Theorem of the Heart in Negative K-Theory for Weight Structures

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Abstract. We construct the strong weight complex functor for a stable infinity-category $\mathcal{C}$ equipped with a bounded weight structure w. Along the way we prove that $\mathcal{C}$ is determined by the infinity-categorical heart of w. This allows us to compare the K-theory of $\mathcal{C}$ and the K-theory of Hw, the classical heart of w. In particular, we prove that $\mathrm{K}_n(\mathcal{C}) \to \mathrm{K}_n(Hw)$ are isomorphisms for $n \leq 0$.

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Contents

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Introduction

The concept of weight structures on triangulated categories was introduced by Bondarko in [Bon10a] and also independently by Pauksztello in [Pau08] under the name of co-t-structures. Bondarko's reason for introducing them was to study various triangulated categories of motives. There exist Chow weight structures on the categories $\mathcal{DM}(S;R)^c$, $\mathcal{DK}(S;R)^c$ for nice pairs (S,R) where S is a scheme and R is a ring (see §4.1 of [BoS18] for a survey). This weight structure also exists on $DM_{gm}^{eff}(k;R)$ for perfect fields k if the characteristic of k is either 0 or is invertible in R (see [Bon10a], [Bon11])[2]. The heart of this weight structure is the category of effective Chow motives. There also exist Gersten weight structures on certain categories of pro-motives containing as a full subcategory either $SH^{S^1}(k)^c$, $SH(k)^c$, $DM_{gm}(k)$ or $SH^{MGL}(k)^c$ (see [Bon18] and [Bon10b]).
A weight structure on a triangulated category consists of two subclasses of "non-positive" and "non-negative" objects of the category that satisfy certain axioms similar to the axioms of t-structures. Both t-structures and weight structures have the associated hearts and both are used to reduce studying arbitrary objects of a triangulated category to studying objects of the heart. However, while t-structures let one work with a triangulated category as with the derived category of some abelian category, weight structures are designed to let one work with a triangulated category as with the homotopy category of complexes over some additive category. With a weight structure w on a triangulated category $\mathcal{C}$ one gets a lot of methods to study the category. As the easiest application of the theory one gets spectral sequences $E(F,M)$ for any homological functor F on $\mathcal{C}$ and any object M of $\mathcal{C}$. The spectral sequences are functorial in M starting from the second page and for weight-bounded objects M they converge to $F(M)$. For example, in the case of the Gersten weight structures these spectral sequences are exactly the Coniveau spectral sequences. If $\mathcal{C}$ satisfies the Brown representability theorem and the classes defining the weight structure are closed under taking all coproducts then $\mathcal{C}$ admits a certain t-structure called adjacent to w.
One of the greatest features of the theory is the existence of the so-called weak weight complex functor. Ideally we would like to have a conservative triangulated functor from $\mathcal{C}$ to $K(Hw)$, the homotopy category of complexes over the heart of w, mapping non-positive (resp. non-negative) objects into complexes homotopy equivalent to complexes concentrated in non-positive (resp. non-negative) degrees and inducing an equivalence of the hearts. In [Bon09] Bondarko constructed this strong weight complex functor for categories that admit a so-called negative dg-enhancement or a filtered enhancement (see also [Sch11] for a detailed proof in the latter case) and conjectured that it exists in general. Besides, in [Bon10a] he was also able to construct a weak version of this functor for any $\mathcal{C}$. It's defined as follows. The weak category of complexes $K_w(Hw)$ is the quotient of $K(Hw)$ by a certain ideal of morphisms weakly

[2] a certain way to avoid the condition on characteristic is studied in [BoK17]

homotopic to 0. Unfortunately this category is not even triangulated, but there are still notions of non-positive and non-negative objects and of distinguished triangles. The weak weight complex functor is a conservative functor $\mathcal{C} \to K_w(Hw)$ satisfying analogous properties to that of the strong weight complex functor. Using this functor one can extend any additive functor from Hw to an abelian category to a homological functor on $\mathcal{C}$ (see Theorem 2.1.2(1) of [Bon19] and Theorem 2.3 of [KeSa17]). Its existence is also used to show that $\mathrm{K}_0(\mathcal{C})$ is isomorphic to $\mathrm{K}_0(Hw)$ if w is a bounded weight structure (see Theorem 5.3.1 of [Bon10a]). Although this weak weight complex functor is already a very useful technique, we still want to construct the strong weight complex functor. In this paper we do that in the case when $\mathcal{C}$ has an ∞-categorical enhancement and w is either bounded or compactly generated.

Let $\mathcal{C}$ be a stable infinity-category endowed with a bounded t-structure t on its homotopy category. In this setting Clark Barwick proved an analogue of Neeman's theorem of the heart (see [Bar15]). More precisely, he showed that the natural map $\mathrm{K}^{con}(Ht) \to \mathrm{K}^{con}(\mathcal{C})$ is a homotopy equivalence of connective K-spectra, where Ht is the heart of the t-structure. Moreover, in [AGH18] the map $\mathrm{K}(Ht) \to \mathrm{K}(\mathcal{C})$ of nonconnective K-theory spectra was also shown to be an equivalence if the heart Ht is noetherian. They also conjecture that the map should be an equivalence in general. Besides, they prove that the map $\mathrm{K}_{-1}(Ht) \to \mathrm{K}_{-1}(\mathcal{C})$ is an isomorphism without any extra assumptions.

Now assume $\mathcal{C}$ is a stable infinity-category endowed with a bounded weight structure w on its homotopy category. As it has already been mentioned there always exists an isomorphism $\mathrm{K}_0(Hw) \cong \mathrm{K}_0(\mathcal{C})$. It is natural to ask whether an analogue of the theorem of the heart holds for weight structures. The answer to this question turns out to be no in general as there are counterexamples. However, in this paper we construct natural maps $\mathrm{K}_n(\mathcal{C}) \to \mathrm{K}_n(Hw)$ for all n and prove that they are isomorphisms for $n \leq 0$. This theorem in particular implies the following result, that has been previously proved in [BGT13]. For a connective ring spectrum R the maps $\mathrm{K}_n(R) \to \mathrm{K}_n(\pi_0(R))$ are isomorphisms for $n \leq 0$. Indeed, the stable ∞-category $Perf(R)$ of perfect complexes over R admits a bounded weight structure whose heart is the category of finitely generated projective $\pi_0(R)$-modules. So our theorem says that the maps $K_n(R) = K_n(Perf(R)) \to K_n(proj(\pi_0(R))) = K_n(\pi_0(R))$ are isomorphisms for $n \leq 0$. In case $R = \mathbb{S}$, this tells us that $\mathrm{K}_n(\mathbb{S})$ vanishes for negative n.

To construct the maps in the theorem we first construct the strong weight complex functor on the ∞-categorical level and then we take the induced maps in K-theory. To construct the strong weight complex functor we introduce a new concept of the ∞-heart of a weight structure w on a stable ∞-category $\mathcal{C}$. It is an additive ∞-category Hw_∞ whose homotopy category is the classical heart Hw. For $Com(Hw)$ it is also equivalent to the nerve of the classical heart but in general it is not discrete. The main advantage of this new concept is that it determines canonically $(\mathcal{C}, w)$. Moreover, any weight exact functor between stable ∞-categories with weight structures is uniquely determined by its restriction to the ∞-hearts. This allows us to construct the strong weight complex functor on the level of ∞-categories $\mathcal{C} \to Com^b(Hw)$ as the unique (up to equivalence) functor corresponding to the additive functor $Hw_\infty \to Nerve(h(Hw_\infty))$.

Combining our result with the recent result of [AGH18] in the setting of triangulated categories of motives we obtain that the non-triviality of the negative K-groups of the additive category $Chow(S)$ would yield an obstruction to having a motivic t-structure on DM_{gm}.

The paper is organized as follows. In sections 1 and 2 we remind the reader of basic notions that we use in the paper, their basic properties, and state the main theorems that we will use. We also introduce some notations there. In section 3 we consider the functor from the ∞-category $WCat_\infty^{st,b}$ of stable ∞-categories equipped with a weight structure to the ∞-category Cat_∞^{add} of additive ∞-categories given by taking the ∞-heart Hw_∞ of a weight structure. We prove that this functor is an equivalence onto its image. The latter consists of those additive ∞-categories in which idempotents of a certain type split. This allows us to construct the weight complex functor for any $\mathcal{C}$ with a bounded weight structure w as the functor corresponding to the map $Hw_\infty \to Nerve(Hw)$ via this equivalence. In section 4 we prove that the weight complex induces isomorphisms in negative K-groups. In the last section we discuss relations between the negative K-theory of the category of Chow motives, the existence of the motivic t-structure, and also the smash-nilpotence conjecture.

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1 Reminder on weight structures and infinity-categories

Notation and conventions. First we fix some notations on basic category theory notions. Categories can be large or small (e.g. in the sense of a Grothendieck universe of large sets containing the Grothendieck universe of small sets).

- $Sets$ is the category of sets.
- Cat is the category of small categories.
- $K(\mathcal{A})$ is the homotopy category of complexes for any additive category $\mathcal{A}$. We use the homological grading for complexes.
- We use quasi-categories as models for $(\infty, 1)$-categories, although any other reasonable model would serve our properties. By an ∞-category we mean a quasi-category. 1-simplices of a quasi-category will be called

morphisms and its 0-simplices will be called objects. We also exploit many usual notions of the theory of quasi-categories such as a subcategory, a functor, a limit, et cetera. We refer to [Lur17a] and [Joy04] for details.

- $sSets_\bullet$ is the ∞-category of pointed simplicial sets.
- Spt is the ∞-category of spectra. Spt^{cn} is the subcategory of connective spectra.
- $Com(\mathcal{A})$ is the ∞-category of complexes for any additive category $\mathcal{A}$.
- $Fun(K, \mathcal{A})$ is the ∞-category of functors for any simplicial set K and an ∞-category $\mathcal{A}$.
- We call the Yoneda embedding the fully faithful left exact functor $\mathcal{C} \to Fun(\mathcal{C}^{op}, sSets_\bullet)$ given by Proposition 5.1.3.1 of [Lur17a]. Usually in the text it will be denoted by j.
- We always use the term "homotopy equivalence" for homotopy equivalence of simplicial sets or spaces. We use the term "equivalence" for equivalence of quasi-categories.
- If $\mathcal{C}$ is a category (resp. a quasi-category), then $\mathrm{Obj}\,\mathcal{C}$ denotes the class of objects of $\mathcal{C}$. For any objects X, Y of $\mathcal{C}$ the set of morphisms (resp. the mapping space of morphisms) is denoted by $\mathcal{C}(X, Y)$.
- The nerve of a small category $\mathcal{A}$ is denoted by $Nerve(\mathcal{A})$. It's a quasi-category.
- The homotopy category of an ∞-category $\mathcal{C}$ is denoted by $h(\mathcal{C})$.
- We say that a category $\mathcal{C}$ admits finite limits (resp. colimits) if any diagram $K \to \mathcal{C}$ has a limit (resp. a colimit), where K is a simplicial set having only finitely many non-degenerate simplices.

  Let $\mathcal{A}, \mathcal{B}$ be quasi-categories that have finite colimits (resp. limits, resp. both). Functors between $\mathcal{A}$ and $\mathcal{B}$ preserving finite colimits (resp. limits, resp. both) form a subcategory of the category of functors which we denote by $Fun_{rex}(\mathcal{A}, \mathcal{B})$ (resp. $Fun_{lex}(\mathcal{A}, \mathcal{B})$, resp. $Fun_{ex}(\mathcal{A}, \mathcal{B})$).

  Let $\mathcal{A}, \mathcal{B}$ be quasi-categories that have small colimits (resp. limits). Functors between $\mathcal{A}$ and $\mathcal{B}$ that preserve the colimits (resp. limits) form a subcategory of the category of functors which we denote by $Fun^L(\mathcal{A}, \mathcal{B})$ (resp. $Fun^R(\mathcal{A}, \mathcal{B})$).
- Let $\mathcal{A}$, $\mathcal{B}$ be ∞-categories that admit finite products. Then the category $Fun_{add}(\mathcal{A}, \mathcal{B})$ is the full subcategory of $Fun(\mathcal{A}, \mathcal{B})$ whose objects are product-preserving functors.

  Sometimes we will denote the ∞-categories $Fun_{add}(\mathcal{A}^{op}, sSets_\bullet)$ and $Fun_{add}(\mathcal{A}^{op}, Spt)$ by $P(\mathcal{A})$ and $SP(\mathcal{A})$, respectively.

1.1 Stable quasi-categories

Definition 1.1. An ∞-category $\mathcal{A}$ is called stable if it admits a zero object, contains all finite limits and colimits, and any commutative square is a pullback if and only if it is a pushout.

Stable ∞-categories are important because of the following theorem.

Theorem 1.2 ([Lur17b], 1.1.2.14). *The homotopy category of a stable ∞-category $\mathcal{C}$ admits a natural structure of a triangulated category. The shift functor and its inverse are induced by the adjoint pair of functors $\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega$. For any pullback square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ pt & \longrightarrow & Z \end{array}$$

there is a triangle $X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \to \Sigma X$.

Due to this theorem stable ∞-categories are used extensively as enhancements for triangulated categories. Most of the known triangulated categories admit such an enhancement (although there are counterexamples and such an enhancement does not have to be unique; see [MSS07] and [Sch02], respectively). We also point out the following easy properties.

Proposition 1.3 ([Lur17b], 1.1.3.1). *If K is any simplicial set and $\mathcal{C}$ is a stable ∞-category then $Fun(K, \mathcal{C})$ is stable.*

Proposition 1.4 ([Lur17b], 1.1.4.1). *Let $\mathcal{C}$, $\mathcal{C}'$ be stable ∞-categories. Then there are equalities $Fun_{lex}(\mathcal{C}, \mathcal{C}') = Fun_{ex}(\mathcal{C}, \mathcal{C}') = Fun_{rex}(\mathcal{C}, \mathcal{C}')$ of subsets of the simplicial set $Fun(\mathcal{C}, \mathcal{C}')$.*

The following two propositions give us the two main examples of stable ∞-categories.

Proposition 1.5 ([Lur17b], 1.4.3.6(1)). *The ∞-category Spt is stable. Its homotopy category is equivalent as a triangulated category to the stable homotopy category SH.*

Proposition 1.6 ([Lur17b], 1.3.2.10). *The ∞-category $Com(\mathcal{A})$ is stable. Its homotopy category is equivalent as a triangulated category to the homotopy category of complexes $K(\mathcal{A})$.*

There is also a well-developed theory of localizations of stable ∞-categories.

Definition 1.7. Let $\mathcal{C}$ be a stable ∞-category. Let $\mathcal{D} \subset \mathcal{C}$ be a full stable subcategory. Then the bottom right vertex in a pushout diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathcal{C}/\mathcal{D} \end{array}$$

is called the localization of $\mathcal{C}$ by $\mathcal{D}$.

By Proposition 5.14 of [BGT13] the homotopy category $h(\mathcal{C}/\mathcal{D})$ is canonically equivalent to the Verdier quotient $h(\mathcal{C})/h(\mathcal{D})$.

1.2 Additive quasi-categories

Definition 1.8. An ∞-category $\mathcal{A}$ is called additive if its homotopy category is additive.
The subcategory of Cat_∞ whose objects are additive ∞-categories and morphisms are product preserving (i.e. additive) functors is denoted by Cat_∞^{add}.

In particular, the nerve of a classical additive category is an additive ∞-category. Moreover, any stable ∞-category is additive.
Now we introduce the notions of an *idempotent-complete* additive ∞-category, the *Karoubization* of an additive ∞-category, and the *small Karoubization* of an additive category. An example to keep in mind is the category of free modules over a ring R. It is always additive but it is idempotent complete if and only if every projective module over R is free. The Karoubization of this category is the category of projective modules whereas the small Karoubization of this category is the category of stably free modules.

Definition 1.9. Let $\mathcal{A}$ be an additive infinity-category.

1. $\mathcal{A}$ is called *idempotent-complete* (or *absolutely Karoubi-closed*) if its homotopy category is idempotent complete, that is every idempotent $X \xrightarrow{p} X$ in $h(\mathcal{A})$ has the form
$$X \cong X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} \mathrm{id}_{X_1} & 0 \\ 0 & 0 \end{pmatrix}} X_1 \oplus X_2 \cong X$$
for some $X_1, X_2 \in \operatorname{Obj} \mathcal{A}$.

2. There exists a universal additive functor $\mathcal{A} \to \operatorname{Kar}(\mathcal{A})$ to an idempotent complete additive ∞-category called the *Karoubization* (or *idempotent completion*) of $\mathcal{A}$ (see Proposition 5.1.4.2 of [Lur17a]).

3. A full subcategory $\mathcal{B}$ of an additive infinity-category $\mathcal{A}$ is called *Karoubi-closed* (in $\mathcal{B}$) if any objects $X, Y \in \operatorname{Obj} \mathcal{B}$ such that $X \oplus Y \in \operatorname{Obj} \mathcal{A}$ also belong to $\operatorname{Obj} \mathcal{B}$.

4. The full subcategory of $\operatorname{Kar}(\mathcal{A})$ whose objects are such X that there exist $X', Y \in \operatorname{Obj} \mathcal{A}$ with $X \oplus X' \cong Y$ is called the *small Karoubization*[3] of $\mathcal{A}$.

[3]Bondarko introduced the notion under the name small envelope

As we will see the less common notion of small Karoubization is of special importance for the theory of weight structures. The essence of the relationship between the two can be illustrated by the following statement, whose more general form is proved in Theorem 4.3.2(II.2) of [Bon10a]: the minimal Karoubi-closed subcategory of the homotopy category of complexes $K^b(\mathcal{A})$ that contains $\mathcal{A}$ is equivalent to the small Karoubization of $\mathcal{A}$.

Proposition 1.10 ([Lur18], Proposition C.1.5.7, Remark C.1.5.9). *For an additive ∞-category $\mathcal{A}$ the functor $\Omega^\infty : Fun_{add}(\mathcal{A}^{op}, Spt^{cn}) \to Fun_{add}(\mathcal{A}^{op}, sSets_\bullet)$ is an equivalence. In particular, there is a fully faithful functor $j : \mathcal{A} \to Fun(\mathcal{A}^{op}, Spt)$ such that the usual Yoneda embedding functor is the composition of Ω^∞ with j.*
For any objects X, Y of $\mathcal{A}$ we denote the spectrum object $j(X)(Y)$ by $\mathcal{M}(X, Y)$.

Proposition 1.10 allows us to write the following definition.

Definition 1.11. Let $\mathcal{A}$ be an additive ∞-category. The category of *finite cell $\mathcal{A}$-modules* $Fun^{fin}(\mathcal{A}^{op}, Spt)$ is the minimal full subcategory of $Fun_{add}(\mathcal{A}^{op}, Spt)$ containing the image of the Yoneda embedding functor $\mathcal{A} \xrightarrow{j} Fun_{add}(\mathcal{A}^{op}, Spt)$ and closed under finite limits and colimits.
We will also sometimes denote it by $SP^{fin}(\mathcal{A})$.

Remark 1.12. The quasi-category $SP(\mathcal{A})$ is stable (see Remark C.1.5.9 of [Lur18]). So the functor j from the Proposition 1.10 gives an embedding of $\mathcal{A}$ into a stable category. Note that the ∞-category $Fun^{fin}(\mathcal{A}^{op}, Spt)$ is also stable since all finite limits and colimits in $Fun^{fin}(\mathcal{A}^{op}, Spt)$ coincide with finite limits and colimits in the ∞-category $Fun_{add}(\mathcal{A}^{op}, Spt)$.

1.3 Weight structures

Now we recall the definition and some basic properties of weight structures.

Definition 1.13. A pair of subclasses $\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0} \subset \operatorname{Obj}\mathcal{C}$ will be said to define a weight structure w for a triangulated category $\mathcal{C}$ if they satisfy the following conditions:
(i) $\mathcal{C}_{w\ge 0}, \mathcal{C}_{w\le 0}$ are Karoubi-closed in $\mathcal{C}$.
(ii) **Semi-invariance with respect to translations.**
$\mathcal{C}_{w\le 0} \subset \mathcal{C}_{w\le 0}[1]$, $\mathcal{C}_{w\ge 0}[1] \subset \mathcal{C}_{w\ge 0}$.
(iii) **Orthogonality.**
$\mathcal{C}(X, Y) = 0$ for any $X \in \mathcal{C}_{w\le 0}$ and $Y \in \mathcal{C}_{w\ge 0}[1]$.
(iv) **Weight decompositions**.
For any $M \in \operatorname{Obj}\mathcal{C}$ there exists a distinguished triangle

$$X \to M \to Y \to X[1] \tag{1}$$

such that $X \in \mathcal{C}_{w\le 0}$, $Y \in \mathcal{C}_{w\ge 0}[1]$.

The basic example of a weight structure is $\mathcal{C} = K(\mathcal{A})$, the homotopy category of complexes over an additive category. In this case $K(\mathcal{A})_{w\leq 0}$ (resp., $K(\mathcal{A})_{w\geq 0}$) is the class of complexes homotopy equivalent to complexes concentrated in non-positive (resp. non-negative) degrees. The weight decomposition axiom is then given by the stupid filtrations of a complex. Moreover, bounded from below, from above, or from both sides complexes also give examples of categories with weight structures. These Unlike the case of t-structures already in these simple examples weight decompositions are not functorial and not even unique.
Another example that we keep in mind is the spherical weight structure on the stable homotopy category SH. The classes $SH_{w\leq 0}$ and $SH_{w\geq 0}$ are defined as the minimal subcategories containing the sphere spectrum, closed under extensions, under taking small coproducts, and under taking the negative (resp. positive) triangulated shift. This weight structure restricts to the subcategory of compact objects SH^c. We refer to section 4.6 of [Bon10a] and §4.2 of [Bon19] for details about this example.

NOTATION.

- The full subcategory $Hw \subset \mathcal{C}$ whose objects are $\mathcal{C}_{w=0} = \mathcal{C}_{w\geq 0} \cap \mathcal{C}_{w\leq 0}$ will be called the *heart* of w.
- $\mathcal{C}_{w\geq i}$ (resp. $\mathcal{C}_{w\leq i}$, resp. $\mathcal{C}_{w=i}$) will denote $\mathcal{C}_{w\geq 0}[i]$ (resp. $\mathcal{C}_{w\leq 0}[i]$, resp. $\mathcal{C}_{w=0}[i]$).
- The class $\mathcal{C}_{w\geq i} \cap \mathcal{C}_{w\leq j}$ will be denoted by $\mathcal{C}_{[i,j]}$.

  $\mathcal{C}^b \subset \mathcal{C}$ is the full subcategory of $\mathcal{C}$ whose objects are $\cup_{i,j\in\mathbb{Z}}\mathcal{C}_{[i,j]}$.
- We say that $(\mathcal{C}, w)$ is *bounded* if $\mathcal{C}^b = \mathcal{C}$.
- Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories endowed with weight structures w and w', respectively; let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor.

  F will be called *left weight-exact* (with respect to w, w') if it maps $\mathcal{C}_{w\leq 0}$ into $\mathcal{C}'_{w'\leq 0}$; it will be called *right weight-exact* if it maps $\mathcal{C}_{w\geq 0}$ into $\mathcal{C}'_{w'\geq 0}$. F is called *weight-exact* if it is both left and right weight-exact.
- Let $\mathcal{H}$ be a full subcategory of a triangulated category $\mathcal{C}$.

  We will say that $\mathcal{H}$ is *negative* if $\mathcal{C}(X, Y) = 0$ for any $X \in \operatorname{Obj}\mathcal{H}$ and $Y \in (\cup_{i>0} \operatorname{Obj}(\mathcal{H}[i]))$.

In this paper we will mostly focus on weight structures on the homotopy category of a stable ∞-category $\mathcal{C}$. Sometimes we will call a weight structure on $h(\mathcal{C})$ just a weight structure on $\mathcal{C}$.

Remark 1.14. 1. Let w be a bounded weight structure on $h(\mathcal{C})$. Then the heart of w generates $h(\mathcal{C})$ as a triangulated category or, equivalently, $\mathcal{C}$ is the minimal subcategory of $\mathcal{C}$ containing objects of Hw and closed under finite limits and colimits (see Corollary 1.5.7 of [Bon10a]).

2. The heart of a weight structure is a negative subcategory by definition. Moreover, any negative subcategory $\mathcal{H}$ that generates $h(\mathcal{C})$ as a triangulated category yields a bounded weight structure whose heart is equivalent to the small Karoubization of $\mathcal{H}$ (see Definition 4.3.1 and Theorem 4.3.2(II.2) of [Bon10a]).
Consequently, a functor F between $\mathcal{C}$ and $\mathcal{C}'$ with bounded weight structures w and w', respectively, is weight exact if and only if F maps objects of the heart of w into objects of the heart of w'.

NOTATION.

- Let w, w' be bounded weight structures on $h(\mathcal{C})$ and $h(\mathcal{C}')$ for a stable ∞-category $\mathcal{C}'$. Then we denote by $Fun_{w.ex}(\mathcal{C}, \mathcal{C}')$ the full subcategory of $Fun_{ex}(\mathcal{C}, \mathcal{C}')$ whose objects are functors such that their associated functor on the homotopy categories is weight exact.

- We denote by $WCat^{st}_\infty$ the simplicial subcategory of Cat_∞ whose objects are small stable infinity categories together with a weight structure and the simplicial subset of morphisms between $(\mathcal{C}, w)$ and $(\mathcal{C}', w')$ is $Fun_{w.ex}(\mathcal{C}, \mathcal{C}')$. The full subcategory of $WCat^{st}_\infty$ whose objects are stable categories with bounded weight structures is denoted by $WCat^{st,b}_\infty$.

2 REMINDER ON K-THEORY SPECTRA

In the section we recall the definitions of the K-theory spectra and state their main properties.
The connective algebraic K-theory spectrum of an exact category (and in particular, of an additive category) was first introduced by Quillen in [Qui73]. In [Wal85] Waldhausen defines the formalism of categories with cofibrations and weak equivalences (nowadays called Waldhausen categories) and constructs the K-theory spectrum using the so-called S-construction. Any exact category gives rise to a Waldhausen category and in this case the Waldhausen K-theory spectrum is shown to be homotopy equivalent to Quillen's K-theory spectrum (see Appendix 1.9 in ibid.).
The non-connective K-theory of schemes was studied in [TT90] (see chapter 6). Later the non-connective K-theory spectrum of a Frobenius pair (and, in particular, of an exact category) was defined in [Sch06] (see 11.4). Finally, in [BGT13] the connective and nonconnective K-theory spectra of a stable ∞-category were defined in sections 7 and 9, respectively. We exploit their definitions for our purposes.

NOTATION.

1. The K-theory spectrum in [BGT13] is defined to be Morita invariant. So be warned that $\mathrm{K}_0(h(\mathcal{C}))$ (i.e. K_0 of the triangulated category $h(\mathcal{C})$) does not coincide with $\mathrm{K}_0(\mathcal{C}) = \pi_0(\mathrm{K}(\mathcal{C}))$ unless $\mathcal{C}$ is idempotent complete.

2. For a stable ∞-category $\mathcal{C}$ we denote by $\mathrm{K}_i(\mathcal{C})$ the i-th homotopy group of the spectrum $\mathrm{K}(\mathcal{C})$. By $\mathrm{K}(-)$ we always mean the nonconnective K-theory spectrum. We denote the connective K-theory spectrum by K^{con}.

Definition 2.1. For an additive ∞-category $\mathcal{A}$ we define $\mathrm{K}(\mathcal{A})$ to be the spectrum $\mathrm{K}(Fun^{fin}(\mathcal{A}^{op}, Spt))$.

For reader's convenience we review the construction of the nonconnective K-theory spectrum from the connective K-theory spectrum. For that we'll need the following statement, usually called the Eilenberg swindle.

Proposition 2.2 ([Bar16], Proposition 8.1). *Let $\mathcal{C}$ be a stable ∞-category such that all countable coproducts exist in $\mathcal{C}$. Then $\mathrm{K}^{con}(\mathcal{C})$ is contractible.*

As before, let $\mathcal{C}$ be a small stable ∞-category. The idea is to embed $\mathcal{C}$ into a stable category whose K-theory is trivial, take the quotient, and then iterate the procedure. The natural stable category with this property where we can embed $\mathcal{C}$ is the category of ind-objects $Ind(\mathcal{C})$. However, this category is usually large, and to get a small category we will take the full subcategory $(Ind(\mathcal{C}))^\kappa$ of $Ind(\mathcal{C})$ whose objects are κ-compact objects, where κ is one's favorite uncountable cardinal. The full embedding $\mathcal{C} \to Fun(\mathcal{C}^{op}, Spt)$ factors through this subcategory. Moreover, the category $(Ind(\mathcal{C}))^\kappa$ is stable (Corollary 1.1.3.6 of [Lur17b]). Denote by $\Sigma^1(\mathcal{C})$ the localization $(Ind(\mathcal{C}))^\kappa/\mathcal{C}$. Inductively, we define $\Sigma^n(\mathcal{C})$ as $\Sigma(\Sigma^{n-1}(\mathcal{C}))$. By Proposition 2.2 $\mathrm{K}^{con}((Ind(\mathcal{C}))^\kappa)$ is homotopy equivalent to the point. By functoriality properties of the connective K-theory we have the following commutative diagram of spectra

$$\begin{array}{ccc} \mathrm{K}^{con}(\Sigma^n(\mathcal{C})) & \longrightarrow & \mathrm{K}^{con}(Ind(\Sigma^n(\mathcal{C}))^\kappa) \cong pt \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathrm{K}^{con}(\Sigma^{n+1}(\mathcal{C})) \end{array}$$

Since $\Omega\,\mathrm{K}^{con}(\Sigma^{n+1}(\mathcal{C}))$ is the homotopy pullback in the diagram above, there is a canonical map $\mathrm{K}^{con}(\Sigma^n(\mathcal{C})) \to \Omega\,\mathrm{K}^{con}(\Sigma^{n+1}(\mathcal{C}))$. We define the non-connective K-theory spectrum $\mathrm{K}(\mathcal{C})$ as the colimit of these maps $\mathrm{colim}_{n\in\mathbb{N}}\,\mathrm{K}^{con}(\Sigma^n(\mathcal{C}))$.

The main property of the non-connective algebraic K-theory spectrum is the following localization theorem.

Theorem 2.3 ([BGT13], 9.8). *Let $\mathcal{C}_1 \to \mathcal{C} \to \mathcal{C}_2$ be an exact sequence of idempotent complete stable ∞-categories, that is $\mathcal{C}_1 \to \mathcal{C}_2$ is a full embedding, the composition is trivial and the induced map $Kar(\mathcal{C}/\mathcal{C}_1) \to \mathcal{C}_2$ is an equivalence. Then the sequence $\mathrm{K}(\mathcal{C}_1) \to \mathrm{K}(\mathcal{C}) \to \mathrm{K}(\mathcal{C}_2)$ is a cofiber sequence.*

3 WEIGHT COMPLEX FUNCTOR

From now on $\mathcal{C}$ will be a stable ∞-category together with a weight structure w on $h(\mathcal{C})$.

DEFINITION 3.1. The ∞-heart Hw_∞ of w is the full subcategory of $\mathcal{C}$ whose objects are those of Hw. It is an additive ∞-category.

The latter definition doesn't make sense for t-structures. Indeed, the mapping spaces between objects of the heart of a t-structure do not have higher homotopy groups and therefore the ∞-heart of a t-structure would be equivalent to its heart in the usual sense.
The situation is totally different for weight structures. For instance, the ∞-heart of the spherical weight structure on the ∞-category of spectra consists of objects of the form $\bigoplus\limits_I \mathbb{S}$. Already the mapping space $Hw_\infty(\mathbb{S},\mathbb{S})$ is homotopy equivalent to $\Omega^\infty\mathbb{S}$. In contrast, the classical heart of the weight structure is equivalent to the category of free $\mathbb{Z}$-modules.
In this section we will see that the ∞-heart determines a stable ∞-category together with a bounded weight structure.

LEMMA 3.2. *For any additive ∞-category $\mathcal{A}$ and any stable ∞-category $\mathcal{C}$ the restriction functor $Fun^L(Fun_{add}(\mathcal{A}^{op}, Spt), \mathcal{C}) \to Fun_{add}(\mathcal{A}, \mathcal{C})$ is an equivalence.*

Proof. By Remark C.1.5.9 of [Lur18] we can identify $Fun_{add}(\mathcal{A}^{op}, Spt)$ with the stabilization of $Fun_{add}(\mathcal{A}^{op}, sSets_\bullet)$. Hence by Corollary 1.4.4.5 of [Lur17b] the restriction functor

$$Fun^L(Fun_{add}(\mathcal{A}^{op}, Spt), \mathcal{C}) \to Fun^L(Fun_{add}(\mathcal{A}^{op}, sSets_\bullet), \mathcal{C})$$

is an equivalence.
Now let $\mathcal{K}$ be the collection of all small simplicial sets and $\mathcal{R}$ be the collection of all maps from finite discrete simplicial sets to $\mathcal{A}$. Then by Proposition 5.3.6.2(2) of [Lur17a] applied to this setting the restriction functor

$$Fun^L(Fun_{add}(\mathcal{A}^{op}, sSets_\bullet), \mathcal{C}) \to Fun_{add}(\mathcal{A}, \mathcal{C})$$

is also an equivalence. □

We now prove that a stable ∞-category with a bounded weight structure is determined by its heart.

PROPOSITION 3.3. *Let w be bounded.*
1. The essential image of the composition functor

$$F' : \mathcal{C} \xrightarrow{j} Fun_{ex}(\mathcal{C}^{op}, Spt) \xrightarrow{res} Fun_{add}(Hw_\infty^{op}, Spt)$$

lies in the full subcategory $Fun^{fin}(Hw_\infty^{op}, Spt)$ and the functor $\mathcal{C} \xrightarrow{F} Fun^{fin}(Hw_\infty^{op}, Spt)$ is an equivalence of ∞-categories.

2. Let $\mathcal{C}'$ be a stable ∞-category with a bounded weight structure w'. Then the restriction functor $Fun_{w.ex}(\mathcal{C},\mathcal{C}') \to Fun_{add}(Hw_\infty, Hw'_\infty)$ is an equivalence of ∞-categories.

Proof. 1. Let X, Y be objects of Hw_∞. By definition F' maps X to the functor $\mathcal{M}_\mathcal{C}(-,X)$ restricted to the subcategory Hw_∞^{op} of $\mathcal{C}^{op}$. By the axiom (iii) of weight structures $\mathcal{M}_\mathcal{C}(-,X)$ is a connective spectrum. Clearly, $\Omega^\infty\mathcal{M}_\mathcal{C}(-,X) = Map_\mathcal{C}(-,X) = j(X)$. By Proposition 1.10 the map $Map_{SP(Hw_\infty)}(F'(X),F'(Y)) \stackrel{\Omega^\infty}{\to} Map_{P(Hw_\infty)}(j(X),j(Y))$ is an equivalence. By the Yoneda Lemma the map $Map_\mathcal{C}(X,Y) \to Map_{P(Hw_\infty)}(j(X),j(Y))$ is an equivalence. Since $\Omega^\infty \circ F'|_{Hw_\infty} = j$ the map $Map_\mathcal{C}(X,Y) \to Map_{SP(Hw_\infty)}(F'(X),F'(Y))$ is also an equivalence. Now it is clear that $F'|_{Hw_\infty}$ is a full embedding whose essential image is equal to $j(Hw_\infty)$.
By Proposition 1.1.4.1 of [Lur17b] F' commutes with finite limits and finite colimits. By Remark 1.14(1) any object of $\mathcal{C}$ can be obtained from objects of Hw_∞ by taking finite limits and colimits. Hence the essential image of F' lies in $SP^{fin}(Hw_\infty)$.
Since any object of $SP^{fin}(Hw_\infty)$ can be obtained from objects of Hw_∞ by taking finite limits and colimits it suffices to prove that F is a full embedding. The morphism

$$\mathcal{M}_\mathcal{C}(X,Y) \stackrel{F_{X,Y}}{\to} \mathcal{M}_{SP^{fin}(Hw_\infty)}(F(X),F(Y))$$

is an equivalence for any $(X,Y) \in \operatorname{Obj} Hw_\infty^{op} \times Hw_\infty$. Since any object of $\mathcal{C}^{op} \times \mathcal{C}$ can be obtained from objects of $Hw_\infty^{op} \times Hw_\infty$ by taking finite limits and finite colimits and since F is exact, the morphism $F_{X,Y}$ is an equivalence for any $(X,Y) \in \operatorname{Obj} \mathcal{C}^{op} \times \mathcal{C}$, which means that F is a full embedding.
2. By assertion 1 and since equivalences induce equivalences of the corresponding functor categories $Fun(-,-)$ (Proposition 1.2.7.3 of [Lur17a]) we can assume $\mathcal{C} = SP^{fin}(Hw)$, $\mathcal{C}' = SP^{fin}(Hw'_\infty)$.
By Proposition 5.5.8.10(6) of [Lur17a] together with Proposition 1.4.3.7 of [Lur17b] the ∞-category $SP(Hw_\infty)$ is compactly generated. Proposition 5.3.4.17 of [Lur17a] implies that the subcategory of compact objects is equivalent to the Karoubization of $SP^{fin}(Hw_\infty)$. By definition of a compactly generated ∞-category $Ind(SP^{fin}(Hw_\infty)) = SP(Hw_\infty)$ (see 5.5.7.1 of [Lur17a]). Now the restriction functor

$$Fun^L(Ind(SP^{fin}(Hw_\infty)), SP(Hw'_\infty)) \to Fun_{ex}(SP^{fin}(Hw_\infty), SP(Hw'_\infty))$$

is an equivalence of ∞-categories by Propositions 5.3.5.10 and 5.3.5.14 of [Lur17a] and Proposition 1.4.4.1(2) of [Lur17b].
The following diagram commutes

$$\begin{array}{ccc} Fun^L(SP(Hw_\infty), SP(Hw'_\infty)) & \xrightarrow{\cong} & Fun_{ex}(SP^{fin}(Hw_\infty), SP(Hw'_\infty)) \\ & \searrow^{\cong} & \downarrow \\ & & Fun_{add}(Hw_\infty, SP(Hw'_\infty)) \end{array}$$

where arrows are restriction functors. Above we've proved that the top horizontal functor is an equivalence. Moreover, by Lemma 3.2 the diagonal functor is an equivalence. Hence the functor

$$Fun_{ex}(SP^{fin}(Hw_\infty), SP(Hw'_\infty)) \stackrel{Res}{\to} Fun_{add}(Hw_\infty, SP(Hw'_\infty))$$

is also an equivalence.
We denote the full embedding $Hw'_\infty \to SP(Hw'_\infty)$ by i and the full embedding $SP^{fin}(Hw'_\infty) \to SP(Hw'_\infty)$ by i'. We define $\mathcal{K} \subset Fun_{ex}(SP^{fin}(Hw_\infty), SP(Hw'_\infty))$ to be the full subcategory whose objects are functors G such that $Res(G) \cong i \circ U$ for some $U : Hw_\infty \to Hw'_\infty$. Since Res is an equivalence the map $\mathcal{K} \to Fun_{add}(Hw_\infty, Hw'_\infty)$ is also an equivalence.
Certainly, the full subcategory

$$Fun_{w.ex}(SP^{fin}(Hw_\infty), SP^{fin}(Hw'_\infty)) \subset Fun_{ex}(SP^{fin}(Hw_\infty), SP(Hw'_\infty))$$

is a subcategory of $\mathcal{K}$. But also the converse is true. Indeed, let G be a functor $SP^{fin}(Hw_\infty) \to SP(Hw'_\infty)$ such that the image of any object of Hw_∞ is an object of Hw'_∞. By definition any object X of $SP^{fin}(Hw_\infty)$ can obtained via a combination of finite limits and finite colimits from objects of Hw_∞. Since the functor G is exact, $G(X)$ is an object of the subcategory $SP^{fin}(Hw'_\infty)$. So we obtain a weight exact functor $G' : SP^{fin}(Hw_\infty) \to SP^{fin}(Hw'_\infty)$ such that $G = i' \circ G'$.
Now the restriction map

$$Fun_{w.ex}(SP^{fin}(Hw_\infty), SP^{fin}(Hw'_\infty)) \stackrel{i'}{\cong} \mathcal{K} \to Fun_{add}(Hw_\infty, Hw'_\infty)$$

is an equivalence. □

Corollary 3.4. *The functor $WCat^{st,b}_\infty \to Cat^{add}_\infty$ that sends a stable ∞-category with a bounded weight structure to its heart, is a full embedding of categories enriched over quasi-categories. The essential image consists of those additive categories whose homotopy categories coincide with their small Karoubization.*

Proof. The functor is a full embedding by Proposition 3.3(2).
All the additive ∞-categories in the image coincide with their small Karoubization by Theorem 4.3.2(II.2) of [Bon10a]. Conversely, let $\mathcal{H}$ be an additive infinity-category. The functor $\mathcal{H} \stackrel{j}{\to} Fun^{fin}(\mathcal{H}^{op}, Spt)$ is a full embedding by Proposition 1.10. The subcategory $h(j(\mathcal{H}))$ is negative and it generates $h(Fun^{fin}(\mathcal{H}^{op}, Spt))$. So, by Remark 1.14(2) there exists a bounded weight structure on $Fun^{fin}(\mathcal{H}^{op}, Spt)$ whose ∞-heart is equivalent to the small Karoubization of $j(\mathcal{H})$. □

Corollary 3.5. *Let w be bounded. Then there exists an exact functor $\mathcal{C} \xrightarrow{t} Com^b(Hw)$ unique up to equivalence that induces a weight exact functor $h(\mathcal{C}) \xrightarrow{h(t)} K^b(Hw)$ that is the identity restricted to the hearts of the weight structures.*
The functor $h(\mathcal{C}) \xrightarrow{t} K^b_w(Hw) \to K^b_w(Hw)$ is the weight complex functor in the sense of [Bon10a](§3) (where $K^b_w(Hw)$ is the weak homotopy category of complexes).
For any stable ∞-category with a bounded weight structure w' and an exact functor $\mathcal{C} \xrightarrow{G} \mathcal{C}'$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{t} & Com^b(Hw) \\ \downarrow{\scriptstyle G} & & \downarrow \\ \mathcal{C}' & \xrightarrow{t} & Com^b(Hw') \end{array}$$

Proof. Consider the functor $u : Cat^{add}_\infty \overset{Nerve(h(-))}{\to} Cat^{add}_\infty$ which is the unit $\mathrm{id}_{Cat^{add}_\infty} \to Nerve(h(-))$ of the adjunction $Cat^{add}_\infty \underset{Nerve}{\overset{h}{\rightleftarrows}} Cat^{add}$.
By the equivalence from Corollary 3.4 we know that the restriction map $Fun_{w.ex}(\mathcal{C}, Com^b(Hw)) \to Fun_{add}(Hw_\infty, Nerve(Hw))$ is an equivalence of infinity-categories for any $\mathcal{C}$ with a bounded weight structure w. In particular there exists a functor $\mathcal{C} \xrightarrow{t} Com^b(Hw)$ whose restriction to the hearts is u. The functor $h(\mathcal{C}) \xrightarrow{t} K^b(Hw) \to K^b_w(Hw)$ is isomorphic to the functor defined in [Bon10a](§3) because t is compatible with weight Postnikov towers.
Moreover, the functor $\mathcal{A} \to Nerve(h(\mathcal{A}))$ is the unique functor up to homotopy that induces the identity functor $h(\mathcal{A}) \to h(\mathcal{A})$. Thus the uniqueness of t follows. □

Remark 3.6. The corollary above proves Conjecture 3.3.3 of [Bon10a] for triangulated categories with a bounded weight structure that have an ∞-categorical enhancement.
Moreover, it enables us to solve the conjecture for enhanced triangulated categories with a compactly-generated weight structure. Let $\mathcal{C}$ be a κ-compactly generated triangulated category with a compactly generated weight structure w on it (i.e. the heart contains the set of compact generators of the category). We assume that it has an ∞-categorical model $\underline{\mathcal{C}}$. By Remark 1.4.4.3 of [Lur17b] $\underline{\mathcal{C}}$ is compactly generated and in particular, the functor $Ind(\underline{\mathcal{C}}^c) \xrightarrow{e} \underline{\mathcal{C}}$ is an equivalence. The triangulated functor $h(Ind(\underline{\mathcal{C}}^c)) \xrightarrow{h(e)} \mathcal{C}$ is now also an equivalence.
Using Corollary 3.5 and the fact that $Com(Hw)$ is compactly generated we obtain the desired functor

$$\mathcal{C} \cong h(Ind(\underline{\mathcal{C}}^c)) \to h(Ind(Com^b(Hw^c))) \cong h(Com(Hw)) \hookrightarrow K(Hw).$$

4 Theorems of the heart

Now we want to relate the K-theory of $\mathcal{C}$ and the K-theory of its heart Hw. Unlike the situation with t-structures, the mapping spaces in the ∞-heart Hw_∞ might not be equivalent from Hw. So, an ∞-category theorist would say that the correct statement of the theorem of the heart should be the following

Corollary 4.1 (The stupid theorem of the heart for weight structures). *There is a canonical homotopy equivalence* $\mathrm{K}(Hw_\infty) \to \mathrm{K}(\mathcal{C})$.

Since we already know that $\mathcal{C} \cong Fun^{fin}(Hw_\infty^{op}, Spt)$ (see Proposition 3.3(1)) the statement follows from the definitions[4].
However, we still want to compare $\mathrm{K}(\mathcal{C})$ and $\mathrm{K}(Hw)$ because $\mathrm{K}(Hw)$ is a priori something easier.
Note that in general the spectrum $\mathrm{K}(Hw_\infty)$ is quite distinct from $\mathrm{K}(Hw)$. For example, the subcategory of compact objects of the category of spectra possesses a weight structure whose ∞-heart Hw_∞ is the additive subcategory generated by the sphere spectrum. So $\mathrm{K}(Hw_\infty) \cong \mathrm{K}(\mathbb{S}-mod^{fin}) = \mathrm{K}(\mathbb{S})$. The classical heart of this weight structure is equivalent to the category of finitely generated free $\mathbb{Z}$-modules, so $\mathrm{K}(Hw) \cong \mathrm{K}(\mathbb{Z})$. The groups $\mathrm{K}_i(\mathbb{S})$ and $\mathrm{K}_i(\mathbb{Z})$ are well-known to not be isomorphic. For instance, it was shown that the p-torsion of $\mathrm{K}_i(\mathbb{S})$ contains the p-torsion of $\pi_i(\mathbb{S})$ for an odd prime p (Theorem 1.2 in [BM14], see also [Rog03]) while $\mathrm{K}_i(\mathbb{Z})$.
However, in some cases $\mathrm{K}_i(Hw)$ is actually isomorphic to $\mathrm{K}_i(Hw_\infty)$. Consider the map of K-theory spectra $\mathrm{K}(\mathcal{C}) \to \mathrm{K}(Hw)$ induced by the weight complex functor (constructed in Corollary 3.5). First note that for $i = 0$ the induced map $\pi_0(\mathrm{K}(\mathcal{C})) \to \mathrm{K}_0(Kar(h(\mathcal{C})))$ is an isomorphism. This together with Theorem 5.3.1 of [Bon10a] implies the following.

Proposition 4.2. *The map of K-theory spectra* $\mathrm{K}(\mathcal{C}) \to \mathrm{K}(Hw)$ *induces isomorphism in* π_0.

Now we generalize this result to all the negative K-groups.

Theorem 4.3 (The theorem of the heart for weight structures in negative K-theory). *The map* $\mathrm{K}_i(\mathcal{C}) \to \mathrm{K}_i(Hw)$ *is an isomorphism for* $i \leq 0$.

Proof. The proof goes by decreasing induction over i. For $i = 0$ the statement follows from Proposition 4.2.
Assume now the theorem is known for $n \geq i+1$. Denote by Hw_∞^{big} the full subcategory of $Fun(\mathcal{C}^{op}, Spt)$ that contains $\bigoplus_{i\in\mathbb{N}} X_i$ for any sequence of objects X_i of Hw_∞. Next denote by $\mathcal{C}^{big}$ the smallest full subcategory of $Fun(\mathcal{C}^{op}, Spt)$ containing Hw_∞^{big} and closed under taking finite limits and colimits. For any object X of Hw_∞^{big} the coproduct $\bigoplus_{i\in\mathbb{N}} X$ exists in $\mathcal{C}^{big}$. Indeed, let X be

[4] Ernest Fontes constructed a related homotopy equivalence in his thesis [Fon18] (see also Theorem 9 in [Hel19]). However, the K-theory of an additive ∞-category is defined there in a different way, using the language of Waldhausen ∞-categories.

an object of $\mathcal{C}^{big}$. For some n its shift $\Sigma^n X$ is a colimit of objects of Hw_∞^{big}. Coproducts commute with colimits and with Σ^n (since Σ is an equivalence), hence the coproduct $\bigoplus_{i\in\mathbb{N}} X$ also exists in $\mathcal{C}^{big}$. The proof of Proposition 8.1 of [Bar16] only uses the existence of such coproducts in the ∞-category. So, it yields that $\mathrm{K}(\mathcal{C}^{big})$ is homotopy equivalent to the point. Denote by Hw^{big} the homotopy category of Hw_∞^{big}. Using the same argument we obtain that $\mathrm{K}(Com^b(Hw^{big}))$ is homotopy equivalent to the point. By definition the ∞-category $\mathcal{C}^{big}$ admits a weight structure whose heart is Hw_∞^{big}.
Now we use Corollary 3.5 to form the diagram

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{C}^{big} & \longrightarrow & \mathrm{Kar}(\mathcal{C}^{big}/\mathcal{C}) \\ \downarrow{\scriptstyle t_{\mathcal{C}}} & & \downarrow{\scriptstyle t_{\mathcal{C}^{big}}} & & \downarrow{\scriptstyle t'} \\ Com^b(Hw) & \longrightarrow & Com^b(Hw^{big}) & \longrightarrow & \mathcal{C}' = \mathrm{Kar}(\frac{Com^b(Hw^{big})}{Com^b(Hw)}) \end{array}$$

By Theorem 2.3 it induces the following diagram of K-groups whose rows are exact sequences

$$\begin{array}{ccccccc} \mathrm{K}_{i+1}(\mathcal{C}^{big}) & \longrightarrow & \mathrm{K}_{i+1}(\mathrm{Kar}(\mathcal{C}^{big}/\mathcal{C})) & \xrightarrow{d_1} & \mathrm{K}_i(\mathcal{C}) & \longrightarrow & \mathrm{K}_i(\mathcal{C}^{big}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{K}_{i+1}(t')} & & \downarrow{\scriptstyle \mathrm{K}_i(t_{\mathcal{C}})} & & \downarrow \\ \mathrm{K}_{i+1}(Hw^{big}) & \longrightarrow & \mathrm{K}_{i+1}(\mathcal{C}') & \xrightarrow{d_2} & \mathrm{K}_i(Hw) & \longrightarrow & \mathrm{K}_i(Hw^{big}) \end{array}$$

By Proposition 8.1.1 of [Bon10a] $h(\mathcal{C}^{big}/\mathcal{C})$ and $h(\mathcal{C}')$ admit weight structures and their hearts are both equivalent to $\frac{Hw^{big}}{Hw}$. The functor t' is weight-exact and it induces an identity functor on the hearts. One may notice that $t_{\mathcal{C}'} \circ t' \cong t_{\mathcal{C}^{big}/\mathcal{C}}$ (see the uniqueness statement in Corollary 3.5) where $t_{\mathcal{C}^{big}/\mathcal{C}}$ and $t_{\mathcal{C}'}$ are corresponding weight complex functors. Then by the inductive assumption $\mathrm{K}_{i+1}(t_{\mathcal{C}'})$ and $\mathrm{K}_{i+1}(t_{\mathcal{C}^{big}/\mathcal{C}})$ are isomorphisms, hence so is $\mathrm{K}_{i+1}(t')$. Since $\mathrm{K}_n(\mathcal{C}^{big}) \cong \mathrm{K}_n(Hw^{big}) \cong 0$ for any n the maps d_1 and d_2 are also isomorphisms. Hence $\mathrm{K}_i(t_{\mathcal{C}})$ is also an isomorphism. □

Remark 4.4. 1. Theorem 4.3 can be seen as a generalization of Theorem 9.53 of [BGT13] from additive ∞-categories generated by one object to general additive ∞-categories. Moreover, modulo Corollary 4.1 or the main result of [ScSh03] one could derive our result from their theorem and from the fact that K-theory commutes with filtered colimits. The ideas of the two proofs are essentially the same.
2. Presumably the ∞-category $Com^b(Hw^{big})/Com^b(Hw)$ appearing in the proof of 4.3 is equivalent to $Com^b(Hw^{big}/Hw)$. That is, not only $\mathrm{K}_{i+1}(t')$ is an isomorphism but also t' itself is an equivalence. However, since the proof of this fact is unnecessary and requires some work on localizations of triangulated categories, we don't include it into the exposition.

5 Negative K-theory of motivic categories

The groups $\mathrm{K}_i(Hw)$ are theoretically much easier to compute than $\mathrm{K}_i(\mathcal{C})$. Indeed, let A be an idempotent complete additive category. For any object $M \in \operatorname{Obj} A$ the full additive subcategory generated by M is equivalent to the category $free(\operatorname{End}_A(M))$ of finitely-generated right free modules over $\operatorname{End}_A(M)$. Moreover, the full additive subcategory closed under retracts generated by M is equivalent to the category $proj(\operatorname{End}_A(M))$ of finitely-generated right projective modules over $\operatorname{End}_A(M)$. This certainly implies that the additive category A is equivalent to the filtered colimit of categories $proj(\operatorname{End}_A(M))$. By Corollary 6.4 of [Sch06] $\mathrm{K}_i(A) \cong \operatorname{colim}_{M \in \operatorname{Obj} A} \mathrm{K}_i(\operatorname{End}_A(M))$ for any idempotent complete additive category A, where the colimit is taken with respect to the maps induced by the embeddings of the minimal Karoubi-closed additive subcategories of A containing M.

Now let $DM^{eff}_{gm}(k;R)$ denote the category of compact objects in the category of effective Voevodsky motives over a field k with coefficients in a (noetherian) ring R. Assume also that $\operatorname{char}(k)$ is invertible in R. By the results of section 6.5 of [Bon10a] there exists a bounded weight structure on this category whose heart is the category of Chow motives.

From Voevodsky's construction it is clear that $DM^{eff}_{gm}(k;R)$ admits an ∞-enhancement (see also the paper [BeVo08] for the thorough construction of a dg-enhancement). We denote the corresponding stable ∞-category by $\mathcal{DM}^{eff}_{gm}(k;R)$. Now Theorem 4.3 yields the following generalization of Theorem 6.4.2 of [Bon09].

Proposition 5.1. *The weight complex functor for $\mathcal{DM}^{eff}_{gm}(k;R)$ induces isomorphisms $\mathrm{K}_n(\mathcal{DM}^{eff}_{gm}(k;R)) \to \mathrm{K}_n(Chow^{eff}(k;R))$ for any $n \leq 0$.*

A hard conjecture predicts that for $R = \mathbb{Q}$ there exists a bounded t-structure on $DM^{eff}_{gm}(k;R)$ with certain properties called the motivic t-structure. It's known that for general rings of coefficients the conjecture doesn't hold (see Proposition 4.3.8 of [Voe00]).

Corollary 5.2. *If the motivic t-structure exists on $DM^{eff}_{gm}(k;R)$ then $\mathrm{K}_n(Chow^{eff}(k;R)) = 0$ for all $n < 0$.*

Proof. The heart of the motivic t-structure is expected to be noetherian. By the main result of [AGH18] the groups $\mathrm{K}_n(\mathcal{C})$ are zero for $n < 0$. Hence by Proposition 5.1 $\mathrm{K}_n(Chow^{eff}(S;R))$ are zero. ☐

Remark 5.3. Note that the homotopy t-structure on $DM^{eff}(k)$ does not restrict to the subcategory $DM^{eff}_{gm}(k)$, so our theorem cannot be applied.

Remark 5.4. For $R = \mathbb{Q}$ the vanishing of the negative K-groups of $Chow(k;\mathbb{Q})$ also follows from the smash-nilpotence conjecture of Voevodsky. Indeed, assume the smash-nilpotence conjecture holds. Then by Corollary 3.3 of [Voe95] the ring of endomorphisms of any motive $M \in \operatorname{Obj} Chow(k;\mathbb{Q})$ is a nil-extension of the ring of endomorphisms of the image of M in the category of Grothendieck motives $GM(k;\mathbb{Q})$. Since $GM(k;\mathbb{Q})$ is semisimple abelian, $\mathrm{End}_{GM(k;\mathbb{Q})}(M)$ and $\mathrm{End}_{Chow(k;\mathbb{Q})}(M)$ are artinian rings. Thus, their negative K-theory vanishes (see Proposition 10.1 of [Bass68].XII). Now using the representation of $\mathrm{K}_i(Chow(k;\mathbb{Q}))$ as the colimit of $\mathrm{K}_i(\mathrm{End}_{Chow(k;\mathbb{Q})}(M))$ we obtain that $\mathrm{K}_i(Chow(k;\mathbb{Q})) \cong 0$.

Question 5.5. *Voevodsky has shown that the motivic t-structure does not exist on $DM^{eff}_{gm}(k;\mathbb{Z})$ if there is a conic without rational points over k (see Proposition 4.3.8 of [Voe00]). Is it also possible to construct a non-trivial element in $\mathrm{K}_{-1}(Chow(k;\mathbb{Z}))$?*

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