Non-Exactness of Direct Products of Quasi-Coherent Sheaves

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Received: March 19, 2019
Revised: September 4, 2019

Communicated by Henning Krause

Abstract. For a noetherian scheme that has an ample family of invertible sheaves, we prove that direct products in the category of quasi-coherent sheaves are not exact unless the scheme is affine. This result can especially be applied to all quasi-projective schemes over commutative noetherian rings. The main tools of the proof are the Gabriel-Popescu embedding and Roos' characterization of Grothendieck categories satisfying Ab6 and Ab4*.

2010 Mathematics Subject Classification: 14F05 (Primary), 18E20, 16D90, 16W50, 13C60 (Secondary)
Keywords and Phrases: Quasi-coherent sheaf, divisorial scheme, invertible sheaf, direct product, Gabriel-Popescu embedding, Grothendieck category

Contents

1 Introduction

The class of Grothendieck categories is a large framework that includes

- the category Mod R of right modules over a ring R,
- the category QCoh X of quasi-coherent sheaves on a scheme X, and
- the category of sheaves of abelian groups on a topological space.

One of the significant properties of Mod R for rings R among Grothendieck categories is the exactness of direct products, which is known as Grothendieck's condition Ab4*. This is immediately verified by direct computation, but it is also a consequence of the fact that Mod R has enough projectives. In general, for a Grothendieck category, the exactness of direct products is equivalent to the category having projective effacements, which is a weak lifting property that resembles the property of having enough projectives. However, it is known that there exists a Grothendieck category that has exact direct products but does not have any nonzero projective objects (see Remark 2.8). The main source of Grothendieck categories with exact direct products is a pair of a ring and an idempotent ideal of it (Remark 2.28). Such a pair is used as the *basic setup* of almost ring theory ([GR03, 2.1.1]).
For a scheme X, it is apparently rare that QCoh X has exact direct products. Indeed, it is known that direct products in QCoh X are not exact when X is either

- the projective line over a field ([Kra05]), or
- the punctured spectrum of a regular local (commutative noetherian) ring with Krull dimension at least two ([Roo66]);

see Theorems 2.6 and 2.7.
The aim of this paper is to generalize these observations to a wide class of schemes:

Theorem 1.1 (Theorem 3.15). *Let X be a divisorial noetherian scheme. Then the following conditions are equivalent:*

(1) Direct products in QCoh X *are exact.*

(2) QCoh X *has enough projectives.*

(3) X is an affine scheme.

A noetherian scheme is called *divisorial* if it admits an ample family of invertible sheaves (Definition 3.1). Since the exactness of direct products is inherited by closed subschemes, we obtain the following corollary:

Corollary 1.2 (Corollary 3.16). *Let X be a scheme that contains a non-affine divisorial noetherian scheme as a closed subscheme. Then direct products in* QCoh X *are not exact.*

Remark 1.3. Since a divisorial noetherian scheme is a generalization of a noetherian scheme having an ample invertible sheaf, Theorem 1.1 can be applied to quasi-projective schemes over commutative noetherian rings. Therefore the aforementioned results of [Kra05] and [Roo66] can be derived from Theorem 1.1.

Acknowledgments

The author thanks Shinnosuke Okawa and Akiyoshi Sannai for providing the idea of Corollary 3.16, and the anonymous referee for their valuable comments. The author was a JSPS Overseas Research Fellow. This work was supported by JSPS KAKENHI Grant Numbers JP17K14164 and JP16H06337.

2 Gabriel-Popescu embedding and Roos' theorem

2.1 Preliminaries

Convention 2.1.

(1) Throughout this paper, we fix a Grothendieck universe. A *small set* is an element of the universe. For objects X and Y in a category $\mathcal{C}$, the set $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is assumed to be small. Colimits and limits always mean those whose index sets are in bijection with small sets. The index set of a generating set is also in bijection with a small set. All rings, schemes, and modules are assumed to be small.

(2) Since we mainly work on Grothendieck categories (which resemble the category of modules over a ring), coproducts of objects are called direct sums, and products are called direct products. For a family $\{M_i\}_{i\in I}$ of objects in a category, its direct sum and direct product are denoted by $\bigoplus_{i\in I} M_i$ and $\prod_{i\in I} M_i$, respectively. A direct limit (resp. inverse limit) is a colimit (resp. limit) of a direct system (resp. inverse system) indexed by a directed set.

We recall Grothendieck's conditions on exactness of colimits and limits and the definition of a generating set:

Definition 2.2.

(1) Let $\mathcal{A}$ be an abelian category that admits direct sums (resp. direct products).

  (a) We say that $\mathcal{A}$ *satisfies Ab4* (resp. *Ab4**) if direct sums (resp. *direct products*) are exact in $\mathcal{A}$, that is, for every family of short exact sequences

$$0 \to L_i \to M_i \to N_i \to 0 \quad (i \in I)$$

in $\mathcal{A}$ (where I is in bijection with a small set), the termwise direct sum (resp. direct product)

$$0 \to \bigoplus_{i \in I} L_i \to \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} N_i \to 0$$

is again a short exact sequence.

(b) We say that $\mathcal{A}$ *satisfies Ab5* (resp. *Ab5**) if direct limits (resp. *inverse limits*) are exact in $\mathcal{A}$, that is, for every direct system (resp. inverse system) of short exact sequences in $\mathcal{A}$, the termwise direct limit (resp. inverse limit) is again exact.

(2) Let $\mathcal{A}$ be an abelian category. A set $\mathcal{U}$ of objects in $\mathcal{A}$ that is in bijection with a small set is called a *generating set* if for every nonzero morphism $f\colon X \to Y$ in $\mathcal{A}$, there exists $U \in \mathcal{U}$ and a morphism $g\colon U \to X$ such that $fg \neq 0$. An object $U \in \mathcal{A}$ is called a *generator* if the singleton $\{U\}$ is a generating set.

(3) An abelian category is called a *Grothendieck category* if it satisfies Ab5 and has a generator.

Remark 2.3.

(1) If an abelian category admits direct sums (resp. direct products), then it admits colimits (resp. limits) (see [KS06, Proposition 2.2.9]). Direct sums and direct limits (resp. direct products and inverse limits) are always right exact (resp. left exact). So the conditions in Definition 2.2 (1) only require left exactness (resp. right exactness).

(2) For an abelian category with direct sums, Ab5 implies Ab4 ([Pop73, Corollary 2.8.9]). See [Pop73, Theorem 2.8.6] for conditions equivalent to Ab5.

(3) It is known that every Grothendieck category admits direct products ([Pop73, Corollary 3.7.10]).

(4) If an abelian category $\mathcal{A}$ admits direct sums and has a generating set $\{U_i\}_{i \in I}$, then the direct sum $\bigoplus_{i \in I} U_i$ is a generator in $\mathcal{A}$ ([Pop73, Proposition 2.8.2]).

Example 2.4. Let R be a ring. Then the category $\operatorname{Mod} R$ of right R-modules is a Grothendieck category satisfying Ab4*.

Example 2.5. Let X be a scheme. Then the category $\operatorname{QCoh} X$ of quasi-coherent sheaves on X is an abelian category satisfying Ab5. It was shown by Gabber that $\operatorname{QCoh} X$ has a generator (see [Bra18, Remarks A.1 and A.2]). Hence $\operatorname{QCoh} X$ is a Grothendieck category.

Example 2.5 implies that $\operatorname{QCoh} X$ for a scheme X admits direct products. However, direct products are not necessarily exact as the following two results show (see also [Wu88] and [Kan19, Example 3.10] for related results):

THEOREM 2.6 (Keller; see [Kra05, Example 4.9]). *Let $X := \mathbb{P}^1_k$ be the projective line over a field k. Then $\operatorname{QCoh} X$ does not satisfy Ab4*.*

THEOREM 2.7 (Roos [Roo66, Example 3]). *Let R be a regular local (commutative noetherian) ring with maximal ideal $\mathfrak{m}$. Define $X := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ as an open subscheme of $\operatorname{Spec} R$. Then $\operatorname{QCoh} X$ satisfies Ab4* if and only if $\dim R \leq 1$.*

REMARK 2.8. Recall that an abelian category $\mathcal{A}$ is said to have *enough projectives* if each object in $\mathcal{A}$ is a quotient object of some projective object. Every Grothendieck category that has enough projectives satisfies Ab4* (the dual of [Pop73, Corollary 3.2.9]). The converse does not hold. Indeed, it is shown in [Roo06, Example 4.2] that there exists a nonzero Grothendieck category that satisfies Ab4* but has no nonzero projective objects.
It is known that a Grothendieck category satisfies Ab4* if and only if it has *projective effacements* ([Gro57, Remark 1 in p. 137]; see also [Roo06, Corollary 1.4]), which can be regarded as a weak form of having enough projectives.

REMARK 2.9. The category of sheaves of abelian groups on a topological space is also a typical example of a Grothendieck category. The exactness of direct products in such a category is characterized in [Roo66, Corollary 1] (see also [Roo06, Theorem 1.7]).

We recall the definitions and basic properties of some classes of subcategories.

DEFINITION 2.10. Let $\mathcal{G}$ be a Grothendieck category.

(1) A *Serre subcategory* of $\mathcal{G}$ is a full subcategory of $\mathcal{G}$ closed under subobjects, quotient objects, and extensions. If $\mathcal{X} \subset \mathcal{G}$ is a Serre subcategory, then we have the *quotient category* of $\mathcal{G}$ by $\mathcal{X}$, which is denoted by $\mathcal{G}/\mathcal{X}$, together with a canonical functor $\mathcal{G} \to \mathcal{G}/\mathcal{X}$ (see [Pop73, Section 4.3]).

(2) A Serre subcategory $\mathcal{X} \subset \mathcal{G}$ is called a *localizing subcategory* if the canonical functor $\mathcal{G} \to \mathcal{G}/\mathcal{X}$ admits a right adjoint.

(3) A localizing subcategory $\mathcal{X} \subset \mathcal{G}$ is called a *bilocalizing subcategory* if the canonical functor $\mathcal{G} \to \mathcal{G}/\mathcal{X}$ also admits a left adjoint.

REMARK 2.11. If $\mathcal{X}$ is a localizing subcategory of a Grothendieck category $\mathcal{G}$, then $\mathcal{G}/\mathcal{X}$ is again a Grothendieck category ([Pop73, Corollary 4.6.2]). The right adjoint $G\colon \mathcal{G}/\mathcal{X} \to \mathcal{G}$ of the canonical functor $F\colon \mathcal{G} \to \mathcal{G}/\mathcal{X}$ is fully faithful, and thus the counit $FG \to 1_{\mathcal{G}/\mathcal{X}}$ is an isomorphism ([Pop73, Proposition 4.4.3]).
See [Pop73, Section 4.3] or [Kan15b, Theorem 5.11] for basic properties of the quotient category $\mathcal{G}/\mathcal{X}$.

Example 2.12 (See [Bra18, Example 4.3]). Let X be a quasi-separated scheme and let $i\colon U \hookrightarrow X$ be an open immersion from a quasi-compact open subscheme U. Then $i_*\colon \operatorname{QCoh} U \to \operatorname{QCoh} X$ and its left adjoint $i^*\colon \operatorname{QCoh} X \to \operatorname{QCoh} U$ induces an equivalence

$$\frac{\operatorname{QCoh} X}{\mathcal{Y}} \xrightarrow{\sim} \operatorname{QCoh} U,$$

where $\mathcal{Y} \subset \operatorname{QCoh} X$ is the localizing subcategory consisting of all objects $\mathcal{M} \in \operatorname{QCoh} X$ with $i^*\mathcal{M} = 0$.

Proposition 2.13. *Let $\mathcal{G}$ be a Grothendieck category.*

(1) Let $\mathcal{X} \subset \mathcal{G}$ be a Serre subcategory. Then the following conditions are equivalent:

(a) $\mathcal{X}$ is a localizing subcategory.

(b) $\mathcal{X}$ is closed under direct sums.

(c) Every object $M \in \mathcal{G}$ has a largest subobject belonging to $\mathcal{X}$.

(2) Let $\mathcal{X} \subset \mathcal{G}$ be a localizing subcategory. Then the following conditions are equivalent:

(a) $\mathcal{X}$ is a bilocalizing subcategory.

(b) $\mathcal{X}$ is closed under direct products.

(c) Every object $M \in \mathcal{G}$ has a largest quotient object belonging to $\mathcal{X}$, that is, M has a smallest subobject among those L satisfying $M/L \in \mathcal{X}$.

Proof. (1) [Pop73, Theorem 4.5.2 and Proposition 4.6.3].
(2) This can be shown in a similar way to the proof of [Pop73, Theorem 4.21.1] for the category of modules over a ring. □

Definition 2.14. Let $\mathcal{G}$ be a Grothendieck category. A *closed subcategory* of $\mathcal{G}$ is a full subcategory closed under subobjects, quotient objects, direct sums, and direct products.

Remark 2.15. Since the direct sum $\bigoplus_{i\in I} M_i$ of objects in a Grothendieck category can be regarded as a subobject of the direct product $\prod_{i\in I} M_i$, the condition of being closed under direct sums in Definition 2.14 can be omitted. By Proposition 2.13 (2), a full subcategory of a Grothendieck category is bilocalizing if and only if it is localizing and closed.

Proposition 2.16. *Let $\mathcal{G}$ be a Grothendieck category and let $\mathcal{C} \subset \mathcal{G}$ be a full subcategory closed under subobjects and quotient objects. Then the following conditions are equivalent:*

(1) $\mathcal{C}$ is a closed subcategory.

(2) Every object $M \in \mathcal{G}$ has a largest quotient object belonging to $\mathcal{C}$.

Proof. [Kan15a, Proposition 11.2]. ☐

Remark 2.17. For a ring R, there exists a bijective correspondence between the two-sided ideals of R and the closed subcategories of $\operatorname{Mod} R$ that sends each I to $\operatorname{Mod}(R/I)$. The bilocalizing subcategories correspond to the idempotent ideals. See [Kan15a, Theorem 11.3 and Proposition 12.6] for more details.
For a scheme X, it is known that the closed subcategories of $\operatorname{QCoh} X$ are in bijection with the closed subschemes of X provided that one of the following conditions holds:

- X is locally noetherian ([Kan15a, Theorem 1.3]).
- X is separated ([Bra18, Proposition A.5]).

2.2 Gabriel-Popescu embedding

A generalization of the Gabriel-Popescu embedding is one of the main tools to prove Theorem 1.1. First we recall the original version:

Theorem 2.18 (Gabriel-Popescu embedding [PG64]). *Let $\mathcal{G}$ be a Grothendieck category and let $U \in \mathcal{G}$ be a generator. Then the functor $\operatorname{Hom}_{\mathcal{G}}(U,-)\colon \mathcal{G} \to \operatorname{Mod}\operatorname{End}_{\mathcal{G}}(U)$ induces an equivalence*

$$\mathcal{G} \xrightarrow{\sim} \frac{\operatorname{Mod}\operatorname{End}_{\mathcal{G}}(U)}{\mathcal{X}},$$

where $\mathcal{X} \subset \operatorname{Mod}\operatorname{End}_{\mathcal{G}}(U)$ is the localizing subcategory consisting of all $M \in \operatorname{Mod}\operatorname{End}_{\mathcal{G}}(U)$ annihilated by the left adjoint of $\operatorname{Hom}_{\mathcal{G}}(U,-)$.

If we have a generating set $\{U_i\}_{i\in I}$ in a Grothendieck category $\mathcal{G}$, then the direct sum $\bigoplus_{i\in I} U_i$ is a generator in $\mathcal{G}$ and we can apply the Gabriel-Popescu embedding. On the other hand, there is a generalized version of the embedding that respects the structure of the given generating set (Theorem 2.21). To state the result, we recall some basic facts on rings that do not necessarily have an identity element.

Definition 2.19. Let R be a ring not necessarily with identity.

(1) A *complete set of orthogonal idempotents* in R is a set of idempotents $\{e_i\}_{i\in I} \subset R$ such that

- $\{e_i\}_{i\in I}$ is orthogonal, that is, $e_ie_j = 0$ for $i \neq j$, and
- $R = \bigoplus_{i,j\in I} e_iRe_j$ (or equivalently, $R = \bigoplus_{i\in I} e_iR = \bigoplus_{i\in I} Re_i$).

We say that R *has enough idempotents* if it admits a complete set of orthogonal idempotents.

(2) Suppose that R has enough idempotents. The category of all right R-modules is denoted by $\operatorname{Mod} R$. Define $\operatorname{MOD} R \subset \operatorname{Mod} R$ to be the full subcategory consisting of right R-modules M with $MR = M$.

Remark 2.20. Let R be a ring not necessarily with identity and let $\{e_i\}_{i\in I}$ be a complete set of orthogonal idempotents.

(1) For every $M \in \operatorname{Mod} R$, the condition $M = MR$ is equivalent to $M = \bigoplus_{i\in I} Me_i$.

(2) The *Dorroh overring* R^* of R is $\mathbb{Z} \times R$ as an abelian group and has multiplication

$$(n_1, r_1) \cdot (n_2, r_2) := (n_1 n_2, n_1 r_2 + n_2 r_1 + r_1 r_2).$$

The Dorroh overring R^* is a ring with identity $(1, 0)$, and R is identified with $0 \times R \subset R^*$ (see [Wis91, Sections 1.5 and 6.3]). Since the forgetful functor $\operatorname{Mod} R^* \to \operatorname{Mod} R$ is an equivalence, the category $\operatorname{Mod} R$ is a Grothendieck category satisfying Ab4*, and colimits and limits in $\operatorname{Mod} R$ can be computed in the same way as the category of right modules over a ring with identity.

(3) $\operatorname{MOD} R \subset \operatorname{Mod} R$ is closed under subobjects, quotient objects, extensions, and direct sums ([CIENT04, Proposition 0.1]) and R is a projective generator in $\operatorname{MOD} R$. Hence $\operatorname{MOD} R$ is also a Grothendieck category satisfying Ab4*, but limits in $\operatorname{MOD} R$ are different from those computed in $\operatorname{Mod} R$ in general.

Theorem 2.21 (Năstăsescu and Chiteş [NC10, Theorem 2.1]). *Let $\mathcal{G}$ be a Grothendieck category and let $\{U_i\}_{i\in I}$ be a generating set in $\mathcal{G}$. Then $R := \bigoplus_{i,j\in I} \operatorname{Hom}_{\mathcal{G}}(U_i, U_j)$ is a ring with enough idempotents, and the functor $G := \bigoplus_{i\in I} \operatorname{Hom}_{\mathcal{G}}(U_i, -)\colon \mathcal{G} \to \operatorname{MOD} R$ induces an equivalence*

$$\mathcal{G} \xrightarrow{\sim} \frac{\operatorname{MOD} R}{\mathcal{X}},$$

where $\mathcal{X} \subset \operatorname{MOD} R$ is the localizing subcategory consisting of all $M \in \operatorname{MOD} R$ annihilated by the left adjoint of G.

Remark 2.22. For $\mathbb{Z}$-linear categories $\mathcal{C}$ and $\mathcal{D}$, let $\operatorname{Func}_{\mathbb{Z}}(\mathcal{C}, \mathcal{D})$ denote the category of $\mathbb{Z}$-functors from $\mathcal{C}$ to $\mathcal{D}$. In the setting of Theorem 2.21, we have an equivalence $\operatorname{Func}_{\mathbb{Z}}(\mathcal{U}^{\mathrm{op}}, \operatorname{Mod} \mathbb{Z}) \xrightarrow{\sim} \operatorname{MOD} R$ given by $F \mapsto \bigoplus_{i\in I} F(U_i)$, where $\mathcal{U} := \{U_i\}_{i\in I}$ is regarded as a full subcategory of $\mathcal{G}$. Hence Theorem 2.21 can be interpreted in terms of the functor category.

Theorem 2.23 (Prest [Pre80, Theorem 1.1]; see also [NC10, Corollary 2.5]). *Let $\mathcal{G}$ be a Grothendieck category and let $\mathcal{U}$ be a generating set in $\mathcal{G}$. Then the functor $G\colon \mathcal{G} \to \operatorname{Func}_{\mathbb{Z}}(\mathcal{U}^{\mathrm{op}}, \operatorname{Mod} \mathbb{Z})$ defined by $M \mapsto \operatorname{Hom}_{\mathcal{G}}(-, M)$ induces an equivalence*

$$\mathcal{G} \xrightarrow{\sim} \frac{\operatorname{Func}_{\mathbb{Z}}(\mathcal{U}^{\mathrm{op}}, \operatorname{Mod} \mathbb{Z})}{\mathcal{X}},$$

where $\mathcal{X} \subset \operatorname{Func}_{\mathbb{Z}}(\mathcal{U}^{\mathrm{op}}, \operatorname{Mod} \mathbb{Z})$ is the localizing subcategory consisting of all objects annihilated by the left adjoint of G.

2.3 Roos' theorem

A generalization of Roos' theorem (Theorem 2.30) is another main ingredient of the proof of Theorem 1.1. To state the result, we recall Grothendieck's condition Ab6:

DEFINITION 2.24. We say that an abelian category $\mathcal{A}$ with direct sums *satisfies Ab6* if the following assertion holds for every object $M \in \mathcal{A}$: For every family $\{\{L_i^j\}_{i\in I_j}\}_{j\in J}$ of directed sets of subobjects of M with respect to inclusion, we have

$$\bigcap_{j\in J}\left(\sum_{i\in I_j} L_i^j\right) = \sum_{(i_j)_{j\in J}\in\prod_{j\in J} I_j}\left(\bigcap_{j\in J} L_{i_j}^j\right).$$

REMARK 2.25. For an abelian category with direct sums, Ab6 implies Ab5 ([Pop73, Corollary 2.8.13]).

REMARK 2.26. Roos [Roo67, Theorem 1] showed that condition Ab6 has the following characterization (see also [Pop73, Exercise 3.5.7]): Let $\mathcal{G}$ be a Grothendieck category. A subobject L of an object $M \in \mathcal{G}$ is said to be *of finite type relative to* M if for every directed set $\{L_i\}_{i\in I}$ of subobjects of M with respect to inclusion satisfying $\sum_{i\in I} L_i = M$, the directed set $\{L_i \cap L\}_{i\in I}$ eventually stabilizes, that is, $L_i \cap L = L$ for some $i \in I$. Then $\mathcal{G}$ satisfies Ab6 if and only if every object $M \in \mathcal{G}$ is the sum of all subobjects of finite type relative to M.

REMARK 2.27. Grothendieck categories that we encounter in practice often satisfy Ab6:

(1) Let $\mathcal{U}$ be a small $\mathbb{Z}$-category and let $\mathcal{G}$ be a Grothendieck category satisfying Ab6. Then $\mathrm{Func}_{\mathbb{Z}}(\mathcal{U}, \mathcal{G})$ is a Grothendieck category satisfying Ab6 ([Pop73, Theorem 3.4.2]). In particular, for every ring R, $\mathrm{Mod}\, R$ is a Grothendieck category satisfying Ab6.

(2) A Grothendieck category is called *locally noetherian* if admits a generating set consisting of noetherian objects. Every locally noetherian Grothendieck category satisfies Ab6. This follows from Remark 2.26 since all noetherian subobjects of an object M are of finite type relative to M.

(3) Let $\mathcal{G}$ be a Grothendieck category satisfying Ab6 (resp. Ab4*) and let $\mathcal{X} \subset \mathcal{G}$ be a bilocalizing subcategory. Then $\mathcal{G}/\mathcal{X}$ is a Grothendieck category satisfying Ab6 (resp. Ab4*). This follows because the canonical functor $\mathcal{G} \to \mathcal{G}/\mathcal{X}$ preserves all colimits and limits.

REMARK 2.28. Let R be a ring and let $I \subset R$ be an idempotent ideal. Then, as in Remark 2.17, $\mathrm{Mod}(R/I)$ is a bilocalizing subcategory of $\mathrm{Mod}\, R$. Thus the quotient category of $\mathrm{Mod}\, R$ by $\mathrm{Mod}(R/I)$ is a Grothendieck category satisfying Ab6 and Ab4* by Remark 2.27.

Roos' theorem shows that all Grothendieck categories satisfying Ab6 and Ab4* arise in the way of Remark 2.28:

THEOREM 2.29 (Roos [Roo65, Theorem 1]). *Let $\mathcal{G}$ be a Grothendieck category and let $U \in \mathcal{G}$ be a generator. Define the localizing subcategory $\mathcal{X} \subset \operatorname{Mod}\operatorname{End}_{\mathcal{G}}(U)$ as in Theorem 2.18. Then the following conditions are equivalent:*

(1) $\mathcal{G}$ satisfies Ab6 and Ab4.*

(2) $\mathcal{X}$ is closed under direct products, that is, $\mathcal{X} \subset \operatorname{Mod}\operatorname{End}_{\mathcal{G}}(U)$ is a bilocalizing subcategory.

Roos' theorem can be generalized so that it fits into the setting of the generalized Gabriel-Popescu embedding:

THEOREM 2.30. *Let $\mathcal{G}$ be a Grothendieck category and let $\{U_i\}_{i\in I}$ be a generating set in $\mathcal{G}$. Let $R := \bigoplus_{i,j\in I} \operatorname{Hom}_{\mathcal{G}}(U_i, U_j)$. Define the localizing subcategory $\mathcal{X} \subset \operatorname{MOD} R$ as in Theorem 2.21. Then the following conditions are equivalent:*

(1) $\mathcal{G}$ satisfies Ab6 and Ab4.*

(2) $\mathcal{X}$ is closed under direct products, that is, $\mathcal{X} \subset \operatorname{MOD} R$ is a bilocalizing subcategory.

Proof. The proof of Theorem 2.29 written in [Pop73, Theorem 4.21.6] also works in this setting. The proof is modified as follows:

(a) Use the generating set $\{U_i\}_{i\in I}$ instead of the generator U. Use $\operatorname{MOD} R$ instead of $\operatorname{Mod} A$.

(b) Define S and $\mathcal{F}$ to be G and $\mathcal{X}$ in Theorem 2.21, respectively.

(c) In the conclusion of [Pop73, Lemma 4.21.3], f should run over all elements of M that are homogeneous in the sense that each of them belongs to $\operatorname{Hom}_{\mathcal{G}}(U_i, X)$ for some $i \in I$.

(d) In [Pop73, Lemma 4.21.4], define X' to be $\bigoplus_{i\in I} U_i^{\oplus \operatorname{Hom}_{\mathcal{G}}(U_i, X)}$.

(e) In [Pop73, Lemma 4.21.5], define $R(X)$ to be the submodule consisting of all finite sums of elements $f\colon U_i \to X$ for various i satisfying the same property with U replaced by U_i. The conclusion of [Pop73, Lemma 4.21.5] is modified in the same way as (c). ☐

Theorem 2.30 can also be stated in terms of a functor category:

COROLLARY 2.31. *Let $\mathcal{G}$ be a Grothendieck category and let $\mathcal{U}$ be a generating set in $\mathcal{G}$. Define the localizing subcategory $\mathcal{X} \subset \operatorname{Func}_{\mathbb{Z}}(\mathcal{U}^{\mathrm{op}}, \operatorname{Mod}\mathbb{Z})$ as in Theorem 2.23. Then the following conditions are equivalent:*

(1) $\mathcal{G}$ satisfies Ab6 and Ab4.*

(2) *$\mathcal{X}$ is closed under direct products, that is, $\mathcal{X} \subset \operatorname{Func}_{\mathbb{Z}}(\mathcal{U}^{\mathrm{op}}, \operatorname{Mod}\mathbb{Z})$ is a bilocalizing subcategory.*

Proof. This is immediate from Theorem 2.30 in view of Remark 2.22. ◻

3 Divisorial noetherian schemes

In this section, we prove the main results. Whenever we consider a scheme X, unadorned tensor products are tensor products of quasi-coherent sheaves on X. The structure sheaf of X is denoted by $\mathcal{O}_X$, and $\Gamma(X, -)$ is the global section functor.
We recall the definition of a divisorial scheme:

Definition 3.1 ([BS03, Proposition 1.1]; see also [Bor67, Definition 3.1]). Let X be a quasi-compact and quasi-separated scheme.

(1) A finite family $\{\mathcal{L}_1, \dots, \mathcal{L}_r\}$ of invertible sheaves on X is called an *ample family* if the set

$$\{\, X_s \mid s \in \Gamma(X, \mathcal{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes d_r}),\ d_1, \dots, d_r \geq 0 \text{ are integers} \,\}$$

is an open basis of X, where $X_s \subset X$ is the open subset consisting of all $x \in X$ such that s_x does not belong the unique maximal ideal of $(\mathcal{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes d_r})_x$.

(2) X is called *divisorial* if it admits an ample family of invertible sheaves.

Remark 3.2. An ample family of invertible sheaves is a generalization of an ample invertible sheaf (see [Gro61, Section 4.5]). In particular, every quasi-projective scheme over a commutative noetherian ring is divisorial.

The following fact is essential for our proof:

Proposition 3.3 ([Bor67, Theorem 3.3]). *Let X be a divisorial noetherian scheme. Then every coherent sheaf on X is isomorphic to a quotient of a direct sum of invertible sheaves.*

Remark 3.4. For a noetherian scheme X, the category $\operatorname{QCoh} X$ is a locally noetherian Grothendieck category ([Gab62, Theorem 1 in p. 443]). An object in $\operatorname{QCoh} X$ is noetherian if and only if it is a coherent sheaf on X.
Hence Proposition 3.3 implies that if X is a divisorial noetherian scheme, then $\operatorname{QCoh} X$ has a generating set consisting of invertible sheaves.

Setting 3.5. In the rest of this section, let X be a divisorial noetherian scheme. We use the following notations:

(1) Fix a generating set $\{\mathcal{L}_\lambda\}_{\lambda \in \Lambda}$ in $\operatorname{QCoh} X$ consisting of invertible sheaves (see Remark 3.4).

(2) Let I be the free abelian group generated by Λ, that is, $I = \bigoplus_{\lambda \in \Lambda} \mathbb{Z}\lambda$. For each $i = \sum_j n_j \lambda_j \in I$, where $n_j \in \mathbb{Z}$ and $\lambda_j \in \Lambda$, define the invertible sheaf

$$\mathcal{L}_i := \bigotimes_j \mathcal{L}_{\lambda_j}^{\otimes n_j}.$$

Then $\{\mathcal{L}_i\}_{i \in I} = \{\mathcal{L}_{-i}\}_{i \in I}$ is also a generating set in $\operatorname{QCoh} X$. Define the ring not necessarily with identity

$$R := \bigoplus_{i,j \in I} \operatorname{Hom}_X(\mathcal{L}_{-i}, \mathcal{L}_{-j}).$$

Let $e_i \in \operatorname{Hom}_X(\mathcal{L}_{-i}, \mathcal{L}_{-i})$ be the identity morphism for every $i \in I$. Then $\{e_i\}_{i \in I}$ is a complete set of orthogonal idempotents of R.

(3) Define the I-graded ring

$$S := \bigoplus_{i \in I} \Gamma(X, \mathcal{L}_i) = \bigoplus_{i \in I} \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{L}_i)$$

with the following multiplication: For each $f \in \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{L}_i)$ and $g \in \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{L}_j)$, $gf \in \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{L}_{i+j})$ is defined to be the composite

$$\begin{array}{ccccc} \mathcal{O}_X & \xrightarrow{f} & \mathcal{L}_i & & \mathcal{L}_{i+j} \\ & & \downarrow\wr & & \wr\uparrow \\ & & \mathcal{L}_i \otimes \mathcal{O}_X & \xrightarrow[\mathcal{L}_i \otimes g]{} & \mathcal{L}_i \otimes \mathcal{L}_j, \end{array}$$

where the isomorphisms are the canonical ones. It is straightforward to see that S is a commutative ring (*with* identity). Denote by $\operatorname{Mod}^I S$ the category of I-graded S-modules whose morphisms are homogeneous S-homomorphisms of degree 0. For an object $M \in \operatorname{Mod}^I S$ and $j \in I$, define the degree shift $M(j) \in \operatorname{Mod}^I S$ to be the same S-module with new grading $M(j)_i = M_{i+j}$. This defines the equivalence

$$(j)\colon \operatorname{Mod}^I S \xrightarrow{\sim} \operatorname{Mod}^I S.$$

Remark 3.6. The *I-algebra* associated to the I-graded ring S is the ring A not necessarily with identity defined by

$$A := \bigoplus_{i,j \in I} A_{i,j}, \quad \text{where} \quad A_{i,j} := S_{j-i},$$

The multiplication $A_{i,j} \times A_{j',k} \to A$ is given by that of S for $j = j'$ and the zero map for $j \neq j'$ (see the last paragraph in [VdB11, p. 3988]). There is an isomorphism $A \xrightarrow{\sim} R$ of rings not necessarily with identities given by

$$- \otimes \mathcal{L}_{-j}\colon A_{i,j} = \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{L}_{j-i}) \xrightarrow{\sim} \operatorname{Hom}_X(\mathcal{L}_{-j}, \mathcal{L}_{-i}) = e_i R e_j.$$

There is an equivalence $\operatorname{MOD} R \xrightarrow{\sim} \operatorname{Mod}^I S$ that sends each $M \in \operatorname{MOD} R$ to $\bigoplus_{i\in I} Me_i$, where the S-action $Me_i \times S_j \to Me_{i+j}$ is induced from the A-action $Me_i \times A_{i,i+j} \to Me_{i+j}$, or the R-action $Me_i \times \operatorname{Hom}_X(\mathcal{L}_{-i-j}, \mathcal{L}_{-i}) \to Me_{i+j}$.

Lemma 3.7. *Assume that* $\operatorname{QCoh} X$ *satisfies Ab4*. Then there exists an equivalence*

$$\operatorname{QCoh} X \xrightarrow{\sim} \frac{\operatorname{Mod}^I S}{\mathcal{Y}},$$

where $\mathcal{Y} \subset \operatorname{Mod}^I S$ *is a bilocalizing subcategory closed under degree shifts, that sends* $\mathcal{O}_X \in \operatorname{QCoh} X$ *to an object isomorphic to the image of* $S \in \operatorname{Mod}^I S$ *by the canonical functor to* $(\operatorname{Mod}^I S)/\mathcal{Y}$.

Proof. $\operatorname{QCoh} X$ satisfies Ab6 since it is locally noetherian (Remark 2.27 (2)). Applying Theorem 2.30 to the generating set $\{\mathcal{L}_{-i}\}_{i\in I}$, we deduce that the functor $\bigoplus_{i\in I} \operatorname{Hom}_X(\mathcal{L}_{-i}, -)\colon \operatorname{QCoh} X \to \operatorname{MOD} R$ induces an equivalence

$$\operatorname{QCoh} X \xrightarrow{\sim} \frac{\operatorname{MOD} R}{\mathcal{Y}'}$$

for some bilocalizing subcategory $\mathcal{Y}' \subset \operatorname{MOD} R$. The equivalence $\operatorname{MOD} R \xrightarrow{\sim} \operatorname{Mod}^I S$ in Remark 3.6 induces an equivalence

$$\frac{\operatorname{MOD} R}{\mathcal{Y}'} \xrightarrow{\sim} \frac{\operatorname{Mod}^I S}{\mathcal{Y}}$$

for some bilocalizing subcategory $\mathcal{Y} \subset \operatorname{Mod}^I S$.
Denote by G the composite

$$\operatorname{QCoh} X \to \operatorname{MOD} R \xrightarrow{\sim} \operatorname{Mod}^I S$$

and let F be the left adjoint of G. Then $G(\mathcal{O}_X) = \bigoplus_{i\in I} \operatorname{Hom}_X(\mathcal{L}_{-i}, \mathcal{O}_X)$, which is isomorphic to S via

$$- \otimes \mathcal{L}_i\colon \operatorname{Hom}_X(\mathcal{L}_{-i}, \mathcal{O}_X) \xrightarrow{\sim} \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{L}_i).$$

Let $j \in I$. For every object $\mathcal{M} \in \operatorname{QCoh} X$,

$$\begin{aligned} G(\mathcal{M} \otimes \mathcal{L}_j)_i &= \operatorname{Hom}_X(\mathcal{L}_{-i}, \mathcal{M} \otimes \mathcal{L}_j) \\ &\cong \operatorname{Hom}_X(\mathcal{L}_{-i-j}, \mathcal{M}) \\ &= G(\mathcal{M})_{i+j} = G(\mathcal{M})(j)_i, \end{aligned}$$

and it is straightforward to see that this gives an isomorphism $G(\mathcal{M} \otimes \mathcal{L}_j) \cong G(\mathcal{M})(j)$ that is functorial in $\mathcal{M}$. Thus the diagram

$$\begin{array}{ccc} \operatorname{QCoh} X & \xrightarrow{G} & \operatorname{Mod}^I S \\ {\scriptstyle -\otimes\mathcal{L}_j}\downarrow\wr & & \wr\downarrow{\scriptstyle (j)} \\ \operatorname{QCoh} X & \xrightarrow[G]{} & \operatorname{Mod}^I S \end{array}$$

commutes up to isomorphism. The adjoint property implies that the diagram

$$\begin{array}{ccc} \operatorname{QCoh} X & \xleftarrow{F} & \operatorname{Mod}^I S \\ {\scriptstyle -\otimes\mathcal{L}_j}\downarrow\wr & & \wr\downarrow{\scriptstyle (j)} \\ \operatorname{QCoh} X & \xleftarrow[F]{} & \operatorname{Mod}^I S \end{array}$$

also commutes up to isomorphism. Since $\mathcal{Y}$ consists of all objects in $\operatorname{Mod}^I S$ annihilated by F, it is closed under degree shifts. □

Setting 3.8. In the subsequent lemmas, we assume that $\operatorname{QCoh} X$ satisfies Ab4*, and use the following notations in addition to Setting 3.5:

(1) Let $\mathcal{Y} \subset \operatorname{Mod}^I S$ be the bilocalizing subcategory closed under degree shifts obtained in Lemma 3.7. Let $F\colon \operatorname{Mod}^I S \to \operatorname{QCoh} X$ be the composite

$$\operatorname{Mod}^I S \to \frac{\operatorname{Mod}^I S}{\mathcal{Y}} \xrightarrow{\sim} \operatorname{QCoh} X$$

of the canonical functor and the equivalence obtained in Lemma 3.7. Let G be its right adjoint. These are the same functors as those appeared in the proof of Lemma 3.7. Note that $G(\mathcal{O}_X) \cong S$ by the proof of Lemma 3.7, and $F(S) \cong FG(\mathcal{O}_X) \cong \mathcal{O}_X$ is a noetherian object by Remarks 2.11 and 3.4.

(2) Define $\mathcal{Z} \subset \operatorname{Mod}^I S$ to be the full subcategory consisting of all objects $M \in \operatorname{Mod}^I S$ such that none of the nonzero subquotients of M belong to $\mathcal{Y}$.

We will show that $\mathcal{Y} = 0$ and $\mathcal{Z} = \operatorname{Mod}^I S$ in Lemma 3.14. So Lemmas 3.9, 3.10, 3.12 and 3.13 are only used to prove Lemma 3.14 and they will eventually become trivial.

Lemma 3.9. *Let $M \in \operatorname{Mod}^I S$ be an object that belongs to $\mathcal{Y}$. Then every $N \in \operatorname{Mod}^I S$ satisfying $N \operatorname{Ann}_S(M) = 0$ belongs to $\mathcal{Y}$.*

Proof. Since $\operatorname{Ann}_S(M) = \bigcap_x \operatorname{Ann}_S(x)$, where x runs over all homogeneous elements of M, we have the canonical monomorphism

$$\frac{S}{\operatorname{Ann}_S(M)} \to \prod_x \frac{S}{\operatorname{Ann}_S(x)},$$

and $S/\operatorname{Ann}_S(x) \cong (xS)(\deg x) \subset M(\deg x) \in \mathcal{Y}$. Hence $S/\operatorname{Ann}_S(M)$ belongs to $\mathcal{Y}$. The condition $N \operatorname{Ann}_S(M) = 0$ implies that N is a quotient of a direct sum of copies of $S/\operatorname{Ann}_S(M)$. Therefore $N \in \mathcal{Y}$. □

Lemma 3.10.

(1) $\mathcal{Z} \subset \operatorname{Mod}^I S$ is a localizing subcategory closed under degree shifts.

(2) *Let* $M \in \operatorname{Mod}^I S$ *be a noetherian object that belongs to* $\mathcal{Z}$. *Then every* $N \in \operatorname{Mod}^I S$ *satisfying* $N \operatorname{Ann}_S(M) = 0$ *belongs to* $\mathcal{Z}$.

Proof. (1) It is obvious that $\mathcal{Z}$ is closed under subobjects and quotient objects. By [Kan12, Proposition 2.4 (4)], every nonzero subquotient of an extension of two objects in $\mathcal{Z}$ is a nonzero extension of subquotients of objects in $\mathcal{Z}$, which does not belong to $\mathcal{Y}$. Hence $\mathcal{Z}$ is closed under extensions. In a similar way to the proof of [Kan15b, Proposition 2.12 (1)], we deduce that $\mathcal{Z}$ is also closed under direct sums. Since $\mathcal{Y}$ is closed under degree shifts, $\mathcal{Z}$ is also closed under degree shifts.
(2) Since M is noetherian, there are a finite number of elements $x_1, \dots, x_n \in S$ such that $\operatorname{Ann}_S(M) = \bigcap_{i=1}^n \operatorname{Ann}_S(x_i)$. Therefore the claim can be shown similarly to Lemma 3.9. □

REMARK 3.11. If an object $M \in \operatorname{Mod}^I S$ has an ascending chain of subobjects $L_0 \subset L_1 \subset \cdots$ such that $L_{i+1}/L_i \notin \mathcal{Y}$ for all $i \geq 0$, then we obtain the strictly ascending chain $F(L_0) \subsetneq F(L_1) \subsetneq \cdots$ of subobjects of $F(M)$ since $F(L_{i+1})/F(L_i) \cong F(L_{i+1}/L_i) \neq 0$. Hence, if $F(M) \in \operatorname{QCoh} X$ is noetherian, then there are no such chains of subobjects of M.
In particular, if $M \in \mathcal{Z}$, then M is noetherian if and only if $F(M)$ is noetherian (see [Pop73, Lemma 5.8.3] for the “only if” part).

LEMMA 3.12. *Let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence in $\operatorname{Mod}^I S$ *such that* $F(M) \in \operatorname{QCoh} X$ *is noetherian and one of the following conditions is satisfied:*

(1) $L \in \mathcal{Y}$ *and* $N \in \mathcal{Z}$.

(2) $L \in \mathcal{Z}$ *and* $N \in \mathcal{Y}$.

Then the exact sequence splits.

Proof. Since S is a commutative ring, we have

$$M \operatorname{Ann}_S(L) \operatorname{Ann}_S(N) = M \operatorname{Ann}_S(N) \operatorname{Ann}_S(L) = 0.$$

Assume (1). Then N is noetherian by Remark 3.11, and Lemma 3.10 implies $L' := M \operatorname{Ann}_S(L) \in \mathcal{Z}$. Since $N' := M/L'$ is annihilated by $\operatorname{Ann}_S(L)$, Lemma 3.9 implies $N' \in \mathcal{Y}$.
Let K be the kernel of the composite $L \to M \to N'$. Then the composite $K \hookrightarrow L \to M$ factors through some morphism $K \to L'$. Since $L \in \mathcal{Y}$ and $\mathcal{Y}$ is closed under subobjects, $K \in \mathcal{Y}$. Thus the only morphism from K to $L' \in \mathcal{Z}$ is zero. This means $K = 0$. The dual argument shows that the cokernel of the composite $L \to M \to N'$ is also zero. Therefore it is an isomorphism, and the given exact sequence splits.
The proof for (2) is similar. □

Lemma 3.13. *Let $\mathcal{C} \subset \operatorname{Mod}^I S$ be the collection of objects H such that*

(1) no nonzero subobjects of H belong to $\mathcal{Y}$, and

(2) no nonzero subobjects of H belong to $\mathcal{Z}$.

If $M \in \operatorname{Mod}^I S$ does not have any nonzero subquotient that belongs to $\mathcal{C}$ and $F(M) \in \operatorname{QCoh} X$ is a noetherian object, then M is a direct sum of an object in $\mathcal{Y}$ and an object in $\mathcal{Z}$.

Proof. Assume that M satisfies the assumption but it is not a direct sum of an object in $\mathcal{Y}$ and an object in $\mathcal{Z}$. Since $F(M)$ is noetherian, we can assume that for every nonzero subobject $L \subset M$ with $F(L) \neq 0$, the quotient M/L is a direct sum of an object in $\mathcal{Y}$ and an object in $\mathcal{Z}$. Indeed, if it is not the case, we can replace M by M/L since M/L satisfies the same assumption. This procedure eventually terminates due to Remark 3.11.
Take the largest subobject $L \subset M$ belonging to $\mathcal{Y}$ using Proposition 2.13 (1). Assume $L = 0$. Then M satisfies (1), and hence it does not satisfy (2). M has a nonzero subobject $N \subset M$ that belongs to $\mathcal{Z}$. Since $\mathcal{Z} \subset \operatorname{Mod}^I S$ is a localizing subcategory by Lemma 3.10 (1), N can be taken to be largest among the subobjects belonging to $\mathcal{Z}$. Since $N \neq 0$, we have $M/N \cong M_1 \oplus M_2$ for some $M_1 \in \mathcal{Y}$ and $M_2 \in \mathcal{Z}$, and the maximality of N implies that $M_2 = 0$. By Lemma 3.12, the short exact sequence

$$0 \to N \to M \to M_1 \to 0$$

splits. This contradicts the assumption that $L = 0$.
Assume $L \neq 0$. Then the argument for $L = 0$ shows that $M/L \cong M_1 \oplus M_2$ for some $M_1 \in \mathcal{Y}$ and $M_2 \in \mathcal{Z}$. By the maximality of L, we have $M_1 = 0$. By Lemma 3.12, the short exact sequence

$$0 \to L \to M \to M_2 \to 0$$

splits. This is again a contradiction. □

Lemma 3.14. *$\mathcal{Y} = 0$ and $\mathcal{Z} = \operatorname{Mod}^I S$.*

Proof. Assume that $S \in \operatorname{Mod}^I S$ has a nonzero subquotient H that belongs to the collection $\mathcal{C}$ defined in Lemma 3.13. If there is a nonzero subobject $L \subset H$ such that H/L has a nonzero subquotient belonging to $\mathcal{C}$, then replace H by H/L. Since $F(S) \cong \mathcal{O}_X$ is noetherian and L does not belong to $\mathcal{Y}$, this procedure eventually terminates by Remark 3.11. Thus we can assume that for every nonzero subobject $L \subset H$, the quotient H/L does not have any nonzero subquotient belonging to $\mathcal{C}$.
Since $\mathcal{Y} \subset \operatorname{Mod}^I S$ is a bilocalizing subcategory, H has the smallest subobject $H' \subset H$ among those satisfying $H/H' \in \mathcal{Y}$ by Proposition 2.13 (2). Since $H' \neq 0$ and H' also belongs to $\mathcal{C}$, we can assume that no nonzero quotient object of H belong to $\mathcal{Y}$ by replacing H by H'.

By property (2) in the definition of $\mathcal{C}$, there exist subobjects $L \subsetneq L' \subset H$ such that $L'/L \in \mathcal{Y}$. Then H/L meets the requirement on M in Lemma 3.13. Hence $H/L \cong M_1 \oplus M_2$ for some $M_1 \in \mathcal{Y}$ and $M_2 \in \mathcal{Z}$. Since H/L has a nonzero subobject $L'/L \in \mathcal{Y}$, the direct summand M_1 is nonzero. This contradicts to that H has no nonzero quotient object that belongs to $\mathcal{Y}$.

Therefore $S \in \operatorname{Mod}^I S$ does not have any nonzero subquotient that belongs to $\mathcal{C}$. Again by Lemma 3.13, $S \cong N_1 \oplus N_2$ for some $N_1 \in \mathcal{Y}$ and $N_2 \in \mathcal{Z}$. Since $F(N_1) = 0$, we have

$$\operatorname{Hom}_S(N_1, S) \cong \operatorname{Hom}_S(N_1, G(\mathcal{O}_X)) \cong \operatorname{Hom}_S(F(N_1), \mathcal{O}_X) = 0.$$

Hence $N_1 = 0$, and $S = N_2 \in \mathcal{Z}$. Since $\{S(i)\}_{i\in I}$ is a generating set in $\operatorname{Mod}^I S$ and $\mathcal{Z} \subset \operatorname{Mod}^I S$ is a localizing subcategory closed under degree shifts by Lemma 3.10 (1), we obtain $\mathcal{Z} = \operatorname{Mod}^I S$. This implies that $\mathcal{Y} = 0$. □

We prove our main results:

THEOREM 3.15 (Theorem 1.1). *Let X be a divisorial noetherian scheme. Then the following conditions are equivalent:*

(1) $\operatorname{QCoh} X$ *satisfies Ab4*.*

(2) $\operatorname{QCoh} X$ *has enough projectives.*

(3) X *is an affine scheme.*

Proof. (3)⇒(2): Since $\operatorname{QCoh} X \cong \operatorname{Mod} \Gamma(X, \mathcal{O}_X)$, it has enough projectives.
(2)⇒(1): See Remark 2.8.
(1)⇒(3): By Lemma 3.7 and Lemma 3.14, we have an equivalence $\operatorname{QCoh} X \xrightarrow{\sim} \operatorname{Mod}^I S$ that sends $\mathcal{O}_X$ to an object isomorphic to S. Hence $\mathcal{O}_X \in \operatorname{QCoh} X$ is a projective object, and we obtain

$$H^d(X, -) \cong \operatorname{Ext}^d_X(\mathcal{O}_X, -) = 0$$

for all integers $d \geq 1$. By Serre's criterion of affineness ([Har77, Theorem III.3.7]), we conclude that X is an affine scheme. □

COROLLARY 3.16 (Corollary 1.2). *Let X be a scheme that contains a non-affine divisorial noetherian scheme as a closed subscheme. Then* $\operatorname{QCoh} X$ *does not satisfy Ab4*.*

Proof. Let $Y \subset X$ be a closed subscheme with the stated property. It is shown in the proof of [BCJF15, Corollary 3.9] that the closed immersion $i\colon Y \hookrightarrow X$ induces the fully faithful functor $i_*\colon \operatorname{QCoh} Y \to \operatorname{QCoh} X$ whose essential image is the full subcategory $\mathcal{C}$ of $\operatorname{QCoh} X$ consisting of all objects $\mathcal{N} \in \operatorname{QCoh} X$ annihilated by the quasi-coherent subsheaf $\mathcal{I}_Y \subset \mathcal{O}_X$ corresponding to the closed subscheme Y. For every object $\mathcal{M} \in \operatorname{QCoh} X$, the quotient object $\mathcal{M}/\mathcal{M}\mathcal{I}_Y$ is largest among those belonging to $\mathcal{C}$. Hence $\mathcal{C} \subset \operatorname{QCoh} X$ is a closed subcategory by Proposition 2.16.

By Theorem 3.15, $\operatorname{QCoh} Y$ does not satisfy Ab4*. Since condition Ab4* on a Grothendieck category is inherited by its closed subcategories, $\operatorname{QCoh} X$ does not satisfy Ab4*, either. □

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