

PULLBACKS OF SAITO–KUROKAWA LIFTS  
AND A CENTRAL VALUE FORMULA FOR DEGREE 6  $L$ -SERIES

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**ABSTRACT.** We prove an explicit central value formula for a family of complex  $L$ -series of degree 6 for  $\mathrm{GL}_2 \times \mathrm{GL}_3$  which arise as factors of certain Garret–Rankin triple product  $L$ -series associated with modular forms. Our result generalizes a previous formula of Ichino involving Saito–Kurokawa lifts, and as an application we prove Deligne’s conjecture stating the algebraicity of the central values of the considered  $L$ -series up to the relevant periods.

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## 1 INTRODUCTION

Explicit central value formulas for  $L$ -series associated with modular forms have always been of interest in number theory. In this paper we prove a central value formula for certain  $L$ -series of degree 6, generalizing a result of Ichino [Ich05], which involves pullbacks of Saito–Kurokawa lifts. This can be seen as yet another evidence of the key role that pullbacks of Siegel Eisenstein series or cusp forms play in the proof of the algebraicity of critical values of certain automorphic  $L$ -functions. Previous instances of this phenomenon are found for example in the works of Garrett [Gar84, Gar87], Böcherer [Böc85], or Böcherer–Furusawa–Schulze-Pillot [BFSP04], or also in a similar flavour in Ichino–Ikeda [II08], where special values of certain triple product  $L$ -series are related with pullbacks of hermitian Maass lifts. All these results, as well as the result in this paper, fit within the range of the ‘refined global Gross–Prasad

conjecture' (cf. [HI10]), reflecting the intimate relation between certain periods of automorphic forms on special orthogonal groups and  $L$ -values.

In order to describe the setting considered in this article, let  $k \geq 1$  be an odd integer, and let  $f \in S_{2k}^{new}(\Gamma_0(N_f))$  and  $g \in S_{k+1}^{new}(\Gamma_0(N_g), \chi)$  be two normalized newforms of weights  $2k$  and  $k + 1$ , and levels  $N_f$  and  $N_g$ , respectively. We assume  $f$  has trivial nebentypus, whereas  $g$  has nebentypus  $\chi$  (hence  $\chi$  is a Dirichlet character modulo  $N_g$ ). Write  $V_\ell(f)$  (resp.  $V_\ell(g)$ ) for the compatible system of  $\ell$ -adic Galois representations attached to  $f$  (resp.  $g$ ), and denote by  $\text{Ad}(V_\ell(g))$  the so-called *adjoint representation* of  $V_\ell(g)$ . In this paper we are concerned with the complex  $L$ -series  $L(f \otimes \text{Ad}(g), s)$  of degree 6 for  $\text{GL}_2 \times \text{GL}_3$  associated with the tensor product  $V_\ell(f) \otimes \text{Ad}(V_\ell(g))$ . This  $L$ -series can be defined by an Euler product for  $\text{Re}(s) \gg 0$ , whose local factors at primes  $p \nmid N_f N_g$  are given as in [Ich05, p. 559]. The completed  $L$ -series

$$\Lambda(f \otimes \text{Ad}(g), s) := \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s + k)\Gamma_{\mathbb{C}}(s - k + 1)L(f \otimes \text{Ad}(g), s),$$

where  $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$  is the usual complex Gamma function, admits analytic continuation to the whole complex plane and satisfies a functional equation relating its values at  $s$  and  $2k - s$ , with sign  $\varepsilon(f \otimes \text{Ad}(g)) \in \{\pm 1\}$ . Under certain hypotheses, which in particular guarantee that the sign  $\varepsilon(f \otimes \text{Ad}(g))$  is  $+1$ , the main result of this paper is an *explicit central value formula* for  $\Lambda(f \otimes \text{Ad}(g), k)$ . As an immediate corollary, we deduce the algebraicity of such value up to a suitable period, as expected by Deligne's conjecture.

As the eager reader might have already suspected, the  $L$ -series  $L(f \otimes \text{Ad}(g), s)$  is closely related to a suitable triple product Garret–Rankin  $L$ -series. Indeed, let  $f' := f \otimes \chi^{-1}$  be the twist of  $f$  by the inverse of the character  $\chi$ . By construction, the motive associated to the triple tensor product  $V_\ell(f') \otimes V_\ell(g) \otimes V_\ell(g)$  is self-dual, and hence the Garret–Rankin  $L$ -series  $L(f' \otimes g \otimes g, s)$  attached to it (or rather, its completed  $L$ -series) satisfies a functional equation relating its values at  $s$  and  $4k - s$ , with sign  $\varepsilon(f', g, g) \in \{\pm 1\}$ . In view of the isomorphisms

$$V_\ell(g) \otimes V_\ell(g) \simeq \det(V_\ell(g)) \oplus \text{Sym}^2(V_\ell(g)) \simeq \det(V_\ell(g)) \otimes (\mathbf{1} \oplus \text{Ad}(V_\ell(g))),$$

where  $\text{Sym}^2(V_\ell(g))$  stands for the *symmetric square* representation of  $V_\ell(g)$  and we use that  $\text{Sym}^2(V_\ell(g)) \simeq \text{Ad}(V_\ell(g)) \otimes \det(V_\ell(g))$ , Artin formalism provides a factorization of complex  $L$ -series

$$L(f' \otimes g \otimes g, s) = L(f, s - k)L(f \otimes \text{Ad}(g), s - k). \tag{1}$$

It is well-known that the completed  $L$ -series  $\Lambda(f, s) := \Gamma_{\mathbb{C}}(s)L(f, s)$  satisfies a functional equation relating its values at  $s$  and  $2k - s$ , with sign  $\varepsilon(f) \in \{\pm 1\}$ , thus the central critical point for the shifted  $L$ -series  $\Lambda(f, s - k)$  is at  $s = 2k$ . Concerning the central values, suppose that  $\Lambda(f \otimes \text{Ad}(g), k)$  is non-zero. If  $\Lambda(f, k)$  is non-zero as well, then one can use (1) straightforward to express the central value  $\Lambda(f \otimes \text{Ad}(g), k)$  as a ratio between the central values  $\Lambda(f' \otimes g \otimes g, 2k)$  and  $\Lambda(f, k)$ . When  $\Lambda(f, k)$  vanishes, however, the identity in (1) is not directly giving a way to obtain an expression for  $\Lambda(f \otimes \text{Ad}(g), k)$ .

Despite the above relation to triple product central  $L$ -values, the approach in this article to obtain an *explicit* central value formula for  $\Lambda(f \otimes \text{Ad}(g), k)$  does not require determining triple product central  $L$ -values. Instead, as pointed out at the beginning of the introduction, we generalize a result of Ichino involving Saito–Kurokawa lifts. As a motivation towards our result, suppose in the previous discussion that  $N_f = N_g = 1$ , hence  $\chi$  is trivial as well and  $f$  and  $g$  are normalized newforms for the full modular group  $\Gamma_0(1) = \text{SL}_2(\mathbb{Z})$ . In this case,  $f' = f$ , and (1) reads

$$L(f \otimes g \otimes g, s) = L(f, s - k)L(f \otimes \text{Ad}(g), s).$$

Our choice of weights makes that  $\varepsilon(f, g, g) = \varepsilon(f) = -1$ , and therefore the sign in the functional equation for  $\Lambda(f \otimes \text{Ad}(g), s)$  is  $+1$ . In this particular setting, Ichino proved in [Ich05] an explicit formula for  $\Lambda(f \otimes \text{Ad}(g), k)$ , involving a half-integral weight modular form  $h \in S_{k+1/2}(\Gamma_0(4))$  associated with  $f$  by the Shimura correspondence and its Saito–Kurokawa lift  $F \in S_{k+1}(\text{Sp}_2(\mathbb{Z}))$ , which is a Siegel modular form of degree 2. In terms of these lifts, Ichino’s formula reads

$$\Lambda(f \otimes \text{Ad}(g), k) = 2^{k+1} \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{|\langle F|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2}{\langle g, g \rangle^2}, \quad (2)$$

where  $F|_{\mathcal{H} \times \mathcal{H}}$  denotes the restriction (or ‘pullback’) of  $F$  to  $\mathcal{H} \times \mathcal{H}$ , embedded ‘diagonally’ in Siegel’s upper half space  $\mathcal{H}_2$ . Our main result can be seen as a generalization of (2), when removing the assumption that  $N_f = N_g = 1$ . However, instead of extending Ichino’s arguments, our strategy relies on a more recent result by Qiu [Qiu14].

Indeed, there is a decomposition formula for the  $\text{SO}(4)$ -period  $\mathcal{P}$  associated with (the restriction of) a Saito–Kurokawa representation of  $\text{PGSp}_2$  and an irreducible cuspidal unitary representation of  $\text{GSO}(4)$ . Here,  $\text{GSO}(4)$  and  $\text{SO}(4)$  stand for the group of similitudes and the special orthogonal group of a certain 4-dimensional (split) quadratic space, and  $\text{PGSp}_2$  is identified with the special orthogonal group of a suitable 5-dimensional quadratic space. The proof of this decomposition result by Qiu in fact reduces to a decomposition formula for a global  $\text{SL}_2$ -period  $\mathcal{Q}$ , proved in the same article. This  $\text{SL}_2$ -period, and the interplay between  $\mathcal{P}$  and  $\mathcal{Q}$ , plays a central role in the proof of our main result.

To illustrate our strategy, consider again the general setting in which  $f$  and  $g$  are of level  $N_f$  and  $N_g$ , respectively, and  $\chi$  is not necessarily trivial. Then let  $\pi$  (resp.  $\tau$ ) be the automorphic representation of  $\text{PGL}_2(\mathbb{A})$  (resp.  $\text{GL}_2(\mathbb{A})$ ) associated with  $f$  (resp.  $g$ ). The Shimura correspondence, settled and investigated in detail by Waldspurger [Wal80, Wal81] as a theta correspondence for the pair  $(\text{PGL}_2, \widetilde{\text{SL}}_2)$ , associates to  $\pi$  (and a choice of non-trivial additive character of  $\mathbb{A}/\mathbb{Q}$ ) a near equivalence class of automorphic representations  $\tilde{\pi}$  of the double metaplectic cover  $\widetilde{\text{SL}}_2(\mathbb{A})$  of  $\text{SL}_2(\mathbb{A})$ . Classical Shimura lifts of  $f$  give rise to automorphic forms in the representations  $\tilde{\pi}$  arising in this theta correspondence. Associated with the representations  $\tilde{\pi}$ ,  $\tau$ , and a Weil representation  $\omega$

depending on the fixed additive character of  $\mathbb{A}/\mathbb{Q}$ , there is a (global)  $\mathrm{SL}_2$ -period functional

$$\mathcal{Q} : \tilde{\pi} \otimes \tilde{\pi} \otimes \tau \otimes \tau \otimes \omega \otimes \omega \longrightarrow \mathbb{C}$$

(cf. Section 6 for its precise definition), which by virtue of [Qiu14, Theorem 4.5] decomposes (when it is non-zero) up to certain special  $L$ -values as a product of local periods

$$\mathcal{I}_v : \tilde{\pi}_v \otimes \tilde{\pi}_v \otimes \tau_v \otimes \tau_v \otimes \omega_v \otimes \omega_v \longrightarrow \mathbb{C}$$

defined by integrating local matrix coefficients. Among the  $L$ -values showing up in this decomposition formula, one finds  $L(1/2, \pi \times \mathrm{ad}\tau)$ , which corresponds with the central value  $\Lambda(f \otimes \mathrm{Ad}(g), k)$  that we are interested in. Moreover, the non-vanishing of the functional  $\mathcal{Q}$  is essentially controlled by the non-vanishing of the special value  $L(1/2, \pi \times \mathrm{ad}\tau)$  (cf. Propositions 6.2 and 6.3). Hence, one can obtain an explicit expression for  $\Lambda(f \otimes \mathrm{Ad}(g), k)$  by finding a *test vector* on which  $\mathcal{Q}$  does not vanish, and computing the local periods  $\mathcal{I}_v$  when evaluated at such test vector. Besides, as hinted above, the global period  $\mathcal{Q}$  is related to the  $\mathrm{SO}(4)$ -period  $\mathcal{P}$ , when replacing the automorphic representations  $\tilde{\pi}$  and  $\tau$  with automorphic representations  $\Pi$  and  $\Upsilon$ , of  $\mathrm{GSp}_2$  and  $\mathrm{GSO}(4)$  respectively, obtained from  $\tilde{\pi}$  and  $\tau$  via theta correspondence. It is via this relation with  $\mathcal{P}$  that the global period  $\mathcal{Q}$  evaluated at the test vector can be interpreted as a classical Petersson product, therefore leading to the aimed expression for  $\Lambda(f \otimes \mathrm{Ad}(g), k)$  in purely classical terms. For example, in Ichino's setting described above, this global automorphic period is the responsible of the factor  $|\langle F|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2$  appearing in (2). The main novelty of our work is the computation of the above mentioned local  $\mathrm{SL}_2$ -periods at ramified primes. It is important to remark that these local  $\mathrm{SL}_2$ -periods have their own interest, and their computation has potential applications in the study of the subconvexity problem for the family of automorphic  $L$ -functions of the form  $L(s, \pi \times \mathrm{ad}\tau)$ . This will be explored in a forthcoming work.

Although the strategy that we have just sketched works in a rather general setting, for the sake of clarity and to simplify the (already involved) local computations we will impose some assumptions on  $f$  and  $g$ . Most importantly, we will assume that

$$N = N_f = N_g \text{ is odd and square-free.} \quad (\mathrm{SF})$$

One could easily relax this assumption to require only that  $N_f$  and  $N_g$  are square-free (but not necessarily equal), at the cost of dealing with more cases when performing the computation of the local periods  $\mathcal{I}_v$  alluded to above. However, we content ourselves with illustrating the method under the assumption (SF).

Besides, let  $M$  denote the conductor of the Dirichlet character  $\chi$ . Thus  $M$  is a positive divisor of  $N$ , and by (SF)  $M$  is square-free as well. If we write  $\chi = \prod_{p|M} \chi_{(p)}$ , where  $\chi_{(p)}$  is a Dirichlet character modulo  $p$  for each prime  $p | M$ ,

then we assume that

$$\chi_{(p)}(-1) = -1 \text{ for all primes } p \mid M. \quad (\text{H1})$$

In particular, this implies that  $M$  is the product of an even number of primes. We use hypothesis (H1) to apply a generalized Kohnen formula due to Baruch and Mao [BM07], recalled in Theorem 2.1 below. Finally, it is well-known that the sign  $\varepsilon(f) \in \{\pm 1\}$  in the functional equation for  $L(f, s)$  might be written as a product of local signs  $\varepsilon(f) = \prod_v \varepsilon_v(f)$ , where  $v$  varies over the rational places,  $\varepsilon_v(f) = \varepsilon(1/2, \pi_v) \in \{\pm 1\}$  for all  $v$ , and  $\varepsilon_v(f) = +1$  for all  $v \nmid N\infty$ . We will assume that

$$\varepsilon_p(f) = -1 \text{ for all primes } p \mid M. \quad (\text{H2})$$

Under our previous assumptions, hypothesis (H2) becomes crucial for the non-vanishing of the period functional  $\mathcal{Q}$  (cf. Section 6). Observe that if  $\chi$  is assumed to be trivial, then  $M = 1$  and hypotheses (H1) and (H2) are empty. In fact, for  $\chi$  trivial, the arguments in this paper would be much less technical (for example, it would be enough to use Kohnen's formula instead of its generalization by Baruch–Mao, computations with the Saito–Kurokawa lift would be simpler, and the whole Section 8 would not be needed) and consequently the length of this note would be also considerably reduced.

A comment on signs is now in order. Indeed, the assumption that  $k$  is *odd* implies  $\varepsilon_\infty(f) = -1$ , and since the triple of weights  $(2k, k+1, k+1)$  is ‘balanced’ (i.e. none of the three weights is at least the sum of the other two) one also has  $\varepsilon_\infty(f' \otimes g \otimes g) = -1$ . Besides, the assumption (SF) together with hypothesis (H2) imply, by [Pra90, Section 8], that  $\prod_{p \mid N} \varepsilon_p(f) = \prod_{p \mid N} \varepsilon_p(f' \otimes g \otimes g)$ . Therefore,  $\varepsilon(f) = \varepsilon(f' \otimes g \otimes g)$ , and it follows from (1) that the sign in the functional equation for  $\Lambda(f \otimes \text{Ad}(g), s)$  is  $+1$ .

Now we can finally state our main result. To do so, let  $S_{k+1/2}^{+,new}(4NM, \chi)$  denote Kohnen's subspace of newforms of weight  $k + 1/2$ , level  $4NM$  and character  $\chi$  (see Section 2.2 for details). Under assumptions (SF) and (H1), it follows from [BM07, Theorem 10.1] (cf. Theorem 2.1 below) that the subspace of  $S_{k+1/2}^{+,new}(4NM, \chi)$  consisting of newforms whose eigenvalues for the Hecke operators at primes  $p \nmid 2N$  coincide with those of  $f \otimes \chi$  is one-dimensional. Let  $h$  be any non-trivial element in this one-dimensional subspace, thus  $h$  is a *Shimura lift* of  $f$  (or of  $f \otimes \chi$ ), and let  $F_\chi \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$  be the Saito–Kurokawa lift of  $h$ , as defined in Section 2.3; this is a Siegel cusp form of degree 2, weight  $k + 1$ , Hecke-type level  $\Gamma_0^{(2)}(N)$ , and character  $\chi$ . One may think of  $F_\chi$  as a Saito–Kurokawa lift of  $f \otimes \chi$ .

**THEOREM 1.1.** *Let  $k, N \geq 1$  be odd integers. Let  $f \in S_{2k}^{new}(\Gamma_0(N))$  and  $g \in S_{k+1}^{new}(\Gamma_0(N), \chi)$  be normalized newforms, and assume (SF), (H1), and (H2). If  $h$  and  $F_\chi$  denote a Shimura lift of  $f$  and its Saito–Kurokawa lift as explained*

above, then

$$\Lambda(f \otimes \text{Ad}(g), k) = 2^{k+1-\nu(M)} C(N, M, \chi) \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{|\langle (\text{id} \otimes U_M) F_\chi|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2}{\langle g, g \rangle^2}, \tag{3}$$

where  $U_M = \prod_{p|M} U_p$  denotes the product of the usual  $p$ -th Hecke operators  $U_p$ ,  $\nu(M)$  denotes the number of prime divisors of  $M$ , and

$$C(N, M, \chi) = |\chi(2)|^{-2} M^{3-k} N^{-1} \prod_{p|N} (p+1)^2 \prod_{p|M} (p+1).$$

In particular, under the assumptions of the theorem  $\Lambda(f \otimes \text{Ad}(g), k) \geq 0$ .

REMARK 1.2. *Let  $f$  and  $g$  be as in Theorem 1.1, and suppose that  $\chi$  is trivial. Then  $M = 1$ , and hypotheses (H1) and (H2) hold trivially, hence the central value formula (3) reads*

$$\Lambda(f \otimes \text{Ad}(g), k) = 2^{k+1} N^{-1} \prod_{p|N} (p+1)^2 \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{|\langle F|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2}{\langle g, g \rangle^2}.$$

*This formula coincides with the one obtained by S.-Y. Chen in [Che19], which appeared after a first version of this paper was made available. Instead of using Qiu’s decomposition theorems and computing local  $\text{SL}_2$ -periods, Chen generalizes straightforward the original strategy of Ichino. Besides considering non-trivial nebentype character, as commented above the novelty of our approach is precisely the computation of local  $\text{SL}_2$ -periods at ramified primes, which have their own interest and applications to other problems. If we further restrict to  $N = 1$ , observe that we obviously recover Ichino’s formula in (2).*

REMARK 1.3. *If the weight of  $g$  is assumed to be  $\ell + 1$  with  $\ell \geq k$  odd, instead of  $k + 1$ , then all the arguments to prove the above central value formula work through by replacing  $h$  and  $F_\chi$  with suitable nearly holomorphic forms obtained from these by applying the relevant derivative operators, and modifying accordingly the local computation for the archimedean period  $\mathcal{I}_\infty$  (cf. Section 9.2).*

Theorem 1.1 has an immediate application to Deligne’s conjecture [Del79]:

COROLLARY 1.4. *Let  $k, N \geq 1$  be odd integers, and  $f, g$  be as in Theorem 1.1. If  $\mathbb{Q}(f, g)$  denotes the number field generated by the Fourier coefficients of  $f$  and  $g$ , then*

$$\Lambda(f \otimes \text{Ad}(g), k)^{\text{alg}} := \frac{\Lambda(f \otimes \text{Ad}(g), k)}{\langle g, g \rangle^2 c^+(f)} \in \mathbb{Q}(f, g),$$

where  $c^+(f)$  denotes the period associated with the cuspidal form  $f$  by Shimura as in [Shi77].

REMARK 1.5. *When  $N = 1$ , this corollary follows of course from Ichino’s formula. And in line with Remark 1.3, when  $N = 1$  and  $f$  and  $g$  have weights  $2k$  and  $\ell+1$ , respectively, with  $\ell \geq k$  odd integers, the algebraicity of the relevant central value has been recently shown by H. Xue in [Xue], working with Jacobi forms instead of Saito–Kurokawa lifts.*

Let us close this introduction by pointing out some applications of the present work. One direction that we want to explore aims for a  $p$ -adic analogue of the factorization of complex  $L$ -series given in (1). Even though one could define a two-variable  $p$ -adic  $L$ -function associated with  $f \otimes \text{Ad}(g)$  by making use of the above hinted relation of  $L(f \otimes \text{Ad}(g), s)$  with suitable triple product  $L$ -series, the explicit central value formula provided by Theorem 1.1 gives an alternative approach for the construction of such a  $p$ -adic  $L$ -function, which is independent of any triple product  $p$ -adic  $L$ -function. Namely, one can directly interpolate the explicit expression in Theorem 1.1 (or rather the algebraic parts as in Corollary 1.4) as  $f$  and  $g$  vary in Hida families. With this  $p$ -adic  $L$ -function at hand, the proof of a factorization of  $p$ -adic  $L$ -functions parallel to (1) will require comparing two Euler systems. Namely, the one associated to generalized Gross–Kudla–Schoen diagonal cycles in the product of three Kuga–Sato varieties, and the one arising from Heegner points. This would extend the factorization result of Dasgupta in [Das16] establishing Greenberg’s conjecture for the adjoint representation, where instead of a triple product  $L$ -series one considers a Rankin  $L$ -series of two cusp forms twisted by a Dirichlet character. In a completely different direction, the computation of local  $\text{SL}_2$ -periods leading to the explicit central value formula in Theorem 1.1, when relaxing the assumption (SF) to requiring only that  $N_f$  and  $N_g$  are square-free, provides an important tool in the study of the subconvexity problem for the family of automorphic  $L$ -functions of the form  $L(s, \pi \times \text{ad}\tau)$ . This problem is related to the limiting mass distribution of automorphic forms (‘arithmetic quantum unique ergodicity’). See for instance [Nel11], [Nel19], and references therein.

## 1.1 OUTLINE OF THE PAPER

Although the strategy of the proof of Theorem 1.1 has already been sketched above, let us briefly explain the organization of this paper. Sections 2 and 3 are devoted to recall and set some necessary background material concerning classical modular forms and automorphic forms. Theorem 2.1, due to Baruch–Mao, will play an important role later on in the paper. After this, we review in Section 4 the theory of quadratic spaces, theta functions, and theta lifts, describing also explicit models for quadratic spaces in low dimension that are used in this paper. Section 5 focuses in the three theta correspondences that are involved in the proof of the main result of this paper. In particular, we prove two explicit identities for theta lifts (see Propositions 5.1 and 5.10) for the theta correspondences between  $\text{GL}_2$  and  $\text{GSO}(4)$ , and between  $\widetilde{\text{SL}}_2$  and  $\text{PGSp}_2$ , respectively, by adapting the ones in [II08, Section 5] and [Ich05, Section 7]. In Section 6 we can already prove Theorem 1.1, and deduce Corollary 1.4,

although the computation of the local periods  $\mathcal{I}_v$  is relegated to Sections 7, 8, and 9, which constitute the most technical part of this article.

## 1.2 NOTATION AND MEASURES

Before closing this introduction, we collect here some notations to be used throughout the paper. We will denote by  $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$  the ring of adèles over  $\mathbb{Q}$ . We will also write  $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$  for the profinite completion of  $\mathbb{Z}$ , which will be regarded as a subring of  $\mathbb{A}$ .

We write  $\zeta$  for Riemann's zeta function, with the usual Euler product  $\zeta = \prod_p \zeta_p$ , and  $\zeta_{\mathbb{Q}}$  will stand for the completed Riemann zeta function defined by  $\zeta_{\mathbb{Q}}(s) := \Gamma_{\mathbb{R}}(s)\zeta(s)$ , where  $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$  and  $\Gamma(s)$  denotes the Gamma function. We also put  $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ .

If  $G$  is a connected reductive group over  $\mathbb{Q}$ , we equip  $G(\mathbb{A})$  with the Tamagawa measure  $dg$ . The volume of  $[G] := G(\mathbb{Q})\backslash G(\mathbb{A})$  with respect to this measure is usually referred to as the Tamagawa number of  $G$ . For  $\mathrm{GL}_2(\mathbb{A})$ , we write  $dg = \prod_v dg_v$  as a product of local Haar measures, satisfying  $\mathrm{vol}(\mathrm{GL}_2(\mathbb{Z}_p), dg_p) = 1$  for all finite primes  $p$ . In the case of  $\mathrm{SL}_2(\mathbb{A})$ , we choose the local Haar measures to satisfy  $\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p)) = \zeta_p(2)^{-1}$  for all finite primes  $p$ .

If  $V$  is a finite-dimensional quadratic space over  $\mathbb{Q}$ , with bilinear form  $(\cdot, \cdot)$ , and  $\psi$  is an additive character of  $\mathbb{A}/\mathbb{Q}$ , then we consider the Haar measure on  $V(\mathbb{A})$  which is self-dual with respect to  $\psi$ , unless otherwise stated. That is to say, the Haar measure such that  $\mathcal{F}(\mathcal{F}(\phi))(x) = \phi(-x)$ , where  $\mathcal{F}(x) = \int_{V(\mathbb{A})} \phi(y)\psi((x, y))dy$  is the Fourier transform of  $\phi$ . The orthogonal group  $\mathrm{O}(V)$  is *not* connected. We choose a measure on  $\mathrm{O}(V)(\mathbb{A})$  as follows: first, we equip  $\mathrm{SO}(V)(\mathbb{A})$  with the Tamagawa measure; secondly, at each place  $v$  we extend the local measure on  $\mathrm{SO}(V)(\mathbb{Q}_v)$  to the non-identity component of  $\mathrm{O}(V)(\mathbb{Q}_v)$ ; and finally, we consider the measure  $dh_v$  on  $\mathrm{O}(V)(\mathbb{Q}_v)$  to be half of this extended measure, and define  $dh = \prod_v dh_v$ . This is the Tamagawa measure on  $\mathrm{O}(V)(\mathbb{A})$ , and  $[\mathrm{O}(V)] = \mathrm{O}(V)(\mathbb{Q})\backslash \mathrm{O}(V)(\mathbb{A})$  has volume 1 with respect to  $dh$ .

Continue to consider a finite-dimensional quadratic space  $V$  as before, and a non-trivial additive character of  $\mathbb{A}/\mathbb{Q}$ . If  $\mathcal{S}(V(\mathbb{A}))$  denotes the space of Bruhat–Schwartz functions on  $V(\mathbb{A})$ , and  $\phi_1, \phi_2 \in \mathcal{S}(V(\mathbb{A}))$ , we set  $\langle \phi_1, \phi_2 \rangle = \int_{V(\mathbb{A})} \phi_1(x)\overline{\phi_2(x)}dx$ , where  $dx$  is the Haar measure that is self-dual with respect to  $\psi$ . If  $\pi$  is an irreducible cuspidal unitary representation of  $G(\mathbb{A})$ , and  $f_1, f_2 \in \pi$ , we define the pairing  $\langle f_1, f_2 \rangle$  to be:

- i)  $\int_{[\mathrm{SL}_2]} f_1(g)\overline{f_2(g)}dg$ , if  $G = \widetilde{\mathrm{SL}}_2$ ;
- ii)  $\int_{[\mathrm{PGL}_2]} f_1(g)\overline{f_2(g)}dg$ , if  $G = \mathrm{GL}_2$ ;
- iii)  $\int_{[G]} f_1(g)\overline{f_2(g)}dg$ , if  $G = \mathrm{SO}(V)$  or  $\mathrm{O}(V)$ .

Finally, let  $p$  be a prime, and fix a non-trivial additive character  $\psi_p : \mathbb{Q}_p \rightarrow \mathbb{C}^{\times}$  of conductor  $\mathbb{Z}_p$ . If  $\mu : \mathbb{Q}_p^{\times} \rightarrow \mathbb{C}^{\times}$  is a character such that  $\mu(p) = 1$  (or



equivalently, a character of  $\mathbb{Z}_p^\times$ ), and  $a \in \mathbb{Q}_p$ , we define the Gauss sum

$$\mathfrak{G}(a, \mu) := \int_{\mathbb{Z}_p^\times} \psi(ax) \mu(x) dx,$$

where  $dx$  is the Haar measure which is self-dual with respect to  $\psi$ , normalized so that  $\mathbb{Z}_p$  has volume 1. It is well-known that  $\mathfrak{G}(a, \mu) = 0$  unless  $\mu = \mathbf{1}$  or  $\text{ord}_p(a) = -\text{cond}(\mu)$ , where  $c = \text{cond}(\mu)$  is the smallest positive integer such that  $\mu$  is trivial on  $1 + p^c \mathbb{Z}_p$ . More precisely, we have

$$\mathfrak{G}(a, \mu) = \begin{cases} 1 - p^{-1} & \text{if } \mu = \mathbf{1}, \text{ord}_p(a) \geq 0, \\ -p^{-1} & \text{if } \mu = \mathbf{1}, \text{ord}_p(a) = -1, \\ 0 & \text{if } \mu = \mathbf{1}, \text{ord}_p(a) < -1, \\ |a|^{-1/2} \varepsilon(1/2, \mu^{-1}) \mu^{-1}(a) & \text{if } \mu \neq \mathbf{1}, \text{ord}_p(a) = -\text{cond}(\mu), \\ 0 & \text{if } \mu \neq \mathbf{1}, \text{ord}_p(a) \neq -\text{cond}(\mu). \end{cases}$$

Here,  $\varepsilon(1/2, \mu^{-1})$  is the (local)  $\varepsilon$ -factor associated with  $\mu^{-1}$ , which satisfies (among other properties)  $\varepsilon(1/2, \mu) \varepsilon(1/2, \mu^{-1}) = \mu(-1)$ .

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## 2 MODULAR FORMS

The main purpose of this section is to recall and set up the notation regarding classical modular forms, with a particular focus on some features concerning distinct liftings between spaces of classical modular forms of different nature (namely, modular forms of integral weight, modular forms of half-integral weight, and Siegel modular forms of degree 2) that will be used in the paper.

### 2.1 INTEGRAL WEIGHT MODULAR FORMS

Let  $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$  be the complex upper half plane, on which the group  $\text{GL}_2^+(\mathbb{R})$  of real 2-by-2 matrices with positive determinant acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Let  $\ell \geq 1$  be an integer, and consider the usual action of  $\mathrm{GL}_2^+(\mathbb{R})$  on the space of holomorphic functions  $g : \mathcal{H} \rightarrow \mathbb{C}$  defined by

$$g|_{\ell}[\gamma](z) = j(\gamma, z)^{-\ell} \det(\gamma)^{\ell/2} g(\gamma z),$$

where  $j(\gamma, z) = cz + d$  if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . If  $N \geq 1$  is an integer and  $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$  is a Dirichlet character modulo  $N$ , we write  $M_{\ell}(N, \chi)$  for the  $\mathbb{C}$ -vector space of modular forms of weight  $\ell$ , level  $N$  and nebentype character  $\chi$ . Namely, the space of holomorphic functions  $g : \mathcal{H} \rightarrow \mathbb{C}$  satisfying

$$g|_{\ell}[\gamma] = \chi(\gamma)g \quad \text{for all } \gamma \in \Gamma_0(N),$$

together with the usual holomorphicity condition at the cusps. Here,  $\Gamma_0(N)$  denotes the level  $N$  Hecke-congruence subgroup of  $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$ ,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

on which  $\chi$  induces a character, which we still denote  $\chi$  by a slight abuse of notation, through the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(d).$$

We denote by  $S_{\ell}(N, \chi) \subseteq M_{\ell}(N, \chi)$  the subspace of cusp forms, namely the subspace of those modular forms which vanish at all the cusps. If  $g_1, g_2 \in M_{\ell}(N, \chi)$ , and at least one of them belongs to  $S_{\ell}(N, \chi)$ , then we consider the Petersson product of  $g_1$  and  $g_2$  defined as

$$\langle g_1, g_2 \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathcal{H}} g_1(z) \overline{g_2(z)} y^{\ell-2} dx dy \quad (z = x + iy).$$

If  $d$  is a positive integer, we define operators  $V_d$  and  $U_d$  by

$$V_d g(z) := d g(dz), \quad U_d g(z) := \frac{1}{d} \sum_{j=0}^{d-1} g\left(\frac{z+j}{d}\right).$$

Then the classical Hecke operators  $T_p$ , for primes  $p \nmid N$ , are expressed in terms of  $V_p$  and  $U_p$ : if  $g \in M_{\ell}(N, \chi)$ , then  $T_p g = U_p g + \chi(p) p^{\ell-2} V_p g$ . A cusp form  $g \in S_{\ell}(N, \chi)$  is said to be a Hecke eigenform if  $g$  is an eigenvector for all the Hecke operators  $T_p$ , for  $p \nmid N$ , and  $U_p$ , for  $p \mid N$ .

If  $g \in M_{\ell}(N, \chi)$ , recall that one disposes of a  $q$ -expansion (at the infinity cusp)

$$g(q) = \sum_{n \geq 0} a(g, n) q^n.$$

If  $g$  is a cusp form, then  $a(g, 0) = 0$ , and we say  $g$  is normalized if  $a(g, 1) = 1$ . If  $g$  is a normalized, new cuspidal Hecke eigenform, then  $T_p g = a(g, p)g$  for

all primes  $p$ . And furthermore,  $g$  is also an eigenvector for the Atkin–Lehner involutions  $W_p$  at each prime  $p \mid N$  (cf. [AL70] for details, especially Theorem 3).

If  $\lambda$  is a Dirichlet character modulo  $M$  and  $g \in M_\ell(N, \chi)$ , then we write  $g \otimes \lambda$  for the unique modular form in  $M_\ell(NM^2, \chi\lambda^2)$  with  $q$  expansion

$$\sum_{n \geq 0} \lambda(n) a(g, n) q^n.$$

For a careful study of the minimal level of  $g \otimes \lambda$ , at least for new forms, we refer the reader to [AL78].

When  $\chi = \mathbf{1}$  is the trivial character, we will write  $S_\ell(N) := S_\ell(N, \mathbf{1})$ . If  $f \in S_\ell(N)$  has  $q$ -expansion  $\sum_{n > 0} a_n q^n$ , then we write

$$L(f, s) = \sum_{n > 0} a_n n^{-s}$$

for the  $L$ -series associated with  $f$ . This Dirichlet series is well-known to converge for  $\operatorname{Re}(s) > 1 + \ell/2$ , and further it extends analytically to a holomorphic function on  $\mathbb{C}$ . The completed  $L$ -series  $\Lambda(f, s) := \Gamma_{\mathbb{C}}(s)L(f, s)$  satisfies a functional equation of the form

$$\Lambda(f, s) = \varepsilon(f) N^{\ell/2-s} \Lambda(f, \ell - s), \tag{4}$$

where  $\varepsilon(f) \in \{\pm 1\}$ . If  $f$  is a new eigenform, then  $L(f, s)$  can be expressed as an Euler product, namely

$$L(f, s) = \prod_p (1 - a_p p^{-s} + e_p p^{\ell-1} p^{-2s})^{-1},$$

where  $e_p = 0$  for  $p \mid N$ , and  $e_p = 1$  otherwise.

## 2.2 HALF-INTEGRAL WEIGHT MODULAR FORMS

We review briefly some aspects of the theory of half-integral weight modular forms, initiated by Shimura [Shi73] and further studied by Kohnen [Koh80, Koh82, Koh85] and many others (see, e.g. [KT04, BM07]).

For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \subset \operatorname{SL}_2(\mathbb{Z})$  and  $z \in \mathbb{C}$ , define

$$\tilde{j}(\gamma, z) = \left(\frac{c}{d}\right) \epsilon(d)(cz + d)^{1/2},$$

where  $(\cdot)$  is the Kronecker symbol as defined in [Shi73, p.442], and  $\epsilon(d)$  equals 1 if  $d \equiv 1 \pmod{4}$  and  $-\sqrt{-1}$  if  $d \equiv 3 \pmod{4}$ . Observe that  $\tilde{j}(\gamma, z)^4 = j(\gamma, z)^2$ . Let  $k \geq 1$  be an integer,  $N$  be an odd positive integer, and  $\chi$  be an even Dirichlet character modulo  $N$ . Consider the space  $S_{k+1/2}(4N, \chi)$  of holomorphic functions  $h : \mathcal{H} \rightarrow \mathbb{C}$  satisfying

$$h(\gamma z) = \tilde{j}(\gamma, z)^{2k+1} \chi(d) h(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$$

and vanishing at the cusps of  $\Gamma_0(4N)$ . The Petersson product of two cusp forms  $h_1$  and  $h_2$  in  $S_{k+1/2}(4N, \chi)$  is defined, similarly as in the case of integral weight, as

$$\langle h_1, h_2 \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]} \int_{\Gamma_0(4N) \backslash \mathcal{H}} h_1(z) \overline{h_2(z)} y^{k-3/2} dx dy \quad (z = x + iy).$$

It is well-known that every cusp form  $h \in S_{k+1/2}(4N, \chi)$  admits a Fourier expansion of the form

$$h(z) = \sum_{n \geq 1} c(n) q^n.$$

One defines *Kohnen's plus subspace*  $S_{k+1/2}^+(4N, \chi) \subset S_{k+1/2}(4N, \chi)$  as the subspace of those cusp forms whose Fourier expansion is of the form

$$h(z) = \sum_{\substack{n \geq 1, \\ (-1)^k n \equiv 0, 1 \pmod{4}}} c(n) q^n.$$

By virtue of [Koh82, Proposition 1], the space  $S_{k+1/2}^+(4N, \chi)$  can be characterized as the eigenspace of a certain hermitean operator satisfying a quadratic equation.

Finally, we recall also that for each prime  $p \nmid 2N$  there is a Hecke operator  $T_{p^2}$  acting on the space  $S_{k+1/2}(4N, \chi)$  (see [Koh82, Eq. (11)], [KT04, Eq. (1-6)]); on Fourier expansions, it is given by sending  $\sum_{n \geq 1} c(n) q^n$  to

$$\sum_{n \geq 1} \left( c(p^2 n) + \left( \frac{(-1)^k n}{p} \right) \chi(p) p^{k-1} c(n) + \chi(p)^2 p^{2k-1} c(n/p^2) \right) q^n,$$

where we read  $c(n/p^2) = 0$  if  $p^2 \nmid n$ .

Shimura's correspondence establishes lifting maps  $\zeta_{k,N,\chi}^D$  from  $S_{k+1/2}^+(4N, \chi)$  to the space  $S_{2k}(N, \chi^2)$  of cusp forms of weight  $2k$ , level  $N$  and nebentype character  $\chi^2$ , which depend on the choice of a (fundamental) discriminant  $D$ . When the character  $\chi$  is trivial and  $N$  is square-free, there is a well-behaved theory of new forms of half-integral weight, and a linear combination of the lifting maps  $\zeta_{k,N}^D$  provides an isomorphism

$$S_{k+1/2}^{+,new}(4N) \xrightarrow{\cong} S_{2k}^{new}(N)$$

commuting with the action of Hecke operators. In particular, for each normalized newform  $f \in S_{2k}^{new}(N)$  there is a unique half-integral weight cusp form  $h \in S_{k+1/2}^{+,new}(4N)$ , up to constant multiples, such that  $h|T_{p^2} = a(p)h$ , where  $a(p)$  is the  $p$ -th Fourier coefficient of  $f$ . Moreover, If  $h \in S_{k+1/2}^{+,new}(4N)$  corresponds to  $f \in S_{2k}^{new}(N)$  under this isomorphism, then the  $|D|$ -th Fourier coefficient  $c(|D|)$  of  $h$  is related to the special value  $L(f, D, k)$  of the complex  $L$ -series

associated to  $f$  twisted by the quadratic character  $\chi_D$ , by Kohnen’s formula (cf. [Koh85, Corollary 1]):

$$\frac{|c(|D|)|^2}{\langle h, h \rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f \rangle}, \tag{5}$$

where  $\nu(N)$  is the number of prime divisors of  $N$ . However, this formula is valid *only* for discriminants  $D$  such that  $(-1)^k D > 0$ , and  $\left(\frac{D}{\ell}\right) = w_\ell$  for all prime divisors  $\ell$  of  $N$ . Here,  $w_\ell$  denotes the eigenvalue of the  $\ell$ -th Atkin–Lehner involution acting on  $f$ .

In general, the lifting maps  $\zeta_{k,N,\chi}^D : S_{k+1/2}^+(4N, \chi) \rightarrow S_{2k}(N, \chi^2)$  are given in terms of Fourier expansions by the recipe (see [KT04, Section 3], for instance)

$$\sum_{\substack{n \geq 1, \\ (-1)^k n \equiv 0,1 \pmod{4}}} c(n)q^n \mapsto \sum_{n \geq 1} \left( \sum_{d|n} \left(\frac{D}{d}\right) \chi(d) d^{k-1} c(n^2|D|/d^2) \right) q^n. \tag{6}$$

Suppose from now on that  $f \in S_{2k}^{new}(N)$  is as in the introduction; in particular,  $k, N \geq 1$  are odd, and  $N$  is square-free. Let also  $\chi$  be an even Dirichlet character modulo  $N$ , of conductor  $M \mid N$ , and write  $\chi = \prod_{p|M} \chi(p)$  through the canonical isomorphism  $(\mathbb{Z}/M\mathbb{Z})^\times \simeq \prod_p (\mathbb{Z}/p\mathbb{Z})^\times$ , where each  $\chi(p)$  is a Dirichlet character modulo  $p$ . For each prime  $p \mid N$ , let  $w_p \in \{\pm 1\}$  be the eigenvalue of the  $p$ -th Atkin–Lehner involution acting on  $f$ , and define  $\mathfrak{D}(N, M)$  to be the set of fundamental discriminants

$$\left\{ D < 0 \text{ fund. discr.} : \left(\frac{D}{p}\right) = w_p \text{ if } p \mid \frac{N}{M}, \left(\frac{D}{p}\right) = -w_p \text{ if } p \mid M \right\}. \tag{7}$$

Besides, consider the twisted cusp form  $f \otimes \chi$ , which by [AL78, Theorem 3.1] belongs to  $S_{2k}^{new}(NM, \chi^2)$ , and define a subspace of  $S_{k+1/2}^+(4NM, \chi)$  by setting

$$S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi) := \{h \in S_{k+1/2}^+(4NM, \chi) : h|T_{p^2} = a_\chi(p)h \text{ for } p \nmid 2N\},$$

where  $a_\chi(p)$  denotes the  $p$ -th Fourier coefficient of  $f \otimes \chi$ . The lifting maps  $\zeta_{k,N,\chi}^D$  map  $S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi)$  to the one-dimensional subspace of  $S_{2k}(NM, \chi^2)$  spanned by the new form  $f \otimes \chi$ . The theorem below follows from [BM07, Theorem 10.1] (cf. loc. cit. for a slightly more general statement).

**THEOREM 2.1 (Baruch–Mao).** *Suppose that  $\chi(p)(-1) = -1$  for all  $p \mid M$ . Then the space*

$$S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi) \subset S_{k+1/2}^+(4NM, \chi)$$

*is one-dimensional. Moreover, if  $h \in S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi)$  is a non-zero element, with Fourier expansion  $h = \sum c(n)q^n$ , and  $D \in \mathfrak{D}(N, M)$ , then*

$$\frac{|c(|D|)|^2}{\langle h, h \rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \left( \prod_{p|M} \frac{p}{p+1} \right) \frac{L(f, D, k)}{\langle f, f \rangle}. \tag{8}$$

The identity in (8) is a generalization of Kohnen’s formula (5), relating the special values  $L(f, D, k)$  to the  $|D|$ -th Fourier coefficients of certain cusp forms of half-integral weight, for discriminants  $D$  to which (5) does not apply. At the end of Section 5.2 we will explain the meaning of the local sign assumptions on  $D$  in terms of Waldspurger’s theta correspondence between automorphic forms of  $\mathrm{PGL}_2$  and  $\widetilde{\mathrm{SL}}_2$ .

REMARK 2.2. *In terms of the completed  $L$ -series  $\Lambda(f, D, s) = \Gamma_{\mathbb{C}}(s)L(f, D, s)$ , the identity in (8) can be rewritten as*

$$\Lambda(f, D, k) = 2^{1-k-\nu(N)} |D|^{1/2-k} |c(|D|)|^2 \left( \prod_{p|M} \frac{p+1}{p} \right) \frac{\langle f, f \rangle}{\langle h, h \rangle}$$

Continue to consider a new form  $f \in S_{2k}^{new}(N)$  as above, and let  $\chi$  be an even Dirichlet character of conductor  $M \mid N$  satisfying the hypothesis of Theorem 2.1. In particular, notice that  $M$  must be the (square-free) product of an even number of prime divisors of  $N$ . Let

$$f = \sum_{n \geq 1} a(n)q^n, \quad f \otimes \chi = \sum_{n \geq 1} a_{\chi}(n)q^n$$

be the Fourier expansions of  $f$  and  $f \otimes \chi$ , respectively, so that  $a_{\chi}(n) = \chi(n)a(n)$ . We further assume that  $f$  is normalized, i.e.  $a(1) = 1$ , hence  $f \otimes \chi$  is normalized as well. By virtue of the above theorem, we can then choose a non-zero cusp form

$$h \in S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi).$$

Now fix a fundamental discriminant  $D \in \mathfrak{D}(N, M)$  such that  $L(f, D, k) \neq 0$ . By (8),  $c(|D|)$  is non-zero, and this implies that  $\zeta_{k,N,\chi}^D(h) \neq 0$ . Indeed, (6) shows that the first Fourier coefficient of  $\zeta_{k,N,\chi}^D(h)$  equals  $c(|D|)$ . It follows that  $\zeta_{k,N,\chi}^D(h) = c(|D|)f \otimes \chi$ . For later purposes, we determine in the next lemma the Fourier coefficients of  $h$  in terms of the Fourier coefficients of  $f \otimes \chi$  (hence, in terms of the Fourier coefficients of  $f$  as well).

LEMMA 2.3. *With the above notation and assumptions, for every integer  $n \geq 1$  one has*

$$c(n^2|D|) = c(|D|) \sum_{\substack{d|n, \\ d>0}} \mu(d) \left( \frac{D}{d} \right) \chi(d) d^{k-1} a_{\chi}(n/d). \tag{9}$$

*Proof.* As observed above,  $\zeta_{k,N,\chi}^D$  maps  $h$  to  $c(|D|)f \otimes \chi$ . From (6), we thus have for all  $n \geq 1$

$$c(|D|)a_{\chi}(n) = \sum_{\substack{d|n, \\ d>0}} \left( \frac{D}{d} \right) \chi(d) d^{k-1} c(n^2|D|/d^2).$$

Suppose first that  $n \geq 1$  satisfies  $(n, ND) = 1$ . Then we can rewrite this last identity as

$$c(|D|) \left(\frac{D}{n}\right) \chi(n)^{-1} n^{1-k} a_\chi(n) = \sum_{0 < t|n} \left(\frac{D}{t}\right) \chi(t)^{-1} t^{1-k} c(t^2|D|).$$

By applying the Möbius inversion formula, one gets

$$c(n^2|D|) = c(|D|) \sum_{0 < d|n} \mu(d) \left(\frac{D}{d}\right) \chi(d) d^{k-1} a_\chi(n/d).$$

Now suppose that  $n \geq 1$ , and write  $n = n_0 n_N n_D$  where  $n_0, n_N, n_D \geq 1$  are integers such that  $(n_0, ND) = 1$  and every prime divisor of  $n_N$  (resp.  $n_D$ ) divides  $N$  (resp.  $D$ ). From (6), we see that  $c(|D|) a_\chi(n_N) = c(n_N^2|D|)$  and  $c(|D|) a_\chi(n_D) = c(n_D^2|D|)$ . Also, for integers  $r, s \geq 1$  with  $(r, s) = 1$  one has  $c(r^2|D|) c(s^2|D|) = c(|D|) c(r^2 s^2|D|)$ . In particular,  $c(n_0^2|D|) c(n_N^2|D|) c(n_D^2|D|) = c(|D|)^2 c(n^2|D|)$ , which together with the above relations imply that

$$c(n^2|D|) = \frac{c(n_N^2|D|)}{c(|D|)} \frac{c(n_D^2|D|)}{c(|D|)} c(n_0^2|D|) = a_\chi(n_N) a_\chi(n_D) c(n_0^2|D|).$$

Now one can apply the previous argument for  $c(n_0^2|D|)$ , since  $(n_0, ND) = 1$ , to eventually conclude that

$$c(n^2|D|) = c(|D|) \sum_{0 < d|n} \mu(d) \left(\frac{D}{d}\right) \chi(d) d^{k-1} a_\chi(n/d).$$

□

### 2.3 SIEGEL MODULAR FORMS OF DEGREE 2 AND SAITO–KUROKAWA LIFTS

Let

$$\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}) : Z = {}^t Z, \text{Im}(Z) \text{ positive definite}\}$$

denote Siegel’s upper half-space of degree 2,

$$\text{GSp}_2^+(\mathbb{R}) := \{g \in M_4(\mathbb{R}) : g J_2 {}^t g = \nu(g) J_2, \nu(g) > 0\}, \quad J_2 = \begin{pmatrix} 0 & \text{Id}_2 \\ -\text{Id}_2 & 0 \end{pmatrix},$$

be the group of symplectic similitudes with positive multipliers, and let  $\text{Sp}_2(\mathbb{R}) = \{g \in \text{GSp}_2^+(\mathbb{R}) : \nu(g) = 1\} \subseteq \text{GL}_4(\mathbb{R})$  be the symplectic group. The group  $\text{GSp}_2^+(\mathbb{R})$  acts on  $\mathcal{H}_2$  by

$$gZ = (AZ + B)(CZ + D)^{-1} \quad \text{if } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{10}$$

Put  $\Gamma_2 := \text{Sp}_2(\mathbb{Z}) = \text{Sp}_2(\mathbb{R}) \cap M_4(\mathbb{Z})$ . If  $N \geq 1$  is an integer, one defines a Hecke-type congruence subgroup of level  $N \geq 1$  of  $\Gamma_2$  by

$$\Gamma_0^{(2)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 : C \equiv 0 \pmod{N} \right\}. \tag{11}$$

Given a Dirichlet character  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ , by a slight abuse of notation we will continue to write  $\chi : \Gamma_0^{(2)} \rightarrow \mathbb{C}^\times$  for the character on  $\Gamma_0^{(2)}(N)$  defined by the rule

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \chi(\det(D)). \quad (12)$$

Fix now an integer  $k \geq 1$ , and for a function  $F : \mathcal{H}_2 \rightarrow \mathbb{C}$  and an element  $g \in \mathrm{GSp}_2^+(\mathbb{R})$ , define

$$(F|_{k+1}[g])(Z) = J(g, Z)^{-k-1} F(gZ),$$

where  $J(g, Z) = \det(CZ + D)$  if  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ . The space  $M_{k+1}(\Gamma_0^{(2)}(N), \chi)$  of Siegel modular forms of weight  $k+1$ , level  $N$ , and character  $\chi$  is the space of holomorphic functions  $F : \mathcal{H}_2 \rightarrow \mathbb{C}$  such that

$$F|_{k+1}[\gamma] = \chi(\gamma)F \quad \text{for all } \gamma \in \Gamma_0^{(2)}(N).$$

Notice that there are no additional conditions of holomorphicity at the cusps ('Koecher's principle'). We will write  $S_{k+1}(\Gamma_0^{(2)}(N), \chi) \subseteq M_{k+1}(\Gamma_0^{(2)}(N), \chi)$  for the subspace of Siegel cusp forms. Given  $F \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$ , one has a Fourier expansion of the form

$$F(Z) = \sum_B A_F(B) e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)},$$

where  $B$  runs over the set of positive definite, half-integral 2-by-2 symmetric matrices. If

$$B = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}, \quad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix},$$

with  $n, r, m \in \mathbb{Z}$  such that  $4nm - r^2 > 0$  and  $\tau, \tau' \in \mathcal{H}$ ,  $z \in \mathbb{C}$  with  $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tau')$ , then notice that  $e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)} = e^{2\pi\sqrt{-1}(n\tau + rz + m\tau')}$ , so that we can rewrite the Fourier expansion as

$$F(\tau, z, \tau') = \sum_{\substack{n, r, m \in \mathbb{Z}, \\ 4nm - r^2 > 0}} A_F(n, r, m) e^{2\pi\sqrt{-1}(n\tau + rz + m\tau')}.$$

Given a Siegel modular form  $F$  for  $\Gamma_0^{(2)}(N)$ , one can restrict it to  $\mathcal{H} \times \mathcal{H}$ , diagonally embedded into  $\mathcal{H}_2$ . This way, we obtain a modular form on  $\mathcal{H} \times \mathcal{H}$  for  $\Gamma_0(N) \times \Gamma_0(N)$ , usually referred to as the "pullback" of  $F$  to the diagonal. Explicitly, this pullback or restriction is obtained by setting  $z = 0$ , hence

$$F|_{\mathcal{H} \times \mathcal{H}}(\tau, \tau') = \sum_{n, m \in \mathbb{Z}} \left( \sum_{\substack{r \in \mathbb{Z}, \\ r^2 < 4nm}} A_F(n, r, m) \right) e^{2\pi\sqrt{-1}(n\tau + m\tau')}.$$



Later we will also need a more technical operation on Siegel forms, which we define now. Let  $F \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$  be a Siegel modular form of degree 2 as before, and let  $p$  be a prime dividing  $N$ . For each  $j \in \mathbb{Z}/p\mathbb{Z}$ , put

$$u_j := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & p^{-1}j \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{Q}),$$

and define a function  $\mathfrak{R}_p F$  on  $\mathcal{H}_2$  by setting (notice that  $J(u_j, Z) = 1$  for all  $j$ )

$$\mathfrak{R}_p F(Z) := \frac{1}{p} \sum_{j \in \mathbb{Z}/p\mathbb{Z}} F(u_j Z) = \frac{1}{p} \sum_{j \in \mathbb{Z}/p\mathbb{Z}} F|_{k+1}[u_j](Z). \tag{13}$$

For any integer  $m \neq 0$ , we write  $\Gamma^{\mathrm{param}}(m) \subseteq \mathrm{Sp}_2(\mathbb{Q})$  for the paramodular subgroup defined by

$$\Gamma^{\mathrm{param}}(m) := \mathrm{Sp}_2(\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & m\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & m^{-1}\mathbb{Z} \\ \mathbb{Z} & m\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ m\mathbb{Z} & m\mathbb{Z} & m\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

LEMMA 2.4. *Let  $F \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$  be as before, and  $p$  be a prime with  $p \mid N$ . Then*

$$\mathfrak{R}_p F \in S_{k+1}(\Gamma_0^{(2)}(Np) \cap \Gamma^{\mathrm{param}}(p), \chi).$$

*If  $p^2 \mid N$ , then  $\mathfrak{R}_p F \in S_{k+1}(\Gamma_0^{(2)}(N) \cap \Gamma^{\mathrm{param}}(p), \chi)$ .*

*Proof.* Let  $\gamma \in \Gamma_0^{(2)}(Np) \cap \Gamma^{\mathrm{param}}(p)$  be written as a block matrix

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where  $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ ,  $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ ,  $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ ,  $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ . Notice that all entries are integral, and we have  $c_1, c_2, c_3, c_4 \in Np\mathbb{Z}$  and  $a_2, d_3 \in p\mathbb{Z}$ . Moreover, observe that  $a_4, d_4$  are invertible modulo  $p$ .

Similarly, write  $u_j$  as a block matrix,

$$u_j = \begin{pmatrix} \mathrm{Id}_2 & E_j \\ 0 & \mathrm{Id}_2 \end{pmatrix}.$$

Choosing  $i \in \mathbb{Z}/p\mathbb{Z}$  such that  $ia_4 \equiv jd_4 \pmod{p}$ , a bit of algebra shows that  $u_j \gamma = \gamma' u_i$ , where  $\gamma' = \begin{pmatrix} A' & B' \\ C & D' \end{pmatrix} \in \Gamma_0^{(2)}(N)$  is such that  $A' \equiv A$  and  $D' \equiv D$  modulo  $N$ . In particular,  $\chi(\gamma') = \chi(\gamma)$  and it follows from the very definition that

$$\mathfrak{R}_p F|_{[\gamma]} = \sum F|_{[u_j \gamma]} = \sum F|_{[\gamma' u_i]} = \chi(\gamma') \sum F|_{[u_i]} = \chi(\gamma) \mathfrak{R}_p F.$$

The second part of the statement follows by checking carefully the omitted algebra.  $\square$

REMARK 2.5. We might observe that  $u_j \in \mathrm{GSp}_2(\mathbb{Q})$  is the image of

$$\left( \left( \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right), \left( \begin{matrix} 1 & p^{-1}j \\ 0 & 1 \end{matrix} \right) \right) \in \mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q}),$$

and hence the pullback  $(\mathfrak{R}_p F)|_{\mathcal{H} \times \mathcal{H}}$  coincides with  $(\mathrm{id} \otimes V_p U_p) F|_{\mathcal{H} \times \mathcal{H}}$ .

Finally, we recall the classical construction of Saito–Kurokawa lifts that we need in this paper. Assume as in the introduction that  $k \geq 1$  is odd,  $N \geq 1$  is an odd square-free integer, and  $f \in S_{2k}^{\mathrm{new}}(N)$  is a normalized new cusp form of weight  $2k$  and level  $N$ . Let  $\chi$  be an even Dirichlet character of conductor  $M$ , for some  $M \mid N$ , satisfying the hypothesis of Theorem 2.1, and let

$$h \in S_{k+1/2}^{+, \mathrm{new}}(4NM, \chi; f \otimes \chi) \subset S_{k+1/2}^{+, \mathrm{new}}(4NM, \chi)$$

be a Shimura lift of  $f \otimes \chi \in S_{2k}^{\mathrm{new}}(NM, \chi^2)$  as in Theorem 2.1. There is an Eichler–Zagier isomorphism

$$\mathbf{Z} : S_{k+1/2}^{+, \mathrm{new}}(4NM, \chi) \xrightarrow{\simeq} J_{k+1,1}^{\mathrm{cusp}, \mathrm{new}}(\Gamma_0(NM)^J, \chi)$$

between  $S_{k+1/2}^{+, \mathrm{new}}(4NM, \chi)$  and the space  $J_{k+1,1}^{\mathrm{cusp}, \mathrm{new}}(\Gamma_0(NM)^J, \chi)$  of Jacobi new cusp forms of weight  $k + 1$ , index 1, level  $\Gamma_0(NM)^J$  and character  $\chi$  (see [EZ85, Ibu12, MR00]), which together with Maaß’ lift

$$\mathbf{M} : J_{k+1,1}^{\mathrm{cusp}, \mathrm{new}}(\Gamma_0(NM)^J, \chi) \hookrightarrow S_{k+1}(\Gamma_0^{(2)}(NM), \chi)$$

gives an injective homomorphism from  $S_{k+1/2}^{+, \mathrm{new}}(4NM, \chi)$  into  $S_{k+1}(\Gamma_0^{(2)}(NM), \chi)$ . We will refer to the Siegel modular form  $F_\chi := \mathbf{M}(\mathbf{Z}(h)) \in S_{k+1}(\Gamma_0^{(2)}(NM), \chi)$  associated with  $h$  under the composition of  $\mathbf{Z}$  and  $\mathbf{M}$ , as a *Saito–Kurokawa lift* of  $f \otimes \chi$ . If we continue to denote by  $c(n)$  the Fourier coefficients of  $h$ , then the Fourier expansion of  $F_\chi$  reads

$$F_\chi(Z) = \sum_{B = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}} \left( \sum_{\substack{a \mid \gcd(n,r,m) \\ \gcd(a,N)=1}} a^k \chi(a) c\left(\frac{4nm - r^2}{a^2}\right) \right) e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)}. \tag{14}$$

If  $p$  is a prime dividing  $M$ , notice that  $p^2 \mid NM$ , hence our discussion above implies that  $\mathfrak{R}_p F_\chi \in S_{k+1}(\Gamma_0^{(2)}(NM) \cap \Gamma^{\mathrm{param}}(p), \chi)$ . By defining the operator  $\mathfrak{R}_M := \prod_{p \mid M} \mathfrak{R}_p$  as the compositum of the operators  $\mathfrak{R}_p$  for primes  $p \mid M$ , we thus have

$$\mathfrak{R}_M F_\chi \in S_{k+1}(\Gamma_0^{(2)}(NM) \cap \Gamma^{\mathrm{param}}(M), \chi).$$

3 AUTOMORPHIC FORMS AND REPRESENTATIONS

Similarly as in the previous one, the aim of this section is to set up some notation and recall some results concerning the theory of automorphic forms for  $GL_2$ , for the metaplectic cover  $\widetilde{SL}_2$  of  $SL_2$ , and for the symplectic similitude group  $GSp_2$  of degree 4. Most of this section can therefore be seen as an automorphic rephrasing of the previous one. We will abbreviate  $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$  for the ring of adèles of  $\mathbb{Q}$ , and  $\mathbb{A}_f$  will denote the subring of finite adèles. Then  $\mathbb{Q}$  sits diagonally in  $\mathbb{A}$ , and for every rational place  $v$ , the local field  $\mathbb{Q}_v$  embeds as a subfield of  $\mathbb{A}$  in the  $v$ -th component. If  $\chi$  is a Dirichlet character modulo  $N \geq 1$ , we will write  $\underline{\chi}$  for the adelization of  $\chi$ . Namely,  $\underline{\chi} : \mathbb{A}^{\times} / \mathbb{Q}^{\times} \rightarrow \mathbb{C}^{\times}$  is the unique Hecke character such that  $\underline{\chi}(\varpi_q) = \chi(q)$  for every prime  $q \nmid N$  and every uniformizer  $\varpi_q \in \mathbb{Q}_q^{\times} \hookrightarrow \mathbb{A}^{\times}$  at  $q$ . For every finite prime  $p$ , we shall denote by  $\underline{\chi}_p$  the restriction of  $\underline{\chi}$  to  $\mathbb{Z}_p^{\times}$ . At primes  $p \mid N$ ,  $\underline{\chi}_p$  coincides with the *inverse* of the character  $\mathbb{Z}_p^{\times} \rightarrow \mathbb{C}^{\times}$  inflated from the  $p$ -th component  $\chi_{(p)}$  of  $\chi$ .

3.1 AUTOMORPHIC FORMS FOR  $GL_2$

Let us briefly recall how classical modular forms of integral weight give rise to automorphic forms and representations of  $GL_2$ . In the following, we identify  $\mathbb{Q}^{\times}$  and  $\mathbb{A}^{\times}$  with the centers of  $GL_2(\mathbb{Q})$  and  $GL_2(\mathbb{A})$ , respectively. Let  $N \geq 1$  be an integer, and consider the compact open subgroup

$$\mathbf{K}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) : c \equiv 0 \pmod{N} \right\}$$

of  $GL_2(\mathbb{A}_f)$ . By strong approximation, one has  $GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) GL_2^+(\mathbb{R}) \mathbf{K}_0(N)$ , where  $GL_2^+(\mathbb{R})$  stands for the subgroup of 2-by-2 real matrices with positive determinant. If  $\chi$  is a Dirichlet character modulo  $N$ , then  $\underline{\chi}$  induces a character of  $\mathbf{K}_0(N)$ , which by a slight abuse of notation we still denote  $\underline{\chi}$ , by

$$\underline{\chi} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \underline{\chi}(d).$$

Observe that this agrees with the Hecke character  $\underline{\chi}$  when restricted to  $\mathbb{A}^{\times} \cap \mathbf{K}_0(N)$ .

Let  $g \in S_{\ell}(N, \chi)$  be a cusp form of weight  $\ell$ , level  $N$  and nebentype character  $\chi$ . Then it is well-known that  $g$  induces an automorphic form  $\mathbf{g}$  for  $GL_2(\mathbb{A})$ , by setting

$$\mathbf{g}(\gamma \gamma_{\infty} k_0) = g(\gamma_{\infty}(i))(ci + d)^{-\ell} (\det \gamma_{\infty})^{\ell/2} \underline{\chi}(k_0) \tag{15}$$

whenever  $\gamma \in GL_2(\mathbb{Q})$ ,  $\gamma_{\infty} \in GL_2^+(\mathbb{R})$  and  $k_0 \in \mathbf{K}_0(N)$ . This gives indeed a well-defined function on  $GL_2(\mathbb{A})$  because  $GL_2(\mathbb{Q}) \cap GL_2^+(\mathbb{R}) \mathbf{K}_0(N) = \Gamma_0(N)$ ,  $GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) GL_2^+(\mathbb{R}) \mathbf{K}_0(N)$ , and  $g$  satisfies

$$g \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right) = \chi(d)(cz + d)^{\ell} g(z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Further, it is clear that  $\mathbf{g}$  satisfies  $\mathbf{g}(\gamma z) = \underline{\chi}(z)\mathbf{g}(\gamma)$  for all  $z \in \mathbb{A}^\times$ . The function  $\mathbf{g}$  just defined belongs in fact to the space of automorphic forms on  $\mathrm{GL}_2(\mathbb{A})$  with central character  $\underline{\chi}$ . If  $\pi = \pi_g$  denotes the linear span of the right translates of  $\mathbf{g}$  under  $\mathrm{GL}_2(\mathbb{A})$ , then  $\pi$  is an admissible smooth representation of  $\mathrm{GL}_2(\mathbb{A})$ . Since  $g$  is an eigenform, it is well-known that  $\pi$  is irreducible and decomposes as a restricted tensor product  $\otimes_v \pi_v$  of admissible representations of  $\mathrm{GL}_2(\mathbb{Q}_v)$ , and  $\mathbf{g} = \otimes_v \mathbf{g}_v$  with  $\mathbf{g}_v \in \pi_v$  for each place  $v$  of  $\mathbb{Q}$ . When the nebentype character  $\chi$  is trivial,  $g$  gives rise to an automorphic representation  $\pi$  of  $\mathrm{GL}_2(\mathbb{A})$  with trivial central character, so that we can regard  $\pi$  as an automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$ . Again,  $\pi$  decomposes as a restricted tensor product  $\pi = \otimes_v \pi_v$  of admissible representations of  $\mathrm{PGL}_2(\mathbb{Q}_v)$ . We refer the reader to [Sch02] for a good account on local types, newforms, and a careful study of local  $\varepsilon$ -factors at non-archimedean places. If  $f \in S_\ell^{\mathrm{new}}(N)$  is a newform of weight  $\ell \geq 1$ , level  $N$ , and trivial nebentype character, then the completed complex  $L$ -series  $\Lambda(f, s)$  associated with  $f$  coincides with the  $L$ -series  $L(\pi, s)$  associated with the automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$  corresponding to  $f$ . The root number  $\varepsilon(f)$  appearing in (4) can be therefore written as the product of local  $\varepsilon$ -factors  $\varepsilon(\pi_v, 1/2)$  at places  $v \mid N\infty$ . At the finite places  $p \mid N$ ,  $\varepsilon(\pi_p, 1/2)$  coincides with the eigenvalue  $w_p \in \{\pm 1\}$  of the  $p$ -th Atkin–Lehner involution acting on  $f$  (cf. [Sch02, Theorem 3.2.2]).

### 3.2 AUTOMORPHIC FORMS FOR $\widetilde{\mathrm{SL}}_2$

We start by setting down some notation concerning metaplectic groups. If  $v$  is a place of  $\mathbb{Q}$ , we write  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$  for the metaplectic cover (of degree 2) of  $\mathrm{SL}_2(\mathbb{Q}_v)$ , and similarly, we denote by  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  the metaplectic cover (of degree 2) of  $\mathrm{SL}_2(\mathbb{A})$ . We will identify  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ , resp.  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ , with  $\mathrm{SL}_2(\mathbb{Q}_v) \times \{\pm 1\}$ , resp.  $\mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ , where the product is given by the rule

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon(g_1, g_2)\epsilon_1 \epsilon_2].$$

At each place  $v$ ,  $\epsilon_v(g_1, g_2)$  is defined as follows. First one defines  $x : \mathrm{SL}_2(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$  by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

Then,  $\epsilon_v(g_1, g_2) = (x(g_1)x(g_1 g_2), x(g_2)x(g_1 g_2))_v$ , where  $(, )_v$  denotes the Hilbert symbol. When  $v$  is a finite place, set also

$$s_v \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} (c, d)_v & \text{if } cd \neq 0, \text{ ord}_v(c) \text{ odd,} \\ 1 & \text{otherwise,} \end{cases}$$

for  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$ , and  $s_\infty(g) = 1$  for all  $g \in \mathrm{SL}_2(\mathbb{R})$ . Then, for each place  $v$ ,  $\mathrm{SL}_2(\mathbb{Q}_v)$  embeds as a subgroup of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$  through  $g \mapsto [g, s_v(g)]$ . If  $v$  is an odd finite prime, then this homomorphism gives a splitting of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$

over the maximal compact subgroup  $SL_2(\mathbb{Z}_p)$ , while for  $v = 2$  this is only a splitting over  $\Gamma_1(4; \mathbb{Z}_2) \subset SL_2(\mathbb{Z}_2)$ . If  $p$  is an odd prime (resp.  $p = 2$ ), and  $G$  is a subgroup of  $SL_2(\mathbb{Z}_p)$  (resp.  $\Gamma_1(4; \mathbb{Z}_2)$ ), then we will write  $\tilde{G} \subseteq \tilde{SL}_2(\mathbb{Z}_p)$  for the image of  $G$  under the previous splitting. We will also regard  $SL_2(\mathbb{Q})$  as a subgroup of  $\tilde{SL}_2(\mathbb{A})$  through the homomorphism  $g \mapsto [g, \prod_v s_v(g)]$ .

To simplify notation, if  $v$  is a place of  $\mathbb{Q}$  and  $x \in \mathbb{Q}_v$ ,  $\alpha \in \mathbb{Q}_v^\times$ , we will write  $u(x)$ ,  $n(x)$ ,  $t(\alpha)$  for the elements of  $SL_2(\mathbb{Q}_v)$  given by

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, n(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, t(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

We will slightly abuse of notation and still write  $u(x)$ ,  $n(x)$ ,  $t(\alpha)$  for the elements  $[u(x), 1]$ ,  $[n(x), 1]$ ,  $[t(\alpha), 1]$  in  $\tilde{SL}_2(\mathbb{Q}_v)$  (notice that these coincide with the images of  $u(x)$ ,  $n(x)$ ,  $t(\alpha)$ , respectively, under the splitting  $g \mapsto [g, s_v(g)]$ , as  $s_v(g) = 1$  in the three cases). We will also write  $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Q}_v)$  (or in  $\tilde{SL}_2(\mathbb{Q}_v)$ ). Then observe that

$$u(x) = t(-1) \cdot s \cdot n(-x) \cdot s \quad \text{for all } x \in \mathbb{Q}_v.$$

Let  $k \geq 1$  be an integer,  $N$  be a positive integer, and  $\chi$  be an even Dirichlet character modulo  $4N$  as before. Write  $\chi_0 = \left(\frac{-1}{\cdot}\right)^k \cdot \chi$ , and let  $\underline{\chi}_0$  denote the associated Hecke character.

Let  $\tilde{\mathcal{A}}_0$  denote the space of cusp forms on  $SL_2(\mathbb{Q}) \backslash \tilde{SL}_2(\mathbb{A})$ , and  $\tilde{\rho}$  denote the right regular representation of the Hecke algebra of  $\tilde{SL}_2(\mathbb{A})$  on  $\tilde{\mathcal{A}}_0$ . Following Waldspurger [Wal81], we define  $\tilde{\mathcal{A}}_{k+1/2}(4N, \underline{\chi}_0)$  to be the subspace of  $\tilde{\mathcal{A}}_0$  consisting of elements  $\varphi$  satisfying the following properties, where  $\tilde{\rho}_v$  denotes the restriction of  $\tilde{\rho}$  to  $SL_2(\mathbb{Q}_v)$ .

- i) For each prime  $q \nmid 2N$ ,  $\tilde{\rho}_q(\gamma)\varphi = \varphi$  for all  $\gamma \in SL_2(\mathbb{Z}_q)$ .
- ii) For each prime  $q \mid N$ ,  $q \neq 2$ ,  $\tilde{\rho}_q(\gamma)\varphi = \underline{\chi}_{0,q}(d)\varphi$  for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q^{\text{ord}_q(N)}) \subset SL_2(\mathbb{Z}_q)$ .
- iii) For all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2^{\text{ord}_2(4N)}) \subset SL_2(\mathbb{Z}_2)$ ,  $\tilde{\rho}_2(\gamma)\varphi = \tilde{\epsilon}_2(\gamma)\underline{\chi}_{0,2}(d)\varphi$ .
- iv) For all  $\theta \in \mathbb{R}$ ,  $\tilde{\rho}_\infty(\tilde{\kappa}(\theta))\varphi = e^{i(k+1/2)\theta}\varphi$ .
- v)  $\tilde{\rho}_\infty(\tilde{D})\varphi = \frac{1}{2}(k+1/2)(k-3/2)\varphi$ .

Here,  $\tilde{D}$  denotes the Casimir element for  $\tilde{SL}_2(\mathbb{R})$ , and the element  $\tilde{\kappa}(\theta)$ , for  $\theta \in \mathbb{R}$ , and the character  $\tilde{\epsilon}_2$  of  $\tilde{\Gamma}_0(4) \subseteq \tilde{SL}_2(\mathbb{Q}_2)$  are defined as in [Wal81, p. 382].

Let  $h \in S_{k+1/2}(4N, \chi)$  be a cusp form of half-integral weight as in Section 2.2. Given  $z = u + iv \in \mathcal{H}$ , let  $b(z) \in \tilde{SL}_2(\mathbb{A})$  be the element which is 1 at all the finite places and equal to

$$\begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix}$$

at the real place. Then there exists a unique continuous function  $\mathbf{h}$  on  $\mathrm{SL}_2(\mathbb{Q}) \backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})$  such that

$$\mathbf{h}(b(z)\tilde{\kappa}(\theta)) = v^{k/2+1/4} e^{i(k+1/2)\theta} h(z)$$

for all  $z \in \mathcal{H}$ ,  $\theta \in \mathbb{R}$ . The assignment  $h \mapsto \mathbf{h}$  gives an *adelization map*

$$S_{k+1/2}(4N, \chi) \longrightarrow \tilde{\mathcal{A}}_{k+1/2}(4N, \underline{\chi}_0), \tag{16}$$

which is an isomorphism (see [Wal81, Proposition 3, p. 386]). Under this isomorphism, the Fourier coefficients of the classical modular form  $h$  are related to the Fourier coefficients of  $\mathbf{h}$  as follows. Let  $\psi$  be the standard additive character of  $\mathbb{Q} \backslash \mathbb{A}$ . That is,  $\psi_\infty(x) = e^{2\pi i x}$  and, for each finite prime  $q$ ,  $\psi_q$  is the unique character of  $\mathbb{Q}_q$  with kernel  $\mathbb{Z}_q$  and such that  $\psi_q(x) = e^{-2\pi i x}$  for  $x \in \mathbb{Z}[1/q]$ . If  $\varphi \in \tilde{\mathcal{A}}_0$  and  $\xi \in \mathbb{Q}$ , then the  $\xi$ -th Fourier coefficient of  $\varphi$  is defined to be the function on  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  given by

$$W_{\varphi, \xi}(\gamma) = \int_{\mathbb{Q} \backslash \mathbb{A}} \varphi(u(x)\gamma) \overline{\psi(\xi x)} dx.$$

If  $h \in S_{k+1/2}(4N, \chi)$  has Fourier expansion

$$h(z) = \sum_{\xi \in \mathbb{Z}, \xi > 0} c(\xi) q^\xi,$$

and  $h \mapsto \mathbf{h}$  under the above isomorphism, then one has (cf. [Wal81, Lemme 3])

$$c(\xi) = e^{2\pi \xi} W_{\mathbf{h}, \xi}(1). \tag{17}$$

For the rest of this subsection, assume that  $k \geq 1$  is odd,  $N \geq 1$  is odd and square-free, and  $\chi$  is an even Dirichlet character satisfying the hypothesis of Theorem 2.1. Let  $f \in S_{2k}^{new}(N)$  and  $h \in S_{k+1/2}^{+, new}(4NM, \chi; f \otimes \chi)$  be chosen as in the discussion after Theorem 2.1. By using (17), the relation (9) proved in Lemma 2.3 can be rephrased in automorphic terms as we will now explain. Given any  $\xi \in \mathbb{Q}_+$ , set  $\xi = \mathfrak{d}_\xi f_\xi^2$ , where  $\mathfrak{d}_\xi \in \mathbb{N}$  is such that  $-\mathfrak{d}_\xi$  equals the discriminant of  $\mathbb{Q}(\sqrt{-\xi})/\mathbb{Q}$ . By (9), we have

$$c(\xi) = c(\mathfrak{d}_\xi) \sum_{\substack{d|f_\xi, \\ d > 0}} \mu(d) \chi_{-\xi}(d) \chi(d) d^{k-1} a_\chi(f_\xi/d). \tag{18}$$

For each prime  $p \nmid N$ , let  $\{\alpha_p, \alpha_p^{-1}\}$  be the Satake parameter of  $f$  at  $p$ , so that

$$(1 - p^{k-1/2} \alpha_p X)(1 - p^{k-1/2} \alpha_p^{-1} X) = 1 - a(p)X + p^{2k-1} X^2.$$

In particular, notice that  $a(p) = p^{k-1/2}(\alpha_p + \alpha_p^{-1})$ . More generally, for each integer  $e \geq 0$  one has

$$a(p^e) = p^{e(k-1/2)} \sum_{i=0}^e \alpha_p^{e-2i}.$$

In contrast, if  $p$  is an (odd) prime dividing  $N$ , being  $N$  square-free we have  $a(p) = -p^{k-1}w_p$ , where  $w_p = w_p(f) \in \{\pm 1\}$  is the eigenvalue of the Atkin–Lehner involution at  $p$  acting on  $f$ . Also, one has  $a(p^e) = a(p)^e$  for all integers  $e \geq 0$  in this case. For each prime  $p \mid N$ , define  $\alpha_p := p^{1/2-k}a(p) = -p^{-1/2}w_p$ . In a similar spirit as for primes not dividing  $N$ , now  $a(p^e) = p^{e(k-1/2)}\alpha_p^e$  for  $e \geq 0$ .

Given a rational prime  $p$ , put  $e_p := \text{ord}_p(\mathfrak{f}_\xi)$  and define a function  $\Psi_p(\xi; X) \in \mathbb{C}[X, X^{-1}]$  by

$$\Psi_p(\xi; X) = \begin{cases} \frac{X^{e_p+1}-X^{-e_p-1}}{X-X^{-1}} - p^{-1/2}\chi_{-\xi}(p)\frac{X^{e_p}-X^{-e_p}}{X-X^{-1}}, & \text{if } p \nmid N, e_p \geq 0, \\ \chi_{-\xi}(p)(\chi_{-\xi}(p) + w_p)X^{e_p}, & \text{if } p \mid N/M, e_p \geq 0, \\ \chi_{-\xi}(p)(\chi_{-\xi}(p) - w_p)X^{e_p}, & \text{if } p \mid M, e_p \geq 0, \\ 0, & e_p < 0. \end{cases}$$

Observe first of all that, fixed  $\xi$ , there are only finitely many primes  $p$  with  $\text{ord}_p(\xi) \neq 0$ . Since  $\Psi_p(\xi; X) = 1$  whenever  $p \nmid N$  and  $e_p = 0$ , we see that  $\Psi_p(\xi; X) = 1$  for almost all primes. Secondly, at a prime  $p \mid N/M$  (resp.  $p \mid M$ ) we see that  $\Psi_p(\xi; X) \neq 0$  if and only if  $\chi_{-\xi}(p) = w_p$  (resp.  $\chi_{-\xi}(p) = -w_p$ ). Hence,  $\prod_p \Psi_p(\xi; X) \neq 0$  if and only if

$$\xi \in \mathbb{Z}, \left(\frac{-\xi}{p}\right) = w_p \text{ for all } p \mid \frac{N}{M}, \text{ and } \left(\frac{-\xi}{p}\right) = -w_p \text{ for all } p \mid M.$$

LEMMA 3.1. *If  $\xi \in \mathbb{Z}$ , and  $\nu(N)$  denotes the number of prime factors of  $N$ , then*

$$c(\xi) = 2^{-\nu(N)}c(\mathfrak{d}_\xi)\chi(\mathfrak{f}_\xi)\mathfrak{f}_\xi^{k-1/2} \prod_p \Psi_p(\xi; \alpha_p). \tag{19}$$

*Proof.* First notice that (19) holds if  $c(\xi) = 0$  by our above observation, so we may assume that  $c(\xi) \neq 0$ . Secondly, both sides in (18) are zero if  $\xi$  is not an integer, so we may assume that  $\xi \in \mathbb{Z}_+$ . Writing  $\xi = \mathfrak{d}_\xi \mathfrak{f}_\xi^2$  as before, and setting  $\mathfrak{f}_\xi = \mathfrak{f}_{\xi,N} \mathfrak{f}_{\xi,0}$ , with  $\mathfrak{f}_{\xi,N}, \mathfrak{f}_{\xi,0}$  integers such that  $(\mathfrak{f}_{\xi,0}, N) = 1$  and every prime divisor of  $\mathfrak{f}_{\xi,N}$  divides  $N$ , equation (18) can be rewritten as

$$c(\xi) = c(\mathfrak{d}_\xi)\chi(\mathfrak{f}_\xi)a(\mathfrak{f}_{\xi,N}) \sum_{\substack{d \mid \mathfrak{f}_{\xi,0} \\ d > 0}} \mu(d)\chi_{-\xi}(d)d^{k-1}a(\mathfrak{f}_{\xi,0}/d).$$

Using the definition of the functions  $\Psi_p(\xi; X)$ , we deduce that

$$\begin{aligned} c(\xi) &= c(\mathfrak{d}_\xi)\chi(\mathfrak{f}_\xi)a(\mathfrak{f}_{\xi,N}) \prod_{p \mid \mathfrak{f}_{\xi,0}} (a(p^{e_p}) - p^{k-1}\chi_{-\xi}(p)a(p^{e_p-1})) = \\ &= c(\mathfrak{d}_\xi)\chi(\mathfrak{f}_\xi)a(\mathfrak{f}_{\xi,N})\mathfrak{f}_{\xi,0}^{k-1/2} \prod_{p \mid \mathfrak{f}_{\xi,0}} \Psi_p(\xi; \alpha_p). \end{aligned}$$

Since  $c(\xi) \neq 0$ , in particular  $\Psi_p(\xi; \alpha_p) \neq 0$  for all  $p \mid N$ . At each prime  $p \mid \mathfrak{f}_{\xi, N}$ , we thus have  $\Psi_p(\xi; \alpha_p) = 2\alpha_p^{e_p}$ . We therefore deduce that

$$a(\mathfrak{f}_{\xi, N}) = \prod_{p \mid \mathfrak{f}_{\xi, N}} a(p^{e_p}) = \mathfrak{f}_{\xi, N}^{k-1/2} \prod_{p \mid \mathfrak{f}_{\xi, N}} \alpha_p^{e_p} = 2^{-\nu(\mathfrak{f}_{\xi, N})} \mathfrak{f}_{\xi, N}^{k-1/2} \prod_{p \mid \mathfrak{f}_{\xi, N}} \Psi_p(\xi; \alpha_p).$$

At primes  $p \mid N$  with  $p \nmid \mathfrak{f}_{\xi, N}$  (if any), we have  $\Psi_p(\xi; \alpha_p) = 2$ , hence we can rewrite the above identity as

$$a(\mathfrak{f}_{\xi, N}) = 2^{-\nu(N)} \mathfrak{f}_{\xi, N}^{k-1/2} \prod_{p \mid N} \Psi_p(\xi; \alpha_p).$$

Since  $\Psi_p(\xi; \alpha_p) = 1$  for all primes  $p \nmid N$ , we deduce that (19) holds when  $c(\xi) \neq 0$ . □

### 3.3 AUTOMORPHIC FORMS FOR $\mathrm{GSp}_2$

We will now set the basic notation and definitions concerning automorphic forms for  $\mathrm{GSp}_2(\mathbb{A})$ , which will naturally arise in this paper by adelization of Siegel modular forms as the ones considered in Section 2.3. Write

$$\mathrm{GSp}_2 = \{g \in \mathrm{GL}_4 : {}^t g J_2 g = \nu(g) J_2 : \nu(g) \in \mathbb{G}_m\}, \quad J_2 = \begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix},$$

for the general symplectic group of degree 2, and recall that  $\mathrm{GSp}_2^+(\mathbb{R})$  acts on Siegel’s upper half-space as in (10). Here,  $\nu : \mathrm{GSp}_2 \rightarrow \mathbb{G}_m$  is the so-called *similitude* (or *scale*) *morphism*. If  $N \geq 1$  is an integer, we set  $K_0^{(2)}(N; \hat{\mathbb{Z}}) = \prod_p K_0^{(2)}(N; \mathbb{Z}_p) \subseteq \mathrm{GSp}_2(\hat{\mathbb{Z}})$ , where for each prime  $p$ ,

$$K_0^{(2)}(N; \mathbb{Z}_p) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_2(\mathbb{Z}_p) : C \equiv 0 \pmod{N} \right\}$$

is the local analogue of the congruence subgroup  $\Gamma_0^{(2)}(N)$  introduced in (11). Observe that  $K_0^{(2)}(N; \mathbb{Z}_p) = \mathrm{GSp}_2(\mathbb{Z}_p)$  for all primes  $p \nmid N$ . Compact open subgroups of  $\mathrm{GSp}_2(\mathbb{A})$  of the form  $K_0^{(2)}(N)$  will play a special role in the paper, although we will also consider certain subgroups of them.

Let  $\mathbf{F} : \mathrm{GSp}_2(\mathbb{A}) \rightarrow \mathbb{C}$  be an automorphic cusp form, and  $\Pi$  be the automorphic representation of  $\mathrm{GSp}_2(\mathbb{A})$  associated with  $\mathbf{F}$  (i.e., the closure of the span of all the right-translates of  $\mathbf{F}$ ). We suppose that  $\Pi$  is irreducible and unitary. By Schur’s lemma,  $\mathbf{F}$  has a central character: there exists a Hecke character  $\lambda : \mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$  such that  $\mathbf{F}(zg) = \lambda(z)\mathbf{F}(g)$  for all  $z \in \mathbb{A}^\times = Z(\mathrm{GSp}_2(\mathbb{A}))$ ,  $g \in \mathrm{GSp}_2(\mathbb{A})$ . If  $\lambda$  is trivial, then  $\mathbf{F}$  is trivial on the center of  $\mathrm{GSp}_2(\mathbb{A})$ , and hence one can regard  $\mathbf{F}$  as an automorphic cusp form on  $\mathrm{PGSp}_2(\mathbb{A})$ .

Besides, suppose that we are given an automorphic cusp form  $\mathbf{F} : \mathrm{PGSp}_2(\mathbb{A}) \rightarrow \mathbb{C}$ , and let  $\lambda : \mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$  be a Hecke character. Then one can define an



automorphic cusp form  $\mathbf{F} \otimes \lambda : \mathrm{GSp}_2(\mathbb{A}) \rightarrow \mathbb{C}$  by the rule  $(\mathbf{F} \otimes \lambda)(g) := \mathbf{F}(g)\lambda(\nu(g))$ . It is readily seen that the central character of  $\mathbf{F} \otimes \lambda$  is  $\lambda^2$ . If  $B \in \mathrm{Sym}_2(\mathbb{Q})$  is a symmetric 2-by-2 matrix, then the  $B$ -th Fourier coefficient of  $\mathbf{F}$  is by definition the function on  $\mathrm{GSp}_2(\mathbb{A})$  given by

$$\mathcal{W}_{\mathbf{F},B}(g) = \int_{\mathrm{Sym}_2(\mathbb{Q}) \backslash \mathrm{Sym}_2(\mathbb{A})} \mathbf{F}(n(X)g) \overline{\psi(\mathrm{Tr}(BX))} dX, \quad g \in \mathrm{GSp}_2(\mathbb{A}), \quad (20)$$

where  $n(X) = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$ . If  $\mathbf{F}$  is right invariant by some subgroup  $K \subseteq \mathrm{GSp}_2(\hat{\mathbb{Z}})$ , then notice that the Fourier coefficients  $\mathcal{W}_{\mathbf{F},B}$  enjoy also the same invariant property. It is well-known that the collection of all Fourier coefficients  $\mathcal{W}_{\mathbf{F},B}$  determine the automorphic form  $\mathbf{F}$ .

We will not develop the general theory of automorphic forms and representations of  $\mathrm{GSp}_2(\mathbb{A})$ , but rather we will focus on the automorphic forms for  $\mathrm{GSp}_2(\mathbb{A})$  that appear by adelization of Siegel modular forms of degree 2 as the ones considered in Section 2.3. Hence, suppose that  $k \geq 1$  is an odd integer,  $N \geq 1$  is an integer, and  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  is a Dirichlet character. As usual, let  $\underline{\chi} : \mathbb{A}^\times \rightarrow \mathbb{C}^\times$  be the Hecke character associated with  $\chi$  as in previous sections. If  $F \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$  is a Siegel modular form of weight  $k + 1$ , level  $\Gamma_0^{(2)}(N)$  and character  $\chi$ , then  $F$  defines an automorphic form  $\mathbf{F}$  for  $\mathrm{GSp}_2(\mathbb{A})$  by setting

$$\mathbf{F}(g) = \det(g_\infty)^{(k+1)/2} \det(C\sqrt{-1} + D)^{-k-1} F(g_\infty\sqrt{-1}) \underline{\chi}(k),$$

whenever  $g = \gamma g_\infty k$  with  $\gamma \in \mathrm{GSp}_2(\mathbb{Q})$ ,  $k \in K_0^{(2)}(N)$ , and

$$g_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{R}).$$

Here,  $\underline{\chi}$  induces a character on  $K_0^{(2)}(N)$  similarly as in the classical situation explained in (12). Since  $\mathrm{GSp}_2(\mathbb{A}) = \mathrm{GSp}_2(\mathbb{Q}) \mathrm{GSp}_2^+(\mathbb{R}) K_0^{(2)}(N)$  by strong approximation, this gives indeed a well-defined function on  $\mathrm{GSp}_2(\mathbb{A})$ . The fact that  $F$  is a Siegel cusp form in  $S_{k+1}(\Gamma_0^{(2)}(N), \chi)$  easily implies that  $\mathbf{F}$  is  $\mathrm{GSp}_2(\mathbb{Q})$ -invariant on the left, that  $\mathbf{F}(gk) = \underline{\chi}(k)\mathbf{F}(g)$  for all  $g \in \mathrm{GSp}_2(\mathbb{A})$  and  $k \in K_0^{(2)}(N)$ , and that the center of  $\mathrm{GSp}_2(\mathbb{A})$  acts through  $\underline{\chi} \circ \nu$ , where  $\nu : \mathrm{GSp}_2(\mathbb{A}) \rightarrow \mathbb{A}^\times$  denotes the similitude morphism. That is to say,  $F(zg) = \underline{\chi}(\nu(z))\mathbf{F}(g)$  for all  $z \in Z(\mathrm{GSp}_2(\mathbb{A})) = \mathbb{A}^\times$ ,  $g \in \mathrm{GSp}_2(\mathbb{A})$ , hence the central character of  $\mathbf{F}$  is  $\underline{\chi}^2$ . We write  $\mathcal{S}_{k+1}(K_0^{(2)}(N), \underline{\chi})$  for the space of automorphic cusp forms on  $\mathrm{GSp}_2(\mathbb{A})$  arising by adelization of Siegel cusp forms in  $S_{k+1}(\Gamma_0^{(2)}(N), \chi)$ .

The level-raising operator  $\mathfrak{R}_p$  introduced classically in (13) can be defined analogously in the automorphic setting. Indeed, let  $p$  be a prime dividing  $N$ , and define

$$\mathfrak{R}_p \mathbf{F} := \sum_{j=0}^{p-1} \Pi(u_j) \mathbf{F},$$

where  $\Pi$  is the automorphic representation of  $\mathrm{GSp}_2(\mathbb{A})$  associated with  $\mathbf{F}$ , acting on  $\mathbf{F}$  by right translation:  $\Pi(h)\mathbf{F} : g \mapsto \mathbf{F}(gh)$ . In line with the classical definition, we now define  $K^{\mathrm{param}}(p; \mathbb{Z}_p)$  to be the subgroup

$$\left\{ \gamma \in \mathrm{GSp}_2(\mathbb{Q}_p) \cap \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} : \det(\gamma) \in \mathbb{Z}_p^\times \right\}.$$

Then we write  $K(N, p; \mathbb{Z}_p) := K_0^{(2)}(N; \mathbb{Z}_p) \cap K^{\mathrm{param}}(p; \mathbb{Z}_p)$ , and

$$K(N, p) = K(N, p; \mathbb{Z}_p) \times \prod_{q \neq p} K_0^{(2)}(N; \mathbb{Z}_q) \subseteq \mathrm{GSp}_2(\hat{\mathbb{Z}}).$$

If  $F \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$  is as above, recall that  $\mathfrak{R}_p F$  belongs to  $S_{k+1}(\Gamma_0^{(2)}(N') \cap \Gamma^{\mathrm{param}}(p), \chi)$ , where  $N' = N$  if  $p^2 \mid N$ , and  $N' = Np$  otherwise. We have  $K(N', p) \cap \mathrm{GSp}_2(\mathbb{Q}) = \Gamma_0^{(2)}(N') \cap \Gamma^{\mathrm{param}}(p)$ , and hence Siegel forms in  $S_{k+1}(\Gamma_0^{(2)}(N') \cap \Gamma^{\mathrm{param}}(p), \chi)$  induce by adalization automorphic cusp forms on  $\mathrm{GSp}_2(\mathbb{A})$  on which  $K(N', p)$  acts on the right through the character  $\underline{\chi} : K(N', p) \rightarrow \mathbb{C}^\times$ . We write  $\mathcal{S}_{k+1}(K(N', p), \underline{\chi})$  for the space of automorphic forms on  $\mathrm{GSp}_2(\mathbb{A})$  obtained by adalization of Siegel forms in  $S_{k+1}(\Gamma_0^{(2)}(N') \cap \Gamma^{\mathrm{param}}(p), \chi)$ .

LEMMA 3.2. *Let  $F \in S_{k+1}(\Gamma_0^{(2)}(N), \chi)$ , and  $\mathbf{F} \in \mathcal{S}_{k+1}(K_0^{(2)}(N), \underline{\chi})$  be its adalization as above. If  $p$  is a prime dividing  $N$ , then  $\mathfrak{R}_p \mathbf{F}$  is the adalization of  $\mathfrak{R}_p F$ . In particular,  $\mathfrak{R}_p \mathbf{F} \in \mathcal{S}_{k+1}(K(Np, p), \underline{\chi})$ . If  $p^2$  divides  $N$ , then one actually has  $\mathfrak{R}_p \mathbf{F} \in \mathcal{S}_{k+1}(K(N, p), \underline{\chi})$ .*

We now particularize the above discussion to a situation which is of particular interest in this note. Continue to assume that  $k \geq 1$  is an odd integer, and let  $N \geq 1$  be an odd square-free integer, and  $\chi$  be an even Dirichlet character of conductor  $M \mid N$ . Let  $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$  be a normalized new cusp form of weight  $2k$ , level  $N$  and trivial nebentype, and let  $F_\chi \in S_{k+1}(\Gamma_0^{(2)}(NM), \chi)$  be the Saito–Kurokawa lift of  $f \otimes \chi$  defined in Section 2.3. Write  $\mathbf{F}_\chi \in \mathcal{S}_{k+1}(K_0^{(2)}(NM), \underline{\chi})$  for the adalization of  $F_\chi$ , hence

$$\mathbf{F}_\chi(g) = \det(g_\infty)^{(k+1)/2} \det(C\sqrt{-1} + D)^{-k-1} F_\chi(g_\infty \sqrt{-1}) \underline{\chi}(k), \tag{21}$$

for all  $g = \gamma g_\infty k$  with  $\gamma \in \mathrm{GSp}_2(\mathbb{Q})$ ,  $k \in K_0^{(2)}(N)$ , and

$$g_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{R}).$$

Let  $B \in \mathrm{Sym}_2(\mathbb{Q})$  be a symmetric 2-by-2 matrix, and  $W_{\mathbf{F}_\chi, B}$  denote the  $B$ -th Fourier coefficient of  $\mathbf{F}_\chi$  as defined above. By strong approximation, together

with (21),  $\mathcal{W}_{\mathbf{F}_\chi, B} : \mathrm{GSp}_2(\mathbb{A}) \rightarrow \mathbb{C}$  is determined by the values  $\mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty)$  for  $g_\infty \in \mathrm{GSp}_2^+(\mathbb{R})$ . Every element  $g_\infty \in \mathrm{GSp}_2^+(\mathbb{R})$  can be written as

$$g_\infty = \begin{pmatrix} z\mathbf{1}_2 & 0 \\ 0 & z\mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_2 & X \\ 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where  $z \in \mathbb{R}_+^\times$ ,  $X \in \mathrm{Sym}_2(\mathbb{R})$ ,  $A \in \mathrm{GL}_2^+(\mathbb{R})$ , and  $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$  with  $\mathbf{k} = \alpha + \sqrt{-1}\beta \in U(2)$ . Since

$$\mathbf{F}_\chi \left( \begin{pmatrix} z\mathbf{1}_2 & 0 \\ 0 & z\mathbf{1}_2 \end{pmatrix} g \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \right) = \det(\mathbf{k})^{k+1} \mathbf{F}_\chi(g)$$

for  $z \in \mathbb{R}_+^\times$  and  $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$  as before, we see that  $\mathcal{W}_{\mathbf{F}_\chi, B}$  is actually determined by the values  $\mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty)$  for elements  $g_\infty \in \mathrm{GSp}_2^+(\mathbb{R})$  of the form

$$g_\infty = n(X)m(A, 1) = \begin{pmatrix} \mathbf{1}_2 & X \\ 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix},$$

with  $X \in \mathrm{Sym}_2(\mathbb{R})$ , and  $A \in \mathrm{GL}_2^+(\mathbb{R})$ . And for  $g_\infty = n(X)m(A, 1)$ , one checks from the definitions that  $\mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty) = 0$  unless  $B$  is positive definite and half-integral, in which case one has

$$\mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty) = A_\chi(B) \det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)}, \tag{22}$$

where  $Y = A^t A$ ,  $Z = X + \sqrt{-1}Y \in \mathcal{H}_2$ , and  $A_\chi(B)$  is the  $B$ -th Fourier coefficient of  $F_\chi$  (cf. (14)).

Finally, let  $p$  be a prime dividing  $M$ . In particular,  $p^2 \mid NM$ , and the adelization of  $\mathfrak{R}_p F_\chi \in S_{k+1}(\Gamma_0^{(2)}(NM) \cap \Gamma^{\mathrm{param}}(p), \chi)$  is precisely (cf. Lemma 3.2)

$$\mathfrak{R}_p \mathbf{F}_\chi \in \mathcal{S}_{k+1}(K(NM, p), \underline{\chi}).$$

It is not hard to see from the definitions that the Fourier coefficients of  $\mathfrak{R}_p \mathbf{F}_\chi$  are closely related to those of  $\mathbf{F}_\chi$ . More precisely, one can prove the following lemma, whose proof is left for the reader.

LEMMA 3.3. *With notation as above, let  $B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix}$  be a positive definite, half-integral symmetric matrix, and let  $g_\infty = n(X)m(A, 1) \in \mathrm{GSp}_2(\mathbb{R})$  be as before. Then  $\mathcal{W}_{\mathfrak{R}_p \mathbf{F}_\chi, B}(g_\infty) = 0$  unless  $b_3 \in p\mathbb{Z}$ , and if this holds then*

$$\mathcal{W}_{\mathfrak{R}_p \mathbf{F}_\chi, B}(g_\infty) = \mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty).$$

If we repeat the above for all primes  $p$  dividing  $M$ , or in other words, if we apply the operator  $\mathfrak{R}_M := \prod_{p \mid M} \mathfrak{R}_p$  to  $\mathbf{F}_\chi$ , we obtain an automorphic cusp form

$$\mathfrak{R}_M \mathbf{F}_\chi \in \mathcal{S}_{k+1}(K(NM, M), \underline{\chi}),$$

and directly from the previous lemma we deduce the following:

COROLLARY 3.4. *Let  $B$  be a positive definite, half-integral symmetric matrix as in the previous lemma, and let  $g_\infty = n(X)m(A, 1) \in \mathrm{GSp}_2(\mathbb{R})$  be as above. Then  $\mathcal{W}_{\mathfrak{R}_M \mathbf{F}_\chi, B}(g_\infty) = 0$  unless  $b_3 \in M\mathbb{Z}$ , and if this holds then*

$$\mathcal{W}_{\mathfrak{R}_M \mathbf{F}_\chi, B}(g_\infty) = \mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty).$$

#### 4 QUADRATIC SPACES AND THETA LIFTS

This section is devoted to briefly recall the essentials on quadratic spaces and theta lifts. We focus especially in the three theta correspondences that will be considered in this paper, which explain the different lifts of classical modular forms described in Section 2 in the language of automorphic forms given in Section 3.

##### 4.1 QUADRATIC SPACES

Let  $F$  be a field with  $\mathrm{char}(F) \neq 2$ , and  $V$  be a quadratic space over  $F$ . That is to say,  $V$  is a finite dimensional vector space over  $F$  equipped with a non-degenerate symmetric bilinear form  $(\ , \ )$ . We denote by  $Q$  the associated quadratic form on  $V$ , given by

$$Q(x) = \frac{1}{2}(x, x), \quad x \in V.$$

If  $m = \dim(V)$ , fixing a basis  $\{v_1, \dots, v_m\}$  of  $V$  and identifying  $V$  with the space of column vectors  $F^m$ , the bilinear form  $(\ , \ )$  determines a matrix  $Q \in \mathrm{GL}_m(F)$  by setting  $Q = ((v_i, v_j))_{i,j}$ . Then we have

$$(x, y) = {}^t x Q y \quad \text{for } x, y \in V.$$

We define  $\det(V)$  to be the image of  $\det(Q)$  in  $F^\times / F^{\times, 2}$ . The orthogonal similitude group of  $V$  is

$$\mathrm{GO}(V) = \{h \in \mathrm{GL}_m : {}^t h Q h = \nu(h)Q, \nu(h) \in \mathbb{G}_m\},$$

where  $\nu : \mathrm{GO}(V) \rightarrow \mathbb{G}_m$  is the so-called similitude morphism (or scale map). From the very definition, observe that  $\det(h)^2 = \nu(h)^m$  for every  $h \in \mathrm{GO}(V)$ . When  $m$  is even, set

$$\mathrm{GSO}(V) = \{h \in \mathrm{GO}(V) : \det(h) = \nu(h)^{m/2}\}.$$

Finally, we let  $\mathrm{O}(V) = \ker(\nu)$  denote the orthogonal group of  $V$ , and write  $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}_m$  for the special orthogonal group.

##### 4.2 EXPLICIT REALIZATIONS IN LOW RANK

In this paper, we are particularly interested in orthogonal groups for vector spaces of dimension 3, 4 and 5. For this reason, we fix here certain explicit

realizations that will be used later on to describe automorphic representations for  $\mathrm{SO}(V)(\mathbb{A})$  and  $\mathrm{GSO}(V)(\mathbb{A})$ . Our choices follow quite closely the ones in [Ich05, Qiu14].

When  $\dim(V) = 3$ , one can show that there is a unique quaternion algebra  $B$  over  $F$  and an element  $a \in F^\times$  such that  $(V, q) \simeq (V_B, aq_B)$ , where  $V_B = \{x \in B : \mathrm{Tr}_B(x) = 0\}$  is the subspace of elements in  $B$  with zero trace (sometimes called ‘pure quaternions’), and  $q_B(x) = -\mathrm{Nm}_B(x)$ . The group of invertible elements  $B^\times$  acts on  $V_B$  by conjugation,  $b \cdot x = bxb^{-1}$ , and this action gives rise to an isomorphism

$$PB^\times \xrightarrow{\simeq} \mathrm{SO}(V_B, q_B) \simeq \mathrm{SO}(V, q).$$

When  $B = M_2$  is the split algebra of 2-by-2 matrices, then notice that  $PB^\times = \mathrm{PGL}_2$ , thus the above identifies  $\mathrm{PGL}_2$  with the special orthogonal group of a three-dimensional quadratic space.

In dimension 4, consider the vector space  $V_4 := M_2(F)$  of 2-by-2 matrices, equipped with the quadratic form  $q(x) = \det(x)$ . The associated non-degenerate bilinear form is  $(x, y) = \mathrm{Tr}(xy^*)$ , where

$$x^* = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in M_2(F).$$

There is an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \mathrm{GL}_2 \times \mathrm{GL}_2 \xrightarrow{\rho} \mathrm{GSO}(V_4) \longrightarrow 1, \tag{23}$$

where  $\iota(a) = (a\mathbf{1}_2, a^{-1}\mathbf{1}_2)$  and  $\rho(h_1, h_2)x = h_1xh_2^*$  for  $a \in \mathbb{G}_m$  and  $h_1, h_2 \in \mathrm{GL}_2$ . One has  $\nu(\rho(h_1, h_2)) = \det(h_1h_2) = \det(h_1)\det(h_2)$ . In particular, when  $F$  is a number field, automorphic representations of  $\mathrm{GSO}(V_4)$  can be seen as automorphic representations of  $\mathrm{GL}_2 \times \mathrm{GL}_2$  through the homomorphism  $\rho$  in the above short exact sequence. Here we might warn the reader that our choice for the homomorphism  $\rho$  in (23) agrees with the one on [Qiu14] and [GT11], but differs from the one considered in [Ich05] (or [II08]), which leads to a slightly different model for  $\mathrm{GSO}(V_4)$ .

Finally, in dimension 5 we will describe a realization of  $\mathrm{SO}(3, 2)$ , the special orthogonal group of a 5-dimensional quadratic space  $(V, q)$  of Witt index 2. Although the isomorphism class of such a quadratic space depends on  $\det(V)$ , the group  $\mathrm{SO}(V, q)$  does not. We describe a model  $V_5$  of such a quadratic space, with determinant 1 (modulo  $F^{\times, 2}$ ). Namely, start considering the 4-dimensional space  $F^4$  of column vectors, on which  $\mathrm{GSp}_2 \subset \mathrm{GL}_4$  acts on the left. Let

$$e_1 = {}^t(1, 0, 0, 0), \dots, e_4 = {}^t(0, 0, 0, 1)$$

be the standard basis on  $F^4$ , and equip  $\tilde{V} := \wedge^2 F^4$  with the non-degenerate symmetric bilinear form  $(, )$  defined by

$$x \wedge y = (x, y) \cdot (e_1 \wedge e_2 \wedge e_3 \wedge e_4), \quad \text{for all } x, y \in \tilde{V}.$$

Set  $x_0 := e_1 \wedge e_3 + e_2 \wedge e_4$ , and define the 5-dimensional subspace  $V_5 \subset \tilde{V}$  to be the orthogonal complement of the span of  $x_0$ , i.e.

$$V_5 := \{x \in \tilde{V} : (x, x_0) = 0\}.$$

Then the homomorphism  $\tilde{\rho} : \text{GSp}_2 \rightarrow \text{SO}(\tilde{V})$  given by  $\tilde{\rho}(h) = \nu(h)^{-1} \wedge^2(h)$  satisfies  $\tilde{\rho}(h)x_0 = x_0$ , and therefore induces an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \text{GSp}_2 \xrightarrow{\rho} \text{SO}(V_5) \longrightarrow 1, \tag{24}$$

where  $\iota(a) = a\mathbf{1}_4$  for  $a \in \mathbb{G}_m$ . This short exact sequence induces an identification

$$\text{PGSp}_2 \simeq \text{SO}(V_5).$$

We fix an identification of  $V_5$  with the 5-dimensional space  $F^5$  of column vectors by

$$\sum_{i=1}^5 x_i v_i \longmapsto {}^t(x_1, x_2, x_3, x_4, x_5),$$

where  $v_1 = e_2 \wedge e_1$ ,  $v_2 = e_1 \wedge e_4$ ,  $v_3 = e_1 \wedge e_3 - e_2 \wedge e_4$ ,  $v_4 = e_2 \wedge e_3$ ,  $v_5 = e_3 \wedge e_4$ . Upon this identification, we consider the non-degenerate bilinear symmetric form  $(, )$  on  $V$  defined by  $(x, y) = {}^t x Q y$  for  $x, y \in F^5$ , where

$$Q = \begin{pmatrix} & & & & -1 \\ & & & & \\ & & Q_1 & & \\ & & & & \\ -1 & & & & \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We shall distinguish the 3-dimensional subspace  $V_3 \subset V_5$  spanned by  $v_2, v_3, v_4$ , equipped with the bilinear form  $(x, y) = {}^t x Q_1 y$ , for  $x, y \in F^3$ , under the identification  $V_3 = F^3$  induced by restricting the above one for  $V = F^5$ . Notice that  $V_5 = \langle v_1 \rangle \oplus V_3 \oplus \langle -v_5 \rangle$ , where  $v_1$  and  $-v_5$  are isotropic vectors with  $(v_1, -v_5) = 1$ , and  $V_3$  is the orthogonal complement of  $\langle v_1, -v_5 \rangle = \langle v_1, v_5 \rangle$ . Also, we distinguish a 4-dimensional subspace of  $V_5$ . Indeed, the subspace  $\{x \in V : (x, v_3) = 0\} = \langle v_3 \rangle^\perp \subset V_5$  is a quadratic 4-dimensional subspace of  $V_5$ , and it can be identified with the space  $V_4$  defined above by means of the linear map

$$\langle v_3 \rangle^\perp \longrightarrow V_4, \quad x_1 v_2 + x_2 v_1 + x_3 v_5 + x_4 v_4 \longmapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

By restricting the homomorphism  $\rho$  from the exact sequence in (23) to

$$G(\text{SL}_2 \times \text{SL}_2)^- := \{(h_1, h_2) \in \text{GL}_2 \times \text{GL}_2 : \det(h_1) \det(h_2) = 1\},$$

one gets an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} G(\text{SL}_2 \times \text{SL}_2)^- \xrightarrow{\rho} \text{SO}(V_4) \longrightarrow 1. \tag{25}$$

Now notice that  $G(\mathrm{SL}_2 \times \mathrm{SL}_2)^-$  is isomorphic to

$$G(\mathrm{SL}_2 \times \mathrm{SL}_2) := \{(h_1, h_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det(h_1) \det(h_2)^{-1} = 1\}$$

through the morphism  $(h_1, h_2) \mapsto (h_1, \det(h_2)^{-1}h_2)$ . Composing this isomorphism with the natural embedding  $G(\mathrm{SL}_2 \times \mathrm{SL}_2) \hookrightarrow \mathrm{GSp}_2$  given by

$$\left( \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix},$$

one gets a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\iota} & G(\mathrm{SL}_2 \times \mathrm{SL}_2)^- & \xrightarrow{\rho} & \mathrm{SO}(V_4) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\iota} & \mathrm{GSp}_2 & \xrightarrow{\rho} & \mathrm{SO}(V_5) \longrightarrow 1 \end{array}$$

and hence an embedding  $\mathrm{SO}(V_4) \subset \mathrm{SO}(V_5)$ . This embedding will be of crucial interest later on.

### 4.3 WEIL REPRESENTATIONS

Let now  $F$  be a local field with  $\mathrm{char}(F) \neq 2$  (for our purposes, one can think of  $F$  being  $\mathbb{Q}_v$  for a rational place  $v$ ), and  $(V, Q)$  be a quadratic space over  $F$  of dimension  $m$  as above. Let  $\mathcal{S}(V)$  denote the space of locally constant and compactly supported complex-valued functions on  $V$ . This is usually referred to as the space of Bruhat–Schwartz functions on  $V$ . If  $F$  is archimedean, we rather consider  $\mathcal{S}(V)$  to be the Fock model (which is a smaller subspace, see [YZZ13, Section 2.1.2]).

We fix a non-trivial additive character  $\psi$  of  $F$ . The Weil representation  $\omega_{\psi, V}$  of  $\widetilde{\mathrm{SL}}_2(F) \times \mathrm{O}(V)$  on  $\mathcal{S}(V)$ , which depends on the choice of the character  $\psi$ , is given by the following formulae. If  $a \in F^\times$ ,  $b \in F$ ,  $h \in \mathrm{O}(V)$ , and  $\phi \in \mathcal{S}(V)$ , then

$$\begin{aligned} \omega_{\psi, V}(h)\phi(x) &= \phi(h^{-1}x), \\ \omega_{\psi, V}\left(\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \epsilon\right)\right)\phi(x) &= \epsilon^m \chi_{\psi, V}(a) |a|^{m/2} \phi(ax) \\ \omega_{\psi, V}\left(\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1\right)\right)\phi(x) &= \psi(Q(x)b)\phi(x), \\ \omega_{\psi, V}\left(\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\right)\phi(x) &= \gamma(\psi, V) \int_F \phi(y)\psi((x, y))dy. \end{aligned}$$

Here,  $\gamma(\psi, V)$  is the Weil index, which is an 8-th root of unity. When  $m = 1$  and  $Q(x) = x^2$ , we will simply write  $\omega_\psi, \chi_\psi$ , and  $\gamma(\psi)$  for  $\omega_{\psi,V}, \chi_{\psi,V}$ , and  $\gamma(\psi, V)$ , respectively. In this case, one has

$$\chi_\psi(a) = (a, -1)_F \frac{\gamma(\psi^a)}{\gamma(\psi)},$$

where  $(\cdot, \cdot)_F$  denotes the Hilbert symbol. This satisfies  $\chi_\psi(ab) = (a, b)_F \chi_\psi(a) \chi_\psi(b)$ ,  $\chi_\psi(a^2) = 1$ , and  $\chi_{\psi^a} = \chi_\psi \cdot \chi_a$ , where  $\chi_a$  stands for the quadratic character  $(a, \cdot)_F$ .

For a general  $V$ , if  $Q(x) = a_1 x_1^2 + \dots + a_m x_m^2$  with respect to some orthogonal basis, then  $\gamma(\psi, V) = \prod_i \gamma_{\psi^{a_i}}$ , and  $\chi_{\psi,V} = \prod_i \chi_{\psi^{a_i}}$ . This does not depend on the chosen basis.

When  $m$  is even, the above simplifies considerably. Indeed, if  $m$  is even, it is clear from the above description that the Weil representation descends to a representation of  $\mathrm{SL}_2(F) \times \mathrm{O}(V)$  on  $\mathcal{S}(V)$ . Further, the Weil index  $\gamma(\psi, V)$  is a 4-th root of unity in this case, and  $\chi_{\psi,V}$  becomes the quadratic character associated with the quadratic space  $(V, Q)$ . This means that

$$\chi_{\psi,V}(a) = (a, (-1)^{m/2} \det(V))_F, \quad a \in F^\times.$$

It will be useful in some settings to extend the Weil representation  $\omega_{\psi,V}$  described above. If  $m$  is even, one defines

$$R = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V)) = \{(g, h) \in \mathrm{GL}_2 \times \mathrm{GO}(V) : \det(g) = \nu(h)\},$$

and then  $\omega_{\psi,V}$  extends to a representation of  $R(F)$  on  $\mathcal{S}(V)$  by setting

$$\omega_{\psi,V}(g, h)\phi = \omega_{\psi,V} \left( g \begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}, 1 \right) L(h)\phi$$

for  $(g, h) \in R(F)$ ,  $\phi \in \mathcal{S}(V)$ , where  $L(h)\phi(x) = |\nu(h)|_F^{-m/4} \phi(h^{-1}x)$  for  $x \in V$ .

#### 4.4 THETA FUNCTIONS AND THETA LIFTS

Now let  $F$  be a number field (for our purposes, we will just consider  $F = \mathbb{Q}$ ), and consider a quadratic space  $V$  over  $F$  of dimension  $m$ . Fix a non-trivial additive character  $\psi$  of  $\mathbb{A}_F/F$  and let  $\omega = \omega_{\psi,V}$  be the Weil representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$  on  $\mathcal{S}(V(\mathbb{A}_F))$  with respect to  $\psi$ . Given  $(g, h) \in \widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$  and  $\phi \in \mathcal{S}(V(\mathbb{A}_F))$ , let

$$\theta(g, h; \phi) := \sum_{x \in V(F)} \omega(g, h)\phi(x).$$

Then  $(g, h) \mapsto \theta(g, h; \phi)$  defines an automorphic form on  $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$ , called a *theta function*. When  $m$  is even, this may be regarded as an automorphic form on  $\mathrm{SL}_2(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$ .



Let  $f$  be a cusp form on  $\mathrm{SL}_2(\mathbb{A}_F)$  if  $m$  is even, and a genuine cusp form on  $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F)$  if  $m$  is odd. If  $\phi \in \mathcal{S}(V(\mathbb{A}_F))$ , put

$$\theta(h; f, \phi) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A}_F)} \theta(g, h; \phi) f(g) dg, \quad h \in \mathrm{O}(V)(\mathbb{A}_F).$$

Then  $\theta(f, \phi) : h \mapsto \theta(h; f, \phi)$  defines an automorphic form on  $\mathrm{O}(V)(\mathbb{A}_F)$ . If  $m$  is even and  $\pi$  is an irreducible cuspidal automorphic representation of  $\mathrm{SL}_2(\mathbb{A}_F)$ , or if  $m$  is odd and  $\pi$  is an irreducible genuine cuspidal automorphic representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F)$ , put

$$\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{O}(V)}(\pi) := \{\theta(f, \phi) : f \in \pi, \phi \in \mathcal{S}(V(\mathbb{A}_F))\}.$$

Then  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{O}(V)}(\pi)$  is an automorphic representation of  $\mathrm{O}(V)(\mathbb{A}_F)$ , called the *theta lift of  $\pi$* . Going in the opposite direction, one defines similarly the theta lift  $\theta(f', \phi)$  of a cusp form  $f'$  on  $\mathrm{O}(V)(\mathbb{A}_F)$  and the theta lift  $\Theta_{\mathrm{O}(V) \times \widetilde{\mathrm{SL}}_2}(\pi')$  of an irreducible cuspidal automorphic representation  $\pi'$  of  $\mathrm{O}(V)(\mathbb{A})$ .

Suppose that  $m$  is even. As we did for the Weil representation, theta lifts can also be extended as follows. If  $(g, h) \in R(\mathbb{A}_F)$  and  $\phi \in \mathcal{S}(V(\mathbb{A}_F))$ , one defines  $\theta(g, h; \phi)$  via the same expression as above (using the extended Weil representation). Then, if  $f$  is a cusp form on  $\mathrm{GL}_2(\mathbb{A}_F)$  and  $h \in \mathrm{GO}(V)(\mathbb{A}_F)$ , choose  $g' \in \mathrm{GL}_2(\mathbb{A}_F)$  with  $\det(g') = \nu(h)$  and set

$$\theta(h; f, \phi) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A}_F)} \theta(gg', h; \phi) f(gg') dg.$$

The integral does not depend on the choice of the auxiliary element  $g'$ , and  $\theta(f, \phi) : h \mapsto \theta(h; f, \phi)$  defines now an automorphic form on  $\mathrm{GO}(V)(\mathbb{A}_F)$ . If  $\pi$  is an irreducible cuspidal automorphic representation of  $\mathrm{GL}_2(\mathbb{A}_F)$ , then its theta lift  $\Theta_{\mathrm{GL}_2 \times \mathrm{GO}(V)}(\pi)$  is formally defined exactly as before (and the same applies for  $\Theta_{\mathrm{GO}(V) \times \mathrm{GL}_2}(\pi')$  if  $\pi'$  is an irreducible cuspidal automorphic representation of  $\mathrm{GO}(V)$ ).

## 5 THREE THETA CORRESPONDENCES

### 5.1 THE PAIR $(\mathrm{GL}_2, \mathrm{GO}_{2,2})$

Let  $V_4$  be the (split) four-dimensional quadratic space as above, and write from now on  $\mathrm{GSO}_{2,2} \subset \mathrm{GO}_{2,2}$  for the groups  $\mathrm{GSO}(V_4) \subset \mathrm{GO}(V_4)$ , and likewise  $\mathrm{O}_{2,2}$  for  $\mathrm{O}(V_4)$ . The theta correspondence for the pair  $(\mathrm{GL}_2, \mathrm{GO}_{2,2})$  is sometimes referred to as the Jacquet–Langlands–Shimizu correspondence [Shi72] (cf. also [II08, Section 5], [Ich05, Section 6]). We will be interested in the restriction of automorphic forms on  $\mathrm{GO}_{2,2}(\mathbb{A})$  to  $\mathrm{GSO}_{2,2}(\mathbb{A})$  (particularly in those arising as theta lifts from automorphic forms on  $\mathrm{GL}_2(\mathbb{A})$ ).

By virtue of the exact sequence in (23), automorphic forms on  $\mathrm{GSO}_{2,2}(\mathbb{A})$  might be seen through the homomorphism  $\rho$  as representations  $\tau_1 \boxtimes \tau_2$  of  $\mathrm{GL}_2 \times \mathrm{GL}_2$

with  $\tau_1$  and  $\tau_2$  having *the same* central character. The involution  $x \mapsto x^*$  induces an element of order two  $c \in \mathrm{GO}_{2,2}$ , and  $\mathrm{GO}_{2,2} = \mathrm{GSO}_{2,2} \rtimes \langle c \rangle$ . Notice that for the theta correspondence to hold between  $\mathrm{GL}_2$  and  $\mathrm{GO}_{2,2}$  it does not suffice to consider the Weil representation of  $\mathrm{SL}_2(\mathbb{A}) \times \mathrm{O}_{2,2}$ ; one needs to consider the Weil representation extended to the group  $R(\mathbb{A}) = \{(g, h) \in \mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}_{2,2}(\mathbb{A}) : \det(g) = \nu(h)\}$  as explained above. If  $\tau$  is an irreducible cuspidal unitary representation of  $\mathrm{GL}_2(\mathbb{A})$ , then one has  $\Theta_{\mathrm{GL}_2 \times \mathrm{GSO}_{2,2}}(\tau) = \tau \boxtimes \tau$ , where in line with the above comment  $\Theta_{\mathrm{GL}_2 \times \mathrm{GSO}_{2,2}}(\tau)$  might be read as the restriction to  $\mathrm{GSO}_{2,2}(\mathbb{A})$  of the theta lift  $\Theta_{\mathrm{GL}_2 \times \mathrm{GO}_{2,2}}(\tau)$ . Conversely, suppose that  $\Upsilon_0$  is an irreducible cuspidal unitary  $\mathrm{GSO}_{2,2}(\mathbb{A})$ -representation. Then there is a unique extension of  $\Upsilon_0$  to an automorphic representation  $\Upsilon$  of  $\mathrm{GO}_{2,2}(\mathbb{A})$  on which there is a non-zero  $\mathrm{O}(V'_4)(\mathbb{A})$ -invariant distribution, where  $V'_4 = \{x \in V_4 : \mathrm{tr}(x) = 0\}$ . If  $\Theta_{\mathrm{GO}_{2,2} \times \mathrm{GL}_2}(\Upsilon) \neq 0$ , then  $\Upsilon_0$  is of the form  $\tau \otimes \tau$  for some irreducible cuspidal unitary representation  $\tau$  of  $\mathrm{GL}_2(\mathbb{A})$  and  $\Theta_{\mathrm{GO}_{2,2} \times \mathrm{GL}_2}(\Upsilon) = \tau$ .

Let us consider a normalized newform  $g \in S_{k+1}(N, \chi)$  as in the introduction, hence  $k \geq 1$  is an odd integer,  $N \geq 1$  is an odd square-free integer, and  $\chi$  is an even Dirichlet character modulo  $N$ . Write  $M$  for the conductor of  $\chi$  (in particular,  $M$  is also odd and square-free). Let  $\tau$  be the irreducible cuspidal automorphic  $\mathrm{GL}_2(\mathbb{A})$ -representation associated with  $g$ , with central character  $\underline{\chi}$ . Then  $\tau \boxtimes \tau$  can be regarded as a representation of  $\mathrm{GSO}_{2,2}(\mathbb{A})$  and it extends to a unique automorphic representation  $\Upsilon$  of  $\mathrm{GO}_{2,2}(\mathbb{A})$  on which there is a non-zero  $\mathrm{O}(V'_4)(\mathbb{A})$ -invariant distribution. Then one has

$$\Theta(\tau) = \Upsilon, \quad \Theta(\Upsilon) = \tau.$$

Let  $\mathbf{g} \in \tau$  be the adelization of  $g$ . Then the cusp form  $\mathbf{g} \otimes \mathbf{g} \in \tau \boxtimes \tau$  extends to a cusp form  $\mathbf{G} \in \Upsilon$  on  $\mathrm{GO}_{2,2}(\mathbb{A})$  such that  $\mathbf{G}(hh') = \mathbf{G}(h)$  for all  $h \in \mathrm{GO}_{2,2}(\mathbb{A})$  and  $h' \in \mu_2(\mathbb{A})$ , where  $\mu_2$  is the subgroup of  $\mathrm{O}_{2,2}$  generated by the involution  $*$  on  $V_4$ .

Define  $\mathbf{g}^\sharp = \otimes_v \mathbf{g}_v^\sharp$  by setting  $\mathbf{g}_v^\sharp = \mathbf{g}_v$  for all places  $v \neq 2$ , and  $\mathbf{g}_2^\sharp = \tau_2(t(2^{-1})_2)\mathbf{g}_v$ , where

$$t(2^{-1})_2 = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_2).$$

Further, consider the  $M$ -th level raising operator  $\mathbf{V}_M$  acting on  $\tau$  by  $\varphi \mapsto \tau(\varpi_M)\varphi$ , where  $\varpi_M \in \mathrm{GL}_2(\mathbb{A})$  is 1 at every place  $v \nmid M$ , and equals  $\varpi_p = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$  at each prime  $p \mid M$ . Then define  $\mathfrak{g} := \mathbf{V}_M \mathbf{g}^\sharp = (\mathbf{V}_M \mathbf{g})^\sharp$ . The cusp form  $\mathfrak{g}$  thus obtained is one of the factors in our test vector.

Accordingly, we also modify slightly the cusp form  $\mathbf{G}$  in the following manner. For each prime  $p \mid M$ , consider the element  $h_p = (1, \varpi_p) \in \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{Q}_p)$ , which we identify with its image  $\rho(h_p) \in \mathrm{GSO}_{2,2}(\mathbb{Q}_p) \subseteq \mathrm{GO}_{2,2}(\mathbb{Q}_p)$ . Let  $\mathbf{Y}_p$  denote the operator acting on  $\Upsilon$  given by  $\Upsilon(h_p)$ , and  $\mathbf{Y}_M := \prod_{p \mid M} \mathbf{Y}_p$ . Similarly as above, if  $h_M$  denotes the element in  $\mathrm{GO}_{2,2}(\mathbb{A})$  whose entries are trivial

at every place  $v$  away from  $M$ , and equals  $h_p$  at each prime  $p \mid M$ , then we consider the cusp form  $\mathbf{Y}_M \mathbf{G} = \Upsilon(h_M) \mathbf{G} \in \Upsilon$ . With this definition, observe that

$$\mathbf{Y}_M \mathbf{G}|_{\mathrm{GL}_2 \times \mathrm{GL}_2} = \mathbf{g} \otimes \mathbf{V}_M \mathbf{g} \in \tau \boxtimes \tau.$$

With the above definitions, the main purpose of this paragraph is to prove the following explicit identity between the cusp forms  $\check{\mathbf{g}}$  and  $\mathbf{Y}_M \mathbf{G}$ . This is made upon the choice of a Bruhat–Schwartz function  $\phi_{\check{\mathbf{g}}} \in \mathcal{S}(V_4(\mathbb{A}))$  to be defined below.

PROPOSITION 5.1. *With the above notation, one has*

$$\theta(\mathbf{Y}_M \mathbf{G}, \phi_{\check{\mathbf{g}}}) = 2^{k-1} M^{-1} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-2} \langle g, g \rangle \check{\mathbf{g}}.$$

Following the approach of Ichino–Ikeda [II08, Section 5], it is useful to consider a different model of the Weil representation. If  $\varphi \in \mathcal{S}(V_4(\mathbb{A}))$ , one defines its partial Fourier transform  $\check{\varphi} \in \mathcal{S}(V_4(\mathbb{A}))$  by

$$\check{\varphi} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = \int_{\mathbb{A}^2} \varphi \left( \begin{pmatrix} x_1 & y_2 \\ x_3 & y_4 \end{pmatrix} \right) \psi(x_2 y_4 - x_4 y_2) dy_2 dy_4, \quad (26)$$

where  $dy_2, dy_4$  are the self-dual measure on  $\mathbb{A}$  with respect to our fixed non-trivial additive character  $\psi$  of  $\mathbb{A}/\mathbb{Q}$ . Then, one defines a representation  $\check{\omega}$  of  $R(\mathbb{A}) = \{(g, h) \in \mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}_{2,2}(\mathbb{A}) : \det(g) = \nu(h)\}$  on  $\mathcal{S}(V_4(\mathbb{A}))$  by setting

$$\check{\omega}(g, h)\check{\varphi} = (\omega(g, h)\varphi).$$

Observe that  $\check{\omega}(g, 1)\check{\varphi}(x) = \check{\varphi}(xg)$  for  $g \in \mathrm{SL}_2(\mathbb{A})$ .

We start defining a Bruhat–Schwartz function  $\phi_{\mathbf{g}} = \otimes_v \phi_{\mathbf{g},v} \in \mathcal{S}(V_4(\mathbb{A}))$  associated with  $\mathbf{g}$  as follows:

- i) At primes  $q \nmid N$ ,  $\phi_{\mathbf{g},q}$  equals the characteristic function on  $M_2(\mathbb{Z}_q)$ .
- ii) At primes  $p \mid N$ ,  $\phi_{\mathbf{g},p}$  is determined by requiring that  $\check{\phi}_{\mathbf{g},p}$  is given by

$$\check{\phi}_{\mathbf{g},p} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = \mathbf{1}_{\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_2) \mathbf{1}_{p\mathbb{Z}_p}(x_3) \mathbf{1}_{\mathbb{Z}_p^\times}(x_4) \chi_p^{-1}(x_4).$$

- iii) At the archimedean place,

$$\phi_{\mathbf{g},\infty} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = (x_1 + \sqrt{-1}x_2 + \sqrt{-1}x_3 - x_4)^{k+1} \exp(-\pi \mathrm{tr}(x^t x)).$$

Notice that the local components at primes dividing  $N$  are defined through their partial Fourier transforms. For later use, we provide an explicit recipe for  $\phi_{\mathbf{g},p}$  at such primes:

LEMMA 5.2. *Let  $p$  be a prime dividing  $N$ , and write  $\varepsilon(1/2, \underline{\chi}_p)$  for the root number of the character  $\underline{\chi}_p : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ . If  $p \mid M$ , then*

$$\phi_{\mathbf{g},p} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = p^{-1/2} \varepsilon(1/2, \underline{\chi}_p) \mathbf{1}_{\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_4) \mathbf{1}_{p\mathbb{Z}_p}(x_3) \mathbf{1}_{p^{-1}\mathbb{Z}_p^\times}(x_2) \underline{\chi}_p(x_2),$$

while if  $p \nmid N/M$  one has

$$\phi_{\mathbf{g},p} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = \mathbf{1}_{\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_4) \mathbf{1}_{p\mathbb{Z}_p}(x_3) (\mathbf{1}_{\mathbb{Z}_p}(x_2) - p^{-1} \mathbf{1}_{p^{-1}\mathbb{Z}_p}(x_2)).$$

The next statement adapts [II08, Proposition 5.2] to our slightly different model for  $\text{GSO}_{2,2}$ .

PROPOSITION 5.3. *With the above notation,*

$$\theta(\mathbf{g}, \phi_{\mathbf{g}})|_{\text{GL}_2 \times \text{GL}_2} = 2^{k+1} \zeta_{\mathbb{Q}}(2)^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \mathbf{g} \otimes \mathbf{g}.$$

*Proof.* We know that  $\theta(\mathbf{g}, \phi_{\mathbf{g}}) \in \tau \boxtimes \tau$ . A routine calculation shows the following assertions.

- i) If  $p \nmid N$ , and  $(g, h) \in R(\mathbb{Z}_p)$ , then  $\omega(g, h) \phi_{\mathbf{g},p} = \phi_{\mathbf{g},p}$ .
- ii) If  $p \mid N$ , and  $h_1, h_2 \in K_0(N; \mathbb{Z}_p)$ , then

$$\omega \left( \begin{pmatrix} \det(h_1 h_2) & 0 \\ 0 & 1 \end{pmatrix}, (h_1, h_2) \right) \phi_{\mathbf{g},p} = \underline{\chi}_p(h_1 h_2) \phi_{\mathbf{g},p}.$$

- iii) If  $k_\theta, k_{\theta_1}, k_{\theta_2} \in \text{SO}(2)$ , then

$$\omega(k_\theta, (k_{\theta_1}, k_{\theta_2})) \phi_{\mathbf{g},\infty} = \exp(\sqrt{-1}(k+1)(-\theta + \theta_1 + \theta_2)) \phi_{\mathbf{g},\infty}.$$

It follows from these properties that there is a constant  $C$  satisfying

$$\theta(\mathbf{g}, \phi_{\mathbf{g}})|_{\text{GL}_2 \times \text{GL}_2} = C \mathbf{g} \otimes \mathbf{g},$$

and one finds  $C = 2^{k+1} \text{vol}(\hat{\Gamma}_0(N))$  by comparing the Fourier coefficients  $W_{\mathbf{g},1}(1)$  and  $W_{\theta(\mathbf{g}, \phi_{\mathbf{g}}),1,1}(1)$  (cf. loc. cit. for details), where  $\hat{\Gamma}_0(N) = \text{SO}(2) \text{SL}_2(\hat{\mathbb{Z}}) \cap \text{SO}(2) \Gamma_0(N; \hat{\mathbb{Z}})$ . The statement then follows by using that  $\text{vol}(\hat{\Gamma}_0(N)) = \zeta_{\mathbb{Q}}(2)^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1}$ .  $\square$

COROLLARY 5.4. *With the above notation,*

$$\theta(\mathbf{G}, \phi_{\mathbf{g}}) = 2^{k+1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-2} \langle g, g \rangle \mathbf{g}. \tag{27}$$

*Proof.* The invariance properties of  $\phi_{\mathbf{g}}$  imply that there is a constant  $C$  such that  $\theta(\mathbf{G}, \phi_{\mathbf{g}}) = C \mathbf{g}$ . Thus we need to determine the precise value of  $C$ . On the one hand we have  $\langle \theta(\mathbf{G}, \phi_{\mathbf{g}}), \mathbf{g} \rangle = C \langle \mathbf{g}, \mathbf{g} \rangle$ . And on the other hand, by the seesaw principle together with the previous proposition we also have

$$\begin{aligned} \langle \theta(\mathbf{G}, \phi_{\mathbf{g}}), \mathbf{g} \rangle &= \langle \mathbf{G}, \theta(\mathbf{g}, \phi_{\mathbf{g}}) \rangle = 2^{k+1} \zeta_{\mathbb{Q}}(2)^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \langle \mathbf{G}, \mathbf{G} \rangle = \\ &= 2^{k+1} \zeta_{\mathbb{Q}}(2)^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \langle \mathbf{g}, \mathbf{g} \rangle^2. \end{aligned}$$

Hence, using that  $\langle \mathbf{g}, \mathbf{g} \rangle = \zeta_{\mathbb{Q}}(2)^{-1} \langle g, g \rangle$  we conclude that  $C = 2^{k+1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-2} \langle g, g \rangle$ .  $\square$

Proposition 5.1 will follow straightforward from the identity in Corollary 5.4. We need to suitably modify the Bruhat–Schwartz function  $\phi_{\mathbf{g}}$  in order to translate (27) into an explicit analogous relation between  $\check{\mathbf{g}}$  and  $\mathbf{Y}_M \mathbf{G}$ . To do so, define  $\phi_{\mathbf{g}^\sharp} = \otimes_v \phi_{\mathbf{g}^\sharp, v}$  by setting  $\phi_{\mathbf{g}^\sharp, v} = \phi_{\mathbf{g}, v}$  at all places  $v \neq 2$ , and  $\phi_{\mathbf{g}^\sharp, 2} = 2^{-2} \omega_2(t(2^{-1})_2, 1) \phi_{\mathbf{g}, 2}$ . From the definition of  $\phi_{\mathbf{g}, 2}$ , one can easily check that  $\phi_{\mathbf{g}^\sharp, 2}(x) = \mathbf{1}_{V_4(2\mathbb{Z}_2)}(x)$  for  $x \in V_4(\mathbb{Q}_2)$ . With this slight modification at  $p = 2$ , Corollary (5.4) gets easily rephrased:

COROLLARY 5.5. *With the above notation,*

$$\theta(\mathbf{G}, \phi_{\mathbf{g}^\sharp}) = 2^{k-1} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-2} \langle g, g \rangle_{\mathbf{g}^\sharp}.$$

*Proof.* This follows from the very definitions. Indeed, recall that for  $x \in \mathrm{GL}_2(\mathbb{A})$  one has

$$\begin{aligned} \theta(\mathbf{G}, \phi_{\mathbf{g}})(x) &= \int_{[\mathcal{O}_{2,2}]} \Theta(x, y'y; \phi_{\mathbf{g}}) \mathbf{G}(y'y) dy = \\ &= \int_{[\mathcal{O}_{2,2}]} \left( \sum_{v \in V_4(\mathbb{Q})} \omega(x, y'y) \phi_{\mathbf{g}}(v) \right) \mathbf{G}(y'y) dy, \end{aligned}$$

where  $y' \in \mathrm{GO}_{2,2}(\mathbb{A})$  is any element with  $\nu(y') = \det(x)$ . From the last expression, observe that if we replace  $\phi_{\mathbf{g}}$  by  $\omega(g, 1) \phi_{\mathbf{g}}$  with  $g \in \mathrm{SL}_2(\mathbb{A})$ , then

$$\begin{aligned} \theta(\mathbf{G}, \omega(g, 1) \phi_{\mathbf{g}})(x) &= \int_{[\mathcal{O}_{2,2}]} \left( \sum_{v \in V_4(\mathbb{Q})} \omega(x, y'y) \omega(g, 1) \phi_{\mathbf{g}}(v) \right) \mathbf{G}(y'y) dy = \\ &= \int_{[\mathcal{O}_{2,2}]} \left( \sum_{v \in V_4(\mathbb{Q})} \omega(xg, y'y) \phi_{\mathbf{g}}(v) \right) \mathbf{G}(y'y) dy = \\ &= \theta(\mathbf{G}, \phi_{\mathbf{g}})(xg) = \tau(g) \theta(\mathbf{G}, \phi_{\mathbf{g}})(x). \end{aligned}$$

Applying this for  $g = t(2^{-1})_2 \in \mathrm{SL}_2(\mathbb{Q}_2) \hookrightarrow \mathrm{SL}_2(\mathbb{A}) \subseteq \mathrm{GL}_2(\mathbb{A})$ , we deduce that

$$\theta(\mathbf{G}, \phi_{\mathbf{g}^\sharp}) = \theta(\mathbf{G}, 2^{-2} \omega(t(2^{-1})_2, 1) \phi_{\mathbf{g}}) = 2^{-2} \tau(t(2^{-1})_2) \theta(\mathbf{G}, \phi_{\mathbf{g}}),$$

and the statement follows from the previous corollary together with the definition of  $\mathbf{g}^\sharp$ . □

Finally, we define  $\phi_{\check{\mathbf{g}}} = \otimes_v \phi_{\check{\mathbf{g}}, v}$  by keeping  $\phi_{\check{\mathbf{g}}, v} = \phi_{\mathbf{g}^\sharp, v}$  for all places  $v \nmid M$ , and setting

$$\phi_{\check{\mathbf{g}}, p} = p^{-1} \omega_p(\varpi_p, h_p) \phi_{\mathbf{g}^\sharp, p} = p^{-1} \omega_p(\varpi_p, h_p) \phi_{\mathbf{g}, p}$$

at each prime  $p \mid M$ . In other words, if  $\varpi_M$  and  $h_M$  are as before, we see that  $\phi_{\check{\mathbf{g}}} = M^{-1} \omega(\varpi_M, h_M) \phi_{\mathbf{g}^\sharp}$ .

*Proof of Proposition 5.1.* As above, if  $x \in \mathrm{GL}_2(\mathbb{A})$  notice that

$$\theta(\mathbf{G}, \phi_{\mathbf{g}^\sharp})(x) = \int_{\mathrm{O}_{2,2}(\mathbb{Q}) \backslash \mathrm{O}_{2,2}(\mathbb{A})} \left( \sum_{v \in V_4(\mathbb{Q})} \omega(x, y'y) \phi_{\mathbf{g}^\sharp}(v) \right) \mathbf{G}(y'y) dy,$$

where  $y' \in \mathrm{GO}_{2,2}(\mathbb{A})$  is such that  $\det(x) = \nu(y')$ . In particular, observe that  $\det(x\varpi_M) = \nu(y'h_M)$ , hence

$$\begin{aligned} \theta(\mathbf{Y}_M \mathbf{G}, \phi_{\check{\mathbf{g}}})(x) &= \frac{1}{M} \int_{[\mathrm{O}_{2,2}]} \left( \sum_{v \in V_4(\mathbb{Q})} \omega(x, y'y) \omega(\varpi_M, h_M) \phi_{\mathbf{g}^\sharp}(v) \right) \mathbf{G}(y'y h_M) dy = \\ &= \frac{1}{M} \int_{[\mathrm{O}_{2,2}]} \left( \sum_{v \in V_4(\mathbb{Q})} \omega(x\varpi_M, y'h_M y) \phi_{\mathbf{g}^\sharp}(v) \right) \mathbf{G}(y'h_M y) dy. \end{aligned}$$

From this we see that  $\theta(\mathbf{Y}_M \mathbf{G}, \phi_{\check{\mathbf{g}}}) = M^{-1} \tau(\varpi_M) \theta(\mathbf{G}, \phi_{\mathbf{g}^\sharp})$ , and hence the statement follows from the previous corollary, together with the fact that  $\check{\mathbf{g}} = \tau(\varpi_M) \mathbf{g}^\sharp$ .  $\square$

For later use, we will need an explicit description of the Bruhat–Schwartz function  $\phi_{\check{\mathbf{g}}}$ . At places  $v \nmid 2M$ , observe that  $\phi_{\check{\mathbf{g}},v} = \phi_{\mathbf{g},v}$ , thus we have:

- i) If  $p \nmid 2M$  is a finite prime, then  $\phi_{\check{\mathbf{g}},p}(x) = \mathbf{1}_{V_4(\mathbb{Z}_p)}(x)$  for all  $x \in V_4(\mathbb{Q}_p)$ .
- ii) If  $p \mid N/M$  is an (odd) prime, then

$$\phi_{\check{\mathbf{g}},p} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = \mathbf{1}_{\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_4) \mathbf{1}_{p\mathbb{Z}_p}(x_3) \left( \mathbf{1}_{\mathbb{Z}_p}(x_2) - p^{-1} \mathbf{1}_{p^{-1}\mathbb{Z}_p}(x_2) \right).$$

- iii) At  $v = \infty$ ,

$$\phi_{\check{\mathbf{g}},\infty} \left( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) = (x_1 + \sqrt{-1}x_2 + \sqrt{-1}x_3 - x_4)^{k+1} \exp(-\pi \mathrm{tr}(x^t x)).$$

At primes  $p \mid 2M$ , we describe  $\phi_{\check{\mathbf{g}},p}$  in the following lemma.

LEMMA 5.6. *With the above notation, the following assertions hold.*

- i) At  $p = 2$ ,  $\phi_{\check{\mathbf{g}},2}(x) = \mathbf{1}_{V_4(2\mathbb{Z}_2)}(x)$  for all  $x \in V_4(\mathbb{Q}_2)$ .
- ii) At (odd) primes  $p \mid M$ , for  $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in V_4(\mathbb{Q}_p)$  we have

$$\phi_{\check{\mathbf{g}},p}(x) = p^{-1/2} \varepsilon(1/2, \lambda_p) \mathbf{1}_{p\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_4) \mathbf{1}_{p^2\mathbb{Z}_p}(x_3) \mathbf{1}_{p^{-1}\mathbb{Z}_p^\times}(x_2) \chi_p(x_2).$$

*Proof.* One just have to compute  $\phi_{\check{\mathbf{g}},2}$  and  $\phi_{\check{\mathbf{g}},p}$  ( $p \mid M$ ) using the definitions of  $\phi_{\mathbf{g},2}$  and  $\phi_{\mathbf{g},p}$  together with the properties of the Weil representation, since  $\phi_{\check{\mathbf{g}},2} = 2^{-2} \omega_2(t(2^{-1})_2, 1) \phi_{\mathbf{g},2}$  and  $\phi_{\check{\mathbf{g}},p} = p^{-1} \omega_p(\varpi_p, h_p) \phi_{\mathbf{g},p}$  for primes  $p \mid M$ . We omit the details and leave them to the reader.  $\square$

5.2 THE PAIR  $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$

Shimura’s correspondence between half-integral weight modular forms and integral weight modular forms was investigated by Waldspurger as a theta correspondence between automorphic representations of  $\mathrm{PGL}_2(\mathbb{A})$  and automorphic representations of  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ . Here,  $\mathrm{PGL}_2$  is identified as the special orthogonal group  $\mathrm{SO}(V)$  of a three-dimensional quadratic space  $V$  as in Section 4.2.

The theta correspondence for the pair  $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$  depends on the choice of an additive character  $\psi$  of  $\mathbb{Q} \backslash \mathbb{A}$ . To emphasize this dependence, we will write

$$\Theta_{\mathrm{PGL}_2 \times \widetilde{\mathrm{SL}}_2}(\pi, \psi) \quad (\text{resp. } \Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGL}_2}(\tilde{\pi}, \psi))$$

for the automorphic representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  (resp.  $\mathrm{PGL}_2(\mathbb{A})$ ) obtained as the theta lift of the automorphic representation  $\pi$  (resp.  $\tilde{\pi}$ ) of  $\mathrm{PGL}_2(\mathbb{A})$  (resp.  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ ). On the local setting, we write

$$\Theta_{\mathrm{PGL}_2 \times \widetilde{\mathrm{SL}}_2}(\pi_v, \psi_v) \quad \text{and} \quad \Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGL}_2}(\tilde{\pi}_v, \psi_v)$$

for the local theta lifts of  $\pi_v$  and  $\tilde{\pi}_v$ , respectively. We will omit the subscripts  $\mathrm{PGL}_2 \times \widetilde{\mathrm{SL}}_2$  or  $\widetilde{\mathrm{SL}}_2 \times \mathrm{PGL}_2$  if the direction of the theta lift is clear.

In the following, for a fixed irreducible cuspidal automorphic representation  $\pi$  of  $\mathrm{PGL}_2(\mathbb{A})$ , and  $D$  varying over the set of fundamental discriminants, the representations  $\Theta(\pi \otimes \chi_D, \psi^D)$  (and their local counterparts  $\Theta(\pi_v \otimes \chi_D, \psi_v^D)$ ) will play a crucial role. Waldspurger’s description of the theta correspondence for  $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$  tells us that the set  $\{\Theta(\pi \otimes \chi_D, \psi^D) : D \in \mathbb{Q}^\times \text{ fund. discr.}\}$  is finite.

In order to describe the local theory, fix a place  $v$  of  $\mathbb{Q}$  and let  $\mathcal{P}_{0,v}$  denote the set of special or supercuspidal representations (or discrete series representations if  $v = \infty$ ) of  $\mathrm{PGL}_2(\mathbb{Q}_v)$ . For  $D \in \mathbb{Q}_v^\times$ , define the symbol

$$\left(\frac{D}{\pi_v}\right) = \chi_D(-1)\epsilon(\pi_v, 1/2)\epsilon(\pi_v \otimes \chi_D, 1/2),$$

and consider the associated partition  $\mathbb{Q}_v^\times = \mathbb{Q}_v^+(\pi_v) \sqcup \mathbb{Q}_v^-(\pi_v)$ , where

$$\mathbb{Q}_v^\pm(\pi_v) := \left\{ D \in \mathbb{Q}_v^\times : \left(\frac{D}{\pi_v}\right) = \pm 1 \right\}.$$

The next statement summarizes Waldspurger’s local theory.

**THEOREM 5.7 (Waldspurger).** *With the above notation, the following assertions hold.*

- i) *If  $\pi_v \notin \mathcal{P}_{0,v}$ , then  $\mathbb{Q}_v^+(\pi_v) = \mathbb{Q}_v^\times$  and  $\tilde{\pi}_v := \Theta(\pi_v, \psi_v) = \Theta(\pi_v \otimes \chi_D, \psi_v^D)$  for all  $D \in \mathbb{Q}_v^\times$ .*
- ii) *If  $\pi_v \in \mathcal{P}_{0,v}$ , then there are two representations  $\tilde{\pi}_v^+$  and  $\tilde{\pi}_v^-$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$  such that*

$$\Theta(\pi_v \otimes \chi_D, \psi_v^D) = \begin{cases} \tilde{\pi}_v^+ & \text{if } D \in \mathbb{Q}_v^+(\pi_v), \\ \tilde{\pi}_v^- & \text{if } D \in \mathbb{Q}_v^-(\pi_v). \end{cases}$$

iii) The equality  $\Theta(\pi_v \otimes \chi_D, \psi_v^D) = \Theta(\pi_v, \psi_v)$  holds if and only if  $\Theta(\pi_v, \psi_v)$  admits a non-trivial  $\psi_v^D$ -Whittaker model.

REMARK 5.8. When  $\pi_v \in \mathcal{P}_{0,v}$ , one has  $\tilde{\pi}_v^+ = \Theta(\pi_v, \psi_v)$  and  $\tilde{\pi}_v^- = \Theta(\text{JL}(\pi_v), \psi_v)$ , where  $\text{JL}(\pi_v)$  is the Jacquet–Langlands lift of  $\pi_v$  to an admissible representation of  $PB_v^\times$ , with  $B_v$  the unique division quaternion algebra over  $\mathbb{Q}_v$ . We warn the reader that the labelling  $+/-$  in ii) depends on the choice of the additive character  $\psi_v$ .

Altogether, the local Waldspurger packet  $\text{Wald}_{\psi_v}(\pi_v)$  is defined to be the singleton  $\{\tilde{\pi}_v\}$  if  $\pi_v \notin \mathcal{P}_{0,v}$ , and the set  $\{\tilde{\pi}_v^+, \tilde{\pi}_v^-\}$  if  $\pi_v \in \mathcal{P}_{0,v}$ . Although the labelling  $+/-$  in the set  $\text{Wald}_{\psi_v}(\pi_v)$  depends on  $\psi_v$ , the packet  $\text{Wald}_{\psi_v}(\pi_v)$  itself does not.

To state the global side of this theory, write  $\tilde{\mathcal{A}}_{00}$  for the subspace of cuspidal automorphic forms on  $\widetilde{\text{SL}}_2(\mathbb{A})$  which are orthogonal to the theta series generated by quadratic forms of one variable. Let  $\mathcal{A}_{0,i}$  be the subspace of cuspidal automorphic forms on  $\text{PGL}_2(\mathbb{A})$  such that for any subrepresentation  $\pi$  of  $\mathcal{A}_{0,i}$  there exists some  $D \in \mathbb{Q}^\times$  with  $L(\pi \otimes \chi_D, 1/2) \neq 0$ .

Let  $\pi$  be an irreducible cuspidal automorphic representation of  $\text{PGL}_2(\mathbb{A})$ , and let  $\Sigma(\pi)$  denote the set of rational places  $v$  such that  $\pi_v \in \mathcal{P}_{0,v}$ . For each  $D \in \mathbb{Q}^\times$ , let  $\epsilon(D, \pi) \in \{\pm 1\}^{|\Sigma(\pi)|}$  be the tuple determined by setting  $\epsilon(D, \pi)_v = (\frac{D}{\pi_v})$  for each  $v \in \Sigma(\pi)$ . Observe that one has by construction

$$\epsilon(\pi \otimes \chi_D, 1/2) = \epsilon(\pi, 1/2) \prod_{v \in \Sigma(\pi)} \left( \frac{D}{\pi_v} \right).$$

For an arbitrary tuple  $\epsilon = (\epsilon_v)_{v \in \Sigma(\pi)} \in \{\pm 1\}^{|\Sigma(\pi)|}$ , define the set  $\mathbb{Q}^\epsilon(\pi) = \{D \in \mathbb{Q}^\times : \epsilon(D, \pi) = \epsilon\}$ . In particular,  $\pi$  determines a partition

$$\mathbb{Q}^\times = \bigsqcup_{\epsilon} \mathbb{Q}^\epsilon(\pi).$$

Having settled this notation, we summarize Waldspurger’s global theory as follows. Below, two irreducible subrepresentations of  $\tilde{\mathcal{A}}_{00}$  are called *near equivalent*, denoted  $\tilde{\pi}_1 \sim \tilde{\pi}_2$ , if it holds that  $\tilde{\pi}_{1,v} \simeq \tilde{\pi}_{2,v}$  for almost all places.

THEOREM 5.9 (Waldspurger). *With the above notation, the following assertions hold.*

- i) *The global theta correspondence between  $\text{PGL}_2$  and  $\widetilde{\text{SL}}_2$  is compatible with the local correspondence. That is, if  $\Theta(\tilde{\pi}, \psi) \neq 0$ , then  $\Theta(\tilde{\pi}, \psi) \simeq \otimes_v \Theta(\tilde{\pi}_v, \psi_v)$ . And analogously, if  $\Theta(\pi, \psi) \neq 0$ , then  $\Theta(\pi, \psi) \simeq \otimes_v \Theta(\pi_v, \psi_v)$ .*
- ii)  *$\Theta(\pi, \psi) \neq 0$  if and only if  $L(\pi, 1/2) \neq 0$ . And  $\Theta(\tilde{\pi}, \psi) \neq 0$  if and only if  $\tilde{\pi}$  has a non-trivial  $\psi$ -Whittaker model.*



- iii) If  $\tilde{\pi}$  is an irreducible subrepresentation of  $\tilde{\mathcal{A}}_{00}$ , then there is a unique irreducible automorphic representation  $\pi = Sh_\psi(\tilde{\pi})$  of  $PGL_2(\mathbb{A})$  associated to  $\tilde{\pi}$  such that  $\Theta(\tilde{\pi}, \psi^D) \neq 0 \Rightarrow \Theta(\tilde{\pi}, \psi^D) \otimes \chi_D = \pi$ . The association  $\tilde{\pi} \rightarrow Sh_\psi(\tilde{\pi})$  defines a bijection between  $\tilde{\mathcal{A}}_{00}/\sim$  and  $\mathcal{A}_{0,i}$ .
- iv) If  $\pi = Sh_\psi(\tilde{\pi})$ , then the near equivalence class of  $\tilde{\pi}$  consists of all the non-zero lifts of the form  $\Theta(\pi \otimes \chi_D, \psi^D)$ .
- v) Let  $\epsilon \in \{\pm 1\}^{|\Sigma(\pi)|}$ . If  $\prod_{v \in \Sigma(\pi)} \epsilon_v \neq \epsilon(\pi, 1/2)$ , then  $\Theta(\pi \otimes \chi_D, \psi^D) = 0$  for all  $D \in \mathbb{Q}^\epsilon(\pi)$ . If  $\prod_{v \in \Sigma(\pi)} \epsilon_v = \epsilon(\pi, 1/2)$ , then there is a unique  $\tilde{\pi}^\epsilon$  such that for every  $D \in \mathbb{Q}^\epsilon(\pi)$  it holds

$$\Theta(\pi \otimes \chi_D, \psi^D) = \begin{cases} \tilde{\pi}^\epsilon & \text{if } L(\pi \otimes \chi_D, 1/2) \neq 0, \\ 0 & \text{if } L(\pi \otimes \chi_D, 1/2) = 0. \end{cases}$$

In the previous theorem, if  $\epsilon = (\epsilon_v)_v \in \{\pm 1\}^{|\Sigma(\pi)|}$ , then  $\tilde{\pi}^\epsilon$  denotes the irreducible cuspidal automorphic representation of  $\widetilde{SL}_2(\mathbb{A})$  whose local components equal  $\tilde{\pi}_v$  at all rational places  $v \notin \Sigma(\pi)$ , and whose local components at places  $v \in \Sigma(\pi)$  equal  $\tilde{\pi}_v^{\epsilon_v}$ . Together with the local theory, the above result motivates the definition of the (global) Waldspurger packet  $Wald_\psi(\pi)$  associated with  $\pi$  as the finite set

$$Wald_\psi(\pi) = \left\{ \tilde{\pi}^\epsilon : \prod_{v \in \Sigma(\pi)} \epsilon_v = \epsilon(\pi, 1/2) \right\}.$$

Notice that  $Wald_\psi(\pi) = Wald_{\psi^D}(\pi \otimes \chi_D)$  for all  $D \in \mathbb{Q}^\times$ , similarly as locally at each place  $v$  one has  $Wald_{\psi^a}(\pi_v) = Wald_{\psi_v^a}(\pi_v \otimes \chi_a)$  for all  $a \in \mathbb{Q}_v^\times$ .

### 5.2.1 ON THE RESULT OF BARUCH–MAO

Having recalled Waldspurger’s theory, we explain briefly how Baruch–Mao’s result stated in Theorem 2.1, leading to a generalization of Kohnen’s formula, fits in this theory. So let  $f \in S_{2k}^{new}(N)$  and  $\chi$  be as in Theorem 2.1, and let  $\pi$  be the irreducible cuspidal automorphic representation of  $PGL_2(\mathbb{A})$  associated with  $f$ . Choose once and for all a fundamental discriminant  $D \in \mathfrak{D}(N, M)$ , where the set  $\mathfrak{D}(N, M)$  is defined in (7), and assume further that  $L(f, D, k) \neq 0$ . Let  $\psi$  be the standard additive character of  $\mathbb{A}/\mathbb{Q}$ , and write  $\overline{\psi} = \psi^{-1}$  and  $\overline{\psi}^{-D} = \psi^{-D}$  for the  $(-1)$ -th and  $(-D)$ -th twists of  $\psi$ , respectively. Then, consider the theta lift  $\tilde{\pi} := \Theta(\pi \otimes \chi_D, \overline{\psi}^{-D})$  of  $\pi \otimes \chi_D$  with respect to the additive character  $\overline{\psi}^{-D}$ . Because of the assumption  $L(f, D, k) \neq 0$ , we have  $\Theta(\pi \otimes \chi_D, \overline{\psi}^{-D}) \neq 0$ , and so  $\tilde{\pi} = \otimes_v \tilde{\pi}_v$  with  $\tilde{\pi}_v \simeq \Theta(\pi_v \otimes \chi_D, \overline{\psi}_v^{-D})$ . Let  $\epsilon := \epsilon(D, \pi) \in \{\pm 1\}^{|\Sigma(\pi)|}$  be defined as above, so that  $\mathfrak{D}(N, M)$  is the set of fundamental discriminants in  $\mathbb{Q}^\epsilon(\pi)$ . It is proved in [BM07, Section 10] that  $\tilde{\pi} = \tilde{\pi}^\epsilon \in Wald_{\overline{\psi}^{-D}}(\pi)$ . In other words, the automorphic representation

$\tilde{\pi} = \Theta(\pi \otimes \chi_D, \overline{\psi}^D)$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  corresponds to the element in the Waldspurger packet  $\mathrm{Wald}_{\overline{\psi}}(\pi)$  whose labelling coincides with  $\epsilon$  (recall that the labelling of  $\tilde{\pi}$  depends on the choice of  $\overline{\psi}$ ). Moreover, one has  $\epsilon_\infty = -1$ , and for each prime  $p \mid N$

$$\epsilon_p = \left(\frac{D}{\pi_p}\right) = \left(\frac{D}{p}\right) = \begin{cases} w_p & \text{if } p \mid N/M, \\ -w_p & \text{if } p \mid M. \end{cases}$$

If  $V_{\tilde{\pi}}$  denotes the representation space for  $\tilde{\pi}$ , then Baruch and Mao show that  $V_{\tilde{\pi}} \cap \tilde{\mathcal{A}}_{k+1/2}^+(4N, \underline{\chi}_0)$  is one-dimensional. The one-dimensional subspace of  $S_{k+1/2}^{+,new}(4NM; \chi)$  denoted  $S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi)$  in Theorem 2.1 is then the preimage of  $V_{\tilde{\pi}} \cap \tilde{\mathcal{A}}_{k+1/2}^+(4N, \underline{\chi}_0)$  under the adelization map (16), and  $h$  can be taken to be the de-adelization of any new vector in  $V_{\tilde{\pi}} \cap \tilde{\mathcal{A}}_{k+1/2}^+(4N, \underline{\chi}_0)$ .

Summing up, the set  $\mathfrak{D}(N, M)$  in Theorem 2.1 singles out a precise element  $\tilde{\pi}$  in the Waldspurger packet  $\mathrm{Wald}_{\overline{\psi}}(\pi) = \mathrm{Wald}_{\overline{\psi}^D}(\pi \otimes \chi_D)$ , where the adelizations of the classical half-integral modular forms in  $S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi)$  belong to. Together with the local assumptions on  $\chi$ , this allows Baruch and Mao to have a clean description of the local types for  $\tilde{\pi}$  at primes  $p$  dividing  $N$ . For later purposes, we briefly describe these local types, according to whether  $p$  divides  $M$  or not.

First suppose that  $p$  is a prime dividing  $N/M$ . The local representation  $\pi_p$  is a quadratic twist of the Steinberg representation, say  $\pi_p = \mathrm{St}_p \cdot \chi_u$ , for some  $u \in \mathbb{Z}_p^\times$ . If  $w_p = 1$  (resp.  $w_p = -1$ ), then  $u$  is a non-quadratic residue (resp. quadratic residue) modulo  $p$ . We have

$$\tilde{\pi}_p \simeq \Theta(\pi_p \otimes \chi_D, \overline{\psi}_p^D) = \Theta(\mathrm{St}_p \cdot \chi_{uD}, \overline{\psi}_p^D),$$

and notice that  $\delta := uD \in \mathbb{Z}_p^\times$  is a non-square in  $\mathbb{Z}_p^\times$ , since  $(\frac{D}{p}) = w_p$ . In this case, it follows that  $\tilde{\pi}_p$  is a special representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ , denoted  $\tilde{\sigma}^\delta(\psi_p^{-D})$  in [BM07, (10.5.3)]. This representation is realized as the space of functions  $\tilde{\varphi} : \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  such that

$$\tilde{\varphi} \left( \left[ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}, \epsilon \right] g \right) = \epsilon \chi_{\overline{\psi}_p^D}(a) \chi_\delta(a) |a|_p^{3/2} \tilde{\varphi}(g)$$

for all  $g \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  and  $a \in \mathbb{Q}_p^\times$ , and satisfying also the vanishing condition

$$\int_{\mathbb{Q}_p} \tilde{\varphi}(\tilde{w}\tilde{n}(x)) \overline{\psi}_p^D(-\delta \Delta^2 x) dx = 0 \quad \text{for all } \Delta \in \mathbb{Q}_p.$$

Here, notice that  $\chi_\delta$  is the unique non-trivial quadratic character of  $\mathbb{Q}_p^\times$ . A newvector  $\tilde{\varphi}_p \in \tilde{\pi}_p$  can be chosen as in [BM07, Lemma 8.3], see also Lemma 7.1 below.

Secondly, suppose that  $p$  is a prime dividing  $M$ . We have again  $\pi_p = \mathrm{St}_p \cdot \chi_u$ , for some  $u \in \mathbb{Z}_p^\times$ , and therefore  $\tilde{\pi}_p \simeq \Theta(\mathrm{St}_p \cdot \chi_{uD}, \overline{\psi}_p^D)$ . But now,  $\delta := uD \in \mathbb{Z}_p^\times$  is a

square in  $\mathbb{Z}_p^\times$  because  $(\frac{D}{p}) = -w_p$ . Then  $\tilde{\pi}_p$  is a supercuspidal representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ . More precisely, it is the *odd* Weil representation  $r_{\tilde{\psi}_p}^-$  associated with the character  $\tilde{\psi}_p^D$  (cf. [BM07, (10.5.4)]). In this case, a choice of newvector  $\tilde{\varphi}_p \in \tilde{\pi}_p$  is described in [BM07, Proposition 8.5], see also Lemma 8.3 below.

### 5.3 THE PAIR $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}_2)$

Now we focus on the theta correspondence for the pair  $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}_2)$ , which explains the classical Saito–Kurokawa lift introduced in Section 2.3. We identify  $\mathrm{PGSp}_2$  with the special orthogonal group  $\mathrm{SO}(3, 2) = \mathrm{SO}(V_5)$ , where  $V_5$  is the five-dimensional quadratic space of determinant 1 and Witt index 2 as in Section 4.2. As we did for the pair  $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$ , let us fix a non-trivial additive character  $\psi$  of  $\mathbb{A}/\mathbb{Q}$ .

Global theta lifts can be defined in the same fashion as we have already explained in the previous two instances, so that for an irreducible cuspidal representation  $\Pi$  of  $\mathrm{PGSp}_2(\mathbb{A})$  and an irreducible component  $\tilde{\pi} \subset \mathcal{A}_{00}(\widetilde{\mathrm{SL}}_2)$ , one can define their lifts  $\Theta_{\mathrm{PGSp}_2 \times \widetilde{\mathrm{SL}}_2}(\Pi; \psi)$  and  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGSp}_2}(\tilde{\pi}; \psi)$ , respectively. The following assertions concerning this theta correspondence can be found in [PSS3].

- a) If  $\Theta_{\mathrm{PGSp}_2 \times \widetilde{\mathrm{SL}}_2}(\Pi; \psi)$  is not zero, then it is irreducible cuspidal.
- b) If  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGL}_2}(\tilde{\pi}; \psi) = 0$ , then  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGSp}_2}(\tilde{\pi}; \psi)$  is irreducible cuspidal.
- c) If  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGL}_2}(\tilde{\pi}; \psi)$  is not zero, then  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGSp}_2}(\tilde{\pi}; \psi)$  is irreducible non-cuspidal and occurs in the discrete spectrum of  $\mathrm{PGSp}_2$ .
- d)  $\Theta_{\mathrm{PGSp}_2 \times \widetilde{\mathrm{SL}}_2}(\Pi; \psi) = \tilde{\pi}$  if and only if  $\Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGSp}_2}(\tilde{\pi}; \psi) = \Pi$ .

Similarly as for Waldspurger packets, one can introduce the notion of local and global *Saito–Kurokawa packets*. Indeed, let  $v$  be a place of  $\mathbb{Q}$ , and  $\pi_v$  be an infinite-dimensional irreducible admissible representation of  $\mathrm{PGL}_2(\mathbb{Q}_v)$ . If  $\epsilon_v \in \{\pm 1\}$  and  $\tilde{\pi}_v^{\epsilon_v} \in \mathrm{Wald}_{\psi_v}(\pi_v)$ , write  $\Pi_v^{\epsilon_v} := \Theta_{\widetilde{\mathrm{SL}}_2 \times \mathrm{PGSp}_2}(\tilde{\pi}_v^{\epsilon_v}; \psi_v)$ . Then the local Saito–Kurokawa packet of  $\pi_v$  is defined to be

$$\mathrm{SK}(\pi_v) := \{\Pi_v^{\epsilon_v} : \tilde{\pi}_v^{\epsilon_v} \in \mathrm{Wald}_{\psi_v}(\pi_v)\}.$$

Now if  $\pi$  is an irreducible cuspidal automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$ , the associated global Saito–Kurokawa packet is just

$$\mathrm{SK}(\pi) := \{\Theta(\tilde{\pi}; \psi) : \tilde{\pi} \in \mathrm{Wald}_\psi(\pi)\}.$$

Given a tuple  $\epsilon = (\epsilon_v)_v$  such that  $\epsilon_v \in \{\pm 1\}$  for every place  $v$ , and  $\epsilon_v = +1$  for all  $v$  such that  $\pi_v$  is not square-integrable, set  $\Pi^\epsilon = \otimes_v \Pi_v^{\epsilon_v}$ . Then we have

$$\mathrm{SK}(\pi) := \{\Pi^\epsilon : \prod_v \epsilon_v = \varepsilon(1/2, \pi)\}.$$

Although the Saito–Kurokawa packet  $\text{SK}(\pi)$  associated with  $\pi$  is defined through  $\text{Wald}_\psi(\pi)$ , it turns out that  $\text{SK}(\pi)$  does not depend on the choice of the additive character  $\psi$ . Also, it is well-known that  $\text{SK}(\pi)$  consists only of cuspidal members when  $L(1/2, \pi) = 0$ .

In the rest of this section, we focus on an explicit relation between a newform  $\mathbf{h} \in \tilde{\pi}$ , where  $\tilde{\pi}$  is as in Section 5.2.1, and a theta lift of  $\mathbf{h}$ . From now on, we fix  $\psi$  to be the standard additive character of  $\mathbb{A}/\mathbb{Q}$ .

If  $\mathbf{F}$  is a cuspidal automorphic form on  $\text{SO}(V_5)(\mathbb{A}) \simeq \text{PGSp}_2(\mathbb{A})$  and  $B \in \text{Sym}_2(\mathbb{Q})$ , we will regard  $\mathbf{F}$  as an automorphic form on  $\text{GSp}_2(\mathbb{A})$  trivial on the center. Then its  $B$ -th Fourier coefficient  $\mathcal{W}_{\mathbf{F}, B} : \text{GSp}_2(\mathbb{A}) \rightarrow \mathbb{C}$  is defined as in (20). In particular, for an automorphic form  $\mathbf{h}$  on  $\widetilde{\text{SL}}_2(\mathbb{A})$  and a Bruhat–Schwartz function  $\varphi \in \mathcal{S}(V_5(\mathbb{A}))$ , the  $B$ -th Fourier coefficient of the theta lift  $\theta(\mathbf{h}, \varphi)$  is the function (of  $g \in \text{GSp}_2(\mathbb{A})$ )

$$g \mapsto \mathcal{W}_{\theta(\mathbf{h}, \varphi), B}(g) = \int_{\text{Sym}_2(\mathbb{Q}) \backslash \text{Sym}_2(\mathbb{A})} \theta(\mathbf{h}, \varphi)(n(X)g) \overline{\psi(\text{tr}(BX))} dX.$$

As in the introduction, let  $k, N \geq 1$  be odd integers, with  $N$  square-free, and let  $\chi$  be an even Dirichlet character modulo  $N$ , of conductor  $M \mid N$ . Let  $f \in S_{2k}^{\text{new}}(N)$  be a normalized newform of weight  $2k$  and level  $N$ , and let  $\pi$  be the automorphic representation of  $\text{PGL}_2$  associated with  $f$ . We assume Hypotheses (H1) and (H2), so that  $\chi_{(p)}(-1) = -1$  and  $w_p = -1$  for all primes  $p \mid M$ .

Let  $\tilde{\pi} \in \text{Wald}_{\overline{\psi}}(\pi)$  be the automorphic representation of  $\widetilde{\text{SL}}_2(\mathbb{A})$  obtained by theta correspondence as explained in Section 5.2.1. Let  $h \in S_{k+1/2}^{+, \text{new}}(4NM, \chi; f \otimes \chi)$  be a Shimura lift of  $f$  as in Theorem 2.1, and let  $\mathbf{h} \in \tilde{\pi}$  be its adelization. Besides, let  $\mathbf{F}_\chi \in \mathcal{S}_{k+1}(K_0^{(2)}(NM), \underline{\chi})$  be the adelization of the Saito–Kurokawa lift  $F_\chi = \mathbf{M}(\mathbf{Z}(h)) \in S_{k+1}(\Gamma_0^{(2)}(NM), \chi)$  as defined in (21) (cf. also Section 2.3). Recall the operator  $\mathfrak{R}_M$ , and consider the automorphic form

$$\mathfrak{R}_M \mathbf{F}_\chi \in \mathcal{S}_{k+1}(K(NM, M), \underline{\chi}),$$

which by Lemma 3.2 is the adelization of  $\mathfrak{R}_M F_\chi \in S_{k+1}(\Gamma_0^{(2)}(NM) \cap \Gamma^{\text{param}}(M), \chi)$ .

The main purpose for the rest of this section is to prove the following identity, where the Bruhat–Schwartz function  $\phi_{\mathbf{h}} \in \mathcal{S}(V_5(\mathbb{A}))$  will be defined below.

PROPOSITION 5.10. *With the above notation,*

$$\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi} = 2^{-2} \chi(2)^{-1} M^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-1} \mathfrak{R}_M \mathbf{F}_\chi. \tag{28}$$

Here, observe that  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$  is an automorphic cusp form on  $\text{PGSp}_2(\mathbb{A})$ , and as in Section 3.3, the automorphic cusp form  $\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}$  on  $\text{GSp}_2(\mathbb{A})$  is defined by

$$(\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi})(g) = \theta(\mathbf{h}, \phi_{\mathbf{h}})(g) \underline{\chi}(\nu(g)),$$

where  $\nu : \mathrm{GSp}_2(\mathbb{A}) \rightarrow \mathbb{A}^\times$  is the similitude morphism. In particular, the above proposition says that the automorphic form  $\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}$  is *classical*, in the sense that it is obtained by adelization of  $2^{-2}\chi(2)^{-1}M^{-1}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1}\zeta_{\mathbb{Q}}(2)^{-1}\mathfrak{R}_M F_\chi$ .

The proof of Proposition 5.10 will proceed by comparing the Fourier coefficients  $W_{\theta(\mathbf{h}, \varphi) \otimes \underline{\chi}, B}$  and  $W_{\mathfrak{R}_M \mathbf{F}_\chi, B}$  of the automorphic forms  $\theta(\mathbf{h}, \varphi) \otimes \underline{\chi}$  and  $\mathfrak{R}_M \mathbf{F}_\chi$  appearing in (28), for arbitrary symmetric matrices  $B \in \mathrm{Sym}_2(\mathbb{Q})$ . Concerning  $\mathfrak{R}_M \mathbf{F}_\chi$ , we know from Lemma 3.4 that  $W_{\mathfrak{R}_M \mathbf{F}_\chi, B}$  is zero unless  $B$  is positive definite, half-integral and  $b_3 \in M\mathbb{Z}$  if

$$B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix}.$$

And in that case,  $W_{\mathfrak{R}_M \mathbf{F}_\chi, B}$  is uniquely determined by the values  $W_{\mathfrak{R}_M \mathbf{F}_\chi, B}(g_\infty)$  at the elements

$$g_\infty = n(X)m(A, 1) = \begin{pmatrix} \mathbf{1}_2 & X \\ 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \in \mathrm{GSp}_2(\mathbb{R}) \quad (29)$$

with  $X \in \mathrm{Sym}_2(\mathbb{R})$ , and  $A \in \mathrm{GL}_2^+(\mathbb{R})$ , for which we know (cf. Corollary 3.4 and equation (22)) that

$$W_{\mathfrak{R}_M \mathbf{F}_\chi, B}(g_\infty) = W_{\mathbf{F}_\chi, B}(g_\infty) = A_\chi(B) \det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1}\mathrm{Tr}(BZ)},$$

where  $Y = A^t A$ ,  $Z = X + \sqrt{-1}Y \in \mathcal{H}_2$ . Here,  $A_\chi(B)$  is the classical  $B$ -th Fourier coefficient of  $F_\chi$ , which can be made precise as in (14).

Regarding  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$ , it will follow from Lemma 5.12 below that  $\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}$  satisfies the same invariance properties with respect to  $K(NM, M) \subseteq \mathrm{GSp}_2(\hat{\mathbb{Z}})$  as  $\mathfrak{R}_M \mathbf{F}_\chi$  does (namely,  $K(NM, M)$  acts through  $\underline{\chi}$  on both  $\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}$  and  $\mathfrak{R}_M \mathbf{F}_\chi$ ). Therefore, by comparing the Fourier coefficients  $W_{\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}, B}$  with  $W_{\mathbf{F}_\chi, B}$  at elements  $g_\infty \in \mathrm{GSp}_2(\mathbb{R})$  as above we will be able to deduce a relation between  $W_{\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}, B}$  and  $W_{\mathbf{F}_\chi, B}$  as functions on  $\mathrm{GSp}_2(\mathbb{A})$ , leading to the identity claimed in (28). Furthermore, observe that from the very definitions we have

$$W_{\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}, B}(g_\infty) = W_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty).$$

For this reason, we will focus on the computation of Fourier coefficients of the automorphic form  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$  on  $\mathrm{PGSp}_2(\mathbb{A})$  obtained as a theta lift from  $\mathbf{h}$ .

In order to determine the Fourier coefficients  $W_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}$ , as in the previous section it is useful to consider another model for the Weil representation. Recall the 3-dimensional quadratic subspace  $V_3 \subset V_5$  on which the quadratic form is given by  $Q_1$ , i.e.  $V_3 = \langle v_2, v_3, v_4 \rangle$ . We identify  $V_3^\perp$  with  $F^2$ , in a compatible way with the fact that

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & Q_1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We consider the partial Fourier transform

$$\mathcal{S}(V_5(\mathbb{A})) \longrightarrow \mathcal{S}(V_3(\mathbb{A})) \otimes \mathcal{S}(\mathbb{A}^2), \quad \phi \longmapsto \hat{\phi},$$

defined by setting

$$\hat{\phi}(x; y) = \int_{\mathbb{A}} \phi(z; x; y_1) \psi(-y_2 z) dz, \tag{30}$$

where  $x \in V_3(\mathbb{A})$  and  $y = (y_1, y_2) \in \mathbb{A}^2$ . As usual, here  $dz$  is the self-dual measure with respect to the additive character  $\psi$ . The Weil representation  $\omega$  of  $\widetilde{\text{SL}}_2(\mathbb{A}) \times \text{O}(V_5)(\mathbb{A})$  on  $\mathcal{S}(V_5(\mathbb{A}))$  gives rise then to a representation  $\hat{\omega}$  of  $\widetilde{\text{SL}}_2(\mathbb{A}) \times \text{O}(V_5)(\mathbb{A})$  on  $\mathcal{S}(V_3(\mathbb{A})) \otimes \mathcal{S}(\mathbb{A}^2)$  by setting

$$\hat{\omega}(g, h)\hat{\phi} = (\omega(g, h)\phi)^\wedge.$$

If  $\hat{\phi} = \phi_1 \otimes \phi_2$  with  $\phi_1 \in \mathcal{S}(V_3(\mathbb{A}))$  and  $\phi_2 \in \mathcal{S}(\mathbb{A}^2)$ , then one has

$$\hat{\omega}((g, \epsilon), 1)\hat{\phi}(x; y) = \omega((g, \epsilon), 1)\phi_1(x) \cdot \phi_2(yg) \tag{31}$$

for  $(g, \epsilon) \in \widetilde{\text{SL}}_2(\mathbb{A})$ . This change of polarization helps to get simpler computations, and the identity in (31), which we will use later, is an instance of this. Most importantly, in terms of this new model one can express the Fourier coefficients of  $\theta(\mathbf{h}, \varphi)$ , for a given  $\varphi \in \mathcal{S}(V_5(\mathbb{A}))$ , in terms of the Fourier coefficients of  $\mathbf{h}$ . Recall that if  $\xi \in \mathbb{Q}$ , then the  $\xi$ -th Fourier coefficient of  $\mathbf{h}$  is by definition the function

$$g \mapsto W_{\mathbf{h}, \xi}(g) = \int_{\mathbb{Q} \backslash \mathbb{A}} \mathbf{h}(u(x)g) \overline{\psi(\xi x)} dx, \quad g \in \widetilde{\text{SL}}_2(\mathbb{A}).$$

As quoted in (17), one has  $c(n) = e^{2\pi n} W_{\mathbf{h}, n}(1)$  for all integers  $n \geq 1$ . With this, the following is proved in [Ich05, Lemma 4.2].

LEMMA 5.11. *If  $\varphi \in \mathcal{S}(V_5(\mathbb{A}))$ , then for  $B \neq 0$  one has*

$$\mathcal{W}_{\theta(\mathbf{h}, \varphi), B}(h) = \int_{U(\mathbb{A}) \backslash \text{SL}_2(\mathbb{A})} \hat{\omega}(g, h)\hat{\varphi}(\beta; 0, 1)W_{\mathbf{h}, \xi}(g)dg, \tag{32}$$

where  $\xi = \det(B)$  and  $\beta = (b_3, b_2/2, -b_1)$  if  $B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix}$ .

The identity in (32) is a crucial ingredient in our computation of the  $B$ -th Fourier coefficients of  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$  towards the proof of Proposition 5.10. To proceed with this computation, we still need to address two tasks. First, we must describe an explicit choice of Bruhat–Schwartz function  $\phi_{\mathbf{h}} \in \mathcal{S}(V_5(\mathbb{A}))$ . And second, we must express the integral on the right hand side of (32) as a product of local integrals, one for each rational place  $v$ . After this is done, we will be able to proceed with the computation of the Fourier coefficients of  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$  by performing local computations prime by prime.

Concerning the choice of  $\phi_{\mathbf{h}} \in \mathcal{S}(V_5(\mathbb{A}))$ , recall the Bruhat–Schwartz function  $\phi_{\mathfrak{g}} \in \mathcal{S}(V_4(\mathbb{A}))$  considered in the previous section, which is involved in the explicit formula in Proposition 5.1. For our proof of the main theorem, it is crucial that  $\phi_{\mathbf{h}}$  is chosen so that its restriction to  $V_4$  coincides with  $\phi_{\mathfrak{g}}$ . Since we have an explicit description of  $\phi_{\mathfrak{g}}$ , we only need to define  $\phi_{\mathbf{h}}$  on the orthogonal complement of  $V_4$ , which is the one-dimensional quadratic subspace  $V_1 := V_4^\perp$  of  $V_5$  spanned by  $v_3$ . Notice that we may identify the space  $\mathcal{S}(V_1(\mathbb{A}))$  with  $\mathcal{S}(\mathbb{A})$ , by identifying  $V_1$  with the one-dimensional quadratic space over  $\mathbb{Q}$  endowed with the quadratic form  $q(x) = x^2$ . In  $\mathcal{S}(V_1(\mathbb{A}))$ , we consider the Bruhat–Schwartz function  $\phi = \otimes_v \phi_v$  determined by its local components as follows:

$$\phi_v(x) = \begin{cases} \mathbf{1}_{\mathbb{Z}_q}(x) & \text{if } v = q \neq \infty, \\ e^{-2\pi x^2} & \text{if } v = \infty. \end{cases}$$

Considering the basis  $v_1, \dots, v_5$  of  $V_5$  as we did above, so that  $V_1 = V_4^\perp$  is generated by  $v_3$ , and taking into account the embedding  $V_4 \subset V_5$  explained in Section 4.2, then  $\phi_{\mathbf{h}}$  is defined by setting

$$\phi_{\mathbf{h}}(z) := \phi(x_3)\phi_{\mathfrak{g}}\left(\begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}\right).$$

for  $z = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5 \in V_5(\mathbb{A})$ . By construction, the local components of  $\phi_{\mathbf{h}} = \otimes_v \phi_{\mathbf{h},v}$  can be easily given in terms of the local components of  $\phi$  and  $\phi_{\mathfrak{g}}$ . For the reader’s convenience, we describe such local components: if  $v$  is a rational place,  $z = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5 \in V_5(\mathbb{Q}_v)$  and we put  $X = \begin{pmatrix} x_2 & x_1 \\ x_5 & x_4 \end{pmatrix}$ , then  $\phi_{\mathbf{h},v}(z)$  is given as detailed below.

- i) If  $v = q \nmid N$  is an odd prime, then  $\phi_{\mathbf{h},q}$  is the characteristic function of  $V_5(\mathbb{Z}_q)$ . Indeed,

$$\phi_{\mathbf{h},q}(z) = \phi_q(x_3)\phi_{\mathfrak{g},q}(X) = \mathbf{1}_{\mathbb{Z}_q}(x_1)\mathbf{1}_{\mathbb{Z}_q}(x_2)\mathbf{1}_{\mathbb{Z}_q}(x_3)\mathbf{1}_{\mathbb{Z}_q}(x_4)\mathbf{1}_{\mathbb{Z}_q}(x_5).$$

- ii) If  $v = 2$ ,  $\phi_{\mathbf{h},2}$  is the characteristic function of  $\mathbb{Z}_2v_3 + V_5(2\mathbb{Z}_2)$ . Indeed,

$$\phi_{\mathbf{h},2}(z) = \phi_2(x_3)\phi_{\mathfrak{g},2}(X) = \mathbf{1}_{2\mathbb{Z}_2}(x_1)\mathbf{1}_{2\mathbb{Z}_2}(x_2)\mathbf{1}_{\mathbb{Z}_2}(x_3)\mathbf{1}_{2\mathbb{Z}_2}(x_4)\mathbf{1}_{2\mathbb{Z}_2}(x_5).$$

- iii) If  $v = p \mid M$  is prime, then  $\phi_{\mathbf{h},p}(z) = \phi_p(x_3)\phi_{\mathfrak{g},p}(X)$  equals

$$p^{-1/2}\varepsilon(1/2, \underline{\chi}_p)\mathbf{1}_{p^{-1}\mathbb{Z}_p^\times}(x_1)\mathbf{1}_{p\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{p^2\mathbb{Z}_p}(x_5)\underline{\chi}_p(x_1).$$

- iv) If  $v = p \mid N/M$  is prime, then  $\phi_{\mathbf{h},p}(z) = \phi_p(x_3)\phi_{\mathfrak{g},p}(X)$  equals

$$(\mathbf{1}_{\mathbb{Z}_p}(x_1) - p^{-1}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(x_1))\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{\mathbb{Z}_p}(x_4)\mathbf{1}_{p\mathbb{Z}_p}(x_5).$$

- v) For the archimedean prime  $v = \infty$ ,

$$\phi_{\mathbf{h},\infty}(z) = (x_2 + \sqrt{-1}x_1 + \sqrt{-1}x_5 - x_4)^{k+1} \exp(-\pi(x_1^2 + x_2^2 + 2x_3^2 + x_4^2 + x_5^2)).$$

In particular, observe that  $\phi_{\mathbf{h}} = \otimes_v \phi_{\mathbf{h},v}$  coincides with the Bruhat–Schwartz function  $\varphi^{(5)} = \otimes_v \varphi_v^{(5)}$  defined in [Ich05, Section 7] locally at every place  $v \nmid N$ . Therefore, we can use Ichino’s computations in loc. cit. at all such places. We need to understand the action of the Weil representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A}) \times \mathrm{O}(V_5)(\mathbb{A})$  on  $\phi_{\mathbf{h}} \in \mathcal{S}(V_5(\mathbb{A}))$ . For later purposes, the properties we are interested in are collected in the following lemma.

LEMMA 5.12. *Let  $\phi_{\mathbf{h}}$  be defined as above, and  $v$  be a rational place. Then the following assertions hold:*

- i) *If  $v = q$  is an odd prime not dividing  $N$ , then  $\hat{\phi}_{\mathbf{h},q} = \phi_{\mathbf{h},q}^1 \otimes \phi_{\mathbf{h},q}^2$ , where  $\phi_{\mathbf{h},q}^1 \in \mathcal{S}(V_3(\mathbb{Q}_p))$  and  $\phi_{\mathbf{h},q}^2 \in \mathcal{S}(\mathbb{Q}_p^2)$  are the characteristic functions of  $V_3(\mathbb{Z}_q)$  and  $\mathbb{Z}_q^2$ , respectively. Besides,  $\omega_q((k, s_q(k)), k')\phi_{\mathbf{h},q} = \phi_{\mathbf{h},q}$  for all  $k \in \mathrm{SL}_2(\mathbb{Z}_q)$  and all  $k' \in \mathrm{GSp}_2(\mathbb{Z}_q)$ .*
- ii) *If  $v = 2$ , then  $\omega_2(k, k')\phi_{\mathbf{h},2} = \tilde{\epsilon}_2(k)\phi_{\mathbf{h},2}$  for all  $k \in \Gamma_0(4; \mathbb{Z}_2)$  and all  $k' \in \mathrm{GSp}_2(\mathbb{Z}_2)$ .*
- iii) *If  $v = \infty$ , then*

$$\omega_{\infty}(\tilde{k}_{\theta}, k')\phi_{\mathbf{h},\infty} = e^{-\sqrt{-1}(k+1/2)\theta} \det(\mathbf{k})^{k+1} \phi_{\mathbf{h},\infty}$$

for all  $\tilde{k}_{\theta} \in \widetilde{\mathrm{SO}}(2)$  and  $k' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$ , with  $\mathbf{k} = \alpha + \sqrt{-1}\beta \in \mathrm{U}(2)$ .

- iv) *If  $p \mid N/M$ , then  $\omega_p((k, s_p(k)), k')\phi_{\mathbf{h},p} = \phi_{\mathbf{h},p}$  for all  $k \in \Gamma_0(NM; \mathbb{Z}_p) = \Gamma_0(p; \mathbb{Z}_p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$  and all  $k' \in K_0^{(2)}(NM; \mathbb{Z}_p) = K_0^{(2)}(p; \mathbb{Z}_p) \subseteq \mathrm{GSp}_2(\mathbb{Z}_p)$ . And if  $p \mid M$ , then  $\omega_p((k, s_p(k)), k')\phi_{\mathbf{h},p} = \phi_{\mathbf{h},p}$  for all  $k \in \Gamma_0(NM; \mathbb{Z}_p) = \Gamma_0(p^2; \mathbb{Z}_p) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$  and all  $k' \in K(NM, p; \mathbb{Z}_p) = K(p^2, p; \mathbb{Z}_p) \subseteq \mathrm{GSp}_2(\mathbb{Z}_p)$ . Explicit expressions for  $\hat{\phi}_{\mathbf{h},p}$  in these cases are given in Lemma 5.13 below.*

*Proof.* Part i) is as in [Ich05, Section 7.3]; part ii) is worked out in [Ich05, Section 7.4], where one also finds an explicit expression for  $\hat{\omega}(r, 1)\hat{\phi}_{\mathbf{h},2}$ , where  $r$  varies over a set of representatives for  $\mathrm{SL}_2(\mathbb{Z}_2)/K_0(4; \mathbb{Z}_2)$  (which consists only of 3 elements); part iii) is covered in [Ich05, Section 7.5]. As for part iv), one can check it by routine (and tedious) computation using the explicit description of  $\phi_{\mathbf{h},p}$  together with the rules for the Weil representation and the explicit model of  $\mathrm{SO}(V_5)$  that we are using (cf. Section 4.2). We omit this computation and leave it for the reader. □

At primes  $p \mid N$ , we will also need the partial Fourier transforms  $\hat{\phi}_{\mathbf{h},p}$  of each local component  $\phi_{\mathbf{h},p}$ , which we collect in the next lemma.

LEMMA 5.13. *With notation as above, let  $p$  be a prime dividing  $N$ , and let  $x = (x_1, x_2, x_3) \in V_3(\mathbb{Q}_p)$ ,  $y = (y_1, y_2) \in \mathbb{Q}_p^2$ . Then one has*

$$\hat{\phi}_{\mathbf{h},p}(x; y) = \begin{cases} \mathbf{1}_{\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{p\mathbb{Z}_p}(y_1)\mathbf{1}_{\mathbb{Z}_p^\times}(y_2) & \text{if } p \mid N/M, \\ \mathbf{1}_{p\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)\mathbf{1}_{p^2\mathbb{Z}_p}(y_1)\mathbf{1}_{\mathbb{Z}_p^\times}(y_2)\chi_p^{-1}(y_2) & \text{if } p \mid M. \end{cases}$$



*Proof.* We consider first the case  $p \mid M$ . By applying the definition of the partial Fourier transform and the recipe in iii) above we have, for  $x = (x_1, x_2, x_3) \in V_3(\mathbb{Q}_p)$  and  $(y_1, y_2) \in \mathbb{Q}_p^2$ ,

$$\begin{aligned} \hat{\phi}_{\mathbf{h},p}(x; y) &= p^{-1/2} \varepsilon(1/2, \underline{\chi}_p) \mathbf{1}_{p\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_2) \mathbf{1}_{\mathbb{Z}_p}(x_3) \mathbf{1}_{p^2\mathbb{Z}_p}(y_1) \times \\ &\quad \times \int_{p^{-1}\mathbb{Z}_p^\times} \underline{\chi}_p(z) \psi_p(-y_2 z) dz = \\ &= \mathbf{1}_{p\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_2) \mathbf{1}_{\mathbb{Z}_p}(x_3) \mathbf{1}_{p^2\mathbb{Z}_p}(y_1) \mathbf{1}_{\mathbb{Z}_p^\times}(y_2) \underline{\chi}_p^{-1}(y_2). \end{aligned}$$

Similarly, for primes  $p \mid N/M$ , we find for  $x = (x_1, x_2, x_3) \in V_3(\mathbb{Q}_p)$  and  $(y_1, y_2) \in \mathbb{Q}_p^2$  that

$$\begin{aligned} \hat{\phi}_{\mathbf{h},p}(x; y) &= \mathbf{1}_{\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_2) \mathbf{1}_{\mathbb{Z}_p}(x_3) \mathbf{1}_{p\mathbb{Z}_p}(y_1) \times \\ &\quad \times \int_{\mathbb{Q}_p} (\mathbf{1}_{\mathbb{Z}_p}(z) - p^{-1} \mathbf{1}_{p^{-1}\mathbb{Z}_p}(z)) \psi_p(-y_2 z) dz. \end{aligned}$$

The last integral is easily seen to equal  $\mathbf{1}_{\mathbb{Z}_p^\times}(y_2)$ , hence the result follows.  $\square$

Having described our choice for  $\phi_{\mathbf{h}}$ , together with its main properties, we now focus on the right hand side of (32). We want to decompose the Fourier coefficients  $W_{\mathbf{h},\xi}$  as a product of local Whittaker functions  $W_{\mathbf{h}_v,\xi}$ , so that the integral on the right hand side of (32) decomposes as a product of local integrals that will be eventually computed place by place. By multiplicity one, any decomposition of  $W_{\mathbf{h},\xi}$  as a product of local Whittaker functions will differ from a fixed one by a non-zero scalar factor. Our choice will follow closely the discussion in [BM07, Section 8], with slight renormalizations so that our decomposition will reflect the identity proved in (19). Let  $\xi \in \mathbb{Q}^+$ , and write  $\xi = \mathfrak{d}_\xi \mathfrak{f}_\xi^2$  with  $\mathfrak{d}_\xi \in \mathbb{N}$  and  $\mathfrak{f}_\xi \in \mathbb{Q}^+$ , so that  $-\mathfrak{d}_\xi$  is the discriminant of  $\mathbb{Q}(\sqrt{-\xi})/\mathbb{Q}$ . Write  $e_p := \text{ord}_p(\mathfrak{f}_\xi)$ , and recall the functions  $\Psi_p(\xi; X) \in \mathbb{C}[X, X^{-1}]$  defined in Section 3.2. For each rational place  $v$ , we define the local Whittaker function  $W_{v,\xi} = W_{\mathbf{h}_v,\xi}$  attached to  $\mathbf{h}_v$  and  $\xi$  as follows (we do not enter here in a description of the local types of  $\tilde{\pi}_p$  at primes  $p \mid N$ , which has been given in Section 5.2.1 and will be recalled and used again in Sections 7 and 8, where we will need them to perform the computation of local  $\text{SL}_2$ -periods).

- i) If  $v = p$  is a finite prime with  $p \nmid N$ , then we define  $W_{p,\xi}$  as in [Ich05, Section 7.2, Appendix A.3]. In particular, for all primes  $p \nmid N$  we have  $W_{p,\xi}(1) = \Psi_p(\xi; \alpha_p)$ .
- ii) If  $v = p$  is a finite prime with  $p \mid N/M$ , then  $\tilde{\pi}_p$  is the *special representation*  $\tilde{\sigma}^\delta(\overline{\psi}_p^D)$  as explained in Section 5.2.1, where  $\delta \in \mathbb{Z}_p^\times$  is any non-square unit. The  $p$ -th component  $\mathbf{h}_p \in \tilde{\pi}_p$  of  $\mathbf{h}$  lies in the one-dimensional subspace of vectors fixed by  $\tilde{\Gamma}_0(p) \subseteq \widetilde{\text{SL}}_2(\mathbb{Z}_p)$ , and hence it is a multiple

of the newvector  $\tilde{\varphi}_p$  chosen as in [BM07, Lemma 8.3] (see Lemma 7.1 below). We consider the local Whittaker function

$$W_{\tilde{\varphi}_p, \xi} : g \mapsto \int_{\mathbb{Q}_p} \tilde{\varphi}_p(s^{-1}u(x)g) \overline{\psi_p(\xi x)} dx$$

associated with  $\tilde{\varphi}_p$ , with respect to  $\psi_p^\xi$ . We may assume that  $p \nmid \mathfrak{d}_\xi$ , as otherwise  $W_{\mathfrak{h}, \xi}(1) = 0$ . Then, it follows from the computations in [BM07, Section 8] that

$$W_{\tilde{\varphi}_p, \xi}(1) = \begin{cases} 2p^{-e_p} & \text{if } \chi_{-\xi}(p) + w_p \neq 0, \\ 0 & \text{if } \chi_{-\xi}(p) + w_p = 0. \end{cases}$$

From the definition of the function  $\Psi_p(\xi; X)$  in Section 3.2, we see that

$$\Psi_p(\xi; \alpha_p) = p^{e_p(3/2-k)} a(p^{e_p}) W_{\tilde{\varphi}_p, \xi}(1).$$

We define  $W_{p, \xi} := p^{e_p(3/2-k)} a(p^{e_p}) W_{\tilde{\varphi}_p, \xi}$ , so that we have  $W_{p, \xi}(1) = \Psi_p(\xi; \alpha_p)$ .

- iii) If  $v = p$  is a finite prime with  $p \mid M$ , then  $\tilde{\pi}_p$  is the *supercuspidal* odd Weil representation  $r_{\tilde{\psi}_p}^-$  explained in Section 5.2.1. The subspace of vectors in  $\tilde{\pi}_p$  on which  $\tilde{\Gamma}_0(p^2) \subseteq \widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)$  acts through  $\underline{\chi}_{0,p}$  ( $= \underline{\chi}_p$ ) is one-dimensional, and hence the  $p$ -th component  $\mathfrak{h}_p \in \tilde{\pi}_p$  of  $\mathfrak{h}$  is a multiple of the newvector  $\tilde{\varphi}_p$  chosen as in [BM07, Proposition 8.5] (see Lemma 8.3 below). The representation  $\tilde{\pi}_p = r_{\tilde{\psi}_p}^-$  is distinguished, in the sense that it only has non-trivial  $\psi_p^\xi$ -Whittaker functionals for  $\xi$  in the same square class as  $-D$ . Equivalently, this holds if and only if  $-\xi$  is in the same square class as  $D$ . If this is the case, we choose the local Whittaker function  $W_{\tilde{\varphi}_p, \xi}$  associated with  $\tilde{\varphi}_p$  to satisfy  $W_{\tilde{\varphi}_p, \xi}(1) = \mathbf{1}_{\mathbb{Z}_p^\times}(\mathfrak{f}_\xi) \underline{\chi}_p(\mathfrak{f}_\xi)^{-1}$  (this normalization differs slightly from the one chosen in [BM07, Section 8]). From the definition of the function  $\Psi_p(\xi; X)$  in Section 3.2, we now have

$$\Psi_p(\xi; \alpha_p) = 2p^{e_p(1/2-k)} a(p^{e_p}) \underline{\chi}_p(\mathfrak{f}_\xi) W_{\tilde{\varphi}_p, \xi}(1).$$

We define  $W_{p, \xi} := 2p^{e_p(1/2-k)} a(p^{e_p}) W_{\tilde{\varphi}_p, \xi}$ , so that  $\Psi_p(\xi; \alpha_p) = \underline{\chi}_p(\mathfrak{f}_\xi) W_{p, \xi}(1)$ .

- iv) At the archimedean place  $v = \infty$ , again as in [Ich05, Section 7.2] we define  $W_{\infty, \xi}$  by setting

$$W_{\infty, \xi}(u(x)t(a)k_\theta) = e^{2\pi\sqrt{-1}\xi x} a^{k+1/2} e^{-2\pi\xi a^2} e^{\sqrt{-1}(k+1/2)\theta}$$

for  $x \in \mathbb{R}$ ,  $a \in \mathbb{R}_+^\times$ ,  $\theta \in \mathbb{R}/4\pi\mathbb{Z}$  and  $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \mathrm{SO}(2)$ . In particular,  $W_{\infty, \xi}(1) = e^{-2\pi\xi}$ .

With these choices, we see from equation (17) and Lemma 3.1 that

$$\begin{aligned} W_{\mathbf{h},\xi}(1) &= e^{-2\pi\xi} 2^{-\nu(N)} c(\mathfrak{d}_\xi) \chi(\mathfrak{f}_\xi) \mathfrak{f}_\xi^{k-1/2} \prod_p \Psi_p(\xi; \alpha_p) = \\ &= 2^{-\nu(N)} c(\mathfrak{d}_\xi) \mathfrak{f}_\xi^{k-1/2} \prod_v W_{v,\xi}(1), \end{aligned} \tag{33}$$

and so the next lemma follows immediately by combining (33) with (32).

LEMMA 5.14. *With the above notations, for  $B \neq 0$  we have*

$$\mathcal{W}_{\theta(\mathbf{h},\phi_{\mathbf{h}}),B} = \begin{cases} 2^{-\nu(N)} c(\mathfrak{d}_\xi) \mathfrak{f}_\xi^{k-1/2} \zeta_{\mathbb{Q}}(2)^{-1} \prod_v \mathcal{W}_{B,v} & \text{if } \xi > 0, \\ 0 & \text{if } \xi \leq 0, \end{cases} \tag{34}$$

where  $\mathcal{W}_{B,v}(h)$  equals

$$\int_{U(\mathbb{Q}_v) \backslash \mathrm{SL}_2(\mathbb{Q}_v)} \hat{\omega}_v(g, h) \hat{\phi}_{\mathbf{h},v}(\beta; 0, 1) W_{v,\xi}(g) dg \times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SO}(2))^{-1} & \text{if } v = \infty. \end{cases}$$

As before, here  $\xi = \det(B)$  and  $\beta = (b_3, b_2/2, -b_1)$  if  $B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix}$ .

By using this lemma, we can now determine the Fourier coefficients  $\mathcal{W}_{\theta(\mathbf{h},\phi_{\mathbf{h}}),B}$  by computing the local functions  $\mathcal{W}_{B,v}$ . Because of the invariance properties spelled out in Lemma 5.12, we see that  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$  is right invariant for  $K(NM, M)$ . In particular, as commented above the Fourier coefficients  $\mathcal{W}_{\theta(\mathbf{h},\phi_{\mathbf{h}}),B}$  are determined by the values  $\mathcal{W}_{\theta(\mathbf{h},\phi_{\mathbf{h}}),B}(g_\infty)$  with  $g_\infty = n(X)m(A, 1)$  as in (29). Hence we only need to determine the values  $\mathcal{W}_{B,p}(1)$  at finite primes  $p$ , together with  $\mathcal{W}_{B,\infty}(n(X)m(A, 1))$ . We discuss case by case such computations.

### 5.3.1 COMPUTATION AT PRIMES $p \nmid N$

At primes  $p \nmid N$ , we can compute  $\mathcal{W}_{B,p}(1)$  literally as in [Ich05, Sections 7.3, 7.4]. We summarize the outcome of such computation. At each prime  $p \nmid N$ , we continue to denote by  $\{\alpha_p, \alpha_p^{-1}\}$  the Satake parameter of  $f$  at  $p$ , and consider the function  $\Psi_p(\xi; X) \in \mathbb{C}[X + X^{-1}]$  as above. Recall that  $\phi_{\mathbf{h},p}$  is the characteristic function of  $V_3(\mathbb{Z}_p)$ . Further, from Lemma 5.12 we have

$$\omega_p((k, s_p(k)), k') \phi_{\mathbf{h},p} = \phi_{\mathbf{h},p}$$

for all  $k \in \mathrm{SL}_2(\mathbb{Z}_p)$  and  $k' \in \mathrm{GSp}_2(\mathbb{Z}_p)$ , and  $\hat{\phi}_{\mathbf{h},p} = \phi_{p,1} \otimes \phi_{p,2}$ , where  $\phi_{p,1} \in \mathcal{S}(V_3(\mathbb{Q}_p))$  and  $\phi_{p,2} \in \mathcal{S}(\mathbb{Q}_p^2)$  are the characteristic functions of  $V_3(\mathbb{Z}_p)$  and  $\mathbb{Z}_p^2$ , respectively. In this case, one finds out that for  $\xi \neq 0$

$$\mathcal{W}_{B,p}(1) = \begin{cases} \sum_{n=0}^{\min(\mathrm{ord}_p(b_i))} p^{n/2} \Psi_p(p^{-2n}\xi; \alpha_p) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

At the prime  $p = 2$ , the 2-component  $\phi_{\mathbf{h},2}$  of the Bruhat–Schwartz function  $\phi_{\mathbf{h}}$  is the characteristic function of  $\mathbb{Z}_2 v_3 + V_5(2\mathbb{Z}_2)$ , which satisfies  $\omega_2(k, k')\phi_{\mathbf{h},2} = \tilde{\epsilon}_2(k)\phi_{\mathbf{h},2}$  for all  $k \in \Gamma_0(4; \mathbb{Z}_2)$ ,  $k' \in \mathrm{GSp}_2(\mathbb{Z}_2)$  (cf. Lemma 5.12). In this case one finds, for  $\xi \neq 0$ ,

$$\mathcal{W}_{B,2}(1) = \begin{cases} 2^{-7/2} \sum_{n=0}^{\min(\mathrm{ord}_2(b_i))} 2^{n/2} \Psi_2(2^{-2n+2}\xi; \alpha_2) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

5.3.2 COMPUTATION AT PRIMES  $p \mid N/M$

Let  $p$  be a prime dividing  $N/M$ . Now  $\hat{\phi}_{\mathbf{h},p}$  is not  $\mathrm{SL}_2(\mathbb{Z}_p)$ -invariant, but only  $\Gamma_0(p)$ -invariant. Let  $R_p$  be a set of representatives for  $\mathrm{SL}_2(\mathbb{Z}_p)/\Gamma_0(p)$ . Then, using that  $\mathrm{SL}_2(\mathbb{Q}_p) = \dot{U}(\mathbb{Q}_p)T(\mathbb{Q}_p)\mathrm{SL}_2(\mathbb{Z}_p)$  we have

$$\begin{aligned} \mathcal{W}_{B,p}(h) &= \frac{1}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))} \int_{\mathbb{Q}_p^\times} \int_{\mathrm{SL}_2(\mathbb{Z}_p)} \hat{\omega}_p(t(a)k, h) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) \frac{W_{p,\xi}(t(a)k)}{|a|_p^2} dk d^\times a = \\ &= \mathfrak{c}_p \int_{\mathbb{Q}_p^\times} \sum_{r \in R_p} \hat{\omega}_p(t(a)r, h) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) W_{p,\xi}(t(a)r) |a|_p^{-2} d^\times a, \end{aligned}$$

where  $\mathfrak{c}_p := [\mathrm{SL}_2(\mathbb{Z}_p) : \Gamma_0(p)]^{-1} = \frac{\mathrm{vol}(\Gamma_0(p))}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))}$ . We will compute  $\mathcal{W}_{B,p}(1)$ . If  $a \in \mathbb{Q}_p^\times$  and  $r \in R_p$ , recall from (31) that

$$\hat{\omega}_p(t(a)r, 1) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) = \omega_p((t(a)r, 1)) \phi_{1,p}(\beta) \phi_{2,p}((0, 1)t(a)r),$$

where  $\phi_{1,p}(x) = \mathbf{1}_{\mathbb{Z}_p}(x_1) \mathbf{1}_{\mathbb{Z}_p}(x_2) \mathbf{1}_{\mathbb{Z}_p}(x_3)$  and  $\phi_{2,p}(y) = \mathbf{1}_{p\mathbb{Z}_p}(y_1) \mathbf{1}_{\mathbb{Z}_p^\times}(y_2)$ . We take  $R_p$  to be the set consisting of the elements

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \text{ with } b \in \mathbb{Z}_p/p\mathbb{Z}_p, \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, the elements  $t(a)r$  with  $a \in \mathbb{Q}_p^\times$ ,  $r \in R_p$ , are precisely the elements of the form

$$\begin{pmatrix} a & 0 \\ a^{-1}b & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}, \quad a \in \mathbb{Q}_p^\times, b \in \mathbb{Z}_p/p\mathbb{Z}_p.$$

From the very definition of  $\phi_{2,p}$ , it is immediate to see that  $\phi_{2,p}((0, 1)t(a)r) = 0$  unless  $r = 1$  and  $a \in \mathbb{Z}_p^\times$ . Therefore, we deduce that

$$\begin{aligned} \mathcal{W}_{B,p}(1) &= \mathfrak{c}_p \int_{\mathbb{Z}_p^\times} \hat{\omega}_p(t(a), 1) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) W_{p,\xi}(t(a)) d^\times a = \\ &= \mathfrak{c}_p \phi_{1,p}(\beta) \int_{\mathbb{Z}_p^\times} W_{p,\xi}(t(a)) d^\times a = \\ &= \mathfrak{c}_p \mathbf{1}_{\mathbb{Z}_p}(b_3) \mathbf{1}_{\mathbb{Z}_p}(b_2) \mathbf{1}_{\mathbb{Z}_p}(b_1) \int_{\mathbb{Z}_p^\times} W_{p,\xi}(t(a)) d^\times a = \\ &= \mathfrak{c}_p \mathbf{1}_{\mathbb{Z}_p}(b_3) \mathbf{1}_{\mathbb{Z}_p}(b_2) \mathbf{1}_{\mathbb{Z}_p}(b_1) \Psi_p(\xi; \alpha_p), \end{aligned}$$

where we have used that for  $a \in \mathbb{Z}_p^\times$  one has (here, recall that  $\psi = \psi_p^{-D}$ , where  $\psi_p$  is the standard additive character of  $\mathbb{Q}_p$ , and  $\delta \in \mathbb{Z}_p^\times$  is any non-square unit)

$$\begin{aligned} W_{p,\xi}(t(a)) &= \chi_\psi(a^{-1})\chi_\delta(a^{-1})|a|_p^{1/2}W_{p,a^2\xi}(1) = \\ &= W_{p,a^2\xi}(1) = \Psi_p(a^2\xi; \alpha_p) = \Psi_p(\xi; \alpha_p). \end{aligned}$$

We conclude that

$$\mathcal{W}_{B,p}(1) = \begin{cases} [\mathrm{SL}_2(\mathbb{Z}_p) : \Gamma_0(p)]^{-1}\Psi_p(\xi; \alpha_p) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \quad (37)$$

### 5.3.3 COMPUTATION AT PRIMES $p \mid M$

We proceed similarly as in the previous case. But now if  $p$  is a prime dividing  $M$ , then  $\hat{\phi}_{\mathbf{h},p}$  is only  $\Gamma_0(p^2)$ -invariant. If we denote by  $R_{p^2}$  a set of representatives for  $\mathrm{SL}_2(\mathbb{Z}_p)/\Gamma_0(p^2)$ , then we have

$$\begin{aligned} \mathcal{W}_{B,p}(h) &= \frac{1}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))} \int_{\mathbb{Q}_p^\times} \int_{\mathrm{SL}_2(\mathbb{Z}_p)} \hat{\omega}_p(t(a)k, h) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) \frac{W_{p,\xi}(t(a)k)}{|a|_p^2} dk d^\times a = \\ &= \mathfrak{c}_{p^2} \int_{\mathbb{Q}_p^\times} \sum_{r \in R_{p^2}} \hat{\omega}_p(t(a)r, h) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) W_{p,\xi}(t(a)r) |a|_p^{-2} d^\times a, \end{aligned}$$

where now  $\mathfrak{c}_{p^2} := [\mathrm{SL}_2(\mathbb{Z}_p) : \Gamma_0(p^2)]^{-1} = \frac{\mathrm{vol}(\Gamma_0(p^2))}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))}$ . To compute  $\mathcal{W}_{B,p}(1)$ , as before we notice that for  $a \in \mathbb{Q}_p^\times$  and  $r \in R_{p^2}$  we have

$$\hat{\omega}_p(t(a)r, 1) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) = \omega_p((t(a)r, 1)) \phi_{1,p}(\beta) \phi_{2,p}((0, 1)t(a)r),$$

where now  $\phi_{1,p}(x) = \mathbf{1}_{p\mathbb{Z}_p}(x_1)\mathbf{1}_{\mathbb{Z}_p}(x_2)\mathbf{1}_{\mathbb{Z}_p}(x_3)$  and  $\phi_{2,p}(y) = \mathbf{1}_{p^2\mathbb{Z}_p}(y_1)\mathbf{1}_{\mathbb{Z}_p^\times}(y_2)\chi_{0,p}^{-1}(y_2)$ . We might take as a set of representatives for  $\mathrm{SL}_2(\mathbb{Z}_p)/\Gamma_0(p^2)$  the set  $R_{p^2}$  consisting of the elements

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \text{ with } b \in \mathbb{Z}_p/p^2\mathbb{Z}_p, \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

hence the elements  $t(a)r$  with  $a \in \mathbb{Q}_p^\times$ ,  $r \in R_{p^2}$ , are precisely the elements of the form

$$\begin{pmatrix} a & 0 \\ a^{-1}b & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}, \quad a \in \mathbb{Q}_p^\times, b \in \mathbb{Z}_p/p^2\mathbb{Z}_p.$$

As in the previous case, one easily checks that  $\phi_{2,p}((0, 1)t(a)r) = 0$  unless  $r = 1$  and  $a \in \mathbb{Z}_p^\times$ , hence it follows that

$$\begin{aligned} \mathcal{W}_{B,p}(1) &= \mathfrak{c}_{p^2} \int_{\mathbb{Z}_p^\times} \hat{\omega}_p(t(a), 1) \hat{\phi}_{\mathbf{h},p}(\beta; 0, 1) W_{p,\xi}(t(a)) d^\times a = \\ &= \mathfrak{c}_{p^2} \phi_{1,p}(\beta) \int_{\mathbb{Z}_p} \chi_{0,p}(a) W_{p,\xi}(t(a)) d^\times a = \\ &= \mathfrak{c}_{p^2} \mathbf{1}_{p\mathbb{Z}_p}(b_3) \mathbf{1}_{\mathbb{Z}_p}(b_2) \mathbf{1}_{\mathbb{Z}_p}(b_1) \int_{\mathbb{Z}_p} \chi_{0,p}(a) W_{p,\xi}(t(a)) d^\times a = \\ &= \mathfrak{c}_{p^2} \mathbf{1}_{p\mathbb{Z}_p}(b_3) \mathbf{1}_{\mathbb{Z}_p}(b_2) \mathbf{1}_{\mathbb{Z}_p}(b_1) \chi_{0,p}(\mathfrak{f}_\xi)^{-1} \Psi_p(\xi; \alpha_p), \end{aligned}$$

where we have used that for  $a \in \mathbb{Z}_p^\times$  one has  $W_{p,\xi}(t(a)) = \chi_p(a\mathfrak{f}_\xi)^{-1} \Psi_p(\xi; \alpha_p)$  for  $a \in \mathbb{Z}_p^\times$ . We thus obtain

$$\mathcal{W}_{B,p}(1) = \begin{cases} [\mathrm{SL}_2(\mathbb{Z}_p) : \Gamma_0(p^2)]^{-1} \chi_{(p)}(\mathfrak{f}_\xi) \Psi_p(\xi; \alpha_p) & \text{if } b_1, b_2 \in \mathbb{Z}_p, b_3 \in p\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

5.3.4 COMPUTATION AT THE ARCHIMEDEAN PLACE

At the archimedean place, the determination of  $\mathcal{W}_{B,\infty}$  is carried out in [Ich05, Section 7.5]. Recall that the  $\infty$ -component  $\phi_{\mathbf{h},\infty} \in \mathcal{S}(V_5(\mathbb{R}))$  of  $\phi_{\mathbf{h}}$  is given by

$$\phi_{\mathbf{h},\infty}(x) = (x_2 + \sqrt{-1}x_1 + \sqrt{-1}x_5 - x_4)^{k+1} e^{-\pi(x_1^2+x_2^2+2x_3^2+x_4^2+x_5^2)},$$

and it satisfies (cf. Lemma 5.12)

$$\omega_\infty(\tilde{k}_\theta, k') \phi_{\mathbf{h},\infty} = e^{-\sqrt{-1}(k+1/2)\theta} \det(\mathbf{k})^{k+1} \phi_{\mathbf{h},\infty}$$

for  $\tilde{k}_\theta \in \widetilde{\mathrm{SO}}(2)$  and  $k' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$  with  $\mathbf{k} = \alpha + \sqrt{-1}\beta \in \mathrm{U}(2)$ . Then it is proved in [Ich05, Lemma 7.6] that for  $\xi > 0$ ,  $A \in \mathrm{GL}_2^+(\mathbb{R})$  and  $X \in \mathrm{Sym}_2(\mathbb{R})$ , one has

$$\mathcal{W}_{B,\infty}(n(X) \mathrm{diag}(A, {}^t A^{-1})) = \begin{cases} 2^{k+1} \det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1} \mathrm{Tr}(BZ)} & \text{if } B > 0, \\ 0 & \text{if } B < 0, \end{cases} \tag{39}$$

where  $Y = A^t A$ ,  $Z = X + \sqrt{-1}Y$ .

5.3.5 PROOF OF PROPOSITION 5.10

We are finally in position to compute the Fourier coefficients for  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$ . Let us fix  $B \in \mathrm{Sym}_2(\mathbb{Q})$ , and

$$g_\infty = n(X)m(A, 1) = n(X) \mathrm{diag}(A, {}^t A^{-1}) \in \mathrm{Sp}_2(\mathbb{R})$$

with  $X \in \text{Sym}_2(\mathbb{R})$  and  $A \in \text{GL}_2^+(\mathbb{R})$ . By using (35)-(39) above, we may assume that  $B > 0$ , and that  $b_1, b_2, b_3 \in \mathbb{Z}$  with  $b_3 \in M\mathbb{Z}$ ; otherwise we have  $\mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}})}(g_\infty) = 0$ . In that case, by virtue of (34) we have

$$\mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty) = 2^{-\nu(N)} \zeta_{\mathbb{Q}}(2)^{-1} c(\mathfrak{d}_\xi) \mathfrak{f}_\xi^{k-1/2} \mathcal{W}_{B, \infty}(g_\infty) \prod_p \mathcal{W}_{B, p}(1),$$

where  $\mathcal{W}_{B, \infty}(g_\infty) = 2^{k+1} \det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1}\text{Tr}(BZ)}$  as in (39). For ease of notation, let us write  $\mathcal{W}_\infty := 2^{-k-1} \mathcal{W}_{B, \infty}(g_\infty) = \det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1}\text{Tr}(BZ)}$ , and to abbreviate put

$$\mu_{N, M} := [\text{SL}_2(\mathbb{Z}) : \Gamma_0(NM)]^{-1} = M^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1}.$$

Then, using (35)-(38),  $\mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty)$  equals

$$\begin{aligned} \mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty) &= 2^{k-\nu(N)-5/2} \zeta_{\mathbb{Q}}(2)^{-1} \mu_{N, M} c(\mathfrak{d}_\xi) \chi(\mathfrak{f}_\xi) \mathfrak{f}_\xi^{k-1/2} \mathcal{W}_\infty \times \\ &\quad \times \prod_{p \nmid N} \sum_{n=0}^{\min(\text{ord}_p(b_i))} p^{n/2} \Psi_p \left( \frac{4\xi}{p^{2n}}; \alpha_p \right) \prod_{p|N} \Psi_p(\xi; \alpha_p). \end{aligned}$$

Next, observe that  $\mathfrak{d}_{4\xi} = \mathfrak{d}_\xi$ , hence  $c(\mathfrak{d}_\xi) = c(\mathfrak{d}_{4\xi})$ , and that  $\mathfrak{f}_\xi^{k-1/2} = 2^{-k+1/2} \mathfrak{f}_{4\xi}^{k-1/2}$ . Therefore, the previous expression can be rewritten as

$$\begin{aligned} \mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty) &= 2^{-\nu(N)-2} \zeta_{\mathbb{Q}}(2)^{-1} \mu_{N, M} c(\mathfrak{d}_{4\xi}) \chi(\mathfrak{f}_\xi) \mathfrak{f}_{4\xi}^{k-1/2} \mathcal{W}_\infty \times \\ &\quad \times \sum_{\substack{d|(b_1, b_2, b_3), \\ (d, N)=1}} d^{1/2} \prod_{p \nmid N} \Psi_p \left( \frac{4\xi}{d^2}; \alpha_p \right) \prod_{p|N} \Psi_p(\xi; \alpha_p). \end{aligned}$$

Now, for every integer  $d$  with  $(d, N) = 1$ , we have  $c(\mathfrak{d}_{4\xi}) = c(\mathfrak{d}_{4\xi/d^2})$ ,  $\mathfrak{f}_{4\xi}^{k-1/2} = d^{k-1/2} \mathfrak{f}_{4\xi/d^2}$ , and  $\Psi_p(\xi; \alpha_p) = \Psi_p\left(\frac{4\xi}{d^2}; \alpha_p\right)$  for every prime  $p \mid N$ . Hence,  $\mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty)$  equals

$$\begin{aligned} &\frac{\mu_{N, M} \mathcal{W}_\infty}{2^2 \zeta_{\mathbb{Q}}(2)} \sum_{\substack{d|(b_1, b_2, b_3), \\ (d, N)=1}} d^k 2^{-\nu(N)} c(\mathfrak{d}_{4\xi/d^2}) \chi(\mathfrak{f}_\xi) \mathfrak{f}_{4\xi/d^2}^{k-1/2} \prod_p \Psi_p \left( \frac{4\xi}{d^2}; \alpha_p \right) = \\ &= \frac{\mu_{N, M} \mathcal{W}_\infty}{2^2 \zeta_{\mathbb{Q}}(2)} \sum_{\substack{d|(b_1, b_2, b_3), \\ (d, N)=1}} \chi(d/2) d^k 2^{-\nu(N)} c(\mathfrak{f}_{4\xi/d^2}) \chi(\mathfrak{f}_{4\xi/d^2}) \mathfrak{f}_{4\xi/d^2}^{k-1/2} \prod_p \Psi_p \left( \frac{4\xi}{d^2}; \alpha_p \right) = \\ &= \frac{\mu_{N, M} \mathcal{W}_\infty}{2^2 \chi(2) \zeta_{\mathbb{Q}}(2)} \sum_{\substack{d|(b_1, b_2, b_3), \\ (d, N)=1}} \chi(d) d^k c(4\xi/d^2) = \\ &= \frac{\mu_{N, M}}{2^2 \chi(2) \zeta_{\mathbb{Q}}(2)} \det(Y)^{(k+1)/2} e^{2\pi\sqrt{-1}\text{Tr}(BZ)} A_\chi(B). \end{aligned}$$

Thus, we conclude that when  $B > 0$  and  $b_1, b_2, b_3 \in \mathbb{Z}$  with  $b_3 \in M\mathbb{Z}$ ,

$$\mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}, B}(g_\infty) = \mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}), B}(g_\infty) = C\mathcal{W}_{\mathbf{F}_\chi, B}(g_\infty) = C\mathcal{W}_{\mathfrak{R}_M \mathbf{F}_\chi, B}(g_\infty),$$

where  $C = 2^{-2}\chi(2)^{-1}\mu_{N, M}\zeta_{\mathbb{Q}}(2)^{-1}$ . Since both  $\mathcal{W}_{\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}, B}$  and  $\mathcal{W}_{\mathfrak{R}_M \mathbf{F}_\chi, B}$  vanish when either  $B$  is not positive definite or  $(b_1, b_2, b_3) \notin \mathbb{Z} \times \mathbb{Z} \times M\mathbb{Z}$ , it follows that  $\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi} = C\mathfrak{R}_M \mathbf{F}_\chi$  as desired.

### 6 THE MAIN RESULT

In this section we finally state precisely and prove the main result of this paper, relegating the technical local computations to the subsequent sections. Since our approach relies crucially on a decomposition formula of Qiu [Qiu14] for an automorphic  $\mathrm{SL}_2$ -period, we first explain how such period is related to the central  $L$ -value that we want to compute.

#### 6.1 QIU’S DECOMPOSITION FORMULA FOR THE $\mathrm{SL}_2$ -PERIOD

Let  $\pi$  (resp.  $\tau$ ) be an irreducible cuspidal automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$  (resp.  $\mathrm{GL}_2(\mathbb{A})$ ). Fix a non-trivial additive character  $\psi$  of  $\mathbb{A}/\mathbb{Q}$ , and let  $\tilde{\pi} \in \mathrm{Wald}_\psi(\pi)$  be an irreducible cuspidal automorphic representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ , belonging to the Waldspurger packet of  $\pi$  with respect to  $\psi$  as explained in Section 5.2. Let also  $\omega = \omega_\psi$  be the Weil representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  acting on the space  $\mathcal{S}(\mathbb{A})$  of Bruhat–Schwartz functions (for the one dimensional quadratic space endowed with bilinear form  $(x, y) = 2xy$ ) with respect to  $\psi$ . Associated with  $\tilde{\pi}$ ,  $\tau$  and  $\omega$ , there is a (global)  $\mathrm{SL}_2$ -period functional

$$\mathcal{Q} : \tilde{\pi} \otimes \tilde{\pi} \otimes \tau \otimes \tau \otimes \omega \otimes \omega \longrightarrow \mathbb{C}$$

defined by associating to each choice of decomposable vectors  $\mathbf{h}_1, \mathbf{h}_2 \in \tilde{\pi}$ ,  $\mathbf{g}_1, \mathbf{g}_2 \in \tau$ ,  $\phi_1, \phi_2 \in \omega$ , the product of integrals

$$\mathcal{Q}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{g}_1, \mathbf{g}_2, \phi_1, \phi_2) := \int_{[\mathrm{SL}_2]} \overline{\mathbf{h}_1(g)} \mathbf{g}_1(g) \Theta_{\phi_1}(g) dg \cdot \overline{\int_{[\mathrm{SL}_2]} \mathbf{h}_2(g) \mathbf{g}_2(g) \Theta_{\phi_2}(g) dg}.$$

It is proved in [Qiu14, Theorem 4.5] that this global period, if it is non-vanishing, decomposes as a product of local  $\mathrm{SL}_2$ -periods up to certain  $L$ -values. Namely, one has

$$\begin{aligned} \mathcal{Q}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{g}_1, \mathbf{g}_2, \phi_1, \phi_2) &= \frac{1}{4} \frac{L(1/2, \pi \times \mathrm{ad}\tau)}{L(1, \pi, \mathrm{ad})L(1, \tau, \mathrm{ad})} \times \\ &\quad \times \prod_v \mathcal{I}_v(\mathbf{h}_{1,v}, \mathbf{h}_{2,v}, \mathbf{g}_{1,v}, \mathbf{g}_{2,v}, \phi_{1,v}, \phi_{2,v}), \end{aligned} \tag{40}$$

where for each place  $v$ , the local period  $\mathcal{I}_v(\mathbf{h}_{1,v}, \mathbf{h}_{2,v}, \mathbf{g}_{1,v}, \mathbf{g}_{2,v}, \phi_{1,v}, \phi_{2,v})$  is defined by integrating a product of matrix coefficients, and equals

$$\frac{L(1, \pi_v, \mathrm{ad})L(1, \tau_v, \mathrm{ad})}{L(1/2, \pi_v \times \mathrm{ad}\tau_v)} \int_{\mathrm{SL}_2(\mathbb{Q}_v)} \overline{\langle \tilde{\pi}(g_v) \mathbf{h}_{1,v}, \mathbf{h}_{2,v} \rangle} \langle \tau(g_v) \mathbf{g}_{1,v}, \mathbf{g}_{2,v} \rangle \langle \omega_v(g_v) \phi_{1,v}, \phi_{2,v} \rangle dg_v.$$



When  $\pi$  (resp.  $\tau$ ) is the automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$  (resp.  $\mathrm{GL}_2(\mathbb{A})$ ) associated with the newform  $f$  (resp.  $g$ ) as in the introduction, notice that  $L(1/2, \pi \times \mathrm{ad}\tau)$  coincides indeed with the special value  $\Lambda(f \otimes \mathrm{Ad}(g), k) = \Lambda(f' \otimes \mathrm{Sym}^2(g), 2k)$  that we are concerned with.

REMARK 6.1. *Qiu's definition of the local periods  $\mathcal{I}_v$  includes a factor  $\zeta_v(2)^{-1}$ , and then the decomposition formula in (40) has accordingly a factor  $\zeta_{\mathbb{Q}}(2)$  on the right hand side. We have chosen to redefine the local periods by the above expression due to our different choice of local measures  $dg_v$ .*

The proof of the central value formula in Theorem 1.1 will rest crucially on this decomposition result, thus it is essential to characterize the conditions under which  $\mathcal{Q}$  does not vanish. In this sense, we shall note the following (see [Qiu14, Proposition 4.1], and [GG09, Theorem 7.1]):

PROPOSITION 6.2. *The functional  $\mathcal{Q}$  is non-vanishing on  $\tilde{\pi} \otimes \tilde{\pi} \otimes \tau \otimes \tau \otimes \omega \otimes \omega$  if and only if the following conditions hold:*

- i)  $L(1/2, \pi \times \mathrm{ad}\tau) \neq 0$ ;
- ii)  $\tilde{\pi} = \tilde{\pi}^\epsilon$  with  $\epsilon_v = \epsilon(1/2, \pi_v \otimes \tau_v \otimes \tau_v^\vee)$ ;
- iii)  $\epsilon(1/2, \pi_v \otimes \tau_v \otimes \tau_v^\vee) = 1$  when  $\pi_v$  is not square-integrable.

In condition ii),  $\tilde{\pi}^\epsilon$  refers to the automorphic representation in  $\mathrm{Wald}_{\psi}(\pi)$  labelled by the tuple  $\epsilon = (\epsilon_v)_v \in \{\pm 1\}^{|\Sigma(\pi)|}$  as in Theorem 5.9. In particular, notice that fixed the automorphic representation  $\pi$  there is *only one* automorphic representation for  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  in the (finite) set  $\mathrm{Wald}_{\psi_v}(\pi_v)$  which makes the period  $\mathcal{Q}$  non-vanishing. When  $\pi_v$  is not square-integrable, recall from Theorem 5.7 that the local Waldspurger packet  $\mathrm{Wald}_{\psi_v}(\pi_v)$  consists of a single element, labelled  $\tilde{\pi}_v$ . Therefore, condition iii) is meant to ensure that condition ii) is not failing by obvious reasons. Also, condition i) implies that  $\epsilon(1/2, \pi \times \mathrm{ad}\tau) = 1$ , and hence  $\prod_v \epsilon_v = \epsilon(1/2, \pi_v \otimes \tau_v \otimes \tau_v^\vee) = \epsilon(1/2, \pi)$ , ensuring that  $\tilde{\pi}^\epsilon$  is a global automorphic representation.

## 6.2 THE CENTRAL VALUE FORMULA

Now we put ourselves in the setting of interest in this paper. Let  $k \geq 1$  be an odd integer,  $N \geq 1$  be an odd square-free integer, and let  $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$  be a normalized newform of weight  $2k$ , level  $N$ , and trivial nebentype. Let also  $\chi$  be a Dirichlet character modulo  $N$ , and  $g \in S_{k+1}^{\mathrm{new}}(\Gamma_0(N), \chi)$  be a normalized newform of weight  $k+1$ , level  $N$ , and nebentype character  $\chi$ . Write  $M \geq 1$  for the conductor of  $\chi$ , which divides  $N$ , and assume Hypotheses (H1) and (H2). That is to say,  $\chi_{(p)}(-1) = -1$  for all  $p \mid M$  and  $\epsilon_p(f) = -1$  for all  $p \mid M$ . Here, the Dirichlet character  $\chi_{(p)} : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  is the  $p$ -th component of  $\chi$ . Let  $\pi$  and  $\tau$  be the cuspidal automorphic representations of  $\mathrm{PGL}_2(\mathbb{A})$  and  $\mathrm{GL}_2(\mathbb{A})$ , respectively, associated with  $f$  and  $g$ . Fix a fundamental discriminant

$D \in \mathfrak{D}(N, M)$  as in Theorem 2.1, and assume further that  $L(f, D, k) \neq 0$ . Let  $\psi$  be the standard additive character of  $\mathbb{A}/\mathbb{Q}$ , and let

$$\tilde{\pi} = \Theta_{\mathrm{PGL}_2 \times \widetilde{\mathrm{SL}}_2}(\pi \otimes \chi_D, \psi^{-D}) \in \mathrm{Wald}_{\overline{\psi}}(\pi)$$

be the automorphic representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{A})$  as explained at the end of Section 5.2, where  $\overline{\psi} = \psi^{-1}$  is the  $(-1)$ -th twist (equivalently, the inverse, or the complex conjugate) of  $\psi$ . Recall that, as an element of  $\mathrm{Wald}_{\overline{\psi}}(\pi)$ ,  $\tilde{\pi}$  corresponds to the automorphic representation labelled by the tuple  $\epsilon = (\epsilon_v)_v \in \{\pm 1\}^{|\Sigma(\pi)|}$  with  $\epsilon_\infty = -1$  and

$$\epsilon_p = w_p \text{ for all primes } p \mid N/M, \quad \epsilon_p = -w_p \text{ for all primes } p \mid M.$$

Under the hypothesis (H2), we thus have  $\epsilon_p = 1$  for all primes  $p \mid M$ . Recall also that the adelization of the half-integral weight cuspidal forms  $h \in S_{k+1/2}^{+,new}(4NM, \chi; f \otimes \chi)$  as in Theorem 2.1 belong to this particular element  $\tilde{\pi}$  in the Waldspurger packet  $\mathrm{Wald}_{\overline{\psi}}(\pi)$ .

In this setting and under our assumptions, the criterion for the non-vanishing of the  $\mathrm{SL}_2$ -period

$$\mathcal{Q} : \tilde{\pi} \otimes \tilde{\pi} \otimes \tau \otimes \tau \otimes \omega_{\overline{\psi}} \otimes \omega_{\overline{\psi}} \longrightarrow \mathbb{C}$$

is reduced to the following statement:

PROPOSITION 6.3. *With the above choices, the functional  $\mathcal{Q}$  is non-vanishing if and only if*

$$\Lambda(f \otimes \mathrm{Ad}(g), k) \neq 0.$$

*Proof.* In the current setting, condition iii) in Proposition 6.2 obviously holds, and we claim that condition ii) holds if and only if hypothesis (H2) is satisfied. Indeed, we only need to take care of condition ii) in Proposition 6.2 at places  $v \mid N\infty$ . First of all, at the archimedean place  $v = \infty$  we have  $\epsilon_\infty = -1$ , and our choice of weights implies that  $\varepsilon(1/2, \pi_\infty \otimes \tau_\infty \otimes \tau_\infty^\vee) = -1$  as well. Secondly, suppose that  $p$  is a prime dividing  $N/M$ . Then both  $\pi_p$  and  $\tau_p$  are quadratic twists of the Steinberg representation, and [Pra90, Proposition 8.6] shows that  $\varepsilon(1/2, \pi_p \otimes \tau_p \otimes \tau_p^\vee) = \varepsilon(1/2, \pi_p) = w_p$ , which agrees with  $\epsilon_p$ . And finally, suppose that  $p$  is a prime factor of  $M$ . In this case,  $\pi_p$  is again a quadratic twist of the Steinberg representation, but  $\tau_p$  is now a (ramified) principal series representation. Then, [Pra90, Proposition 8.4] implies that  $\varepsilon(1/2, \pi_p \otimes \tau_p \otimes \tau_p^\vee) = 1$ . Under hypothesis (H2), this indeed agrees with  $\epsilon_p$  as pointed out above, and therefore the statement follows from Proposition 6.2.  $\square$

In light of this proposition, the period functional  $\mathcal{Q}$  is therefore non-vanishing if we assume that  $\Lambda(f \otimes \mathrm{Ad}(g), k) \neq 0$ . When this holds, the strategy for proving our central value formula is now clear. Indeed, let  $\mathbf{h} \in \tilde{\pi}$ ,  $\mathbf{g} \in \tau$ , and

$\phi \in \omega_{\bar{\psi}}$  be decomposable vectors, and write  $\mathcal{Q}(\mathbf{h}, \mathbf{g}, \phi) := \mathcal{Q}(\mathbf{h}, \mathbf{h}, \mathbf{g}, \mathbf{g}, \phi, \phi)$ , and  $\mathcal{I}_v(\mathbf{h}, \mathbf{g}, \phi) := \mathcal{I}_v(\mathbf{h}_v, \mathbf{h}_v, \mathbf{g}_v, \mathbf{g}_v, \phi_v, \phi_v)$ , so that

$$\mathcal{I}_v(\mathbf{h}, \mathbf{g}, \phi) = \frac{L(1, \pi_v, \text{ad})L(1, \tau_v, \text{ad})}{L(1/2, \pi_v \times \text{ad}\tau_v)} \times \int_{\text{SL}_2(\mathbb{Q}_v)} \overline{\langle \tilde{\pi}(g_v)\mathbf{h}_v, \mathbf{h}_v \rangle} \langle \tau(g_v)\mathbf{g}_v, \mathbf{g}_v \rangle \langle \omega_{\bar{\psi}_v}(g_v)\phi_v, \phi_v \rangle dg_v.$$

Regularizing these local periods by setting

$$\mathcal{I}_v^\#(\mathbf{h}, \mathbf{g}, \phi) := \frac{\mathcal{I}_v(\mathbf{h}, \mathbf{g}, \phi)}{\langle \mathbf{h}_v, \mathbf{h}_v \rangle \langle \mathbf{g}_v, \mathbf{g}_v \rangle \langle \phi_v, \phi_v \rangle} = \frac{\mathcal{I}_v(\mathbf{h}, \mathbf{g}, \phi)}{\|\mathbf{h}_v\|^2 \|\mathbf{g}_v\|^2 \|\phi_v\|^2},$$

we will write

$$\mathcal{I}_v^\#(\mathbf{h}, \mathbf{g}, \phi) = \frac{L(1, \pi_v, \text{ad})L(1, \tau_v, \text{ad})}{L(1/2, \pi_v \times \text{ad}\tau_v)} \alpha_v^\#(\mathbf{h}, \mathbf{g}, \phi), \tag{41}$$

where

$$\alpha_v^\#(\mathbf{h}, \mathbf{g}, \phi) := \int_{\text{SL}_2(\mathbb{Q}_v)} \frac{\overline{\langle \tilde{\pi}(g_v)\mathbf{h}_v, \mathbf{h}_v \rangle}}{\|\mathbf{h}_v\|^2} \frac{\langle \tau(g_v)\mathbf{g}_v, \mathbf{g}_v \rangle}{\|\mathbf{g}_v\|^2} \frac{\langle \omega_{\bar{\psi}_v}(g_v)\phi_v, \phi_v \rangle}{\|\phi_v\|^2} dg_v. \tag{42}$$

Then, if one can choose  $\mathbf{h}, \mathbf{g}$  and  $\phi$  so that  $\mathcal{Q}(\mathbf{h}, \mathbf{g}, \phi)$  does not vanish, then one might rewrite (40) as

$$\Lambda(f \otimes \text{Ad}(g), k) = \frac{4L(1, \pi, \text{ad})L(1, \tau, \text{ad})}{\langle \mathbf{h}, \mathbf{h} \rangle \langle \mathbf{g}, \mathbf{g} \rangle \langle \phi, \phi \rangle} \left( \prod_v \mathcal{I}_v^\#(\mathbf{h}, \mathbf{g}, \phi)^{-1} \right) \mathcal{Q}(\mathbf{h}, \mathbf{g}, \phi). \tag{43}$$

By virtue of this identity, our proof of Theorem 1.1 consists essentially in choosing an appropriate *test vector*  $\mathbf{h} \otimes \mathbf{g} \otimes \phi \in \tilde{\pi} \otimes \tau \otimes \omega_{\bar{\psi}}$  for which the right hand side of (43) does not vanish. Computing the regularized local periods  $\mathcal{I}_v^\#(\mathbf{h}, \mathbf{g}, \phi)$  and translating the global period  $\mathcal{Q}(\mathbf{h}, \mathbf{g}, \phi)$  into classical terms, leads eventually to the desired explicit central value formula.

**THEOREM 6.4.** *Let  $k, N \geq 1$  be odd integers. Let  $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$  and  $g \in S_{k+1}^{\text{new}}(\Gamma_0(N), \chi)$  be normalized newforms, and assume (SF), (H1), and (H2). If  $h \in S_{k+1/2}^{+, \text{new}}(4NM, \chi; f \otimes \chi)$  and  $F_\chi$  denote a Shimura lift of  $f$  as above and its Saito–Kurokawa lift, then*

$$\Lambda(f \otimes \text{Ad}(g), k) = 2^{k+1-\nu(M)} C(N, M, \chi) \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{|\langle (\text{id} \otimes U_M)F_\chi|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2}{\langle g, g \rangle^2}, \tag{44}$$

where  $\nu(M)$  denotes the number of prime divisors of  $M$ , and

$$C(N, M, \chi) = |\chi(2)|^{-2} M^{3-k} N^{-1} \prod_{p|N} (p+1)^2 \prod_{p|M} (p+1).$$

*Proof.* Suppose first that  $\Lambda(f \otimes \text{Ad}(g), k) \neq 0$ . By Proposition 6.3, the functional period  $\mathcal{Q}$  does not vanish, and so we might use (43) for a suitable choice of *test vector*. Keeping the notations as above, consider the pure tensor

$$\mathbf{h} \otimes \check{\mathbf{g}} \otimes \phi \in \tilde{\pi} \otimes \tau \otimes \omega_{\check{\psi}},$$

where  $\mathbf{h} \in \tilde{\mathcal{A}}_{k+1/2}(4NM, \chi_0)$  is the adelization of  $h$ ,  $\check{\mathbf{g}} = V_M \mathbf{g}^\# \in \tau$  is the automorphic cusp form obtained from the adelization  $\mathbf{g} \in \tau$  of  $g$  as in Section 5.1, and  $\phi \in \omega_{\check{\psi}}$  is the one-dimensional Bruhat–Schwartz function determined by requiring that  $\phi_q = \mathbf{1}_{\mathbb{Z}_q}$  at all finite primes  $p$ , and  $\phi_\infty(x) = e^{-2\pi x^2}$  for all  $x \in \mathbb{R}$ .

With this choice, we have

$$\langle \mathbf{h}, \mathbf{h} \rangle = \frac{1}{2\zeta_{\mathbb{Q}}(2)} \langle h, h \rangle, \quad \langle \check{\mathbf{g}}, \check{\mathbf{g}} \rangle = \langle \mathbf{g}, \mathbf{g} \rangle = \frac{1}{\zeta_{\mathbb{Q}}(2)} \langle g, g \rangle, \quad \langle \phi, \phi \rangle = \|\phi_\infty\|^2 = \frac{1}{2}.$$

And by [Hid00, Theorem 5.15], [Wat02, §3.2.1], it is known that

$$\begin{aligned} L(1, \pi, \text{ad}) &= 2^{2k} N^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \langle f, f \rangle, \\ L(1, \tau, \text{ad}) &= 2^{k+1} N^{-1} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \langle g, g \rangle. \end{aligned}$$

Therefore, the first factor on the right hand side of (43) for our choice of test vector reads

$$\frac{4L(1, \pi, \text{ad})L(1, \tau, \text{ad})}{\langle \mathbf{h}, \mathbf{h} \rangle \langle \mathbf{g}, \mathbf{g} \rangle \langle \phi, \phi \rangle} = \frac{2^{3k+5} \zeta_{\mathbb{Q}}(2)^2 [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^2 \langle f, f \rangle}{N^2 \langle h, h \rangle}.$$

Because of our choice of  $\mathbf{h}$ ,  $\check{\mathbf{g}}$ , and  $\phi$ , it follows from [Qiu14, Lemma 4.4] that  $\mathcal{I}_q^\#(\mathbf{h}, \check{\mathbf{g}}, \phi) = 1$  for all finite primes  $q \nmid 2N$  (notice that our choice for the local measure  $dg_q$  on  $\text{SL}_2(\mathbb{Q}_q)$  is different to Qiu’s choice, but we have modified accordingly the definition of the local periods  $\mathcal{I}_v$  after (40), cf. Remark 6.1). In the next sections we will compute the regularized local periods at the remaining places: from Propositions 7.15, 8.19, 9.2, and 9.4, we have:

$$\mathcal{I}_v^\#(\mathbf{h}, \check{\mathbf{g}}, \phi)^{-1} = \begin{cases} p & \text{if } v = p, p \mid N/M, \\ \frac{p(p+1)}{2} & \text{if } v = p, p \mid M, \\ 1 & \text{if } v = 2 \text{ or } v = \infty. \end{cases}$$

Therefore, in (43) we have

$$\mathcal{I} := \prod_v \mathcal{I}_v^\#(\mathbf{h}, \check{\mathbf{g}}, \phi)^{-1} = N \prod_{p \mid M} \frac{(p+1)}{2} = 2^{-\nu(M)} N \prod_{p \mid M} (p+1).$$

Finally, it remains to deal with the global  $\text{SL}_2$ -period  $\mathcal{Q}(\mathbf{h}, \check{\mathbf{g}}, \phi)$ . Recall from Sections 5.1 and 5.3 that we have associated Bruhat–Schwartz functions  $\phi_{\check{\mathbf{g}}} \in$

$\mathcal{S}(V_4(\mathbb{A}))$  and  $\phi_{\mathbf{h}} = \phi \otimes \phi_{\check{\mathbf{g}}} \in \mathcal{S}(V_5(\mathbb{A}))$  to  $\check{\mathbf{g}}$  and  $\mathbf{h}$ . By virtue of Proposition 5.1, we have

$$\theta(\mathbf{Y}_M \mathbf{G}, \phi_{\check{\mathbf{g}}}) = C_1 \check{\mathbf{g}},$$

where  $C_1 = 2^{k-1} M^{-1} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-2} \langle g, g \rangle$ , and  $\mathbf{Y}_M \mathbf{G} \in \Upsilon = \Theta(\tau)$  is an automorphic form for  $\mathrm{GSO}_{2,2}$  whose restriction to  $\mathrm{GL}_2 \times \mathrm{GL}_2$  coincides with  $\mathbf{g} \otimes \mathbf{V}_M \mathbf{g} \in \tau \boxtimes \tau$ . Write  $\Pi$  for the automorphic representation of  $\mathrm{PGSp}_2(\mathbb{A})$  associated with  $\theta(\mathbf{h}, \phi_{\mathbf{h}})$ . It follows from the proof of [Qiu14, Theorem 5.3] that

$$\mathcal{Q}(\mathbf{h}, \check{\mathbf{g}}, \phi) = C_1^{-2} \mathcal{P}(\theta(\mathbf{h}, \phi_{\mathbf{h}}), \mathbf{Y}_M \mathbf{G}),$$

where on the right hand side  $\mathcal{P} : \Pi \otimes \Pi \otimes \Upsilon \otimes \Upsilon \rightarrow \mathbb{C}$  is an  $\mathrm{SO}(V_4)$ -period defined by associating to any choice of decomposable vectors  $\mathbf{F}_1, \mathbf{F}_2 \in \Pi, \mathbf{G}_1, \mathbf{G}_2 \in \Upsilon$  the value

$$\mathcal{P}(\mathbf{F}_1, \mathbf{F}_2, \mathbf{G}_1, \mathbf{G}_2) := \int_{[\mathrm{SO}(V_4)]} \mathbf{F}_1(h) \overline{\mathbf{G}_1(h)} dh \cdot \int_{[\mathrm{SO}(V_4)]} \overline{\mathbf{F}_2(h)} \mathbf{G}_2(h) dh,$$

and we have abbreviated

$$\mathcal{P}(\theta(\mathbf{h}, \phi_{\mathbf{h}}), \mathbf{Y}_M \mathbf{G}) = \mathcal{P}(\theta(\mathbf{h}, \phi_{\mathbf{h}}), \theta(\mathbf{h}, \phi_{\mathbf{h}}), \mathbf{Y}_M \mathbf{G}, \mathbf{Y}_M \mathbf{G})$$

(cf. [Qiu14, Section 5]). Let  $\Pi_{\underline{\chi}} = \Pi \otimes \underline{\chi}$  be the automorphic representation of  $\mathrm{GSp}_2(\mathbb{A})$  associated with  $\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}$ . Since the similitude morphism is trivial on  $\mathrm{SO}(V_4)$ , if one defines a period functional

$$\mathcal{P}_{\underline{\chi}} : \Pi_{\underline{\chi}} \otimes \Pi_{\underline{\chi}} \otimes \Upsilon \otimes \Upsilon \longrightarrow \mathbb{C}$$

by the same recipe as for  $\mathcal{P}$ , then one has  $\mathcal{P}(\mathbf{F}_1, \mathbf{F}_2, \mathbf{G}_1, \mathbf{G}_2) = \mathcal{P}_{\underline{\chi}}(\mathbf{F}_1 \otimes \underline{\chi}, \mathbf{F}_2 \otimes \underline{\chi}, \mathbf{G}_1, \mathbf{G}_2)$  for all decomposable vectors  $\mathbf{F}_1, \mathbf{F}_2 \in \Pi, \mathbf{G}_1, \mathbf{G}_2 \in \Upsilon$ . In particular,

$$\mathcal{Q}(\mathbf{h}, \check{\mathbf{g}}, \phi) = C_1^{-2} \mathcal{P}_{\underline{\chi}}(\theta(\mathbf{h}, \phi_{\mathbf{h}}) \otimes \underline{\chi}, \mathbf{Y}_M \mathbf{G}),$$

and thanks to Proposition 5.10 we deduce that

$$\mathcal{Q}(\mathbf{h}, \check{\mathbf{g}}, \phi) = C_1^{-2} |C_2|^2 \mathcal{P}_{\underline{\chi}}(\mathfrak{A}_M \mathbf{F}_{\underline{\chi}}, \mathbf{Y}_M \mathbf{G}),$$

where  $C_2 = 2^{-2} \chi(2)^{-1} M^{-1} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1} \zeta_{\mathbb{Q}}(2)^{-1}$ . Now, when restricted to  $\mathrm{SO}(V_4)$ ,  $\mathfrak{A}_M \mathbf{F}_{\underline{\chi}}$  coincides with the adelicization of  $\mathfrak{A}_M F_{\chi|_{\mathcal{H} \times \mathcal{H}}}$ , which in turn equals  $(\mathrm{id} \otimes V_M U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}$ . Besides,  $\mathbf{Y}_M \mathbf{G}$  restricted to  $\mathrm{GL}_2 \times \mathrm{GL}_2$  is the adelicization of  $M^{(k+1)/2-1} g \times V_M g$ . Therefore, we have

$$\mathcal{P}_{\underline{\chi}}(\mathfrak{A}_M \mathbf{F}_{\underline{\chi}}, \mathbf{Y}_M \mathbf{G}) = C_3^2 M^{k-1} | \langle (\mathrm{id} \otimes V_M U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, (\mathrm{id} \otimes V_M) g \times g \rangle |^2,$$

where  $C_3 = 2^{-1} \zeta_{\mathbb{Q}}(2)^{-2}$  (cf. [III10, Section 9]). Furthermore, one can easily check that

$$\langle (\mathrm{id} \otimes V_M U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, (\mathrm{id} \otimes V_M) g \times g \rangle = M^{2-k} \langle (\mathrm{id} \otimes U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, g \times g \rangle,$$

where now the right hand side makes sense as a Petersson product with respect to  $\Gamma_0(NM) \times \Gamma_0(NM)$ . Therefore,

$$\mathcal{P}_\chi(\mathfrak{A}_M \mathbf{F}_\chi, \mathbf{Y}_M \mathbf{G}) = C_3^2 M^{3-k} | \langle (\text{id} \otimes U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, g \times g \rangle |^2.$$

Altogether, we have

$$\Lambda(f \otimes \text{Ad}(g), k) = \mathcal{C} \cdot \mathcal{I} \cdot \frac{\langle f, f \rangle}{\langle h, h \rangle} | \langle (\text{id} \otimes U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, g \times g \rangle |^2,$$

where we put  $\mathcal{C} = 2^{3k+5} \zeta_{\mathbb{Q}}(2)^2 [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^2 M^{3-k} N^{-2} |C_2|^2 C_3^2 C_1^{-2}$ . Plugging the values of the constants, and of the product of local periods  $\mathcal{I}$ , one concludes that

$$\Lambda(f \otimes \text{Ad}(g), k) = 2^{k+1-\nu(M)} C(N, M, \chi) \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{| \langle (\text{id} \otimes U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, g \times g \rangle |^2}{\langle g, g \rangle^2},$$

where  $C(N, M, \chi)$  is as in the statement.

Finally, when  $\Lambda(f \otimes \text{Ad}(g), k) = 0$  the global functional  $\mathcal{Q}$  vanishes by Proposition 6.3. In particular,  $\mathcal{Q}(\mathbf{h}, \mathbf{g}, \mathbf{\phi}) = 0$ , and this implies that  $\langle (\text{id} \otimes U_M) F_{\chi|_{\mathcal{H} \times \mathcal{H}}}, g \times g \rangle = 0$ . Thus we see that the formula in the statement holds trivially in this case.  $\square$

An immediate application of Theorem 6.4 is the following algebraicity result, predicted by Deligne’s conjecture, in which  $c^+(f)$  denotes the period associated with  $f$  by Shimura as in [Shi77].

COROLLARY 6.5. *Let  $f$  and  $g$  be as in Theorem 6.4. If  $\sigma \in \text{Aut}(\mathbb{C})$ , then*

$$\left( \frac{\Lambda(f \otimes \text{Ad}(g), k)}{\langle g, g \rangle^2 c^+(f)} \right)^\sigma = \frac{\Lambda(f^\sigma \otimes \text{Ad}(g^\sigma), k)}{\langle g^\sigma, g^\sigma \rangle^2 c^+(f^\sigma)}.$$

*In particular, if  $\mathbb{Q}(f, g)$  denotes the number field generated by the Fourier coefficients of  $f$  and  $g$ , then*

$$\Lambda(f \otimes \text{Ad}(g), k)^{\text{alg}} := \frac{\Lambda(f \otimes \text{Ad}(g), k)}{\langle g, g \rangle^2 c^+(f)} \in \mathbb{Q}(f, g).$$

*Proof.* First of all, we may assume that the Fourier coefficients of  $h$ , and hence of  $F_\chi$ , belong to the number field  $\mathbb{Q}(f, \chi)$  generated by the Fourier coefficients of  $f$  together with the values of  $\chi$ . This is either a totally real field or a CM field.

Choose a fundamental discriminant  $D < 0$ ,  $D \in \mathfrak{D}(N, M)$ , with  $L(f, D, k) \neq 0$ . By Theorem 2.1 (see also Remark 2.2), if  $c_h(|D|)$  denotes the  $|D|$ -th Fourier coefficient of  $h$ , then

$$\frac{\langle f, f \rangle}{\langle h, h \rangle} = 2^{k-1+\nu(N)} \left( \prod_{p|M} \frac{p}{p+1} \right) \frac{|D|^{k-1/2} \Lambda(f, D, k)}{|c_h(|D|)|^2}.$$

Combined with the central value formula in Theorem 6.4,

$$\Lambda(f \otimes \text{Ad}(g), k)^{\text{alg}} = 2^{2k+\nu(N/M)} \tilde{C}(N, M, \chi) \frac{|D|^{k-1/2} \Lambda(f, D, k)}{c^+(f) |c_h(|D|)|^2} \times \frac{|\langle (\text{id} \otimes U_M) F_\chi|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2}{\langle g, g \rangle^4},$$

where  $\tilde{C}(N, M, \chi) = C(N, M, \chi) \prod_{p|M} \frac{p}{p+1}$ . Now fix  $\sigma \in \text{Aut}(\mathbb{C})$ . One has  $c_h(|D|)^\sigma = c_{h^\sigma}(|D|)$ , and hence by the properties of the period  $c^+(f)$  we have

$$\left( \frac{|D|^{k-1/2} \Lambda(f, D, k)}{c^+(f) c_h(|D|)^2} \right)^\sigma = \frac{|D|^{k-1/2} \Lambda(f^\sigma, D, k)}{c^+(f^\sigma) c_{h^\sigma}(|D|)^2}.$$

Besides,  $F_\chi^\sigma$  is the Saito–Kurokawa lift of  $h^\sigma$  and

$$\left( \frac{|\langle (\text{id} \otimes U_M) F_\chi|_{\mathcal{H} \times \mathcal{H}}, g \times g \rangle|^2}{\langle g, g \rangle^4} \right)^\sigma = \frac{|\langle (\text{id} \otimes U_M) F_\chi^\sigma|_{\mathcal{H} \times \mathcal{H}}, g^\sigma \times g^\sigma \rangle|^2}{\langle g^\sigma, g^\sigma \rangle^4},$$

so the statement follows. □

### 7 COMPUTATION OF LOCAL PERIODS AT PRIMES $p \mid N/M$

This section is devoted to compute the regularized local periods  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi})$  at primes  $p \mid N/M$ . First of all, we will describe the local components  $\check{\mathbf{g}}_p$  and  $\mathbf{h}_p$  (up to scalar multiple), according to the local types of the representations  $\tau$  and  $\tilde{\pi}$  at such primes. Then we will compute the matrix coefficients  $\langle \tau_p(g) \check{\mathbf{g}}_p, \check{\mathbf{g}}_p \rangle$  and  $\langle \tilde{\pi}_p(g) \mathbf{h}_p, \mathbf{h}_p \rangle$ , for  $g \in \text{SL}_2(\mathbb{Q}_p)$ , which together with the Weil parings  $\langle \omega_{\psi_p}^-(g) \boldsymbol{\phi}_p, \boldsymbol{\phi}_p \rangle$  will lead to the determination of  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi})$ . Thus let us fix through all this section a prime factor  $p$  of  $N/M$ , and let  $\psi_p$  denote the  $p$ -th component of the standard additive character  $\psi : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{C}^\times$ .

For the  $\text{GL}_2$  case,  $\tau_p$  a twist of the Steinberg representation  $\text{St}_p$  by some unramified quadratic character  $\xi : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ . That is to say, it is the unique irreducible subrepresentation of the induced representation  $\pi(\xi|\cdot|_p^{1/2}, \xi|\cdot|_p^{-1/2})$ . The representation  $\pi(\xi|\cdot|_p^{1/2}, \xi|\cdot|_p^{-1/2})$  is realized as the space of all locally constant functions  $\varphi : \text{GL}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  satisfying the transformation property

$$\varphi \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x \right) = \xi(ad) \left| \frac{a}{d} \right|_p \varphi(x) \quad \text{for all } a, d \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p, x \in \text{GL}_2(\mathbb{Q}_p), \quad (45)$$

and the subspace corresponding to  $\tau_p$  is that of such functions which, in addition, satisfy a certain vanishing condition.

To describe  $\check{\mathbf{g}}_p \in \tau_p$ , notice first of all that  $\check{\mathbf{g}}_p = \mathbf{g}_p$  because  $p \nmid M$ . Therefore,  $\check{\mathbf{g}}_p = \mathbf{g}_p$  belongs to the space  $\tau_p^{K_0}$  of  $K_0$ -fixed vectors, where we abbreviate

$$K_0 = K_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p} \right\}.$$

The space  $\tau_p^{K_0}$  is well-known to be one-dimensional. When replacing  $\mathbf{g}_p$  by a scalar multiple, the ratio  $\langle \tau_p(g)\check{\mathbf{g}}_p, \check{\mathbf{g}}_p \rangle / \|\mathbf{g}_p\|^2$  remains invariant, so we can freely choose a new vector in  $\tau_p^{K_0}$  and suppose that  $\mathbf{g}_p$  coincides with such choice. Following [Sch02, Section 2.1], we can choose  $\mathbf{g}_p : \mathrm{GL}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  in the induced model to be the local vector characterized by the property that

$$\mathbf{g}_p|_{\mathrm{GL}_2(\mathbb{Z}_p)} = \mathbf{1}_{K_0} - \frac{1}{p}\mathbf{1}_{K_0 w K_0}, \tag{46}$$

where  $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\mathbf{1}_X$  denotes the characteristic function of  $X \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ . Thanks to the Iwasawa decomposition for  $\mathrm{GL}_2(\mathbb{Q}_p)$ , this together with the rule (45) determines uniquely  $\mathbf{g}_p$ . Notice that being  $\xi : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$  unramified means that  $\xi(a) = 1$  for all  $a \in \mathbb{Z}_p^\times$ , hence  $\xi$  is completely determined by the value  $\xi(p)$ . Since  $\xi$  is quadratic, we have  $\xi(p) = \pm 1$ , and it is well-known that  $\xi(p) = -1$  (resp.  $+1$ ) if and only if the local root number (or the Atkin–Lehner eigenvalue) of  $\tau_p$  is 1 (resp.  $-1$ ).

Now we move to the case of  $\widetilde{\mathrm{SL}}_2$ . Recall from Section 5.2.1 that  $\tilde{\pi}_p$  is the special representation  $\tilde{\sigma}^\delta(\overline{\psi}_p^D)$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ , where  $\delta \in \mathbb{Z}_p^\times$  is any non-square unit, and  $D \in \mathbb{Z}_p^\times$  satisfies  $(\frac{D}{p}) = w_p = \varepsilon(1/2, \pi_p)$ . In order to lighten the notation, we will write from now on  $\psi := \overline{\psi}_p^D$ . As explained in Section 5.2.1, the representation space of  $\tilde{\pi}_p$  is then the space of locally constant functions  $\tilde{\varphi} : \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  such that

$$\tilde{\varphi} \left( \left[ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}, \epsilon \right] g \right) = \epsilon \chi_\psi(a) \mu(a) |a|_p \tilde{\varphi}(g) = \epsilon \chi_\psi(a) \chi_\delta(a) |a|_p^{3/2} \tilde{\varphi}(g) \tag{47}$$

for all  $g \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  and  $a \in \mathbb{Q}_p^\times$ , together with a certain vanishing condition that we will not need here. Notice that  $\chi_\delta$  is the unique non-trivial quadratic character of  $\mathbb{Q}_p^\times$  hence its restriction to  $\mathbb{Z}_p^\times$  is trivial, and  $\chi_\delta(p) = -1$ .

In order to describe  $\mathbf{h}_p$ , recall first that  $\mathrm{SL}_2(\mathbb{Z}_p)$  embeds into  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  by  $g \mapsto [g, s_p(g)]$ , and let  $\tilde{\Gamma}_0$  denote the image of  $\Gamma_0$  in  $\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)$  under this embedding. Then let  $\mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)}$  be the (genuine) function on  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  which sends  $[g, \epsilon]$  to 0 if  $g \notin \mathrm{SL}_2(\mathbb{Z}_p)$  and to  $\epsilon s_p(g)$  otherwise (thus it takes value 1 if  $[g, \epsilon]$  lies in the image of  $\mathrm{SL}_2(\mathbb{Z}_p)$ , and  $-1$  if  $g \in \mathrm{SL}_2(\mathbb{Z}_p)$  but  $s_p(g) = -\epsilon$ ). Similarly, let  $\mathbf{1}_{\tilde{\Gamma}_0}$  be the function on  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  which sends  $[g, \epsilon]$  to 0 if  $g \notin \Gamma_0$  and to  $\mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)}([g, \epsilon])$  otherwise. With these notations, the following is proved in [BM07, Lemma 8.3].

LEMMA 7.1. *The space of  $\tilde{\Gamma}_0$ -fixed vectors in  $\tilde{\pi}_p$  is one-dimensional, and a new vector generating such space is given by the function  $\tilde{\varphi}_p : \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  whose restriction to  $\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)$  equals  $\mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)} - (p + 1)\mathbf{1}_{\tilde{\Gamma}_0}$ .*

The condition in the statement determines completely  $\tilde{\varphi}_p$ , thanks to the Iwasawa decomposition of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  (which is lifted from that of  $\mathrm{SL}_2(\mathbb{Q}_p)$ ). The  $p$ -th component  $\mathbf{h}_p$  of the adelization  $\mathbf{h}$  of the half-integral weight modular form  $h$  is



therefore a scalar multiple of  $\tilde{\varphi}_p$ . Since  $\langle \tilde{\pi}_p(g)\mathbf{h}_p, \mathbf{h}_p \rangle / \|\mathbf{h}_p\|^2$  is invariant under replacing  $\mathbf{h}_p$  by a scalar multiple of it, we may assume that  $\mathbf{h}_p = \tilde{\varphi}_p$ .

LEMMA 7.2. *For the above choice of  $\mathbf{h}_p$ , we have  $\|\mathbf{h}_p\|^2 = p^{-1}(p^2 - 1)$ .*

*Proof.* This is a straightforward computation. Indeed, when restricted to  $\mathrm{SL}_2(\mathbb{Z}_p)$  and  $\Gamma_0$  the functions  $\mathbf{1}_{\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)}$  and  $\mathbf{1}_{\widetilde{\Gamma}_0}$  become the characteristic functions of  $\mathrm{SL}_2(\mathbb{Z}_p)$  and  $\Gamma_0$ , respectively. Therefore,

$$\begin{aligned} \|\mathbf{h}_p\|^2 &= \int_{\mathrm{SL}_2(\mathbb{Z}_p)} \mathbf{h}_p(h) \overline{\mathbf{h}_p(h)} dh = \int_{\mathrm{SL}_2(\mathbb{Z}_p) \setminus \Gamma_0} dh + p^2 \int_{\Gamma_0} dh = \\ &= \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p) - \Gamma_0) + p^2 \mathrm{vol}(\Gamma_0). \end{aligned}$$

Since our measure is normalized so that  $\mathrm{SL}_2(\mathbb{Z}_p)$  has volume  $\zeta_p(2)^{-1} = (1 - p^{-1})(1 + p^{-1})$ , and  $\Gamma_0$  has index  $p + 1$  in  $\mathrm{SL}_2(\mathbb{Z}_p)$ , it follows easily that  $\mathrm{vol}(\Gamma_0) = p^{-1}(1 - p^{-1})$  and  $\|\mathbf{h}_p\|^2 = p^{-1}(p^2 - 1)$ .  $\square$

Finally, for simplicity we will write  $\omega_p = \omega_{\tilde{\psi}_p}$  for the local Weil representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  acting on the space of Bruhat–Schwartz functions  $\mathcal{S}(\mathbb{Q}_p)$ , with respect to the character  $\tilde{\psi}_p = \psi_p^{-1}$ . Here,  $\mathbb{Q}_p$  is to be regarded as the one-dimensional quadratic space endowed with the bilinear form  $(x, y) = 2xy$ . In our choice of test vector, the  $p$ -th component of  $\phi$  is  $\phi_p = \mathbf{1}_{\mathbb{Z}_p}$ , the characteristic function of  $\mathbb{Z}_p$ . It is easily checked that  $\phi_p$  is invariant under the action of  $\mathrm{SL}_2(\mathbb{Z}_p)$ . In particular, it is also invariant under  $\Gamma_0$ .

Having described the local components  $\mathfrak{g}_p \in \tau_p$ ,  $\mathbf{h}_p \in \tilde{\pi}_p$ , and  $\phi_p \in \mathcal{S}(\mathbb{Q}_p)$ , observe that the three of them are invariant under the action of  $\Gamma_0$ . It thus follows that for any  $g \in \mathrm{SL}_2(\mathbb{Q}_p)$  the values

$$\Phi_{\check{\mathfrak{g}}_p}(g) := \frac{\langle \tau_p(g)\check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle}{\|\check{\mathfrak{g}}_p\|^2}, \quad \Phi_{\mathbf{h}_p}(g) := \frac{\langle \tilde{\pi}_p(g)\mathbf{h}_p, \mathbf{h}_p \rangle}{\|\mathbf{h}_p\|^2}, \quad \Phi_{\phi_p}(g) := \frac{\langle \omega_p(g)\phi_p, \phi_p \rangle}{\|\phi_p\|^2}$$

depend only on the double coset  $\Gamma_0 g \Gamma_0$ . Thus we only need to compute these values for  $g$  varying in a set of representatives for the double cosets for  $\Gamma_0$  in  $\mathrm{SL}_2(\mathbb{Q}_p)$ . Define elements  $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Q}_p)$  by

$$\alpha := \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}, \quad \beta := s\alpha = \begin{pmatrix} 0 & p^{-1} \\ -p & 0 \end{pmatrix},$$

where  $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Then the Cartan decomposition for  $\mathrm{SL}_2(\mathbb{Q}_p)$  relative to the maximal compact open subgroup  $\mathrm{SL}_2(\mathbb{Z}_p)$  gives

$$G = \bigsqcup_{n \geq 0} \mathrm{SL}_2(\mathbb{Z}_p)\alpha_n \mathrm{SL}_2(\mathbb{Z}_p) = \bigsqcup_{n \geq 0} \mathrm{SL}_2(\mathbb{Z}_p)\alpha_{-n} \mathrm{SL}_2(\mathbb{Z}_p),$$

where we put  $\alpha_n := \alpha^n$  for any integer  $n$ . Combining this with the so-called Bruhat decomposition for  $\mathrm{SL}_2$  over the residue field  $\mathbb{F}_p$ , one obtains also a decomposition for  $\mathrm{SL}_2(\mathbb{Q}_p)$  in terms of  $\Gamma_0$ :

$$\mathrm{SL}_2(\mathbb{Q}_p) = \bigsqcup_{n \in \mathbb{Z}} \Gamma_0 \alpha_n \Gamma_0 \sqcup \bigsqcup_{m \in \mathbb{Z}} \Gamma_0 \beta_m \Gamma_0. \tag{48}$$

where  $\beta_m := s\alpha_m = s\alpha^m$ . By virtue of (48) and our above observation, it will be enough to compute the values  $\Phi_{\mathfrak{g}_p}(g)$ ,  $\Phi_{\mathfrak{h}_p}(g)$ , and  $\Phi_{\mathfrak{f}_p}(g)$  for  $g \in \{\alpha_n, \beta_m : n, m \in \mathbb{Z}\}$ . We will need the volumes of the double cosets  $\Gamma_0\alpha_n\Gamma_0$  and  $\Gamma_0\beta_m\Gamma_0$ , hence we collect them in the following lemma for ease of reference.

LEMMA 7.3. *Keep the same notation as above.*

i)  $\text{vol}(\Gamma_0\alpha_0\Gamma_0) = \text{vol}(\Gamma_0) = p^{-1}(1 - p^{-1})$ , and for  $n \neq 0$ , one has

$$\text{vol}(\Gamma_0\alpha_n\Gamma_0) = \begin{cases} p^{2n-1}(1 - p^{-1}) & \text{if } n > 0, \\ p^{-2n-1}(1 - p^{-1}) & \text{if } n < 0. \end{cases}$$

ii)  $\text{vol}(\Gamma_0\beta_0\Gamma_0) = \text{vol}(\Gamma_0s\Gamma_0) = (1 - p^{-1})$ , and for  $m \neq 0$ , one has

$$\text{vol}(\Gamma_0\beta_m\Gamma_0) = \begin{cases} p^{2m-2}(1 - p^{-1}) & \text{if } m > 0, \\ p^{-2m}(1 - p^{-1}) & \text{if } m < 0. \end{cases}$$

*Proof.* For each  $g \in \{\alpha_n, \beta_m : n, m \in \mathbb{Z}\}$ , one writes the double coset  $\Gamma_0g\Gamma_0$  as a disjoint union of finitely many cosets  $\Gamma_0g_i$ . Then the volume of  $\Gamma_0g\Gamma_0$  equals the volume of  $\Gamma_0$  multiplied by the number of coset representatives  $g_i$ . We omit the details.  $\square$

## 7.1 THE $\text{GL}_2$ CASE

In the case of matrix coefficients for  $\text{GL}_2$ , the values  $\Phi_{\mathfrak{g}_p}(g)$  for  $g \in \text{SL}_2(\mathbb{Q}_p)$  can be easily deduced from the results in [Woo, §3]. We explain briefly how to get such values.

PROPOSITION 7.4. *For an integer  $n$ , we have  $\Phi_{\mathfrak{g}_p}(\alpha_n) = p^{-2|n|}$ .*

*Proof.* Suppose that  $n \geq 0$ , and notice first of all that, as elements of  $\text{GL}_2(\mathbb{Q}_p)$ , we have

$$\alpha_n = p^{-n}\rho_{2n}, \quad \text{where } \rho_{2n} = \begin{pmatrix} p^{2n} & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, it follows that  $\Phi_{\mathfrak{g}_p}(\alpha_n) = \xi(p^{-n})^2\Phi_{\mathfrak{g}_p}(\rho_{2n}) = \xi(p)^{-2n}\Phi_{\mathfrak{g}_p}(\rho_{2n})$ . According to [Woo, Proposition 3.8], for  $n \geq 1$  we have  $\Phi_{\mathfrak{g}_p}(\rho_{2n}) = \xi(p)^{2n}p^{-2n}$ , hence combining the last two identities we find out that  $\Phi_{\mathfrak{g}_p}(\alpha_n) = p^{-2n}$ . For  $n = 0$ ,  $\alpha_0 = \text{Id}_2$  and we trivially find  $\Phi_{\mathfrak{g}_p}(\text{Id}_2) = 1$ . Thus the statement holds as well. When  $n < 0$ , we can proceed similarly. Indeed, we now have the identity

$$\alpha_n = p^n w \rho_{-2n} w, \quad \text{where } w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that  $\Phi_{\mathfrak{g}_p}(\alpha_n) = \xi(p)^{2n}\Phi_{\mathfrak{g}_p}(w\rho_{-2n}w)$ . Now  $-2n > 0$ , and again by [Woo, Proposition 3.8]  $\Phi_{\mathfrak{g}_p}(w\rho_{-2n}w) = \xi(p)^{-2n}p^{2n}$ . Altogether, we conclude as desired that  $\Phi_{\mathfrak{g}_p}(\alpha_n) = p^{2n}$ .  $\square$

The next lemma deals with the case  $g = \beta_m = s\alpha_m$ , for  $m \in \mathbb{Z}$ .

PROPOSITION 7.5. *For an integer  $m$ , we have  $\Phi_{\mathfrak{g}_p}(\beta_m) = -p^{-|2m-1|}$ .*

*Proof.* First suppose that  $m > 0$ , and observe that

$$\beta_m = \begin{pmatrix} 0 & p^{-m} \\ -p^m & 0 \end{pmatrix} = p^{-m} \begin{pmatrix} 0 & 1 \\ p^{2m} & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since  $\tau_p$  is invariant by the rightmost element, we have  $\Phi_{\mathfrak{g}_p}(\beta_m) = \xi(p)^{-2m} \Phi_{\mathfrak{g}_p}(w\rho_{2m})$ . Again by [Woo, Proposition 3.8],  $\Phi_{\mathfrak{g}_p}(w\rho_{2m}) = -\xi(p)^{2m} p^{1-2m}$ , hence we conclude that  $\Phi_{\mathfrak{g}_p}(\beta_m) = -p^{-2m+1}$ . When  $m = 0$ , observe that  $\beta_0 = s$ . Since

$$s = w \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and  $\tau_p$  is invariant by the rightmost element, we see that  $\Phi_{\mathfrak{g}_p}(s) = \Phi_{\mathfrak{g}_p}(w)$ . By [Woo, Proposition 3.8],  $\Phi_{\mathfrak{g}_p}(w) = -p^{-1}$ , hence for  $m = 0$  we have  $\Phi_{\mathfrak{g}_p}(\beta_0) = -p^{-1}$ , which fits into the statement. Finally, suppose that  $m < 0$  and write

$$\beta_m = p^m \begin{pmatrix} 0 & p^{-2m} \\ -1 & 0 \end{pmatrix} = p^m \begin{pmatrix} 0 & p^{-2m} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From this,  $\Phi_{\mathfrak{g}_p}(\beta_m) = \xi(p)^{2m} \Phi_{\mathfrak{g}_p}(\rho_{-2m}w)$ . Since  $-2m > 0$ , now [Woo, Proposition 3.8] implies that  $\Phi_{\mathfrak{g}_p}(\rho_{-2m}w) = -\xi(p)^{-2m} p^{2m-1}$ , and hence for  $m < 0$  we conclude as claimed that  $\Phi_{\mathfrak{g}_p}(\beta_m) = -p^{2m-1}$ .  $\square$

### 7.2 THE $\widetilde{\text{SL}}_2$ CASE

For the computation of matrix coefficients in the  $\widetilde{\text{SL}}_2$  case, it will be useful to introduce the following subsets of  $\text{SL}_2(\mathbb{Z}_p)$ . For each integer  $j \geq 0$ , we define

$$\begin{aligned} \mathcal{L}_j &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p) : \text{ord}_p(c) = j \right\}, \\ \mathcal{R}_j &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p) : \text{ord}_p(d) = j \right\}. \end{aligned}$$

Recall that our measure on  $\text{SL}_2(\mathbb{A})$  is chosen so that  $\text{vol}(\text{SL}_2(\mathbb{Z}_p)) = \zeta_p(2)^{-1} = 1 - p^{-2}$ . By expressing the sets  $\mathcal{L}_j$  and  $\mathcal{R}_j$  in terms of the subgroups  $\Gamma_0(p^j) \subseteq \text{SL}_2(\mathbb{Z}_p)$ , for  $j \geq 0$ , one can easily prove:

LEMMA 7.6. *With the above notation,  $\text{vol}(\mathcal{L}_0) = \text{vol}(\mathcal{R}_0) = 1 - p^{-1}$ ,  $\text{vol}(\mathcal{L}_0 \cap \mathcal{R}_0) = (1 - p^{-1})^2$ , and*

$$\text{vol}(\mathcal{L}_j) = \text{vol}(\mathcal{R}_j) = p^{-j}(1 - p^{-1})^2 \text{ for all } j > 0.$$

For the computation of the (normalized) matrix coefficients  $\Phi_{\mathfrak{h}_p}(\alpha_n)$  and  $\Phi_{\mathfrak{h}_p}(\beta_m)$ , recall that  $\widetilde{\text{SL}}_2(\mathbb{Q}_p) = \text{SL}_2(\mathbb{Q}_p) \times \{\pm 1\}$  as sets. The group operation is given by

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1g_2, \epsilon(g_1, g_2)\epsilon_1\epsilon_2],$$

where  $\epsilon(g_1, g_2) = (x(g_1)x(g_1g_2), x(g_2)x(g_1g_2))_p$ , with  $x : \mathrm{SL}_2(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$  being defined as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

Recall that we regard  $\mathrm{SL}_2(\mathbb{Z}_p)$  as a subgroup of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  via the splitting  $k \mapsto [k, s_p(k)]$ , where

$$s_p \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} (c, d)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

And recall also that  $\mathbf{h}_p \in \tilde{\pi}_p$  is the function  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  described in Lemma 7.1, which satisfies the transformation rule spelled out in (47).

### 7.2.1 COMPUTATION OF $\Phi_{\mathbf{h}_p}(\alpha_n)$

Fix throughout this paragraph  $n \in \mathbb{Z}$ , and identify  $\alpha_n$  with the element  $[\alpha_n, 1] \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ . In the computation of  $\Phi_{\mathbf{h}_p}(\alpha_n)$ , we encounter products of the form  $h\alpha_n$ , with  $h \in \mathrm{SL}_2(\mathbb{Z}_p)$  and  $n \in \mathbb{Z}$ . This is to be seen as the product

$$[h, s_p(h)][\alpha_n, 1] = [h\alpha_n, \epsilon] \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),$$

where using the above recipe we have  $\epsilon = s_p(h)\epsilon(h, \alpha_n)$ . The sign  $s_p(h) \in \{\pm 1\}$  is given as above. And by using the definition of  $\epsilon(\cdot, \cdot)$ , we have

$$\epsilon(h, \alpha_n) = (x(h)x(h\alpha_n), x(\alpha_n)x(h\alpha_n))_p.$$

Write

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h\alpha_n = \begin{pmatrix} ap^n & bp^{-n} \\ cp^n & dp^{-n} \end{pmatrix},$$

so that we have

$$s_p(h) = \begin{cases} (c, d)_p & \text{if } cd \neq 0, \mathrm{ord}_p(c) \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases} \quad x(h) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0, \end{cases}$$

$$x(h\alpha_n) = \begin{cases} cp^n & \text{if } c \neq 0, \\ dp^{-n} & \text{if } c = 0. \end{cases}$$

Then one can easily check that

$$\epsilon = \begin{cases} (d, p^n)_p & \text{if } c = 0, \\ (c, dp^n)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ odd,} \\ (c, p^n)_p & \text{otherwise.} \end{cases} \quad (49)$$

To compute  $\Phi_{\mathbf{h}_p}(\alpha_n)$ , we will need to evaluate  $\mathbf{h}_p$  at the element  $[h\alpha_n, \epsilon]$ . And to do so, we need to write down an Iwasawa decomposition of this element, induced from an Iwasawa decomposition for  $h\alpha_n$  in  $\mathrm{SL}_2(\mathbb{Q}_p)$ . The shape of such an Iwasawa decomposition will vary according to whether  $h$  belongs to the subregions  $\mathcal{A}_1(n)$  or  $\mathcal{A}_2(n)$  of  $\mathrm{SL}_2(\mathbb{Z}_p)$  that we now discuss.

i) Let  $\mathcal{A}_1(n) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$  be the subset of  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p)$  with  $|cp^n|_p < |dp^{-n}|_p$ . Equivalently,  $|c|_p < p^{2n}|d|_p$ . Observe that one always has  $d \neq 0$  in  $\mathcal{A}_1(n)$ . If  $h \in \mathcal{A}_1(n)$ , we have

$$h\alpha_n = \begin{pmatrix} ap^n & bp^{-n} \\ cp^n & dp^{-n} \end{pmatrix} = \begin{pmatrix} d^{-1}p^n & * \\ 0 & dp^{-n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ cd^{-1}p^{2n} & 1 \end{pmatrix} =: g_1g_2,$$

where notice that the rightmost element  $g_2$  belongs to  $\Gamma_0$ . This lifts to an identity

$$[h\alpha_n, \epsilon] = \left[ \begin{pmatrix} d^{-1}p^n & * \\ 0 & dp^{-n} \end{pmatrix}, e \right] \left[ \begin{pmatrix} 1 & 0 \\ cd^{-1}p^{2n} & 1 \end{pmatrix}, 1 \right], \tag{50}$$

where  $e = \epsilon(g_1, g_2)\epsilon$ . Since  $g_1g_2 = h\alpha_n$ , we have

$$\epsilon(g_1, g_2) = (x(g_1)x(h\alpha_n), x(g_2)x(h\alpha_n))_p = \begin{cases} (dc, dp^n)_p & \text{if } c \neq 0, \\ 1 & \text{if } c = 0. \end{cases}$$

From our recipe for  $\epsilon$  in (49), noticing that  $d \neq 0$  in  $\mathcal{A}_1(n)$  and using elementary properties of the Hilbert symbol, the above recipe for  $e = e(h)$  when  $h \in \mathcal{A}_1(n)$  gets simplified to

$$e = \begin{cases} (cd, d)_p(d, p^n)_p & \text{if } c \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ even,} \\ (d, p^n)_p & \text{otherwise.} \end{cases} \tag{51}$$

ii) Let  $\mathcal{A}_2(n) \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$  be the subset of elements  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p)$  with  $|cp^n| \geq |dp^{-n}|_p$ , or equivalently,  $|c|_p \geq p^{2n}|d|_p$ . In  $\mathcal{A}_2(n)$ , observe that one always has  $c \neq 0$ . For  $h \in \mathcal{A}_2(n)$ , we have

$$h\alpha_n = \begin{pmatrix} ap^n & bp^{-n} \\ cp^n & dp^{-n} \end{pmatrix} = \begin{pmatrix} c^{-1}p^{-n} & * \\ 0 & cp^n \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & dc^{-1}p^{-2n} \end{pmatrix} =: g_1g_2,$$

where now observe that  $g_2 \in \mathrm{SL}_2(\mathbb{Z}_p) - \Gamma_0$ . This identity lifts now to

$$[h\alpha_n, \epsilon] = \left[ \begin{pmatrix} c^{-1}p^{-n} & * \\ 0 & cp^n \end{pmatrix}, e \right] \left[ \begin{pmatrix} 0 & -1 \\ 1 & dc^{-1}p^{-2n} \end{pmatrix}, 1 \right], \tag{52}$$

again with  $e = \epsilon(g_1, g_2)\epsilon$ . In this case, since  $c \neq 0$  for all  $h \in \mathcal{A}_2(n)$ , we have

$$\epsilon(g_1, g_2) = (x(g_1)x(h\alpha_n), x(g_2)x(h\alpha_n))_p = (c^2p^{2n}, cp^n)_p = 1,$$

and therefore, when  $h \in \mathcal{A}_2(n)$  the value of  $e = e(h)$  is given as follows:

$$e = \epsilon = \begin{cases} (c, d)_p(c, p^n)_p & \text{if } d \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ odd,} \\ (c, p^n)_p & \text{if } d = 0 \text{ or } \mathrm{ord}_p(c) \text{ even.} \end{cases} \tag{53}$$

As we know  $\|\mathbf{h}_p\|^2$  from Lemma 7.2, to compute  $\Phi_{\mathbf{h}_p}(\alpha_n)$  we need to compute  $\langle \tilde{\pi}_p(\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle$ . And since  $\mathcal{A}_1(n)$  and  $\mathcal{A}_2(n)$  are disjoint by definition, and their union is the whole  $\mathrm{SL}_2(\mathbb{Z}_p)$ , we see that  $\langle \tilde{\pi}_p(\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle$  equals

$$A_1(n) + A_2(n) := \int_{\mathcal{A}_1(n)} \mathbf{h}_p(h\alpha_n)\overline{\mathbf{h}_p(h)}dh + \int_{\mathcal{A}_2(n)} \mathbf{h}_p(h\alpha_n)\overline{\mathbf{h}_p(h)}dh.$$

In the remaining of this section we will compute separately  $A_1(n)$  and  $A_2(n)$ . Observe that  $\mathbf{h}_p(h)$  only takes values 1 and  $-p$  for  $h \in \mathrm{SL}_2(\mathbb{Z}_p)$ , hence we have  $\overline{\mathbf{h}_p(h)} = \mathbf{h}_p(h)$  in the above integrals.

COMPUTATION OF  $A_1(n)$

We proceed now with the computation of  $A_1(n) = \int_{\mathcal{A}_1(n)} \mathbf{h}_p(h\alpha_n)\mathbf{h}_p(h)dh$ . Writing  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as usual, by virtue of (50) and (47) we see that

$$A_1(n) = -p^{1-3n/2}(-1)^n\chi_\psi(p^n) \int_{\mathcal{A}_1(n)} (d, p^n)_p\chi_\psi(d)e(h)\chi_\delta(d)|d|_p^{-3/2}\mathbf{h}_p(h)dh,$$

where  $e(h)$  is given by the recipe in (51). Denote by  $\mathcal{I}_1(n)$  the last integral. Defining

$$\begin{aligned} \mathcal{A}_1^+(n) &:= \{h \in \mathcal{A}_1(n) : c \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ even}\}, \\ \mathcal{A}_1^-(n) &:= \{h \in \mathcal{A}_1(n) : c = 0 \text{ or } \mathrm{ord}_p(c) \text{ odd}\}, \end{aligned}$$

we have, according to (51),

$$\begin{aligned} \mathcal{I}_1(n) &= \int_{\mathcal{A}_1^+(n)} (cd, d)_p\chi_\psi(d)\chi_\delta(d)|d|_p^{-3/2}\mathbf{h}_p(h)dh + \\ &\quad + \int_{\mathcal{A}_1^-(n)} \chi_\psi(d)\chi_\delta(d)|d|_p^{-3/2}\mathbf{h}_p(h)dh. \end{aligned}$$

Observe that  $d$  is always a unit in  $\mathcal{A}_1^-(n)$ , thus the above becomes

$$\mathcal{I}_1(n) = \int_{\mathcal{A}_1^+(n)} (cd, d)_p\chi_\psi(d)\chi_\delta(d)|d|_p^{-3/2}\mathbf{h}_p(h)dh + \int_{\mathcal{A}_1^-(n)} \mathbf{h}_p(h)dh.$$

From this expression we will now easily obtain the value of  $A_1(n)$ .

LEMMA 7.7. *With the above notation,*

$$A_1(n) = \begin{cases} p^{-3n/2}(-1)^n\chi_\psi(p^n)(1-p^{-1})(1+p-p^n) & \text{if } n > 0, \\ p^{3n/2}(-1)^n\chi_\psi(p^n)(1-p^{-1})p^{1-n} & \text{if } n \leq 0. \end{cases}$$

*Proof.* Suppose that  $n > 0$ . First of all, observe that  $\mathcal{A}_1^-(n) = \mathcal{A}_1^-(0)$ , which up to a set of measure zero is the disjoint union of the sets  $\mathcal{L}_{2j+1}$  with  $j \geq 0$ .

Therefore we have

$$\begin{aligned} \int_{\mathcal{A}_1^-(n)} \mathbf{h}_p(h)dh &= \sum_{j \geq 0} \int_{\mathcal{L}_{2j+1}} \mathbf{h}_p(h)dh = -p \sum_{j \geq 0} \text{vol}(\mathcal{L}_{2j+1}) = \\ &= -p \sum_{j \geq 0} p^{-2j-1}(1-p^{-1})^2 = -(1-p^{-1})^2 \sum_{j \geq 0} p^{-2j} = \\ &= \frac{-(1-p^{-1})^2}{1-p^{-2}} = \frac{-(1-p^{-1})}{1+p^{-1}} = \frac{1-p}{p+1}. \end{aligned}$$

On the other hand,  $\mathcal{A}_1^+(n)$  is the disjoint union of the sets  $\mathcal{L}_{2j}$  with  $j > 0$ ,  $\mathcal{L}_0 \cap \mathcal{R}_0$ , and  $\mathcal{R}_j$  with  $1 \leq j \leq 2n-1$ . If  $h \in \mathcal{L}_{2j}$  with  $j > 0$ , then  $h \in \Gamma_0$ ,  $\text{ord}_p(c)$  is even, and  $d$  is a unit. Thus

$$\begin{aligned} \int_{\mathcal{L}_{2j}} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h)dh &= -p \int_{\mathcal{L}_{2j}} 1dh = -p \text{vol}(\mathcal{L}_{2j}) = \\ &= -p^{1-2j}(1-p^{-1})^2, \end{aligned}$$

and one deduces that

$$\begin{aligned} \sum_{j > 0} \int_{\mathcal{L}_{2j}} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h)dh &= -p(1-p^{-1})^2 \sum_{j > 0} p^{-2j} = \\ &= -p^{-1} \frac{(1-p^{-1})^2}{(1-p^{-2})} = \frac{1-p}{p(p+1)}. \end{aligned}$$

When  $h \in \mathcal{L}_0 \cap \mathcal{R}_0$ , both  $c$  and  $d$  are units, and  $\mathbf{h}_p(h) = 1$ , hence

$$\int_{\mathcal{L}_0 \cap \mathcal{R}_0} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h)dh = \text{vol}(\mathcal{L}_0 \cap \mathcal{R}_0) = (1-p^{-1})^2.$$

Finally, if  $h \in \mathcal{R}_j$  with  $j > 0$ , we have  $c \in \mathbb{Z}_p^\times$  and  $\text{ord}_p(d) = j$ . Using elementary properties of the quadratic symbol, it follows that

$$\int_{\mathcal{R}_j} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h)dh = (-1)^j \chi_\psi(p^j) p^{3j/2} \int_{\mathcal{R}_j} (cd, p^j)_p dh.$$

If  $j = 2t$  is even, then  $(cd, p^j)_p = 1$  and we deduce that

$$\int_{\mathcal{R}_{2t}} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h)dh = p^{3t} \text{vol}(\mathcal{R}_{2t}) = p^t(1-p^{-1})^2.$$

In contrast, when  $j$  is odd, by applying the automorphism of  $\mathcal{R}_j$  given by conjugation by  $\gamma_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ , with  $u \in \mathbb{Z}_p^\times$  a non-quadratic residue, one sees that the above integral equals the same integral multiplied by  $-1$ . Therefore, it must vanish. As a consequence, we just have to compute

$$\begin{aligned} \sum_{t=1}^{n-1} \int_{\mathcal{R}_{2t}} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h)dh &= (1-p^{-1})^2 \sum_{t=1}^{n-1} p^t = \\ &= (1-p^{-1})^2 \frac{p-p^n}{1-p} = (1-p^{-1})^2 \frac{p^{n-1}-1}{1-p^{-1}} = (1-p^{-1})(p^{n-1}-1). \end{aligned}$$

Summing all the contributions, we find for  $n > 0$

$$\mathcal{I}_1(n) = -(1-p^{-1}) + (1-p^{-1})^2 + (1-p^{-1})(p^{n-1}-1) = p^{-1}(1-p^{-1})(p^n-p-1),$$

and hence  $A_1(n) = p^{-3n/2}(-1)^n \chi_\psi(p^n)(1-p^{-1})(1+p-p^n)$ .

Next assume that  $n \leq 0$ . In this case, one easily checks that  $\mathcal{A}_1^+(n)$  is the disjoint union of the sets  $\mathcal{L}_{2t}$  for  $t > -n$ . Therefore, it follows that the integral

$$\int_{\mathcal{A}_1^+(n)} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h) dh$$

equals (notice that  $d$  is a unit)

$$\begin{aligned} \sum_{t > -n} \int_{\mathcal{L}_{2t}} (cd, d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h) dh &= -p \sum_{t > -n} \text{vol}(\mathcal{L}_{2t}) = \\ &= -p(1-p^{-1})^2 \sum_{t > -n} p^{-2t} = -p^{2n-1} \frac{(1-p^{-1})^2}{1-p^{-2}} = -p^{2n} \frac{p-1}{p(p+1)}. \end{aligned}$$

Besides, one can check that up to a set of measure zero the set  $\mathcal{A}_1^-(n)$  equals the disjoint union of the sets  $\mathcal{L}_{2j+1}$  with  $j \geq -n$ . Therefore, the integral over  $\mathcal{A}_1^-(n)$  in the expression for  $\mathcal{I}_1(n)$  equals

$$\begin{aligned} \sum_{j \geq -n} \int_{\mathcal{L}_{2j+1}} \mathbf{h}_p(h) dh &= -p \sum_{j \geq -n} \text{vol}(\mathcal{L}_{2j+1}) = -p(1-p^{-1})^2 \sum_{j \geq -n} p^{-2j-1} = \\ &= -p^{2n} \frac{(1-p^{-1})^2}{1-p^{-2}} = -p^{2n} \frac{p-1}{p+1}. \end{aligned}$$

Altogether, we find for  $n \leq 0$  that

$$\mathcal{I}_1(n) = -p^{2n} \frac{p-1}{p(p+1)} - p^{2n} \frac{p-1}{p+1} = -p^{2n} \frac{p-1}{p+1} (1+p^{-1}) = -p^{2n-1} (p-1)$$

and therefore  $A_1(n) = p^{-3n/2}(-1)^n \chi_\psi(p^n)(p-1)p^{2n} = p^{3n/2}(-1)^n \chi_\psi(p^n)(1-p^{-1})p^{1-n}$ .  $\square$

#### COMPUTATION OF $A_2(n)$

We now deal with the computation of  $A_2(n) = \int_{\mathcal{A}_2(n)} \mathbf{h}_p(h\alpha_n) \mathbf{h}_p(h) dh$ . Writing  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as usual, by virtue of (52) and (47) we see that

$$A_2(n) = p^{3n/2}(-1)^n \chi_\psi(p^n) \int_{\mathcal{A}_2(n)} (c, p^n)_p \chi_\psi(c) e(h) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh,$$

where  $e(h)$  is given by the recipe in (53). Write  $\mathcal{I}_2(n)$  for the integral on the right hand side. Then define

$$\mathcal{A}_2^+(n) := \{h \in \mathcal{A}_2(n) : d = 0 \text{ or } \text{ord}_p(c) \text{ even}\},$$

$$\mathcal{A}_2^-(n) := \{h \in \mathcal{A}_2(n) : d \neq 0 \text{ and } \text{ord}_p(c) \text{ odd}\},$$



so that  $\mathcal{A}_2 = \mathcal{A}_2^+(n) \sqcup \mathcal{A}_2^-(n)$  and

$$\begin{aligned} \mathcal{I}_2(n) &= \int_{\mathcal{A}_2^+(n)} \chi_\psi(c)\chi_\delta(c)|c|_p^{-3/2}\mathbf{h}_p(h)dh + \\ &\quad + \int_{\mathcal{A}_2^-(n)} (c, d)_p\chi_\psi(c)\chi_\delta(c)|c|_p^{-3/2}\mathbf{h}_p(h)dh. \end{aligned}$$

Concerning the second integral, applying conjugation by  $\gamma_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$  shows that such integral equals itself multiplied by  $-1$ , hence it vanishes. Therefore, we have

$$\mathcal{I}_2(n) = \int_{\mathcal{A}_2^+(n)} \chi_\psi(c)\chi_\delta(c)|c|_p^{-3/2}\mathbf{h}_p(h)dh.$$

Using this expression, we can easily compute  $\mathcal{I}_2(n)$ , and hence  $A_2(n)$ .

LEMMA 7.8. *With the above notation,*

$$A_2(n) = \begin{cases} p^{-3n/2}(-1)^n\chi_\psi(p^n)(1-p^{-1})p^n & \text{if } n > 0, \\ p^{3n/2}(-1)^n\chi_\psi(p^n)(1-p^{-1})(1+p-p^{1-n}) & \text{if } n \leq 0. \end{cases}$$

*Proof.* The proof goes along the same lines of the previous lemma, so we omit the computations. We just point out that when  $n > 0$  (resp.  $n \leq 0$ ) the set  $\mathcal{A}_2^+(n)$  coincides, up to a zero measure set, with the disjoint union of the sets  $\mathcal{R}_j$ , with  $j \geq 2n$  (resp. the sets  $\mathcal{L}_{2t}$ , with  $0 \leq t \leq -n$ ).  $\square$

PROPOSITION 7.9. *With the above notation,  $\Phi_{\mathbf{h}_p}(\alpha_n) = p^{-3|n|/2}(-1)^n\chi_\psi(p^n)$ .*

*Proof.* Recalling that  $\Phi_{\mathbf{h}_p}(\alpha_n) = \langle \tilde{\pi}_p(\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle / \|\mathbf{h}_p\|^2$ , and that  $\|\mathbf{h}_p\|^2 = p^{-1}(p^2 - 1)$  from Lemma 7.2, the statement follows by combining Lemmas 7.7 and 7.8 since  $\langle \tilde{\pi}_p(\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle = A_1(n) + A_2(n)$ .  $\square$

### 7.2.2 COMPUTATION OF $\Phi_{\mathbf{h}_p}(\beta_m)$

Now we proceed with the computation of  $\Phi_{\mathbf{h}_p}(\beta_m)$ . Thus fix from now on an integer  $m$ , and let  $\beta_m = s\alpha_m = \begin{pmatrix} 0 & p^{-m} \\ -p^m & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$ . As before, we identify  $\alpha_m$  with the element  $[\alpha_m, 1] \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ . We will need to evaluate  $\mathbf{h}_p$  at products of the form

$$[hs, s_p(hs)][\alpha_m, 1] = [hs\alpha_m, \epsilon] \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),$$

where  $\epsilon = s_p(hs)\epsilon(hs, \alpha_m)$ . If we write  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then

$$hs = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}, \quad h\beta_m = hs\alpha_m = \begin{pmatrix} -bp^m & ap^{-m} \\ -dp^m & cp^{-m} \end{pmatrix},$$

so that

$$s_p(hs) = \begin{cases} (c, -d)_p & \text{if } cd \neq 0, \text{ord}_p(d) \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases} \quad x(hs) = \begin{cases} -d & \text{if } d \neq 0, \\ c & \text{if } d = 0, \end{cases}$$

$$x(h\beta_m) = \begin{cases} -dp^m & \text{if } d \neq 0, \\ cp^{-m} & \text{if } d = 0. \end{cases}$$

Besides,  $x(\alpha_m) = p^{-m}$ , hence we can compute the sign

$$\epsilon(hs, \alpha_m) = (x(hs)x(h\beta_m), x(\alpha_m)x(h\beta_m))_p$$

as follows. If  $d \neq 0$ , we have  $\epsilon(hs, \alpha_m) = (-d, p^m)_p$ , whereas if  $d = 0$ , we have  $\epsilon(hs, \alpha_m) = (c, p^m)_p$ . Together with the recipe for  $s_p(hs)$ , we have

$$\epsilon = s_p(hs)\epsilon(hs, \alpha_m) = \begin{cases} (c, -d)_p(-d, p^m)_p & \text{if } cd \neq 0, \text{ord}_p(d) \text{ is odd,} \\ (-d, p^m)_p & \text{if } cd \neq 0, \text{ord}_p(d) \text{ is even,} \\ (c, p^m)_p & \text{if } d = 0, \\ (-d, p^m)_p & \text{if } c = 0. \end{cases} \quad (54)$$

As in the case of the elements  $\alpha_n$ , now to compute  $\Phi_{\mathbf{h}_p}(\beta_m)$  we need an Iwasawa decomposition for  $[h\beta_m, \epsilon]$ , induced from the Iwasawa decomposition in  $\text{SL}_2(\mathbb{Q}_p)$ . Such a decomposition will depend on two subregions  $\mathcal{B}_1(m)$  and  $\mathcal{B}_2(m)$  of  $\text{SL}_2(\mathbb{Z}_p)$ . The discussion is analogous to the one we did above for the sets  $\mathcal{A}_1(n)$  and  $\mathcal{A}_2(n)$ , thus we omit some details.

- i) Let  $\mathcal{B}_1(m) \subseteq \text{SL}_2(\mathbb{Z}_p)$  be the set of  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p)$  with  $|dp^m|_p < |cp^{-m}|_p$ , or equivalently  $|d|_p < p^{2m}|c|_p$ . Notice that one always has  $c \neq 0$  in  $\mathcal{B}_1(m)$ . If  $h \in \mathcal{B}_1(m)$ , one has

$$[h\beta_m, \epsilon] = \left[ \begin{pmatrix} c^{-1}p^m & * \\ 0 & cp^{-m} \end{pmatrix}, e \right] \left[ \begin{pmatrix} 1 & 0 \\ -\frac{d}{c}p^{2m} & 1 \end{pmatrix}, 1 \right], \quad (55)$$

where one can check that the sign  $e$  is given by the recipe

$$e = \begin{cases} (c, p^m)_p & \text{if } d = 0 \text{ or } \text{ord}_p(d) \text{ is odd,} \\ (c, d)_p(c, p^m)_p & \text{if } d \neq 0 \text{ and } \text{ord}_p(d) \text{ is even.} \end{cases} \quad (56)$$

- ii) Let  $\mathcal{B}_2(m) \subseteq \text{SL}_2(\mathbb{Z}_p)$  be the set of elements  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p)$  such that  $|dp^m|_p \geq |cp^{-m}|_p$ , or equivalently,  $|d|_p \geq p^{2m}|c|_p$ . Now, one always has  $d \neq 0$  in  $\mathcal{B}_2(m)$ . If  $h \in \mathcal{B}_2(m)$ , one has

$$[h\beta_m, \epsilon] = \left[ \begin{pmatrix} -d^{-1}p^{-m} & * \\ 0 & -dp^m \end{pmatrix}, e \right] \left[ \begin{pmatrix} 0 & -1 \\ 1 & \frac{c}{d}p^{-2m} \end{pmatrix}, 1 \right], \quad (57)$$

where the sign  $e$  is now computed according to the recipe

$$e = \begin{cases} (c, -d)_p(-d, p^m)_p & \text{if } \text{ord}_p(d) \text{ is odd,} \\ (-d, p^m)_p & \text{if } \text{ord}_p(d) \text{ is even.} \end{cases} \quad (58)$$

Now, to proceed with the computation of  $\Phi_{\mathbf{h}_p}(\beta_m)$ , we may focus on the computation of the matrix coefficients  $\langle \tilde{\pi}_p(\beta_m)\mathbf{h}_p, \mathbf{h}_p \rangle$ , since we know  $\|\mathbf{h}_p\|^2$  from Lemma 7.2, and clearly  $\langle \tilde{\pi}_p(\beta_m)\mathbf{h}_p, \mathbf{h}_p \rangle$  equals

$$B_1(m) + B_2(m) := \int_{\mathcal{B}_1(m)} \mathbf{h}_p(h\beta_m)\overline{\mathbf{h}_p(h)}dh + \int_{\mathcal{B}_2(m)} \mathbf{h}_p(h\beta_m)\overline{\mathbf{h}_p(h)}dh.$$

Thus from now on, we will focus on the computation of the integrals  $B_1(m)$  and  $B_2(m)$ . Notice again that in the above expression one has  $\overline{\mathbf{h}_p(h)} = \mathbf{h}_p(h)$ .

COMPUTATION OF  $B_1(m)$

We proceed now with the computation of  $B_1(m) = \int_{\mathcal{B}_1(m)} \mathbf{h}_p(h\beta_m)\mathbf{h}_p(h)dh$ . Writing  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as usual, by virtue of (55) and (47) we see that

$$B_1(m) = -p^{1-3m/2}(-1)^m \chi_\psi(p^m) \int_{\mathcal{B}_1(m)} (c, p^m)_p \chi_\psi(c) e(h) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh,$$

where  $e(h)$  is given by the recipe in (56). Let us denote by  $\mathcal{J}_1(m)$  the last integral, and define

$$\begin{aligned} \mathcal{B}_1^+(m) &:= \{h \in \mathcal{B}_1(m) : d = 0 \text{ or } \text{ord}_p(d) \text{ odd}\}, \\ \mathcal{B}_1^-(m) &:= \{h \in \mathcal{B}_1(m) : d \neq 0 \text{ and } \text{ord}_p(d) \text{ even}\}. \end{aligned}$$

Then, according to (56) we have

$$\begin{aligned} \mathcal{J}_1(m) &= \int_{\mathcal{B}_1^+(m)} \chi_\psi(c) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh + \\ &\quad + \int_{\mathcal{B}_1^-(m)} (c, d)_p \chi_\psi(c) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh. \end{aligned}$$

Observe that  $\mathcal{B}_1^+(m) \subseteq \text{SL}_2(\mathbb{Z}_p) - \Gamma_0$ , hence  $\mathbf{h}_p(h) = 1$  and  $c$  is a unit for all  $h \in \mathcal{B}_1^+(m)$ . Therefore, the first integral in the last expression equals  $\text{vol}(\mathcal{B}_1^+(m))$ , and

$$\mathcal{J}_1(m) = \text{vol}(\mathcal{B}_1^+(m)) + \int_{\mathcal{B}_1^-(m)} (c, d)_p \chi_\psi(c) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh.$$

LEMMA 7.10. *With the above notation,*

$$B_1(m) = \begin{cases} p^{1-3m/2}(-1)^{m+1} \chi_\psi(p^m)(1-p^{-1})(1+p-p^m) & \text{if } m > 0, \\ p^{3m/2}(-1)^{m+1} \chi_\psi(p^m)(1-p^{-1})p^{-m} & \text{if } m \leq 0. \end{cases}$$

*Proof.* First, suppose that  $m \leq 0$ . Then, up to a set of measure zero,  $\mathcal{B}_1^+(m)$  equals the union of the sets  $\mathcal{R}_{2j+1}$  with  $j \geq -m$ . Thus

$$\begin{aligned} \text{vol}(\mathcal{B}_1^+(m)) &= \sum_{j \geq -m} \text{vol}(\mathcal{R}_{2j+1}) = p^{-1}(1-p^{-1})^2 \sum_{j \geq -m} p^{-2j} = \\ &= p^{2m-1}(1-p^{-1})^2 \frac{1}{1-p^{-2}} = p^{2m-1} \frac{1-p^{-1}}{1+p^{-1}}. \end{aligned}$$

Besides,  $\mathcal{B}_1^-(m)$  equals the union of the sets  $\mathcal{R}_{2j}$  with  $j > -m$ . When  $h \in \mathcal{R}_{2j}$  we have  $\mathbf{h}_p(h) = 1$  and  $c$  is a unit, and one has

$$\begin{aligned} \int_{\mathcal{B}_1^-(m)} (c, d)_p \chi_\psi(c) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh &= \sum_{j > -m} \text{vol}(\mathcal{R}_{2j}) = (1 - p^{-1})^2 \sum_{j > -m} p^{-2j} = \\ &= p^{2m-2} \frac{1 - p^{-1}}{1 + p^{-1}}. \end{aligned}$$

Altogether, in this case we see that  $\mathcal{J}_1(m) = p^{2m-1}(1 - p^{-1})$ , and hence

$$B_1(m) = -p^{1-3m/2}(-1)^m \chi_\psi(p^m) \mathcal{J}_1(m) = p^{3m/2}(-1)^{m+1} \chi_\psi(p^m)(1 - p^{-1})p^{-m}.$$

Now suppose that  $m > 0$ . One easily checks that  $\mathcal{B}_1^+(m) = \mathcal{B}_1^+(0)$ , and that  $\mathcal{B}_1^-(m)$  is the union of  $\mathcal{B}_1^-(0)$  together with the sets  $\mathcal{L}_0 \cap \mathcal{R}_0$  and the sets  $\mathcal{L}_j$  with  $1 \leq j \leq 2m - 1$ . Integration over  $\mathcal{L}_0 \cap \mathcal{R}_0$  gives  $\text{vol}(\mathcal{L}_0 \cap \mathcal{R}_0) = (1 - p^{-1})^2$ . For *odd* integers  $j$  with  $1 \leq j \leq 2m - 1$ , we see with an already used argument that

$$\int_{\mathcal{L}_j} (c, d)_p \chi_\psi(c) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh$$

vanishes. And for  $j = 2t$  even,  $1 \leq t \leq m - 1$ , we have

$$\int_{\mathcal{L}_{2t}} (c, d)_p \chi_\psi(c) \chi_\delta(c) |c|_p^{-3/2} \mathbf{h}_p(h) dh = -p^{1+3t} \text{vol}(\mathcal{L}_{2t}) = -p^{1+t}(1 - p^{-1})^2.$$

Summing up all the contributions, for  $m > 0$  one checks that  $\mathcal{J}_1(m) = (1 - p^{-1})(1 + p - p^m)$ , and hence in this case  $B_1(m) = p^{1-3m/2}(-1)^{m+1} \chi_\psi(p^m)(1 - p^{-1})(1 + p - p^m)$ .  $\square$

COMPUTATION OF  $B_2(m)$

We now deal with the computation of  $B_2(m) = \int_{\mathcal{B}_2(m)} \mathbf{h}_p(h\beta_m) \mathbf{h}_p(h) dh$ . Writing  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as usual, by virtue of (57) and (47) we see that

$$B_2(m) = p^{3m/2}(-1)^m \chi_\psi(p^m) \int_{\mathcal{B}_2(m)} e(h)(-d, p^m) \chi_\psi(-d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h) dh,$$

where  $e(h)$  is given by the recipe in (58). Let us denote by  $\mathcal{J}_2(m)$  the last integral, and define

$$\begin{aligned} \mathcal{B}_2^-(m) &:= \{h \in \mathcal{B}_2(m) : \text{ord}_p(d) \text{ odd}\}, \\ \mathcal{B}_2^+(m) &:= \{h \in \mathcal{B}_2(m) : \text{ord}_p(d) \text{ even}\}. \end{aligned}$$

Then, by (58) we have

$$\begin{aligned} \mathcal{J}_2(m) &= \int_{\mathcal{B}_2^-(m)} (c, -d)_p \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h) dh + \\ &\quad + \int_{\mathcal{B}_2^+(m)} \chi_\psi(d) \chi_\delta(d) |d|_p^{-3/2} \mathbf{h}_p(h) dh. \end{aligned}$$

LEMMA 7.11. *With the above notation,*

$$B_2(m) = \begin{cases} p^{1-3m/2}(-1)^{m+1}\chi_\psi(p^m)(1-p^{-1})p^m & \text{if } m > 0, \\ p^{3m/2}(-1)^{m+1}\chi_\psi(p^m)(1-p^{-1})(1+p^{-1}-p^{-m}) & \text{if } m \leq 0. \end{cases}$$

*Proof.* The proof goes along the same lines of the previous lemma, and hence we omit the computations. We just stress that when  $m > 0$  the set  $\mathcal{B}_2^-(m)$  is empty, and  $\mathcal{B}_2^+(m)$  is the disjoint union of the sets  $\mathcal{L}_j$  with  $j \geq 2m$ . And for  $m \leq 0$ , the set  $\mathcal{B}_2^-(m)$  equals the disjoint union of the sets  $\mathcal{R}_{2j+1}$  with  $0 \leq j \leq -m-1$ , whereas  $\mathcal{B}_2^+(m)$  is the disjoint union of the sets  $\mathcal{R}_{2j}$  with  $0 \leq j \leq -m$ .  $\square$

PROPOSITION 7.12. *With the above notation,*

$$\Phi_{\mathbf{h}_p}(\beta_m) = p^{-|3m/2-1|}(-1)^{m+1}\chi_\psi(p^m).$$

*Proof.* Recalling that  $\Phi_{\mathbf{h}_p}(\beta_m) = \langle \tilde{\pi}_p(\beta_m)\mathbf{h}_p, \mathbf{h}_p \rangle / \|\mathbf{h}_p\|^2$ , and that  $\|\mathbf{h}_p\|^2 = p^{-1}(p^2-1)$  from Lemma 7.2, the statement follows by combining Lemmas 7.10 and 7.11 since  $\langle \tilde{\pi}_p(\beta_m)\mathbf{h}_p, \mathbf{h}_p \rangle = B_1(m) + B_2(m)$ .  $\square$

### 7.3 WEIL PAIRINGS

Finally, we also need to compute the Weil pairings

$$\Phi_{\phi_p}(\alpha_n) = \langle \omega_{\overline{\psi}_p}(\alpha_n)\phi_p, \phi_p \rangle \quad \text{and} \quad \Phi_{\phi_p}(\beta_m) = \langle \omega_{\overline{\psi}_p}(\beta_m)\phi_p, \phi_p \rangle$$

for  $n, m \in \mathbb{Z}$ . Notice that  $\|\phi_p\|^2 = \int_{\mathbb{Q}_p} \mathbf{1}_{\mathbb{Z}_p}(x)\overline{\mathbf{1}_{\mathbb{Z}_p}(x)}dx = \text{vol}(\mathbb{Z}_p) = 1$ . The next statement is actually valid for all primes  $p$ , and we will use it later not only for primes  $p \mid N/M$ , but also for primes  $p \nmid M$ .

PROPOSITION 7.13. *Let  $p$  be a prime, and  $\phi_p = \mathbf{1}_{\mathbb{Z}_p} \in \mathcal{S}(\mathbb{Q}_p)$  be the characteristic function of  $\mathbb{Z}_p$ . With the above notation, if  $n, m \in \mathbb{Z}$  we have*

$$\frac{\Phi_{\phi_p}(\alpha_n)}{\|\phi_p\|^2} = \chi_{\overline{\psi}_p}(p^n)p^{-|n|/2}, \quad \frac{\Phi_{\phi_p}(\beta_m)}{\|\phi_p\|^2} = \chi_{\overline{\psi}_p}(p^m)p^{-|m|/2}.$$

*Proof.* This follows immediately from the definitions.  $\square$

### 7.4 COMPUTATION OF $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$

Continue to fix a prime  $p \mid N/M$ . Recall from (41), (42) that

$$\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \frac{L(1, \pi_v, \text{ad})L(1, \tau_v, \text{ad})}{L(1/2, \pi_v \times \text{ad}\tau_v)} \alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi),$$

where  $\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \int_{\text{SL}_2(\mathbb{Q}_p)} \Omega_p(g)dg$  and  $\Omega_p(g) := \overline{\Phi_{\mathbf{h}_p}(g)}\Phi_{\check{\mathbf{g}}_p}(g)\Phi_{\phi_p}(g)$  for  $g \in \text{SL}_2(\mathbb{Q}_p)$ . By using the double coset decomposition of  $\text{SL}_2(\mathbb{Q}_p)$  as in (48), we have

$$\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \sum_{n \in \mathbb{Z}} \Omega_p(\alpha_n)\text{vol}(\Gamma_0\alpha_n\Gamma_0) + \sum_{m \in \mathbb{Z}} \Omega_p(\beta_m)\text{vol}(\Gamma_0\beta_m\Gamma_0).$$

PROPOSITION 7.14. *Let  $p$  be a prime dividing  $N/M$ . Then*

$$\alpha_p^\sharp(\mathbf{h}, \mathfrak{g}, \phi) = \frac{p - w_p}{p^2 + w_p} \zeta_p(2)^{-1}.$$

*Proof.* For  $n = 0$  we have  $\Omega_p(\alpha_0) \text{vol}(\Gamma_0 \alpha_0 \Gamma_0) = \text{vol}(\Gamma_0) = p^{-1}(1 - p^{-1})$ . Let  $n \neq 0$  be an integer. From Proposition 7.13, we have  $\Phi_{\phi_p}(\alpha_n) = \chi_{\overline{\psi}_p}(p^n) p^{-|n|/2}$ , whereas from Propositions 7.4 and 7.9 we have

$$\Phi_{\mathfrak{g}_p}(\alpha_n) = p^{-2|n|}, \quad \Phi_{\mathbf{h}_p}(\alpha_n) = p^{-3|n|/2} (-1)^n \chi_\psi(p^n).$$

Since  $\psi = \overline{\psi}_p^D$ , one has  $\chi_\psi = \chi_{\overline{\psi}_p} \cdot \chi_D$ , and therefore  $\chi_\psi(p^n) = (\frac{D}{p})^n \chi_{\overline{\psi}_p}(p^n) = w_p^n \chi_{\overline{\psi}_p}(p^n)$ . From this, using the volumes from Lemma 7.3, we deduce that

$$\Omega_p(\alpha_n) \text{vol}(\Gamma_0 \alpha_n \Gamma_0) = (-w_p)^n p^{-2|n|-1} (1 - p^{-1}),$$

and therefore

$$\sum_{n \in \mathbb{Z}} \Omega_p(\alpha_n) \text{vol}(\Gamma_0 \alpha_n \Gamma_0) = \frac{1 - p^{-1}}{p} \left( 1 + \sum_{n > 0} (-w_p p^{-2})^n + \sum_{n < 0} (-w_p p^{-2})^{-n} \right).$$

The two geometric sums on the right hand side are the same, and equal  $\frac{-w_p}{p^2 + w_p}$ . Hence,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \Omega_p(\alpha_n) \text{vol}(\Gamma_0 \alpha_n \Gamma_0) &= p^{-1} (1 - p^{-1}) \left( 1 - \frac{2w_p}{p^2 + w_p} \right) = \\ &= p^{-1} (1 - p^{-1}) \frac{p^2 - w_p}{p^2 + w_p}. \end{aligned} \quad (59)$$

Besides, for  $m = 0$  we have  $\Omega_p(\beta_0) \text{vol}(\Gamma_0 \beta_0 \Gamma_0) = (-p^{-1})(-p^{-1})(1 - p^{-1}) = p^{-2}(1 - p^{-1})$ . And if  $m \neq 0$ , we have again  $\Phi_{\phi_p}(\beta_m) = \chi_{\overline{\psi}_p}(p^m) p^{-|m|/2}$ , and Propositions 7.5 and 7.12 tell us that

$$\Phi_{\mathfrak{g}_p}(\beta_m) = -p^{-|2m-1|}, \quad \Phi_{\mathbf{h}_p}(\beta_m) = p^{-|3m/2-1|} (-1)^{m+1} \chi_\psi(p^m).$$

Using again that  $\chi_\psi(p^m) = w_p^m \chi_{\overline{\psi}_p}(p^m)$ , and the volumes from Lemma 7.3, it follows that when  $m \neq 0$

$$\Omega_p(\beta_m) \text{vol}(\Gamma_0 \beta_m \Gamma_0) = \begin{cases} p^{-2m} (-w_p)^m (1 - p^{-1}) & \text{if } m > 0, \\ p^{2m-2} (-w_p)^m (1 - p^{-1}) & \text{if } m < 0. \end{cases}$$

Summing up all the terms, one easily checks that

$$\sum_{m \in \mathbb{Z}} \Omega_p(\beta_m) \text{vol}(\Gamma_0 \beta_m \Gamma_0) = (1 - p^{-1}) \frac{1 - w_p}{p^2 + w_p}. \quad (60)$$

Finally, combining (59) and (60) we conclude that

$$\begin{aligned} \alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) &= \frac{1 - p^{-1}}{p^2 + w_p} (p^{-1}(p^2 - w_p) + 1 - w_p) = \\ &= \frac{p - 1}{p^2} \cdot \frac{p^2 + p(1 - w_p) - w_p}{p^2 + w_p} = \frac{p - w_p}{p^2 + w_p} \zeta_p(2)^{-1}. \end{aligned}$$

□

PROPOSITION 7.15. *Let  $p$  be a prime dividing  $N/M$ . Then  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = p^{-1}$ .*

*Proof.* Recall from the definition of  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$  that

$$\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \frac{L(1, \pi_p, \text{ad})L(1, \tau_p, \text{ad})}{L(1/2, \pi_p \times \text{ad}\tau_p)} \alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi).$$

Since  $p \mid N/M$ , both  $\pi_p$  and  $\tau_p$  are (unramified) special representations, say  $\pi_p = \xi_1 \text{St}_p$ ,  $\tau_p = \xi_2 \text{St}_p$ . And recall that  $\varepsilon(1/2, \pi_p) = -\xi_1(p)$  and similarly for  $\tau_p$ . Then (cf. [JL70] and [Kud94, Section 3]) we have

$$L(1/2, \pi_p \otimes \tau_p \otimes \tau_p) = \frac{1}{(1 + w_p p^{-2})(1 + w_p p^{-1})^2} = \frac{p^4}{(p^2 + w_p)(p + w_p)^2},$$

where  $w_p = \varepsilon(1/2, \pi_p)$ . On the other hand, it is well-known that  $L(1/2, \pi_p) = \frac{p}{p + w_p}$ , so that (since  $p \nmid M$ )

$$L(1/2, \pi_p \times \text{ad}\tau_p) = \frac{L(1/2, \pi_p \otimes \tau_p \otimes \tau_p)}{L(1/2, \pi_p)} = \frac{p^3}{(p^2 + w_p)(p + w_p)}.$$

Also, from [Hid86, Section 10] (cf. also [GJ78]), we have  $L(1, \pi_p, \text{ad}) = L(1, \tau_p, \text{ad}) = \zeta_p(2)$ . It thus follows from the previous proposition that

$$\begin{aligned} \mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) &= \frac{\zeta_p(2)^2 (p^2 + w_p)(p + w_p)}{p^3} \frac{p - w_p}{\zeta_p(2)(p^2 + w_p)} = \\ &= \zeta_p(2) \frac{(p + w_p)(p - w_p)}{p^3} = \frac{p^2(p^2 - 1)}{(p^2 - 1)p^3} = \frac{1}{p}. \end{aligned}$$

□

## 8 COMPUTATION OF LOCAL INTEGRALS AT PRIMES $p \mid M$

In this section we focus on the regularized local periods  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$  at primes  $p \mid M$ . First we will recall the local types of the representations  $\tau$  and  $\tilde{\pi}$  at such primes, and describe explicitly the local components  $\check{\mathbf{g}}_p$  and  $\mathbf{h}_p$ . After this, we will be concerned with the matrix coefficients  $\langle \tau_p(g) \check{\mathbf{g}}_p, \check{\mathbf{g}}_p \rangle$  and  $\langle \tilde{\pi}_p(g) \mathbf{h}_p, \mathbf{h}_p \rangle$ ,

together with the Weil pairings  $\langle \omega_{\check{\psi}_p}^{-1}(g)\phi_p, \phi_p \rangle$ , for  $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ , towards obtaining an explicit expression for  $\mathcal{I}_p^{\mathfrak{h}}(\mathfrak{h}, \check{\mathfrak{g}}, \phi)$ . As in the previous section, for  $g \in \mathrm{SL}_2(\mathbb{Q}_p)$  we write

$$\Phi_{\check{\mathfrak{g}}_p}(g) := \frac{\langle \tau_p(g)\check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle}{\|\check{\mathfrak{g}}_p\|^2}, \quad \Phi_{\mathfrak{h}_p}(g) := \frac{\langle \tilde{\tau}_p(g)\mathfrak{h}_p, \mathfrak{h}_p \rangle}{\|\mathfrak{h}_p\|^2}, \quad \Phi_{\phi_p}(g) := \frac{\langle \omega_{\check{\psi}_p}^{-1}(g)\phi_p, \phi_p \rangle}{\|\phi_p\|^2}.$$

When  $p$  divides  $M$ ,  $\check{\mathfrak{g}}_p$  and  $\mathfrak{h}_p$  are not  $\Gamma_0$ -invariant, although our choice guarantees that for any element  $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ , the product  $\Omega_p(g) := \overline{\Phi_{\mathfrak{h}_p}(g)}\Phi_{\check{\mathfrak{g}}_p}(g)\Phi_{\phi_p}(g)$  depends only on the double coset  $\Gamma_{00}g\Gamma_{00}$ . Because of this, we will need to refine our decomposition of  $\mathrm{SL}_2(\mathbb{Q}_p)$  in (48) into a decomposition in terms of double cosets for  $\Gamma_{00}$ . Then, we will need to compute the products  $\Omega_p(g)$  for  $g$  varying in a set of representatives for the double cosets for  $\Gamma_{00}$  in  $\mathrm{SL}_2(\mathbb{Q}_p)$ . As we will see, many of the involved matrix coefficients vanish, so we will not need to compute all of them in order to obtain the quantities  $\Omega_p(g)$ .

Fix a prime  $p \mid M$  through all this section. As before, write  $\psi_p$  for the  $p$ -th component of the standard additive character  $\psi : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{C}^\times$ .

### 8.1 LOCAL TYPES AND EXPLICIT DESCRIPTION OF TEST VECTORS

As in last section, we start considering the  $\mathrm{GL}_2$  case. Now  $\tau_p$  is a (ramified) principal series. More precisely,  $\tau_p = \pi(\xi_1, \xi_2)$  is the principal series representation induced by two characters  $\xi_1, \xi_2 : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ . In the induced model, this space is realized as the space of those functions  $\varphi : \mathrm{GL}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  such that

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x\right) = \xi_1(a)\xi_2(d) \left| \frac{a}{d} \right|_p^{1/2} \varphi(x) \quad \text{for } a, d \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p, x \in \mathrm{GL}_2(\mathbb{Q}_p). \quad (61)$$

Because of our assumption that  $N$  is square-free, we may assume that  $\xi_1$  is unramified (hence  $\xi_1(a) = 1$  for all  $a \in \mathbb{Z}_p^\times$ ) and  $\xi_2$  is ramified of ( $p$ -power) conductor 1 (meaning that  $\xi_2(1 + p\mathbb{Z}_p) = 1$ ). Define

$$K_0^1 = \left\{ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0, d \equiv 1 \pmod{p} \right) \right\},$$

and notice that  $\mathrm{GL}_2(\mathbb{Z}_p) = B(\mathbb{Z}_p)K_0^1 \sqcup B(\mathbb{Z}_p)wK_0^1$ , where  $B(\mathbb{Z}_p) = B(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p)$  is the subgroup of matrices of  $\mathrm{GL}_2(\mathbb{Z}_p)$  which belong to the Borel subgroup of upper triangular matrices in  $\mathrm{GL}_2(\mathbb{Q}_p)$ , and  $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . In the induced model for  $\tau_p = \pi(\xi_1, \xi_2)$ , the subspace  $\tau_p^{K_0^1} \subset \tau_p$  of vectors fixed by  $K_0^1$  turns out to be one-dimensional, and a non-trivial  $K_0^1$ -invariant vector is described in [Sch02, Proposition 2.1.2]. Namely, the vector  $\varphi_p : \mathrm{GL}_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$  characterized by requiring that

$$\varphi_p(h) = \begin{cases} \xi_1(p)^{-1}\xi_1(a)\xi_2(d)|ad^{-1}|^{1/2} & \text{if } h \in \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} K_0^1, \quad a, d \in \mathbb{Q}_p^\times, \\ 0 & \text{if } h \notin B(\mathbb{Q}_p)K_0^1. \end{cases} \quad (62)$$



Since we are only interested in the normalized value  $\Phi_{\mathfrak{g}_p}(g)$ , we may assume that  $\mathfrak{g}_p$  coincides with the local vector  $\varphi_p$  given by the above recipe. In contrast to the case of the previous section, however, when  $p$  divides  $M$  the new vector  $\mathfrak{g}_p$  does not ensure the non-vanishing of the local periods  $\mathcal{I}_p$ . To remedy this, we replace  $\mathfrak{g}_p$  by  $\check{\mathfrak{g}}_p = \mathbf{V}_p \mathfrak{g}_p \in \tau_p$ , where  $\mathbf{V}_p$  is the  $p$ -th level raising operator acting on  $\tau_p$  by  $\varphi \mapsto \tau_p(\varpi_p)\varphi$ , with  $\varpi_p = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$ . The vector  $\check{\mathfrak{g}}_p$  is no longer  $K_0^1$ -invariant, but defining

$$K_{00}^1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0, d \equiv 1 \pmod{p^2} \right\},$$

we have the following:

LEMMA 8.1. *With the above notation,  $\check{\mathfrak{g}}_p$  is  $K_{00}^1$ -invariant. Moreover, we have  $\|\check{\mathfrak{g}}_p\|^2 = (p+1)^{-1}$ .*

*Proof.* It is straightforward to check the first assertion, hence we focus in the computation of  $\|\check{\mathfrak{g}}_p\|^2$ . By definition, we have  $\|\check{\mathfrak{g}}_p\|^2 = \langle \check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle$ . Using the decomposition  $\mathrm{GL}_2(\mathbb{Z}_p) = K_0 \sqcup K_0 w K_0$ , we have

$$\begin{aligned} \langle \check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle &= \int_{\mathrm{GL}_2(\mathbb{Z}_p)} \mathfrak{g}_p(h\varpi_p) \overline{\mathfrak{g}_p(h\varpi_p)} dh = \\ &= \int_{K_0} \mathfrak{g}_p(h\varpi_p) \overline{\mathfrak{g}_p(h\varpi_p)} dh + \int_{K_0 w K_0} \mathfrak{g}_p(h\varpi_p) \overline{\mathfrak{g}_p(h\varpi_p)} dh. \end{aligned}$$

We deal separately each of the two integrals. If  $h = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in K_0$  is an arbitrary element in  $K_0$ , using Iwasawa decomposition in  $K_0$ , we can write

$$h\varpi_p = \begin{pmatrix} xp^{-1} & y \\ zp^{-1} & t \end{pmatrix} = \begin{pmatrix} p^{-1}t^{-1} \det(h) & y \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1}p^{-1}z & 1 \end{pmatrix},$$

where the rightmost element, call it  $h_0$ , belongs to  $\mathrm{GL}_2(\mathbb{Z}_p)$ . Moreover, if  $h \in K_{00}$  then  $h_0$  clearly belongs to  $K_0^1$ ; and one can check that if  $h \notin K_{00}$  then  $h_0 \notin B(\mathbb{Q}_p)K_0^1$ , where recall that  $B(\mathbb{Q}_p)$  stands for the upper triangular Borel subgroup of  $\mathrm{GL}_2(\mathbb{Q}_p)$ . Thus, it follows from (62) (recall that  $\xi_1$  is unramified,  $\xi_2$  is ramified of conductor 1, and  $\xi_1\xi_2 = \underline{\chi}_p$ ) that

$$\mathfrak{g}_p(h\varpi_p) = \begin{cases} p^{1/2}\xi_1(p)^{-2}\underline{\chi}_p(h) & \text{if } h \in K_{00}, \\ 0 & \text{if } h \notin K_{00}. \end{cases}$$

Now suppose that  $h = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in K_0 w K_0$  is an arbitrary element in  $K_0 w K_0$ . Then, again using Iwasawa decomposition and noticing that  $z \in \mathbb{Z}_p^\times$ , we have

$$h\varpi_p = \begin{pmatrix} xp^{-1} & y \\ zp^{-1} & t \end{pmatrix} = \begin{pmatrix} -z^{-1} \det(h) & xp^{-1} \\ 0 & zp^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z^{-1}tp \end{pmatrix}.$$

The rightmost element belongs to  $\mathrm{GL}_2(\mathbb{Z}_p)$ , but not to  $K_0^1$ . And further, it can be easily shown that it does not belong to  $B(\mathbb{Q}_p)K_0^1$  either, hence by applying (62) we deduce that  $\mathbf{g}_p(h\varpi_p) = 0$  for all  $h \in K_0wK_0$ . Therefore, we conclude that  $\langle \check{\mathbf{g}}_p, \check{\mathbf{g}}_p \rangle = p\mathrm{vol}(K_{00}) = (p + 1)^{-1}$ .  $\square$

REMARK 8.2. *The same arguments of the proof of the previous lemma show that, for an arbitrary  $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ , we have*

$$\langle \tau_p(g)\check{\mathbf{g}}_p, \check{\mathbf{g}}_p \rangle = \int_{K_{00}} \mathbf{g}_p(hg\varpi_p)\overline{\mathbf{g}(h\varpi_p)}dh = p^{1/2}\xi_1(p)^2 \int_{K_{00}} \mathbf{g}_p(hg\varpi_p)\check{\chi}_p^{-1}(h)dh.$$

Now we turn our attention to the representation  $\tilde{\pi}_p$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ . In order to lighten the notation, we will write as in the previous section  $\psi = \overline{\psi}_p^D$ , where  $\overline{\psi}_p = \psi_p^{-1}$ . Then, as explained in Section 5.2.1,  $\tilde{\pi}_p$  is the odd Weil representation  $r_{\overline{\psi}}^-$  (which is supercuspidal). The space of  $r_{\overline{\psi}}^-$  is the subspace of *odd* functions in  $\mathcal{S}(\mathbb{Q}_p)$  (where  $\mathbb{Q}_p$  is regarded as a quadratic space endowed with the bilinear form  $(x, y) = 2xy$ ). Recall that the action of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  is determined by the following properties: if  $\varphi \in \mathcal{S}(\mathbb{Q}_p)$  is odd,  $a \in \mathbb{Q}_p^\times$ ,  $x \in \mathbb{Q}_p^\times$ , and we write  $s = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ , then

$$\begin{aligned} r_{\overline{\psi}}^- \left[ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, 1 \right] \varphi(x) &= |a|_p^{1/2} \chi_\psi(a) \varphi(ax), \\ r_{\overline{\psi}}^- \left[ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, 1 \right] \varphi(x) &= \psi(bx^2) \varphi(x), \\ r_{\overline{\psi}}^- [s, 1] \varphi(x) &= \gamma(\psi) \int_{\mathbb{Q}_p} \varphi(y) \psi(2xy) dy. \end{aligned}$$

For our choice of  $\psi$ , we have  $\gamma(\psi) = 1$ , hence the third identity above simplifies to

$$r_{\overline{\psi}}^- [s, 1] \varphi(x) = \int_{\mathbb{Q}_p} \varphi(y) \psi(2xy) dy.$$

Let  $\tilde{\Gamma}_{00}$  denote the image of  $\Gamma_{00}$  in  $\widetilde{\mathrm{SL}}_2(\mathbb{Z}_p)$ . The following is proved in [BM07, Lemma 8.5]:

LEMMA 8.3. *The space of vectors  $\varphi_p$  in  $r_{\overline{\psi}}^-$  satisfying  $r_{\overline{\psi}}^- \varphi_p = \check{\chi}_p(k) \varphi_p$  for all  $k \in \tilde{\Gamma}_{00}$  is one-dimensional, and it is generated by the function  $\mathbf{1}_{\mathbb{Z}_p^\times} \cdot \check{\chi}_p^{-1}$ .*

The  $p$ -th component  $\mathbf{h}_p$  of the vector  $\mathbf{h}$  is thus a scalar multiple of the function given in the lemma. Since the matrix coefficients  $\Phi_{\mathbf{h}_p}(g)$  are normalized so that they are invariant under replacing  $\mathbf{h}_p$  by a scalar multiple, we will assume in the following that  $\mathbf{h}_p = \mathbf{1}_{\mathbb{Z}_p^\times} \cdot \check{\chi}_p^{-1}$ . Notice that

$$\|\mathbf{h}_p\|^2 = \int_{\mathbb{Q}_p} \mathbf{h}_p(x) \overline{\mathbf{h}_p(x)} dx = \int_{\mathbb{Z}_p^\times} dx = \mathrm{vol}(\mathbb{Z}_p^\times) = 1 - p^{-1}. \tag{63}$$

Having described our choices for the  $p$ -th components  $\check{\mathfrak{g}}_p$  and  $\mathbf{h}_p$ , we should note that by construction the function of  $\mathrm{SL}_2(\mathbb{Q}_p)$  defined by  $g \mapsto \Phi_{\check{\mathfrak{g}}_p}(g)\overline{\Phi_{\mathbf{h}_p}(g)}$  is  $\Gamma_{00}$ -biinvariant (and it is not  $\Gamma_0$ -biinvariant). Indeed, this follows immediately from the invariance properties of  $\check{\mathfrak{g}}_p$  and  $\mathbf{h}_p$ , together with the properties of the matrix coefficients  $\langle \tau_p(g)\check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle$  and  $\langle r_{\bar{\psi}}(g)\mathbf{h}_p, \mathbf{h}_p \rangle$ . Finally, as in the previous section we write  $\omega_p = \omega_{\bar{\psi}_p}$  for the Weil representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$  acting on the space of Bruhat–Schwartz functions  $\mathcal{S}(\mathbb{Q}_p)$ , with respect to the character  $\bar{\psi}_p = \psi_p^{-1}$ . As before, by our choice of test vector we have  $\phi_p = \mathbf{1}_{\mathbb{Z}_p}$ , and recall that Proposition 7.13 continues to hold when  $p|M$ .

### 8.2 COMPUTATION OF $\alpha_p^\sharp(\mathbf{h}, \check{\mathfrak{g}}, \phi)$ FOR $p | M$

Recall from last section that

$$\mathrm{SL}_2(\mathbb{Q}_p) = \bigsqcup_{n \in \mathbb{Z}} \Gamma_0 \alpha_n \Gamma_0 \sqcup \bigsqcup_{n \in \mathbb{Z}} \Gamma_0 \beta_n \Gamma_0. \tag{64}$$

By the comment we have just made above, now one cannot compute  $\alpha_p^\sharp(\mathbf{h}, \check{\mathfrak{g}}, \phi)$  by only computing the matrix coefficients for  $\check{\mathfrak{g}}_p$ ,  $\mathbf{h}_p$ , and  $\phi_p$  at the elements  $\alpha_n$  and  $\beta_m$ . However, starting from this decomposition we can refine it to obtain a decomposition in terms of double cosets for  $\Gamma_{00}$ . First of all, one might observe that  $\Gamma_{00}$  is *not normal* in  $\Gamma_0$ . However, one has

$$\Gamma_0 = \bigsqcup_{\gamma \in \mathbb{Z}_p/p\mathbb{Z}_p} \Gamma_{00} \nu_\gamma = \bigsqcup_{\gamma \in \mathbb{Z}_p/p\mathbb{Z}_p} \nu_\gamma \Gamma_{00}, \quad \nu_\gamma = \begin{pmatrix} 1 & 0 \\ \gamma p & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p),$$

so that for each  $n$  and  $m$  we can write

$$\Gamma_0 \alpha_n \Gamma_0 = \bigcup_{\gamma, \delta \in \mathbb{Z}_p/p\mathbb{Z}_p} \Gamma_{00} \nu_\gamma \alpha_n \nu_\delta \Gamma_{00}, \quad \Gamma_0 \beta_m \Gamma_0 = \bigcup_{\gamma, \delta \in \mathbb{Z}_p/p\mathbb{Z}_p} \Gamma_{00} \nu_\gamma \beta_m \nu_\delta \Gamma_{00}. \tag{65}$$

However, these unions are *not* disjoint. Nevertheless, it is not so difficult (although labourious) to reduce these expressions to disjoint unions, so that eventually one obtains a set of representatives for the double cosets for  $\Gamma_{00}$  in  $\mathrm{SL}_2(\mathbb{Q}_p)$ . We describe this set in the following lemma, whose proof is skipped.

LEMMA 8.4. *Fix a non-quadratic residue  $u \in \mathbb{Z}_p^\times$ . Then*

$$\mathrm{SL}_2(\mathbb{Q}_p) = \bigsqcup_{r \in \mathcal{R}} \Gamma_{00} r \Gamma_{00}, \tag{66}$$

where the set  $\mathcal{R}$  is the union of the following sets:

- I)  $\{1, \nu_1, \nu_u\}$ ;
- II)  $\{\alpha_n, \alpha_n \nu_1, \alpha_n \nu_u : n > 0\}$ ;
- III)  $\{\alpha_n, \nu_1 \alpha_n, \nu_u \alpha_n : n < 0\}$ ;

IV)  $\{\beta_m : m \in \mathbb{Z}\};$

V)  $\{\beta_m \nu_1, \beta_m \nu_u : m > 0\} \sqcup \{\nu_1 \beta_m \nu_\delta, \nu_u \beta_m \nu_\delta : m > 0, \delta \in \mathbb{Z}/p\mathbb{Z}\}.$

Now, we can finally proceed with the main goal of this section, namely the computation of  $\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$ , and therefore of the regularized local period  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$ . To ease the notation, for  $g \in \mathrm{SL}_2(\mathbb{Q}_p)$  we will write  $\Omega_p(g) := \overline{\Phi_{\mathbf{h}_p}(g)} \Phi_{\check{\mathbf{g}}_p}(g) \Phi_{\phi_p}(g)$ , so that

$$\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \int_{\mathrm{SL}_2(\mathbb{Q}_p)} \Omega_p(g) dg.$$

By our choice of  $\mathbf{h}_p, \check{\mathbf{g}}_p$ , and  $\phi_p$ , we see that  $\Omega_p(g)$  depends only on the double coset  $\Gamma_{00} g \Gamma_{00}$ . By using the decomposition explained in Lemma 8.4, we see that

$$\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \sum_{r \in \mathcal{R}} \Omega_p(r) \mathrm{vol}(\Gamma_{00} r \Gamma_{00}). \tag{67}$$

Therefore, we will proceed by computing  $\Omega_p(r) \mathrm{vol}(\Gamma_{00} r \Gamma_{00})$  for each  $r \in \mathcal{R}$ . We will deal with the cases I - V listed in Lemma 8.4 one by one. First we will concentrate in computing  $\Omega_p(r)$ .

### 8.2.1 CASE I

We start computing  $\Omega_p(\nu_\gamma)$  for  $\gamma \in \mathbb{Z}_p$ . First of all we have the following vanishing statement for  $\Phi_{\check{\mathbf{g}}_p}$ .

LEMMA 8.5. *If  $\gamma \in \mathbb{Z}_p^\times$ , then  $\Phi_{\check{\mathbf{g}}_p}(\nu_\gamma) = 0$ .*

*Proof.* Let  $\gamma \in \mathbb{Z}_p^\times$ . By Remark 8.2, we have

$$\langle \tau_p(\nu_\gamma) \check{\mathbf{g}}_p, \check{\mathbf{g}}_p \rangle = \int_{K_{00}} \mathbf{g}_p(h \nu_\gamma \varpi_p) \overline{\mathbf{g}_p(h \varpi_p)} dh.$$

But for  $h \in K_{00}$ , an Iwasawa decomposition for  $h \nu_\gamma \varpi_p$  reads

$$h \nu_\gamma \varpi_p = \begin{pmatrix} p^{-1} t^{-1} \det(h) & y \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1} p^{-1} z + \gamma & 1 \end{pmatrix}.$$

Under the assumption that  $\gamma \in \mathbb{Z}_p^\times$ , and taking into account that  $h \in K_{00}$ , we see from this identity that  $h \nu_\gamma \varpi_p$  does not belong to  $B(\mathbb{Q}_p) K_0^1$ , thus it follows from (62) that  $\mathbf{g}_p(h \nu_\gamma \varpi_p) = 0$  for all  $h \in K_{00}$ , and hence the statement follows.  $\square$

With this we can easily deduce  $\Omega_p(r)$  for elements  $r$  as in Case I.

PROPOSITION 8.6. *We have  $\Omega_p(1) = 1$ . And for  $\gamma \in \mathbb{Z}_p^\times$ ,  $\Omega_p(\nu_\gamma) = 0$ .*

*Proof.* Recall that  $\Omega_p(g) := \overline{\Phi_{\mathbf{h}_p}(g)} \Phi_{\check{\mathbf{g}}_p}(g) \Phi_{\phi_p}(g)$ . By our normalization of  $\Phi_{\mathbf{h}_p}, \Phi_{\check{\mathbf{g}}_p}$ , and  $\Phi_{\phi_p}$ , it is clear that  $\Omega_p(1) = 1$ . And for  $g = \nu_\gamma$  with  $\gamma \in \mathbb{Z}_p^\times$ , the previous lemma implies that  $\Omega_p(\nu_\gamma) = 0$ .  $\square$

8.2.2 CASE II

Now we focus on elements of the form  $\alpha_n \nu_c$ , with  $n > 0$  and  $c \in \mathbb{Z}_p$ . When  $c \notin \mathbb{Z}_p^\times$ ,  $\nu_c \in K_{00}$  and therefore  $\Omega_p(\alpha_n \nu_c) = \Omega_p(\alpha_n)$ .

LEMMA 8.7. *The following assertions hold:*

- i)  $\Phi_{\mathbf{h}_p}(\alpha_n) = 0$  for all  $n > 0$ .
- ii) If  $n > 0$  and  $c \in \mathbb{Z}_p^\times$ , then  $\Phi_{\mathbf{h}_p}(\alpha_n \nu_c) = 0$ .

*Proof.* From one of the rules for the odd Weil representation  $r_\psi^-$ , we have

$$(r_\psi^-(\alpha_n)\mathbf{h}_p)(x) = |p|_p^{n/2} \chi_\psi(p^n) \mathbf{h}_p(p^n x) = |p|_p^{n/2} \chi_\psi(p^n) \mathbf{1}_{p^{-n}\mathbb{Z}_p^\times}(x) \underline{\chi}_p^{-1}(p^n x).$$

Therefore, if  $n \neq 0$  we find that

$$\langle r_\psi^-(\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle = |p|_p^{n/2} \chi_\psi(p^n) \int_{\mathbb{Q}_p} \mathbf{1}_{p^{-n}\mathbb{Z}_p^\times}(x) \underline{\chi}_p^{-1}(p^n x) \mathbf{1}_{\mathbb{Z}_p^\times}(x) \underline{\chi}_p(x) dx = 0,$$

and i) follows. Notice that the argument does not require  $n > 0$ , but only  $n \neq 0$ . To show ii), notice that

$$\nu_c = (-1)s \begin{pmatrix} 1 & -cp \\ 0 & 1 \end{pmatrix} s.$$

By applying repeatedly the rules for the Weil representation to the elements on the right hand side of this identity, one arrives to

$$\begin{aligned} r_\psi^-(\nu_c)\mathbf{h}_p(x) &= \int_{\mathbb{Q}_p} \psi(-2xy - cpy^2) \mathfrak{G}(2y, \underline{\chi}_p^{-1}) dy = \\ &= \mathbf{1}_{\mathbb{Z}_p}(x) \int_{\mathbb{Q}_p} \psi(-2xy - cpy^2) \mathfrak{G}(2y, \underline{\chi}_p^{-1}) dy, \end{aligned}$$

and applying  $r_\psi^-(\alpha_n)$  to this expression we get

$$r_\psi^-(\alpha_n \nu_c)\mathbf{h}_p(x) = \chi_\psi(p^n) p^{-n/2} \mathbf{1}_{\mathbb{Z}_p}(xp^n) \int_{\mathbb{Q}_p} \psi(-2xp^n y - cpy^2) \mathfrak{G}(2y, \underline{\chi}_p^{-1}) dy.$$

Completing the squares and with some elementary computation, we find more explicitly

$$\begin{aligned} r_\psi^-(\alpha_n \nu_c)\mathbf{h}_p(x) &= p^{(1-n)/2} \chi_\psi(p^n) \varepsilon(1/2, \underline{\chi}_p) \underline{\chi}_p(2) \mathbf{1}_{p^{-n}\mathbb{Z}_p}(x) \psi\left(\frac{p^{2n}x^2}{cp}\right) \times \\ &\quad \times \int_{\mathbb{Z}_p^\times} \psi\left(\frac{-c}{p} \left(y + \frac{p^n x}{c}\right)^2\right) \underline{\chi}_p(y) dy. \end{aligned}$$

Using that  $\psi(\frac{p^{2n}x^2}{cp}) = 1$  and that

$$\int_{\mathbb{Z}_p^\times} \psi(\frac{-c}{p}(y + \frac{p^n x}{c})^2) \chi_p(y) dy = \int_{\mathbb{Z}_p^\times} \psi(\frac{-cy^2}{p}) \chi_p(y) dy$$

for  $n > 0$  and  $x \in \mathbb{Z}_p^\times$ , it follows that

$$\begin{aligned} \langle r_\psi^-(\alpha_n \nu_c) \mathbf{h}_p, \mathbf{h}_p \rangle &= p^{(1-n)/2} \chi_\psi(p^n) \varepsilon(1/2, \chi_p) \chi_p(2) \times \\ &\quad \times \int_{\mathbb{Z}_p^\times} \psi\left(\frac{-cy^2}{p}\right) \chi_p(y) dy \int_{\mathbb{Z}_p^\times} \chi_p(x) dx. \end{aligned}$$

But the last integral vanishes by orthogonality of Dirichlet characters, hence  $\langle r_\psi^-(\alpha_n \nu_c) \mathbf{h}_p, \mathbf{h}_p \rangle = 0$ , which gives  $\Phi_{h_p}(\alpha_n \nu_c) = 0$  as we wanted to prove.  $\square$

As a consequence, we immediately have the next vanishing statement.

PROPOSITION 8.8. *For all  $n > 0$  and  $c \in \mathbb{Z}_p$ ,  $\Omega_p(\alpha_n \nu_c) = 0$ .*

### 8.2.3 CASE III

Now we consider the case of elements  $\nu_c \alpha_n$  with  $n < 0$  and  $c \in \mathbb{Z}_p$ .

LEMMA 8.9. *The following assertions hold:*

- i)  $\Phi_{h_p}(\alpha_n) = 0$  for all  $n < 0$ .
- ii) If  $n < 0$  and  $c \in \mathbb{Z}_p^\times$ , then  $\Phi_{h_p}(\nu_c \alpha_n) = 0$ .

*Proof.* The first assertion follows as in Lemma 8.7, where we only used  $n \neq 0$ . To prove ii), observe that

$$\begin{aligned} r_\psi^-(\alpha_n) \mathbf{h}_p(x) &= \chi_\psi(p^n) p^{-n/2} \mathbf{1}_{\mathbb{Z}_p^\times}(p^n x) \chi_p^{-1}(p^n x) = \\ &= \chi_\psi(p^n) p^{-n/2} \mathbf{1}_{p^{-n} \mathbb{Z}_p^\times}(x) \chi_p^{-1}(x). \end{aligned}$$

Secondly, using again the decomposition  $\nu_c = (-1)s \begin{pmatrix} 1 & -cp \\ 0 & 1 \end{pmatrix} s$  as in the previous case, one finds

$$r_\psi^-(\nu_c \alpha_n) \mathbf{h}_p(x) = \chi_\psi(p^n) p^{-n/2} \int_{\mathbb{Q}_p} \psi(-2xp^n z - cp^{2n+1} z^2) \mathfrak{G}(2z, \chi_p^{-1}) dz.$$

From this,  $\langle r_\psi^-(\nu_c \alpha_n) \mathbf{h}_p, \mathbf{h}_p \rangle$  equals

$$\chi_\psi(p^n) p^{-n/2} \int_{\mathbb{Z}_p^\times} \left( \int_{\mathbb{Q}_p} \psi(-2xp^n z - cp^{2n+1} z^2) \mathfrak{G}(2z, \chi_p^{-1}) dz \right) \chi_p(x) dx.$$

We have  $\mathfrak{G}(2z, \underline{\chi}_p^{-1}) = p^{-1/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2)\mathbf{1}_{p^{-1}\mathbb{Z}_p^\times}(z)\underline{\chi}_p(z)$ , thus the inner integral in the above expression for  $\langle r_\psi^-(\nu_c\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle$  equals

$$p^{1/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2) \int_{\mathbb{Z}_p^\times} \psi(-2xp^{n-1}z - cp^{2n-1}z^2)\underline{\chi}_p(z)dz.$$

By completing squares and plugging this in the expression for  $\langle r_\psi^-(\nu_c\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle$ , one eventually obtains

$$\begin{aligned} \langle r_\psi^-(\nu_c\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle &= \chi_\psi(p^n)p^{(1-n)/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2) \times \\ &\quad \times \int_{\mathbb{Z}_p^\times} \psi(-cp^{2n-1}z^2)\underline{\chi}_p(z)\mathfrak{G}(-2zp^{n-1}, \underline{\chi}_p)dz. \end{aligned}$$

But notice that  $-2zp^{n-1} \notin p^{-1}\mathbb{Z}_p^\times$  for  $z \in \mathbb{Z}_p^\times$  because  $n < 0$ . Since  $\underline{\chi}_p$  has conductor 1, it follows that  $\mathfrak{G}(-2zp^{n-1}, \underline{\chi}_p) = 0$ , and therefore  $\langle r_\psi^-(\nu_c\alpha_n)\mathbf{h}_p, \mathbf{h}_p \rangle = 0$  as well, which implies  $\Phi_{\mathbf{h}_p}(\nu_c\alpha_n) = 0$ .  $\square$

PROPOSITION 8.10. *For all  $n < 0$  and  $c \in \mathbb{Z}_p$ ,  $\Omega_p(\nu_c\alpha_n) = 0$ .*

### 8.2.4 CASE IV

Now we deal with Case IV in Lemma 8.4, consisting only of elements  $\beta_m$  with  $m \in \mathbb{Z}$ .

LEMMA 8.11. *The following assertions hold:*

- i)  $\Phi_{\mathbf{h}_p}(\beta_m) = 0$  for all  $m \neq 1$ .
- ii)  $\Phi_{\mathfrak{g}_p}(\beta_1) = 0$ .

*Proof.* i) Since  $\beta_m = s\alpha_m$ , we compute  $r_\psi^-(\beta_m)\mathbf{h}_p(x)$  by applying first  $r_\psi^-(\alpha_m)$  and then  $r_\psi^-(s)$ . We have

$$\begin{aligned} r_\psi^-(\alpha_m)\mathbf{h}_p(x) &= \chi_\psi(p^m)p^{-m/2}\mathbf{1}_{\mathbb{Z}_p^\times}(p^m x)\underline{\chi}_p^{-1}(p^m x) = \\ &= \chi_\psi(p^m)p^{-m/2}\mathbf{1}_{p^{-m}\mathbb{Z}_p^\times}(x)\underline{\chi}_p^{-1}(x), \end{aligned}$$

and therefore applying  $r_\psi^-(s)$  gives

$$\begin{aligned} r_\psi^-(\beta_m)\mathbf{h}_p(x) &= \chi_\psi(p^m)p^{-m/2} \int_{p^{-m}\mathbb{Z}_p^\times} \psi(2xy)\underline{\chi}_p^{-1}(y)dy = \\ &= \chi_\psi(p^m)p^{m/2}\mathfrak{G}(2xp^{-m}, \underline{\chi}_p^{-1}). \end{aligned}$$

From this, we have

$$\langle r_\psi^-(\beta_m)\mathbf{h}_p, \mathbf{h}_p \rangle = \chi_\psi(p^m)p^{m/2} \int_{\mathbb{Z}_p^\times} \mathfrak{G}(2xp^{-m}, \underline{\chi}_p^{-1})\underline{\chi}_p(x)dx.$$

But  $\mathfrak{G}(2xp^{-m}, \underline{\chi}_p^{-1}) = p^{-1/2}\varepsilon(1/2, \underline{\chi}_p)\mathbf{1}_{p^{m-1}\mathbb{Z}_p^\times}(x)\underline{\chi}_p(2x)$ . If  $m \neq 1$ , we have  $p^{m-1}\mathbb{Z}_p^\times \cap \mathbb{Z}_p^\times = \emptyset$ , which implies that  $\langle r_\psi^-(\beta_m)\mathbf{h}_p, \mathbf{h}_p \rangle = 0$  and therefore  $\Phi_{\mathbf{h}_p}(\beta_m) = 0$ .

ii) If  $h = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in K_{00}$  is an arbitrary element in  $K_{00}$ , an Iwasawa decomposition for  $h\beta_1\varpi_p$  reads

$$h\beta_1\varpi_p = \begin{pmatrix} -y & xp^{-1} \\ -t & zp^{-1} \end{pmatrix} = \begin{pmatrix} t^{-1}\det(h)p^{-1} & -y \\ 0 & -t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -t^{-1}p^{-1}z \end{pmatrix},$$

and it is easy to check from this expression that  $h\beta_1\varpi_p \notin B(\mathbb{Q}_p)K_0^1$ . Thus we have  $\mathfrak{g}_p(h\beta_1\varpi_p) = 0$  for all  $h \in K_{00}$  by (62), and Remark 8.2 implies that  $\langle \tau_p(\beta_1)\mathfrak{g}_p, \mathfrak{g}_p \rangle = 0$ . Therefore  $\Phi_{\mathfrak{g}_p}(\beta_1) = 0$  as well.  $\square$

Directly from the lemma, and the definition of  $\Omega_p$ , we deduce:

PROPOSITION 8.12.  $\Omega_p(\beta_m) = 0$  for all  $m \in \mathbb{Z}$ .

### 8.2.5 CASE V

Finally, we consider the computation of matrix coefficients for elements in Case V from Lemma 8.4. We start considering the elements of the form  $\beta_m\nu_\delta$  with  $m > 0$  and  $\delta \in \mathbb{Z}_p^\times$ . First of all, we note the following vanishing statement for  $m > 1$ :

LEMMA 8.13. *If  $\delta \in \mathbb{Z}_p^\times$ , then  $\Phi_{\mathbf{h}_p}(\beta_m\nu_\delta) = 0$  for all  $m > 1$ .*

*Proof.* As in previous lemmas, we have

$$r_\psi^-(\alpha_m\nu_\delta)\mathbf{h}_p(x) = \chi_\psi(p^m)p^{-m/2}\mathbf{1}_{p^{-m}\mathbb{Z}_p}(x) \int_{\mathbb{Q}_p} \psi(-2xp^m y - \delta p y^2)\mathfrak{G}(2y, \underline{\chi}_p^{-1})dy.$$

By applying  $r_\psi^-(s)$ , we deduce that  $r_\psi^-(\beta_m\nu_\delta)\mathbf{h}_p(x)$  equals

$$\begin{aligned} & \chi_\psi(p^m)p^{-m/2} \int_{p^{-m}\mathbb{Z}_p} \psi(2xz) \left( \int_{\mathbb{Q}_p} \psi(-2zp^m y - \delta p y^2)\mathfrak{G}(2y, \underline{\chi}_p^{-1})dy \right) dz = \\ & = \chi_\psi(p^m)p^{m/2} \int_{\mathbb{Q}_p} \psi(-\delta p y^2)\mathfrak{G}(2y, \underline{\chi}_p^{-1}) \left( \int_{\mathbb{Z}_p} \psi(2(xp^{-m} - y)z)dz \right) dy. \end{aligned}$$

Using that  $\mathfrak{G}(2y, \underline{\chi}_p^{-1}) = p^{-1/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2)\mathbf{1}_{p^{-1}\mathbb{Z}_p^\times}(y)\underline{\chi}_p(y)$ , one finds

$$\begin{aligned} r_\psi^-(\beta_m\nu_\delta)\mathbf{h}_p(x) &= \chi_\psi(p^m)p^{(m+1)/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2) \times \\ & \quad \times \int_{\mathbb{Z}_p^\times} \psi\left(\frac{-\delta y^2}{p}\right) \left( \int_{\mathbb{Z}_p} \psi\left(\frac{2(xp^{1-m} - y)z}{p}\right) dz \right) \underline{\chi}_p(y)dy. \end{aligned}$$



The inner integral vanishes unless  $x \in p^{m-1}\mathbb{Z}_p^\times$  and  $xp^{1-m} \equiv y \pmod{p}$ , so one easily gets

$$r_\psi^-(\beta_m\nu_\delta)\mathbf{h}_p(x) = \chi_\psi(p^m)p^{(m-1)/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2)\mathbf{1}_{p^{m-1}\mathbb{Z}_p^\times}(x)\psi\left(\frac{-\delta x^2}{p^{2m-1}}\right)\underline{\chi}_p(x).$$

From this, it follows easily that  $\langle r_\psi^-(\beta_m\nu_\delta)\mathbf{h}_p, \mathbf{h}_p \rangle = 0$  for all  $m > 1$ , since  $\mathbb{Z}_p^\times \cap p^{m-1}\mathbb{Z}_p^\times = \emptyset$ .  $\square$

It follows from this lemma that  $\Omega_p(\beta_m\nu_\delta) = 0$  for all  $\delta \in \mathbb{Z}_p^\times$  and  $m > 1$ . So we are left with the case  $m = 1$ . However, looking at  $\Phi_{\mathfrak{g}_p}(\beta_1\nu_\delta)$  we find:

LEMMA 8.14. *For all  $\delta \in \mathbb{Z}_p^\times$ , one has  $\Phi_{\mathfrak{g}_p}(\beta_1\nu_\delta) = 0$ .*

*Proof.* Let  $h = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in K_{00}$  be an arbitrary element. An Iwasawa decomposition for  $h\beta_1\nu_\delta\varpi_p$  reads

$$h\beta_1\nu_\delta\varpi_p = \begin{pmatrix} t^{-1}\det(h)p^{-1} & * \\ 0 & -t \end{pmatrix} \begin{pmatrix} \delta & 1 \\ 1 - \delta t^{-1}p^{-1}z & -t^{-1}p^{-1}z \end{pmatrix}.$$

Using this decomposition we see that  $h\beta_1\nu_\delta\varpi_p \notin B(\mathbb{Q}_p)K_0^1$ . Thus it follows from (62) that  $\mathfrak{g}_p(h\beta_1\nu_\delta\varpi_p) = 0$  for all  $h \in K_{00}$ , and hence by Remark 8.2  $\langle \tau_p(\beta_1\nu_\delta)\mathfrak{g}_p, \mathfrak{g}_p \rangle = 0$ , which implies that  $\Phi_{\mathfrak{g}_p}(\beta_1\nu_\delta) = 0$ .  $\square$

Therefore, together with the above discussion we see that  $\Omega_p(\beta_m\nu_\delta) = 0$  for all  $m > 0$  and all  $\delta \in \mathbb{Z}_p^\times$ . Let us next consider the elements of the form  $\nu_\gamma\beta_m\nu_\delta$  when both  $\gamma, \delta \in \mathbb{Z}_p^\times$ . In the next lemma we will deal with matrix coefficients of the form  $\langle \tau_p(\nu_\gamma\beta_m\nu_\delta)\mathfrak{g}_p, \mathfrak{g}_p \rangle$ , with  $m > 0$ . To compute these, we need to find an Iwasawa decomposition for elements  $h\nu_\gamma\beta_m\nu_\delta\varpi_p$  with  $h \in K_{00}$  (cf. Remark 8.2). Using that  $h\nu_\gamma\beta_m\nu_\delta\varpi_p = h\nu_\gamma\beta_m\varpi_p\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$ , one can first determine an Iwasawa decomposition for

$$h\nu_\gamma\beta_m\varpi_p = \begin{pmatrix} -yp^{m-1} & xp^{-m} + \gamma yp^{1-m} \\ -tp^{m-1} & zp^{-m} + \gamma tp^{1-m} \end{pmatrix},$$

and then multiply it on the right by  $\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$ . This is how we proceed in the proof of the next lemma.

LEMMA 8.15. *Let  $m > 0$  be an integer, and let  $\gamma, \delta \in \mathbb{Z}_p^\times$ . Then  $\Phi_{\mathfrak{g}_p}(\nu_\gamma\beta_m\nu_\delta) = 0$  unless  $m = 1$  and  $\gamma\delta \equiv 1 \pmod{p}$ , and in that case one has  $\Phi_{\mathfrak{g}_p}(\nu_\gamma\beta_1\nu_\delta) = \underline{\chi}_p(\gamma)$ .*

*Proof.* Let us first consider the case  $m = 1$ . In this case, if  $h \in K_{00}$  then one has

$$h\nu_\gamma\beta_1\nu_\delta\varpi_p = \begin{pmatrix} t^{-1}p^{-1}\det(h) & -y \\ 0 & -t \end{pmatrix} \begin{pmatrix} \delta & 1 \\ 1 - \delta\eta(h) & -\eta(h) \end{pmatrix},$$

where  $\eta(h) = t^{-1}p^{-1}z + \gamma \in \mathbb{Z}_p$ . Since  $h \in K_{00}$ , observe that  $\eta(h) \equiv \gamma \pmod{p}$ , and therefore  $1 - \delta\eta(h) \equiv 1 - \gamma\delta \pmod{p}$ . By looking at the right hand side,

it is easy to check that  $h\nu_\gamma\beta_1\nu_\delta\varpi_p$  belongs to  $B(\mathbb{Q}_p)K_0^1$  if and only if  $\gamma\delta \equiv 1 \pmod{p}$ . We deduce from (62) that

$$\mathfrak{g}_p(h\nu_\gamma\beta_1\nu_\delta\varpi_p) = \begin{cases} p^{1/2}\xi_1(p)^{-2}\underline{\chi}_p(\gamma)\underline{\chi}_p(h) & \text{if } \gamma\delta \equiv 1 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it immediately follows from this that  $\langle \tau_p(\nu_\gamma\beta_1\nu_\delta)\check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle = 0$  whenever  $\gamma\delta \not\equiv 1 \pmod{p}$ , and so  $\Phi_{\check{\mathfrak{g}}_p}(\nu_\gamma\beta_1\nu_\delta) = 0$  as well in this case. When  $\gamma\delta \equiv 1 \pmod{p}$ , the above tells us that

$$\begin{aligned} \langle \tau_p(\nu_\gamma\beta_1\nu_\delta)\check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle &= p^{1/2}\xi_1(p)^2 \int_{K_{00}} \mathfrak{g}_p(h\nu_\gamma\beta_1\nu_\delta\varpi_p)\underline{\chi}_p^{-1}(h)dh = \\ &= p\underline{\chi}_p(\gamma)\text{vol}(K_{00}) = \frac{\underline{\chi}_p(\gamma)}{p+1}. \end{aligned}$$

Dividing by  $\|\check{\mathfrak{g}}_p\|^2 = (p+1)^{-1}$  (cf. Lemma 8.1), we get  $\Phi_{\check{\mathfrak{g}}_p}(\nu_\gamma\beta_1\nu_\delta) = \underline{\chi}_p(\gamma)$  when  $\gamma\delta \equiv 1 \pmod{p}$ .

Now suppose that  $m > 1$ . Then an Iwasawa decomposition for  $h\nu_\gamma\beta_m\nu_\delta\varpi_p$  reads

$$h\nu_\gamma\beta_m\nu_\delta\varpi_p = \begin{pmatrix} p^{m-2}\frac{\det(h)}{t\eta(h)} & xp^{-m} + \gamma yp^{1-m} \\ 0 & p^{1-m}t\eta(h) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta - \eta(h)^{-1}p^{2m-2} & 1 \end{pmatrix},$$

where  $\eta(h) = t^{-1}p^{-1}z + \gamma$  as before. Since  $h \in K_{00}$ , we have  $\eta(h) \in \mathbb{Z}_p^\times$ , and since we are now assuming  $m > 1$ , we see that  $\delta - \eta(h)^{-1}p^{2m-2} \in \mathbb{Z}_p^\times$  because  $\delta \in \mathbb{Z}_p^\times$ . Therefore, the rightmost element in the above identity does not belong to  $K_0^1$ , and it is not difficult to see that it does not belong to  $B(\mathbb{Q}_p)K_0^1$  either. Hence, by applying (61) we deduce that  $\mathfrak{g}_p(h\nu_\gamma\beta_m\nu_\delta\varpi_p) = 0$  for all  $h \in K_{00}$ . It follows that  $\langle \tau_p(\nu_\gamma\beta_m\nu_\delta)\check{\mathfrak{g}}_p, \check{\mathfrak{g}}_p \rangle = 0$ , and hence  $\Phi_{\check{\mathfrak{g}}_p}(\nu_\gamma\beta_m\nu_\delta) = 0$  as well.  $\square$

By virtue of the last lemma, for any integer  $m > 0$  we have  $\Omega_p(\nu_\gamma\beta_m\nu_\delta) = 0$  unless  $m = 1$  and  $\gamma\delta \equiv 1 \pmod{p}$ . Therefore, we only need to compute  $\Phi_{\mathfrak{h}_p}(\nu_\gamma\beta_m\nu_\delta)$  for  $m = 1$  and  $\gamma, \delta \in \mathbb{Z}_p^\times$  with  $\gamma\delta \equiv 1 \pmod{p}$ . Further, we see from Case V in Lemma 8.4 that the latter condition is only satisfied for the representatives  $\nu_1\beta_1\nu_1$  and  $\nu_u\beta_1\nu_{u^{-1}}$ . That is to say, there will be only two elements  $r$  arising in Case V for which  $\Omega_p(r)$  might be non-zero. The next lemma addresses the computation of  $\Phi_{\mathfrak{h}_p}(\nu_\gamma\beta_1\nu_{\gamma^{-1}})$  for  $\gamma \in \mathbb{Z}_p^\times$ ; for  $\gamma = 1, u$  it thus provides the values that we are looking for. In the proof we will need to deal with certain quadratic Gauss sums. For each  $a \in \mathbb{Q}_p^\times$ , we write  $\mathcal{G}(a) := \int_{\mathbb{Z}_p} \psi(az^2)dz$ . Here, recall that  $\psi = \overline{\psi}_p^D = \psi_p^{-D}$ , and that  $D \in \mathbb{Z}_p^\times$  is a square thanks to hypothesis (H2), together with the fact that  $(\frac{D}{p}) = -w_p$ . In particular, if  $a \in \mathbb{Z}_p^\times$ , then we have  $\mathcal{G}(a/p) = p^{-1}G(a, p)$ , where  $G(a, p)$  denotes the usual quadratic Gauss sum defined by

$$G(a, p) = \sum_{j=0}^{p-1} \zeta^{ax^2}, \quad \text{with } \zeta = e^{2\pi\sqrt{-1}/p}.$$

Recall that one has  $G(a, p) = \left(\frac{a}{p}\right) G(1, p)$ , and  $G(1, p)^2 = \left(\frac{-1}{p}\right) p$ .

LEMMA 8.16. *Let  $\gamma \in \mathbb{Z}_p^\times$ , and put  $\nu = \nu_\gamma$  and  $\nu' = \nu_{\gamma^{-1}}$ . Then*

$$\Phi_{\mathbf{h}_p}(\nu\beta_1\nu') = \frac{\chi_\psi(p)\underline{\chi}_p(\gamma)p^{-1/2}}{p-1}(p - G(-\gamma, p)).$$

*Proof.* From previous lemmas, we know that

$$r_\psi^-(\alpha_1\nu')\mathbf{h}_p(x) = \chi_\psi(p)p^{-1/2}\mathbf{1}_{p^{-1}\mathbb{Z}_p}(x) \int_{\mathbb{Q}_p} \psi(-2xyp - \gamma^{-1}y^2p)\mathfrak{G}(2y, \underline{\chi}_p^{-1})dy.$$

Observing that

$$\nu\beta_1\nu' = (-1)s \begin{pmatrix} 1 & -\gamma p \\ 0 & 1 \end{pmatrix} ss\alpha_1\nu' = s \begin{pmatrix} 1 & -\gamma p \\ 0 & 1 \end{pmatrix} \alpha_1\nu'$$

we compute  $(r_\psi^-(\nu\beta_1\nu')\mathbf{h}_p)(x)$  by applying  $r_\psi^-(s \begin{pmatrix} 1 & -\gamma p \\ 0 & 1 \end{pmatrix})$  to the above expression. This gives

$$\chi_\psi(p)p^{1/2} \int_{\mathbb{Z}_p} \psi\left(\frac{z(2x - \gamma z)}{p}\right) \left( \int_{\mathbb{Q}_p} \psi(-y(2z + \gamma^{-1}yp))\mathfrak{G}(2y, \underline{\chi}_p^{-1})dy \right) dz.$$

Using that  $\mathfrak{G}(2y, \underline{\chi}_p^{-1}) = p^{-1/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2)\mathbf{1}_{p^{-1}\mathbb{Z}_p^\times}(y)\underline{\chi}_p(y)$ , the inner integral becomes

$$p^{1/2}\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2) \int_{\mathbb{Z}_p^\times} \psi\left(\frac{-y(2z + \gamma^{-1}y)}{p}\right) \underline{\chi}_p(y)dy.$$

Setting  $C = p\chi_\psi(p)\varepsilon(1/2, \underline{\chi}_p)\underline{\chi}_p(2)$ , one can then rewrite  $r_\psi^-(\nu\beta_1\nu')\mathbf{h}_p(x)$  as

$$r_\psi^-(\nu\beta_1\nu')\mathbf{h}_p(x) = C \int_{\mathbb{Z}_p^\times} \psi\left(\frac{-\gamma^{-1}y^2}{p}\right) \underline{\chi}_p(y) \left( \int_{\mathbb{Z}_p} \psi\left(\frac{2xz - 2zy - \gamma z^2}{p}\right) dz \right) dy.$$

Completing squares, the inner integral is essentially a (translated) quadratic Gauss sum, and it is not difficult to see that the above expression can be simplified to

$$r_\psi^-(\nu\beta_1\nu')\mathbf{h}_p(x) = C\mathcal{G}\left(\frac{-\gamma}{p}\right) \mathbf{1}_{\mathbb{Z}_p}(x)\psi\left(\frac{x^2}{\gamma p}\right) \mathfrak{G}\left(\frac{-2x}{\gamma p}, \underline{\chi}_p\right).$$

Replacing  $\mathfrak{G}\left(\frac{-2x}{\gamma p}, \underline{\chi}_p\right)$  and  $C$  by their value, we eventually find that

$$r_\psi^-(\nu\beta_1\nu')\mathbf{h}_p(x) = \chi_\psi(p)p^{1/2}\underline{\chi}_p(\gamma)\mathcal{G}\left(\frac{-\gamma}{p}\right) \mathbf{1}_{\mathbb{Z}_p^\times}(x)\psi\left(\frac{x^2}{\gamma p}\right) \underline{\chi}_p^{-1}(x),$$

and using this expression, we deduce that

$$\begin{aligned} \langle r_{\overline{\psi}}^-(\nu\beta_1\nu')\mathbf{h}_p, \mathbf{h}_p \rangle &= \chi_\psi(p)p^{1/2}\underline{\chi}_p(\gamma)\mathcal{G}\left(\frac{-\gamma}{p}\right)\int_{\mathbb{Z}_p^\times}\psi\left(\frac{x^2}{\gamma p}\right)dx = \\ &= \chi_\psi(p)p^{1/2}\underline{\chi}_p(\gamma)\mathcal{G}\left(\frac{-\gamma}{p}\right)\left(\mathcal{G}\left(\frac{1}{\gamma p}\right)-\frac{1}{p}\right). \end{aligned}$$

Dividing by  $\|\mathbf{h}_p\|^2 = 1 - p^{-1} = p^{-1}(p - 1)$ , we obtain

$$\Phi_{\mathbf{h}_p}(\nu\beta_1\nu') = \frac{\chi_\psi(p)\underline{\chi}_p(\gamma)p^{3/2}}{p - 1} \cdot \mathcal{G}\left(\frac{-\gamma}{p}\right)\left(\mathcal{G}\left(\frac{\gamma^{-1}}{p}\right) - \frac{1}{p}\right).$$

Finally, using that  $\mathcal{G}\left(\frac{-\gamma}{p}\right) = \frac{1}{p}\left(\frac{-\gamma}{p}\right)G(1, p)$ , and  $\mathcal{G}\left(\frac{\gamma^{-1}}{p}\right) = \frac{1}{p}\left(\frac{\gamma}{p}\right)G(1, p)$ , we conclude that

$$\Phi_{\mathbf{h}_p}(\nu\beta_1\nu') = \frac{\chi_\psi(p)\underline{\chi}_p(\gamma)p^{-1/2}}{p - 1}(p - G(-\gamma, p)).$$

□

Combining the previous four lemmas, we summarize our discussion for Case V:

PROPOSITION 8.17. *Let  $m > 0$  be an integer, and let  $\gamma, \delta \in \mathbb{Z}_p^\times$ . Then  $\Omega_p(\beta_m\nu_\delta) = 0$ , and*

$$\Omega_p(\nu_\gamma\beta_m\nu_\delta) = \begin{cases} \frac{p-G(\gamma, p)}{p(p-1)} & \text{if } m = 1, \gamma\delta \equiv 1 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The first assertion follows immediately from Lemmas 8.13 and 8.14. As for the second one, combining Lemmas 8.15 and 8.16 yields that  $\Omega_p(\nu_\gamma\beta_m\nu_\delta) = 0$  unless  $m = 1$  and  $\gamma\delta \equiv 1 \pmod{p}$ . And in that case, the same lemmas together with Proposition 7.13 tell us that

$$\begin{aligned} \Phi_{\mathbf{h}_p}(\nu_\gamma\beta_1\nu_\delta) &= \frac{\chi_\psi(p)\underline{\chi}_p(\gamma)p^{-1/2}}{p - 1}(p - G(-\gamma, p)), \quad \Phi_{\phi_p}(\nu_\gamma\beta_1\nu_\delta) = \chi_{\overline{\psi}_p}(p)p^{-1/2}, \\ \Phi_{\mathfrak{g}_p}(\nu_\gamma\beta_1\nu_\delta) &= \underline{\chi}_p(\gamma). \end{aligned}$$

Notice that  $\chi_\psi(p) = \chi_{\overline{\psi}_p}(p)\chi_D(p) = \chi_{\overline{\psi}_p}(p)$  because  $D \in \mathbb{Z}_p^\times$  is a square. Therefore,

$$\begin{aligned} \Omega_p(\nu_\gamma\beta_1\nu_\delta) &= \overline{\Phi_{\mathbf{h}_p}(\nu_\gamma\beta_1\nu_\delta)}\Phi_{\mathfrak{g}_p}(\nu_\gamma\beta_1\nu_\delta)\Phi_{\phi_p}(\nu_\gamma\beta_1\nu_\delta) = \\ &= \frac{p - G(-\gamma, p)}{p(p - 1)} = \frac{p - G(\gamma, p)}{p(p - 1)}. \end{aligned}$$

□

8.3 COMPUTATION OF  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$

Finally, we conclude with the computation of the regularized local periods  $\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$  at primes  $p \mid M$ . Recall from (67) that

$$\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \sum_{r \in \mathcal{R}} \Omega_p(r) \text{vol}(\Gamma_{00} r \Gamma_{00}),$$

where  $\mathcal{R}$  is the set described in Lemma 8.4. However, by Propositions 8.6, 8.8, 8.10, 8.12, 8.17,

$$\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \Omega_p(1) \text{vol}(\Gamma_{00}) + \Omega_p(r_1) \text{vol}(\Gamma_{00} r_1 \Gamma_{00}) + \Omega_p(r_u) \text{vol}(\Gamma_{00} r_u \Gamma_{00}), \tag{68}$$

where  $u \in \mathbb{Z}_p^\times$  is a fixed non-quadratic residue, and  $r_\gamma = \nu_\gamma \beta_1 \nu_{\gamma-1}$  for  $\gamma = 1, u$ . It is not hard to see that  $\text{vol}(\Gamma_{00}) = p^{-3}(p-1)$ , and that

$$\text{vol}(\Gamma_{00} r_1 \Gamma_{00}) = \text{vol}(\Gamma_{00} r_u \Gamma_{00}) = \frac{p^{-3}(p-1)^2}{2}.$$

PROPOSITION 8.18. *Let  $p$  be a prime dividing  $M$ . Then  $\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = 2p^{-3}(p-1)$ .*

*Proof.* We just need to compute the three terms on the right hand side of (68). Clearly, we have  $\Omega_p(1) \text{vol}(\Gamma_{00}) = \text{vol}(\Gamma_{00}) = p^{-3}(p-1)$ . As for the other two terms, it follows from Proposition 8.17 that

$$\begin{aligned} \Omega_p(r_1) \text{vol}(\Gamma_{00} r_1 \Gamma_{00}) &= \frac{p - G(1, p)}{p(p-1)} \text{vol}(\Gamma_{00} r_1 \Gamma_{00}), \\ \Omega_p(r_u) \text{vol}(\Gamma_{00} r_u \Gamma_{00}) &= \frac{p - G(u, p)}{p(p-1)} \text{vol}(\Gamma_{00} r_u \Gamma_{00}). \end{aligned}$$

Therefore, the sum of these two equals

$$\frac{p^{-3}(p-1)}{2p} (2p - G(1, p) - G(u, p)) = p^{-3}(p-1),$$

since  $G(u, p) = -G(1, p)$ . It follows that  $\alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = 2p^{-3}(p-1)$ . □

PROPOSITION 8.19. *Let  $p$  be a prime dividing  $M$ . Then*

$$\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \frac{2}{p(p+1)}.$$

*Proof.* From the definition of the regularized local period, we have

$$\mathcal{I}_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = \frac{L(1, \pi_p, \text{ad})L(1, \tau_p, \text{ad})}{\zeta_{\mathbb{Q}_p}(2)L(1/2, \pi_p \times \text{ad}\tau_p)} \alpha_p^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi).$$

As in the case where  $p$  divides  $N/M$ ,  $\pi_p$  is an unramified Steinberg representation, so that we have again

$$L(1, \pi_p, \text{ad}) = \frac{p^2}{p^2 - 1}, \quad L(1/2, \pi_p) = \frac{1}{(1 + w_p p^{-1})} = \frac{p}{p + w_p} = \frac{p}{p - 1},$$

where now we are using that  $w_p = -1$  by hypothesis (H2). In contrast, as already explained,  $\tau_p = \pi(\xi_1, \xi_2)$  is a (ramified) principal series representation induced by a pair of characters  $\xi_1, \xi_2 : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$  (with  $\xi_1$  unramified and  $\xi_2$  of conductor 1). In this case, from [Hid86, Section 10] (or [GJ78]) we have

$$L(1, \tau_p, \text{ad}) = \frac{1}{1 - p^{-1}} = \frac{p}{p - 1}.$$

Besides, one can now check using [JL70] and [Kud94, Section 3] that

$$L(1/2, \pi_p \otimes \tau_p \otimes \tau_p^\vee) = L(1/2, \pi_p)^2 = \frac{p^2}{(p - 1)^2}.$$

Therefore,  $L(1/2, \pi_p \times \text{ad}\tau_p) = \frac{p}{(p-1)}$  and the statement follows.  $\square$

## 9 COMPUTATION OF LOCAL INTEGRALS AT $p = 2$ AND $v = \infty$

Let  $f \in S_{2k}(N)$ ,  $g \in S_{k+1}(N, \chi)$ , and  $h \in S_{k+1/2}^+(4N, \chi)$  be as usual, and let  $\mathbf{f}$ ,  $\mathbf{g}$ , and  $\mathbf{h}$  be their adelizations. Write  $\pi$ ,  $\tau$  and  $\tilde{\pi}$  for the corresponding automorphic representations as in Section 6. Continue to consider our test vector  $\mathbf{h} \otimes \check{\mathbf{g}} \otimes \check{\phi} \in \tilde{\pi} \otimes \tau \otimes \omega$ , where  $\omega = \omega_{\check{\psi}}$ . Recall that  $\psi$  denotes the standard additive character of  $\mathbb{A}/\mathbb{Q}$ , and  $\bar{\psi} = \psi^{-1}$ .

### 9.1 COMPUTATION AT $p = 2$

For simplicity, in the following we write  $\psi$  for the local additive character  $\bar{\psi}_2 = \psi_2^{-1}$  of  $\mathbb{Q}_2$ , so that  $\omega_2 = \omega_\psi$ . Write also  $t(2) = \begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix} \in \text{SL}_2(\mathbb{Q}_2)$ , which we identify with  $[t(2), 1] \in \widetilde{\text{SL}}_2(\mathbb{Q}_2)$ , and recall that  $\check{\mathbf{g}}_2 = \mathbf{g}_2^\# = \tau_2(t(2)^{-1})\mathbf{g}_2$ . Then define

$$\mathbf{h}_2^{(2)} := \tilde{\pi}_2(t(2))\mathbf{h}_2, \quad \phi_2^{(2)} := 2^{1/2}\omega_2(t(2))\phi_2.$$

LEMMA 9.1. *With the above notation, we have  $\phi_2^{(2)} = \mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$ . Furthermore, the following identities hold:*

$$\langle \mathbf{h}_2^{(2)}, \mathbf{h}_2^{(2)} \rangle = \langle \mathbf{h}_2, \mathbf{h}_2 \rangle, \quad \langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle = \langle \mathbf{g}_2, \mathbf{g}_2 \rangle, \quad \langle \phi_2^{(2)}, \phi_2^{(2)} \rangle = 2\langle \phi_2, \phi_2 \rangle.$$

*Proof.* Recall that  $\phi_2 = \mathbf{1}_{\mathbb{Z}_2}$ . Then by applying the rules of the Weil representation we have

$$\omega_2(t(2))\phi_2 = \omega_2(t(2))\mathbf{1}_{\mathbb{Z}_2} = \chi_\psi(2)|2|_2^{1/2}\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}.$$

One can check that  $\chi_\psi(2) = 1$  for our choice of  $\psi$ , and hence  $\omega_2(t(2))\phi_2 = 2^{-1/2}\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$ . It follows that  $\phi_2^{(2)} = \mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$  as stated. Furthermore, by definition we have

$$\langle \phi_2^{(2)}, \phi_2^{(2)} \rangle = \int_{\mathbb{Q}_p} \mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}(x) \overline{\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}(x)} dx = \text{vol}(2^{-1}\mathbb{Z}_2) = 2 = 2\text{vol}(\mathbb{Z}_2) = 2\langle \phi_2, \phi_2 \rangle.$$

As for the other identities, we show only the one concerning  $\mathbf{g}$  (the other one can be dealt with similarly). First of all,  $\mathbf{g}_2$  is a  $\text{GL}_2(\mathbb{Z}_2)$ -fixed vector in the unramified principal series representation  $\tau_2 = \pi(\xi, \xi^{-1})$ , where  $\xi : \mathbb{Q}_2^\times \rightarrow \mathbb{C}^\times$  is an unramified character. The space of such vectors is known to be one-dimensional, and a non-trivial choice is given by the function  $\varphi_2 : \text{GL}_2(\mathbb{Z}_2) \rightarrow \mathbb{C}$  defined by (cf. [Sch02, Section 2])

$$\varphi_2(\gamma) = \begin{cases} \xi(ad^{-1})|ad^{-1}|_2^{1/2} & \text{if } \gamma \in \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \text{GL}_2(\mathbb{Z}_2), \quad a, d \in \mathbb{Q}_2^\times, \\ 0 & \text{otherwise.} \end{cases} \quad (69)$$

Since we are only interested in the ratio  $\langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle / \langle \mathbf{g}_2, \mathbf{g}_2 \rangle$ , we may assume that  $\mathbf{g}_2 = \varphi_2$ . Then, one has  $\langle \mathbf{g}_2, \mathbf{g}_2 \rangle = \int_{\text{GL}_2(\mathbb{Z}_2)} \mathbf{g}_2(x) \overline{\mathbf{g}_2(x)} dx = \text{vol}(\text{GL}_2(\mathbb{Z}_2)) = 1$ . To compute

$$\langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle = \langle \mathbf{g}_2^\sharp, \mathbf{g}_2^\sharp \rangle = \int_{\text{GL}_2(\mathbb{Z}_2)} \mathbf{g}_2(xt(2^{-1})) \overline{\mathbf{g}_2(xt(2^{-1}))} dx,$$

we divide  $\text{GL}_2(\mathbb{Z}_2)$  into three subregions, namely

$$\begin{aligned} L_0 &:= \{x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_2) : c \in \mathbb{Z}_2^\times\}, \\ L_1 &:= \{x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_2) : c \in 2\mathbb{Z}_2^\times\}, \end{aligned}$$

and  $L_2 := \text{GL}_2(\mathbb{Z}_2) - L_0 - L_1 = K_0(4)$ . Working separately each of these regions, finding an Iwasawa decomposition for  $xt(2^{-1})$  one checks that

$$\mathbf{g}_2(xt(2^{-1})) \overline{\mathbf{g}_2(xt(2^{-1}))} = \begin{cases} 2^{-2} & \text{if } x \in L_0, \\ 1 & \text{if } x \in L_1, \\ 2^2 & \text{if } x \in L_2. \end{cases}$$

Therefore,  $\langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle = 2^{-4}\text{vol}(L_0) + \text{vol}(L_1) + 2^4\text{vol}(L_2)$ . Since  $L_0 = \text{GL}_2(\mathbb{Z}_2) - K_0(2)$  and  $L_2 = K_0(4)$ , and  $K_0(2)$  (resp.  $K_0(4)$ ) has index 3 (resp. 6) in  $\text{GL}_2(\mathbb{Z}_2)$ , one can easily compute  $\langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle = 1 = \langle \mathbf{g}_2, \mathbf{g}_2 \rangle$ .  $\square$

The next proposition computes the local regularized period  $\mathcal{I}_2^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi)$ , by relating it to the period  $\mathcal{I}_2^\sharp(\mathbf{h}^{(2)}, \mathbf{g}, \phi^{(2)})$  and invoking the computation done in [Xue18, Section 6].

PROPOSITION 9.2.  $\mathcal{I}_2^\sharp(\mathbf{h}, \check{\mathbf{g}}, \phi) = 1$ .

*Proof.* Recall that by definition we have

$$\mathcal{I}_2^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) = \frac{\mathcal{I}_2(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi})}{\langle \mathbf{h}_2, \mathbf{h}_2 \rangle \langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle \langle \boldsymbol{\phi}_2, \boldsymbol{\phi}_2 \rangle},$$

where

$$\begin{aligned} \mathcal{I}_2(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) &= \frac{L(1, \pi_2, \text{ad})L(1, \tau_2, \text{ad})}{L(1/2, \pi_2 \times \text{ad}\tau_2)} \times \\ &\quad \times \int_{\text{SL}_2(\mathbb{Q}_2)} \overline{\langle \tilde{\pi}_2(g)\mathbf{h}_2, \mathbf{h}_2 \rangle \langle \tau_2(g)\check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle \langle \omega_2(g)\boldsymbol{\phi}_2, \boldsymbol{\phi}_2 \rangle} dg. \end{aligned}$$

Denote by  $\alpha_2(\mathbf{h}_2, \check{\mathbf{g}}_2, \boldsymbol{\phi}_2)$  the integral on the right hand side. By replacing  $g$  with  $t(2)^{-1}gt(2)$ , one checks

$$\begin{aligned} \alpha_2(\mathbf{h}_2, \check{\mathbf{g}}_2, \boldsymbol{\phi}_2) &= \alpha_2(\tilde{\pi}_2(t(2))\mathbf{h}_2, \tau_2(t(2))\check{\mathbf{g}}_2, \omega_2(t(2))\boldsymbol{\phi}_2) = \\ &= \alpha_2(\tilde{\pi}_2(t(2))\mathbf{h}_2, \mathbf{g}_2, \omega_2(t(2))\boldsymbol{\phi}_2), \end{aligned}$$

using that  $\check{\mathbf{g}}_2 = \mathbf{g}_2^\sharp = \tau_2(t(2)^{-1})\mathbf{g}_2$ . Since  $\tilde{\pi}_2(t(2))\mathbf{h}_2 = \mathbf{h}_2^{(2)}$  and  $\omega_2(t(2))\boldsymbol{\phi}_2 = 2^{-1/2}\boldsymbol{\phi}_2^{(2)}$  by definition, the above shows that  $\mathcal{I}_2(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) = 2^{-1}\mathcal{I}_2(\mathbf{h}^{(2)}, \mathbf{g}, \boldsymbol{\phi}^{(2)})$ . Hence, by the previous lemma,

$$\begin{aligned} \mathcal{I}_2^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) &= \frac{\mathcal{I}_2(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi})}{\langle \mathbf{h}_2, \mathbf{h}_2 \rangle \langle \check{\mathbf{g}}_2, \check{\mathbf{g}}_2 \rangle \langle \boldsymbol{\phi}_2, \boldsymbol{\phi}_2 \rangle} = \frac{2^{-1}\mathcal{I}_2(\mathbf{h}^{(2)}, \mathbf{g}, \boldsymbol{\phi}^{(2)})}{\langle \mathbf{h}_2^{(2)}, \mathbf{h}_2^{(2)} \rangle \langle \mathbf{g}_2, \mathbf{g}_2 \rangle 2^{-1} \langle \boldsymbol{\phi}_2^{(2)}, \boldsymbol{\phi}_2^{(2)} \rangle} = \\ &= \mathcal{I}_2^\sharp(\mathbf{h}^{(2)}, \mathbf{g}, \boldsymbol{\phi}^{(2)}). \end{aligned}$$

Finally, it follows from Xue’s computation in [Xue18, Section 6] that the right hand side equals 1.  $\square$

## 9.2 COMPUTATION AT THE ARCHIMEDEAN PLACE

Finally, we deal with the computation of the regularized local period  $\mathcal{I}_\infty^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi})$  at the real place  $v = \infty$ . The approach we follow here has already been considered in [Xue] (where the case that  $g$  has weight  $\ell+1$  with  $\ell \geq k$  odd is also covered). For simplicity, in what follows we will write  $\psi = \psi_\infty^{-1}$  for the twist of the standard additive character  $\psi_\infty$  on  $\mathbb{R}$  by  $-1$ , so that  $\psi(x) = e^{-2\pi\sqrt{-1}x}$  for all  $x \in \mathbb{R}$ . Then  $\omega_\infty = \omega_\psi$ . By Iwasawa decomposition, every element  $g \in \text{SL}_2(\mathbb{R})$  can be written as

$$g = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k$$

for some  $y \in \mathbb{R}_{>0}$ ,  $x \in \mathbb{R}$  and  $k \in \text{SO}(2)$ . We consider the Haar measure  $dg = y^{-2}dx dy dk$ , where  $dx$  and  $dy$  are the Lebesgue measure on  $\mathbb{R}$ , and  $dk$  is the Haar measure on  $\text{SO}(2)$  with  $\text{vol}(\text{SO}(2)) = \pi$ .

Observe that  $\tau_\infty$  (resp.  $\pi_\infty$ ) is a discrete series representation of  $\text{PGL}_2(\mathbb{R})$  of weight  $k+1$  (resp.  $2k$ ). The archimedean component  $\check{\mathbf{g}}_\infty = \mathbf{g}_\infty$  of  $\check{\mathbf{g}}$  is a lowest



weight vector in  $\tau_\infty$ . Similarly,  $\tilde{\pi}_\infty$  is a discrete series representation of  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  of lowest  $K$ -type  $k + 1/2$ , and  $\mathbf{h}_\infty$  is a lowest weight vector in  $\tilde{\pi}_\infty$ . Let  $J$  be the Jacobi group, which arises as the semidirect product of  $\mathrm{SL}_2$  with the so-called Heisenberg group  $H$ , and it can be realized as a subgroup of  $\mathrm{Sp}_4$  (see [BS98, Section 1.1]). In explicit terms, elements in  $J$  can be written as products

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\lambda, \mu, \xi) = \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & & \xi \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

with  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2, (\lambda, \mu, \xi) \in H$ .

By virtue of [BS98, Theorem 7.3.3],  $\tilde{\pi}_\infty \otimes \omega_\infty$  is isomorphic to a discrete series representation  $\rho_\infty$  of  $J(\mathbb{R})$  of lowest  $K$ -type  $k + 1$ . In particular, the vector  $\mathbf{h}_\infty \otimes \phi_\infty \in \tilde{\pi}_\infty \otimes \omega_\infty$  is then identified under the previous isomorphism with a lowest weight vector  $\mathbf{J}_\infty \in \rho_\infty$ . Being the isomorphism  $\tilde{\pi}_\infty \otimes \omega_\infty \simeq \rho_\infty$  an isometry (see loc. cit.), we have

$$\begin{aligned} \alpha_\infty^\#(\mathbf{h}, \check{\mathbf{g}}, \phi) &= \int_{\mathrm{SL}_2(\mathbb{R})} \frac{\langle \tau(g)\check{\mathbf{g}}_\infty, \check{\mathbf{g}}_\infty \rangle \overline{\langle \tilde{\pi}(g)\mathbf{h}_\infty, \mathbf{h}_\infty \rangle} \langle \omega_\infty(g)\phi_\infty, \phi_\infty \rangle}{\|\check{\mathbf{g}}_\infty\|^2 \|\mathbf{h}_\infty\|^2 \|\phi_\infty\|^2} dg = \\ &= \int_{\mathrm{SL}_2(\mathbb{R})} \frac{\langle \tau(g)\check{\mathbf{g}}_\infty, \check{\mathbf{g}}_\infty \rangle \overline{\langle \rho(g)\mathbf{J}_\infty, \mathbf{J}_\infty \rangle}}{\|\check{\mathbf{g}}_\infty\|^2 \|\mathbf{J}_\infty\|^2} dg =: \alpha_\infty^\#(\check{\mathbf{g}}_\infty, \mathbf{J}_\infty). \end{aligned}$$

To compute  $\alpha_\infty^\#(\mathbf{h}, \check{\mathbf{g}}, \phi)$ , we will consider the explicit model  $D(k + 1, N)$  of the discrete series representation  $\rho_\infty$  that can be found in [BS98, Chapter 3]. As vector spaces, one has

$$D(k + 1, N) = \bigoplus_{m, \ell \geq 0, \ell \text{ even}} \mathbb{C} \cdot v_{m, \ell},$$

and  $\mathrm{SO}_2(\mathbb{R})$  acts on  $v_{m, \ell}$  through the character  $u \mapsto u^{k+1+m+\ell}$ . The element  $v_{0,0}$  is a lowest weight vector, and  $\mathrm{SO}_2(\mathbb{R})$  acts on the line spanned by  $v_{0,0}$  through the character  $u \mapsto u^{k+1}$ . Let  $\mathfrak{r}$  be the Lie algebra of  $R(\mathbb{R})$ , and denote by  $\mathfrak{r}_\mathbb{C}$  its complexification. Then there are certain operators  $X_+, X_-, Y_+, Y_-$  acting on  $\mathfrak{r}_\mathbb{C}$  (see loc. cit.). One has that  $\mathfrak{sl}_2$  is a Lie subalgebra of  $\mathfrak{r}$ , and  $X_\pm \in \mathfrak{sl}_{2, \mathbb{C}}$ . Also, one has  $d\rho_\infty X_- \mathbf{J}_\infty = d\rho_\infty Y_- \mathbf{J}_\infty = 0$ . The action of these operators is given by the following recipe:

$$\begin{aligned} d\rho_\infty Y_+ v_{m, \ell} &= v_{m+1, \ell}, & d\rho_\infty X_+ v_{m, \ell} &= -\frac{1}{2\pi N} v_{m+2, \ell}, \\ d\rho_\infty Y_- v_{m, \ell} &= -2\pi N m v_{m-1, \ell}, \\ d\rho_\infty X_- v_{m, \ell} &= \pi N m(m - 1) v_{m-2, \ell} - \frac{\ell}{4} (2k + \ell - 1) v_{m, \ell-2}. \end{aligned}$$

The space  $D(k+1, N)$  is endowed with an inner product  $\langle \cdot, \cdot \rangle$ , and the vectors  $v_{m,\ell}$  form an orthogonal basis with respect to this inner product. We put  $\|v\|^2 = \langle v, v \rangle$ . For each pair of integers  $m, \ell \geq 0$  with  $\ell$  even, one can compute  $\|v_{m,\ell}\|^2$  in a recursive manner from  $\|v_0\|^2$ , where we abbreviate  $v_0 := v_{0,0}$ . We shall normalize the inner product  $\langle \cdot, \cdot \rangle$  so that  $\|v_0\|^2 = 1$ .

LEMMA 9.3. *With the above notation,  $\alpha_\infty^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) = 2\pi^2 k^{-1}$ .*

*Proof.* With respect to the above model,  $\tau_\infty$  might be realized as a subrepresentation of  $\rho_\infty|_{\mathrm{SL}_2(\mathbb{R})}$ , spanned by  $v_0$ . Then we can assume the inner product for  $\tau_\infty$  to be given by the restriction of the inner product for  $\rho_\infty$ . As  $\alpha_\infty^\sharp(\check{\mathbf{g}}_\infty, \mathbf{J}_\infty)$  is normalized so that it is invariant under replacing  $\check{\mathbf{g}}_\infty$  and  $\mathbf{J}_\infty$  by scalar multiples of them, we can therefore assume that  $\check{\mathbf{g}}_\infty = \mathbf{J}_\infty = v_0$ . Then we have

$$\begin{aligned} \alpha_\infty^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) &= \alpha_\infty^\sharp(\check{\mathbf{g}}_\infty, \mathbf{J}_\infty) = \frac{1}{\|v_0\|^4} \int_{\mathrm{SL}_2(\mathbb{R})} |\langle \tau_\infty(g)v_0, v_0 \rangle|^2 dg = \\ &= \int_{\mathrm{SL}_2(\mathbb{R})} |\langle \tau_\infty(g)v_0, v_0 \rangle|^2 dg. \end{aligned}$$

Write  $A^+ := \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} : t \geq 0 \right\}$ , and consider the map

$$\begin{aligned} (\mathrm{SO}_2(\mathbb{R}) \times A^+ \times \mathrm{SO}_2(\mathbb{R})) / \{\pm 1\} &\longrightarrow \mathrm{SL}_2(\mathbb{R}) / \{\pm 1\}, \\ \left( k, \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}, k' \right) &\longmapsto k \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} k', \end{aligned}$$

where on the left hand side  $-1$  is identified with the element  $(-1, 1, -1)$ . This map is bijective outside the boundary of  $A^+$ , by virtue of Cartan decomposition. Using a similar argument to the one in [II10, Section 12], one deduces that  $dg = 2\sinh(2t) dt dk dk'$ , where  $dk$  and  $dk'$  are the Haar measure as above for which  $\mathrm{SO}_2(\mathbb{R})$  has volume  $\pi$ , and  $dt$  is the Lebesgue measure. It is well-known (cf. [Kna79]) that

$$\left\langle \tau_\infty \left( \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \right) v_0, v_0 \right\rangle = \cosh(t)^{-(k+1)},$$

and hence it follows that

$$\begin{aligned} \alpha_\infty^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) &= \int_{\mathrm{SL}_2(\mathbb{R})} |\langle \tau_\infty(g)v_0, v_0 \rangle|^2 dg = \\ &= \mathrm{vol}(\mathrm{SO}_2(\mathbb{R}))^2 \int_0^\infty \cosh(t)^{-2(k+1)} 2\sinh(2t) dt = \\ &= 2\pi^2 \int_0^\infty \cosh(t)^{-2(k+1)} \sinh(2t) dt = 2\pi^2 k^{-1}. \end{aligned}$$

□

PROPOSITION 9.4. *We have  $\mathcal{I}_\infty^\sharp(\mathbf{h}, \check{\mathbf{g}}, \boldsymbol{\phi}) = 1$ .*

*Proof.* We have seen in the previous lemma that  $\alpha_{\infty}^{\sharp}(\mathbf{h}, \check{\mathfrak{g}}, \phi) = 2\pi^2 k^{-1}$ . Besides, the ratio

$$\frac{L(1, \pi_{\infty}, \text{Ad})L(1, \tau_{\infty}, \text{Ad})}{L(1/2, \pi_{\infty} \times \text{Ad}(g))}$$

equals

$$\frac{2(2\pi)^{-k-1}\Gamma(k+1)\pi^{-1}\Gamma(1) \cdot 2(2\pi)^{-2k}\Gamma(2k)\pi^{-1}\Gamma(1)}{2^2(2\pi)^{-2k-1}\Gamma(2k)\Gamma(k-k+1) \cdot 2(2\pi)^{-k}\Gamma(k)} = \frac{k}{2\pi^2},$$

and thus it follows from the definition of  $\mathcal{I}_{\infty}^{\sharp}(\mathbf{h}, \check{\mathfrak{g}}, \phi)$  that

$$\mathcal{I}_{\infty}^{\sharp}(\mathbf{h}, \check{\mathfrak{g}}, \phi) = \frac{L(1, \pi_{\infty}, \text{Ad})L(1, \tau_{\infty}, \text{Ad})}{L(1/2, \pi_{\infty} \times \text{Ad}(g))} \alpha_{\infty}^{\sharp}(\mathbf{h}, \check{\mathfrak{g}}, \phi) = 1.$$

□

#### REFERENCES

- [AL70] A. O. L. Atkin and J. Lehner. Hecke operators on  $\Gamma_0(m)$ . *Math. Ann.*, 185:134–160, 1970.
- [AL78] A. O. L. Atkin and Wen Ch'ing Winnie Li. Twists of newforms and pseudo-eigenvalues of  $W$ -operators. *Invent. Math.*, 48(3):221–243, 1978.
- [BFSP04] S. Böcherer, M. Furusawa, and R. Schulze-Pillot. On the global Gross-Prasad conjecture for Yoshida liftings. In *Contributions to automorphic forms, geometry, and number theory*, pages 105–130. Johns Hopkins Univ. Press, Baltimore, MD, 2004.
- [BM07] E.-M. Baruch and Z. Mao. Central value of automorphic  $L$ -functions. *Geom. Funct. Anal.*, 17(2):333–384, 2007.
- [Böc85] S. Böcherer. Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen. II. *Math. Z.*, 189(1):81–110, 1985.
- [BS98] R. Berndt and R. Schmidt. *Elements of the representation theory of the Jacobi group*, volume 163 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1998.
- [Che19] S.-Y. Chen. Pullback formulae for nearly holomorphic Saito–Kurokawa lifts. *Manuscripta Math.* (2019), <https://doi.org/10.1007/s00229-019-01111-2>.
- [Das16] S. Dasgupta. Factorization of  $p$ -adic Rankin  $L$ -series. *Invent. Math.*, 205(1):221–268, 2016.

- [Del79] P. Deligne. Valeurs de fonctions  $L$  et périodes d'intégrales. In *Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 313–346. Amer. Math. Soc., Providence, R.I., 1979. With an appendix by N. Koblitz and A. Ogus.
- [EZ85] M. Eichler and D. Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.
- [Gar84] P. B. Garrett. Pullbacks of Eisenstein series; applications. In *Automorphic forms of several variables (Katata, 1983)*, volume 46 of *Progr. Math.*, pages 114–137. Birkhäuser Boston, Boston, MA, 1984.
- [Gar87] P. B. Garrett. Decomposition of Eisenstein series: Rankin triple products. *Ann. of Math. (2)*, 125(2):209–235, 1987.
- [GG09] W. T. Gan and N. Gurevich. Restrictions of Saito-Kurokawa representations. In *Automorphic forms and  $L$ -functions. I. Global aspects*, volume 488 of *Contemp. Math.*, pages 95–124. Amer. Math. Soc., Providence, RI, 2009. With an appendix by Gordan Savin.
- [GT11] W.-T. Gan and S. Takeda. Theta correspondences for  $\mathrm{GSp}(4)$ . *Represent. Theory*, 15:670–718, 2011.
- [GJ78] S. Gelbart and H. Jacquet. A relation between automorphic representations of  $\mathrm{GL}(2)$  and  $\mathrm{GL}(3)$ . *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.
- [Hid86] H. Hida. Galois representations into  $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$  attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.
- [Hid00] H. Hida. *Modular forms and Galois cohomology*, volume 69 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2000.
- [Ibu12] T. Ibukiyama. Saito-Kurokawa liftings of level  $N$  and practical construction of Jacobi forms. *Kyoto J. Math.*, 52(1):141–178, 2012.
- [Ich05] A. Ichino. Pullbacks of Saito-Kurokawa lifts. *Invent. Math.*, 162(3):551–647, 2005.
- [II08] A. Ichino and T. Ikeda. On Maass lifts and the central critical values of triple product  $L$ -functions. *Amer. J. Math.*, 130(1):75–114, 2008.
- [II10] A. Ichino and T. Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. *Geom. Funct. Anal.*, 19(5):1378–1425, 2010.

- [JL70] H. Jacquet and R. P. Langlands. *Automorphic forms on  $GL(2)$* . Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.
- [Kna79] A. W. Knap. Representations of  $GL_2(\mathbb{R})$  and  $GL_2(\mathbb{C})$ . In A. Borel and W. Casselman (editors), *Automorphic Forms, Representations and L-functions, Part 1*, volume 33, pages 87–91. Amer. Math. Soc., 1979.
- [Koh80] W. Kohlen. Modular forms of half-integral weight on  $\Gamma_0(4)$ . *Math. Ann.*, 248(3):249–266, 1980.
- [Koh82] W. Kohlen. Newforms of half-integral weight. *J. Reine Angew. Math.*, 333:32–72, 1982.
- [Koh85] W. Kohlen. Fourier coefficients of modular forms of half-integral weight. *Math. Ann.*, 271(2):237–268, 1985.
- [KT04] H. Kojima and Y. Tokuno. On the Fourier coefficients of modular forms of half integral weight belonging to Kohlen’s spaces and the critical values of zeta functions. *Tohoku Math. J. (2)*, 56(1):125–145, 2004.
- [Kud94] S. S. Kudla. The local Langlands correspondence: the non-Archimedean case. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 365–391. Amer. Math. Soc., Providence, RI, 1994.
- [MR00] M. Manickam and B. Ramakrishnan. On Shimura, Shintani and Eichler-Zagier correspondences. *Trans. Amer. Math. Soc.*, 352(6):2601–2617, 2000.
- [Nel19] P. D. Nelson. Subconvex equidistribution of cusp forms: reduction to Eisenstein observables. *Duke Math. J.*, 168(9):1665–1722, 2019.
- [Nel11] P. D. Nelson. Equidistribution of cusp forms in the level aspect. *Duke Math. J.*, 160(3):467–501, 2011.
- [Pra90] D. Prasad. Trilinear forms for representations of  $GL(2)$  and local  $\epsilon$ -factors. *Compositio Math.*, 75(1):1–46, 1990.
- [PS83] I. I. Piatetski-Shapiro. On the Saito-Kurokawa lifting. *Invent. Math.*, 71(2):309–338, 1983.
- [Qiu14] Y. Qiu. Periods of Saito-Kurokawa representations. *Int. Math. Res. Not. IMRN*, (24):6698–6755, 2014.
- [Sch02] R. Schmidt. Some remarks on local newforms for  $GL(2)$ . *J. Ramanujan Math. Soc.*, 17(2):115–147, 2002.

- [Shi72] H. Shimizu. Theta series and automorphic forms on  $GL_2$ . *J. Math. Soc. Japan*, 24:638–683, 1972.
- [Shi73] G. Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.
- [Shi77] G. Shimura. On the periods of modular forms. *Math. Ann.*, 229(3):211–221, 1977.
- [Wal80] J.-L. Waldspurger. Correspondance de Shimura. *J. Math. Pures Appl. (9)*, 59(1):1–132, 1980.
- [Wal81] J.-L. Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl. (9)*, 60(4):375–484, 1981.
- [Wat02] T. C. Watson. *Rankin triple products and quantum chaos*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.), Princeton University.
- [Woo] M. Woodbury. Trilinear forms and subconvexity of the triple product L-function. Preprint.
- [Xue] H. Xue. Central values of degree 6 L-functions. To appear in *Journal of Number Theory*.
- [Xue18] H. Xue. Fourier–Jacobi periods of classical Saito–Kurokawa lifts. *Ramanujan J.*, 45(1):111–139, 2018.
- [YZZ13] X. Yuan, S.-W. Zhang, and W. Zhang. *The Gross-Zagier formula on Shimura curves*, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

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