

PARABOLIC CONFORMALLY SYMPLECTIC STRUCTURES. III:
INVARIANT DIFFERENTIAL OPERATORS AND COMPLEXES

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ABSTRACT. This is the last part of a series of articles on a family of geometric structures (PACS-structures) which all have an underlying almost conformally symplectic structure. While the first part of the series was devoted to the general study of these structures, the second part focused on the case that the underlying structure is conformally symplectic (PCS-structures). In that case, we obtained a close relation to parabolic contact structures via a concept of parabolic contactification. It was also shown that special symplectic connections (and thus all connections of exotic symplectic holonomy) arise as the canonical connection of such a structure.

In this last part, we use parabolic contactifications and constructions related to Bernstein–Gelfand–Gelfand (BGG) sequences for parabolic contact structures, to construct sequences of differential operators naturally associated to a PCS-structure. In particular, descending BGG sequences gives rise to a large family of complexes of differential operators associated to a special symplectic connection. For some types of structures, descending relative BGG sequences and subcomplexes in BGG sequences provides large families of complexes for more general instances of PCS-structures. In the last part of the article we prove several results on the cohomology of the complexes obtained by descending BGG sequences.

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1. INTRODUCTION

This article is the last part in a series of three which aims at constructing a large family of differential complexes naturally associated to certain geometric structures. These structures are associated to certain parabolic subalgebras in simple Lie algebras and they come with an underlying almost conformally symplectic structure, so we call them *parabolic almost conformally symplectic structures* or PACS-structures for short. The precise definition of these structures was given in the first part [8] of the series, where we also showed that any such structure gives rise to a canonical connection on the tangent bundle of the underlying manifold. Hence the torsion and the curvature of this canonical connection are natural invariants of a PACS-structure. The torsion naturally splits into two components, one of which is exactly the obstruction to the underlying structure being conformally symplectic. If this obstruction vanishes, the structure is called a PCS-structure, and if the remaining component of torsion also vanishes, one talks about a torsion-free PCS-structure.

Using the classification of simple Lie algebras, one can give an explicit description of the PACS-structures. In [8] it was shown that, on the one hand, these structures provide natural extensions of several well known and interesting geometries. For example, for any Kähler metric, the Kähler form and the complex structure define a torsion-free PCS-structure corresponding to the simple Lie algebra $\mathfrak{su}(n+1, 1)$. Indeed, torsion-free PCS-structures of this type are equivalent to Kähler metrics. Allowing torsion for the PCS-structure, one obtains certain more general types of almost Hermitian manifolds, while for PACS-structures of that type, there is no obvious description in terms of Hermitian metrics. Things work similarly in indefinite signatures and for para-Hermitian metrics. Another type of PCS-structure is closely related to almost quaternionic manifolds endowed with a conformally symplectic structure that is Hermitian in the quaternionic sense.

On the other hand, there is a close relation between PCS-structures and special symplectic connections in the sense of [3]. Indeed any special symplectic connection turns out to be the canonical connection of a torsion-free PCS-structure, so in particular, this applies to all connections of exotic symplectic holonomy. There is a nice characterization of the PCS-structures whose canonical connection is special symplectic using local parabolic contactifications (see below), which is crucial for the developments in this article.

The algebraic data which determine a type of PACS-structures at the same time determine another geometric structure in one higher dimension. Any of these structures comes with an underlying contact structure and they are called *parabolic contact structures*, see the discussion in Section 4.2 of [10]. Now on a contact manifold, the Reeb field of any contact form defines a *transversal infinitesimal automorphism* of the contact structure. In particular it gives rise to a one-dimensional foliation and any local space of leaves for this foliation naturally inherits a conformally symplectic structure. As discussed in [7], any

conformally symplectic structure can be locally realized in this way (“local contactification”) and this realization is unique up to local contactomorphism. For parabolic contact structures, transversal infinitesimal automorphisms are much more rare (and don’t exist generically). Still, as shown in the second part [9] of this series there is a perfect analog of the above constructions in the setting of parabolic contact structures and PCS-structures. For any transversal infinitesimal automorphism of a parabolic contact structure of any type, a local leaf space naturally inherits a PCS-structure of the corresponding type (“PCS-quotients”). Locally, any PCS-structure can be realized in this way (“parabolic contactification”) and this realization is unique up to local isomorphism (of parabolic contact structures). So one can view PCS-structures as geometric structures characterizing reductions of parabolic contact structures by a transversal infinitesimal symmetry.

In the language of parabolic contactifications, one can also deal with geometries corresponding to Lie algebras of type C_n (which are excluded in [8]), see Section 3 of [9]. Here the parabolic contact structure is a *contact projective structure* (see [18]) with vanishing contact torsion, while the analog of a PCS-structure is a conformally Fedosov structure as introduced in [17] (with slight modifications). Finally, in all cases, the PCS-structures for which the distinguished connection is special symplectic (which in the conformally Fedosov case means that it is of Ricci-type) are exactly those, for which any local parabolic contactification is locally isomorphic to the homogeneous model. This puts the construction for special symplectic connections from [3] into a broader context.

As stated above, this last part of the series aims at constructing differential complexes, which are naturally associated to special symplectic connections or more general PCS-structures. The original motivation for the series were the differential complexes on $\mathbb{C}P^n$ constructed in [16] and applied to problems in integral geometry there. In that construction, it was not clear what kind of geometric structure on $\mathbb{C}P^n$ is “responsible” for the existence of the complexes. A surprising feature is that these complexes are one step longer than the de-Rham complex, which suggests that they have their origin in one higher dimension. The simplest instance of such a complex is the so-called co-effective complex on a conformally symplectic manifold, which looks like the Rumin complex associated to a contact structure in one higher dimension. Indeed, as a “proof of concept” for the current series, it was shown in [7] that the co-effective complex can be constructed from the Rumin complex on local contactifications.

Basically, we carry out a similar procedure in this article, starting from a large family of differential complexes that are naturally associated to parabolic contact structures. These are derived from BGG sequences as introduced in [11]. Standard BGG sequences are complexes only on locally flat geometries, so pushing them down, one obtains sequences of differential operators naturally associated to a PCS-structure, which are complexes provided that the canonical connection of the PCS-structure is special symplectic. For certain geometries, it has been shown in [12] that certain parts of BGG sequences are subcomplexes

under weaker assumptions than local flatness. For parabolic contact structures, this only applies in the case of structures of type A_n , so this gives rise to a construction of complexes (of unusual length) for certain PCS-structures of Kähler and para-Kähler type. In the para-Kähler case, one can also start from the relative version of BGG sequences which were constructed in the recent article [14], and which are complexes under much weaker assumptions than local flatness.

In the situation of the co-effective complex and the Rumin complex, both the construction of the upstairs complex and the procedure of pushing down can be phrased in terms of differential forms. In the general situation of BGG sequences and their variants, one has to deal with differential forms with values in a tractor bundle on the level of parabolic contact structures, and constructing the BGG sequence is much more involved. Therefore, we use a slightly different approach than in [7]. The main observation here is that for a completely reducible natural bundle on a parabolic contact structure (i.e. a bundle induced by a completely reducible representations of the parabolic subgroup) there is an obvious counterpart for PCS-structures of the corresponding type. For a local contactification, there is a rather simple relation between sections upstairs and downstairs, which allows one to directly descend invariant differential operators acting between sections of such bundles. Hence we can directly descend the operators in the BGG sequence to any PCS-quotient, without the need to think about descending tractor bundles or tractor connections. It should be remarked that also an approach via downstairs tractor bundles should be feasible. For the case of conformally Fedosov structures, this has been carried out in [17].

After a short review of the geometric structure involved and the parabolic version of contactification, the push down procedure for invariant operators acting between sections of completely reducible bundles is described in Section 2; the main results are Theorems 2.4 and 2.5. Section 3 discusses the applications of this technique to BGG sequences and the related constructions described above. We describe in detail the complexes associated to connections of Ricci type in Theorem 3.1 and those associated to Bochner-bi-Lagrangian metrics in Theorem 3.3. The complexes for para-Kähler manifolds obtained from relative BGG sequences are described in detail in Theorem 3.4. The cases of complexes for Bochner-Kähler metrics (of any signature) coming from BGG sequences and for Kähler metrics coming from subcomplexes in BGG sequences are briefly outlined in Section 3.4 and Remark 3.5.

In Section 4, we describe results on the cohomology of the descended version of BGG sequences. This is similar to the results for the co-effective complex in [7], but this time, the main work is done on the level of the parabolic contact structure. The basic ingredient here is that on that level, the cohomology of a BGG sequence can be described as a twisted de-Rham cohomology. A detailed analysis of the construction of BGG sequences shows that there is a sequence of subsheaves in the upstairs sheaves of tractor-bundle-valued differential forms which computes the cohomology of the descended complex. In Theorem 4.8 we construct a long exact sequence involving the cohomology groups of that sheaf.

Specializing to the case of the homogeneous model, Theorem 4.10 then allows one to interpret this sequence in terms of “downstairs” data. The results are analyzed locally as well as for the global contactification of $\mathbb{C}P^n$ by the sphere S^{2n+1} , where we obtain a vast generalization of the results on cohomology needed for the applications in [16].

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2. PUSHING DOWN INVARIANT OPERATORS

We first review PCS–structures and their relation to parabolic contact structures. Then we show that each invariant differential operator acting between sections of irreducible natural bundles on the parabolic contact structure descends to a natural differential operator on the corresponding PCS–structure.

2.1. THE TYPES OF GEOMETRIC STRUCTURES

To specify a type of PCS–structure and corresponding parabolic contact structure, we have to choose some algebraic data, see Section 2.1 of [9] for more details. We first need a semisimple Lie group G , whose Lie algebra \mathfrak{g} admits a so–called contact grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Next, we have to choose a parabolic subgroup $P \subset G$ corresponding to the Lie subalgebra $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then we define a closed subgroup $G_0 \subset P$ with Lie algebra \mathfrak{g}_0 as consisting of those elements of P whose adjoint action preserves the grading of \mathfrak{g} . It turns out that the exponential mapping restricts to a diffeomorphism from $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \mathfrak{g}_2$ onto a closed normal subgroup $P_+ \subset P$ such that P is the semi–direct product of G_0 and P_+ . In particular, $P/P_+ \cong G_0$.

By definition, $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a Heisenberg algebra, so its Lie bracket defines a non–degenerate line in $\Lambda^2(\mathfrak{g}_{-1})^*$. This in turn defines the conformally symplectic group $CSp(\mathfrak{g}_{-1}) \subset GL(\mathfrak{g}_{-1})$. It turns out that for any element $\varphi \in CSp(\mathfrak{g}_{-1})$, there is a unique linear isomorphism $\psi : \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_{-2}$ such that (ψ, φ) defines an automorphism of the graded Lie algebra \mathfrak{g}_- , so one obtains an isomorphism $\text{Aut}_{gr}(\mathfrak{g}_-) \cong CSp(\mathfrak{g}_{-1})$. By definition, the adjoint action of G_0 restricts to an action by automorphisms on the graded Lie algebra \mathfrak{g}_- , so one obtains a homomorphism $G_0 \rightarrow CSp(\mathfrak{g}_{-1})$ which turns out to be infinitesimally injective.

If \mathfrak{g} is not of type C_n , then both geometric structures we need are determined by this homomorphism. In [8], we have defined the *PACS–structure* associated to (G, P) as the first order structure on manifolds of dimension $\dim(\mathfrak{g}_{-1})$ determined by the homomorphism $G_0 \rightarrow GL(\mathfrak{g}_{-1})$. Hence such a structure on a smooth manifold M is given by a principal G_0 –bundle together with a *soldering form*, a strictly horizontal, G_0 –equivariant \mathfrak{g}_{-1} –valued one–form on the total space of this bundle. Since the image of our homomorphism is contained in

$CSp(\mathfrak{g}_{-1})$, any such structure induces an underlying almost conformally symplectic structure. A *PCS-structure of type (G, P)* is then such a first order structure for which this underlying structure is conformally symplectic.

On the other hand, for a contact manifold of dimension $\dim(\mathfrak{g}_-)$, one considers the associated graded to the tangent bundle, which has a natural frame bundle with structure group $\text{Aut}_{gr}(\mathfrak{g}_-) \cong CSp(\mathfrak{g}_{-1})$. A *parabolic contact structure of type (G, P)* is then given by a reduction of structure group of this frame bundle corresponding to the homomorphism $G_0 \rightarrow CSp(\mathfrak{g}_{-1})$, respectively by the canonical Cartan geometry that such a reduction determines, see Section 4.2 of [10].

If \mathfrak{g} is of type C_n , then it turns out that the homomorphism $G_0 \rightarrow CSp(\mathfrak{g}_{-1})$ induces an isomorphism between the Lie algebras of the two groups. Thus reductions of structure group as considered above carry very little information. Nonetheless, there are analogs for both types of geometries in the C_n -case. On the parabolic contact side, these are *contact projective structures* (with vanishing contact torsion) as discussed in [18] and in Section 4.2.6 of [10]. On the conformally symplectic side, these are the *conformally Fedosov structures* discussed in Section 3 of [9] based on the earlier treatment in the first preprint version of [17].

Treating the structures in terms of principal bundles and soldering forms (of appropriate type) the C_n -case looks essentially the same as the other cases. Hence we will give a uniform treatment below and also refer to conformally Fedosov structures as PCS-structures of type C_n .

2.2. INVARIANT OPERATORS ON PARABOLIC CONTACT STRUCTURES

The uniform description of parabolic contact structures is via Cartan geometries of type (G, P) . A parabolic contact structure of type (G, P) on a manifold $M^\#$ determines a principal P -bundle $p^\# : \mathcal{G}^\# \rightarrow M^\#$ and a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}^\#, \mathfrak{g})$. Factoring by the free action of $P_+ \subset P$, we obtain a principal G_0 -bundle $\mathcal{G}_0^\# \rightarrow M^\#$. The Cartan connection ω descends to a soldering form $\theta^\#$ that defines a reduction of the frame bundles of the associated graded to the tangent bundle, see Section 2.3 of [9] for more details. If \mathfrak{g} is not of type C_n , then this is the equivalent encoding of the geometry as discussed in Section 2.1.

Any representation of the group P gives rise to a natural vector bundle on parabolic contact structures of type (G, P) via forming associated bundles to the Cartan bundle. We will only meet natural bundles obtained in this way in this article. The general representation theory of P is rather complicated, but irreducible and hence completely reducible representations of P are easy to understand. If \mathbb{W} is such a representation, then the nilpotent normal subgroup $P_+ \subset P$ acts trivially on \mathbb{W} , so we obtain a representation of G_0 . This immediately implies that the associated bundle $\mathcal{G}^\# \times_P \mathbb{W}$ can be naturally identified with $\mathcal{G}_0^\# \times_{G_0} \mathbb{W}$. Hence natural bundles associated to completely reducible representations can be readily understood in terms of the underlying structure.

Finally, the group G_0 is always reductive, so its representation theory is well understood.

The equivalence between parabolic contact structures and Cartan geometries in particular implies that any automorphism of a parabolic contact structure on $M^\#$ lifts to a bundle-automorphism of $\mathcal{G}^\#$ which preserves ω . This implies an analogous result for an infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$, i.e. a vector field whose local flows are automorphisms. Such a vector field always uniquely lifts to a P -invariant vector field $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ such that for the Lie derivative \mathcal{L} , we get $\mathcal{L}_{\tilde{\xi}}\omega = 0$. Invariance of $\tilde{\xi}$ implies that there is an intermediate vector field $\xi_0 \in \mathfrak{X}(\mathcal{G}_0^\#)$ whose flow preserves the soldering form. We will be particularly interested in the case of *transverse* infinitesimal automorphisms, i.e. the case that all values of $\xi \in \mathfrak{X}(M^\#)$ are transverse to the contact subbundle (so in particular, ξ is nowhere vanishing).

Given a representation \mathbb{W} of P , the space of sections of the natural bundle $\mathcal{G}^\# \times_P \mathbb{W}$ can be naturally identified with the space of P -equivariant smooth functions $\mathcal{G}^\# \rightarrow \mathbb{W}$. Given a vector field $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$, we can differentiate such equivariant functions and if $\tilde{\xi}$ is P -invariant, then the resulting function will be equivariant and hence correspond to a section. Hence an equivariant vector field acts on sections of any natural vector bundle, and we will denote this action by $\mathcal{L}_{\tilde{\xi}}$.

If \mathbb{W} is completely reducible then P -equivariance of a function $\mathcal{G}^\# \rightarrow \mathbb{W}$ implies invariance under the group P_+ . Hence such a function descends to $\mathcal{G}^\# / P_+ = \mathcal{G}_0^\#$ and is G_0 -equivariant there. A P -invariant vector field $\tilde{\xi}$ as above induces a G_0 -invariant vector field ξ_0 on $\mathcal{G}_0^\#$ and we can use ξ_0 to differentiate sections as above, thus obtaining the same action as above.

There is a general concept of invariant differential operators acting between sections of natural vector bundles over manifolds endowed with a parabolic contact structure of some fixed type. For our purposes it suffices to know that such an operator is defined on any manifold endowed with a structure of the given type and that these operators are compatible with the inclusion of open subsets (endowed with the restricted structure) and with the action of isomorphisms. Hence they are compatible with the action of local isomorphism and in particular of local automorphisms. Applying this to local flows, we conclude that for an infinitesimal automorphism ξ of the Cartan geometry determined by a parabolic contact structure, any invariant differential operator D commutes with the action of $\mathcal{L}_{\tilde{\xi}}$. In the case of completely reducible bundles, one may as well work on the G_0 -principal bundle $\mathcal{G}_0^\#$ using \mathcal{L}_{ξ_0} .

2.3. PCS-QUOTIENTS

The fundamental notion for the study of contactifications in the realm of PCS-structures and parabolic contact structures in [9] is a PCS-quotient. Suppose that we have given a parabolic contact structure of type (G, P) on $M^\#$ together with a transverse infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ of this geometry. Then ξ is nowhere vanishing and hence defines a one-dimensional foliation

of $M^\#$. Basically, a PCS–quotient is a global space of leaves for this foliation which is endowed with a PCS–structure of type (G, P) that can be viewed as a quotient of the parabolic contact structure.

To formulate the precise definition, let $p_0^\# : \mathcal{G}_0^\# \rightarrow M^\#$ be the G_0 –bundle determined by the parabolic contact structure and let $\theta^\#$ be its soldering form. Now for a PCS–quotient, one requires

- a surjective submersion $q : M^\# \rightarrow M$ onto a smooth manifold M such that the fibers of q are connected and their tangent spaces are spanned by ξ
- a PCS–structure of type (G, P) on M with G_0 –bundle $p : \mathcal{G}_0 \rightarrow M$ and soldering form θ
- a lift $q_0 : \mathcal{G}_0^\# \rightarrow \mathcal{G}_0$ of q to a morphism of principal bundles which is a surjective submersion with connected fibers, whose tangent spaces are spanned by ξ_0 and such that $q_0^*\theta$ coincides with the component $\theta_{-1}^\#$ of the “upstairs” soldering form (see Sections 2.3 and 2.4 of [9] for details)

Remark 2.1. If \mathfrak{g} is not of type C_n , then the above is exactly the definition of a PCS–quotient from Section 2.4 of [9]. If \mathfrak{g} is of type C_n , then the discussion in Section 3.3 of [9] shows that the same setup is available for a projective contact structure (with vanishing contact torsion) on $M^\#$ and a conformally Fedosov structure on M , see in particular the proof of the first part of Theorem 8 of [9].

As it stands, the concept of a PCS–quotient may look rather restrictive and one might doubt whether there are many examples. However, the results of [7] and of [9] imply that there are lots of examples. This is best formulated in the language of *parabolic contactifications* of PCS–structures. By a parabolic contactifications of a PCS–structure M , we simply mean a realization of M as a PCS–quotient of a parabolic contact structure of type (G, P) (respectively that parabolic contact structure). For later use, let us collect the fundamental results on parabolic contactifications:

THEOREM 2.2. (1) *Let M be a PCS–structure with underlying conformally symplectic structure $\ell \subset \Lambda^2 T^*M$. Then any open subset $U \subset M$ over which ℓ admits a nowhere–vanishing section which is exact as a two–form on M admits a parabolic contactification.*

(2) *Let M and N be two PCS–structures endowed with fixed parabolic contactifications. Then locally any morphism of PCS–structures (compare with Section 2.5 below) lifts to a contactomorphism between the contactifications.*

(3) *Any lift of a morphism of PCS–structures to a contactomorphism of parabolic contactifications is automatically compatible with the infinitesimal automorphisms up to a nowhere–vanishing, locally constant factor and a morphism of parabolic contact structures.*

Proof. (1) By Lemma 3.1 of [7], U can be realized as the quotient $q : U^\# \rightarrow U$ of a contact manifold $U^\#$ by a transverse infinitesimal contactomorphism. By Theorem 4 (for \mathfrak{g} not of type C_n) respectively part 2 of Theorem 8 (for \mathfrak{g} of

type C_n) of [9], a PCS-structure of type (G, P) on U lifts to a parabolic contact structure of type (G, P) on $U^\#$, thus providing the required PCS-quotient.

The statement of (3) is proved in Theorem 5 (for \mathfrak{g} not of type C_n) respectively in part 3 of Theorem 8 (for \mathfrak{g} of type C_n) of [9]. Moreover, by Proposition 3.1 of [7], the assumption of part (3) is locally satisfied for each morphism between PCS-structures endowed with contactifications, so (2) follows. \square

We want to remark that in [9] we also constructed examples of global contactifications which will play an important role later on.

2.4. DESCENDING INVARIANT DIFFERENTIAL OPERATORS TO PCS-QUOTIENTS

Now suppose that \mathbb{W} is a representation of G_0 , which we can also view as a completely reducible representation of P . Then as discussed in Section 2.2, this gives rise to a (completely reducible) natural vector bundle on parabolic contact structures of type (G, P) . Given such a geometry $(p^\# : \mathcal{G}^\# \rightarrow M^\#, \omega)$ we denote the resulting bundle by $\mathcal{WM}^\# := \mathcal{G}^\# \times_P \mathbb{W}$. As noted in Section 2.2, we can also view $\mathcal{WM}^\#$ as the bundle $\mathcal{G}_0^\# \times_{G_0} \mathbb{W}$ associated to the underlying G_0 -bundle.

On the other hand, one also obtains a natural vector bundle on PCS-structures of type (G, P) , since they are also defined by a principal G_0 -bundle. Given such a geometry $(p : \mathcal{G}_0 \rightarrow M, \theta)$ we write $\mathcal{WM} := \mathcal{G}_0 \times_{G_0} \mathbb{W}$ for this bundle. Now it is well known that sections of an associated bundle can be viewed as equivariant functions on the total space of the inducing principal bundle. Explicitly, the space $\Gamma(\mathcal{WM} \rightarrow M)$ of sections is naturally isomorphic to

$$C^\infty(\mathcal{G}_0, \mathbb{W}) = \{f \in C^\infty(\mathcal{G}_0, \mathbb{W}) : f(u \cdot g) = g^{-1} \cdot f(u) \quad \forall g \in G_0\}.$$

Evidently, such a function can be pulled back via $q_0 : \mathcal{G}_0^\# \rightarrow \mathcal{G}_0$ to a smooth equivariant function $\mathcal{G}_0^\# \rightarrow \mathbb{W}$, which then defines a smooth section of $\mathcal{WM}^\# \rightarrow M^\#$. This defines an injection $\Gamma(\mathcal{WM}) \hookrightarrow \Gamma(\mathcal{WM}^\#)$, which we denote by q_0^* .

LEMMA 2.3. *Let $q : M^\# \rightarrow M$ be a PCS-quotient by a transversal infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ of a parabolic contact structure of type (G, P) with bundle map $q_0 : \mathcal{G}_0^\# \rightarrow \mathcal{G}_0$. Let \mathbb{W} be a representation of G_0 and consider the corresponding induced bundles $\mathcal{WM}^\#$ and \mathcal{WM} as above. Let $\xi_0 \in \mathfrak{X}(\mathcal{G}_0^\#)$ be the G_0 -invariant vector field induced by ξ and consider the induced map \mathcal{L}_{ξ_0} on the space $\Gamma(\mathcal{WM}^\#)$. Then the the image of $q_0^* : \Gamma(\mathcal{WM}) \rightarrow \Gamma(\mathcal{WM}^\#)$ coincides with the kernel of \mathcal{L}_{ξ_0} .*

Proof. In the language of equivariant functions, \mathcal{L}_{ξ_0} is simply given by differentiating vector valued functions using the vector field ξ_0 . (This preserves the space of equivariant functions since ξ_0 is G_0 -invariant.) In view of this, the result follows from the description of q_0^* in terms of equivariant functions, since the fibers of q_0 are connected by assumption and their tangent spaces are spanned by ξ_0 . \square

Having this at hand, we can formulate the fundamental results about descending invariant differential operators.

THEOREM 2.4. *Let $q : M^\# \rightarrow M$ be a PCS-quotient by a transversal infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ of a parabolic contact structure of type (G, P) with bundle map $q_0 : \mathcal{G}_0^\# \rightarrow \mathcal{G}_0$, and let \mathbb{W} and $\tilde{\mathbb{W}}$ be representations of G_0 . Further, let $D^\# : \Gamma(\mathcal{W}M^\#) \rightarrow \Gamma(\tilde{\mathcal{W}}M^\#)$ be a linear invariant differential operator for the given parabolic contact structure.*

Then there is a unique linear differential operator $D : \Gamma(\mathcal{W}M) \rightarrow \Gamma(\tilde{\mathcal{W}}M)$ such that $q_0^ \circ D = D^\# \circ q_0^*$.*

Proof. Since $D^\#$ is an invariant differential operator, it commutes with pullback along the flow of ξ_0 . Infinitesimally, this means that $D^\# \circ \mathcal{L}_{\xi_0} = \mathcal{L}_{\xi_0} \circ D^\#$, so in particular $D^\#$ maps $\ker(\mathcal{L}_{\xi_0}) \subset \Gamma(\mathcal{W}M^\#)$ to $\ker(\mathcal{L}_{\xi_0}) \subset \Gamma(\tilde{\mathcal{W}}M^\#)$. Using Lemma 2.3, we conclude that given $\sigma \in \Gamma(\mathcal{W}M)$, there is a unique section $\tilde{\sigma}$ such that $D^\#(q_0^*\sigma) = q_0^*\tilde{\sigma}$, so we can define $D(\sigma) := \tilde{\sigma}$ to obtain an operator with the desired property. Clearly, D is linear, and in local coordinates it is evident that D is a differential operator. Alternatively, one may observe that if σ vanishes on an open subset $U \subset M$, then $q_0^*\sigma$ vanishes on $(q_0)^{-1}(U)$. Since $D^\#$ is a differential operator, the same holds for $D^\#(q_0^*\sigma)$ so $D(\sigma)$ vanishes on U . Thus D is a local operator and thus a differential operator by the Peetre theorem. \square

2.5. NATURALITY OF THE DESCENDED OPERATORS

We next show that pushing down to PCS-quotients can be used to construct natural operators on the category of PCS-structures from invariant differential operators for the corresponding parabolic contact structure. To explain the meaning of “natural operator”, we have to recall some concepts.

First a morphism of PCS-structures is defined to be a principal bundle morphism φ which covers a local diffeomorphism $\underline{\varphi}$ of the base manifolds and is compatible with the soldering forms. Next, we need the concept of a natural vector bundle on the category of PCS-structures, but here we restrict to bundles associated to the defining principal bundle. So as in Section 2.4, we take a representation \mathbb{W} of G_0 and for a PCS-structure $(\mathcal{G}_0 \rightarrow M, \theta)$ we define $\mathcal{W}M := \mathcal{G}_0 \times_{G_0} \mathbb{W}$. The soldering form θ can then be used to identify natural bundles of this type with more traditional natural bundles like tensor bundles. This implies that any morphism φ of PCS-structures, say on M and N , induces a vector bundle map $\mathcal{W}\varphi : \mathcal{W}M \rightarrow \mathcal{W}N$, which restricts to a linear isomorphism on each fiber. Compatibility with the soldering forms implies that this is compatible with the identifications with tensor bundles, i.e. one obtains the usual induced bundle maps there. The induced vector bundle maps can then be used to pull back sections of associated bundles: For $\sigma \in \Gamma(\mathcal{W}N)$, there is a unique section $\varphi^*\sigma \in \Gamma(\mathcal{W}M)$ such that $\sigma \circ \underline{\varphi} = \mathcal{W}\varphi \circ \varphi^*\sigma$.

Now given a second representation $\tilde{\mathbb{W}}$, a natural operator between sections of the corresponding associated bundles is defined as a family of differential operators $D_M : \Gamma(\mathcal{W}M) \rightarrow \Gamma(\tilde{\mathcal{W}}M)$ which is compatible with the actions of

all pullback operators associated to morphisms of PCS-structures. Hence for any morphism φ to a PCS-structure over N , and any section $\sigma \in \Gamma(\mathcal{W}N)$ we require $D_M(\varphi^*\sigma) = \varphi^*(D_N(\sigma))$, where (as usual) we denote all pullback operators by the same symbol.

THEOREM 2.5. *Let \mathbb{W} and $\tilde{\mathbb{W}}$ be two representations of G_0 , which we also view as completely reducible representations of P . Then any invariant operator on the category of parabolic contact structures of type (G, P) between sections of the natural bundles induced by the two representations canonically induces a natural differential operator on the category of PCS-structures of type (G, P) acting between sections of the induced bundles corresponding to the two representations.*

Proof. This follows rather easily from the results on PCS-contactifications in Theorem 2.2. Let us start with a PCS-structure on M , a section $\sigma \in \Gamma(\mathcal{W}M)$ and a point $x \in M$. By part (1) of Theorem 2.2, there is an open neighborhood U of x in M which can be realized as a PCS-quotient $q : U^\# \rightarrow U$. Given the invariant operator $D^\#$ on parabolic contact structures, we can use Theorem 2.4 to obtain an operator $D_U : \Gamma(\mathcal{W}U) \rightarrow \Gamma(\tilde{\mathcal{W}}U)$. In particular, we can apply this to $\sigma|_U$ to obtain a section of $\tilde{\mathcal{W}}M$ defined over U .

To complete the proof, we need a fact on pullbacks of sections. (This may look rather obvious in written form, but this is slightly deceiving, since this relates two different concepts of pullback, which are denoted in the same way. In particular, one of this is non-standard since it relates bundles over different manifolds.) Suppose that M and \tilde{M} are PCS-structures endowed with parabolic contactifications $q : M^\# \rightarrow M$ and $\tilde{q} : \tilde{M}^\# \rightarrow \tilde{M}$, and let us denote the corresponding bundles by $\mathcal{G}_0, \tilde{\mathcal{G}}_0, \mathcal{G}_0^\#$ and $\tilde{\mathcal{G}}_0^\#$, respectively. Assume further that $\Phi : \mathcal{G}_0 \rightarrow \tilde{\mathcal{G}}_0$ is a morphism of PCS-structures and that $\Psi : \mathcal{G}_0^\# \rightarrow \tilde{\mathcal{G}}_0^\#$ is a lift to a morphism of parabolic contact structures which is compatible with the infinitesimal automorphisms up to a constant multiple. Then for any section σ of a natural bundle over \tilde{M} , we have $\Psi^*\tilde{q}_0^*\sigma = q_0^*\Phi^*\sigma$.

To prove this claim, observe that if f is the equivariant function on $\tilde{\mathcal{G}}_0$ corresponding to σ , then $\tilde{q}_0^*\sigma$ and $\Psi^*\tilde{q}_0^*\sigma$ correspond to $f \circ \tilde{q}_0$ and $f \circ \tilde{q}_0 \circ \Psi$, respectively. Since Ψ is compatible with the infinitesimal automorphisms up to constant multiple, the fact that $\tilde{q}_0^*\sigma$ lies in the kernel of $\mathcal{L}_{\tilde{\xi}_0}$ implies that $\Psi^*\tilde{q}_0^*\sigma$ lies in the kernel of \mathcal{L}_{ξ_0} . Thus it must be of the form $q_0^*\tau$ for some section τ and then $\tilde{q}_0 \circ \Psi = \Phi \circ q_0$ implies that $\tau = q_0^*\Phi^*\sigma$, which completes the proof of the claim.

Returning to $\mathcal{W}M$ and $\tilde{\mathcal{W}}M$, we can carry out the above construction for the elements of an open covering $\{U_i : i \in I\}$ of M , so for each i we descend $D^\#(q_i^*(\sigma|_{U_i}))$ to a section $D_i(\sigma)$ of $\tilde{\mathcal{W}}M$ defined over U_i . If $U_i \cap U_j = U_{ij} \neq \emptyset$, then by parts (2) and (3) of Theorem 2.2, the identity on U_{ij} locally lifts to an isomorphism Ψ_{ij} of the parabolic contact structures which is compatible with the transversal infinitesimal automorphisms up to a constant multiple. By the claim, this implies that over the open subset in question, we have $\Psi_{ij}^*q_j^*\sigma = q_i^*\sigma$. Invariance of $D^\#$ now implies that $D^\#(q_i^*\sigma) = \Psi_{ij}^*D^\#(q_j^*\sigma)$,

so $q_i^*(D_i(\sigma)) = \Psi_{ij}^* q_j^*(D_j(\sigma))$. Again by the claim, the right hand side equals $q_i^*(D_j(\sigma))$. Thus, locally on U_{ij} , we have $D_i(\sigma) = D_j(\sigma)$, so this has to hold on all of U_{ij} .

On the one hand, this shows that the sections $D_i(\sigma)$ can be pieced together to define a global section $D_M(\sigma)$. On the other hand, applying the argument to the union of two coverings, we see that $D_M(\sigma)$ is independent of the choice of covering, so we have obtained a well defined linear differential operator D_M . Thus it remains to prove that the D_M define a natural operator. Naturality of a differential operator can be verified locally, so in view of Theorem 2.2, it suffices to do this for PCS-structures admitting a global contactification and for morphisms which lift to the contactifications. But in this case, the required property follows immediately from the claim. \square

3. (RELATIVE) BGG COMPLEXES AND SUBCOMPLEXES

Since PCS-structures admit canonical connections, constructing differential operators, which are intrinsic to such structures, is not difficult. As in Riemannian geometry, one can simply form iterated covariant derivatives with respect to induced linear connections, combine them with iterated covariant derivatives of the torsion and the curvature of the canonical connection and then apply tensorial operations. Constructing differential *complexes* naturally associated to such geometries is a completely different issue, and it is not at all clear, how to do this “by hand”.

On the other hand, there are general constructions for a large number of differential complexes on locally flat parabolic contact structures as well as on certain non-flat structures of type A_n . All these complexes can be pushed down to PCS-quotients thus providing a large number of differential complexes, which are naturally associated to such structures.

3.1. BGG SEQUENCES

Let us start with a type (G, P) of parabolic geometries and a finite dimensional representation \mathbb{V} of G . Associated to these data there is a sequence of invariant differential operators acting on sections of certain irreducible natural vector bundles over parabolic geometries of type (G, P) . A construction for these sequences was given in [11] and improved in [4]. More recently, the construction was generalized and substantially improved in [14]. In the case of the homogeneous model G/P of the geometry, the resulting sequence turns out to be a complex and a fine resolution of the locally constant sheaf \mathbb{V} . In a certain sense this resolution is dual to Lepowsky’s generalization (see [20]) of the Bernstein–Gelfand–Gelfand resolution of \mathbb{V} by homomorphisms of Verma modules. The fact that the BGG sequence is a complex, extends from the homogeneous model G/P to all parabolic contact structures which are locally isomorphic to G/P , i.e. to the locally flat geometries. Also in this more general case, the BGG complex is a fine resolution of a sheaf. This can be described explicitly as the sheaf of those sections of the vector bundle induced by \mathbb{V} (a

so-called tractor bundle) which are parallel for a canonical flat connection on that bundle.

Applying the push-down construction from Section 2 to a BGG-sequence, one therefore obtains a complex if all local contactifications of a given PCS-structure are locally flat. The latter property is analyzed in Theorem 7 and Corollary 1 of [9], where it is shown to be equivalent to the fact that the canonical connection associated to the PCS-structure is a special symplectic connection in the sense of [3]. Since conversely any special symplectic connection is the canonical connection associated to a PCS-structure, we conclude that the pushed down versions of BGG complexes are associated to special symplectic connections. We should point out that global contactifications as discussed in Sections 2.6 and 3.4 of [9] are of particular interest here. These are contactifications of compact PCS-structures, which are circle bundles, and in this case it is possible to analyze the cohomology of the resulting complexes, see Section 4.

The bundles showing up in a BGG sequence are associated to the representations of P on the Lie algebra homology groups $H_*(\mathfrak{p}_+, \mathbb{V})$. Here \mathfrak{p}_+ is the nilradical of the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, and there is a general result that these homology representations are always completely reducible. Hence in the complex case, they can be described in terms of weights and Kostant's theorem (see [19]) gives an explicit way to compute the relevant weights algorithmically. These results can be extended to the real case using complexifications. In what follows, we will usually suppress such computations and just state the descriptions of the resulting bundles.

Invariance of the operators in a BGG sequence can also be used to determine the principal part of the operator. By construction, the principal symbol of any operator showing up in a BGG sequence has to be a natural bundle map and thus is induced by a P -homomorphism between the inducing representations. As noted above, the inducing representations are completely reducible, so the action of P comes from a representation of the reductive group G_0 , whose representation theory is well understood.

To describe the BGG complexes associated to special symplectic connections, the main task therefore is to convert the representation theory information available for the parabolic contact structures into information on bundles on PCS-quotients. We will do this in a bit more detail for the C_n and A_n types and sketch how things look for the other types.

3.2. EXAMPLE: COMPLEXES ASSOCIATED TO CONNECTIONS OF RICCI TYPE
This is the case discussed in Section 3 of [9]. One starts with a conformally Fedosov structure on a smooth manifold M , which is given by a conformally symplectic structure $\ell \in \Lambda^2 T^*M$ and a projective class of torsion free linear connections on TM , which satisfy a certain compatibility condition. Locally, this structure determines a symplectic form ω on M (up to a constant multiple) and a unique connection ∇ in the projective class such that $\nabla\omega = 0$. So locally, the structure is just given by a torsion-free symplectic connection, see

Proposition 2 in [9]. Any local contactification of M then inherits a canonical contact projective structure, which is locally flat if and only if the connection ∇ is of Ricci type, see Theorem 8 and Corollary 1 in [9]. This means that in the decomposition of the curvature of ∇ into irreducible pieces, the trace-free part vanishes. Thus all the curvature is concentrated in the trace part, which encodes the Ricci curvature of ∇ .

The irreducible natural bundles available in this situation are easy to describe. They are equivalent to irreducible representation of $\mathfrak{g}_0 \cong \mathfrak{osp}(2n)$, where $2n = \dim(\mathfrak{g}_{-1})$. Irreducible representations of the center are one-dimensional and thus give rise to natural line bundles. On the other hand the irreducible representations of $\mathfrak{sp}(2n)$ can all be constructed from the standard representation by tensorial operations. For a contact projective structure on $M^\#$, the standard representation of \mathfrak{g}_0 corresponds to the contact subbundle $H \subset TM^\#$. The basic natural line bundle in this case is the quotient $Q := TM^\#/H$, and one can construct density bundles as (real) roots of the line bundle $Q \otimes Q$, which has to be trivial.

For a PCS-structure on M , the standard representation of \mathfrak{g}_0 corresponds to the tangent bundle TM . Now on $M^\#$, the Levi bracket induces an isomorphism $H \cong H^* \otimes Q$, whereas on M , inserting vector fields into elements of ℓ defines an isomorphism $TM \otimes \ell \rightarrow T^*M$, which shows that the representation \mathfrak{g}_{-2} of \mathfrak{g}_0 , which gives rise to Q on $M^\#$ corresponds to ℓ^* on M , compare with Section 3.2 of [7]. This is sufficient to explicitly associate to any irreducible representation of \mathfrak{g}_0 a weighted tensor bundle (a tensor product of a natural line bundle with a natural subbundle of a tensor bundle) on M .

Using this, we can give a description of the resulting sequences in the spirit of the parametrization of BGG sequences for AHS-structures introduced in [1]. To do this, we make one more observation. Suppose that V and W are two irreducible representations of \mathfrak{g}_0 . Then in the tensor product $V \otimes W$, there is a specific irreducible component called the *Cartan product*, which we denote by $V \odot W$. This is the component of maximal highest weight respectively the subrepresentation generated by the tensor product of two highest weight vectors. Given two natural tensor bundles E and F , we denote by $E \odot F \subset E \otimes F$ the irreducible tensor subbundle corresponding to the Cartan product of the inducing representations. Since $V \odot W$ occurs in $V \otimes W$ with multiplicity one, there is a unique (up to scale) natural bundle map $E \otimes F \rightarrow E \odot F$, which we call the canonical projection onto the Cartan product.

THEOREM 3.1. *Let E be an irreducible tensor bundle on conformally symplectic manifolds of dimension $2n \geq 4$ and let $k \geq 0$ be an integer, which, depending on E has to be even or odd. Then pushing down an appropriate BGG sequence on local contactifications, one obtains on any conformally Fedosov manifold M of dimension $2n$ a sequence of weighted irreducible tensor bundles and invariant differential operators of the form*

$$\Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \dots \xrightarrow{D_{2n-1}} \Gamma(E_{2n}) \xrightarrow{D_{2n}} \Gamma(E_{2n+1}).$$

*This sequence is a complex if the canonical connection ∇ of M is of Ricci-type. Moreover, E_0 is a tensor product of E with some density bundle, and E_1 is the Cartan product $S^{k+1}T^*M \odot E_0$. The operator D_0 has order $k+1$ and its principal part is given by forming the $k+1$ -fold covariant derivative with respect to (the connection induced by) ∇ , symmetrizing and then projecting to the Cartan product.*

Proof. We have $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$ and $\mathfrak{p} \subset \mathfrak{g}$ is the stabilizer of a line in the standard representation. The semisimple part \mathfrak{g}_0^0 of \mathfrak{g}_0 , which is isomorphic to $\mathfrak{sp}(2n, \mathbb{R})$, can be identified with the space of those maps, which vanish on a non-degenerate plane containing this line. Since \mathfrak{g} is a split real form, it has a root decomposition and there is a simple root α_1 , such that a root space \mathfrak{g}_α lies in \mathfrak{g}_0^0 if and only if α is linear combination of the other simple roots $\alpha_2, \dots, \alpha_{n+1}$. Now let $\omega_1, \dots, \omega_{n+1}$ be the corresponding fundamental weights, so dominant integral weights for \mathfrak{g} are linear combinations of these weights with non-negative integral coefficients. If such a linear combination does not involve ω_1 , then it can naturally be viewed as a weight of \mathfrak{g}_0^0 , and all weights of \mathfrak{g}_0^0 arise in this way.

For our purposes, it is better to describe representations by the negatives of lowest weights rather than using the usual description in terms of highest weights, but this causes only small differences. The irreducible tensor bundle E then corresponds to a weight of \mathfrak{g}_0^0 . Representing this weight as a linear combination of $\omega_2, \dots, \omega_{n+1}$ and adding $k\omega_1$ (where k is the chosen integer), we obtain a dominant integral weight. This corresponds to a finite dimensional irreducible representation \mathbb{V} of \mathfrak{g} , which integrates to the group $Sp(2n+2, \mathbb{R})$. Now let us in addition assume that the sum of the coefficients of those ω_i with odd i is even, which, depending on E , means that k has to be even or that k has to be odd. Then the homomorphism $Sp(2n, \mathbb{R}) \rightarrow GL(\mathbb{V})$ defining the representation factorizes to $G := PSp(2n+2, \mathbb{R})$, and hence \mathbb{V} gives rise to a BGG-sequence on parabolic geometries of type (G, P) which are equivalent to contact projective structures.

Via the construction in Section 2 we can descend this to a sequence of invariant differential operators on conformally Fedosov structures. The BGG sequence is a complex if the contact projective structure is locally flat, so we obtain a complex if the canonical connection of the conformally Fedosov structure is of Ricci type. To prove the rest of the theorem we need some information on the bundles occurring in the BGG sequence, which all follow from the description of the homology groups $H_*(\mathfrak{p}_+, \mathbb{V})$ via Kostant's theorem. The homology groups split into a direct sum of different irreducible representations of \mathfrak{g}_0 , and the corresponding weights are obtained from the weight determined by \mathbb{V} by the affine action of a certain subset $W^{\mathfrak{p}}$ of the Weyl group W of \mathfrak{g} . The homology degree in which an irreducible component occurs is given by the length of the corresponding Weyl group element.

Using the algorithms from Section 3.2.16 of [10], one easily verifies that $W^{\mathfrak{p}}$ consists of $2n+2$ elements, which have length $0, 1, \dots, 2n+1$, respectively.

This implies that the BGG sequence and hence the descended sequence has the claimed form with an irreducible bundle in each degree between 0 and $2n + 1$. The unique element of length zero in $W^{\mathfrak{p}}$ is the identity, so $H_0(\mathfrak{p}_+, \mathbb{V})$ is the representation of \mathfrak{g}_0 corresponding to the same weight as \mathbb{V} . But this exactly says that E_0 is the tensor product of E with a natural line bundle. The unique element of length one in $W^{\mathfrak{p}}$ is the simple reflection corresponding to α_1 . The affine action by this reflection maps $k\omega_1 + a_2\omega_2 + \sum_{i \geq 3} a_i\omega_i$ to $(-k - 2)\omega_1 + (a_2 + k + 1)\omega_2 + \sum_{i \geq 3} a_i\omega_i$, which is just the sum of the initial weight with $-(k + 1)\alpha_1$. Now α_1 is the lowest weight of $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}^*$, so $-(k + 1)\alpha_1$ is the negative of the lowest weight of the representation inducing $S^{k+1}T^*M$, which implies the claim on E_1 . The claim on the principal part of D_0 then follows from invariance on the level of contact projective structures. \square

Via Kostant's theorem, the representations inducing the bundles E_i in the sequence can be determined explicitly and algorithmically. However, for $i \geq 2$, the explicit form of the bundles depends on the initial representation in a more complicated way. Let us just describe one situation in a bit more detail, which in particular covers the complexes used in [16]. To obtain these complexes, we use the global contactification $S^{2n+1} \rightarrow \mathbb{C}P^n$ defined by the Hopf-fibration. This can be interpreted as a PCS-contactification of the conformally Fedosov structure on $\mathbb{C}P^n$ defined by the Levi-Civita connection of the Fubini-Study metric by the flat contact projective structure on S^{2n+1} , see Section 3.4 in [9]. In the language of Theorem 3.1, the relevant complexes correspond to the case that $E_0 = S^\ell T^*M$ for some $\ell \in \mathbb{N}$ and to $k = 0$. Hence one starts with a completely symmetric covariant tensor of valence ℓ and D_0 is given by taking a covariant derivative and then completely symmetrizing the result. For the applications in [16] one mainly needs the principal part of the operator D_1 in that complex and the information that, on $\mathbb{C}P^n$, one has $\ker(D_1) = \text{im}(D_0) \subset \Gamma(E_1)$. Here we indicate how to get the necessary information on the principal part, the cohomology of the sequence will be discussed in Section 4 below, see in particular Theorem 4.13.

We actually discuss a slightly more general setting, looking at the case that $E_0 = S^\ell T^*M$ with an arbitrary even number k . In the language of the proof of Theorem 3.1, the weight determining \mathbb{V} is $\lambda := k\omega_1 + \ell\omega_2$. The unique element of length two in $W^{\mathfrak{p}}$ is given by $\sigma_1 \circ \sigma_2$, where we write σ_i for the reflection corresponding to the i th simple root. The affine action of this composition maps λ to $(-k - \ell - 3)\omega_1 + k\omega_2 + (\ell + 1)\omega_3$, so this is the weight corresponding to the bundle E_2 , which we denote by E_2^k to indicate the dependence on k . The bundle E_2^0 is described in [16] in detail. Up to a twist by a natural line bundle, this is the Cartan product of $\ell + 1$ copies of the trace-free part $\Lambda_0^2 T^*M$ of $\Lambda^2 T^*M$, i.e. it corresponds to the highest weight subspace in $S^{\ell+1}(\Lambda_0^2 T^*M)$. Hence it can be viewed as tensors with $2\ell + 2$ indices which come up as $\ell + 1$ skew symmetric pairs and the tensor is symmetric under permutations of the pairs of indices.

Representation theory also implies that there is a unique (up to a constant) natural bundle map $S^{\ell+1}T^*M \otimes S^{\ell+1}T^*M \rightarrow E_2^0$ on conformally symplectic manifolds. This is basically given by grouping the indices into pairs and then alternating each pair. By Theorem 3.1, the bundle E_1 is, for $k = 0$, isomorphic to $S^{\ell+1}T^*M$. Hence one can use information on BGG sequences on contact projective structures, to see that D_1 has order $\ell + 1$ and obtain information on its principal part.

For general k , the situation is similar. Up to a twist by a natural line bundle, E_2^k is the Cartan product of S^kT^*M and E_2^0 , while $E_1^k = S^{k+\ell+1}T^*M$. Basic representation theory again shows that there is a unique (up to scale) natural bundle map $S^{\ell+1}T^*M \otimes E_1^k \rightarrow E_2^k$ on conformally symplectic manifolds. This shows that D_1 still has order $\ell + 1$ in the general case, and one can use results on BGG sequences to get information on its principal part.

Remark 3.2. Let us briefly discuss the restriction on the parity of the integer k which determines the order of the first operator in the sequence in Theorem 3.1. From the proof it is clear that this is only needed in order that a certain Lie algebra representation integrates to a group representation of $PSp(2n + 2, \mathbb{R})$. However, for any choice of k , the Lie algebra representations integrate to representations of $Sp(2n + 2, \mathbb{R})$. Hence this restriction could be avoided if one can construct a parabolic geometry of type $(Sp(2n + 2), P)$ on the contactifications, for example by choosing some additional data on the given conformally Fedosov structure.

It seems very plausible that this is possible, at least locally, or provided that the line bundle ℓ defining the conformally symplectic structure is trivial. However, Section 3.4 of [9] shows that this does not work in a straightforward way for the global contactification $S^{2n+1} \rightarrow \mathbb{C}P^n$ defined by the Hopf-fibration. We will not study this question further here.

3.3. COMPLEXES ASSOCIATED TO BOCHNER-BI-LAGRANGEAN METRICS

We next discuss the case of PCS-structures associated to simple Lie algebras of type A_n . Here there are two basic structures related to different real forms of $\mathfrak{sl}(n + 2, \mathbb{C})$, see Section 3.2 of [8]. For the split real form $\mathfrak{sl}(n + 2, \mathbb{R})$, the corresponding PCS-structure is given by a conformally symplectic structure $\ell \subset \Lambda^2T^*M$ and a decomposition $TM = E \oplus F$ into a sum of Lagrangean subbundles. Such a structure is torsion-free if and only if the subbundles E and F in TM are involutive. In this case, one obtains a para-Kähler-metric on M and the canonical connection for the PCS-structure is the Levi-Civita connection of this metric.

Parabolic contactification for PCS-structures of this type produces a so-called Lagrangean contact structure, i.e. a contact structure together with a decomposition of the contact subbundle into a direct sum of Lagrangean subbundles, see Section 4.2.3 of [10]. Torsion-freeness in this picture again is equivalent to involutivity of the two Lagrangean subbundles. To obtain differential complexes

from BGG-sequences on the parabolic contactification, we need this contactification to be locally flat (and thus in particular torsion-free). By Theorem 7 of [9], this is the case if and only if the metric is Bochner-bi-Lagrangian.

To formulate the theorem on BGG sequences in this case, observe that for $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ the algebra \mathfrak{g}_0 has center \mathbb{R}^2 and semi-simple part $\mathfrak{sl}(n, \mathbb{R})$. Up to twisting by natural line bundles the subbundles $E, F \subset TM$ correspond to the standard representation of $\mathfrak{sl}(n, \mathbb{R})$ and its dual. Hence all bundles corresponding to irreducible representations of \mathfrak{g}_0 can be obtained from natural line bundles and these two basic bundles via tensorial constructions. Thus we can use a similar parametrization of BGG sequences as in Theorem 3.1.

THEOREM 3.3. *Let \mathbb{W} be an irreducible representation of $\mathfrak{sl}(n, \mathbb{R})$ and let W be the corresponding natural tensor bundle on a PCS-manifold (M, ℓ, E, F) of para-Kähler type of dimension $2n \geq 6$ (with E playing the role of the standard representation and F playing the role of its dual). Let $k, \ell \geq 0$ be integers such that for even n , the number $k + \ell$ is, depending on W , either even or odd. Then pushing down an appropriate BGG sequence on parabolic contactifications leads to a sequence of tensor bundles and invariant differential operators of the form*

$$\Gamma(W_0) \xrightarrow{D_0} \Gamma(W_1) \xrightarrow{D_1} \dots \xrightarrow{D_{2n-1}} \Gamma(W_{2n}) \xrightarrow{D_{2n}} \Gamma(W_{2n+1}).$$

This sequence is a complex, if (M, ℓ, E, F) is Bochner-bi-Lagrangian. Moreover, the bundles W_0 and W_{2n+1} are irreducible, while for $i = 1, \dots, n$ the bundles W_i and W_{2n+1-i} each split into a direct sum of $i+1$ irreducible tensor bundles. Finally, W_0 is the tensor product of W with a natural line bundle, while $W_1 = W_{(1,0)} \oplus W_{(0,1)}$ with $W_{(1,0)} = S^k E^ \odot W_0$ and $W_{(0,1)} \cong S^\ell F^* \odot W_0$.*

Proof. Put $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ and let $\mathfrak{g}_0^0 \cong \mathfrak{sl}(n, \mathbb{R})$ be the semisimple part of \mathfrak{g}_0 . Then for the standard numbering $\alpha_1, \dots, \alpha_{n+1}$ of simple roots of \mathfrak{g} , a root space \mathfrak{g}_α is contained in \mathfrak{g}_0^0 if and only if α is a linear combination of $\alpha_2, \dots, \alpha_n$ only. Hence these roots form a simple system for \mathfrak{g}_0^0 . Denoting by $\omega_1, \dots, \omega_{n+1}$ the fundamental weights corresponding to the simple system $\{\alpha_1, \dots, \alpha_{n+1}\}$, dominant weights for \mathfrak{g}_0^0 are equivalent to linear combinations of $\omega_2, \dots, \omega_n$ with non-negative integral coefficients. As before, we use negatives of lowest weights rather than highest weights. Anyway, the irreducible representation \mathbb{W} determines a dominant integral weight of \mathfrak{g}_0^0 , which can be written as a linear combination of $\omega_2, \dots, \omega_n$ with non-negative integral coefficients.

Adding $k\omega_1 + \ell\omega_{n+1}$ to this weight, we obtain a dominant integral weight for \mathfrak{g} , which determines an irreducible representation \mathbb{V} of \mathfrak{g} . Now we have to discuss whether the representation \mathbb{V} integrates to the group $G := PGL(n+2, \mathbb{R})$, thus giving rise to a tractor bundle and hence to a BGG sequence on Lagrangian contact structures. For odd n , this is not a problem, since the map $A \mapsto \det(A)^{-1/(n+2)} A$ induces an isomorphism $PGL(n+2, \mathbb{R}) \cong SL(n+2, \mathbb{R})$ in this case. In the case of even n , $PGL(n, \mathbb{R})$ is well known to be isomorphic to $PSL(n, \mathbb{R})$. Hence \mathbb{V} integrates if and only if the center $\{\pm \mathbb{I}\}$ of $SL(n+2, \mathbb{R})$ acts trivially on \mathbb{V} . In terms of the negative of the lowest weight, written as

$a_1\omega_1 + \cdots + a_{n+1}\omega_{n+1}$, this boils down to the condition that the sum of all coefficients with odd indices is even. Depending on \mathbb{W} , this means that $k + \ell$ either has to be even or has to be odd.

Having \mathbb{V} as a representation of G , the existence of a BGG sequence on parabolic geometries of type (G, P) which are equivalent to Lagrangean contact structures follows from the general theory developed in [4, 11, 14]. Using the results from Section 2, this can be pushed down to a sequence of invariant differential operators on PCS-structures of para-Kähler type. The BGG sequence is a complex if the Lagrangean contact structure is locally flat, so we obtain complexes on Bochner-bi-Lagrangean manifolds.

The bundles showing up in the BGG sequence correspond to the Lie algebra homology groups $H_*(\mathfrak{p}_+, \mathbb{V})$, which can be computed using Kostant's theorem. In particular, $H_0(\mathfrak{p}_+, \mathbb{V})$ is the P -irreducible quotient of \mathbb{V} which has the same lowest weight. This is the tensor product of \mathbb{W} with the one-dimensional representation corresponding to $k\omega_1 + \ell\omega_{n+1}$, which shows that W_0 is the tensor product of W with a natural line bundle.

As noted in the proof of Theorem 3.1, the basic structure of $H_*(\mathfrak{p}_+, \mathbb{V})$ is encoded in the Hasse diagram $W^{\mathfrak{p}}$ associated to the parabolic \mathfrak{p} , which is determined in Section 3.6 of [12]. In particular, this contains the information of the number of irreducible components in $H_k(\mathfrak{p}_+, \mathbb{V})$ for each k , thus proving the claims on the number of irreducible summands in each W_i . Finally, the components of $H_1(\mathfrak{p}_+, \mathbb{V})$ correspond to simple reflections contained in $W^{\mathfrak{p}}$. These are exactly the reflections corresponding to α_1 and α_{n+1} , respectively. Their action on the weight $k\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n + \ell\omega_{n+1}$ is given by adding $-2(k+1)\omega_1 + (k+1)\omega_2$ and $(\ell+1)\omega_n - 2(\ell+1)\omega_{n+1}$, respectively. Since these are the negatives of the lowest weights of $S^{k+1}E^*$ and $S^{\ell+1}F^*$, respectively, the claim on W_1 follows. \square

3.4. REMARKS ON BGG SEQUENCES ASSOCIATED TO BOCHNER-KÄHLER METRICS

PACS-structures of Kähler type correspond to the real forms $\mathfrak{su}(p+1, q+1)$ of $\mathfrak{sl}(p+q+2, \mathbb{C})$. Such a structure corresponds to a conformally symplectic structure $\ell \subset \Lambda^2 T^*M$ and an almost complex structure J on M , for which ℓ is Hermitian. Torsion-freeness of the structure is equivalent to J being a complex structure, which then gives rise to a pseudo-Kähler metric of signature (p, q) on M . Parabolic contactifications of PCS-structures of this type are partially integrable almost CR structures of the appropriate signature, see Section 4.2.4 of [10]. Torsion-freeness is equivalent to the structure being integrable and hence a CR structure. It turns out that such a parabolic contactification is locally flat if and only if it is torsion free and the corresponding metric is Bochner-Kähler (of any signature). This is originally due to S.M. Webster ([22]), see [15] for a modern discussion and Theorem 7 of [9].

As the description suggests, there are strong similarities to para-Kähler type as discussed in Section 3.3 above. In view of this similarities, we will only briefly outline the differences to the para-Kähler case.

For $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$, the subalgebra \mathfrak{g}_0 has center \mathbb{C} and semi-simple part $\mathfrak{su}(p, q)$. Up to twisting by natural line bundles, the tangent bundle TM corresponds to the standard representation \mathbb{C}^{p+q} of \mathfrak{g}_0 . The analogy to the para-Kähler case becomes clear after complexification, where we get $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ and the two summands correspond to dual representations of $\mathfrak{su}(p, q)$. This shows that, after complexification, the situation is parallel to the para-Kähler case, with complex linearity and anti-linearity properties replacing the decomposition into E and F .

BGG sequences on partially integrable almost CR structures are associated both to real and to complex representations of the group $G := PSU(p+1, q+1)$ which governs the geometry. A complex representation of \mathfrak{g} is again determined by its restriction to \mathfrak{g}_0 (which also is a complex representation) and two non-negative integers describing the action of the center. However, in this case the center of $SU(p+1, q+1)$ is isomorphic to \mathbb{Z}_{p+q+2} , so the condition that a representation integrates to G imposes more restrictive conditions on the two integers describing the action of the center. Correspondingly, there are less BGG sequences available than in the para-Kähler case, unless it is possible to choose an additional structure as discussed in Remark 3.2.

Once a complex representation \mathbb{W} of \mathfrak{g}_0 and two non-negative integers k and ℓ give rise to a complex representation \mathbb{V} of G , the situation becomes very similar to Theorem 3.3, compare with Sections 3.6 and 3.8 of [12]. There is a sequence $D_i : \Gamma(W_i) \rightarrow \Gamma(W_{i+1})$ differential operators for $i = 0, \dots, 2n$, which is a complex provided that one starts with a Bochner-Kähler metric of any signature. The bundles W_0 and W_{2n+1} are irreducible, whereas W_i and W_{2n+1-i} split into a direct sum of $i+1$ bundles associated to complex irreducible representations for $i = 1, \dots, n$. One may also describe W_1 and the principal parts of the two components of D_0 similarly to Theorem 3.3, with complex linearity and conjugate linearity replacing the appearance of copies of E and F .

There are also BGG sequences of partially integrable almost CR structures induced by real representations of G (which do not admit a G -invariant complex structure). Again, such a representation is determined by its restriction to \mathfrak{g}_0^0 , which is a real irreducible representation, and by the action of the center. Since there is no complex structure available, there are stronger restrictions for the action of the center than in the complex case here. Next, one needs to make sure that the resulting representation \mathbb{V} of \mathfrak{g} integrates to G . Given a real representation of G , there again is a BGG sequence of the same length as in the complex case, see again Section 3.8 of [12]. The main difference to the complex case is the number of irreducible components of the bundles W_i and W_{2n+1-i} , which now is $(i+1)/2$ for odd i and $i/2+1$ for even i . In particular, in this case W_0 and W_1 both are irreducible and the principal part of D_0 is just given by a symmetrized iterated covariant derivative followed by a projection to the Cartan product.

3.5. RELATIVE BGG SEQUENCES

We now turn to a second general construction of sequences and complexes of invariant differential operators on parabolic geometries, which was introduced in the recent article [14]. To simplify comparison to that reference, we briefly change notation and denote by $Q \subset G$ the parabolic subgroup corresponding to a contact grading. The construction of relative BGG sequences on geometries of type (G, Q) in addition needs a second parabolic subgroup P such that $G \supset P \supset Q$. Hence in the case of parabolic contact structures, this construction is only available for the A_n -series, since for the other series the parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ corresponding to the contact grading is maximal. Even in the A_n -case, an intermediate parabolic is only available for the real form $\mathfrak{sl}(n+1, \mathbb{R})$, for the real forms $\mathfrak{su}(p+1, q+1)$ the parabolic \mathfrak{q} corresponding to the contact grading again is maximal. Hence relative BGG sequences can only be used to construct invariant differential operators on PCS-structures of para-Kähler type. However, in this case the resulting sequences are highly interesting since they give rise to differential complexes under much weaker assumptions than coming from a Bochner–bi-Lagrangian metric.

Given a type (G, Q) and an intermediate parabolic P , the input needed to construct a relative BGG sequence is a finite dimensional, completely reducible representation \mathbb{V} of the group P . Complete reducibility means that the nilpotent subgroup $P_+ \subset P$ acts trivially, so \mathbb{V} is a representation of the reductive Levi-factor $P_0 \cong P/P_+$. The bundles showing up in the relative BGG sequence determined by \mathbb{V} are induced by certain Lie algebra homology groups which we describe next. The setup easily implies that $P_+ \subset Q_+$ and that $\mathfrak{p}_+ \subset \mathfrak{q}_+$ is an ideal. Hence $\mathfrak{q}_+/\mathfrak{p}_+$ naturally is a Lie algebra, which acts on \mathbb{V} since the restriction to \mathfrak{q}_+ of the derivative of the P -action descends to the quotient. The homology groups in question then are the groups $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, which can be computed algorithmically using a relative version of Kostant’s theorem, see [13]. As before, we will simply state the resulting descriptions of bundles in the sequence.

There are general results showing that relative BGG sequences are complexes under much weaker assumptions than local flatness. The relevant concept here is called relative curvature. Given a parabolic geometry $p : \mathcal{G} \rightarrow M^\#$ of type (G, Q) and an intermediate parabolic P , the Q -invariant subspace $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ gives rise to a subbundle $T_\rho M^\# \subset TM$ called the *relative tangent bundle*. Likewise, the Q -invariant subspaces $\mathfrak{p}_+ \subset \mathfrak{p} \subset \mathfrak{g}$ corresponds to subbundles $\mathcal{A}_{\mathfrak{p}_+} M^\# \subset \mathcal{A}_{\mathfrak{p}} M^\#$ in the adjoint tractor bundle $\mathcal{A}M^\#$. One defines the relative adjoint tractor bundle $\mathcal{A}_\rho M^\#$ of the geometry to be $\mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{p}_+) \cong \mathcal{A}_{\mathfrak{p}} M^\# / \mathcal{A}_{\mathfrak{p}_+} M^\#$.

Now the first condition one has to impose is that $T_\rho M^\# \subset TM^\#$ is an involutive distribution. This is easily seen to be equivalent to the curvature $\kappa \in \Omega^2(M, \mathcal{A}M^\#)$ having the property that its values on two tangent vectors from $T_\rho M^\# \subset TM^\#$ always lie in $\mathcal{A}_{\mathfrak{p}} M^\#$. Assuming this, the values can be projected to the relative adjoint tractor bundle, thus defining a section κ_ρ of

the bundle $\Lambda^2 T_\rho^* M^\# \otimes \mathcal{A}_\rho M^\#$. This is the relative curvature of the geometry, and if this vanishes identically, any relative BGG sequence on $M^\#$ is a complex and a fine resolution of a certain sheaf on $M^\#$, which locally descends to leaf spaces of the distribution $T_\rho M^\#$.

3.6. RELATIVE BGG COMPLEXES ASSOCIATED TO PARA-KÄHLER METRICS

In the case of PCS-structures of para-Kähler type in dimension $2n$, the group G is $PGL(n+2, \mathbb{R})$ and the parabolic subgroup $Q \subset G$ comes from the stabilizer of a flag consisting of a line contained in a hyperplane in the standard representation. Hence $Q = P \cap \tilde{P}$, where P comes from the stabilizer of the line and \tilde{P} comes from the stabilizer of the hyperplane. Since P and \tilde{P} are maximal parabolic subgroups in G , they are the only two possible choices of intermediate parabolic subgroups in this case. A parabolic geometry $p: \mathcal{G} \rightarrow M^\#$ of type (G, Q) is given by a contact structure $H \subset TM^\#$ and a decomposition $H = E^\# \oplus F^\#$ into a direct sum of Legendrean subbundles. It is easy to see that the relative tangent bundles corresponding to P and \tilde{P} are just the subbundle $E^\#$ and $F^\#$, respectively. In particular, the situation between P and \tilde{P} is completely symmetric, so it suffices to discuss one of the two cases.

On the level of PCS-structures, we have a smooth manifold M of dimension $2n$, a conformally symplectic structure $\ell \subset \Lambda^2 T^*M$ and a decomposition $TM = E \oplus F$ into subbundles which are Lagrangean for ℓ . As discussed in Section 3.2 of [8], this gives rise to a split-signature conformal structure on M , by extending the pairing between E and F induced by a local section of ℓ to a symmetric tensor field, for which the two subbundles are isotropic. In the PCS-case, local closed sections of ℓ are uniquely determined up to constant multiples, so we even get local split signature metrics which are unique up to a constant factor (and hence all have the same Levi-Civita connection). In Section 4.5 of [8], it is shown that torsion freeness of the PCS-structure defined by ℓ , E and F is equivalent to the fact that the subbundles E and F are both involutive. Since the canonical connection of the PCS-structure preserves the distinguished metrics by construction, torsion-freeness shows that it has to coincide with the Levi-Civita connection in this case.

Here we can work in a slightly more general situation, namely that the one of the subbundles, say F , is involutive. Assuming that E is non-involutive, the canonical connection ∇ of the PCS-structure has non-trivial torsion (since it preserves E). More precisely, identifying $\Lambda^2 T^*M$ with $\Lambda^2 E^* \oplus (E^* \otimes F^*) \oplus \Lambda^2 F^*$ the restriction of the torsion to the last two summands has to be trivial, whereas the restriction to the first summand coincides with the negative of the tensorial map $\Lambda^2 E^* \rightarrow F$ induced by projecting the Lie bracket to F . In particular, ∇ has to be different from the Levi-Civita connection of the distinguished metric. However, since ∇ by construction preserves the distinguished metric (since it preserves E , F and ℓ), and its torsion is known, there is an explicit formula relating it to the Levi-Civita connection.

To formulate the result on complexes induced by relative BGG complexes, we need a bit more information on the groups involved. Recall that $G =$

$PGL(n + 2, \mathbb{R})$, $P \subset G$ comes from the stabilizer of a line in the standard representation, while $Q \subset P$ comes from the stabilizer of the flag consisting of that line and a hyperplane containing it. Via the restriction of the adjoint action of G , P acts on $\mathfrak{g}/\mathfrak{p}$ and it is well known that this induces an isomorphism $P/P_+ \cong GL(\mathfrak{g}/\mathfrak{p}) \cong GL(n+1, \mathbb{R})$, compare with Section 4.1.5 of [10]. Moreover, the sum of all but the lowest grading components of \mathfrak{g} with respect to the grading defined by \mathfrak{q} is a codimension-one subspace $\mathfrak{q}^{-1} \subset \mathfrak{g}$ containing \mathfrak{p} . Hence $\mathfrak{q}^{-1}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$ is a hyperplane and $Q \subset P$ can be characterized as those elements whose action on $\mathfrak{g}/\mathfrak{p}$ stabilizes this hyperplane, see Section 4.4.2 of [10]. This gives rise to a surjection $Q \rightarrow GL(\mathfrak{q}^{-1}/\mathfrak{p}) \cong GL(n, \mathbb{R})$ which has Q_+ in its kernel. For a parabolic contact structure $(M^\#, H = E^\# \oplus F^\#)$ of type (G, Q) , the vector bundle induced by this representation is $E^\#$. Viewing the above homomorphism as $Q/Q_+ \rightarrow GL(\mathfrak{q}^{-1}/\mathfrak{p})$ its kernel is isomorphic to $\mathbb{R} \setminus \{0\}$. A faithful representation of this kernel corresponds to $\Lambda^n F^\#$ on each parabolic contact structure.

THEOREM 3.4. *Let \mathbb{W} be an irreducible representation of $GL(n, \mathbb{R})$ and let W be the corresponding natural tensor bundle on a PCS-manifold (M, ℓ, E, F) of para-Kähler type of dimension $2n$ (with E playing the role of the standard representation) and let $k \geq 0$ be an integer.*

Then pushing down an appropriate relative BGG sequence on parabolic contactifications leads to a sequence of irreducible tensor bundles and invariant differential operators of the form

$$\Gamma(W_0) \xrightarrow{D_0} \Gamma(W_1) \xrightarrow{D_1} \dots \xrightarrow{D_{n-2}} \Gamma(W_{n-1}) \xrightarrow{D_{n-1}} \Gamma(W_n).$$

This sequence is a complex, if the subbundle $F \subset TM$ is involutive. The bundle W_0 is the tensor product of W with a real power of the line bundle $(\Lambda^n F)^2$ and $W_1 \cong S^k F^ \odot W_0$.*

Finally suppose that \mathbb{W} is chosen in such a way that W_0 coincides with one of the irreducible summands in the bundles from Theorem 3.3. Then the same holds for all the bundles W_j and the sequence constructed here is a subsequence respectively a subcomplex in the sequence from that Theorem.

Proof. Consider $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ with simple roots α_i and corresponding fundamental weights ω_i as in the proof of Theorem 3.3. The corresponding Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ also is a Cartan subalgebra for the reductive subalgebras $\mathfrak{p} \cong \mathfrak{gl}(n+1, \mathbb{R})$ and $\mathfrak{q}_0 \subset \mathfrak{p}$. Let us decompose $\mathfrak{q}_0 \cong \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}$ as described on the group level before the theorem. Then the negative of the lowest weight of the representation \mathbb{W} can be expressed as a linear combination $a_1\omega_1 + \dots + a_n\omega_n$ with $a_1 \in \mathbb{R}$ and non-negative integers a_2, \dots, a_n . Adding $k\omega_{n+1}$ to this, we obtain a weight which is the negative of the lowest weight of a finite dimensional, irreducible representation \mathbb{V} of \mathfrak{p} . (The part $a_2\omega_2 + \dots + a_n\omega_n + k\omega_{n+1}$ is a dominant integral weight for the semisimple part of \mathfrak{p} , and adding $a_1\omega_1$ corresponds to tensorizing by a one-dimensional representation of the center.) There is no problem with the representation integrating to the group $P \cong GL(n+1, \mathbb{R})$.

Hence the general results of [14] imply the existence of an associated relative BGG–sequence for each parabolic geometry of type (G, Q) . Via the mechanism introduced in Section 2, this sequence can be pushed down a manifold endowed with a PCS–structure from local contactifications. The bundles showing up in the resulting sequence are induced by the Lie algebra homology groups $H_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, in particular the degrees range from 0 to $\dim(\mathfrak{q}_+) - \dim(\mathfrak{p}_+) = n$. To obtain the shape of the sequence, one has to determine the relative Hasse diagram $W_{\mathfrak{p}}^{\mathfrak{q}}$ as described in Lemma 2.6 and Example 3.2 of [13]. It is easy to see $W_{\mathfrak{p}}^{\mathfrak{q}}$ consists of $n+1$ elements of length $0, \dots, n$. This implies the statement on irreducibility of W_i for each i . Moreover, $H_0(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ is the Q –irreducible quotient of \mathbb{V} , which implies the description of W_0 . The unique element of length 1 in $W_{\mathfrak{q}}^{\mathfrak{p}}$ is the simple reflection corresponding to α_{n+1} , from which the description of W_1 follows as in the proof of Theorem 3.3.

Suppose next, that the subbundle $F \subset TM$ is involutive. Then for each local parabolic contactification $(M^{\#}, H = E^{\#} \oplus F^{\#})$, the subbundle $F^{\#} \subset TM^{\#}$ is involutive, too. As we have observed above, this is exactly the relative tangent bundle $T_{\rho}M^{\#}$ for the intermediate parabolic P . By Proposition 4.2.3 of [10] this implies that one of the three harmonic curvature components of the parabolic geometry on $M^{\#}$ vanishes identically. But the discussion of harmonic curvature in Section 4.2.3 of [10] shows that the assumptions of part (1) of Proposition 4.18 of [14] are satisfied, so the relative curvature of the geometry vanishes. By part (1) of Theorem 4.11 of that reference, any relative BGG sequence on $M^{\#}$ is a complex, so the descended sequence is a complex, too.

To prove that last claim, we observe that by Kostant’s theorem, all \mathfrak{p} –dominant weights in the affine Weyl orbit of the negative of the lowest weight of \mathbb{V} are realized by irreducible components of the representations $H_j(\mathfrak{p}_+, \mathbb{V})$ for $j = 0, \dots, \dim(\mathfrak{p}_+)$. Hence our assumptions mean that W_0 occurs as an irreducible component in one of these homology representations (which happen to be irreducible in our case). But by Theorem 3.3 of [13], the homologies $H_i(\mathfrak{q}_+/\mathfrak{p}_+, H_j(\mathfrak{p}_+, \mathbb{V}))$ are contained in $H_{i+j}(\mathfrak{q}_+, \mathbb{V})$, so all bundles W_i occur in the sequence from Theorem 3.3. It is proved in Theorem 5.2 of [14] that then the absolute and the relative BGG constructions produce the same differential operators between these bundles, which implies the last claim. \square

Remark 3.5. As already remarked above, relative BGG sequences are not available for the parabolic contact geometries associated to $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$. However, there is a case in which our methods can produce differential complexes on general Kähler manifolds (i.e. without the assumption on vanishing Bochner curvature). This is related to those cases in Theorem 3.4 in which a relative BGG sequence is included in a proper BGG sequence. In these cases, the existence of subcomplexes can also be proved more directly, see [12]. While these techniques require slightly stronger assumption (involutivity of both E and F), they also work for the other real forms.

In the setting of Section 3.4 this method applies to torsion–free PCS–structures of Kähler type, which are equivalent to pseudo–Kähler metrics of any signature,

see Proposition 4.5 of [8]. The local contactifications of such a geometry carry an (integrable) CR structure of hypersurface type of the same signature. As discussed in Section 3.4 there are BGG sequences on such structures associated to real and complex representations of the groups $PSU(p+1, q+1)$. Theorem 3.8 in [12] shows that, both in the real and in the complex case, there are several subcomplexes in such a BGG sequences, which descend to differential complexes on the underlying pseudo-Kähler manifolds.

4. THE COHOMOLOGY OF DESCENDED BGG COMPLEXES

In this last section, we will derive some results on the cohomology of the differential complexes associated to special symplectic connections via descending BGG sequences. The strongest results are obtained in the case of global contactifications with compact fibers, thus in particular applying to complexes on $\mathbb{C}P^n$ and $Gr(2, \mathbb{C}^n)$ as treated in Sections 2.6 and 3.4 of [9]. Several steps towards these main results are proved in a more general setting.

4.1. THE RELATION TO TWISTED DE-RHAM COHOMOLOGY

For the first step in the description, we need a few details on the construction of BGG sequences. As discussed in Section 3.1, a BGG sequence on parabolic contact structures of type (G, P) is determined by a representation \mathbb{V} of the Lie group G . Via taking the associated bundle to the Cartan bundle determined by \mathbb{V} , this representation gives rise to a natural vector bundle on such geometries. On the homogeneous model G/P , this is just the homogeneous vector bundle $G \times_P \mathbb{V}$. Bundles of this type are called *tractor bundles*. Their main feature is that the Cartan connection induces a canonical linear connection on each tractor bundle, which is flat if and only if either \mathbb{V} is a trivial representation or the geometry is locally flat. Since the case of the trivial representation is treated in [7], we will always assume that \mathbb{V} is a non-trivial, irreducible representation from now on.

Given a parabolic contact geometry $(p: \mathcal{G}^\# \rightarrow M^\#, \omega)$, let us denote by $\mathcal{VM}^\#$ the tractor bundle on $M^\#$ induced by \mathbb{V} and by $\nabla^\mathbb{V}$ the canonical tractor connection induced by ω . Coupling $\nabla^\mathbb{V}$ to the exterior derivative, one obtains the *covariant exterior derivative* $d^\nabla: \Omega^k(M^\#, \mathcal{VM}^\#) \rightarrow \Omega^{k+1}(M^\#, \mathcal{VM}^\#)$ for each k . Obviously, this is a sequence of invariant differential operators and it is well known that they form a complex if and only if the connection $\nabla^\mathbb{V}$ is flat. We will refer to this as the twisted de-Rham sequence respectively the twisted de-Rham complex determined by \mathbb{V} .

Now it turns out that, via the Cartan connection ω , the cotangent bundle $T^*M^\#$ can be naturally identified with the associated bundle $\mathcal{G}^\# \times_P \mathfrak{p}_+$. Hence the bundle of $\mathcal{VM}^\#$ -valued k -forms is induced by the representation $\Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$, which is the space of k -chains in the standard complex computing the Lie algebra homology $H_*(\mathfrak{p}_+, \mathbb{V})$. Since the standard differentials in this complex are P -equivariant, they induce natural bundle maps $\Lambda^k T^*M^\# \otimes \mathcal{VM}^\# \rightarrow \Lambda^{k-1} T^*M^\# \otimes \mathcal{VM}^\#$, which traditionally are denoted by ∂^* . Hence $\text{im}(\partial^*)$ and $\text{ker}(\partial^*)$ are nested natural subbundles in $\Lambda^k T^*M \otimes \mathcal{VM}$

and the quotient $\ker(\partial^*)/\text{im}(\partial^*)$ is by construction isomorphic to the associated bundle $\mathcal{G} \times_P H_k(\mathfrak{p}_+, \mathbb{V})$ which we denote by $\mathcal{H}_k^\vee M^\#$.

By construction, there is a bundle projection $\Pi = \Pi_k : \ker(\partial^*) \rightarrow \mathcal{H}_k^\vee M^\#$ which induces a tensorial operator on the spaces of sections of these bundles that we denote by the same symbol. Now the key to the construction of BGG sequences is that there is an invariant differential operator $S = S_k$ which splits this tensorial projection. Otherwise put, to any section $\sigma \in \Gamma(\mathcal{H}_k^\vee M^\#)$ we can associate $S(\sigma) \in \Omega^k(M^\#, \mathcal{V}M^\#)$ such that $\partial^* \circ S(\sigma) = 0$ and $\Pi(S(\sigma)) = \sigma$. Moreover it turns out that this splitting operator has the property that $\partial^* \circ d^\nabla(S(\sigma)) = 0$ for any σ (which uniquely determines S). Hence one can define an invariant differential operator $D^\# = D_k^\# : \Gamma(\mathcal{H}_k^\vee M^\#) \rightarrow \Gamma(\mathcal{H}_{k+1}^\vee M^\#)$ by $D^\#(\sigma) := \Pi(d^\nabla(S(\sigma)))$, and these operators form the BGG sequence. Moreover, if ∇^\vee is flat, then the splitting operators have the property that $d^\nabla \circ S_k = S_{k+1} \circ D_k^\#$ for all k . This readily implies that the BGG sequence is a complex, and that the S_k define a homomorphism of complexes from the BGG complex to the twisted de–Rham complex.

Now suppose that we have given a PCS–quotient $q : M^\# \rightarrow M$ of type (G, P) , and let $\pi : \mathcal{G}_0 \rightarrow M$ be the corresponding G_0 –principal bundle. Then by construction, the operators D obtained by descending the operators $D^\#$ act on sections of the bundle $\mathcal{H}_k^\vee M := \mathcal{G}_0 \times_{G_0} H_k(\mathfrak{p}_+, \mathbb{V})$ for all k . (Here one uses that each $H_k(\mathfrak{p}_+, \mathbb{V})$ is a completely reducible representation of P , so it descends to $P/P_+ \cong G_0$.) Recall that the infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ giving rise to the PCS–quotient induces a P –invariant vector field $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ which projects onto ξ . Similarly as discussed in Section 2.4, sections of associated bundles to $\mathcal{G}^\#$ can be identified with smooth functions with values in the inducing representation, so there is a natural action of $\tilde{\xi}$ via a Lie derivative $\mathcal{L}_{\tilde{\xi}}$. In particular, we define $\Omega_{\tilde{\xi}}^k(M^\#, \mathcal{V}M^\#)$ as the subspace of those forms φ , for which $\mathcal{L}_{\tilde{\xi}}(\varphi) = 0$. This works in the same way on open subsets of $M^\#$ so that we have actually defined a subsheaf of the sheaf of $\mathcal{V}M^\#$ –valued k –forms.

THEOREM 4.1. *Consider a PCS–quotient $q : M^\# \rightarrow M$ of type (G, P) . Let \mathbb{V} be a representation of G , $\mathcal{V}M^\# \rightarrow M^\#$ the tractor bundle determined by \mathbb{V} and $(\Omega^*(M^\#, \mathcal{V}M^\#), d^\nabla)$ the induced twisted de–Rham sequence. Let $(\mathcal{H}_*^\vee M, D_*)$ be the sequence of differential operators on M obtained by descending the BGG sequence determined by \mathbb{V} as in Theorem 2.4.*

Then d^∇ commutes with $\mathcal{L}_{\tilde{\xi}}$, and hence it preserves the subspaces $\Omega_{\tilde{\xi}}^(M^\#, \mathcal{V}M^\#)$. Moreover, if ∇ is flat, then these subspaces form a subcomplex in the twisted de–Rham complex, whose cohomology is naturally isomorphic to the cohomology of the complex $(\mathcal{H}_*^\vee M, D_*)$.*

Proof. Since any local flow of $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ is an automorphism of the Cartan geometry $(p : \mathcal{G}^\# \rightarrow M^\#, \omega)$, it follows as in Lemma 2.3 that any invariant differential operator commutes with $\mathcal{L}_{\tilde{\xi}}$. In particular, applying this to d^∇ we readily conclude that $d^\nabla(\Omega_{\tilde{\xi}}^k(M^\#, \mathcal{V}M^\#)) \subset \Omega_{\tilde{\xi}}^{k+1}(M^\#, \mathcal{V}M^\#)$. In the case that ∇ is flat, we hence get a subcomplex in the twisted de–Rham complex.

We can also apply this argument to the BGG operators $D^\#$. For a natural vector bundle $\mathcal{W}M^\#$ let us denote by $\Gamma_\xi(\mathcal{W}M^\#) \subset \Gamma(\mathcal{W}M^\#)$ the kernel of $\mathcal{L}_{\tilde{\xi}}$. Naturality of the BGG operators then shows that we get a subcomplex $(\Gamma_\xi(\mathcal{H}_*^\vee M^\#), D_*^\#)$ in the BGG complex. Now since each $H_k(\mathfrak{p}_+, \mathbb{V})$ is a completely reducible representation of P , we can equivalently describe sections of the corresponding associated bundle via the intermediate principal bundle $\mathcal{G}_0^\# := \mathcal{G}^\# / P_+$. Recall from Section 2.2 that there is a vector field $\xi_0 \in \mathfrak{X}(\mathcal{G}_0^\#)$ which lies between $\tilde{\xi}$ and ξ . By construction, identifying $\Gamma(\mathcal{H}_k^\vee M^\#)$ with $C^\infty(\mathcal{G}_0, H_k(\mathfrak{p}_+, \mathbb{V}))^{G_0}$, the subspace Γ_ξ exactly corresponds to the kernel of the Lie derivative \mathcal{L}_{ξ_0} . Theorem 2.4 and Lemma 2.3 thus imply that the complex $(\Gamma_\xi(\mathcal{H}_*^\vee M^\#), D_*^\#)$ is isomorphic to $(\Gamma(\mathcal{H}_*^\vee M), D_*)$.

Thus we can complete the proof by showing that $(\Gamma_\xi(\mathcal{H}_*^\vee M^\#), D_*^\#)$ computes the same cohomology as the subcomplex $(\Omega_\xi^*(M^\#, \mathcal{V}M^\#), d^\nabla)$ in the twisted de–Rham complex. For this, we can adapt the usual proof for BGG sequences from Theorem 2.6 and Lemma 2.7 of [11]. Naturality of the splitting operators implies that for each k we get $S(\Gamma_\xi(\mathcal{H}_k^\vee M^\#)) \subset \Gamma_\xi(\ker(\partial^*)) \subset \Omega_\xi^k(M^\#, \mathcal{V}M^\#)$. In particular, the fact that $d^\nabla \circ S = S \circ D^\#$ verified in Lemma 2.7 of [11] shows that S defines a complex map between the two subcomplexes, and we claim that this induces an isomorphism in cohomology. Suppose that $\varphi \in \Omega_\xi^k(M^\#, \mathcal{V}M^\#)$ satisfies $d^\nabla \varphi = 0$. Then in Lemma 2.7 of [11] it is shown that there is a form $\psi \in \Omega^{k-1}(M^\#, \mathcal{V}M^\#)$ such that $\varphi + d^\nabla \psi \in \Gamma(\ker(\partial^*))$. As shown in Theorems 3.9 and 3.14 of [14], a form with this property can be obtained as the value of an invariant differential operator on φ . Thus we may assume that $\mathcal{L}_{\tilde{\xi}}\psi = 0$ and hence $\varphi + d^\nabla \psi \in \Gamma_\xi(\ker(\partial^*))$. But then naturality of the bundle map Π shows that $\alpha := \Pi(\varphi + d^\nabla \psi) \in \Gamma_\xi(\mathcal{H}_k^\vee M^\#)$. Now by construction $d^\nabla(\varphi + d^\nabla \psi) = 0$ which shows that $\varphi + d^\nabla \psi = S(\alpha)$ and $D^\#(\alpha) = 0$. Hence the cohomology class of α is mapped to the cohomology class of φ , so the induced map in cohomology is surjective.

On the other hand, suppose that $\alpha \in \Gamma_\xi(\mathcal{H}_k^\vee M^\#)$ satisfies $D^\#(\alpha) = 0$ and $S(\alpha) = d^\nabla \psi$ for some $\psi \in \Omega_\xi^{k-1}(M^\#, \mathcal{V}M^\#)$. As in the previous step, we may without loss of generality assume that $\psi \in \Gamma_\xi(\ker(\partial^*))$ and then project this to $\beta = \Pi(\psi) \in \Gamma_\xi(\mathcal{H}_{k-1}^\vee M^\#)$. Then $d^\nabla \psi = S(\alpha) \in \Gamma(\ker(\partial^*))$ shows that $\psi = S(\beta)$ and hence $D^\#(\beta) = \Pi \circ S(\alpha) = \alpha$, which shows injectivity of the induced map in cohomology. \square

4.2. REDUCTION TO HORIZONTAL EQUIVARIANT FORMS

Suppose that $q : M^\# \rightarrow M$ is a PCS–quotient with corresponding infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ corresponding to $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$. Then the forms in $\Omega_\xi^*(M^\#, \mathcal{V}M^\#)$ as studied above, are (in an appropriate sense) equivariant for $\tilde{\xi}$. Similarly to the case of ordinary forms treated in Section 2.3 of [7], it is natural to next look at forms which in addition are horizontal, since these essentially are objects on M already. Hence we define $\Omega_\xi^k(M^\#, \mathcal{V}M^\#)_{\text{hor}}$ to be the space of those $\varphi \in \Omega^k(M^\#, \mathcal{V}M^\#)$, for which $\mathcal{L}_{\tilde{\xi}}\varphi = 0$ and $i_\xi\varphi = 0$. To simplify

notation, we will write $A^k := \Omega_\xi^k(M^\#, \mathcal{V}M^\#)$ and $A_{\text{hor}}^k := \Omega_\xi^k(M^\#, \mathcal{V}M^\#)_{\text{hor}}$ in what follows.

Recall that the infinitesimal automorphism ξ determines a unique contact form $\alpha \in \Omega^1(M^\#)$ for which ξ is the Reeb field, i.e. such that $i_\xi \alpha = 1$ and $i_\xi d\alpha = 0$, see Proposition 2.2 of [7]. Observe also, that there is an obvious wedge product $\Omega^k(M^\#) \times \Omega^\ell(M^\#, \mathcal{V}M^\#) \rightarrow \Omega^{k+\ell}(M^\#, \mathcal{V}M^\#)$. In terms of these operations, we can now derive a description of A^k .

LEMMA 4.2. *Let $q : M^\# \rightarrow M$ be a PCS-quotient of type (G, P) with corresponding infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ and let $\alpha \in \Omega^1(M^\#)$ be the contact form associated to ξ . Then the maps $\varphi \mapsto (\varphi - \alpha \wedge i_\xi \varphi, i_\xi \varphi)$ and $(\varphi_1, \varphi_2) \mapsto (\varphi_1 + \alpha \wedge \varphi_2)$ define inverse isomorphisms between A^k and $A_{\text{hor}}^k \oplus A_{\text{hor}}^{k-1}$.*

Proof. Since $i_\xi \circ i_\xi = 0$, we see that both $i_\xi \varphi$ and $\varphi - \alpha \wedge i_\xi \varphi$ are horizontal, and then one immediately verifies that the two maps in the claim are inverse to each other. So it remains to show that the construction can be restricted to the kernels of $\mathcal{L}_{\tilde{\xi}}$ on both sides.

To do this, we have to derive some results on the operator $\mathcal{L}_{\tilde{\xi}}$, which by definition is given by differentiating the equivariant functions corresponding to sections of natural vector bundles in the direction of $\tilde{\xi}$. Let us denote by $(p : \mathcal{G}^\# \rightarrow M^\#, \omega)$ the Cartan geometry describing the parabolic contact structure on $M^\#$. Then the isomorphism $TM^\# \cong \mathcal{G}^\# \times_P (\mathfrak{g}/\mathfrak{p})$ comes from the fact that for $u \in \mathcal{G}$ the map $\omega(u) : T_u \mathcal{G}^\# \rightarrow \mathfrak{g}$ descends to a linear isomorphism $T_{p(u)} M^\# \rightarrow \mathfrak{g}/\mathfrak{p}$. Otherwise put, the equivariant smooth function $f : \mathcal{G}^\# \rightarrow \mathfrak{g}/\mathfrak{p}$ corresponding to a vector field $\eta \in \mathfrak{X}(M^\#)$ can be written as $\omega(\tilde{\eta}) + \mathfrak{p}$, where $\tilde{\eta} \in \mathfrak{X}(\mathcal{G}^\#)$ is a P -equivariant lift of η . Since ξ is an infinitesimal automorphism, the lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ satisfies $0 = \mathcal{L}_{\tilde{\xi}} \omega$. This implies that for $\tilde{\eta}$ as above, we get $\tilde{\xi} \cdot \omega(\tilde{\eta}) = \omega([\tilde{\xi}, \tilde{\eta}])$. Since $[\tilde{\xi}, \tilde{\eta}]$ is a lift of $[\xi, \eta]$, we conclude that $\mathcal{L}_{\tilde{\xi}} \eta = [\xi, \eta]$. Hence on $\mathfrak{X}(M)$ the operator $\mathcal{L}_{\tilde{\xi}}$ coincides with the usual Lie derivative \mathcal{L}_ξ along ξ , and in particular, $\mathcal{L}_{\tilde{\xi}} \xi = 0$.

By construction, $\mathcal{L}_{\tilde{\xi}}$ satisfies the usual compatibility conditions with tensor products and contractions. Using this, the result for vector fields easily implies that $\mathcal{L}_{\tilde{\xi}}$ coincides with the usual Lie derivative \mathcal{L}_ξ on all tensor fields and in particular on (real valued) differential forms. The definition of the contact form α then implies that $0 = \mathcal{L}_\xi \alpha = \mathcal{L}_{\tilde{\xi}} \alpha$. Together with naturality and $\mathcal{L}_{\tilde{\xi}} \xi = 0$, this now implies that all the maps we have used preserve the kernels of $\mathcal{L}_{\tilde{\xi}}$. \square

For the next step, we have to impose an additional restriction on the infinitesimal automorphism in question. Since $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ is P -invariant vector field, equivariance of the Cartan connection ω implies that $\omega(\tilde{\xi}) : \mathcal{G}^\# \rightarrow \mathfrak{g}$ is a P -equivariant function. Thus it defines a smooth section of the associated bundle $\mathcal{A}M^\# := \mathcal{G}^\# \times_P \mathfrak{g}$, the adjoint tractor bundle of the parabolic geometry $(p : \mathcal{G}^\# \rightarrow M^\#, \omega)$. Indeed, this establishes a bijection between $\Gamma(\mathcal{A}M^\#)$ and the space of P -invariant vector fields on $\mathcal{G}^\#$. It turns out that infinitesimal automorphisms can be nicely characterized in this picture, see [5]. Since

$\mathcal{A}M^\#$ is a tractor bundle, it carries the tractor connection $\nabla^{\mathcal{A}}$. It turns out (see Proposition 3.2 of [5]) that this connection can be naturally modified by a term involving the Cartan curvature to a linear connection $\tilde{\nabla}$ whose parallel sections exactly correspond to the canonical lifts $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ of infinitesimal automorphisms $\xi \in \mathfrak{X}(M^\#)$. This bijection is implemented by an analog of the splitting operator S discussed in Section 4.1 above. Moreover, it turns out that if a section of $\mathcal{A}M^\#$ is parallel for $\nabla^{\mathcal{A}}$, then it is also parallel for $\tilde{\nabla}$, see Corollary 3.5 of [5]. Hence parallel sections of $\nabla^{\mathcal{A}}$ correspond to a subclass of infinitesimal automorphisms.

DEFINITION 4.3. Let $(p : \mathcal{G}^\# \rightarrow M^\#, \omega)$ be a parabolic geometry of type (G, P) . An infinitesimal automorphism $\xi \in \mathfrak{X}(M^\#)$ of the geometry is called *normal* if and only if the induced P -invariant vector field $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$ corresponds to a section of $\mathcal{A}M^\#$ which is parallel for the tractor connection $\nabla^{\mathcal{A}}$.

By Corollary 3.5 of [5] an infinitesimal automorphism ξ is normal if and only if ξ inserts trivially into the curvature two-form of the Cartan connection ω . In particular, any infinitesimal automorphism on a locally flat geometry is normal.

Next, we can use the infinitesimal automorphism ξ and its lift $\tilde{\xi}$ to define a smooth bundle map $\Xi : \mathcal{V}M^\# \rightarrow \mathcal{V}M^\#$ on a tractor bundle $\mathcal{V}M^\# \rightarrow M^\#$. To define this, observe that $\mathcal{V}M^\# = \mathcal{G}^\# \times_P \mathbb{V}$ for a representation \mathbb{V} of G , so we have the infinitesimal representation $\mathfrak{g} \rightarrow L(\mathbb{V}, \mathbb{V})$. This means that any point $u \in \mathcal{G}^\#$ defines a linear isomorphism $\psi_u : \mathbb{V} \rightarrow \mathcal{V}_x M^\#$, where $x = p(u) \in M^\#$. For any $g \in P$ and $v \in \mathbb{V}$, we then get $\psi_{u \cdot g}(v) = \psi_u(g \cdot v)$, so $\psi_{u \cdot g} = \psi_u \circ \rho(g)$, where ρ denotes the representation of G . On the other hand, the function $\omega(\tilde{\xi}) : \mathcal{G}^\# \rightarrow \mathfrak{g}$ satisfies $\omega(\tilde{\xi})(u \cdot g) = \text{Ad}(g^{-1})(\omega(\tilde{\xi})(u))$. This shows that, denoting by ρ' the infinitesimal representation, we conclude that

$$\psi_u \circ \rho'(\omega(\tilde{\xi})(u)) \circ \psi_u^{-1} = \psi_{u \cdot g} \circ \rho'(\omega(\tilde{\xi})(u \cdot g)) \circ \psi_{u \cdot g}^{-1}.$$

Thus we get a well defined linear map $\Xi(x) : \mathcal{V}_x M^\# \rightarrow \mathcal{V}_x M^\#$ and hence a smooth bundle map as claimed.

PROPOSITION 4.4. Let $(p : \mathcal{G}^\# \rightarrow M^\#, \omega)$ be a parabolic geometry of type (G, P) such that $M^\#$ is connected. Let $\xi \in \mathfrak{X}(M^\#)$ be a normal infinitesimal automorphism and let $\Xi : \mathcal{V}M^\# \rightarrow \mathcal{V}M^\#$ be the induced bundle map on a tractor bundle $\mathcal{V}M^\# \rightarrow M^\#$. Then Ξ has constant rank, so its kernel $\ker(\Xi)$ and its image $\text{im}(\Xi)$ are smooth subbundles of $\mathcal{V}M^\#$, and we get a smooth vector bundle $\text{coker}(\Xi) := \mathcal{V}M^\# / \text{im}(\Xi)$.

Proof. In Lemma 2.3 of [6] it is shown that connectedness of $M^\#$ implies that for a normal infinitesimal automorphism ξ , the image of the function $\omega(\tilde{\xi}) : \mathcal{G}^\# \rightarrow \mathfrak{g}$ is contained in a single orbit of the adjoint action of g . Now for $g \in G$ and $X \in \mathfrak{g}$, and the actions ρ of G and ρ' of \mathfrak{g} on \mathbb{V} , it is well known that $\rho'(\text{Ad}(g)(X)) = \rho(g) \circ \rho'(X) \circ \rho(g)^{-1}$. This shows that the maps $\rho'(X)$ and $\rho'(\text{Ad}(g)(X))$ have the same rank, which by construction implies that Ξ has constant rank. All other claims are well known consequences of this fact. \square

Using Ξ , we can now define several subspaces in the space of \mathcal{VM} -valued differential forms. First, we of course have $\Omega^k(M^\#, \ker(\Xi)) \subset \Omega^k(M^\#, \mathcal{VM}^\#)$. Moreover, even though $\ker(\xi)$ is not a natural vector bundle, it makes no problem to require $\mathcal{L}_{\tilde{\xi}}\varphi = 0$ as well as $i_\xi\varphi = 0$ for $\varphi \in \Omega^k(M^\#, \ker(\Xi))$. Thus we can define $K^i := \Omega_\xi^i(M^\#, \ker(\Xi))_{hor} \subset A_{hor}^i$. The cokernel of Ξ is more complicated to deal with, and we need some preliminary results to do this.

4.3. THE CARTAN FORMULA

We next derive an analog of the Cartan formula for the covariant exterior derivative d^∇ on $\Omega^*(M^\#, \mathcal{VM}^\#)$. This will be a crucial step towards the construction of various subcomplexes and to a description of the cohomology of the subcomplex from Theorem 4.1.

LEMMA 4.5. *For any $\varphi \in \Omega^*(M^\#, \mathcal{VM}^\#)$ we have*

$$\mathcal{L}_{\tilde{\xi}}\varphi = i_\xi d^\nabla\varphi + d^\nabla i_\xi\varphi - \Xi_*(\varphi),$$

where $\Xi_* : \Omega^*(M^\#, \mathcal{VM}^\#) \rightarrow \Omega^*(M^\#, \mathcal{VM}^\#)$ is given by applying Ξ to the values of $\mathcal{VM}^\#$ -valued forms.

In particular, for $\varphi \in A_{hor}^*$, we have $i_\xi d^\nabla\varphi = \Xi_*(\varphi)$.

Proof. The first step is as in the proof of Cartan’s formula for the exterior derivative. Using the standard formula for d^∇ , one verifies that the value of $i_\xi d^\nabla\varphi + d^\nabla i_\xi\varphi$ maps vector fields $\eta_1, \dots, \eta_k \in \mathfrak{X}(M^\#)$ to

$$(1) \quad \nabla_\xi^{\mathcal{VM}}\varphi(\eta_1, \dots, \eta_k) + \sum_{i=1}^k (-1)^i \varphi([\xi, \eta_i], \eta_1, \dots, \widehat{\eta}_i, \dots, \eta_k).$$

Let us denote by $s \in \Gamma(\mathcal{AM}^\#)$ the section of the adjoint tractor bundle corresponding to $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)^P$. Then by the construction from Section 4.1, the operator $\mathcal{L}_{\tilde{\xi}}$ coincides with the so-called fundamental derivative D_s , see Section 1.5.8 of [10]. Likewise, the bundle map Ξ by construction coincides with the operation $s\bullet$ from Section 1.5.7 of [10]. Thus the formula for the tractor connection in Theorem 1.5.8 of [10] shows that the first summand in (1) can be rewritten as

$$(2) \quad \mathcal{L}_{\tilde{\xi}}(\varphi(\eta_1, \dots, \eta_k)) + \Xi(\varphi(\eta_1, \dots, \eta_k)).$$

In the second part of (1), we can move the Lie bracket to the i th entry of φ at the expense of a sign $(-1)^{i-1}$. As noted in the proof of Lemma 4.2, we have $[\xi, \eta_i] = \mathcal{L}_{\tilde{\xi}}\eta_i$, so the second term in (1) can be written as

$$- \sum_{i=1}^k \varphi(\eta_1, \dots, \mathcal{L}_{\tilde{\xi}}\eta_i, \dots, \eta_k),$$

and naturality of $\mathcal{L}_{\tilde{\xi}}$ implies that this adds up with the first term in (2) to $(\mathcal{L}_{\tilde{\xi}}\varphi)(\eta_1, \dots, \eta_k)$. □

The last statement of the Lemma shows that d^∇ does not preserve the subspace $\Omega_\xi^*(M^\#, \mathcal{VM}^\#)_{hor}$. Of course there is the possibility of combining d^∇ with the projection to horizontal forms from Lemma 4.2:

DEFINITION 4.6. We define the *horizontal derivative*

$$\hat{d} : \Omega^k(M^\#, \mathcal{V}M^\#) \rightarrow \Omega^{k+1}(M^\#, \mathcal{V}M^\#)$$

by $\hat{d}\varphi = d^\nabla\varphi - \alpha \wedge i_\xi d^\nabla\varphi$.

Notice that by definition $i_\xi \hat{d}\varphi = 0$ for all $\varphi \in \Omega^*(M^\#, \mathcal{V}M^\#)$. Moreover, all the operations used in the definition are compatible with $\mathcal{L}_{\tilde{\xi}}$, so $\hat{d}(A^k) \subset A_{\text{hor}}^{k+1}$.

PROPOSITION 4.7. *Assuming that $\xi \in \mathfrak{X}(M^\#)$ is a normal infinitesimal automorphism, we have:*

- (1) *The operator Ξ_* commutes with d^∇ and with \hat{d} .*
- (2) *If $d^\nabla \circ d^\nabla = 0$, then the subspaces $K^k = \Omega_\xi^k(M^\#, \ker(\Xi))_{\text{hor}}$ form a sub-complex of (A^*, d^∇) .*
- (3) *Denoting by C^k the quotient $A_{\text{hor}}^k / \Xi_*(A_{\text{hor}}^k)$, the horizontal derivative induces a well defined operator $\hat{d} : C^k \rightarrow C^{k+1}$ for each k . If $d^\nabla \circ d^\nabla = 0$, then (C^*, \hat{d}) is a complex.*

Proof. (1) Let $s \in \Gamma(\mathcal{A}M^\#)$ be the section corresponding to $\tilde{\xi} \in \mathfrak{X}(\mathcal{G}^\#)$, so by assumption $\nabla^{\mathcal{A}}s = 0$. As we have noted in the proof of Lemma 4.5 above, we get $\Xi_*\varphi(\eta_1, \dots, \eta_k) = s \bullet (\varphi(\eta_1, \dots, \eta_k))$ for arbitrary vector fields $\eta_1, \dots, \eta_k \in \mathfrak{X}(M^\#)$. Using Proposition 1.5.7 of [10], this shows that

$$\nabla_\eta^\mathcal{V}(\Xi_*\varphi(\eta_1, \dots, \eta_k)) = \Xi_*(\nabla_\eta^\mathcal{V}\varphi(\eta_1, \dots, \eta_k))$$

holds for arbitrary vector fields $\eta, \eta_1, \dots, \eta_k$. Using the standard formula for d^∇ , this readily implies that $d^\nabla \circ \Xi_* = \Xi_* \circ d^\nabla$. Since we evidently get $\Xi_*(\alpha \wedge i_\xi\varphi) = \alpha \wedge i_\xi(\Xi_*\varphi)$, this also implies $\hat{d} \circ \Xi_* = \Xi_* \circ \hat{d}$.

(2) Theorem 4.1 and the last part of Lemma 4.5 show that for $\varphi \in K^k$, we get $d^\nabla\varphi \in A_{\text{hor}}^{k+1}$. By part (1), we also get $\Xi_*(d^\nabla\varphi) = d^\nabla(\Xi_*\varphi) = 0$, so indeed $d^\nabla(K^k) \subset K^{k+1}$, and the last claim is obvious.

(3) For $\psi \in A_{\text{hor}}^k$ we get $\hat{d}\Xi_*\psi = \Xi_*(\hat{d}\psi)$ by part (1). But as observed above, $\hat{d}\psi \in A_{\text{hor}}^{k+1}$, so we conclude that \hat{d} induces a well defined operator $C^k \rightarrow C^{k+1}$.

Next, for $\varphi \in A_{\text{hor}}^k$, the definition of \hat{d} and the last part of Lemma 4.5 show that $\hat{d}\varphi = d^\nabla\varphi - \alpha \wedge \Xi_*\varphi$. The standard formula for d^∇ easily implies that we can compute $d^\nabla(\alpha \wedge \Xi_*\varphi)$ as $d\alpha \wedge \Xi_*\varphi - \alpha \wedge d^\nabla(\Xi_*\varphi)$. Since ξ is the Reeb field for α , the first summand is horizontal already. On the other hand, the second summand lies in the kernel of the projection to horizontal forms, so $\hat{d}(\alpha \wedge \Xi_*\varphi) = \Xi_*(d\alpha \wedge \varphi)$. But assuming $d^\nabla \circ d^\nabla = 0$, the fact that $\hat{d}\varphi = d^\nabla\varphi - \alpha \wedge \Xi_*\varphi$ implies that

$$\hat{d}\hat{d}\varphi = -\hat{d}(\alpha \wedge \Xi_*\varphi) = -\Xi_*(d\alpha \wedge \varphi),$$

so the last claim follows. □

4.4. A LONG EXACT SEQUENCE

We are now ready to construct a long exact sequence of cohomology groups, which will be the fundamental tool to compute the cohomology of descended BGG sequences. To define the necessary maps, let us first make the definition

of the cohomology of (C^*, \hat{d}) more explicit. We assume that $d^\nabla \circ d^\nabla = 0$ from now on. A k -cocycle in the complex (C^*, \hat{d}) by definition is represented by a form $\varphi \in A_{\text{hor}}^k$ for which there is a form $\psi \in A_{\text{hor}}^{k+1}$ such that $\hat{d}\varphi = \Xi_*\psi$. We then simply write $[\varphi] \in H^k(C^*, \hat{d})$ for the cohomology class represented by φ , and $[\varphi] = [\tilde{\varphi}]$ if and only if there are forms $\psi_1 \in A_{\text{hor}}^{k-1}$ and $\psi_2 \in A_{\text{hor}}^k$ such that $\tilde{\varphi} = \varphi + \hat{d}\psi_1 + \Xi_*\psi_2$.

Now let us assume that $\tau \in A^k$ such that $d^\nabla \tau = 0$. Then by Lemma 4.2 we get $i_\xi \tau \in A_{\text{hor}}^{k-1}$ and Lemma 4.5 shows that $d^\nabla i_\xi \tau = \Xi_*\tau$. By definition, this implies that $\hat{d}i_\xi \tau = \Xi_*(\tau - \alpha \wedge i_\xi \tau)$, so we can form the class $[i_\xi \tau] \in H^{k-1}(C^*, \hat{d})$. We obtain a map π from $\ker(d^\nabla) \subset A^k$ to $H^{k-1}(C^*, \hat{d})$.

On the other hand, suppose that we have given $\varphi \in A_{\text{hor}}^{k-1}$ and $\psi \in A_{\text{hor}}^k$ such that $\hat{d}\varphi = \Xi_*\psi$. Then we can form $d\alpha \wedge \varphi + \hat{d}\psi$ and using that ξ is the Reeb field for α , we see that this lies in A_{hor}^{k+1} . Moreover by part (1) of Proposition 4.7 we get $\Xi_*(\hat{d}\psi) = \hat{d}(\Xi_*\psi) = \hat{d}\hat{d}\varphi$. In the proof of part (3) of that Proposition, we have seen that $\hat{d}\hat{d}\varphi = -\Xi_*(d\alpha \wedge \varphi)$, which shows that actually $d\alpha \wedge \varphi + \hat{d}\psi \in K^{k+1}$.

By the last part of Lemma 4.5, $i_\xi d^\nabla \psi = \Xi_*\psi = \hat{d}\varphi$, so $\hat{d}\psi = d^\nabla \psi - \alpha \wedge \hat{d}\varphi$, and in the last term we can replace $\hat{d}\varphi$ by $d^\nabla \varphi$. Using this and $(d^\nabla)^2 = 0$, we get $d^\nabla \hat{d}\psi = -d\alpha \wedge d^\nabla \varphi$. But this clearly cancels with $d^\nabla(d\alpha \wedge \varphi)$, so $d\alpha \wedge \varphi + \hat{d}\psi$ is a cocycle in K^{k+1} and we can form the cohomology class $[d\alpha \wedge \varphi + \hat{d}\psi] \in H^{k+1}(K^*, d^\nabla)$.

In the beginning we had fixed a form ψ such that $\hat{d}\varphi = \Xi_*\psi$. Of course this pins down ψ up to adding an element of K^k . This shows that the cohomology class $[d\alpha \wedge \varphi + \hat{d}\psi]$ depends only on φ , so we get a well defined map

$$\delta : \{\varphi \in A_{\text{hor}}^{k-1} : \hat{d}\varphi \in \Xi_*(A_{\text{hor}}^k)\} \rightarrow H^{k+1}(K^*, d^\nabla).$$

THEOREM 4.8. *The maps π and δ induce well defined maps in cohomology, which we denote by the same symbols, i.e. $\pi : H^k(A^*, d^\nabla) \rightarrow H^{k-1}(C^*, \hat{d})$ and $\delta : H^{k-1}(C^*, \hat{d}) \rightarrow H^{k+1}(K^*, d^\nabla)$. Together with the map j induced by the inclusion $K^* \hookrightarrow A^*$, these fit into a long exact sequence of the form*

$$\dots \xrightarrow{\delta} H^k(K^*, d^\nabla) \xrightarrow{j} H^k(A^*, d^\nabla) \xrightarrow{\pi} H^{k-1}(C^*, \hat{d}) \xrightarrow{\delta} H^{k+1}(K^*, d^\nabla) \xrightarrow{j} \dots$$

Proof. Since both π and δ are evidently linear, we have to show that they vanish on elements representing trivial cohomology classes to obtain well defined maps in cohomology. If $\tau \in A^{k-1}$, then Lemma 4.5 shows that $i_\xi d^\nabla \tau = -d^\nabla i_\xi \tau + \Xi_*\tau$. Writing $-d^\nabla i_\xi \tau$ as $-\hat{d}i_\xi \tau - \alpha \wedge i_\xi d^\nabla i_\xi \tau$, the second summand can be rewritten as $-\alpha \wedge \Xi_*(i_\xi \tau)$ by Lemma 4.5. Hence we see that $\pi(d^\nabla \tau) = \Xi_*(\tau - \alpha \wedge i_\xi \tau)$, and since $\tau - \alpha \wedge i_\xi \tau \in A_{\text{hor}}^{k-1}$, this has trivial class in $H^k(C^*, \hat{d})$.

On the other hand, take $\psi_1, \psi_2 \in A_{\text{hor}}^*$ of degrees $k-2$ and $k-1$ respectively. To determine $\delta(\hat{d}\psi_1 + \Xi_*\psi_2)$, we first have to compute the image of this element under \hat{d} . This gives $-d\alpha \wedge \Xi_*\psi_1 + \Xi_*\hat{d}\psi_2$, so

$$\delta(d^\nabla \psi_1 + \Xi_*\psi_2) = d\alpha \wedge \hat{d}\psi_1 + d\alpha \wedge \Xi_*\psi_2 + \hat{d}(-d\alpha \wedge \psi_1 + \hat{d}\psi_2).$$

Now the second and last term in the right hand side clearly cancel, and a short computation shows that $\hat{d}(d\alpha \wedge \psi_1) = d\alpha \wedge \hat{d}\psi_1$, so the other two terms cancel, too. This shows that δ induces a well defined map in cohomology.

To prove exactness of the sequence, we first observe that for $\varphi \in K^k$, we have $i_\xi \varphi = 0$ by definition, so $\pi \circ j = 0$. On the other hand, suppose that $\tau \in A^k$ satisfies $d^\nabla \tau = 0$ and $\pi([\tau]) = 0$. Then $i_\xi \tau = \hat{d}\psi_1 + \Xi_* \psi_2$ for elements $\psi_1, \psi_2 \in A_{\text{hor}}^*$ of degree $k - 2$ and $k - 1$, respectively. Then consider the form

$$\tilde{\tau} := \tau + d^\nabla(\alpha \wedge \psi_1 - \psi_2) = \tau + d\alpha \wedge \psi_1 - \alpha \wedge d^\nabla \psi_1 - d^\nabla \psi_2,$$

which represents the same cohomology class as τ . Under insertion of ξ , the second summand in the right hand side vanishes, while the third summand produces $-d^\nabla \psi_1 + \alpha \wedge i_\xi d^\nabla \psi_1 = -\hat{d}\psi_1$ and the last one gives $-\Xi_* \psi_2$. Hence $i_\xi \tilde{\tau} = 0$ and since also $d^\nabla \tilde{\tau} = 0$, Lemma 4.5 shows that $\Xi_* \tilde{\tau} = 0$. Hence $\tilde{\tau}$ is a cocycle in K^k and $\ker(\pi) = \text{im}(j)$.

Next, we claim that $\delta \circ \pi = 0$. Taking $\tau \in A^k$ with $d^\nabla \tau = 0$, we have observed above that $\hat{d}i_\xi \tau = \Xi_*(\tau - \alpha \wedge i_\xi \tau)$. So by definition, $\delta([i_\xi \tau])$ is the cohomology class of $d\alpha \wedge i_\xi \tau + \hat{d}(\tau - \alpha \wedge i_\xi \tau)$. Computing $d^\nabla(\tau - \alpha \wedge i_\xi \tau)$ using that τ is closed, we get $-d\alpha \wedge i_\xi \tau + \alpha \wedge d^\nabla i_\xi \tau$. Projecting to the horizontal part leaves the first term untouched and kills the second term, so $\delta([i_\xi \tau]) = 0$.

Conversely, let us assume that $\varphi \in A_{\text{hor}}^{k-1}$ has the property that $\hat{d}\varphi = \Xi_* \psi$ for some $\psi \in A_{\text{hor}}^k$ and that $\delta([\varphi]) = 0$. This means that there is $\tilde{\psi} \in K^k$ such that $d\alpha \wedge \varphi + \hat{d}\psi = d^\nabla \tilde{\psi}$. Taking into account that $i_\xi d^\nabla \tilde{\psi} = 0$, we may simply replace ψ by $\psi - \tilde{\psi}$, and assume that $\hat{d}\varphi = \Xi_* \psi$ and $d\alpha \wedge \varphi + \hat{d}\psi = 0$. Now consider $\tau := \alpha \wedge \varphi + \psi \in A^k$, which evidently satisfies $i_\xi \tau = \varphi$. Now

$$d^\nabla \tau = d\alpha \wedge \varphi - \alpha \wedge d^\nabla \varphi + d^\nabla \psi$$

Now in the second summand, we can clearly replace d^∇ by \hat{d} . In the third summand, we rewrite $d^\nabla \psi = \hat{d}\psi + \alpha \wedge i_\xi d^\nabla \psi$. Rewriting the last term as $\alpha \wedge \Xi_* \psi$ we conclude that $d^\nabla \tau = 0$, so $[\varphi] = \pi([\tau])$ and $\ker(\delta) = \text{im}(\pi)$.

Finally, if $\varphi \in A_{\text{hor}}^{k-1}$ has the property that $\hat{d}\varphi = \Xi_* \psi$ for some $\psi \in A_{\text{hor}}^k$, then $\delta([\varphi]) = d\alpha \wedge \varphi + \hat{d}\psi$. But then $\tau := \alpha \wedge \varphi + \psi \in A^k$ and we get

$$d^\nabla \tau = d\alpha \wedge \varphi - \alpha \wedge d^\nabla \varphi + \hat{d}\psi + \alpha \wedge \Xi_* \psi.$$

This vanishes since in the second term we may replace d^∇ by \hat{d} , and we see that $j \circ \delta = 0$.

Conversely, assume that $\varphi \in K^{k+1}$ has the property that $\varphi = d^\nabla \tau$ for some $\tau \in A^k$. Then by assumption, we have $0 = i_\xi d^\nabla \tau$ and Lemma 4.5 shows that $d^\nabla i_\xi \tau = \Xi_* \tau$. Thus we get $\hat{d}i_\xi \tau = \Xi_*(\tau - \alpha \wedge i_\xi \tau)$ and we may form $[i_\xi \tau] \in H^{k-1}(C^*, \hat{d})$. But then $\delta([i_\xi \tau])$ is the class of $d\alpha \wedge i_\xi \tau - \hat{d}(\tau - \alpha \wedge i_\xi \tau)$. Now $\hat{d}\tau = d^\nabla \tau = \varphi$, while $d^\nabla(\alpha \wedge i_\xi \tau) = d\alpha \wedge i_\xi \tau - \alpha \wedge d^\nabla i_\xi \tau$. As before, projecting the right hand side to the horizontal part leaves the first term unchanged and kills the second term, so $\delta([i_\xi \tau]) = [\varphi]$, which completes the proof. \square

4.5. THE CASE OF THE HOMOGENEOUS MODEL

As a last step, we specialize further to the case of PCS-quotients of connected open subsets of the homogeneous model G/P of a parabolic contact structure. In particular, this includes the global contactification $S^{2n+1} \rightarrow \mathbb{C}P^n$ in the two geometric interpretations discussed in Proposition 1 and Section 3.4 of [9]. In the second case, we obtain generalizations of all results on cohomology needed for the applications in [16].

Observe that all restrictions we have imposed so far are satisfied for a connected open subset in G/P , since any infinitesimal automorphism of a locally flat geometry is normal. The crucial additional ingredient we get for the homogeneous model is that any tractor bundle admits a global parallel frame.

LEMMA 4.9. *Let $M^\# = G/P$ be the homogeneous model of a parabolic contact structure, let \mathbb{V} be a representation of G and $\mathcal{VM}^\# = G \times_P \mathbb{V}$ the corresponding tractor bundle. For $v \in \mathbb{V}$ consider the section $\sigma_v \in \Gamma(\mathcal{VM}^\#)$ corresponding to the P -equivariant function $f_v : G \rightarrow \mathbb{V}$ defined by $f_v(g) := g^{-1} \cdot v$. Then $\nabla^\mathbb{V} \sigma_v = 0$, so starting from a basis of \mathbb{V} , we obtain a global parallel frame for $\mathcal{VM}^\#$.*

Proof. It is well known that the extended principal bundle $G \times_P G$ is canonically trivialized by the map sending the class of (g_1, g_2) to $(g_1 P, g_1 g_2) \in G/P \times G$. Now we can also realize $\mathcal{VM}^\#$ as $(G \times_P G) \times_G \mathbb{V}$, thus obtaining a canonical trivialization of $\mathcal{VM}^\#$. The canonical connection $\nabla^\mathbb{V}$ is well known to be the flat connection induced by this trivialization which implies the result. \square

Now of course we also get a global parallel trivialization of $\mathcal{VM}^\#$ in the case that $M^\#$ is a connected open subset in G/P . In this case $\mathcal{G}^\# \subset G$ is the (open) pre-image of $M^\# \subset G/P$ in G . Now assume that $q : M^\# \rightarrow M$ is a PCS-quotient, and let $\tilde{\xi} \in \mathfrak{X}(M^\#)$ be the corresponding infinitesimal automorphism. Since any infinitesimal automorphism corresponds to a parallel section of \mathcal{AM} , we see that $\tilde{\xi}$ must be the restriction to $\mathcal{G}^\#$ of a right invariant vector field R_X on G . Using this, one easily verifies that the corresponding bundle map Ξ on $\mathcal{VM}^\#$ satisfies $\Xi \circ \sigma_v = \sigma_{-X \cdot v}$. Now let $\mathbb{W}_1 \subset \mathbb{V}$ be the kernel and \mathbb{W}_2 the cokernel of the map $\mathbb{V} \rightarrow \mathbb{V}$ defined by $v \mapsto X \cdot v$. Then of course mapping (x, w) to $\sigma_w(x)$ defines a trivialization $M^\# \times \mathbb{W}_1 \cong \ker(\Xi)$ and similarly we get a trivialization of $\text{coker}(\Xi)$.

THEOREM 4.10. *Suppose that $M^\#$ is a connected open subset of the homogeneous model G/P of some parabolic contact structure and that $q : M^\# \rightarrow M$ is a PCS-quotient for which the infinitesimal automorphism defining the quotient corresponds to $X \in \mathfrak{g}$. Let \mathbb{V} be a representation of \mathfrak{g} , let $\rho_X : \mathbb{V} \rightarrow \mathbb{V}$ be the action of X and put $\mathbb{W}_1 := \ker(\rho_X)$ and $\mathbb{W}_2 := \mathbb{V}/\text{im}(\rho_X)$.*

(1) *For the complex (K^*, d^∇) from Proposition 4.7, the cohomology is given by $H^k(K^*, d^\nabla) \cong H^k(M) \otimes \mathbb{W}_1$, where $H^k(M)$ is the k -th de-Rham cohomology of M .*

(2) *Suppose further that $\ker(\rho_X) \cap \text{im}(\rho_X) = \{0\}$. Then there is a natural isomorphism $\mathbb{W}_1 \cong \mathbb{W}_2$ and also for the complex (C^*, \hat{d}) from Proposition 4.7,*

the cohomology is given by $H^k(C^*, \hat{d}) \cong H^k(M) \otimes \mathbb{W}_1$. Moreover, under this identification and the one from part (1), the homomorphism δ in the long exact sequence from Theorem 4.8 corresponds to map $H^{i-1}(M) \otimes \mathbb{W}_1 \rightarrow H^{i+1}(M) \otimes \mathbb{W}_1$ is given by taking the wedge product with the cohomology class $[\omega] \in H^2(M)$, where $\omega \in \Omega^2(M)$ is characterized by $q^*\omega = d\alpha$.

Proof. The global trivialization $\mathcal{V}M^\# \cong M^\# \times \mathbb{V}$ constructed above of course defines an isomorphism

$$(3) \quad \Omega^k(M^\#, \mathcal{V}M^\#) \cong \Omega^k(M^\#) \otimes \mathbb{V}.$$

By definition, the map Ξ_* corresponds to $\text{id} \otimes \rho_X$ under this isomorphism while i_ξ corresponds to $i_\xi \otimes \text{id}_\mathbb{V}$. Moreover, the fact that the trivializing frame consists of parallel sections implies that d^∇ corresponds to $d \otimes \text{id}_\mathbb{V}$ under the isomorphism (3). Finally, the considerations about naturality of $\mathcal{L}_{\tilde{\xi}}$ from Section 4.2 together with the observations on the trivializing sections above show that $\mathcal{L}_{\tilde{\xi}}$ corresponds to $\mathcal{L}_\xi \otimes \text{id} - \text{id} \otimes \rho_X$ under the isomorphism (3).

Now by definition $K^k \subset \Omega^k(M^\#, \mathcal{V}M^\#)$ consists of those forms φ such that $\mathcal{L}_\xi \varphi = 0$, $i_\xi \varphi = 0$ and $\Xi_*(\varphi) = 0$. Hence we see restricting the above map, we obtain an isomorphism between K^k and the joint kernel of $\text{id} \otimes \rho_X$, $i_\xi \otimes \text{id}$ and $\mathcal{L}_\xi \otimes \text{id}$. Of course, this joint kernel is exactly $\Omega^*(M) \otimes \mathbb{W}_1$, and d^∇ corresponds to $d \otimes \text{id}$, so (1) follows.

In the setting of (2), we first observe that restricting the projection $\mathbb{V} \rightarrow \mathbb{W}_2$ to \mathbb{W}_1 , we obtain an injection by assumption, so this must be a linear isomorphism for dimensional reasons. Now we can compose the isomorphism (3) with the projection onto $\Omega^k(M^\#) \otimes \mathbb{W}_2$ and restrict the resulting map to $A_{hor}^k \subset \Omega^k(M^\#, \mathcal{V}M^\#)$. By the above observations on compatibility, the values of this map lie in the kernels of $\mathcal{L}_\xi \otimes \text{id}$ and $i_\xi \otimes \text{id}$, so we actually land in $\Omega^k(M) \otimes \mathbb{W}_2$.

Moreover, by assumption, any form in $\Omega^k(M^\#, \mathcal{V}M^\#)$ can be written as $\varphi = \varphi_1 + \varphi_2$, where φ_1 has values in $\ker(\Xi)$ while φ_2 has values in $\text{im}(\Xi)$. From above, we see that $\mathcal{L}_{\tilde{\xi}}$ preserves these two subspaces, so we see that $\mathcal{L}_{\tilde{\xi}}\varphi = 0$ if and only if $\mathcal{L}_{\tilde{\xi}}\varphi_i = 0$ for $i = 1, 2$. The same result trivially holds for i_ξ so we see that $\varphi \in A_{hor}^k$ implies $\varphi_i \in A_{hor}^k$ for $i = 1, 2$, so in particular $\varphi_1 \in K^k$. Again by assumption Ξ restricts to an isomorphism on $\text{im}(\Xi)$, which shows that $\varphi_2 \in \Xi_*(A_{hor}^k)$, so the class of φ in C^k coincides with the class of φ_1 .

On the other hand, given $\tau \in \Omega^k(M)$ and $w \in \mathbb{W}_2$, we can find an element $\tilde{w} \in \mathbb{W}_1$ projecting onto w and then consider $q^*\tau \otimes \sigma_{\tilde{w}} \in \Omega^k(M^\#, \mathcal{V}M^\#)$. Since $\Xi_*(\sigma_{\tilde{w}}) = 0$, we see that this lies in A_{hor}^k , so we can look at its class in C^k . Together with the above, this shows that we get an inverse, so $C^k \cong \Omega^k(M) \otimes \mathbb{W}_2$. Of course, $d^\nabla(q^*\tau \otimes \sigma_{\tilde{w}}) = (q^*d\tau) \otimes \sigma_{\tilde{w}}$, and since the pullback is horizontal, this coincides with $\hat{d}(q^*\tau \otimes \sigma_{\tilde{w}})$. Hence under our isomorphism \hat{d} on C^* again corresponds to $d \otimes \text{id}$. Finally, if $d\tau = 0$, then $\hat{d}(q^*\tau \otimes \sigma_{\tilde{w}}) = 0$, which readily implies the claim about δ . □

4.6. EXAMPLES

Let us first observe that the conditions of part (2) of Theorem 4.10 are often satisfied.

PROPOSITION 4.11. *Let \mathfrak{g} be a simple Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$ and suppose that $X \in \mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ is semisimple, i.e. such that ad_X is diagonalizable on $\mathfrak{g}_{\mathbb{C}}$. Then the assumption of part (2) of Theorem 4.10 is satisfied for any finite dimensional representation of \mathfrak{g} .*

Proof. It is a classical result of Lie theory that X acts diagonalizably on any complex representation of $\mathfrak{g}_{\mathbb{C}}$. But for a diagonalizable map, the kernel is the eigenspace for the eigenvalue 0, while the image coincides with the sum of all other eigenspaces. Hence $\ker(\rho_X) \cap \text{im}(\rho_X) = \{0\}$ on such representations, and via complexifications, this easily extends to all real representations of \mathfrak{g} . \square

Next, we can sort out the local case.

COROLLARY 4.12. *Suppose that the assumptions of part (2) of Theorem 4.10 are satisfied and that $q : M^{\#} \rightarrow M$ has the property that the form $\omega \in \Omega^2(M)$ such that $q^*\omega = d\alpha$ is exact. Then for each k , the cohomology \mathcal{H}_k in degree k of the descended BGG sequence fits into an exact sequence*

$$0 \rightarrow \Omega^k(M) \otimes \mathbb{W}_1 \rightarrow \mathcal{H}_k \rightarrow \Omega^{k-1}(M) \otimes \mathbb{W}_1 \rightarrow 0.$$

In particular, the local cohomology of the complex vanishes except in degrees 0 and 1, where it is isomorphic to \mathbb{W}_1 .

Proof. Part (2) of Theorem 4.10 gives an interpretation of the cohomology groups of K^* and C^* showing up in the long exact sequence from Theorem 4.8 and shows that the connecting homomorphisms δ in that sequence are all 0. Hence the sequence decomposes into short exact sequences as claimed. The result on local cohomology follows readily. \square

Finally, we can sort out the case of complex projective space in either of the two interpretations from [9]. Note that the only information needed for the applications in [16] is vanishing of the first cohomology for a class of descended BGG sequences.

THEOREM 4.13. *For $n \geq 2$ consider the global PCS-quotient $q : M^{\#} := S^{2n+1} \rightarrow \mathbb{C}P^n =: M$, either for the PCS-structure of Kähler type on $\mathbb{C}P^n$ as discussed in Proposition 1 of [9] or the induced conformal Fedosov structure as in Section 3.4 of that reference. Let \mathbb{V} be a representation of the corresponding group G , let $X \in \mathfrak{g}$ be the element generating the parallel section of $AM^{\#}$ giving rise to the PCS-quotient and put $\mathbb{W} := \{v \in \mathbb{V} : X \cdot v = 0\}$. Let $\mathcal{V}M^{\#} \rightarrow M^{\#}$ be the tractor bundle induced by \mathbb{V} .*

Then the cohomology of the sequence of differential operators on M obtained by descending the BGG sequence induced by \mathbb{V} vanishes in degrees different from 0 and $2n + 1$, while in degrees 0 and $2n + 1$ it is isomorphic to \mathbb{W} .

Proof. The Lie algebra \mathfrak{g} of G either equals $\mathfrak{su}(n+1, 1)$ or $\mathfrak{sp}(2n+2, \mathbb{R})$. In the first case, \mathfrak{g} naturally acts on \mathbb{C}^{n+2} and in the second case we consider it as acting on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. In both cases, the discussion in [9] shows that X acts diagonalizably (over \mathbb{C}) in this representation. Thus we can apply Proposition 4.11 to see that the assumptions of part (2) of Theorem 4.10 are satisfied for \mathbb{V} . Together with the well known description of $H^*(\mathbb{C}P^n)$, Theorem 4.10 shows that both (K^*, d^∇) and (C^*, \hat{d}) have vanishing cohomology in odd degrees. Moreover, all the connecting homomorphisms $\delta : H^{k-1}(K^*, d^\nabla) \rightarrow H^{k+1}(C^*, \hat{d})$ in the long exact sequence from Theorem 4.8 are isomorphisms whenever $1 \leq k \leq 2n-1$. Using this, the long exact sequence readily implies vanishing of the cohomology of (A^*, d^∇) in degrees different from 0 and $2n+1$. For these two degrees the long exact sequence contains the parts $0 \rightarrow H^0(K^*, d^\nabla) \rightarrow H^0(A^*, d^\nabla) \rightarrow 0$ and $0 \rightarrow H^{2n+1}(A^*, d^\nabla) \rightarrow H^{2n}(C^*, \hat{d}) \rightarrow 0$ which together with Theorem 4.10 completes the proof. \square

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