and for $c \otimes b_0 + c_0 \otimes b \in ({}_{\mathcal{C}}\mathcal{C}_{\mathcal{B}}^{\psi} \otimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}^{\phi})_0 = {}_{\mathcal{C}}\mathcal{C}_N^{\psi} \otimes {}_{\mathcal{B}_0}\mathcal{B}_{\mathcal{A}}^{\phi} + {}_{\mathcal{C}_0}\mathcal{C}_{\mathcal{B}}^{\psi} \otimes {}_{\mathcal{B}_N}\mathcal{B}_{\mathcal{A}}^{\phi}$ we have $m(\psi, \phi)(c \otimes b_0 + c_0 \otimes b) = c\psi(b_0) + c_0\psi(b) \in {}_{\mathcal{C}_0}$. The composition and identity coherences from Definition A.2 are easy computations. Finally, L is clearly injective on objects and also on 1-morphisms, since changing $\phi \in C_3\text{Alg}(\mathcal{B}, \mathcal{A})$ will lead to different bimodule structures on $L_{\mathcal{B}\mathcal{A}}(\phi)$. \square

This embedding of $C_3\text{Alg}$ in $C_3\text{Bimod}$ can also be defined by omitting the tot-component, giving an embedding of $C_2\text{Alg}$ in $C_3\text{Alg}$.

REMARK 4.10 *Note that also the projections onto the tot-component and the N-component yield functors of bicategories*

$$\text{tot}: C_3\text{Bimod} \longrightarrow \text{Bimod} \tag{57}$$

and

$$N: C_3\text{Bimod} \longrightarrow \text{Bimod}, \tag{58}$$

respectively. Note that for these functors the natural isomorphisms from Definition A.2 are in fact identities, simplifying the situation.

5 MORITA EQUIVALENCE OF COISOTROPIC ALGEBRAS

In classical Morita theory two algebras (or rings) are Morita equivalent if and only if they are isomorphic in the bicategory Bimod of algebras, bimodules and bimodule morphisms. Recall that two objects in a bicategory are called *isomorphic* if there exists an (up to 2-morphisms) invertible 1-morphism between them. Having defined the bicategories $C_3\text{Bimod}$ and $C_2\text{Bimod}$ we can now give a definition of Morita equivalence of coisotropic triples and pairs of algebras.

DEFINITION 5.1 (MORITA EQUIVALENCE OF COISOTROPIC ALGEBRAS) *Two coisotropic triples \mathcal{A}, \mathcal{B} of algebras are called Morita equivalent if they are isomorphic in the bicategory $C_3\text{Bimod}$. Similarly, two coisotropic pairs \mathcal{A}, \mathcal{B} are called Morita equivalent if they are isomorphic in the bicategory $C_2\text{Bimod}$. An invertible bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ implementing a Morita equivalence of \mathcal{A} and \mathcal{B} is called Morita equivalence bimodule in both cases.*

Thus Morita equivalence of coisotropic algebras is completely encoded in the so called Picard bigroupoids $\text{Pic}(C_3\text{Bimod})$ and $\text{Pic}(C_2\text{Bimod})$, given by all coisotropic algebras and all corresponding invertible 1- and 2-morphisms.

Let $\mathcal{A}, \mathcal{B} \in C_3\text{Alg}$ be Morita equivalent, and let furthermore $\mathcal{E} \in C_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be a $(\mathcal{B}, \mathcal{A})$ -bimodule implementing Morita equivalence of \mathcal{A} and \mathcal{B} . Thus we assume that there exists a $(\mathcal{A}, \mathcal{B})$ -bimodule $\mathcal{E}' \in C_3\text{Bimod}(\mathcal{A}, \mathcal{B})$ and isomorphisms

$$\phi: \mathcal{E}' \otimes_{\mathcal{B}} \mathcal{E} \longrightarrow \mathcal{A} \quad \text{and} \quad \psi: \mathcal{E} \otimes_{\mathcal{B}} \mathcal{E}' \longrightarrow \mathcal{B} \tag{59}$$

such that

$$\psi(x \otimes x') \cdot y = x \cdot \phi(x' \otimes y) \tag{60}$$

holds for all $x, y \in \mathcal{E}$ and $x' \in \mathcal{E}'$. Note that (60) can always be achieved by turning a usual equivalence into an adjoint equivalence, see [27, A.1.3]. The

fact that (59) are morphisms of coisotropic bimodules means in particular that the diagram

$$\begin{array}{ccc}
 \mathcal{E}_{\text{tot}} \otimes \mathcal{E}'_{\text{tot}} & \xrightarrow{\psi_{\text{tot}}} & \mathcal{B}_{\text{tot}} \\
 \uparrow \iota_{\mathcal{E}} \otimes \iota_{\mathcal{E}'} & & \uparrow \iota_{\mathcal{B}} \\
 \mathcal{E}_{\text{N}} \otimes \mathcal{E}'_{\text{N}} & \xrightarrow{\psi_{\text{N}}} & \mathcal{B}_{\text{N}}
 \end{array} \tag{61}$$

commutes. Then, since ψ_{N} and $\iota_{\mathcal{B}}$ are injective, so is $\psi_{\text{tot}} \circ \iota_{\mathcal{E}} \otimes \iota_{\mathcal{E}'} = \iota_{\mathcal{B}} \circ \psi_{\text{N}}$ and hence also $\iota_{\mathcal{E}} \otimes \iota_{\mathcal{E}'}$.

Since by projecting onto the tot- and N-components yields functors of bicategories $\text{tot}: \mathbf{C}_3\text{Bimod} \rightarrow \mathbf{Bimod}$ and $\text{N}: \mathbf{C}_3\text{Bimod} \rightarrow \mathbf{Bimod}$ to the bicategory of algebras and bimodules according to Remark 4.10 we know that Morita equivalence bimodules get mapped to Morita equivalence bimodules. Hence, by the classical theory of Morita equivalence for algebras we know, in particular, that \mathcal{E}_{tot} and \mathcal{E}_{N} are finitely generated projective modules, thus we have

$$\mathcal{E}_{\text{tot}} \simeq \mathbf{e}_{\text{tot}} \mathcal{A}_{\text{tot}}^n \tag{62}$$

and

$$\mathcal{E}_{\text{N}} \simeq \mathbf{e}_{\text{N}} \mathcal{A}_{\text{N}}^m \tag{63}$$

for some $n, m \in \mathbb{N}$ and full idempotents $\mathbf{e}_{\text{tot}} \in \text{End}(\mathcal{A}_{\text{tot}}^n)$ and $\mathbf{e}_{\text{N}} \in \text{End}(\mathcal{A}_{\text{N}}^m)$. The following lemma gives a way to relate these two finitely generated projective modules:

LEMMA 5.2 *Let $\mathcal{E} \in \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be a Morita equivalence bimodule of coisotropic algebras $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\text{Alg}$. Then every dual basis $\{e_j, e^j\}_{j=1, \dots, n}$ of the finitely generated projective module \mathcal{E}_{N} gives rise to a dual basis $\{e_j^{\text{tot}}, e^j_{\text{tot}}\}_{j=1, \dots, m}$ for \mathcal{E}_{tot} , given by*

$$e_j^{\text{tot}} = \iota_{\mathcal{E}}(e_j) \tag{64}$$

and

$$e^j_{\text{tot}}(x) = \left(\sum_{i=1}^k (\iota_{\mathcal{A}} \circ e^j)(x_i^{\mathbb{1}_{\text{N}}}) \cdot \phi_{\text{tot}}(\iota_{\mathcal{E}'}(y_i^{\mathbb{1}_{\text{N}}}) \otimes x) \right), \tag{65}$$

where $\mathbb{1}_{\mathcal{B}_{\text{N}}} = \psi_{\text{N}}\left(\sum_{i=1}^k x_i^{\mathbb{1}_{\text{N}}} \otimes y_i^{\mathbb{1}_{\text{N}}}\right)$. For $x_{\text{N}} \in \mathcal{E}_{\text{N}}$ the dual basis (65) simplifies to

$$e^j_{\text{tot}}(\iota_{\mathcal{E}}(x_{\text{N}})) = \iota_{\mathcal{A}}(e^j(x_{\text{N}})). \tag{66}$$

PROOF: First we note that we actually find elements $x_i^{\mathbb{1}_{\text{N}}} \in \mathcal{E}_{\text{N}}$ and $y_i^{\mathbb{1}_{\text{N}}} \in \mathcal{E}'_{\text{N}}$ such that $\mathbb{1}_{\mathcal{B}_{\text{N}}} = \psi_{\text{N}}\left(\sum_{i=1}^k x_i^{\mathbb{1}_{\text{N}}} \otimes y_i^{\mathbb{1}_{\text{N}}}\right)$ since ψ_{N} is surjective. Since $\mathcal{B}_{\text{N}} \subseteq \mathcal{B}_{\text{tot}}$ is a unital subalgebra it follows that $\mathbb{1}_{\mathcal{B}_{\text{tot}}} = \psi_{\text{tot}}\left(\sum_{i=1}^k \iota_{\mathcal{E}}(x_i^{\mathbb{1}_{\text{N}}}) \otimes \iota_{\mathcal{E}'}(y_i^{\mathbb{1}_{\text{N}}})\right)$ by

using the commutativity of (61). Now fix a dual basis of \mathcal{E}_N such that for any $x_N \in \mathcal{E}_N$ we have $x_N = \sum_{j=1}^m e_j \cdot e^j(x)$. Then for $x \in \mathcal{E}_{\text{tot}}$ we get

$$\begin{aligned} x &= \mathbb{1}_{\mathcal{B}_{\text{tot}}} \cdot x \\ &= \sum_{i=1}^k \psi_{\text{tot}} (\iota_{\mathcal{E}}(x_i^{\mathbb{1}_N}) \otimes \iota_{\mathcal{E}'}(y_i^{\mathbb{1}_N})) \cdot x \\ &\stackrel{(a)}{=} \sum_{i=1}^k \iota_{\mathcal{E}}(x_i^{\mathbb{1}_N}) \cdot \phi_{\text{tot}} (\iota_{\mathcal{E}'}(y_i^{\mathbb{1}_N}) \otimes x) \\ &= \sum_{i=1}^k \iota_{\mathcal{E}} \left(\sum_{j=1}^m e_j \cdot e^j(x_i^{\mathbb{1}_N}) \right) \cdot \phi_{\text{tot}} (\iota_{\mathcal{E}'}(y_i^{\mathbb{1}_N}) \otimes x) \\ &= \sum_{i=1}^k \left(\sum_{j=1}^m \iota_{\mathcal{E}}(e_j) \cdot (\iota_{\mathcal{A}} \circ e^j)(x_i^{\mathbb{1}_N}) \right) \cdot \phi_{\text{tot}} (\iota_{\mathcal{E}'}(y_i^{\mathbb{1}_N}) \otimes x) \\ &= \sum_{j=1}^m \underbrace{\iota_{\mathcal{E}}(e_j)}_{=: e_j^{\text{tot}}} \cdot \underbrace{\left(\sum_{i=1}^k (\iota_{\mathcal{A}} \circ e^j)(x_i^{\mathbb{1}_N}) \cdot \phi_{\text{tot}} (\iota_{\mathcal{E}'}(y_i^{\mathbb{1}_N}) \otimes x) \right)}_{=: e_{\text{tot}}^j(x)}, \end{aligned}$$

where we used (60) in (a). Note that indeed $e_j^{\text{tot}} \in \mathcal{E}_{\text{tot}}$ and $e_{\text{tot}}^j \in \mathcal{E}_{\text{tot}}^*$. Now for $x_N \in \mathcal{E}_N$ we compute using (61) that $e_{\text{tot}}^j(\iota_{\mathcal{E}}(x_N)) = \iota_{\mathcal{A}}(e^j(x_N))$ holds. \square

Thus we can choose the isomorphisms (62) and (63) such that $n = m$. This leads us to the next lemma, showing that in addition for a Morita equivalence bimodule the projectors for the tot- and N-components agree:

LEMMA 5.3 *Let $\mathcal{E} \in \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be a Morita equivalence bimodule of coisotropic algebras $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\text{Alg}$. Then we can choose the isomorphisms $\mathcal{E}_{\text{tot}} \simeq \mathbf{e}_{\text{tot}} \mathcal{A}_{\text{tot}}^n$ and $\mathcal{E}_N \simeq \mathbf{e}_N \mathcal{A}_N^n$ such that $\mathbf{e}_{\text{tot}} = \mathbf{e}_N \in \mathbf{M}_n(\mathcal{A}_N)$.*

PROOF: First we fix a dual basis of \mathcal{E}_N with corresponding dual basis of \mathcal{E}_{tot} according to Lemma 5.2. Then the components of $\mathbf{e}_{\text{tot}} \in \mathbf{M}_n(\mathcal{A}_{\text{tot}})$ are given by

$$(\mathbf{e}_{\text{tot}})_{ij} = e_{\text{tot}}^i(e_j^{\text{tot}}) = e_{\text{tot}}^i(\iota_{\mathcal{E}}(e_j)) = \iota_{\mathcal{A}}(e^i(e_j)) = \iota_{\mathcal{A}}((\mathbf{e}_N)_{ij}).$$

Thus viewing elements of \mathcal{A}_N as elements in \mathcal{A}_{tot} via the embedding $\iota_{\mathcal{A}}$ gives the statement. \square

As a first consequence, given a classical Morita equivalence bimodule $\mathcal{E}_{\text{tot}} = \mathbf{e}_{\text{tot}} \mathcal{A}_{\text{tot}}^n$ for the tot-components \mathcal{B}_{tot} and \mathcal{A}_{tot} we know that necessarily this can only be a Morita equivalence bimodule for coisotropic triples of algebras $\mathcal{A} = (\mathcal{A}_{\text{tot}}, \mathcal{A}_N, \mathcal{A}_0)$ and $\mathcal{B} = (\mathcal{B}_{\text{tot}}, \mathcal{B}_N, \mathcal{B}_0)$ if $\mathbf{e}_{\text{tot}} \in \mathbf{M}_n(\mathcal{A}_N)$.

Now that we clarified the structure and relation of the N- and tot-components of coisotropic Morita equivalence bimodules let us turn to a description of the 0-part. Here by definition of a coisotropic module we have $\mathcal{E}_N \cdot \mathcal{A}_0 \subseteq \mathcal{E}_0$. For Morita equivalence bimodules we get in fact equality.

LEMMA 5.4 *Let $\mathcal{E} \in \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be a Morita equivalence bimodule of coisotropic algebras $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\text{Alg}$. Then $\mathcal{E}_N \cdot \mathcal{A}_0 = \mathcal{E}_0$.*

PROOF: Note that the inclusion $\mathcal{E}_N \cdot \mathcal{A}_0 \subseteq \mathcal{E}_0$ holds by definition. For the other inclusion we use $\mathcal{A}_0 \simeq (\mathcal{E}' \otimes_{\mathcal{B}} \mathcal{E})_0 = \mathcal{E}'_N \otimes_{\mathcal{B}_N} \mathcal{E}_0 + \mathcal{E}'_0 \otimes_{\mathcal{B}_N} \mathcal{E}_N$ by (59) and the definition of the tensor product. Thus we get

$$\mathcal{E}_N \cdot \mathcal{A}_0 \simeq \mathcal{E}_N \otimes \mathcal{E}'_0 \otimes \mathcal{E}_N + \mathcal{E}_N \otimes \mathcal{E}'_N \otimes \mathcal{E}_0 \simeq \mathcal{E}_N \otimes \mathcal{E}'_0 \otimes \mathcal{E}_N + \mathcal{E}_0$$

showing that $\mathcal{E}_0 \subseteq \mathcal{E}_N \cdot \mathcal{A}_0$. □

Putting these previous statements together we get a quite explicit description of Morita equivalence bimodules for coisotropic algebras, similar to the well-known description of Morita equivalence bimodules for classical rings by Morita’s theorems.

THEOREM 5.5 *Let $\mathcal{E} \in \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be a coisotropic triple of bimodules over coisotropic triples of algebras \mathcal{A} and \mathcal{B} . The bimodule \mathcal{E} is a Morita equivalence bimodule if and only if there exists an isomorphism of coisotropic bimodules such that*

$$\mathcal{E}_{\text{tot}} \simeq \mathbf{e}\mathcal{A}_{\text{tot}}^n, \tag{67}$$

$$\mathcal{E}_N \simeq \mathbf{e}\mathcal{A}_N^n, \tag{68}$$

$$\mathcal{E}_0 \simeq \mathbf{e}\mathcal{A}_0^n \tag{69}$$

with a full projection $\mathbf{e} \in M_n(\mathcal{A}_N)$ and

$$\mathcal{B}_{\text{tot}} \simeq \text{End}_{\mathcal{A}_{\text{tot}}}(\mathcal{E}_{\text{tot}}), \tag{70}$$

$$\mathcal{B}_N \simeq \text{End}_{\mathcal{A}_N}(\mathcal{E}_N), \tag{71}$$

$$\mathcal{B}_0 \simeq \text{Hom}_{\mathcal{A}_N}(\mathcal{E}_N, \mathcal{E}_0), \tag{72}$$

where all isomorphisms are given by left multiplication and we view $\text{Hom}_{\mathcal{A}_N}(\mathcal{E}_N, \mathcal{E}_0)$ as a subset of $\text{End}_{\mathcal{A}_N}(\mathcal{E}_N)$.

PROOF: Let \mathcal{E} be a Morita equivalence bimodule. Fix a dual basis $\{e_j, e^j\}_{j=1, \dots, n}$ for \mathcal{E}_N and consider the dual basis $\{e_j^{\text{tot}}, e_{\text{tot}}^j\}_{j=1, \dots, m}$ of \mathcal{E}_{tot} as constructed in Lemma 5.2. These dual bases give rise to isomorphisms

$$g_N : \mathcal{E}_N \ni \sum_{i=1}^n e_i e^i(x) \mapsto \sum_{i=1}^n b_i e^i(x) \in \mathbf{e}\mathcal{A}_N^n,$$

where b_i is the standard basis of \mathcal{A}_N^n , and similarly $g_{\text{tot}} : \mathcal{E}_{\text{tot}} \rightarrow \mathbf{e}\mathcal{A}_{\text{tot}}^n$. The idempotent \mathbf{e} is full since \mathcal{E}_N is a classical Morita equivalence bimodule. A straightforward computation shows that $\iota_{\mathcal{A}} \circ g_N = g_{\text{tot}} \circ \iota_{\mathcal{E}}$. The compatibility of g_N with \mathcal{E}_0 is clear by Lemma 5.4. Thus we get an isomorphism of coisotropic bimodules. Moreover, since \mathcal{E}_{tot} and \mathcal{E}_N are classical Morita equivalence bimodules of the tot- and N-components, respectively, we immediately get (70) and

(71). For (72) we only need to show that $\text{im}(\mathcal{B}_0) = \text{Hom}_{\mathcal{A}_N}(\mathcal{E}_N, \mathcal{E}_0)$ under left multiplication. For this let $\xi \in \text{Hom}_{\mathcal{A}_N}(\mathcal{E}_N, \mathcal{E}_0)$ and $\mathbb{1}_{\mathcal{B}_N} = \psi_N(\sum_{i=1}^k x_i^{\mathbb{1}_N} \otimes y_i^{\mathbb{1}_N})$ as before. Then $\psi_N(\xi(x_i^{\mathbb{1}_N}) \otimes y_i^{\mathbb{1}_N}) \in \mathcal{B}_0$ and

$$\begin{aligned} \psi_N\left(\sum_{i=1}^k \xi(x_i^{\mathbb{1}_N}) \otimes y_i^{\mathbb{1}_N}\right) \cdot x &= \sum_{i=1}^k \xi(x_i^{\mathbb{1}_N}) \phi_N(y_i^{\mathbb{1}_N} \otimes x) \\ &= \xi\left(\sum_{i=1}^k \psi_N(x_i^{\mathbb{1}_N} \otimes y_i^{\mathbb{1}_N})x\right) = \xi(x) \end{aligned}$$

shows that $\mathcal{B}_0 \simeq \text{Hom}_{\mathcal{A}_N}(\mathcal{E}_N, \mathcal{E}_0)$. Now let $\mathbf{e} \in M_n(\mathcal{A}_N)$ be given such that (67) to (72) hold. Then since \mathcal{A}_N is unital we have $\mathcal{A}_{\text{tot}}\mathbf{e}\mathcal{A}_{\text{tot}} = \mathcal{A}_{\text{tot}}$, i.e. $\mathbf{e} \in M_n(\mathcal{A}_{\text{tot}})$ is full. Moreover, since \mathcal{E}_{tot} and \mathcal{E}_N are Morita equivalence bimodules in the classical sense they have inverse bimodules $(\mathcal{E}_{\text{tot}})'$ and $(\mathcal{E}_N)'$. Then setting $(\mathcal{E}')_{\text{tot}} = (\mathcal{E}_{\text{tot}})'$, $(\mathcal{E}')_N = (\mathcal{E}_N)'$ and $(\mathcal{E}')_0 = \mathcal{A}_0 \cdot \mathcal{E}'_N$ defines an inverse coisotropic bimodule for \mathcal{E} . Hence \mathcal{E} is a Morita equivalence bimodule. \square

From this it directly follows that for an equivalence bimodule the map from the N-component to the tot-component is in fact injective.

COROLLARY 5.6 *Let $\mathcal{E} \in \mathbf{C}_3\text{Bimod}(\mathcal{B}, \mathcal{A})$ be a Morita equivalence bimodule for $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\text{Alg}$. Then $\iota_{\mathcal{E}} : \mathcal{E}_N \rightarrow \mathcal{E}_{\text{tot}}$ is injective, i.e. $\mathcal{E}_N \subseteq \mathcal{E}_{\text{tot}}$ is a submodule.*

REMARK 5.7 *On the one hand, the theorem gives a complete picture of how the equivalence bimodules for coisotropic triples of algebras look like. On the other hand, it is quite bad news that the N-component controls and determines the other components of the bimodule. It will be the one which is the most inaccessible in the examples of deformation quantization.*

EXAMPLE 5.8 (STANDARD EXAMPLE) *From the above characterization we obtain the first standard example: for a coisotropic triple \mathcal{A} the matrices*

$$M_n(\mathcal{A}) = (M_n(\mathcal{A}_{\text{tot}}), M_n(\mathcal{A}_N), M_n(\mathcal{A}_0)) \tag{73}$$

form again a coisotropic triple of algebras which is now Morita equivalent to \mathcal{A} for all $n \in \mathbb{N}$. As equivalence bimodule we can take

$$\mathcal{A}^n = (\mathcal{A}_{\text{tot}}^n, \mathcal{A}_N^n, \mathcal{A}_0^n). \tag{74}$$

6 REDUCTION FOR BIMODULES

Following the idea of constructing vector bundles on the reduced manifold by reducing a vector bundle on the manifold we started with, we want to turn bimodules over coisotropic algebras into bimodules over the reduced algebras. The idea is to proceed similarly to the algebra case and consider the quotient $\mathcal{E}_N/\mathcal{E}_0$, see Proposition 3.5. Again this construction uses only the information of

the N - and 0 -components. Therefore we only consider reduction for bimodules over coisotropic pairs. Reduction for bimodules over coisotropic triples is then given by first forgetting about the tot -component. This construction indeed yields bimodules over the reduced algebras and better still it is compatible with the bicategory structure of $\mathbf{C}_2\text{Bimod}$ in the best possible way, i.e. we get a functor of bicategories, called *reduction functor*:

THEOREM 6.1 (REDUCTION IN $\mathbf{C}_2\text{Bimod}$) *A functor of bicategories $\text{red}: \mathbf{C}_2\text{Bimod} \rightarrow \mathbf{Bimod}$ is given by the following data:*

1. A map $\text{red}: \text{Obj}(\mathbf{C}_2\text{Bimod}) \rightarrow \text{Obj}(\mathbf{Bimod})$ on objects, given by

$$\mathcal{A} \mapsto \mathcal{A}_{\text{red}}. \tag{75}$$

2. For any two coisotropic pairs of algebras \mathcal{B} and \mathcal{A} a functor

$$\text{red}_{\mathcal{B}\mathcal{A}}: \mathbf{C}_2\text{Bimod}(\mathcal{B}, \mathcal{A}) \rightarrow \mathbf{C}_2\text{Bimod}(\mathcal{B}_{\text{red}}, \mathcal{A}_{\text{red}}), \tag{76}$$

given by

$$\mathcal{E}_{\text{red}} = \mathcal{E}_N / \mathcal{E}_0 \tag{77}$$

on objects and

$$\Phi_{\text{red}}: \mathcal{E}_{\text{red}} \ni [x] \mapsto [\Phi(x)] \in \mathcal{F}_{\text{red}} \tag{78}$$

on morphisms $\Phi: \mathcal{E} \rightarrow \mathcal{F}$.

3. For any three coisotropic pairs of algebras \mathcal{A} , \mathcal{B} , and \mathcal{C} a natural isomorphism $\mathfrak{m}_{\mathcal{C}\mathcal{B}\mathcal{A}}: \otimes_{\mathcal{B}_{\text{red}}} \circ (\text{red}_{\mathcal{C}\mathcal{B}} \times \text{red}_{\mathcal{B}\mathcal{A}}) \implies \text{red}_{\mathcal{C}\mathcal{A}} \circ \otimes_{\mathcal{B}}$ given by a family of maps determined by

$$\mathfrak{m}(\mathcal{F}, \mathcal{E}): \mathcal{F}_{\text{red}} \otimes_{\mathcal{B}_{\text{red}}} \mathcal{E}_{\text{red}} \ni [y] \otimes [x] \mapsto [y \otimes x] \in (\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_{\text{red}} \tag{79}$$

with $\mathcal{F} \in \mathbf{C}_2\text{Bimod}(\mathcal{C}, \mathcal{B})$, $\mathcal{E} \in \mathbf{C}_2\text{Bimod}(\mathcal{B}, \mathcal{A})$.

4. For any coisotropic pair of algebras \mathcal{A} the identity 2-isomorphism

$$\text{id}: {}_{\mathcal{A}_{\text{red}}}\mathcal{A}_{\text{red}}\mathcal{A}_{\text{red}} \rightarrow {}_{\text{red}\mathcal{A}\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}). \tag{80}$$

PROOF: First note that (75) is well-defined since \mathcal{A}_0 is a two-sided ideal in \mathcal{A}_N . Furthermore, (77) gives a well-defined $(\mathcal{B}_{\text{red}}, \mathcal{A}_{\text{red}})$ -bimodule by the definition of modules over coisotropic algebras, and (78) is well-defined since morphisms of modules over coisotropic algebras preserve the submodules. It is a standard computation to check that (79) is a well-defined natural isomorphism. Note, that here we crucially need that $(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_0 = \mathcal{F}_N \otimes_{\mathcal{B}_N} \mathcal{E}_0 + \mathcal{F}_0 \otimes_{\mathcal{B}_N} \mathcal{E}_N$. For the last part it is clear that ${}_{\text{red}\mathcal{A}\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}) = \mathcal{A}_N / \mathcal{A}_0 = {}_{\mathcal{A}_{\text{red}}}\mathcal{A}_{\text{red}}\mathcal{A}_{\text{red}}$. \square

This reduction functor for $C_2\text{Bimod}$ is also compatible with the reduction in $C_2\text{Alg}$ in the sense that

$$\begin{array}{ccc}
 C_2\text{Alg} & \xleftarrow{L} & C_2\text{Bimod} \\
 \text{red} \downarrow & & \downarrow \text{red} \\
 \text{Alg} & \xleftarrow{\quad} & \text{Bimod}
 \end{array} \tag{81}$$

commutes. Indeed, on coisotropic algebras both reduction functors are defined the same and for a morphism $\phi \in C_2\text{Alg}(\mathcal{B}, \mathcal{A})$ we clearly have $\mathcal{B}_{\text{red}}^{\phi_{\text{red}}} = (\mathcal{B}^{\phi})_{\text{red}}$ as sets and the $(\mathcal{B}_{\text{red}}, \mathcal{A}_{\text{red}})$ -bimodule structures also coincide.

Moreover, by identifying isomorphic coisotropic bimodules we can construct the classifying categories $[C_2\text{Bimod}]$ and $[\text{Bimod}]$. Since $\text{red}: C_2\text{Bimod} \rightarrow \text{Bimod}$ is a functor of bicategories we get a well-defined functor $\text{red}: [C_2\text{Bimod}] \rightarrow [\text{Bimod}]$, such that

$$\begin{array}{ccccc}
 C_2\text{Alg} & \xleftarrow{L} & C_2\text{Bimod} & \xrightarrow{[\cdot]} & [C_2\text{Bimod}] \\
 \text{red} \downarrow & & \downarrow \text{red} & & \downarrow \text{red} \\
 \text{Alg} & \xleftarrow{\quad} & \text{Bimod} & \xrightarrow{[\cdot]} & [\text{Bimod}]
 \end{array} \tag{82}$$

commutes.

Also, red maps invertible morphisms to invertible morphisms, thus it restricts to a functor

$$\text{red}: \text{Pic}(C_2\text{Bimod}) \rightarrow \text{Pic}(\text{Bimod}) \tag{83}$$

between the corresponding Picard bigroupoids. Similarly, we get a functor $\text{red}: [\text{Pic}(C_2\text{Bimod})] \rightarrow [\text{Pic}(\text{Bimod})]$ between the Picard groupoids, leading to the commutative diagram

$$\begin{array}{ccc}
 \text{Pic}(C_2\text{Bimod}) & \xrightarrow{[\cdot]} & [\text{Pic}(C_2\text{Bimod})] \\
 \text{red} \downarrow & & \downarrow \text{red} \\
 \text{Pic}(\text{Bimod}) & \xrightarrow{[\cdot]} & [\text{Pic}(\text{Bimod})]
 \end{array} \tag{84}$$

This means that reduction of coisotropic algebras preserves Morita equivalence.

EXAMPLE 6.2 *Let $\mathcal{A} \in C_3\text{Alg}$ be a commutative coisotropic triple of algebras. Then we can define a coisotropic bimodule $\text{Der}(\mathcal{A}) \in C_3\text{Bimod}(\mathcal{A})$ by*

$$\text{Der}(\mathcal{A})_{\text{tot}} = \text{Der}(\mathcal{A}_{\text{tot}}) \tag{85}$$

$$\text{Der}(\mathcal{A})_{\text{N}} = \{\Phi \in \text{Der}(\mathcal{A}_{\text{tot}}) \mid \Phi(\mathcal{A}_{\text{N}}) \subseteq \mathcal{A}_{\text{N}}, \Phi(\mathcal{A}_0) \subseteq \mathcal{A}_0\} \tag{86}$$

$$\text{Der}(\mathcal{A})_0 = \{\Phi \in \text{Der}(\mathcal{A}_{\text{tot}}) \mid \Phi(\mathcal{A}_{\text{N}}) \subseteq \mathcal{A}_0\}, \tag{87}$$

giving directly an injective module homomorphism $\text{Der}(\mathcal{A})_{\text{red}} \rightarrow \text{Der}(\mathcal{A}_{\text{red}})$. In most geometric examples this is in fact an isomorphism. Following Example 2.3 we have $\text{Der}(\mathcal{A}_{\text{tot}}) = \Gamma^\infty(TM)$ and we can extend every vector field on

the reduced manifold to a vector field on M by choosing a connection on the principal fibre bundle $C \rightarrow M_{\text{red}}$ and a tubular neighbourhood of C . Thus we get $\text{Der}(\mathcal{A})_{\text{red}} \simeq \Gamma^\infty(TM_{\text{red}})$ and therefore can construct the tangent bundle on M_{red} as a reduction of a coisotropic triple of modules.

7 CLASSICAL LIMIT

In formal deformation quantization one is interested in algebras of formal power series over a ring $\mathbb{R}[[\lambda]]$, e.g. $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ as algebra over $\mathbb{C}[[\lambda]]$ for a Poisson manifold M . Given such an $\mathbb{R}[[\lambda]]$ -algebra \mathcal{A} we can construct an \mathbb{R} -algebra, called the *classical limit*, by taking the quotient $\text{cl}(\mathcal{A}) = \mathcal{A}/\lambda\mathcal{A}$. The crucial property of cl is now that all multiples of λ will vanish, i.e. we have $\text{cl}(\lambda a) = 0$ for all $a \in \mathcal{A}$.

In the following we set the underlying ring as subscript for all involved categories in order to distinguish coisotropic triples and pairs of algebras over $\mathbb{R}[[\lambda]]$ from the ones over \mathbb{R} .

In order to define a classical limit for a coisotropic triple $\mathcal{A} \in \text{C}_3\text{Alg}_{\mathbb{R}[[\lambda]]}$ we can not simply set $\text{cl}(\mathcal{A})_{\mathbb{N}} = \text{cl}(\mathcal{A}_{\mathbb{N}})$, since this would not be a subset of $\text{cl}(\mathcal{A}_{\text{tot}})$ directly. Instead, we have to take its image in the classical limit of the tot-component, leading to the following definition for the classical limit of a coisotropic triple:

DEFINITION 7.1 (CLASSICAL LIMIT OF COISOTROPIC TRIPLES) *Let \mathcal{A} be a coisotropic triple over $\mathbb{R}[[\lambda]]$. Then the coisotropic triple*

$$\text{cl}(\mathcal{A})_{\text{tot}} = \text{cl}(\mathcal{A}_{\text{tot}}) \tag{88}$$

$$\text{cl}(\mathcal{A})_{\mathbb{N}} = \mathcal{A}_{\mathbb{N}}/(\lambda\mathcal{A}_{\text{tot}} \cap \mathcal{A}_{\mathbb{N}}) \subseteq \text{cl}(\mathcal{A})_{\text{tot}} \tag{89}$$

$$\text{cl}(\mathcal{A})_0 = \mathcal{A}_0/(\lambda\mathcal{A}_{\text{tot}} \cap \mathcal{A}_0) \subseteq \text{cl}(\mathcal{A})_{\mathbb{N}} \tag{90}$$

is called the classical limit of \mathcal{A} .

Note that $\text{cl}(\mathcal{A})_0$ is indeed a two-sided ideal in $\text{cl}(\mathcal{A})_{\mathbb{N}}$. In addition to the classical limit of deformed coisotropic triples we can also take the classical limit of morphisms of coisotropic triples. We define for a morphism $T: \mathcal{A} \rightarrow \mathcal{B}$ of coisotropic triples the classical limit $\text{cl}(T): \text{cl}(\mathcal{A}) \rightarrow \text{cl}(\mathcal{B})$ by setting $\text{cl}(T)(\text{cl}(\mathbf{a})) = \text{cl}(T(\mathbf{a}))$ for $\mathbf{a} \in \mathcal{A}_{\text{tot}}$. This is just the map defined on the quotient since every morphism $T: \mathcal{A} \rightarrow \mathcal{B}$ maps $\lambda\mathcal{A}_{\text{tot}}$ to $\lambda\mathcal{B}_{\text{tot}}$ by $\mathbb{R}[[\lambda]]$ -linearity.

Let us now check that the classical limit gives a functor from $\text{C}_3\text{Alg}_{\mathbb{R}[[\lambda]]}$ to $\text{C}_3\text{Alg}_{\mathbb{R}}$.

PROPOSITION 7.2 (CLASSICAL LIMIT FUNCTOR OF COISOTROPIC TRIPLES)

The classical limit

$$\text{cl}: \text{C}_3\text{Alg}_{\mathbb{R}[[\lambda]]} \rightarrow \text{C}_3\text{Alg}_{\mathbb{R}} \tag{91}$$

given by the classical limit of coisotropic triples on objects and quotient maps on morphisms is a functor.

PROOF: First we know that for coisotropic $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\mathbf{Alg}_{\mathbb{R}[[\lambda]]}$ and a morphism $T: \mathcal{A} \rightarrow \mathcal{B}$ the classical limits $\text{cl}(\mathcal{A}), \text{cl}(\mathcal{B})$ and $\text{cl}(T)$ are again coisotropic algebras and morphisms, respectively. Moreover, it is clear that $\text{cl}(\text{id}_{\mathcal{A}}) = \text{id}_{\text{cl}(\mathcal{A})}$. Let in addition $\mathcal{C} \in \mathbf{C}_3\mathbf{Alg}_{\mathbb{R}}$ be another coisotropic triple and $S: \mathcal{B} \rightarrow \mathcal{C}$ a morphism, then

$$\begin{aligned} \text{cl}(S \circ T)(\text{cl}(\mathbf{a})) &= \text{cl}((S \circ T)(\mathbf{a})) = \text{cl}(S)(\text{cl}(T(\mathbf{a}))) \\ &= \text{cl}(S)(\text{cl}(T)(\text{cl}(\mathbf{a}))) = (\text{cl}(S) \circ \text{cl}(T))(\text{cl}(\mathbf{a})) \end{aligned}$$

for $\mathbf{a} \in \mathcal{A}$, shows that the classical limit is functorial. □

By viewing $\mathbf{C}_2\mathbf{Alg}_{\mathbb{R}[[\lambda]]}$ as a subcategory of $\mathbf{C}_3\mathbf{Alg}_{\mathbb{R}[[\lambda]]}$ we can also define a classical limit functor $\text{cl}: \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}[[\lambda]]} \rightarrow \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}}$.

Now let us check if this classical limit is compatible with the reduction functor for coisotropic algebras. Since reduction of coisotropic triples is given by forgetting the tot-part and subsequent reduction of coisotropic pairs we only consider pairs from the start. Thus we want to clarify if the diagram

$$\begin{array}{ccc} \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}} \\ \text{red} \downarrow & & \downarrow \text{red} \\ \mathbf{Alg}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \mathbf{Alg}_{\mathbb{R}} \end{array} \tag{92}$$

commutes. Recall, that commutativity of a diagram of categories and functors means that all possible compositions between the same start and end points are the same up to natural isomorphisms.

PROPOSITION 7.3 *There exists a natural isomorphism $\eta: (\text{cl} \circ \text{red}) \implies (\text{red} \circ \text{cl})$ given by*

$$\eta_{\mathcal{A}}: \text{cl}(\mathcal{A}_{\text{red}}) \ni \text{cl}([\mathbf{a}]) \mapsto [\text{cl}(\mathbf{a})] \in \text{cl}(\mathcal{A})_{\text{red}} \tag{93}$$

for $\mathcal{A} \in \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}[[\lambda]]}$.

PROOF: First, an easy computation shows that $\eta_{\mathcal{A}}$ is well-defined for $\mathcal{A} \in \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}[[\lambda]]}$. Similarly one can show that

$$\eta_{\mathcal{A}}^{-1}: \text{cl}(\mathcal{A})_{\text{red}} \ni [\text{cl}(\mathbf{a})] \mapsto \text{cl}([\mathbf{a}]) \in \text{cl}(\mathcal{A}_{\text{red}})$$

is well-defined and is an inverse of $\eta_{\mathcal{A}}$. Therefore we have a family of algebra isomorphisms. Finally for $\mathcal{B} \in \mathbf{C}_2\mathbf{Alg}_{\mathbb{R}[[\lambda]]}$ and $T: \mathcal{A} \rightarrow \mathcal{B}$ we have $(\text{red} \circ \text{cl})(T) \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}} \circ (\text{cl} \circ \text{red})(T)$, as a simple evaluation on elements shows. Thus η is a natural isomorphism. □

Since we are interested in Morita equivalence of deformed coisotropic algebras and the relation to the classical limit we also need to define a classical limit for modules over coisotropic algebras. For a module \mathcal{E} over $\mathbb{R}[[\lambda]]$ we define the classical limit by $\text{cl}(\mathcal{E}) = \mathcal{E}/\lambda\mathcal{E}$, in analogy to the case of algebras over $\mathbb{R}[[\lambda]]$. This yields a functor of bicategories $\text{cl}: \mathbf{C}_3\mathbf{Bimod}_{\mathbb{R}[[\lambda]]} \rightarrow \mathbf{C}_3\mathbf{Bimod}_{\mathbb{R}}$. The following two lemmas will be needed to proof this.

LEMMA 7.4 *Let $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\text{Alg}_{\mathbb{R}[[\lambda]]}$ be coisotropic algebras. Then the classical limit yields a functor*

$$\text{cl}: \mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A}) \rightarrow \mathbf{C}_3\text{Bimod}_{\mathbb{R}}(\text{cl}(\mathcal{B}), \text{cl}(\mathcal{A})) \tag{94}$$

given by

$$\text{cl}(\mathcal{E})_{\text{tot}} = \text{cl}(\mathcal{E}_{\text{tot}}) = \mathcal{E}_{\text{tot}} / \lambda \mathcal{E}_{\text{tot}} \tag{95}$$

$$\text{cl}(\mathcal{E})_{\text{N}} = \text{cl}(\mathcal{E}_{\text{N}}) = \mathcal{E}_{\text{N}} / \lambda \mathcal{E}_{\text{N}} \tag{96}$$

$$\text{cl}(\mathcal{E})_0 = \mathcal{E}_0 / (\lambda \mathcal{E}_{\text{N}} \cap \mathcal{E}_0) \subseteq \text{cl}(\mathcal{E})_{\text{N}} \tag{97}$$

for objects $\mathcal{E} \in \mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A})$, and by the usual map on quotients for morphisms.

PROOF: First note that the morphism $\iota_{\mathcal{E}}: \mathcal{E}_{\text{N}} \rightarrow \mathcal{E}_{\text{tot}}$ induces a morphism $\iota_{\text{cl}(\mathcal{E})}: \text{cl}(\mathcal{E})_{\text{N}} \rightarrow \text{cl}(\mathcal{E})_{\text{tot}}$ by the $\mathbb{R}[[\lambda]]$ -linearity of $\iota_{\mathcal{E}}$. By definition, $\text{cl}(\mathcal{E})_0 \subseteq \text{cl}(\mathcal{E})_{\text{N}}$ is a submodule. Moreover, $\text{cl}(\mathcal{A}_0) \cdot \text{cl}(\mathcal{E})_{\text{N}} = \text{cl}(\mathcal{A}_0 \cdot \mathcal{E}_{\text{N}}) \subseteq \text{cl}(\mathcal{E}_0)$ and $\text{cl}(\mathcal{E}_{\text{N}}) \cdot \text{cl}(\mathcal{B}_0) = \text{cl}(\mathcal{E}_{\text{N}} \cdot \mathcal{B}_0) \subseteq \text{cl}(\mathcal{E}_0)$ hold, hence $\text{cl}(\mathcal{E})$ is a coisotropic bimodule. Finally, for a morphism $T: \mathcal{E} \rightarrow \mathcal{E}'$ between coisotropic modules, we have $\text{cl}(T)(\text{cl}(\mathcal{E}_0)) = \text{cl}(T(\mathcal{E}_0)) \subseteq \text{cl}(\mathcal{E}'_0)$. Thus $\text{cl}(T)$ is a morphism indeed. Then the functoriality is clear. \square

Note that in contrast to the classical limit of coisotropic algebras we do not construct $\text{cl}(\mathcal{E})_{\text{N}}$ as a submodule of $\text{cl}(\mathcal{E})_{\text{tot}}$ which is consistent with our requirement that we only need a morphism $\iota_{\text{cl}(\mathcal{E})}: \text{cl}(\mathcal{E})_{\text{N}} \rightarrow \text{cl}(\mathcal{E})_{\text{tot}}$.

To make this into a functor of bicategories we also need two natural isomorphisms taking care of the composition of 1-morphisms and identities.

LEMMA 7.5 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{C}_3\text{Alg}_{\mathbb{R}[[\lambda]]}$ be coisotropic triples of algebras over $\mathbb{R}[[\lambda]]$. Moreover, let $\mathcal{F} \in \mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{C}, \mathcal{B})$ and $\mathcal{E} \in \mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A})$ be coisotropic bimodules. Then*

$$\mathfrak{m}: \text{cl}(\mathcal{F}) \otimes_{\text{cl}(\mathcal{B})} \text{cl}(\mathcal{E}) \ni \text{cl}(y) \otimes \text{cl}(x) \mapsto \text{cl}(y \otimes x) \in \text{cl}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}) \tag{98}$$

defines a natural isomorphism $\mathfrak{m}: \otimes_{\text{cl}(\mathcal{B})} \circ (\text{cl} \times \text{cl}) \implies \text{cl} \circ \otimes_{\mathcal{B}}$.

PROOF: A routine check shows that \mathfrak{m} is a well-defined isomorphism on both the tot- and N-component. Moreover, it is a morphism of coisotropic bimodules since it respects the 0-component, i.e. we have $\mathfrak{m}((\text{cl}(\mathcal{F}) \otimes_{\text{cl}(\mathcal{B})} \text{cl}(\mathcal{E}))_0) \subseteq \text{cl}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E})_0$, and $\mathfrak{m} \circ (\text{cl}(\iota_{\mathcal{F}}) \otimes \text{cl}(\iota_{\mathcal{E}})) = \text{cl}(\iota_{\text{cl}(\mathcal{F})} \otimes \iota_{\text{cl}(\mathcal{E})}) \circ \mathfrak{m}$ holds. Finally, one can easily check that it is indeed a natural isomorphism, i.e. it holds $\mathfrak{m} \circ (\text{cl}(T) \otimes \text{cl}(S)) = \text{cl}(T \otimes S) \circ \mathfrak{m}$, for $T: \mathcal{F} \rightarrow \mathcal{F}'$ and $S: \mathcal{E} \rightarrow \mathcal{E}'$. \square

Putting these lemmas together we finally get the statement we aimed for.

THEOREM 7.6 (CLASSICAL LIMIT FOR $\mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}$) *The classical limit as constructed above is a functor of bicategories*

$$\text{cl}: \mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]} \rightarrow \mathbf{C}_3\text{Bimod}_{\mathbb{R}}. \tag{99}$$

PROOF: On coisotropic algebras we use the classical limit defined in Proposition 7.2. For any two coisotropic triples $\mathcal{A}, \mathcal{B} \in \mathbf{C}_3\text{Alg}_{\mathbb{R}[[\lambda]]}$ there exists a classical limit functor $\text{cl}: \mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A}) \rightarrow \mathbf{C}_3\text{Bimod}_{\mathbb{R}}(\text{cl}(\mathcal{B}), \text{cl}(\mathcal{A}))$ by Lemma 7.4. The natural isomorphisms of composition are given as in Lemma 7.5. The unit 2-isomorphisms are just the identities $u_{\mathcal{A}} = \text{id}_{\text{cl}(\mathcal{A})}$. The coherences can then be checked on elements. \square

Since this classical limit is a functor of bicategories it drops to a functor of the corresponding Picard (bi-)groupoids. Thus Morita equivalent coisotropic algebras get mapped to Morita equivalent coisotropic algebras. As always we can view $\mathbf{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}$ as a sub-bicategory of $\mathbf{C}_3\text{Bimod}_{\mathbb{R}[[\lambda]]}$, thus giving us a classical limit functor for coisotropic pairs $\text{cl}: \mathbf{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]} \rightarrow \mathbf{C}_2\text{Bimod}_{\mathbb{R}}$ as well.

Now the question arises if this classical limit is compatible with reduction, hence, if the diagram

$$\begin{array}{ccc}
 \mathbf{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \mathbf{C}_2\text{Bimod}_{\mathbb{R}} \\
 \text{red} \downarrow & & \downarrow \text{red} \\
 \mathbf{Bimod}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \mathbf{Bimod}_{\mathbb{R}}
 \end{array} \tag{100}$$

commutes. We only consider coisotropic pairs here, since we know that reduction of triples is simply given by forgetting the tot-component and using the reduction functor on pairs. We have to be careful here, since this is a diagram consisting of functors between bicategories. So instead of checking both compositions for equality we should see if they are equal up to higher morphisms. More precisely, this means we have to find natural transformations $\mu: (\text{cl} \circ \text{red}) \Rightarrow (\text{red} \circ \text{cl})$ and $\hat{\mu}: (\text{red} \circ \text{cl}) \Rightarrow (\text{cl} \circ \text{red})$ of functors between bicategories and invertible modifications $\Gamma: \hat{\mu} \circ \mu \Rightarrow \text{id}_{\text{cl} \circ \text{red}}$ and $\hat{\Gamma}: \mu \circ \hat{\mu} \Rightarrow \text{id}_{\text{red} \circ \text{cl}}$ implementing that $\hat{\mu}$ is the inverse of μ , see Definition A.3 and Definition A.4. As a diagram we get something like

$$\begin{array}{ccc}
 \mathbf{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \mathbf{C}_2\text{Bimod}_{\mathbb{R}} \\
 \text{red} \downarrow & \begin{array}{c} \nearrow \hat{\mu} \\ \nearrow \Gamma \\ \nearrow \mu \end{array} & \downarrow \text{red} \\
 \mathbf{Bimod}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \mathbf{Bimod}_{\mathbb{R}}
 \end{array} \tag{101}$$

There are quite a lot of things to check, so we start with giving some properties that will later be combined to give the commutativity of (100).

In Proposition 7.2 we already showed that on objects the diagram (100) commutes up to a natural isomorphisms. Since we can interpret morphisms of (coisotropic) algebras as (coisotropic) modules we can restate parts of this result as follows:

LEMMA 7.7 *Let $\mathcal{A} \in \mathbf{C}_2\text{Alg}_{\mathbb{R}[[\lambda]]}$. Then $\mu_{\mathcal{A}} = \eta_{\mathcal{A}}^{-1}(\text{cl}(\mathcal{A}_{\text{red}}))$ with $\eta_{\mathcal{A}} : \text{cl}(\mathcal{A}_{\text{red}}) \rightarrow \text{cl}(\mathcal{A})_{\text{red}}$ given by (93) is an invertible $(\text{cl}(\mathcal{A})_{\text{red}}, \text{cl}(\mathcal{A}_{\text{red}}))$ -bimodule in $\mathbf{Bimod}_{\mathbb{R}}$.*

Here $\eta_{\mathcal{A}}^{-1}(\text{cl}(\mathcal{A}_{\text{red}}))$ denotes the algebra $\text{cl}(\mathcal{A}_{\text{red}})$ regarded as a module over itself with left multiplication twisted by the map $\eta_{\mathcal{A}}^{-1}$. By essentially the same computations as in Proposition 7.2 we obtain a similar result for bimodules instead of algebras:

LEMMA 7.8 *For every coisotropic $(\mathcal{B}, \mathcal{A})$ -bimodule $\mathcal{E} \in \mathbf{CoisoBimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A})$ the map*

$$\eta(\mathcal{E}) : \eta_{\mathcal{E}}^{-1}(\text{cl}(\mathcal{E}_{\text{red}})) \ni \text{cl}([X]) \mapsto [\text{cl}(X)] \in (\text{cl}(\mathcal{E})_{\text{red}})^{\eta_{\mathcal{E}}} \tag{102}$$

is a well-defined isomorphism of $(\text{cl}(\mathcal{B})_{\text{red}}, \text{cl}(\mathcal{A}_{\text{red}}))$ -bimodules.

This family of 2-morphisms is in fact a natural transformation:

LEMMA 7.9 *For any two coisotropic algebras $\mathcal{A}, \mathcal{B} \in \mathbf{C}_2\text{Bimod}_0$ there is a natural isomorphism*

$$\mu : (\mu_{\mathcal{B}})_* \circ (\text{cl} \circ \text{red})_{\mathcal{B}\mathcal{A}} \implies (\mu_{\mathcal{A}})^* \circ (\text{red} \circ \text{cl})_{\mathcal{B}\mathcal{A}} \tag{103}$$

between the functors

$$(\mu_{\mathcal{B}})_* \circ (\text{cl} \circ \text{red})_{\mathcal{B}\mathcal{A}} : \mathbf{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathbf{Bimod}_{\mathbb{R}}(\text{cl}(\mathcal{B})_{\text{red}}, \text{cl}(\mathcal{A}_{\text{red}})) \tag{104}$$

and

$$(\mu_{\mathcal{A}})^* \circ (\text{red} \circ \text{cl})_{\mathcal{B}\mathcal{A}} : \mathbf{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathbf{Bimod}_{\mathbb{R}}(\text{cl}(\mathcal{B})_{\text{red}}, \text{cl}(\mathcal{A}_{\text{red}})), \tag{105}$$

given by the family

$$\mu(\mathcal{E}) = \text{right}^{-1} \circ \eta(\mathcal{E}) \circ \text{left} : \mu_{\mathcal{B}} \otimes \text{cl}(\mathcal{E}_{\text{red}}) \longrightarrow \text{cl}(\mathcal{E})_{\text{red}} \otimes \mu_{\mathcal{A}}. \tag{106}$$

of 2-isomorphisms, with $\eta(\mathcal{E})$ as in Lemma 7.8.

PROOF: It is left to show that

$$\begin{array}{ccc} \mu_{\mathcal{B}} \otimes \text{cl}(\mathcal{E}_{\text{red}}) & \xrightarrow{\text{id}_{\mu_{\mathcal{B}}} \otimes \text{cl}(\phi_{\text{red}})} & \mu_{\mathcal{B}} \otimes \text{cl}(\mathcal{F}_{\text{red}}) \\ \mu(\mathcal{E}) \downarrow & & \downarrow \mu(\mathcal{F}) \\ \text{cl}(\mathcal{E})_{\text{red}} \otimes \mu_{\mathcal{A}} & \xrightarrow{\text{cl}(\phi)_{\text{red}} \otimes \text{id}_{\mu_{\mathcal{A}}}} & \text{cl}(\mathcal{F})_{\text{red}} \otimes \mu_{\mathcal{A}} \end{array}$$

commutes for all $\phi : \mathcal{E}_{\mathcal{B}\mathcal{A}} \longrightarrow \mathcal{F}_{\mathcal{B}\mathcal{A}}$. This can be done by a simple computation on elements. □

With all these lemmas we get a natural transformation of functors between bicategories.

LEMMA 7.10 *The 1-morphisms $\mu_{\mathcal{A}} \in \text{Bimod}_R(\text{cl}(\mathcal{A})_{\text{red}}, \text{cl}(\mathcal{A}_{\text{red}}))$ from Lemma 7.7 together with the natural isomorphisms*

$$\mu: (\mu_{\mathcal{B}})_* \circ (\text{cl} \circ \text{red})_{\mathcal{B}\mathcal{A}} \implies (\mu_{\mathcal{A}})^* \circ (\text{red} \circ \text{cl})_{\mathcal{B}\mathcal{A}} \tag{107}$$

from Lemma 7.9 form a natural transformation

$$\mu: (\text{cl} \circ \text{red}) \implies (\text{red} \circ \text{cl}) \tag{108}$$

of functors between bicategories.

PROOF: The only things left to show are the coherence conditions for natural transformations between functors of bicategories, see Definition A.3. Again, this is a simple verification. \square

This is not yet everything we need for (100) to commute. We still need to show that the natural transformation μ is invertible. For this we heavily use the fact that the 1-morphisms $\mu_{\mathcal{A}}$ are given by twisting the left multiplication of $\text{id}_{\text{cl}(\mathcal{A}_{\text{red}})}$ with the algebra isomorphism $\eta_{\mathcal{A}}^{-1}$. Thus we can define

$$\hat{\mu}_{\mathcal{A}} = {}^{\eta_{\mathcal{A}}}(\text{cl}(\mathcal{A})_{\text{red}}) \tag{109}$$

in analogy to Lemma 7.7. Similarly,

$$\hat{\mu}(\mathcal{E}) = \text{right}^{-1} \circ \hat{\eta}(\mathcal{E}) \circ \text{left}: \hat{\mu}_{\mathcal{B}} \otimes \text{cl}(\mathcal{E})_{\text{red}} \longrightarrow \text{cl}(\mathcal{E}_{\text{red}}) \otimes \hat{\mu}_{\mathcal{A}} \tag{110}$$

with

$$\hat{\eta}(\mathcal{E}): {}^{\eta_{\mathcal{B}}} \text{cl}(\mathcal{E})_{\text{red}} \ni [\text{cl}(\mathbf{x})] \longmapsto \text{cl}([\mathbf{x}]) \in \text{cl}(\mathcal{E}_{\text{red}})^{\eta_{\mathcal{B}}^{-1}} \tag{111}$$

gives a natural isomorphism

$$\hat{\mu}: (\hat{\mu}_{\mathcal{B}})_* \circ (\text{red} \circ \text{cl})_{\mathcal{B}\mathcal{A}} \implies (\hat{\mu}_{\mathcal{A}})^* \circ (\text{cl} \circ \text{red})_{\mathcal{B}\mathcal{A}} \tag{112}$$

in analogy to Lemma 7.9. This yields again a natural transformation of functors between bicategories.

LEMMA 7.11 *The 1-morphisms $\hat{\mu}_{\mathcal{A}} \in \text{CoisoBimod}_R(\text{cl}(\mathcal{A}_{\text{red}}), \text{cl}(\mathcal{A})_{\text{red}})$ together with the natural isomorphisms*

$$\hat{\mu}: (\hat{\mu}_{\mathcal{B}})_* \circ (\text{red} \circ \text{cl})_{\mathcal{B}\mathcal{A}} \implies (\hat{\mu}_{\mathcal{A}})^* \circ (\text{cl} \circ \text{red})_{\mathcal{B}\mathcal{A}} \tag{113}$$

form a natural transformation

$$\hat{\mu}: (\text{red} \circ \text{cl}) \implies (\text{cl} \circ \text{red}). \tag{114}$$

of functors between bicategories.

Now the last thing to show is that μ and $\hat{\mu}$ are indeed inverse to each other: this is of course to be understood in the sense of natural transformations between bicategories and hence up to a modification:

LEMMA 7.12 *The natural transformations $\mu: (\text{cl} \circ \text{red}) \implies (\text{red} \circ \text{cl})$ and $\hat{\mu}: (\text{red} \circ \text{cl}) \implies (\text{cl} \circ \text{red})$ are inverse to each other.*

PROOF: We need to show that there are invertible modifications $\Gamma: \hat{\mu} \circ \mu \implies \text{id}_{\text{cl} \circ \text{red}}$ and $\hat{\Gamma}: \mu \circ \hat{\mu} \implies \text{id}_{\text{red} \circ \text{cl}}$. Hence, we need for any $\mathcal{A} \in \text{CoisoBimod}_{\mathbb{R}[[\lambda]]}$ a 2-isomorphism $\Gamma_{\mathcal{A}}: \hat{\mu}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \longrightarrow \text{id}_{\text{cl}(\mathcal{A}_{\text{red}})}$. Recall that $\hat{\mu}_{\mathcal{A}} = \eta_{\mathcal{A}} \text{cl}(\mathcal{A}_{\text{red}})$ and $\mu_{\mathcal{A}} = \eta_{\mathcal{A}}^{-1} \text{cl}(\mathcal{A}_{\text{red}})$, thus we get an isomorphism $\Gamma_{\mathcal{A}}$ by

$$\hat{\mu}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} = \eta_{\mathcal{A}} \text{cl}(\mathcal{A}_{\text{red}}) \otimes \eta_{\mathcal{A}}^{-1} \text{cl}(\mathcal{A}_{\text{red}}) \simeq (\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}}^{-1}) \text{cl}(\mathcal{A}_{\text{red}}) = \text{cl}(\mathcal{A}_{\text{red}}) = \text{id}_{\text{cl}(\mathcal{A}_{\text{red}})},$$

mapping $[\text{cl}(\mathbf{a})] \otimes [\text{cl}(\mathbf{b})]$ to $[\text{cl}(\mathbf{ab})]$ and with inverse mapping $\text{cl}([\mathbf{a}])$ to $[\text{cl}(\mathbb{1}_{\mathcal{A}})] \otimes [\text{cl}(\mathbf{a})]$. With this isomorphism the diagram

$$\begin{array}{ccc} (\hat{\mu}_{\mathcal{B}} \otimes \mu_{\mathcal{B}}) \otimes \text{cl}(\mathcal{E}_{\text{red}}) & \xrightarrow{(\hat{\mu} \circ \mu)(\mathcal{E})} & \text{cl}(\mathcal{E}_{\text{red}}) \otimes (\hat{\mu}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \\ \Gamma_{\mathcal{B}} \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \Gamma_{\mathcal{A}} \\ \text{id}_{\text{cl}(\mathcal{B}_{\text{red}})} \otimes \text{cl}(\mathcal{E}_{\text{red}}) & \xrightarrow{\text{id}_{(\text{cl} \circ \text{red})}(\mathcal{E})} & \text{cl}(\mathcal{E}_{\text{red}}) \otimes \text{id}_{\text{cl}(\mathcal{A}_{\text{red}})} \end{array}$$

commutes. Similarly, we obtain an isomorphism $\hat{\Gamma}_{\mathcal{A}}$ by

$$\mu_{\mathcal{A}} \otimes \hat{\mu}_{\mathcal{A}} = \eta_{\mathcal{A}}^{-1} \text{cl}(\mathcal{A}_{\text{red}}) \otimes \eta_{\mathcal{A}} \text{cl}(\mathcal{A}_{\text{red}}) \simeq \eta_{\mathcal{A}}^{-1} \circ \eta_{\mathcal{A}} \text{cl}(\mathcal{A}_{\text{red}}) = \text{cl}(\mathcal{A}_{\text{red}}) = \text{id}_{\text{cl}(\mathcal{A}_{\text{red}})},$$

mapping $\text{cl}([\mathbf{a}]) \otimes [\text{cl}(\mathbf{b})]$ to $[\text{cl}(\mathbf{ab})]$ and inverse mapping $[\text{cl}(\mathbf{a})]$ to $[\text{cl}(\mathbb{1}_{\mathcal{A}})] \otimes [\text{cl}(\mathbf{a})]$. \square

Thus we finally see that (100) commutes:

THEOREM 7.13 *The classical limit on $\text{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}$ commutes with reduction, i.e. the diagram (100) given as*

$$\begin{array}{ccc} \text{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \text{C}_2\text{Bimod}_{\mathbb{R}} \\ \text{red} \downarrow & & \downarrow \text{red} \\ \text{Bimod}_{\mathbb{R}[[\lambda]]} & \xrightarrow{\text{cl}} & \text{Bimod}_{\mathbb{R}} \end{array} \tag{115}$$

commutes up to the invertible natural transformations μ and $\hat{\mu}$.

Thinking in geometric terms the Morita equivalence on the classical side is well-understood. Moreover, Morita equivalence after reduction is just the classical Morita equivalence. Thus if we want to understand Morita equivalence in $\text{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}$ better it might be helpful to examine the functors cl and red in order to transport knowledge about the classical or reduced side back to $\text{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}$.

REMARK 7.14 (PICARD GROUPOIDS) *A first observation is that taking Picard (bi-)groupoids of all involved bicategories in Theorem 7.13 immediately yields the commutativity of*

$$\begin{array}{ccc}
 \text{Pic}(\text{C}_2\text{Bimod}_{\mathbb{R}[[\lambda]]}) & \xrightarrow{\text{cl}} & \text{Pic}(\text{C}_2\text{Bimod}_{\mathbb{R}}) \\
 \text{red} \downarrow & & \downarrow \text{red} \\
 \text{Pic}(\text{Bimod}_{\mathbb{R}[[\lambda]]}) & \xrightarrow{\text{cl}} & \text{Pic}(\text{Bimod}_{\mathbb{R}}).
 \end{array} \tag{116}$$

This diagram now opens the doors to investigate the Morita theory of coisotropic triples both from the deformed and classical point of view. In previous works like [12, 30, 13, 31] similar diagrams have been used to understand the Morita equivalence of deformed algebras in terms of their undeformed classical limits as well as algebras having additional structure preserved by re-fined versions of Morita theory and their counterparts after forgetting the additional structure. This proved to be a very efficient way to encode the relevant questions, which, of course, then have to be answered for classes of examples admitting more specific information. Thus the above diagram can be understood as the starting point for investigations of Morita theory of e.g. star products before (the triples) and after reduction.

A BICATEGORIES

For the convenience of the reader and to explain our conventions, we collect some basic definitions concerning bicategories, see [3] or [32] for a more modern treatment.

DEFINITION A.1 (BICATEGORY) *A bicategory \mathfrak{B} consists of the following data:*

1. *A class \mathfrak{B}_0 , the objects of \mathfrak{B} .*
2. *For any two objects $A, B \in \mathfrak{B}_0$ a category $\mathfrak{B}(B, A)$. The objects $\mathfrak{B}_1(B, A) = \text{Obj}(\mathfrak{B}(B, A))$ of this category are called 1-morphisms from A to B . Morphisms $\phi: f \rightarrow g$ between 1-morphisms $f, g \in \mathfrak{B}_1(B, A)$ are called 2-morphisms from f to g . The set of such 2-morphisms is denoted by $\mathfrak{B}_2(g, f)$.*
3. *For any three objects $A, B, C \in \mathfrak{B}_0$ a functor*

$$\otimes_{CBA}: \mathfrak{B}(C, B) \times \mathfrak{B}(B, A) \longrightarrow \mathfrak{B}(C, A), \tag{117}$$

called the composition or tensor product of 1-morphisms.

4. *For each object $A \in \mathfrak{B}_0$ a 1-morphism $\text{Id}_A \in \mathfrak{B}_1(A, A)$, called the identity at A .*

5. For any four objects $A, B, C, D \in \mathfrak{B}_0$ a natural isomorphism

$$\text{asso}_{DCBA}: \otimes_{CBA} \circ (\otimes_{DCB} \times \text{id}) \implies \otimes_{DCB} \circ (\text{id} \times \otimes_{CBA}), \tag{118}$$

called the associativity.

6. For any two objects $A, B \in \mathfrak{B}_0$ natural isomorphisms

$$\text{left}_{BA}: \otimes_{BBA} \circ (\text{id}_B \times \text{id}) \implies \text{id} \tag{119}$$

and

$$\text{right}_{BA}: \otimes_{BAA} \circ (\text{id} \times \text{id}_A) \implies \text{id}, \tag{120}$$

called the left and right identity, respectively.

These data are required to fulfill the following coherence conditions:

1. Associativity coherence: the diagram

$$\begin{array}{ccc} ((k \otimes_D h) \otimes_C g) \otimes_B f & \xrightarrow{\text{asso}(k,h,g) \otimes_B \text{id}} & (k \otimes_D (h \otimes_C g)) \otimes_B f \\ \downarrow \text{asso}(k \otimes_D h, g, f) & & \text{asso}(k, h \otimes_C g, f) \downarrow \\ (k \otimes_D h) \otimes_C (g \otimes_B f) & & k \otimes_D ((h \otimes_C g) \otimes_B f) \\ \searrow \text{asso}(k, h, g \otimes_B f) & & \swarrow \text{id} \otimes_D \text{asso}(h, g, f) \\ & k \otimes_D (h \otimes_C (g \otimes_B f)) & \end{array} \tag{121}$$

commutes for all $k \in \mathfrak{B}_1(E, D)$, $h \in \mathfrak{B}_1(D, C)$, $g \in \mathfrak{B}_1(C, B)$ and $f \in \mathfrak{B}_1(B, A)$.

2. Identity coherence: the diagram

$$\begin{array}{ccc} (g \otimes_B \text{Id}_B) \otimes_B f & \xrightarrow{\text{asso}(g, \text{Id}_B, f)} & g \otimes_B (\text{Id}_B \otimes_B f) \\ \searrow \text{right}(g) \otimes_B \text{id} & & \swarrow \text{id} \otimes_B \text{left}(f) \\ & g \otimes_B f & \end{array} \tag{122}$$

commutes for all $g \in \mathfrak{B}_1(C, B)$ and $f \in \mathfrak{B}_1(B, A)$.

Note that we simplify \otimes_{CBA} to \otimes_B and drop indices of the involved natural isomorphisms whenever there is no possibility of confusion. Recall that in bicategories there is a way to compose 1-morphisms with 2-morphisms. Let

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{g} \\ \Downarrow \phi \\ \xrightarrow{g'} \end{array} C \tag{123}$$

be given, then we get a 2-morphism $f^*\phi = \phi \otimes_B \text{id}_f: g \otimes f \rightarrow g' \otimes f$ between the (horizontal) compositions of f and g , and f and g' , respectively. In the same way, given

$$\begin{array}{ccc}
 & g & \\
 & \curvearrowright & \\
 B & \Downarrow \phi & C \xrightarrow{h} D \\
 & \curvearrowleft & \\
 & g' &
 \end{array} \tag{124}$$

one defines a 2-morphism $h_*\phi = \text{id}_h \otimes_C \phi: h \otimes g \rightarrow h \otimes g'$ between the (horizontal) compositions. These compositions can also be seen as functors between the appropriate hom-categories and are sometimes called *whiskering*.

As morphisms of bicategories we use what is often called a weak (2-)functor or pseudofunctor. Note that there are also weaker notions like lax and oplax functor which, however, will not suffice for our purposes. In the original work Benabou calls the following version a homomorphism of bicategories [2]:

DEFINITION A.2 (FUNCTOR OF BICATEGORIES) *Let \mathfrak{A} and \mathfrak{B} be two bicategories. A functor F from \mathfrak{A} to \mathfrak{B} , written $F: \mathfrak{A} \rightarrow \mathfrak{B}$, consists of the following data:*

1. A map $F: \mathfrak{A}_0 \rightarrow \mathfrak{B}_0$ mapping objects of \mathfrak{A} to objects of \mathfrak{B} .
2. For any two objects $A, B \in \mathfrak{A}_0$ a functor

$$F_{BA}: \mathfrak{A}(B, A) \rightarrow \mathfrak{B}(FB, FA). \tag{125}$$

3. For each three objects $A, B, C \in \mathfrak{A}_0$ a natural isomorphism

$$m_{CBA}: \otimes_{FB} \circ (F_{CB} \times F_{BA}) \implies F_{CA} \circ \otimes_B. \tag{126}$$

4. For any object $A \in \mathfrak{A}_0$ a 2-isomorphism

$$u_A: \text{Id}_{FA} \rightarrow F_{AA}(\text{Id}_A). \tag{127}$$

These data are required to fulfil the following coherence conditions:

1. *Composition coherence: the diagram*

$$\begin{array}{ccc}
 Fh \otimes_{FC} (Fg \otimes_{FB} Ff) & \xrightarrow{\text{asso}} & (Fh \otimes_{FC} Fg) \otimes_{FB} Ff \\
 \text{id}_{Fh} \otimes m(g,f) \downarrow & & \downarrow m(h,g) \otimes \text{id}_{Ff} \\
 Fh \otimes_{FC} F(g \otimes_B f) & & F(h \otimes_C g) \otimes_{FB} Ff \\
 m(h,g \otimes_B f) \downarrow & & \downarrow m(h \otimes_C g, f) \\
 F(h \otimes_C (g \otimes_B f)) & \xrightarrow{F(\text{asso})} & F((h \otimes_C g) \otimes_B f)
 \end{array} \tag{128}$$

commutes for all $h \in \mathfrak{A}_1(D, C)$, $g \in \mathfrak{A}_1(C, B)$ and $f \in \mathfrak{A}_1(B, A)$.

2. Identity coherence: the diagram

$$\begin{array}{ccccc}
 \text{Id}_{\mathbb{F}B} \otimes_{\mathbb{F}B} \mathbb{F}f & \xrightarrow{\text{left}(\mathbb{F}f)} & \mathbb{F}f & \xleftarrow{\text{right}(\mathbb{F}f)} & \mathbb{F}f \otimes_{\mathbb{F}A} \text{Id}_{\mathbb{F}A} \\
 \downarrow \text{u}_B \otimes_{\mathbb{F}B} \text{id}_{\mathbb{F}f} & & \downarrow \text{id} & & \downarrow \text{id}_{\mathbb{F}f} \otimes_{\mathbb{F}A} \text{u}_A \\
 \mathbb{F}(\text{Id}_B) \otimes_{\mathbb{F}B} \mathbb{F}f & & \mathbb{F}f & & \mathbb{F}f \otimes_{\mathbb{F}A} \mathbb{F}(\text{Id}_A) \\
 \downarrow \text{m}(\text{Id}_B, f) & & \downarrow \text{id} & & \downarrow \text{m}(f, \text{Id}_A) \\
 \mathbb{F}(\text{Id}_B \otimes_B f) & \xrightarrow{\mathbb{F}(\text{left}(f))} & \mathbb{F}f & \xleftarrow{\mathbb{F}(\text{right}(f))} & \mathbb{F}(f \otimes_A \text{Id}_A)
 \end{array} \tag{129}$$

commutes for all $f \in \mathfrak{A}_1(B, A)$.

Composition of functors of bicategories is defined by composing the obvious maps, functors and natural transformations. Similar to usual categories there is also a notion of natural transformation between functors. But now we have to incorporate the higher morphisms.

DEFINITION A.3 (NATURAL TRANSFORMATION) *Let $\mathbb{F}, \mathbb{G}: \mathfrak{A} \rightarrow \mathfrak{B}$ be functors between bicategories \mathfrak{A} and \mathfrak{B} . A natural transformation η from \mathbb{F} to \mathbb{G} , written $\eta: \mathbb{F} \Rightarrow \mathbb{G}$, consists of the following data:*

1. for each $A \in \mathfrak{A}_0$ a 1-morphism $\eta_A: \mathbb{F}A \rightarrow \mathbb{G}A$ in \mathfrak{B} .
2. for each 1-morphism $f \in \mathfrak{A}_1(B, A)$ a 2-isomorphism

$$\eta_f: \eta_B \otimes_{\mathbb{F}B} \mathbb{F}f \longrightarrow \mathbb{G}f \otimes_{\mathbb{G}A} \eta_A, \tag{130}$$

such that for any $A, B \in \mathfrak{A}_0$ the 2-morphisms η_f are the components of a natural isomorphism

$$\eta_{BA}: (\eta_B)_* \circ \mathbb{F}_{BA} \Longrightarrow (\eta_A)^* \circ \mathbb{G}_{BA}. \tag{131}$$

These data are required to fulfil the following coherence conditions:

1. The diagram

$$\begin{array}{ccc}
 \eta_C \otimes_{FC} (Fg \otimes_{FB} Ff) & \xrightarrow{\text{id} \otimes m^F} & \eta_C \otimes_{FC} F(g \otimes f) \\
 \text{asso}^{-1} \downarrow & & \downarrow \eta(g \otimes f) \\
 (\eta_C \otimes_{FC} Fg) \otimes_{FB} Ff & & \\
 \eta(g) \otimes \text{id} \downarrow & & \\
 (Gg \otimes_{GB} \eta_B) \otimes_{FB} Ff & & \\
 \text{asso} \downarrow & & \\
 Gg \otimes_{GB} (\eta_B \otimes_{FB} Ff) & & \\
 \text{id} \otimes \eta(f) \downarrow & & \\
 Gg \otimes_{GB} (Gf \otimes_{GA} \eta_A) & & \\
 \text{asso}^{-1} \downarrow & & \\
 (Gg \otimes_{GB} Gf) \otimes_{GA} \eta_A & \xrightarrow{m^G \otimes \text{id}} & G(g \otimes f) \otimes_{GA} \eta_A
 \end{array} \tag{132}$$

commutes for all $f \in \mathfrak{A}_1(B, A)$ and $g \in \mathfrak{A}_1(C, B)$.

2. The diagram

$$\begin{array}{ccc}
 \eta_A \otimes_{FA} \text{Id}_{FA} & \xrightarrow{\text{right}} & \eta_A & \xrightarrow{\text{left}^{-1}} & \text{Id}_{GA} \otimes_{GA} \eta_A \\
 \text{id} \otimes u_A \downarrow & & & & \downarrow u_A \otimes \text{id} \\
 \eta_A \otimes_{FA} F(\text{Id}_A) & \xrightarrow{\eta_{\text{Id}_A}} & & & G(\text{Id}_A) \otimes_{GA} \eta_A
 \end{array} \tag{133}$$

commutes for all $A \in \mathfrak{A}_0$.

For bicategories there is also the possibility to relate natural transformations via so called modifications:

DEFINITION A.4 (MODIFICATION) *Let \mathfrak{A} and \mathfrak{B} be bicategories. Let furthermore $\eta: F \Rightarrow G$ and $\mu: F \Rightarrow G$ be two natural transformations between functors $F, G: \mathfrak{A} \rightarrow \mathfrak{B}$. A modification $\Gamma: \eta \Rrightarrow \mu$ is an assignment that assigns to every object $A \in \mathfrak{A}_0$ a 2-morphism $\Gamma_A: \eta_A \rightarrow \mu_A$ such that for each morphism $f \in \mathfrak{A}_1(B, A)$ the diagram*

$$\begin{array}{ccc}
 \eta_B \otimes_{FB} Ff & \xrightarrow{\eta(f)} & Gf \otimes_{GA} \eta_A \\
 \Gamma_B \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \Gamma_A \\
 \mu_B \otimes_{FB} Ff & \xrightarrow{\mu(f)} & Gf \otimes_{GA} \mu_A
 \end{array} \tag{134}$$

commutes.

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