On Deligne's Conjecture for Certain Automorphic L-Functions for $\mathrm{GL}(3)\times\mathrm{GL}(2)$ and $\mathrm{GL}(4)$

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Abstract. We prove Deligne's conjecture for certain automorphic L-functions for $\mathrm{GL}(3)\times\mathrm{GL}(2)$ and $\mathrm{GL}(4)$. The proof is based on rationality results for central critical values of triple product L-functions, which follow from establishing explicit Ichino's formulae for trilinear period integrals for Hilbert cusp forms on totally real étale cubic algebras over $\mathbf{Q}$.

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1 Introduction

The purpose of this paper is to establish the explicit Ichino formula for twisted triple product L-functions. As an application of our formula, we establish new cases on the algebraicity of central critical values of certain class of automorphic L-functions for $\mathrm{GL}(3)\times\mathrm{GL}(2)$ divided by the associated Deligne's periods. To begin with, let f and g be elliptic newforms of weights κ' and κ, level $\Gamma_0(N_1)$ and $\Gamma_0(N_2)$, respectively. We let $L(s,\mathrm{Sym}^2(g)\otimes f)$ be the motivic L-function associated with $\mathrm{Sym}^2(g)\otimes f$. Recall we have the functional equation which relates $L(s,\mathrm{Sym}^2(g)\otimes f)$ and $L(w+1-s,\mathrm{Sym}^2(g)\otimes f)$, where $w=2\kappa+\kappa'-3$ (cf. see (7.1)). Put $\epsilon=(-1)^{\kappa'/2-1}$. Denote by $\Omega_f^{\pm}$ the Shimura's periods of f

in [Shi77], and $\Omega_{f,g} \in \mathbf{C}^\times$ be Deligne's period of the tensor motive associated to $\mathrm{Sym}^2(g)\otimes f$ with sign ϵ. Then

$$\Omega_{f,g} = \begin{cases} (2\pi\sqrt{-1})^{3-3\kappa}(\sqrt{-1})^{1-\kappa'}\langle f,f\rangle\Omega_f^\epsilon & \text{if } 2\kappa \leq \kappa', \\ (2\pi\sqrt{-1})^{2-\kappa-\kappa'}\langle g,g\rangle^2\Omega_f^\epsilon & \text{if } 2\kappa > \kappa'. \end{cases} \tag{1.1}$$

Here $\langle f,f\rangle$ and $\langle g,g\rangle$ are the Petersson norms of f and g, respectively. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, let f^σ and g^σ be the Galois conjugates of f and g by σ, respectively. Our main result is as follows.

Theorem A. *(Cor. 7.1) Suppose that N_1 and N_2 are square-free. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, we have*

$$\left(\frac{L((w+1)/2,\mathrm{Sym}^2(g)\otimes f)}{(2\pi\sqrt{-1})^{3(w+1)/2}\Omega_{f,g}}\right)^\sigma = \frac{L((w+1)/2,\mathrm{Sym}^2(g^\sigma)\otimes f^\sigma)}{(2\pi\sqrt{-1})^{3(w+1)/2}\Omega_{f^\sigma,g^\sigma}}.$$

Remark 1.1.

(1) The period $\Omega_{f,g}$ coincides with Deligne's period for the motive attached to $\mathrm{Sym}^2(g)\otimes f$. Indeed, it is a direct consequence of the period calculation due to Blasius in [Bla87].

(2) If the central value $L(\frac{\kappa'}{2}, f)$ is non-zero, then Theorem A follows from the algebraicity of the central value $L(\frac{w+1}{2}, g\otimes g\otimes f)$ proved by Harris-Kudla in [HK91].

(3) If $N_1 = 1$ and $2\kappa > \kappa'$, then the above algebraicity result was obtained by Ichino [Ich05, Corollary 2.6] via the explicit pullback formula for Saito-Kurokawa lifts ($N_2 = 1$ and $\kappa = \kappa'/2+1$) and by Xue ($N_2 = 1$) using a different but closely related approach [Xue19]. The first author has generalized Ichino's pullback formula of Saito-Kurokawa lifts in [Che19] if N_2 is furthter assumed to be odd and cubic-free.

Our result covers the remaining cases and thus settles down Deligne's conjecture for the central value of the L-functions for $\mathrm{Sym}^2(g)\otimes f$ at least when the levels of f and g are square-free.

(4) If $2\kappa > \kappa'+4$, then $\mathrm{Sym}^2(g)\otimes f$ has a non-zero critical value $L(n, \mathrm{Sym}^2(g)\otimes f)$ such that n has the same parity with $(w+1)/2$. In this case, Theorem A also follows from the results of Garrett-Harris and Januszewski in [GH93, Theorem 4.6] and [Jan18, Theorem A], respectively, and the factorization of motivic L-functions

$$L(s, g\otimes g\otimes f) = L(s,\mathrm{Sym}^2(g)\otimes f)L\left(s-\kappa+1, f\right).$$

We remark that Raghuram has proved the algebraicity of the central critical values of the Rankin-Selberg L-functions attached to regular algebraic cuspidal automorphic representations on $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ in a quite general setting [Rag09]. His method is based on a cohomological interpretation of the Rankin-Selberg zeta integral, and specializing the result of Raghuram to $n = 3$, one also obtains the algebraicity of the central critical value of $L(s, \mathrm{Sym}^2(g) \otimes f)$ divided by certain cohomological period for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ in the case $2\kappa > \kappa'$. However, our result in this case is not covered by [Rag09] in the sense that the periods in both results are quite different. More precisely, the periods in our main theorem coincide with Deligne's period described by Blasius in [Bla87] while Raghuram uses the period $p^{\pm}(\Pi)$ obtained from the comparison between the rational structures defined by Whittaker models and relative Lie algebra cohomology groups for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ [Rag09, §3.2.1]. It seems a difficult problem to study directly the relation between Deligne's period and Raghuram's cohomological period. Our result combined with the non-vanishing hypothesis of central L-values would give a comparison between these two periods. In the case $2\kappa > \kappa' + 4$, the comparison of periods follows from the results of Garrett-Harris and Januszewski (cf. Remark 1.1-(4)).

Our approach also offers the algebraicity of the central critical value of symmetric cube L-functions with the assumption on the non-vanishing of L-values with cubic twist. Put

$$w = 3\kappa' - 3; \quad \epsilon = (-1)^{\kappa'/2-1}.$$

Denote $\Omega_{f,\mathrm{Sym}^3} \in \mathbf{C}^\times$ be Deligne's period of the motive associated to $\mathrm{Sym}^3(f)$ with sign ϵ. Then

$$\Omega_{f,\mathrm{Sym}^3} = (2\pi\sqrt{-1})^{1-\kappa'}(\sqrt{-1})^{1-\kappa'}\langle f, f\rangle(\Omega_f^\epsilon)^2. \tag{1.2}$$

Theorem B (Cor. 7.3). *Suppose that $N_1 > 1$ is square-free and there exist a cubic Dirichlet character χ such that $L\left(\frac{\kappa'}{2}, f \otimes \chi\right) \neq 0$. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, we have*

$$\left(\frac{L((w+1)/2, \mathrm{Sym}^3(f))}{(2\pi\sqrt{-1})^{w+1}\Omega_{f,\mathrm{Sym}^3}}\right)^\sigma = \frac{L((w+1)/2, \mathrm{Sym}^3(f^\sigma))}{(2\pi\sqrt{-1})^{w+1}\Omega_{f^\sigma,\mathrm{Sym}^3}}.$$

The hypothesis on the non-vanishing of cubic twists of L-values is expected to hold in general but seems unfortunately a far-reaching problem at this moment. So far this hypothesis is only known to be satisfied for cuspidal automorphic representations on $\mathrm{GL}_2(\mathbf{A}_K)$ when $\mathbf{Q}(\sqrt{-3}) \subset K$ in [BFH05].

Combining with the results of Januszewski in [Jan16] and [Jan18], and Jiang-Sun-Tian in [JST19], we obtain conditional results on Deligne's conjecture for arbitrary critical values with abelian twists.

Theorem C (Cor. 7.2 and 7.4). *For a Dirichlet character χ, denote by $G(\chi)$ the Gauss sum associated to χ.*

(1) *Suppose that* N_1 *and* N_2 *are square-free,* $2\kappa > \kappa'$*, and* $L(\frac{w+1}{2}, \mathrm{Sym}^2(g)\otimes f) \neq 0$*. Let* $n \in \mathbf{Z}$ *be a critical integer for* $L(s, \mathrm{Sym}^2(g)\otimes f)$ *and* χ *be a Dirichlet character such that* $(-1)^n\chi(-1) = \epsilon$*. For* $\sigma \in \mathrm{Aut}(\mathbf{C})$*, we have*

$$\left(\frac{L(n, \mathrm{Sym}^2(g)\otimes f\otimes\chi)}{G(\chi)^3(2\pi\sqrt{-1})^{3n}\Omega_{f,g}}\right)^\sigma = \frac{L(n, \mathrm{Sym}^2(g^\sigma)\otimes f^\sigma\otimes\chi^\sigma)}{G(\chi^\sigma)^3(2\pi\sqrt{-1})^{3n}\Omega_{f^\sigma,g^\sigma}}.$$

(2) *Suppose that* $N_1 > 1$ *is square-free,* $L(\frac{w+1}{2}, \mathrm{Sym}^3(f)) \neq 0$*, and there exist a cubic Dirichlet character* χ *such that* $L\left(\frac{\kappa'}{2}, f\otimes\chi\right) \neq 0$*. Let* $n \in \mathbf{Z}$ *be a critical integer for* $L(s, \mathrm{Sym}^3(f))$ *and* μ *be a Dirichlet character such that* $(-1)^n\mu(-1) = \epsilon$*. For* $\sigma \in \mathrm{Aut}(\mathbf{C})$*, we have*

$$\left(\frac{L(n, \mathrm{Sym}^3(f)\otimes\mu)}{G(\mu)^2(2\pi\sqrt{-1})^{2n}\Omega_{f,\mathrm{Sym}^3}}\right)^\sigma = \frac{L(n, \mathrm{Sym}^3(f^\sigma)\otimes\mu^\sigma)}{G(\mu^\sigma)^2(2\pi\sqrt{-1})^{2n}\Omega_{f^\sigma,\mathrm{Sym}^3}}.$$

Remark 1.2. The algebraicity of the non-central critical values of $L(s, \mathrm{Sym}^3(f))$ was proved by Garrett-Harris in [GH93, Theorem 6.2]. In particular, if $\kappa' \geq 6$, then Theorem C-(2) also follows from the results of Garrett-Harris and Januszewski in [Jan16, Theorem A] and Jiang-Sun-Tian in [JST19, Theorem 1.1].

Our proof of Theorem A is based on an explicit Ichino's central value formula for the twisted triple product L-functions. Let K be a real quadratic field and let g_K be the Hilbert modular newform over K associated to g obtained by the base change lift. Let $L(s, g_K\otimes f)$ be the triple product L-function associated to $g_K\otimes f$. Let τ_K be the quadratic Dirichlet character associated with $K/\mathbf{Q}$. From the following factorization of L-functions

$$L(s, g_K\otimes f) = L(s, \mathrm{Sym}^2(g)\otimes f)L\left(s-\kappa+1, f\otimes\tau_K\right),$$

one can deduce easily the algebraicity of $L\left(\frac{w+1}{2}, \mathrm{Sym}^2(g)\otimes f\right)$ (divided by the associated Deligne's period) from that of the central value $L\left(\frac{w+1}{2}, g_K\otimes f\right)$ of the twisted triple product and that of the central value $L\left(\frac{\kappa'}{2}, f\otimes\tau_K\right)$ of elliptic modular forms whenever $L(\frac{\kappa'}{2}, f\otimes\tau_K)$ does not vanish. The algebraicity of critical L-values of elliptic modular forms with Dirichlet twists is a classical result due to Shimura, so the main task is to choose a nice real quadratic field K with $L(\frac{\kappa'}{2}, f\otimes\tau_K) \neq 0$ and show the algebraicity of the central value $L(\frac{w+1}{2}, g_K\otimes f)$, for which one appeals to Ichinos's formula in [Ich08]. More precisely, if the global sign in the functional equation of the automorphic L-function for the twisted triple product $g_K\otimes f$ is $+1$, then Ichino's formua alluded to above asserts that there exists a quaternion algebra D over $\mathbf{Q}$ so that the central critical value $L\left(\frac{w+1}{2}, g_K\otimes f\right)$ is the ratio between the square of the global trilinear period integral of an automorphic form on $D^\times(\mathbf{A}_K)\times D^\times(\mathbf{A})$

and a product of certain local zeta integrals. Taking into account the functional equation and the Galois invariance of the global sign, we may assume the global sign of $\mathrm{Sym}^2(g) \otimes f$ is $+1$. Then by using a result of Friedberg and Hofffstein [FH95], we can choose a real quadratic field K such that (i) $L(\frac{\kappa'}{2}, f \otimes \tau_K) \neq 0$, (ii) the sign of $g_K \otimes f$ is $+1$ and (iii) the quaternion algebra D in Ichino's formula is the matrix algebra (resp. a definite quaternion algebra) over $\mathbf{Q}$ in the case $2\kappa \leq \kappa'$ (resp. $2\kappa > \kappa'$ if we assume further that $N_1 > 1$). To obtain the explicit Ichino's central value formula, we calculate the local period integral at each place (Theorems 6.2 and 6.3) in terms of global period integral, and as a consequence, we obtain the algebraicity of the central value $L\left(\frac{w+1}{2}, g_K \otimes f\right)$ (cf. Corollary 6.4) by a standard argument. In fact, we prove in Corollary 6.6 that the ratio between the central L-value of the twisted triple product L-function and the Petersson norms is essentially a square in the Hecke field.

The idea of the proof for Theorem B is similar. Assume χ is a cubic Dirichlet character such that $L\left(\frac{\kappa'}{2}, f \otimes \chi\right) \neq 0$. Let E be the totally real cubic Galois extension over $\mathbf{Q}$ cut out by χ and let f_E be the Hilbert modular newform over E associated to f via the base change lift. Consider the degree eight triple product L-function $L(s, f_E)$ associated to f_E. Then we have the factorization of L-functions :

$$L(s, f_E) = L\left(s, \mathrm{Sym}^3(f)\right) L(s - \kappa' + 1, f \otimes \chi) L(s - \kappa' + 1, f \otimes \chi^2).$$

Thus the algebraicity of $L\left(\frac{w+1}{2}, \mathrm{Sym}^3(f)\right)$ is a consequence of the algebraicity of $L\left(\frac{w+1}{2}, f_E\right)$, which again can be deduced from the explicit Ichino central value formula in this case.

This paper is organized as follows. We first study the local zeta integrals in Ichino's formula. In §2, we introduce the local zeta integrals and fix the test vectors used in the subsequent local calculation. After recalling basic properties of local matrix coefficients for GL(2) in §3, we carry out the calculations of local zeta integrals in the cases of the matrix algebra and the division algebra in §4 and §5, respectively. In particular, we compute the archimedean zeta integrals explicitly. In §6, we recall Ichino's central value formula and establish its explicit version in Theorem 6.2 and Theorem 6.3. Finally, we prove our main results in §7 as an application of the explicit central value formulae.

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2 Local zeta integrals

In this section, we setup the notation and assumptions for our local computations in §3 and §4. We also fix the test vectors, raising elements and define the local zeta integrals which appear in our local calculation. These local zeta integrals are used to establish explicit Ichino's formulae in §6.

2.1 Notation and assumptions

Let F be a local field of characteristic zero. When F is non-archimedean, denote by $\mathcal{O}_F$, ϖ_F and q_F, the valuation ring, a uniformizer and the cardinality of the residue field of F, respectively. Moreover, let ord_F be the valuation on F normalized so that $\mathrm{ord}_F(\varpi_F) = 1$, and let $|\cdot|_F$ be the absolute value on F with $|\varpi_F|_F = q_F^{-1}$. When F is archimedean, let $|\cdot|_{\mathbf{R}}$ be the usual absolute value on $\mathbf{R}$ and $|z|_{\mathbf{C}} = z\overline{z}$ on $\mathbf{C}$. Let E be an étale cubic algebra over a local field F. Then E is (i) $F \times F \times F$ three copies of F, or (ii) $K \times F$, where K is a quadratic extension of F, or (iii) is a cubic field extension of F. Let D be a quaternion algebra over F. If L is a F-algebra, let $D^\times(L) := (D \otimes_F L)^\times$. Let $\mathit{\Pi}$ be a unitary irreducible admissible representation of $D^\times(E)$ whose central character we assume to be trivial on $F^\times$. Let $\mathit{\Pi}'$ be the unitary irreducible admissible representation of $\mathrm{GL}_2(E)$ associated to $\mathit{\Pi}$ via the Jacquet-Langlands correspondence. Therefore $\mathit{\Pi}' = \mathit{\Pi}$ if $D = \mathrm{M}_2(F)$ is the matrix algebra. Notice that $\mathit{\Pi}' = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$ (if $E = F \times F \times F$), or $\mathit{\Pi}' = \pi' \boxtimes \pi$ (if $E = K \times F$), where π_j ($j = 1, 2, 3$) and π are unitary irreducible admissible representations of $\mathrm{GL}_2(F)$, and π' is a unitary irreducible admissible representation of $\mathrm{GL}_2(K)$. We make the following assumptions on the triplet $(F, E, \mathit{\Pi})$ in the rest of this section and §4, §5.

- When F is archimedean, we assume $F = \mathbf{R}$ and $E = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$.
- When $F = \mathbf{R}$, we assume $\mathit{\Pi}'$ is a (limit of) discrete series with the minimal weight $\underline{k} = (k_1, k_2, k_3)$ and the central character $\mathrm{Sgn}^{k_1} \boxtimes \mathrm{Sgn}^{k_2} \boxtimes \mathrm{Sgn}^{k_3}$ for some positive integers k_1, k_2, k_3.
- When F is non-archimedean and π'' is a unitary irreducible admissible generic representation of $\mathrm{GL}_2(L)$, where $L \in \{F, K, E\}$ is a field, we assume π'' is either spherical or is a unramified special representation. Moreover, we assume π'' has trivial central character.
- We assume $\Lambda(\mathit{\Pi}') < 1/2$, where $\Lambda(\mathit{\Pi}')$ is defined in [Ich08, pp. 284-285].
- We assume $\mathrm{Hom}_{D^\times(F)}(\mathit{\Pi}, \mathbf{C}) \neq \{0\}$.

There are some remarks.

REMARK 2.1.

(1) By the results of Prasad [Pra90] and [Pra92], the space $\mathrm{Hom}_{D^\times(F)}(\varPi, \mathbf{C})$ has dimension at most one. When $F = \mathbf{R}$, it follows from [Pra90, Theorem 9.5] that $\mathrm{Hom}_{D^\times(\mathbf{R})}(\varPi, \mathbf{C}) \neq \{0\}$ precisely when (i) $D = \mathrm{M}_2(\mathbf{R})$ and $2\max\{k_1, k_2, k_3\} \geq k_1 + k_2 + k_3$, or (ii) D is the division algebra and $2\max\{k_1, k_2, k_3\} < k_1 + k_2 + k_3$. We call the first case the *unbalanced* case, while the second case is called the *balanced* case. Notice that $k_1 + k_2 + k_3 \equiv 0 \pmod 2$ by our assumption.

(2) Suppose L is a non-archimdean local field. By a unramified special representation π'' of $\mathrm{GL}_2(L)$ we mean $\pi'' = \mathrm{St}_L \otimes \chi$, where St_L is the Steinberg representation of $\mathrm{GL}_2(L)$, and χ is a unramified character of $L^\times$. Note that if π'' has trivial central character, then χ is a quadratic character.

2.2 THE NEW LINE

Denote by $V_\varPi$ the representation space of $\varPi$. In what follows, we shall introduce a special one-dimensional subspace in $V_\varPi$, which is called *the new line* $V_\varPi^{\mathrm{new}}$ of $V_\varPi$. If F is non-archimedean and $\mathfrak{a}$ is an ideal of $\mathcal{O}_E$, let

$$\mathcal{U}_0(\mathfrak{a}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_E) \mid c \in \mathfrak{a} \right\}.$$

Suppose that $D = \mathrm{M}_2(F)$. If F is non-archimedean, then by [Cas73], there is a unique ideal $c(\varPi)$ of $\mathcal{O}_E$ such that

$$\dim_{\mathbf{C}} V_\varPi^{\mathcal{U}_0(c(\varPi))} = 1.$$

The ideal $c(\varPi)$ is called the *conductor* of $\varPi$, and define the new line $V_\varPi^{\mathrm{new}} := V_\varPi^{\mathcal{U}_0(c(\varPi))}$. The conductor $c(\varPi)$ is of the following form (i) $C(\varPi) = (\varpi_F^a, \varpi_F^b, \varpi_F^c)\mathcal{O}_E$ when $E = F \times F \times F$; (ii) $C(\varPi) = (\varpi_K^a, \varpi_F^b)\mathcal{O}_E$ when $E = K \times F$; (iii) $C(\varPi) = \varpi_E^a \mathcal{O}_E$ when E is a field. Note that our assumption implies $0 \leq a, b, c \leq 1$. If $F = \mathbf{R}$, then $E = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ according to our assumption. In this case, the new line $V_\varPi^{\mathrm{new}}$ is defined to be the one-dimensional subspace of the minimal weight under the $\mathrm{SO}_2(E)$-action.
Suppose that D is division. If F is non-archimedean, and $E \neq K \times F$, then $V_\varPi$ is already one-dimensional according to our assumption. In this case, we put $V_\varPi^{\mathrm{new}} = V_\varPi$. When $E = K \times F$, we have $\varPi = \pi' \boxtimes \pi$, where π (resp. π') is a unitary irreducible admissible (resp. generic) representation of $D^\times(F)$ (resp. $\mathrm{GL}_2(K)$). Note that π is one-dimensional by our assumption. In this case, we have $V_\varPi = V_{\pi'} \otimes V_\pi$ and we define the new line $V_\varPi^{\mathrm{new}}$ to be $V_{\pi'}^{\mathrm{new}} \otimes V_\pi$. Of course $V_{\pi'}^{\mathrm{new}}$ stands for the one-dimensional subspace spanned by the new vector of π'. Finally, if $F = \mathbf{R}$ and $2\max\{k_1, k_2, k_3\} < k_1 + k_2 + k_3$, we define the new line $V_\varPi^{\mathrm{new}}$ to be the one-dimensional subspace $V_\varPi^{D^\times(\mathbf{R})}$ of $V_\varPi$ [Pra90, Theorem 9.3].

2.3 Raising elements

Let $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathfrak{U}$ be the universal enveloping algebra of $\mathfrak{g}$. We put $\mathfrak{U}_E = \mathfrak{U} \otimes \mathfrak{U} \otimes \mathfrak{U}$. Let

$$\tilde{V}_+ := (-\frac{1}{8\pi}) \cdot V_+ \in \mathfrak{U} \quad \text{with} \quad V_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1}$$

be the weight raising operator [JL70, Lemma 5.6]. Let $\tau_F \in \mathrm{GL}_2(F)$ be given by

$$\tau_F = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } F = \mathbf{R}, \\ 1 & \text{if } F \text{ is nonarchimedean.} \end{cases} \tag{2.1}$$

Define a special element $\mathbf{t} \in \mathfrak{U}_E \times \mathrm{SO}(2, E)$ or $D^\times(E)$ attached to $\mathit{\Pi}$ as follows:

- $F = \mathbf{R}$, $D = \mathrm{M}_2(F)$ and $2 \max\{k_1, k_2, k_3\} \geq k_1 + k_2 + k_3$. Suppose that $k_3 = \max\{k_1, k_2, k_3\}$. Then

$$\mathbf{t} = \left(1 \otimes \widetilde{V}_+^{\frac{k_3 - k_1 - k_2}{2}} \otimes 1, (1, 1, \tau_{\mathbf{R}})\right) \in \mathfrak{U}_E \times \mathrm{SO}(2, E).$$

- F non-archimedean, $E = F \times F \times F$, $D = \mathrm{M}_2(F)$, $\mathit{\Pi} = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$ and precisely one of π_j is unramified special, say π_1:

$$\mathbf{t} = \left(1, \begin{pmatrix} \varpi_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \in \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F).$$

- F non-archimedean, $E = K \times F$, K/F is ramified, $D = \mathrm{M}_2(F)$, $\mathit{\Pi} = \pi' \boxtimes \pi$ with π' spherical and π unramified special:

$$\mathbf{t} = \left(\begin{pmatrix} \varpi_K^{-1} & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \in \mathrm{GL}_2(K) \times \mathrm{GL}_2(F).$$

- F non-archimedean, $E = K \times F$, K/F is ramified, $D = \mathrm{M}_2(F)$, $\mathit{\Pi} = \pi' \boxtimes \pi$ with π' unramified special and π spherical:

$$\mathbf{t} = \left(\begin{pmatrix} \varpi_F & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \in \mathrm{GL}_2(K) \times \mathrm{GL}_2(F).$$

- F non-archimedean, E ramified cubic extension, $D = \mathrm{M}_2(F)$, $\mathit{\Pi}$ unramified special:

$$\mathbf{t} = \begin{pmatrix} \varpi_E^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(E).$$

- For all other cases:

$$\mathbf{t} = 1 \in D^\times(E).$$

2.4 Definition of local zeta integrals

We now define the local zeta integrals in our local computation except for the balanced case, which will be defined by equation (5.6).
Let $\mathcal{J} \in D^\times(E)$ be given as follows:

$$\mathcal{J} = \begin{cases} (\tau_{\mathbf{R}}, \tau_{\mathbf{R}}, \tau_{\mathbf{R}}) \in \mathrm{GL}_2(\mathbf{R})^3 & \text{if } F = \mathbf{R} \text{ and } D = \mathrm{M}_2(\mathbf{R}), \\ 1 & \text{otherwise.} \end{cases}$$

Let $\zeta_F(s)$ denote the local zeta function. Therefore,

$$\zeta_F(s) = \begin{cases} 2(2\pi)^{-s}\Gamma(s) & \text{if } K = \mathbf{C}, \\ \pi^{-s/2}\Gamma(s/2) & \text{if } K = \mathbf{R}, \\ (1 - q_F^{-s})^{-1} & \text{if } K \text{ is nonarchimedean,} \end{cases}$$

where q_F is the cardinality of the residue field of F when F is non-archimedean.

Definition 2.2. Fix a nonzero $D^\times(E)$-invariant pairing $\mathcal{B}_\Pi \colon V_\Pi \times V_\Pi \to \mathbf{C}$. Let $\phi_\Pi \in V_\Pi^{\mathrm{new}}$ be a non-zero vector in the new line. The normalized local zeta integral is defined by

$$I(\Pi, \mathbf{t}) = \int_{F^\times \backslash D^\times(F)} \frac{\mathcal{B}_\Pi(\Pi(h\mathbf{t})\phi_\Pi, \Pi(\mathbf{t})\phi_\Pi)}{\mathcal{B}_\Pi(\Pi(\mathcal{J})\phi_\Pi, \phi_\Pi)}\, dh,$$
$$I^*(\Pi, \mathbf{t}) = \frac{\zeta_F(2)}{\zeta_E(2)} \cdot \frac{L(1, \Pi', \mathrm{Ad})}{L(1/2, \Pi', r)} \cdot I(\Pi, \mathbf{t}).$$

Here the L-factors are defined in [Ich08, pp. 282-283].

Remark 2.3.

(1) Since the central character of Π is trivial on $F^\times$, the integrals are well-defined. Moreover, our assumption $\Lambda(\Pi') < 1/2$ implies these integrals converge absolutely [Ich08, lemma 2.1].

(2) We note that ϕ_Π is unique up to a constant as well as $\mathcal{B}_\Pi$. Thus $I(\Pi, \mathbf{t})$ is independence of the choice of ϕ_Π and $\mathcal{B}_\Pi$. But it does depend on the choice of the measure dh.

3 Matrix coefficients for GL(2)

Let F be either $\mathbf{R}$ or a non-archimdean local field. Let π be a unitary irreducible admissible generic representation of $\mathrm{GL}_2(F)$. Define a non-zero element ϕ_π as follows. When $F = \mathbf{R}$, let ϕ_π be a vector with minimal non-negative weight under the SO(2)-action. When F is non-archimedean, let ϕ_π be a new vector. Fix a non-zero $\mathrm{GL}_2(F)$-invariant bilinear pairing $\mathcal{B}_\pi \colon \pi \times \tilde{\pi} \to \mathbf{C}$, where $\tilde{\pi}$ is the admissible dual of π.

Definition 3.1. We define the matrix coefficient associate with an element $t \in \mathfrak{U} \times \mathrm{O}(2)$ or $t \in \mathrm{GL}_2(F)$ by

$$\Phi_\pi(h;t) = \frac{\mathcal{B}_\pi(\pi(ht)\phi_\pi, \tilde{\pi}(t)\phi_{\tilde{\pi}})}{\mathcal{B}_\pi(\pi(\tau_F)\phi_\pi, \phi_{\tilde{\pi}})}, \quad h \in \mathrm{GL}_2(F).$$

Recall that τ_F is given by (2.1). When $t = 1$, we simply denote $\Phi_\pi(h)$ for $\Phi_\pi(h;t)$.

Remark 3.2. Note that $\Phi_\pi(h;t)$ is independent of the choice of elements ϕ_π and $\phi_{\tilde{\pi}}$ in the one dimensional subspaces of π and $\tilde{\pi}$ which consisting of either minimal non-negative weight elements or new vectors. Moreover, it is also independent of $\mathcal{B}_\pi$ and the models for which we used to realize π and $\tilde{\pi}$.

3.1 A formula of $\Phi_\pi(h,t)$: the archimedean case

Let π be a (limit of) discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with minimal weight $k \geq 1$ and the central character sgn^k. Note that $\pi \cong \tilde{\pi}$. Let ψ be the additive character of $\mathbf{R}$ defined by $\psi(x) = e^{2\pi\sqrt{-1}x}$. Let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π with respect to ψ. Let $\mathcal{B}_\pi : \mathcal{W}(\pi,\psi) \times \mathcal{W}(\pi,\psi) \to \mathbf{C}$ be the $\mathrm{GL}_2(\mathbf{R})$-invariant bilinear pairing given by

$$\mathcal{B}_\pi(W,W') = \int_{\mathbf{R}^\times} W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) W'\left(\begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times t, \tag{3.1}$$

for $W, W' \in \mathcal{W}(\pi,\psi)$. Here $d^\times t = |t|_{\mathbf{R}}^{-1} dt$, and dt is the usual Lebesgue measure on $\mathbf{R}$.
Let $W_\pi \in \mathcal{W}(\pi,\psi)$ be the weight k element characterized by

$$W_\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = a^{\frac{k}{2}} e^{-2\pi a} \cdot \mathbb{I}_{\mathbf{R}_+}(a), \quad a \in \mathbf{R}^\times. \tag{3.2}$$

For each $m \in \mathbf{Z}_{\geq 0}$, we put

$$W_\pi^m = \rho\left(V_+^m\right) W_\pi.$$

Here ρ denotes the right translation. In particular, we have $W_\pi^0 = W_\pi$. We note that W_π^m has weight $k+2m$. The following recursive formula can be deduced from the proof of [JL70, Lemma 5.6]

$$W_\pi^{m+1}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 2a\cdot\frac{d}{da} W_\pi^m\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) + (k+2m-4\pi a)\cdot W_\pi^m\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{3.3}$$

Lemma 3.3. *We have*

$$W_\pi^m\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 2^m P_\pi^m(a) e^{-2\pi a} \cdot \mathbb{I}_{\mathbf{R}_+}(a),$$

where P_π^m is the polynomial given by

$$P_\pi^m(a) = \sum_{j=0}^m (-4\pi)^j \binom{m}{j} \frac{\Gamma(k+m)}{\Gamma(k+j)} \cdot a^{\frac{k}{2}+j}.$$

Proof. This follows from (3.2), (3.3) and the induction on m. ☐

Lemma 3.4. *Let $a \in \mathbf{R}^\times$ and $x \in \mathbf{R}$. Then*

$$\mathcal{B}_\pi\left(\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)W_\pi^m, W_\pi^m\right)$$

is equal to

$$2^{-k+2m}\pi^{-k}\Gamma(k+m)^2\,\mathbb{I}_{\mathbf{R}_-}(a)\sum_{i,j=0}^m(-2)^{i+j}$$
$$\times\binom{m}{i}\binom{m}{j}\frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)}\frac{(-a)^{\frac{k}{2}+i}}{\left[(1-a)+\sqrt{-1}\,x\right]^{k+i+j}}.$$

Proof. By (3.1) and Lemma 3.3 we have

$$\begin{aligned}
&\mathcal{B}_\pi\left(\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)W_\pi^m, W_\pi^m\right)\\
&= \int_{\mathbf{R}^\times} W_\pi^m\left(\begin{pmatrix}at & 0\\ 0 & 1\end{pmatrix}\right)W_\pi^m\left(\begin{pmatrix}-t & 0\\ 0 & 1\end{pmatrix}\right)\psi(xt)d^\times t\\
&= 2^{2m}\int_{\mathbf{R}^\times} P_\pi^m(at)P_\pi^m(-t)e^{-2\pi\left\{(1-a)+\sqrt{-1}\,x\right\}(-t)}\cdot\mathbb{I}_{\mathbf{R}_+}(at)\mathbb{I}_{\mathbf{R}_+}(-t)d^\times t\\
&= 2^{2m}\mathbb{I}_{\mathbf{R}_-}(a)\sum_{i,j=0}^m(-4\pi)^{i+j}\binom{m}{i}\binom{m}{j}\frac{\Gamma(k+m)^2}{\Gamma(k+i)\Gamma(k+j)}\cdot a^{\frac{k}{2}+i}\cdot I_{ij},
\end{aligned}$$

where

$$\begin{aligned}
I_{ij} &= (-1)^{\frac{k}{2}+i}\int_0^\infty t^{k+i+j}e^{-2\pi\left[(1-a)+\sqrt{-1}\,x\right]t}d^\times t\\
&= (-1)^{\frac{k}{2}+i}\left(2\pi\left[(1-a)+\sqrt{-1}\,x\right]\right)^{-(k+i+j)}\Gamma(k+i+j).
\end{aligned}$$

This proves the lemma. ☐

Lemma 3.5. *Let N be a nonnegative integer. We have the following identity*

$$\sum_{i=0}^N(-1)^i\binom{N}{i}\frac{\Gamma(z+i)}{\Gamma(w+i)} = \frac{\Gamma(z)}{\Gamma(w-z)}\cdot\frac{\Gamma(w-z+N)}{\Gamma(w+N)},$$

for every $z, w \in \mathbf{C}$.

Proof. This is [Ike98, Lemma 2.1]. □

LEMMA 3.6. *We have*

$$\mathcal{B}_\pi\left(\rho\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)W_\pi^m, W_\pi^m\right) = 4^{-k+m}\pi^{-k}\Gamma(k+m)\Gamma(m+1).$$

Proof. By Lemma 3.4 we have

$$\begin{aligned}&\mathcal{B}_\pi\left(\rho\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)W_\pi^m, W_\pi^m\right)\\ &= 4^{-k+m}\pi^{-k}\Gamma(k+m)^2\sum_{i,j=0}^m(-1)^{i+j}\binom{m}{i}\binom{m}{j}\frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)}.\end{aligned}$$

Applying Lemma 3.5, we find that

$$\begin{aligned}&\sum_{i,j=0}^m(-1)^{i+j}\binom{m}{i}\binom{m}{j}\frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)}\\ &= \sum_{i=0}^m\binom{m}{i}\frac{(-1)^i}{\Gamma(k+i)}\sum_{j=0}^m(-1)^j\binom{m}{j}\frac{\Gamma(k+i+j)}{\Gamma(k+j)}\\ &= \sum_{i=0}^m(-1)^i\binom{m}{i}\frac{\Gamma(m-i)}{\Gamma(-i)\Gamma(k+m)}\\ &= (-1)^m\frac{\Gamma(0)}{\Gamma(-m)\Gamma(k+m)} = \frac{\Gamma(m+1)}{\Gamma(k+m)}.\end{aligned}$$

This proves the lemma. □

Combining the above results, we obtain the following corollary.

COROLLARY 3.7. *Let $m \in \mathbf{Z}_{\geq 0}$, $x \in \mathbf{R}$ and $a \in \mathbf{R}^\times$. We have*

(1)

$$\begin{aligned}&\Phi_\pi\left(\begin{pmatrix}a & x\\ 0 & 1\end{pmatrix}; V_+^m\right)\\ &= 2^{k+2m}\frac{\Gamma(k+m)^2}{\Gamma(k)}\mathbb{I}_{\mathbf{R}_-}(a)\sum_{i,j=0}^m(-2)^{i+j}\binom{m}{i}\binom{m}{j}\\ &\times\frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)}\frac{(-a)^{\frac{k}{2}+i}}{\left[(1-a)+\sqrt{-1}\,x\right]^{k+i+j}}.\end{aligned}$$

(2)

$$\Phi_\pi\left(\begin{pmatrix}a & x\\ 0 & 1\end{pmatrix}; \tau_{\mathbf{R}}\right) = 2^k\frac{(-a)^{\frac{k}{2}}}{\left[(1-a)-\sqrt{-1}\,x\right]^k}\mathbb{I}_{\mathbf{R}_-}(a).$$

3.2 A formula of $\Phi_\pi(h,t)$: the non-archimedean case

Let F be a non-archimdean local field. Let $B(F)$ be the subgroup of upper triangular matrices in $\mathrm{GL}_2(F)$. Denote by St_F the Steinberg representation of $\mathrm{GL}_2(F)$. Namely, St_F is the unique irreducible subrepresentation in the induced representation

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(|\cdot|_F^{1/2}\boxtimes|\cdot|_F^{-1/2}).$$

Lemma 3.8. *Suppose that* $\pi=\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(|\cdot|_F^{\lambda}\boxtimes|\cdot|_F^{-\lambda}\right)$ *is spherical. Let* $\alpha=|\varpi|_F^{\lambda}$. *Then for* $n\in\mathbf{Z}$, *we have*

$$\Phi_\pi\left(\begin{pmatrix}\varpi_F^n & 0\\ 0 & 1\end{pmatrix}\right)=\frac{q_F^{-|n|/2}}{1+q_F^{-1}}\left(\alpha^{|n|}\cdot\frac{1-\alpha^{-2}q_F^{-1}}{1-\alpha^{-2}}+\alpha^{-|n|}\cdot\frac{1-\alpha^{2}q_F^{-1}}{1-\alpha^{2}}\right)$$

Proof. This is Macdonald's formula. For example, see [Bum98, Theorem 4.6.6]. □

Lemma 3.9. *Suppose* $\pi=\mathrm{St}_F\otimes\chi$, *where* χ *is a unramified quadratic character of* $F^\times$. *Then for* $n\in\mathbf{Z}$, *we have*

$$\Phi_\pi\left(\begin{pmatrix}\varpi_F^n & 0\\ 0 & 1\end{pmatrix}\right)=\chi(\varpi_F^n)q_F^{-|n|}$$

and

$$\Phi_\pi\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varpi_F^n & 0\\ 0 & 1\end{pmatrix}\right)=-\chi(\varpi_F^n)q_F^{-|n-1|}.$$

Proof. See [GJ72, §7]. □

4 The calculation of local zeta integral (I)

In this section, let $D=\mathrm{M}_2(F)$. We compute the normalized local zeta integral $I^*(\varPi,\mathbf{t})$ in Definition 2.2.

4.1 Haar measures

If $F=\mathbf{R}$, let dx be the usual Lebesgue measure on $\mathbf{R}$, and the Haar measure $d^\times x$ on $\mathbf{R}^\times$ is given by $|x|_{\mathbf{R}}^{-1}d^\times x$. The Haar measure dh on $\mathrm{GL}_2(\mathbf{R})$ is given by

$$dh=\frac{dz}{|z|_{\mathbf{R}}}\frac{dxdy}{|y|_{\mathbf{R}}^2}dk$$

for $h=z\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}k$ with $x\in\mathbf{R},y\in\mathbf{R}^\times,z\in\mathbf{R}_+^\times,k\in\mathrm{SO}(2)$, where dx,dy,dz are the usual Lebesgue measures and dk is the Haar measure on $\mathrm{SO}(2)$ such that $\mathrm{Vol}(\mathrm{SO}(2),dk)=2$.

If F is non-archimedean, let dx be the Haar measure on F so that the total volume of $\mathcal{O}_F$ is equal to 1 and let $d^\times x$ on $F^\times$ be the Haar measure on $F^\times$ so that $\mathcal{O}_F^\times$ also has volume 1. On $\mathrm{GL}_2(F)$, we let dh be the Haar measure determined by $\mathrm{Vol}\,(\mathrm{GL}_2(\mathcal{O}_F), dh) = 1$.
The measure on the quotient space $F^\times\backslash\mathrm{GL}_2(F)$ is the unique quotient measure induced from the measure dh on $\mathrm{GL}_2(F)$ and the measure $d^\times x$ on $F^\times$.

4.2 The archimedean case

Let π_j $(j = 1, 2, 3)$ be a (limit of) discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with minimal weight $k_j \geq 1$ and central character sgn^{k_j} such that

$$2\max\{k_1, k_2, k_3\} \geq k_1 + k_2 + k_3.$$

We may assume $k_3 = \max\{k_1, k_2, k_3\}$ and let $2m = k_3 - k_1 - k_2$ for some integer $m \geq 0$.

Proposition 4.1. *We have*

$$I^*(\mathit{\Pi}, \mathbf{t}) = 2^{k_1+k_2-k_3+1}.$$

Proof. Note that the L-factor given by

$$\begin{aligned} L(s, \mathit{\Pi}, r) =& \zeta_{\mathbf{C}}(s + (k_3 + k_2 + k_1 - 3)/2))\zeta_{\mathbf{C}}(s + (k_3 - k_2 - k_1 + 1)/2) \\ & \times \zeta_{\mathbf{C}}(s + (k_3 - k_2 + k_1 - 1)/2)\zeta_{\mathbf{C}}(s + (k_3 + k_2 - k_1 - 1)/2)). \end{aligned}$$

We proceed to compute $I(\mathit{\Pi}, \mathbf{t})$. By definition

$$\begin{aligned} I(\mathit{\Pi}, \mathbf{t}) &= \int_{\mathbf{R}^\times\backslash\mathrm{GL}_2(\mathbf{R})} \Phi_{\pi_1}(h)\Phi_{\pi_2}\left(h; \tilde{V}_+^m\right)\Phi_{\pi_3}(h; \tau_{\mathbf{R}})dh \\ &= \left(\frac{1}{8\pi}\right)^{2m} \int_{\mathbf{R}^\times\backslash\mathrm{GL}_2(\mathbf{R})} \Phi_{\pi_1}(h)\Phi_{\pi_2}\left(h; V_+^m\right)\Phi_{\pi_3}(h; \tau_{\mathbf{R}})dh. \end{aligned}$$

Put

$$\Phi(h) = \Phi_{\pi_1}(h)\Phi_{\pi_2}\left(h; V_+^m\right)\Phi_{\pi_3}(h; \tau_{\mathbf{R}}), \quad h \in \mathrm{GL}_2(\mathbf{R}).$$

We now focus our attention to compute the following integral:

$$I := \int_{\mathbf{R}^\times\backslash\mathrm{GL}_2(\mathbf{R})} \Phi(h)dh.$$

Note that $\Phi(h)$ is right $\mathrm{SO}(2)$-invariant. By the choice of measure, we see that the total volume of $\{\pm 1\}\backslash\mathrm{SO}(2)$ is 1, and it follows that

$$\begin{aligned} &\int_{\mathbf{R}^\times\backslash\mathrm{GL}_2(\mathbf{R})} \Phi(h)dh \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}_+} \left[\Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) + \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \frac{d^\times a}{|a|_{\mathbf{R}}} dx, \end{aligned}$$

by the Iwasawa decomposition. Since $\Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)$ vanishes when $a \in \mathbf{R}_+$, we find that

$$I = \int_{\mathbf{R}^\times\backslash \mathrm{GL}_2(\mathbf{R})} \Phi(h)dh = \int_{\mathbf{R}}\int_{\mathbf{R}_+} \Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix}\right)\frac{d^\times a}{|a|_{\mathbf{R}}}\,dx.$$

By Corollary 3.7, we have $\Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)$ is equal to $2^{2k_3}\frac{\Gamma(k_2+m)^2}{\Gamma(k_2)}$ times

$$\begin{aligned}&\mathbb{I}_{\mathbf{R}_-}(a)\sum_{i,j=0}^{m}(-2)^{i+j}\binom{m}{i}\binom{m}{j}\frac{\Gamma(k_2+i+j)}{\Gamma(k_2+i)\Gamma(k_2+j)}\\ &\times\frac{(-a)^{k_3-m+i}}{[(1-a)-\sqrt{-1}\,x]^{k_3}[(1-a)+\sqrt{-1}\,x]^{k_3-2m+i+j}}.\end{aligned}\tag{4.1}$$

By (4.1) we have

$$I = 2^{2k_3}\frac{\Gamma(k_2+m)^2}{\Gamma(k_2)}\sum_{i,j=0}^{m}(-2)^{i+j}\binom{m}{i}\binom{m}{j}\frac{\Gamma(k_2+i+j)}{\Gamma(k_2+i)\Gamma(k_2+j)}\cdot I_{i,j},$$

where for $0\le i,j\le m$,

$$\begin{aligned}I_{i,j} &:= \int_{\mathbf{R}}\int_{\mathbf{R}_+}\frac{a^{k_3-m+i-1}}{[(1+a)-\sqrt{-1}\,x]^{k_3}[(1+a)+\sqrt{-1}\,x]^{k_3-2m+i+j}}\,d^\times a\,dx\\ &= \int_{\mathbf{R}_+}\frac{a^{k_3-m+i-1}}{(1+a)^{2k_3-2m+i+j-1}}\,d^\times a\\ &\times\int_{\mathbf{R}}\frac{dx}{[1+\sqrt{-1}\,x]^{k_3-2m+i+j}[1-\sqrt{-1}\,x]^{k_3}}\\ &= 2^{2-(2k_3-2m+i+j)}\pi\frac{\Gamma(k_3-m+i-1)\Gamma(k_3-m+j)}{\Gamma(k_3-2m+i+j)\Gamma(k_3)}.\end{aligned}$$

The last equality follows from the following lemma.

Lemma 4.2. *For $|arg\, z|<\pi$, $0<Re(\beta)<Re(\alpha)$, we have*

$$\int_{\mathbf{R}_+}\frac{t^\beta}{(t+z)^\alpha}d^\times t = z^{\beta-\alpha}\cdot\frac{\Gamma(\alpha-\beta)\Gamma(\beta)}{\Gamma(\alpha)}.$$

For $Re(\alpha+\beta)>1$, we have

$$\int_{\mathbf{R}}\frac{dx}{(1+\sqrt{-1}\,x)^\alpha(1-\sqrt{-1}\,x)^\beta} = 2^{2-\alpha-\beta}\cdot\pi\cdot\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}.$$

Proof. These are [Ike98, Lemmas 2.4 and 2.5] ☐

Thus we obtain

$$I = 2^{2+2m}\pi \frac{\Gamma(k_2+m)^2}{\Gamma(k_2)\Gamma(k_3)} \sum_{i,j=0}^{m} (-1)^{i+j} \binom{m}{i}\binom{m}{j} \times \frac{\Gamma(k_2+i+j)}{\Gamma(k_2+i)\Gamma(k_2+j)} \cdot \frac{\Gamma(k_3-m+i-1)\Gamma(k_3-m+j)}{\Gamma(k_3-2m+i+j)}.$$

To simply the above expression of I, we need one more combinatorial identity from [Orl87, Lemma 3].

LEMMA 4.3. *Let $N \in \mathbf{Z}_{\geq 0}$ and $t, \alpha, \beta \in \mathbf{C}$. Then*

$$\Gamma(\alpha+N)\sum_{i=0}^{N}(-1)^i \binom{N}{i} \frac{\Gamma(t+i)}{\Gamma(\alpha+i)} \cdot \frac{\Gamma(t+\beta+\alpha+N-1+i)}{\Gamma(2t+\beta+i)} = (-1)^N \frac{\Gamma(t)\Gamma(t+\beta+\alpha+N-1)}{\Gamma(2t+\beta+N)} \cdot \frac{\Gamma(t+\beta+N)}{\Gamma(t+\beta)} \cdot \frac{\Gamma(t-\alpha+1)}{\Gamma(t-\alpha-N+1)}.$$

Now we write

$$I = 2^{2+2m}\pi \frac{\Gamma(k_2+m)}{\Gamma(k_1)\Gamma(k_2)} \sum_{j=0}^{m}(-1)^j \binom{m}{j} \frac{\Gamma(k_3-m+j)}{\Gamma(k_2+j)} \cdot I',$$

where

$$I' = \Gamma(k_2+m)\sum_{i=0}^{m}(-1)^i \binom{m}{i} \frac{\Gamma(k_2+j+i)}{\Gamma(k_2+i)} \cdot \frac{\Gamma(k_3-m-1+i)}{\Gamma(k_3-2m+j+i)}.$$

Applying Lemma 4.3 to I' with $t = k_2+j$, $\alpha = k_2$ and $\beta = k_3-2m-2k_2-j$, we find that

$$I' = (-1)^m \cdot \frac{\Gamma(k_2+j)\Gamma(k_3-m-1)}{\Gamma(k_3-m+j)} \cdot \frac{\Gamma(k_3-k_2-m)}{\Gamma(k_3-k_2-2m)} \cdot \frac{\Gamma(j+1)}{\Gamma(j-m+1)}.$$

It follows that

$$I = (-1)^m 2^{2+2m}\pi \frac{\Gamma(k_3-m-1)\Gamma(k_2+m)\Gamma(k_1+m)}{\Gamma(k_3)\Gamma(k_2)\Gamma(k_1)} \times \sum_{j=0}^{m}(-1)^j \binom{m}{j} \frac{\Gamma(1+j)}{\Gamma(1-m+j)}.$$

Applying Lemma 3.5, we obtain

$$I = 2^{2+2m}\pi \frac{\Gamma(k_3-m-1)\Gamma(k_2+m)\Gamma(k_1+m)\Gamma(m+1)}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)}.$$

Therefore we find that

$$\begin{aligned} I(\mathit{\Pi},\mathbf{t}) &= \left(\frac{1}{8\pi}\right)^{2m} I \\ &= 2^{2-4m}\pi^{1-2m}\frac{\Gamma(k_3-m-1)\Gamma(k_2+m)\Gamma(k_1+m)\Gamma(m+1)}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)}, \end{aligned}$$

and the proposition follows. □

We deduce a consequence from Proposition 4.1. Let m_1, m_2 be two non-negative integers such that $m_1 + m_2 = m$. Put

$$\mathbf{t}_{m_1,m_2} = \left(\tilde{V}_+^{m_1} \otimes \tilde{V}_+^{m_2} \otimes 1, (1,1,\tau_{\mathbf{R}})\right) \in \mathfrak{U}_E \times \mathrm{O}(2,E).$$

Then our original element $\mathbf{t}$ is $\mathbf{t}_{0,m}$. Put

$$I^*(\mathit{\Pi};\mathbf{t}_{m_1,m_2}) = \frac{L(1,\mathit{\Pi},\mathrm{Ad})}{\zeta_{\mathbf{R}}(2)^2 L(1/2,\mathit{\Pi},r)} \cdot I(\mathit{\Pi},\mathbf{t}_{m_1,m_2}),$$

where

$$I(\mathit{\Pi},\mathbf{t}_{m_1,m_2}) = \int_{\mathbf{R}^\times\backslash \mathrm{GL}_2(\mathbf{R})} \Phi_{\pi_1}\left(h;\tilde{V}_+^{m_1}\right)\Phi_{\pi_2}\left(h;\tilde{V}_+^{m_2}\right)\Phi_{\pi_3}(h;\tau_{\mathbf{R}})dh$$

Then $I^*(\mathit{\Pi},\mathbf{t})$ in Definition 2.2 is nothing but $I^*(\mathit{\Pi},\mathbf{t}_{0,m})$.

Corollary 4.4. *Notation is as above. We have*

$$I^*(\mathit{\Pi},\mathbf{t}_{m_1,m_2}) = I^*(\mathit{\Pi},\mathbf{t})$$

for every non-negative integers m_1, m_2 such that $m_1 + m_2 = m$.

Proof. This is in fact an easy consequence form the multiplicity one result of local trilinear forms, Proposition 4.1 together with the local Rankin-Selberg integral. More precisely, let $\mu_2 = |\cdot|_{\mathbf{R}}^{(k_2-1)/2}$ and $\nu_2 = |\cdot|_{\mathbf{R}}^{(1-k_2)/2}\mathrm{sgn}^{k_2}$ be two characters of $\mathbf{R}^\times$. Then π_2 can be realized as the unique irreducible subrepresentation of $\mathrm{Ind}_{B(\mathbf{R})}^{\mathrm{GL}_2(\mathbf{R})}(\mu_2 \boxtimes \nu_2)$ which we denote by $\mathrm{Ind}_{B(\mathbf{R})}^{\mathrm{GL}_2(\mathbf{R})}(\mu_2 \boxtimes \nu_2)_0$. For every non-negative integer n, we let $f_{\pi_2}^n \in \mathrm{Ind}_{B(\mathbf{R})}^{\mathrm{GL}_2(\mathbf{R})}(\mu_2 \boxtimes \nu_2)_0$ be the element characterized by requiring

$$f_{\pi_2}^n\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{i(k_2+2n)\theta}.$$

We have the following relation, which can be found in [JL70, Lemma 5.6 (iii)]

$$\rho\left(\tilde{V}_+\right) f_{\pi_2}^n = 2(k_2+n) f_{\pi_2}^{n+1}.$$

Inductively we find that

$$\rho\left(\tilde{V}_+^{\ell}\right) f_{\pi_2}^n = c(\pi_2, n, \ell) f_{\pi_2}^{n+\ell},$$

where

$$c(\pi_2, n, \ell) = 2^{\ell} \frac{\Gamma(k_2+n+\ell)}{\Gamma(k_2+n)}, \tag{4.2}$$

for every $\ell \geq 0$.

Let $\Psi : \mathcal{W}(\pi_1, \psi) \boxtimes \mathrm{Ind}_{B(\mathbf{R})}^{\mathrm{GL}_2(\mathbf{R})}(\mu_2 \boxtimes \nu_2)_0 \boxtimes \mathcal{W}(\pi_3, \psi) \to \mathbf{C}$ be the local Rankin-Selberg integral defined by

$$\Psi(W_1 \otimes f_2 \otimes W_3) = \int_{\mathbf{R}^\times N(\mathbf{R}) \backslash \mathrm{GL}_2(\mathbf{R})} W_1(\tau_{\mathbf{R}} g) W_3(g) f_2(g) dg,$$

for $W_1 \in \mathcal{W}(\pi_1, \psi)$, $f_2 \in \mathrm{Ind}_{B(\mathbf{R})}^{\mathrm{GL}_2(\mathbf{R})}(\mu_2 \boxtimes \nu_2)_0$ and $W_3 \in \mathcal{W}(\pi_3, \psi)$. Here

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 \right\}.$$

One check easily that this integral converges absolutely and certainly it defines a $\mathrm{GL}_2(\mathbf{R})$-invariant trilinear form. From the multiplicity one result of such trilinear form and the fact that $I^*(\mathit{\Pi}, \mathbf{t}) \neq 0$, one can deduce that following equality easily

$$\frac{I^*\left(\mathit{\Pi}, \mathbf{t}_{m_1,m_2}\right)}{I^*\left(\mathit{\Pi}, \mathbf{t}\right)} = \left(\frac{c(\pi_2, m_1, m_2)}{c(\pi_2, 0, m)}\right)^2 \left(\frac{\Psi\left(W_{\pi_1}^{m_1} \otimes f_{\pi_2}^{m_2} \otimes \rho(\tau_{\mathbf{R}}) W_{\pi_3}\right)}{\Psi\left(W_{\pi_1} \otimes f_{\pi_2}^{m} \otimes \rho(\tau_{\mathbf{R}}) W_{\pi_3}\right)}\right)^2. \tag{4.3}$$

Recall that $W_{\pi_1}^n = \rho\left(\tilde{V}_+^n\right) W_{\pi_1}$ for every $n \geq 0$. Our task now is to compute the ratio of these two Rankin-Selberg integrals. Since we can let m_1, m_2 vary, it suffices to compute the numerator. Applying Lemma 3.3 and Lemma 3.5, we find that the numerator is

$$\begin{aligned}
&\Psi\left(W_{\pi_1}^{m_1} \otimes f_{\pi_2}^{m_2} \otimes \rho(\tau_{\mathbf{R}}) W_{\pi_3}\right) \\
&= \int_{\mathbf{R}^\times} W_{\pi_1}^{m_1}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right) W_{\pi_3}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|_{\mathbf{R}}^{\frac{k_2}{2}-1} d^\times a \\
&= 2^{k_1+k_3+m_1} \sum_{j=0}^{m_1} (-4\pi)^j \frac{\Gamma(k_1+m_1)}{\Gamma(k_1+j)} \binom{m_1}{j} \int_0^\infty a^{\frac{k_1+k_2+k_3}{2}+j-1} e^{-4\pi a} d^\times a \\
&= 2^{k_1+k_3+m_1} (4\pi)^{1-\frac{k_1+k_2+k_3}{2}} \Gamma(k_1+m_1) \sum_{j=0}^{m_1} (-1)^j \binom{m_1}{j} \frac{\Gamma\left(\frac{k_1+k_2+k_3}{2}+j-1\right)}{\Gamma(k_1+j)} \\
&= (-1)^{m_1} 2^{k_1+k_3+m_1} (4\pi)^{1-\frac{k_1+k_2+k_3}{2}} \frac{\Gamma\left(\frac{k_1+k_2+k_3}{2}-1\right)\Gamma(k_2+m_1+m_2)}{\Gamma(k_2+m_2)}.
\end{aligned}$$

By letting $m_1 = 0$ and $m_2 = m$, we obtain the value of denominator. Combining with equation (4.2), we find that the right hand side of the equation (4.3) is equal to 1. The corollary follows. ☐

4.3 The non-archimedean case

Let F be a non-archimedean local field. Write $\varpi = \varpi_F$ and $q = q_F$ for simplicity. Recall that we have assumed

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathit{\Pi}, \mathbf{C}) \neq \{0\}. \tag{4.4}$$

According to the results of Prasad [Pra90] and [Pra92] and our assumption on $\mathit{\Pi}$, (4.4) holds for the following cases. (i) Suppose $E = F \times F \times F$ so that $\mathit{\Pi} = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$. Then (i-a) one of π_1, π_2, π_3 is spherical; (i-b) $\pi_j = \mathrm{St}_F \otimes \chi_j$ are unramified special representations for $j = 1, 2, 3$ with $\chi_1\chi_2\chi_3(\varpi) = -1$. (ii) Suppose $E = K \times F$ so that $\mathit{\Pi} = \pi' \boxtimes \pi$. Then (ii-a) π is spherical; (ii-b) π' is spherical, $\pi = \mathrm{St}_F \otimes \chi$ is a unramified special representation, K/F is ramified and $\chi(\varpi) = -1$; (ii-c) $\pi' = \mathrm{St}_K \otimes \chi'$, $\pi = \mathrm{St}_F \otimes \chi$ are unramified special representations, K/F is ramified or K/F is unramified and $\chi'\chi(\varpi) = 1$. (iii) Suppose E is a field. Then (iii-a) $\mathit{\Pi}$ is spherical; (iii-b) $\mathit{\Pi} = \mathrm{St}_E \otimes \chi$ is a unramified special representation with $\chi(\varpi) = -1$.
We say that E is unramified over F if either $E = F \times F \times F$, or $E = K \times F$, where K is the unramified extension over F, or E is the unramified cubic extension over F. The evaluation of $I^*(\mathit{\Pi}, \mathbf{t})$ has been carried out in the following cases.

Proposition 4.5.

(1) Suppose E/F is unramified and $\mathit{\Pi}$ is spherical. Then we have

$$I^*(\mathit{\Pi}, \mathbf{t}) = 1.$$

(2) Suppose $E = F \times F \times F$ and $\pi_j = \mathrm{St}_F \otimes \chi_j$, where χ_j are unramified quadratic characters of $F^\times$ for $j = 1, 2, 3$. Then we have

$$I^*(\mathit{\Pi}, \mathbf{t}) = 2q^{-1}(1 + q^{-1}).$$

(3) Suppose $E = F \times F \times F$ and one of π_j $(j = 1, 2, 3)$ is spherical and the other two are unramified special. Then we have

$$I^*(\mathit{\Pi}, \mathbf{t}) = q^{-1}.$$

Here $I^*(\mathit{\Pi}, \mathbf{t})$ is defined in §2.4.

Proof. Part (1) is [Ich08, Lemma 2.2], (2) is in [II10, Section 7] and (3) is a result of [Nel11, Lemma 4.4]. □

We proceed to compute $I^*(\mathit{\Pi}, \mathbf{t})$ in the remaining cases. For $\Phi \in L^1(F^\times\backslash \mathrm{GL}_2(F))$ such that $\Phi(khk') = \Phi(h)$ for every $h \in \mathrm{GL}_2(F)$ and $k, k' \in K_0(\varpi)$, where

$$K_0(\varpi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \mid c \in \varpi\mathcal{O}_F \right\},$$

we have the integration formula

$$\int_{F^\times\backslash \mathrm{GL}_2(F)} \Phi(h)dh = (1+q)^{-1}\sum_{n\in\mathbf{Z}} \Phi\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1\end{pmatrix}\right) q^{|n|} + (1+q)^{-1}\sum_{n\in\mathbf{Z}} \Phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} \varpi^n & 0 \\ 0 & 1\end{pmatrix}\right) q^{|n-1|} \tag{4.5}$$

(*cf.* [II10, Section 7]).

PROPOSITION 4.6. *Let $E = F\times F\times F$. Suppose one of π_j is unramified special and the other two are spherical. Then we have*

$$I^*(\mathit{\Pi},\mathbf{t}) = q^{-1}(1+q^{-1})^{-1}.$$

Proof. In this case, the L-factor is given by

$$L(s,\Pi,r) = (1-\chi(\varpi)\alpha\beta q^{-s-1/2})^{-1}(1-\chi(\varpi)\alpha\beta^{-1}q^{-s-1/2})^{-1} \times (1-\chi(\varpi)\alpha^{-1}\beta q^{-s-1/2})^{-1}(1-\chi(\varpi)\alpha^{-1}\beta^{-1}q^{-s-1/2})^{-1}.$$

We continue to compute $I(\mathit{\Pi},\mathbf{t})$. Assume $\pi_1 = \mathrm{St}_F\otimes\chi$ for some unramified quadratic character χ of $F^\times$, and

$$\pi_j = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(|\cdot|_F^{\lambda_j}\boxtimes|\cdot|_F^{-\lambda_j}\right)$$

for $j=2,3$. Let $\alpha = |\varpi|_F^{\lambda_2}$ and $\beta = |\varpi|_F^{\lambda_3}$. Then we have

$$I(\mathit{\Pi},\mathbf{t}) = \int_{F^\times\backslash\mathrm{GL}_2(F)} \Phi(h)dh,$$

where

$$\Phi(h) = \Phi_{\pi_1}(h)\Phi_{\pi_2}(h)\Phi_{\pi_3}\left(h;\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1\end{pmatrix}\right),\quad h\in\mathrm{GL}_2(F).$$

By (4.5), Lemma 3.8 and Lemma 3.9, we find that

$$\begin{aligned}
&I(\mathit{\Pi},\mathbf{t})\\
&= (1+q)^{-1}\sum_{n=-\infty}^{\infty}\Phi\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1\end{pmatrix}\right) q^{|n|}\\
&+ (1+q)^{-1}\sum_{n=-\infty}^{\infty}\Phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} \varpi^n & 0 \\ 0 & 1\end{pmatrix}\right) q^{|n-1|}\\
&= (1+q)^{-1}\cdot\frac{(1-q^{-1})}{(1+q^{-1})}\cdot(1-\alpha^2q^{-1})(1-\alpha^{-2}q^{-1})(1-\beta^2q^{-1})(1-\beta^{-2}q^{-1})\\
&\times(1-\chi(\varpi)\alpha\beta q^{-1})^{-1}(1-\chi(\varpi)\alpha\beta^{-1}q^{-1})^{-1}\\
&\times(1-\chi(\varpi)\alpha^{-1}\beta q^{-1})^{-1}(1-\chi(\varpi)\alpha^{-1}\beta^{-1}q^{-1})^{-1}.
\end{aligned}$$

This completes the proof. □

Proposition 4.7. *Let $E = K \times F$ and π be a spherical representation of $\mathrm{GL}_2(F)$.*

(1) If K/F is ramified and π' is spherical, then we have

$$I^*(\Pi, \mathbf{t}) = 1.$$

(2) If π' is unramified special, then we have

$$I^*(\Pi, \mathbf{t}) = \begin{cases} q^{-1}(1+q^{-1})^{-2}(1+q^{-2}) & \text{if } K/F \text{ is unramified,} \\ q^{-1}(1+q^{-1})^{-1} & \text{if } K/F \text{ is ramified.} \end{cases}$$

Proof. Let

$$\pi = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(|\cdot|_F^{\lambda} \boxtimes |\cdot|_F^{-\lambda}\right), \quad \beta = |\varpi|_F^{\lambda}.$$

We begin with (1). Let

$$\pi' = \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}\left(|\cdot|_K^{\lambda'} \boxtimes |\cdot|_K^{-\lambda'}\right), \quad \alpha = |\varpi_K|_K^{\lambda'}.$$

For a non-negative integer n, let X_n be the image of

$$\mathrm{GL}_2(\mathcal{O}_F)\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_F)$$

in $F^\times \backslash \mathrm{GL}_2(F)$. Note that

$$\mathrm{vol}(X_n, dh) = \begin{cases} 1 & \text{if } n = 0, \\ q^n(1+q^{-1}) & \text{if } n \geq 1. \end{cases}$$

By Lemma 3.8, we have

$$\begin{aligned} &I(\Pi, \mathbf{t}) \\ &= \sum_{n=0}^{\infty} \Phi_{\pi'}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) \Phi_{\pi}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) \mathrm{vol}(X_n, dh) \\ &= \frac{(1-\alpha^2 q^{-1})(1-\alpha^{-2}q^{-1})(1+\beta q^{-1/2})(1+\beta^{-1}q^{-1/2})}{(1-\alpha^2\beta q^{-1/2})(1-\alpha^{-2}\beta q^{-1/2})(1-\alpha^2\beta^{-1}q^{-1/2})(1-\alpha^{-2}\beta^{-1}q^{-1/2})}. \end{aligned}$$

Recall that the L-factor is given by

$$\begin{aligned} L(s, \Pi, r) = {} & (1-\alpha\beta p^{-s})^{-1}(1-\alpha\beta^{-1}p^{-s})^{-1}(1-\beta p^{-s})^{-1} \\ & \times (1-\beta^{-1}p^{-s})^{-1}(1-\alpha^{-1}\beta p^{-s})^{-1}(1-\alpha^{-1}\beta^{-1}p^{-s})^{-1}. \end{aligned}$$

This shows (1).

Now we consider (2). Let $\pi' = \mathrm{St}_K \otimes \chi'$ for some unramified quadratic character χ' of $K^\times$. Suppose K is unramified over F. By definition

$$I\left(\Pi, \mathbf{t}\right) = \int_{F^\times \backslash \mathrm{GL}_2(F)} \Phi_{\pi'}(h)\Phi_\pi(h)dh.$$

Applying (4.5), Lemma 3.8 and Lemma 3.9,

$$\begin{aligned} I\left(\Pi, \mathbf{t}\right) &= (1+q)^{-1} \sum_{n=-\infty}^{\infty} \chi'(\varpi)^n \Phi_\pi \left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \right) \left\{ q^{-|n|} - q^{-|n-1|} \right\} \\ &= \frac{(1-q^{-1})(1+q^{-2})}{(1+q^{-1})} \cdot \frac{(1-\chi'(\varpi)\alpha q^{-1/2})(1-\chi'(\varpi)\alpha^{-1}q^{-1/2})}{(1-\chi'(\varpi)\alpha q^{-3/2})(1-\chi'(\varpi)\alpha^{-1}q^{-3/2})}. \end{aligned}$$

Suppose K is ramified over F. Similar calculations shows

$$I\left(\Pi, \mathbf{t}\right) = q^{-1}\frac{(1-q^{-1})}{(1+q^{-1})} \cdot \frac{(1-\alpha^2 q^{-1})((1-\alpha^{-2}q^{-1}))}{(1-\alpha^2 q^{-3/2})(1-\alpha^{-2}q^{-3/2})}.$$

Finally, if K/F is unramified, we have

$$\begin{aligned} L(s, \Pi, r) = &(1+\chi'(\varpi)\alpha q^{-s})^{-1}(1-\chi'(\varpi)\alpha q^{-s-1})^{-1} \\ &\times (1+\chi'(\varpi)\alpha^{-1}q^{-s})^{-1}(1-\chi'(\varpi)\alpha^{-1}q^{-s-1})^{-1}, \end{aligned}$$

while if K/F is ramified,

$$L(s, \Pi, r) = (1-\alpha q^{-s-1})^{-1}(1-\alpha^{-1}q^{-s-1})^{-1}.$$

This shows (2) and our proof is complete. □

Proposition 4.8. *Let $E = K \times F$ and $\pi = \mathrm{St}_F \otimes \chi$, where χ is a unramified quadratic character of $F^\times$.*

(1) If π' is spherical, $\chi(\varpi) = -1$ and K/F is ramified, then we have

$$I^*\left(\Pi, \mathbf{t}\right) = 2q^{-1}(1+q^{-1})^{-1}.$$

(2) If $\pi' = \mathrm{St}_K \otimes \chi'$, where χ' is an unramified quadratic character of $K^\times$, then we have

$$I^*(\Pi, \mathbf{t}) = \begin{cases} 2q^{-1}(1+q^{-1})^{-1}(1+q^{-2}) & \text{if } K/F \text{ is unramified and } \chi'\chi(\varpi) = 1, \\ q^{-1} & \text{if } K/F \text{ is ramified.} \end{cases}$$

Proof. We first consider (1). By definition,

$$I\left(\Pi, \mathbf{t}\right) = \int_{F^\times \backslash \mathrm{GL}_2(F)} \Phi(h)dh,$$

where

$$\Phi(h) = \Phi_{\pi'}\left(h; \begin{pmatrix} \varpi_K^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \Phi_\pi(h), \quad h \in \mathrm{GL}_2(F).$$

By (4.5), Lemma 3.8 and Lemma 3.9, we have

$$\begin{aligned} I(\mathit{\Pi}, \mathbf{t}) &= (1+q)^{-1}(1-\chi(\varpi)) \sum_{n=-\infty}^{\infty} \chi(\varpi)^n \Phi_{\pi'}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= 2q^{-1}\frac{(1-q^{-1})}{(1+q^{-1})^2} \cdot \frac{(1+\alpha^2 q^{-1})(1+\alpha^{-2}q^{-1})}{(1-\alpha^2 q^{-1})(1-\alpha^{-2}q^{-1})}. \end{aligned}$$

Notice that

$$\begin{aligned} &L(s, \mathit{\Pi}, r) \\ &= (1-\chi(\varpi)\alpha^2 q^{-s-1/2})^{-1}(1-\chi(\varpi)\alpha^{-2}q^{-s-1/2})^{-1}(1-\chi(\varpi)q^{-s-1/2})^{-1}. \end{aligned}$$

This shows (1).
Now we consider (2). By definition,

$$I(\mathit{\Pi}, \mathbf{t}) = \int_{F^\times \backslash \mathrm{GL}_2(F)} \Phi(h)dh,$$

where

$$\Phi(h) = \Phi_{\pi'}(h)\Phi_\pi(h), \quad h \in \mathrm{GL}_2(F).$$

Suppose K is umramified over F. Applying (4.5), Lemma 3.9, we find that

$$\begin{aligned} I(\mathit{\Pi}, \mathbf{t}) &= (1+q)^{-1} \sum_{n=-\infty}^{\infty} \Phi\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) q^{|n|} \\ &\quad + (1+q)^{-1} \sum_{n=-\infty}^{\infty} \Phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) q^{|n-1|} \\ &= (1+q)^{-1}(1+\chi'\chi(\varpi))\frac{(1+\chi'\chi(\varpi)q^{-2})}{(1-\chi'\chi(\varpi)q^{-2})}. \end{aligned}$$

When K is ramified over F, a similar calculation shows that

$$I^*(\mathit{\Pi}, \mathbf{t}) = q^{-1}\frac{(1+\chi'\chi(\varpi)q^{-1})}{(1-\chi'\chi(\varpi)q^{-2})}.$$

Note that the L-factor $L(s, \mathit{\Pi}, r)$ is equal to

$$(1-\chi'\chi(\varpi)q^{-s-3/2})^{-1}(1-q^{-2s-1})^{-1}$$

or

$$(1-\chi(\varpi)q^{-s-3/2})^{-1}(1-\chi(\varpi)q^{-s-1/2})^{-1}$$

according to E/F is unramified or ramified. This proves the proposition. □

Proposition 4.9. *Let E is a field.*

(1) If E/F is ramified and $\mathit{\Pi}$ is spherical, then we have

$$I^*(\mathit{\Pi}, \mathbf{t}) = 1$$

(2) If $\mathit{\Pi} = \mathrm{St}_E \otimes \chi$, where χ is the non-trivial unramified quadratic character of $E^\times$, then we have

$$I(\mathit{\Pi}, \mathbf{t}) = \begin{cases} 2q^{-1}(1+q^{-1})^{-1}(1-q^{-1}+q^{-2}) & \text{if } E/F \text{ is unramified,} \\ 2q^{-1}(1+q^{-1})^{-1} & \text{if } E/F \text{ is ramified.} \end{cases}$$

Proof. Suppose $\mathit{\Pi}$ is spherical and E/F is ramified. Let

$$\mathit{\Pi} = \mathrm{Ind}_{B(E)}^{\mathrm{GL}_2(E)}\left(|\cdot|_E^{\lambda} \boxtimes |\cdot|_E^{-\lambda}\right), \quad \alpha = |\varpi_E|_E^{\lambda}.$$

For a non-negative integer n, let X_n be the image of

$$\mathrm{GL}_2(\mathcal{O}_F)\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_F)$$

in $F^\times\backslash\mathrm{GL}_2(F)$. Note that

$$\mathrm{Vol}(X_n, dh) = \begin{cases} 1 & \text{if } n = 0, \\ q^n(1+q^{-1}) & \text{if } n \geq 1. \end{cases}$$

Applying Lemma 3.8,

$$\begin{aligned} I(\mathit{\Pi}, \mathbf{t}) &= \sum_{n=0}^{\infty} \Phi_{\mathit{\Pi}}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right)\mathrm{Vol}(X_n, dh) \\ &= \frac{(1-q^{-1})(1+\alpha q^{-1/2})(1+\alpha^{-1}q^{-1/2})}{(1-\alpha^3 q^{-1/2})(1-\alpha^{-3}q^{-1/2})}. \end{aligned}$$

Notice that

$$L(s, \Pi, r) = (1-\alpha^3 p^{-s})^{-1}(1-\alpha p^{-s})^{-1}(1-\alpha^{-1}p^{-s})^{-1}(1-\alpha^{-3}p^{-s})^{-1}.$$

This proves (2).

Suppose $\mathit{\Pi} = \mathrm{St}_E \otimes \chi$, where χ is the non-trivial unramified quadratic character of $E^\times$. If E/F is unramified,

$$I(\mathit{\Pi}, \mathbf{t}) = \int_{F^\times\backslash\mathrm{GL}_2(F)} \Phi_{\mathit{\Pi}}(h)dh.$$

By (4.5) and Lemma 3.9, we obtain

$$I\left(\mathit{\Pi},\mathbf{t}\right)=(1+q)^{-1}\sum_{n=-\infty}^{\infty}\Phi_{\mathit{\Pi}}\left(\begin{pmatrix}\varpi^n & 0\\ 0 & 1\end{pmatrix}\right)q^{|n|}$$

$$+(1+q)^{-1}\sum_{n=-\infty}^{\infty}\Phi_{\mathit{\Pi}}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varpi^n & 0\\ 0 & 1\end{pmatrix}\right)q^{|n-1|}$$
$$=(1+q)^{-1}(1-\chi(\varpi))\frac{(1+\chi(\varpi)q^{-2})}{(1-\chi(\varpi)q^{-2})}.$$

When E/F is ramified, similar calculations show

$$I^*(\mathit{\Pi},\mathbf{t})=\frac{2q^{-1}}{(1+q^{-1})}.$$

On the other hand, the L-factor $L(s,\Pi,r)$ is equal to

$$(1-\chi(\varpi)q^{-s-3/2})^{-1}(1+\chi(\varpi)q^{-s-1/2}+q^{-2s-1})^{-1}$$

or

$$(1-\chi(\varpi)q^{-s-3/2})^{-1}$$

according to E/F is unramified or ramified. This completes the proof. □

5 The calculation of local zeta integral (II)

The purpose of this section is to compute the normalized zeta integral $I^*(\mathit{\Pi},\mathbf{t})$ in Definition 2.2 when D is a division algebra over F.

5.1 Haar measures

Haar measures on F and $F^\times$ are the same as in §4.1. We describe the choice of Haar measures on $D^\times(F)$. When $F=\mathbf{R}$, let dh be the Haar measure on $D^\times(\mathbf{R})$ such that $\mathrm{Vol}(D^\times(\mathbf{R})/\mathbf{R}^\times, dh/d^\times t)=1$, where $d^\times t=|t|_{\mathbf{R}}^{-1}dt$ and dt is the usual Lebesgue measure on $\mathbf{R}$. When F is non-archimedean, let $\mathcal{O}_D$ be its maximal compact subring. Then dh is chosen so that $\mathrm{Vol}\left(\mathcal{O}_D^\times, dh\right)=1$.
In any cases, the measure on the quotient space $F^\times\backslash D^\times(F)$ is the unique quotient measure induced from the measure dh on $D^\times(F)$ and the measure $d^\times x$ on $F^\times$.

5.2 Embeddings

We fix various embeddings in this section. Following results depend on these embeddings. When $F=\mathbf{R}$, we embedded $D(\mathbf{R})$ in $\mathrm{M}_2(\mathbf{C})$ in the usual way.

More precisely, we let

$$D(\mathbf{R}) = \mathbf{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{M}_2(\mathbf{C}) \right\}.$$

When F is non-archimedean and $E = K \times F$, we have $D(E) = \mathrm{M}_2(K) \times D(F)$ and we fix an embedding:

$$\iota : D(F) \to \mathrm{M}_2(K),$$

so that

$$\iota(D(F)) = \left\{ \begin{pmatrix} \alpha & \beta \\ \omega\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in K \right\},$$

where $x \mapsto \bar{x}$ is the non-trivial Galois action on $x \in K$, and ω is either ϖ or a unit u such that $F(\sqrt{u})$ is the unramified extension over F, according to K is unramified or ramified over F. We then identify $D(F)$ with its image under the embedding ι. The maximal order $\mathcal{O}_D$ in $D(F)$ is then

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \omega\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathcal{O}_K \right\}.$$

Let

$$\varpi_D = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \quad \text{or} \quad \varpi_D = \begin{pmatrix} \varpi_K & 0 \\ 0 & -\varpi_K \end{pmatrix},$$

according to K is unramified or ramified over F. We have

$$F^\times \backslash D^\times(F) = \left(\mathcal{O}_F^\times \backslash \mathcal{O}_D^\times\right) \sqcup \varpi_D \left(\mathcal{O}_F^\times \backslash \mathcal{O}_D^\times\right). \tag{5.1}$$

Note that

$$\mathrm{Vol}(\mathcal{O}_F^\times \backslash \mathcal{O}_D^\times, dh) = 1,$$

according to our choice of measures.

5.3 The archimedean case

In this case, we have following realizations

$$\left(\pi_j, V_{\pi_j}\right) = \left(\rho_{k_j}, \mathcal{L}_{k_j}(\mathbf{C})\right)$$

for $j = 1, 2, 3$, where

$$\mathcal{L}_{k_j}(\mathbf{C}) = \bigoplus_{n_j=0}^{k_j-2} \mathbf{C} \cdot X_j^{n_j} Y_j^{k_j-2-n_j}$$

and

$$\rho_{k_j}(g)P(X_j, Y_j) = P((X_j, Y_j)g)\mathrm{det(g)}^{-k_j/2-1},$$

for $g \in D^\times(\mathbf{R})$ and $P(X_j, Y_j) \in \mathcal{L}_{k_j-2}(\mathbf{C})$. The representation space of Π is given by

$$V_\Pi = \mathcal{L}_{k_1}(\mathbf{C}) \otimes \mathcal{L}_{k_2}(\mathbf{C}) \otimes \mathcal{L}_{k_3}(\mathbf{C}). \tag{5.2}$$

Recall the new line $V_\Pi^{\rm new}$ in this case is the one-dimensional subspace fixed by $D^\times(\mathbf{R})$. This is a consequence of the uniqueness of the trilinear form [Pra92, Theorem 9.3]. Let $\mathbf{P}_{\underline{k}}$ be the distinguished vector in $V_\Pi^{\rm new}$ defined by

$$\mathbf{P}_{\underline{k}} = \det\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{k_3^*} \otimes \det\begin{pmatrix} X_2 & X_3 \\ Y_2 & Y_3 \end{pmatrix}^{k_1^*} \otimes \det\begin{pmatrix} X_3 & X_1 \\ Y_3 & Y_1 \end{pmatrix}^{k_2^*} \tag{5.3}$$

where $k_3^* = (k_1+k_2-k_3-2)/2$, $k_1^* = (k_2+k_3-k_1-2)/2$ and $k_2^* = (k_1+k_3-k_2-2)/2$. Its clear that $\mathbf{P}_{\underline{k}}$ is non-zero and invariant by $D^\times(\mathbf{R})$. Therefore, we have

$$V_\Pi^{\rm new} = \mathbf{C}\cdot \mathbf{P}_{\underline{k}}.$$

Let $\langle\, ,\, \rangle_{k_j}$ be the $D^\times(\mathbf{R})$-invariant bilinear pairing on $\mathcal{L}_{k_j-2}(\mathbf{C})$ defined by

$$\begin{aligned} &\langle X_j^{n_j} Y_j^{k_j-2-n_j}, X_j^{m_j} Y_j^{k_j-2-m_j} \rangle_{k_j} \\ &= \begin{cases} (-1)^{n_j} \begin{pmatrix} k_j-2 \\ n_j \end{pmatrix}^{-1} & \text{if } n_j+m_j = k_j-2, \\ 0 & \text{if } n_j+m_j \neq k_j-2, \end{cases} \end{aligned} \tag{5.4}$$

for $0 \leq n_j, m_j \leq k_j - 2$. Let $\langle\, ,\, \rangle_{\underline{k}}$ be the $D^\times(E)$-invariant pairing on V_Π given by

$$\langle\, ,\, \rangle_{\underline{k}} = \langle\, ,\, \rangle_{k_1} \otimes \langle\, ,\, \rangle_{k_2} \otimes \langle\, ,\, \rangle_{k_3}. \tag{5.5}$$

In this case, the normalized local zeta integral $I^*(\Pi, \mathbf{t})$ in Definition 2.2 is equal to

$$I^*(\Pi, \mathbf{t}) = \frac{\zeta_F(2)}{\zeta_E(2)} \cdot \frac{L(1, \Pi', {\rm Ad})}{L(1/2, \Pi', r)} \cdot \langle \mathbf{P}_{\underline{k}}, \mathbf{P}_{\underline{k}} \rangle_{\underline{k}}, \tag{5.6}$$

where Π' is the Jacquet-Langlands lift of Π to $\mathrm{GL}_2(\mathbf{R})$. We proceed to compute the value $\langle \mathbf{P}_{\underline{k}}, \mathbf{P}_{\underline{k}} \rangle_{\underline{k}}$. Let ℓ be the linear map

$$\ell : V_\Pi \to V_\Pi^{D^\times(\mathbf{R})} = V_\Pi^{\rm new}, \quad v \mapsto \ell(v) = \int_{\mathbf{R}^\times \backslash D^\times(\mathbf{R})} \Pi(h) v\, dh.$$

Since $\ell(\mathbf{P}_{\underline{k}}) = \mathbf{P}_{\underline{k}} \neq 0$, we have $\ell \neq 0$ and hence surjective. We have the following equality

$$\langle \mathbf{P}_{\underline{k}}, \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} \cdot \langle \ell(v_1), \ell(v_2) \rangle_{\underline{k}} = \langle v_1, \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} \cdot \langle v_2, \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} \tag{5.7}$$

for every $v_1, v_2 \in V_\Pi$.

Proposition 5.1. *We have*

$$I^*(\Pi, \mathbf{t}) = \frac{(k_1-1)(k_2-1)(k_3-1)}{4\pi^2}.$$

Proof. Note that the L-factor is given by

$$L(s,\Pi,r)=\zeta_{\mathbf{C}}(s+(k_1+k_2+k_3-3)/2))\zeta_{\mathbf{C}}(s+(-k_1+k_2+k_3-1)/2)\\ \times\zeta_{\mathbf{C}}(s+(k_1-k_2+k_3-1)/2)\zeta_{\mathbf{C}}(s+(k_1+k_2-k_3-1)/2)).$$

In view of (5.6), it suffices to show that

$$\langle \mathbf{P}_{\underline{k}},\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}=\frac{\Gamma(k_1^*+k_2^*+k_3^*+2)\Gamma(k_1^*+1)\Gamma(k_2^*+1)\Gamma(k_3^*+1)}{\Gamma(k_1^*+k_2^*+1)\Gamma(k_1^*+k_3^*+1)\Gamma(k_2^*+k_3^*+1)}.$$

By direct computation, we have

$$\mathbf{P}_{\underline{k}}=\sum_{n_1=0}^{k_1^*}\sum_{n_2=0}^{k_2^*}\sum_{n_3=0}^{k_3^*}\binom{k_1^*}{n_1}\binom{k_2^*}{n_2}\binom{k_3^*}{n_3}(-1)^{(k_1^*+k_2^*+k_3^*)-(n_1+n_2+n_3)}\\ \times X_1^{k_2^*-n_2+n_3}Y_1^{k_3^*+n_2-n_3}\otimes X_2^{k_3^*+n_1-n_3}Y_2^{k_1^*-n_1+n_3}\otimes X_3^{k_1^*-n_1+n_2}Y_3^{k_2^*+n_1-n_2}.$$

The coefficient in front of the vector $v_1:=X_1^{k_1-2}\otimes Y_2^{k_2-2}\otimes X_3^{k_1^*}Y_3^{k_2^*}$ in the expression of $\mathbf{P}_{\underline{k}}$ is equal to $(-1)^{k_1^*+k_2^*}$. On the other hand, the coefficient in front of the vector $v_2:=Y_1^{k_1-2}\otimes X_2^{k_2-2}\otimes X_3^{k_2^*}Y_3^{k_1^*}$ is $(-1)^{k_3^*}$. It follows that

$$\langle v_1,\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}\cdot\langle v_2,\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}=(-1)^{k_1^*+k_2^*+k_3^*}\cdot\langle v_1,v_2\rangle_{\underline{k}}^2=(-1)^{k_1^*+k_2^*+k_3^*}\binom{k_1^*+k_2^*}{k_1^*}^{-2}. \tag{5.8}$$

On the other hand, we have

$$\langle \ell(v_1),\ell(v_2)\rangle_{\underline{k}}=\int_{\mathbf{R}^\times\backslash D^\times(\mathbf{R})}\langle \Pi(h)v_1,v_2\rangle_{\underline{k}}dh.$$

Note that

$$\mathbf{R}^\times\backslash D^\times(\mathbf{R})\cong\{\pm1\}\backslash\mathrm{SU}(2),$$

We parametrize $u=\begin{pmatrix}\alpha&\beta\\-\bar\beta&\bar\alpha\end{pmatrix}\in\mathrm{SU}(2)$ by setting $\alpha=\cos\theta\cdot e^{i\varphi}$ and $\beta=\sin\theta\cdot e^{i\chi}$ with $0\le\theta\le\pi/2$ and $0\le\varphi,\chi\le2\pi$. For $\Phi\in\mathrm{L}^1(\mathrm{SU}(2))$, we have

$$\int_{\mathrm{SU}(2)}\Phi(u)\,du=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\Phi(\theta,\varphi,\chi)\cdot\sin2\theta\;d\theta\,d\varphi\,d\chi. \tag{5.9}$$

Our choice of the Haar measure on $\mathbf{R}^\times\backslash D^\times(\mathbf{R})$ implies the total volume of SU(2) is equal to 2.

Let $u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2)$. By (5.9), we have

$$\begin{aligned}
&\int_{\mathbf{R}^\times \backslash D^\times(\mathbf{R})} \langle \Pi(h)v_1, v_2\rangle_{\underline{k}} dh \\
&= (-1)^{k_1^*+k_2^*+k_3^*} \binom{k_1^*+k_2^*}{k_1^*}^{-1} \\
&\times \sum_{j=0}^{k_1^*} \binom{k_1^*}{j}\binom{k_2^*}{j} \frac{(-1)^j}{2} \int_{\mathrm{SU}(2)} |\alpha|_{\mathbf{C}}^{k_1^*+k_1^*+k_3^*-j} |\beta|_{\mathbf{C}}^{j} du \\
&= (-1)^{k_1^*+k_1-2}(k_1^*+k_2^*+k_3^*+1)^{-1}\binom{k_1^*+k_2^*}{k_1^*}^{-1} \sum_{j=0}^{k_1^*}(-1)^j \frac{\binom{k_1^*}{j}\binom{k_2^*}{j}}{\binom{k_1^*+k_2^*+k_3^*}{j}}.
\end{aligned}$$

Using (5.7), (5.8) and the equation above, we obtain

$$\begin{aligned}
\langle \mathbf{P}_{\underline{k}}, \mathbf{P}_{\underline{k}}\rangle_{\underline{k}}^{-1} &= (k_1^*+k_2^*+k_3^*+1)^{-1}\binom{k_1^*+k_2^*}{k_1^*} \sum_{j=0}^{k_1^*}(-1)^j \frac{\binom{k_1^*}{j}\binom{k_2^*}{j}}{\binom{k_1^*+k_2^*+k_3^*}{j}} \\
&= (k_1^*+k_2^*+k_3^*+1)^{-1}\binom{k_1^*+k_2^*}{k_1^*}\binom{k_1^*+k_2^*+k_3^*}{k_2^*+k_3^*}^{-1} \\
&\times \sum_{j=0}^{n}(-1)^j \binom{k_2^*}{j}\binom{k_1^*+k_2^*+k_3^*-j}{k_2^*+k_3^*} \\
&= \frac{\Gamma(k_1^*+k_2^*+1)\Gamma(k_1^*+k_3^*+1)\Gamma(k_2^*+k_3^*+1)}{\Gamma(k_1^*+k_2^*+k_3^*+2)\Gamma(k_1^*+1)\Gamma(k_2^*+1)\Gamma(k_3^*+1)}.
\end{aligned}$$

The last equality follows from Lemma 5.2 below. This completes the proof of Proposition 5.1. □

Lemma 5.2. *Let a, b and n be non-negative integers. Suppose $a \geq n$. Then we have*

$$\sum_{j=0}^{n}(-1)^j \binom{a}{j}\binom{a+b+n-j}{a+b} = \binom{b+n}{b}. \tag{5.10}$$

Proof. Consider the generating function

$$\sum_{j=0}^{a}(-1)^j \binom{a}{j}(1+X)^{a+b+n-j} = X^a(1+X)^{b+n}.$$

The lemma follows at once when one compares the coefficients of the term X^{a+b} on both sides. □

5.4 The non-archimedean case

Let F be a non-archimedean local field and D is the quaternion division algebra over F. Recall that we have assumed $\mathrm{Hom}_{D^\times}(\varPi, \mathbf{C}) \neq \{0\}$. Then by the results of Prasad [Pra90], [Pra92] and our assumption on $\varPi$, this happens precisely for the cases being considered in the following proposition.

Proposition 5.3. *Let $\nu_D : D^\times \to \mathbb{G}_m$ be the reduced norm of D.*

(1) Let $E = F \times F \times F$. If $\pi_j = \chi_j \circ \nu_D$, where χ_j is a unramified quadratic character of $F^\times$ with $\chi_1\chi_2\chi_3(\varpi) = 1$. Then we have

$$I^*(\varPi, \mathbf{t}) = 2(1-q^{-1})^2$$

(2) Let $E = K \times F$ and $\pi = \chi \circ \nu_D$, where χ is a unramified quadratic character of $F^\times$. Then we have

$$I^*(\varPi, \mathbf{t}) = \begin{cases} 1 & \text{if } \pi' \text{ is spherical and } K/F \text{ is unramified,} \\ 2 & \text{if } \pi' \text{ is spherical, } \chi(\varpi) = 1 \text{ and } K/F \text{ is ramified} \\ 2(1+q^{-2}) & \text{if } \pi' = \mathrm{St}_K \otimes \chi' \text{ with } \chi'\chi(\varpi) = -1. \end{cases}$$

Here χ' is a unramified quadratic character of $K^\times$.

(3) Let E be a field. If $\varPi$ is the trivial character of $D^\times(E)$, then we have

$$I^*(\varPi, \mathbf{t}) = \begin{cases} 2(1+q^{-1}+q^{-2}) & \text{if } E/F \text{ is unramified,} \\ 2 & \text{if } E/F \text{ is ramified.} \end{cases}$$

Here $I^*(\varPi, \mathbf{t})$ is the local zeta integral in Definition 2.2.

Proof. We first treat (1). Since $\chi_1\chi_2\chi_3(\varpi) = 1$, we have

$$I(\varPi, \mathbf{t}) = \int_{F^\times \backslash D^\times(F)} \chi_1\chi_2\chi_3(\nu_D(h))dh = \mathrm{Vol}(F^\times \backslash D^\times(F), dh) = 2.$$

The L-factor is

$$L(s, \varPi', r) = (1-\chi_1\chi_2\chi_3(\varpi)q^{-s-1/2})^{-2}(1-\chi_1\chi_2\chi_3(\varpi)q^{-s-3/2})^{-1},$$

where $\varPi'$ is the Jacquet-Langlands lift of $\varPi$ to $\mathrm{GL}_2(F)$. This shows (1).
We proceed to show (2). Suppose π' is spherical. then by Lemma 3.8 and (5.1), we find that

$$\begin{aligned} I(\varPi, \mathbf{t}) &= \int_{F^\times \backslash D^\times(F)} \Phi_{\pi'}(h)\pi(h)dh = 1 + \Phi_{\pi'}(\varpi_D)\chi(\varpi) \\ &= \begin{cases} (1+q^{-2})^{-1}(1+\chi(\varpi)\alpha q^{-1})(1+\chi(\varpi)\alpha^{-1}q^{-1}) & \text{if } K/F \text{ is unramified,} \\ 2 & \text{if } K/F \text{ is ramified.} \end{cases} \end{aligned}$$

Suppose $\pi' = \mathrm{St}_K \otimes \chi'$. In this case,

$$I(\mathit{\Pi}, \mathbf{t}) = \int_{F^\times \backslash D^\times(F)} \Phi_{\pi'}(h)\pi(h)dh = 1 + \Phi_{\pi'}(\varpi_D)\chi(\varpi) = (1 - \chi'\chi(\varpi)) = 2.$$

Here we use Lemma 3.9 and the observation that $\mathcal{O}_D^\times$ is contained in the Iwahori subgroup of $\mathrm{GL}_2(K)$. The L-factors are given in Proposition 4.7 and Proposition 4.8. This shows (2).
For the case (3), we have

$$I(\mathit{\Pi}, \mathbf{t}) = \mathrm{Vol}\left(F^\times \backslash D^\times(F), dh\right) = 2.$$

The L-factors are given in Proposition 4.9. This completes the proof. □

6 Explicit central value formulae and algebraicity for triple product

The purpose of this section is to give explicit central value formulae for the triple product L-functions by combining Ichino's formula [Ich08, Theorem 1.1 and Remark 1.3] with the local calculations in the previous sections. We use these formulae to prove the algebraicity of the central values.
Since the work of Garrett [Gar87], special values of triple product L-functions have been studied extensively by many people such as Orloff [Orl87], Satoh [Sat87], Harris and Kudla [HK91], Garrett and Harris [GH93], Gross and Kudla [GK92], Bocherer and Schulze-Pillot [BSP96], Furusawa and Morimoto [FM14], [FM16].

6.1 Notation

We fix some notations here. If F is a number field, let $\mathcal{O}_F$ be its ring of integers, $\mathcal{D}_F$ be its absolute discriminant, and h_F be its class number. Let $\mathbf{A}$ be the ring of adeles of $\mathbf{Q}$ and $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ be the profinite completion of $\mathbf{Z}$. We will denote by v a place of $\mathbf{Q}$ and by p a finite prime of $\mathbf{Q}$. If R is a $\mathbf{Q}$-algebra, let $\mathbf{A}_R = \mathbf{A} \otimes_{\mathbf{Q}} R$ and $R_v = R \otimes_{\mathbf{Q}} \mathbf{Q}_v$. For an abelian group M, let $\widehat{M} = M \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$.
We fix an additive character $\psi = \prod_v \psi_v : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}^\times$ defined by $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbf{R}$, and $\psi_p(x) = e^{-2\pi\sqrt{-1}x}$ for $x \in \mathbf{Z}[p^{-1}]$.

6.2 Modular forms and Automoprhic forms

We briefly review the definitions of modular forms and automorphic forms on certain quaternion algebras, and we write down an explicit correspondence between them. We follow the exposition of [Shi81, section 1], but with some modifications, so that it will be suitable for our application here.
We first introduce some notations. Let $d \geq 1$ be an integer and $\mathfrak{H}^d$ be the d-fold product of the upper half complex plane $\mathfrak{H}$. Let $\mathrm{GL}_2^+(\mathbf{R})$ be the identity

connect component of $\mathrm{GL}_2(\mathbf{R})$. If $d=1$, we let $h \in \mathrm{GL}_2^+(\mathbf{R})$ act on $z \in \mathfrak{H}$ and we define the factor $J(h,z)$ by

$$h \cdot z = \frac{az+b}{cz+d},$$
$$J(h,z) = \det(g)^{-\frac{1}{2}}(cz+d) \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In general, we let $\mathrm{GL}_2^+(\mathbf{R})^d$ acting on $\mathfrak{H}^d$ component-wise. If $\underline{k} = (k_1, \ldots, k_d) \in \mathbf{Z}^d$, we put

$$J(h,z)^{\underline{k}} = \prod_{j=1}^{d} j(h_j, z_j)^{k_j}.$$

for $h = (h_1, \ldots, h_d) \in \mathrm{GL}_2^+(\mathbf{R})^d$ and $z = (z_1, \ldots, z_d) \in \mathfrak{H}^d$. Let $C^\infty(\mathfrak{H})$ be the space of $\mathbf{C}$-valued smooth functions on $\mathfrak{H}$. Let k be an integer. Recall the Maass-Shimura differential operators δ_k and ε on $C^\infty(\mathfrak{H})$ are given by

$$\delta_k = \frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y}\right) \quad \text{and} \quad \varepsilon = -\frac{1}{2\pi\sqrt{-1}} y^2 \frac{\partial}{\partial \bar{z}} \quad y = \mathrm{Im}(z)$$

(cf. [Hid93, page 310]). If $m \geq 0$ is an integer, we put $\delta_k^m = \delta_{k+2m-2} \cdots \delta_{k+2}\delta_k$. In general, if $\underline{k} = (k_1, \ldots, k_d)$, $\underline{m} = (m_1, \ldots, m_d) \in \mathbf{Z}^d$ with $m_j \geq 0$ for $1 \leq j \leq d$, we let $\delta_{\underline{k}}^{\underline{m}}$ and $\varepsilon^{\underline{m}}$ be given by

$$\delta_{\underline{k}}^{\underline{m}} = (\delta_{k_1}^{m_1}, \ldots, \delta_{k_d}^{m_d}) \quad \text{and} \quad \varepsilon^{\underline{m}} = (\varepsilon^{m_1}, \ldots, \varepsilon^{m_d}),$$

and acting on $f \in C^\infty(\mathfrak{H}^d)$ coordinate-wise.
Let F be a totally real number field over $\mathbf{Q}$ with degree $d = [F : \mathbf{Q}]$. Let $\mathbf{A}_F$ be the ring of adeles of F and $\widehat{F}$ be its finite part. Let $\Sigma_F := \mathrm{Hom}_{\mathbf{Q}}(F, \mathbf{C})$ and $\mathfrak{H}^{\Sigma_F}$ be the d-fold product of $\mathfrak{H}$. Let D be a quaternion algebra over F. Let $G = D^\times$ viewed as an algebraic group defined over F. For any F-algebra L, $G(L) = (D \otimes_F L)^\times$. We assume D is either totally indefinite or totally definite. In other words, we assume either $G(F_\infty) \cong \mathrm{GL}_2(\mathbf{R})^{\Sigma_F}$ or $G(F_\infty) \cong (\mathbf{H}^\times)^{\Sigma_F}$, where $\mathbf{H}$ is the Hamiltonian quaternion algebra.

6.2.1 The totally indefinite case

Let $\underline{k} = (k_\sigma)_{\sigma \in \Sigma_F}, \underline{m} = (m_\sigma)_{\sigma \in \Sigma_F} \in \mathbf{Z}^{\Sigma_F}$ with $k_\sigma > 0$ and $m_\sigma \geq 0$ for all $\sigma \in \Sigma_F$. The zero and the identity element $\mathbf{Z}^{\Sigma_F}$ will be denoted by $\underline{0}$ and $\underline{1}$, respectively. Let $U \subset G(\widehat{F})$ be an open compact subgroup. We assume $\nu_D(U) = \widehat{\mathcal{O}}_F^\times$, where ν_D is the reduced norm of D and we extend it to a map on $D \otimes_F \widehat{F}$ in an obvious way.
We assume that D is totally indefinite. Denote by $\mathcal{N}_{\underline{k}}^{[\underline{m}]}(D, F; U)$ the space of functions $f : \mathfrak{H}^{\Sigma_F} \times G(\widehat{F}) \to \mathbf{C}$ such that $f(z, ahu) = f(z, h)$ for $z \in \mathfrak{H}^{\Sigma_F}$ and $(a, h, u) \in \widehat{F}^\times \times G(\widehat{F}) \times U$, and we require for each $h \in G(\widehat{F})$, the function

$f_h(z) := f(z,h) \in C^\infty(\mathfrak{H}^{\Sigma_F})$ is slowly increasing and $\varepsilon^{\underline{m}+\underline{1}} f_h = 0$, and satisfies the following automorphy condition:

$$f_h(\gamma \cdot z) J(\gamma, z)^{-\underline{k}} = f_h(z), \quad \gamma \in G(F) \cap \left(G^+(F_\infty) \times hUh^{-1}\right), \tag{6.1}$$

where $G^+(F_\infty)$ is the identity connect component of $G(F_\infty)$. We put $\mathcal{N}_{\underline{k}}(D,F;U) = \cup_{\underline{m}} \mathcal{N}_{\underline{k}}^{[\underline{m}]}(D,F;U)$. Notice that if $f \in \mathcal{N}_{\underline{k}}(D,F;U)$, then $\delta_{\underline{k}}^{\underline{m}} f \in \mathcal{N}_{\underline{k}+2\underline{m}}(D,F;U)$ (cf. [Hid93, page 312]). Assume $D = \mathrm{M}_2$ is the matrix algebra. Let $\mathfrak{n} \subset \mathcal{O}_F$ be an ideal. Put

$$K_0(\widehat{\mathfrak{n}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \mid c \in \widehat{\mathfrak{n}} \right\}.$$

Then $\mathcal{N}_{\underline{k}}^{[0]}(\mathrm{M}_2, F; K_0(\widehat{\mathfrak{n}})) = \mathcal{M}_{\underline{k}}(\mathrm{M}_2, F; K_0(\widehat{\mathfrak{n}}))$ is the space of holomorphic Hilbert modular forms of F of weight $\underline{k}$ and level $\mathfrak{n}$. Let $\mathcal{S}_{\underline{k}}(\mathrm{M}_2, F; K_0(\widehat{\mathfrak{n}}))$ be the subspace of holomorphic cusp forms in $\mathcal{M}_{\underline{k}}(\mathrm{M}_2, F; K_0(\widehat{\mathfrak{n}}))$.

We also define a subspace of automorphic forms on $G(\mathbf{A}_F)$ as follows. Let $\underline{k}$ and U be as above. We identify U and $G(F_\infty)$ with subgroups of $G(\mathbf{A}_F)$ in an obvious way. Let $\mathcal{A}_{\underline{k}}(D,F;U)$ be the space of automorphic forms $\mathbf{f} : G(\mathbf{A}_F) \to \mathbf{C}$ (cf. [BJ79, section 4]) such that

$$\mathbf{f}(a\gamma h k(\underline{\theta}) u) = \mathbf{f}(h) e^{\sqrt{-1}\underline{k}\cdot\underline{\theta}}, \quad \underline{k} \cdot \underline{\theta} = \sum_{\sigma \in \Sigma_F} k_\sigma \theta_\sigma$$

for $a \in \mathbf{A}_F^\times,\ \gamma \in G(F),\ u \in U,\ \underline{\theta} = (\theta_\sigma)_{\sigma\in\Sigma_F},\ k(\underline{\theta}) = (k(\theta_\sigma))_{\sigma \in \Sigma_F}$ with

$$k(\theta_\sigma) = \begin{pmatrix} \cos\theta_\sigma & \sin\theta_\sigma \\ -\sin\theta_\sigma & \cos\theta_\sigma \end{pmatrix}.$$

Denote by $\mathcal{A}_{\underline{k}}^0(D,F;U)$ the subspace of cusp forms in $\mathcal{A}_{\underline{k}}(D,F;U)$.

Suppose $F = \mathbf{Q}$. Let $\tilde{V}_\pm : \mathcal{A}_k(D,F;U) \to \mathcal{A}_{k\pm 2}(D,F;U)$ be the normalized weight raising/lowing elements ([JL70, page 165]) given by

$$\tilde{V}_\pm = -\frac{1}{8\pi}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes 1 \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \sqrt{-1} \right) \in \mathrm{Lie}(\mathrm{GL}_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C}.$$

In general, we have $\tilde{V}_\pm^{\underline{m}} : \mathcal{A}_{\underline{k}}(D,F;U) \to \mathcal{A}_{\underline{k}\pm 2\underline{m}}(D,F;U)$, where $\tilde{V}_\pm^{\underline{m}} = (\tilde{V}_\pm^{m_\sigma})_{\sigma\in\Sigma_F}$ acts on the archimedean component of $\mathbf{f} \in \mathcal{A}_{\underline{k}}(D,F;U)$ coordinate-wisely.

We write down an explicit correspondence between the spaces $\mathcal{N}_{\underline{k}}(D,F;U)$ and $\mathcal{A}_{\underline{k}}(D,F;U)$. Fix a set of representatives $\{x_1, \cdots, x_h\}$ for the double cosets $G(F)\backslash G(\mathbf{A}_F)/G^+(F_\infty)U$. Then

$$G(\mathbf{A}_F) = \amalg_{j=1}^h G(F) x_j G^+(F_\infty) U$$

is a disjoint union. We may assume every archimedean component of x_j is one for $1 \le j \le r$, and we regard x_j as elements in $G(\widehat{F})$. For each $f \in \mathcal{N}_{\underline{k}}(D, F; U)$, we define $\Phi(f) \in \mathcal{A}_{\underline{k}}(D, F; U)$ *the adelic lift* of f by the formulae

$$\Phi(f)(\gamma x_j h_\infty u) = f_{x_j}(h_\infty \cdot \mathbf{i}) J(h_\infty, \mathbf{i})^{-\underline{k}}, \quad \mathbf{i} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{H}^{\Sigma_F},$$
$$(\gamma \in G(F),\ h_\infty \in G^+(F_\infty),\ u \in U,\ 1 \le j \le h).$$

Conversely, we can recover f form $\Phi(f)$ by setting

$$f(z, h) = \Phi(f)(h_\infty h) J(h_\infty, \mathbf{i})^{\underline{k}}, \quad h_\infty \in G^+(F_\infty) \text{ with } h_\infty \cdot \mathbf{i} = z.$$

The weight raising/lowering operators are the adelic version of the Maass-Shimura differential operators $\delta_{\underline{k}}^{\underline{m}}$ and $\varepsilon^{\underline{m}}$ on the space of automorphic forms. More precisely, one checks that

$$\tilde{V}_+^{\underline{k}} \Phi(f) = \Phi(\delta_{\underline{k}}^{\underline{m}} f) \quad \text{and} \quad \tilde{V}_-^{\underline{m}} \Phi(f) = \Phi(\varepsilon^{\underline{m}} f). \tag{6.2}$$

In particular, f is holomorphic if and only if $\tilde{V}_-^{\underline{1}} \Phi(f) = 0$.

6.2.2 D is totally definite

Let $\underline{k} = (k_\sigma)_{\sigma \in \Sigma_F}$ and U be as above. We assume that D is totally definite and $k_\sigma \ge 2$ for all $\sigma \in \Sigma_F$. We identify $G(F_\infty)$ with $(\mathbf{H}^\times)^{\Sigma_F} \subset \mathrm{GL}_2(\mathbf{C})^{\Sigma_F}$. Let $(\rho_{k_\sigma}, \mathcal{L}_{k_\sigma}(\mathbf{C}))$ be the $(k_\sigma - 1)$-dimensional irreducible representation of $\mathbf{H}^\times$, and $\langle \cdot, \cdot \rangle_{k_\sigma}$ be the bilinear pairing on $\mathcal{L}_{k_\sigma}(\mathbf{C})$ defined in §5.3, respectively. We form an irreducible representation $(\rho_{\underline{k}}, \mathcal{L}_{\underline{k}}(\mathbf{C}))$ of $G(F_\infty)$ by setting

$$\rho_{\underline{k}} = \boxtimes_{\sigma \in \Sigma_F} \rho_{k_\sigma} \quad \text{and} \quad \mathcal{L}_{\underline{k}}(\mathbf{C}) = \otimes_{\sigma \in \Sigma_F} \mathcal{L}_{k_\sigma}(\mathbf{C}).$$

Then $\langle \cdot, \cdot \rangle_{\underline{k}} = \otimes_{\sigma \in \Sigma_F} \langle \cdot, \cdot \rangle_{k_\sigma}$ defines a bilinear pairing on $\mathcal{L}_{\underline{k}}(\mathbf{C})$.
Let $\mathcal{M}_{\underline{k}}(D, F; U)$ be the space of $\mathcal{L}_{\underline{k}}(\mathbf{C})$-valued atomorphic forms of type $\rho_{\underline{k}}$, which consists of functions $f : G(\mathbf{A}_F) \to \mathcal{L}_{\underline{k}}(\mathbf{C})$ such that

$$f(a\gamma h h_\infty u) = \rho_{\underline{k}}(h_\infty)^{-1} f(h),$$
$$(a \in \mathbf{A}_F^\times,\ \gamma \in G(F),\ h_\infty \in G(F_\infty),\ u \in U)$$

Let $\mathcal{A}(G(\mathbf{A}_F))$ be the space of $\mathbf{C}$-valued automorphic forms on $G(\mathbf{A}_F)$ (cf. [BJ79, section 4]). For $\mathbf{v} \in \mathcal{L}_{\underline{k}}(\mathbf{C})$ and $f \in \mathcal{M}_{\underline{k}}(D, F; U)$, we define a function $\Phi(\mathbf{v} \otimes f) : G(F) \backslash G(\mathbf{A}_F) \to \mathbf{C}$ by

$$\Phi(\mathbf{v} \otimes f)(h) = \langle \mathbf{v}, f(h) \rangle_{\underline{k}}.$$

Then the map $\mathbf{v} \mapsto \Phi(\mathbf{v} \otimes f)$ gives rise to a $G(F_\infty)$-equivalent morphism $\mathcal{L}_{\underline{k}}(\mathbf{C}) \to \mathcal{A}(G(\mathbf{A}_F))$ for every $f \in \mathcal{M}_{\underline{k}}(D, F; U)$. Let $\mathcal{A}_{\underline{k}}(D, F; U)$ the subspace of $\mathcal{A}(G(\mathbf{A}_F))$, consisting of functions $\Phi(\mathbf{v} \otimes f) : G(\mathbf{A}_F) \to \mathbf{C}$ for $\mathbf{v} \in \mathcal{L}_{\underline{k}}(\mathbf{C})$ and $f \in \mathcal{M}_{\underline{k}}(D, F; U)$.

More generally, suppose $F = F_1 \times \cdots \times F_r$, where F_j are totally real number fields. Let D be a quaternion $\mathbf{Q}$-algebra and put $D_{F_j} = D \otimes_{\mathbf{Q}} F_j$, $D_F = D \otimes_{\mathbf{Q}} F$. Let $U_j \subset G_j(\widehat{F}_j)$ be open compact subgroups, where $G_j := D_{F_j}^{\times}$ viewed as an algebraic group defined over F_j. Let $\underline{k}_j \in \mathbf{Z}^{\Sigma_{F_j}}$ be sets of positive integers. Put $U = (U_1, \ldots, U_r)$ and $\underline{k} = (\underline{k}_1, \ldots, \underline{k}_r)$. If D is totally definite, we define

$$\mathcal{M}_{\underline{k}}(D, F; U) = \otimes_{j=1}^{r} \mathcal{M}_{\underline{k}_j}(D_{F_j}, F_j; U_j),$$
$$\mathcal{A}_{\underline{k}}(D, F; U) = \otimes_{j=1}^{r} \mathcal{A}_{\underline{k}_j}(D_{F_j}, F_j; U_j).$$

If D is totally indefinite, similar definitions apply to the spaces $\mathcal{N}_{\underline{k}}^{[m]}(D, F; U)$ and $\mathcal{A}_{\underline{k}}(D, F; U)$.

6.3 Global settings

Let E be an étale cubic $\mathbf{Q}$-algebra. Then E is (i) $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ three copies of $\mathbf{Q}$, or (ii) $F \times \mathbf{Q}$, where F is a quadratic extension of $\mathbf{Q}$, or (iii) E is a field. Let $\mathcal{O}_E$ be the maximal order in E and let $\mathcal{D}_E$ be the absolute discriminant of E. Put

$$c = \begin{cases} 3 & \text{if } E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}, \\ 2 & \text{if } E = F \times \mathbf{Q}, \\ 1 & \text{if } E \text{ is a cubic extension of } \mathbf{Q}. \end{cases} \tag{6.3}$$

Here F is a quadratic extension over $\mathbf{Q}$. We assume

$$E_\infty = E \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbf{R} \times \mathbf{R} \times \mathbf{R}. \tag{6.4}$$

In particular, F is a real quadratic extension over $\mathbf{Q}$, and E is a real cubic extension over $\mathbf{Q}$ if it is a field.

Let $\mathfrak{n} \subset \mathcal{O}_E$ be an ideal. We have $\mathfrak{n} = (N_1\mathbf{Z}, N_2\mathbf{Z}, N_3\mathbf{Z})$ or $\mathfrak{n} = (\mathfrak{n}_F, N\mathbf{Z})$ according to $E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ or $E = F \times \mathbf{Q}$, respectively. Here N_j, N $(j = 1, 2, 3)$ are positive integers and $\mathfrak{n}_F$ is an ideal of $\mathcal{O}_F$. Let $\underline{k} = (k_1, k_2, k_3)$ be a triple of positive even integers with $k_j \geq 2$ for $j = 1, 2, 3$. We put

$$w = k_1 + k_2 + k_3 - 3. \tag{6.5}$$

Let $f_E \in \mathcal{M}_{\underline{k}}(\mathrm{M}_2, E; K_0(\widehat{\mathfrak{n}}))$ be a normalized Hilbert newform of weight $\underline{k}$ and level $K_0(\widehat{\mathfrak{n}})$ (cf. [Shi78, page 652]). More precisely, if $E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$, then $f_E = f_1 \otimes f_2 \otimes f_3$, where $f_j \in \mathcal{S}_{k_j}(M_1, \mathbf{Q}; K_0(N_j\widehat{\mathbf{Z}}))$ is a normalized newform of weight k_j and level $K_0(N_j\widehat{\mathbf{Z}})$. On the other hand, if $E = F \times \mathbf{Q}$, then $f_E = g_F \otimes f$, where $g_F \in \mathcal{S}_{(k_1,k_2)}(\mathrm{M}_2, F; K_0(\widehat{\mathfrak{n}}_F))$ is a normalized Hilbert newform of weight (k_1, k_2) and level $K_0(\widehat{\mathfrak{n}}_F)$, and $f \in \mathcal{S}_{k_3}(\mathrm{M}_2, \mathbf{Q}; K_0(N\widehat{\mathbf{Z}}))$ is a normalized newform of weight k_3 and level $K_0(N\widehat{\mathbf{Z}})$. Let $\mathbf{f}_E = \varPhi(f_E)$ be its adelic lift to $\mathcal{A}_{\underline{k}}(\mathrm{M}_2, E; K_0(\widehat{\mathfrak{n}}))$. Let $\varPi$ be the unitary irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_E)$ generated by $\mathbf{f}_E$. By the tensor product theorem [Fla79], $\Pi \cong \otimes'_v \Pi_v$, where Π_v are irreducible admissible representations of $\mathrm{GL}_2(E_v)$. We define the L-function and ϵ-factor associated to Π and r as product of local L-factors and ϵ-factors. That is, we put

$$L(s,\Pi,r) = \prod_v L(s,\Pi_v,r_v) \quad \text{and} \quad \epsilon(s,\Pi,r) = \prod_v \epsilon(s,\Pi_v,r_v,\psi_v).$$

Note that $L(s,\Pi,r)$ is holomorphic at $s=1/2$.
Ichino's formula relates the period integrals of triple products of certain automorphic forms on quaternion algebras along the diagonal cycles and the central values of triple L-functions. To describe the choice of the quaternion algebra, we define the local root number $\epsilon(\Pi_v) \in \{\pm 1\}$ associated to Π_v for each place v by the following condition

$$\epsilon(\Pi_v) = 1 \Leftrightarrow \mathrm{Hom}_*(\Pi_v, \mathbf{C}) \neq \{0\},$$

where $* = \mathrm{GL}_2(\mathbf{Q}_p)$ or $(\mathfrak{g}, K)$ according to $v=p$ or $v=\infty$, respectively. By the results of Prasad in [Pra90, Theorem 1.4] and [Pra92, Theorem D], we have

$$\epsilon(\Pi_v) = \epsilon\left(\frac{1}{2}, \Pi_v, r_v\right) \chi_{K_v/F_v}(-1),$$

where K_v is the quadratic discriminant algebra of E_v/F_v and χ_{K_v/F_v} is the quadratic character associated with K_v/F_v by the local class field theory. Define the global root number $\epsilon(\Pi)$ associated to Π by

$$\epsilon(\Pi) := \prod_v \epsilon(\Pi_v).$$

Notice that $\epsilon(\Pi_v) = 1$ for almost all v by the results of [Pra90, Theorem 1.2] and [Pra92, Theorem B].
In this paragraph, we assume the global root number $\epsilon(\Pi)$ is equal to 1. By this assumption, there is a unique quaternion $\mathbf{Q}$-algebra D such that D_v is the division $\mathbf{Q}_v$-algebra if and only if $\epsilon(\Pi_v) = -1$. Applying [Pra90, Theorem 1.2] and [Pra92, Theorem B], we see that the Jacquet-Langlands lift $\Pi^D = \otimes'_v \Pi_v^D$ of Π to $D^\times(\mathbf{A}_E)$ exists, where Π_v^D is a unitary irreducible admissible representation of $D^\times(E_v)$. Moreover, by the way we chose D, the following local root number condition is satisfied:

$$\epsilon(\Pi_v) = \begin{cases} 1 & \text{if } D_v \text{ is the matrix algebra,} \\ -1 & \text{if } D_v \text{ is the division algebra.} \end{cases} \tag{6.6}$$

Let Σ_D be the ramification set of D and $\Sigma_D^{(\infty)} \subset \Sigma_D$ be the subset without the infinite place. For each $v \notin \Sigma_D$, we fix an isomorphism $\iota_v : \mathrm{M}_2(\mathbf{Q}_v) \cong D \otimes_{\mathbf{Q}} \mathbf{Q}_v$ once and for all. Let $\mathcal{O}_D$ be the maximal order of D such that $D \otimes_{\mathbf{Z}} \mathbf{Z}_p = \iota_p(\mathrm{M}_2(\mathbf{Z}_p))$ for all $p \notin \Sigma_D$. If R is a $\mathbf{Q}$-algebra, we put $D(R) := D \otimes_{\mathbf{Q}} R$. We introduce following three sets of places of $\mathbf{Q}$:

$$\begin{aligned}\Sigma_3 &= \{v \mid E_v \cong \mathbf{Q}_v \times \mathbf{Q}_v \times \mathbf{Q}_v\}\,,\\ \Sigma_2 &= \{v \mid E_v \cong K_v \times \mathbf{Q}_v, \text{ for some quadratic extension } K_v \text{ of } \mathbf{Q}_v\}\,,\\ \Sigma_1 &= \{v \mid E_v \text{ is a cubic extension of } \mathbf{Q}_v\}\,.\end{aligned} \tag{6.7}$$

Note that by our assumption, we have $\infty \in \Sigma_3$. Also for every $p \notin \Sigma_D$, the map ι_p induces isomorphisms $D(E_p) \cong \mathrm{M}_2(E_p)$ and $\mathcal{O}_D \otimes_{\mathbf{Z}} \mathcal{O}_{E_p} \cong \mathrm{M}_2(\mathcal{O}_{E_p})$, where $\mathcal{O}_{E_p}$ is the maximal order of E_p. For $v \notin \Sigma_2 \cap \Sigma_D$, the canonical diagonal embedding $\mathbf{Q}_v \hookrightarrow E_v$ induces a diagonal embedding $D_v \hookrightarrow D(E_v)$. On the other hand, for each $p \in \Sigma_2 \cap \Sigma_D$, we choose an isomorphism $D(K_p) \cong \mathrm{M}_2(K_p)$ so that the embedding $D_p \hookrightarrow D(E_p) \cong \mathrm{M}_2(K_p) \times D_p$ is the identity map in the second coordinate, and is given by the one in §5.2 for the first coordinate. In any case, we identify D_v as subalgebras of $D(E_v)$ via these embeddings. Suppose E is a field, we note that the finite ramification sets $\Sigma^{(\infty)}_{D(F)}$ and $\Sigma^{(\infty)}_{D(E)}$ of $D(F)$ and $D(E)$ are given by

$$\begin{aligned}\Sigma^{(\infty)}_{D(F)} &= \{\mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal} \mid \mathfrak{p} \text{ divides } p \text{ for some } p \in \Sigma_3 \cap \Sigma_D\}\,,\\ \Sigma^{(\infty)}_{D(E)} &= \{\mathfrak{p} \subset \mathcal{O}_E \text{ prime ideal} \mid \mathfrak{p} \text{ divides } p \text{ for some } p \in (\Sigma_1 \cup \Sigma_3) \cap \Sigma_D\}\,.\end{aligned}$$

We put

$$N^- = \prod_{p \in \Sigma^{(\infty)}_D} p, \quad \mathfrak{N}^-_F = \prod_{\mathfrak{p} \in \Sigma^{(\infty)}_{D(F)}} \mathfrak{p}, \quad \mathfrak{N}^-_E = \prod_{\mathfrak{p} \in \Sigma^{(\infty)}_{D(E)}} \mathfrak{p}. \tag{6.8}$$

Recall that $\mathfrak{n}$ is an ideal in $\mathcal{O}_E$ and $\widehat{\mathfrak{n}} = \prod_p \mathfrak{n}_p$ is the closure of $\mathfrak{n}$ in $\widehat{E}$. In the following, we further assume that

$$\mathfrak{n} \text{ is square-free.} \tag{6.9}$$

More precisely, we assume N_1, N_2 and N_3 are square-free integers if $E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ and $\mathfrak{n}_F \subset \mathcal{O}_F$, $N \in \mathbf{Z}$ are square-free if $E = F \times \mathbf{Q}$. Let

$$M = \prod_{p \mid N^E_{\mathbf{Q}}(\mathfrak{n})} p. \tag{6.10}$$

If $L > 0$ is an integer coprime to N^-, we denote by R'_L the standard Eichler order of level L contained in $\mathcal{O}_D$. Similar notation is used to indicate the standard Eichler orders of $D(F)$ and $D(E)$. We define the order R_{Π^D} of $D(E)$ by

$$R_{\Pi^D} = \begin{cases} R'_{N_1/N^-} \times R'_{N_2/N^-} \times R'_{N_3/N^-} & \text{if } E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q},\\ R'_{\mathfrak{n}_F/\mathfrak{N}^-_F} \times R'_{N/N^-} & \text{if } E = F \times \mathbf{Q},\\ R'_{\mathfrak{n}/\mathfrak{N}^-_E} & \text{if } E \text{ is a field.}\end{cases}$$

We mention that the divisibility of each ideals appeared in the definition of R_{Π^D} follows from the results of [Pra90] and [Pra92]. We also define an order

R_{M/N^-} of D, which is a twist of the standard Eichler order R'_{M/N^-}. More precisely, for p such that $p \in \Sigma_{E,2}$ with $p \mid \mathcal{D}_E$ and $\mathfrak{n}\mathcal{O}_{E_p} = \varpi_{K_p}\mathcal{O}_{K_p} \times \mathbf{Z}_p$, we require

$$R_{M/N^-} \otimes_{\mathbf{Z}} \mathbf{Z}_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_0(p) \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that these are precisely the places p so that $E_p = K_p \times \mathbf{Q}_p$ with $K_p/\mathbf{Q}_p$ is unramified, and $\mathit{\Pi}_p^D = \mathit{\Pi}_p = \pi'_p \boxtimes \pi_p$ where π'_p (resp. π_p) is a special (unramified) representation of $\mathrm{GL}_2(K_p)$ (resp. $\mathrm{GL}_2(\mathbf{Q}_p)$).
To describe our formula, we need a notation. Let $\nu(\mathit{\Pi})$ be the number of prime p such that

- $p \in \Sigma_3$, $\mathit{\Pi}_p = \pi_{1,p} \boxtimes \pi_{2,p} \boxtimes \pi_{3,p}$ and $\pi_{j,p}$ are special representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ for $j = 1, 2, 3$.
- $p \in \Sigma_2$, $\mathit{\Pi}_p = \pi'_p \boxtimes \pi_p$ and π'_p (resp. π_p) is a special representation of $\mathrm{GL}_2(K_p)$ (resp. $\mathrm{GL}_2(\mathbf{Q}_p)$).
- $p \in \Sigma_2$, $K_p/\mathbf{Q}_p$ is ramified, $\mathit{\Pi}_p = \pi'_p \boxtimes \pi_p$ and π'_p (resp. π_p) is a unramified representation (resp. special representation) of $\mathrm{GL}_2(K_p)$ (resp. $\mathrm{GL}_2(\mathbf{Q}_p)$).
- $p \in \Sigma_1$ and $\mathit{\Pi}_p$ is a special representation of $\mathrm{GL}_2(E_p)$.

6.4 Unbalanced case

Assume $\epsilon(\mathit{\Pi}) = 1$ and $\epsilon(\mathit{\Pi}_\infty) = 1$ in this section. We assume without loss of generality that $k_3 = \max\{k_1, k_2, k_3\}$. Then $\epsilon(\mathit{\Pi}_\infty) = 1$ implies $k_3 \geq k_1 + k_2$. In this case, we have

$$D^\times(E_\infty) = \mathrm{GL}_2(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{R}) \quad \text{and} \quad \mathit{\Pi}_\infty^D = \mathit{\Pi}_\infty,$$

is the discrete series representation of $D^\times(E_\infty)$ of minimal weight $\underline{k}$ and trivial central character. Let $\mathcal{A}(D^\times(\mathbf{A}_E))$ be the space of $\mathbf{C}$-valued automorphic forms on $D^\times(\mathbf{A}_E)$ and let $\mathcal{A}(D^\times(\mathbf{A}_E))_{\mathit{\Pi}^D}$ be the underlying space of $\mathit{\Pi}^D$ in $\mathcal{A}(D^\times(\mathbf{A}_E))$. Put

$$\mathcal{A}_{\underline{k}}(D, E; \widehat{R}^\times_{\mathit{\Pi}^D})[\mathit{\Pi}^D] = \mathcal{A}_{\underline{k}}(D, E; \widehat{R}^\times_{\mathit{\Pi}^D}) \cap \mathcal{A}(D^\times(\mathbf{A}_E))_{\mathit{\Pi}^D}.$$

By the multiplicity one theorem and the theory of newform, we have

$$\mathcal{A}_{\underline{k}}(D, E; \widehat{R}^\times_{\mathit{\Pi}^D})[\mathit{\Pi}^D] = \mathbf{C}\,\mathbf{f}_E^D,$$

for some non-zero element $\mathbf{f}_E^D \in \mathit{\Pi}^D$.
Let $f_E^D \in \mathcal{N}_{\underline{k}}(D, E; \widehat{R}^\times_{\mathit{\Pi}^D})$ so that $\mathit{\Phi}(f_E^D) = \mathbf{f}_E^D$. We define the norm $\langle f_E^D, f_E^D \rangle_{\widehat{R}^\times_{\mathit{\Pi}^D}}$ of f_E^D as follows. Fix a set of representatives $\{x_1, \cdots, x_r\}$ for the double cosets $D^\times(E)\backslash D^\times(\mathbf{A}_E)/D^\times(E_\infty)^+\widehat{R}^\times_{\mathit{\Pi}^D}$, where $D^\times(E_\infty)^+$ is the

three-fold product of $\mathrm{GL}_2^+(\mathbf{R})$. We may assume every archimedean component of x_j is one for $1 \le j \le r$. Let

$$\Gamma_j = D^\times(E) \cap \left(D^\times(E_\infty)^+ \times x_j \widehat{R}_{\Pi^D}^\times x_j^{-1} \right), \quad 1 \le j \le r.$$

The functions $f^D_{E,x_j} : \mathfrak{H}^3 \to \mathbf{C}$ satisfy the automorphy condition (6.1) for $\gamma \in \Gamma_j$. We define

$$\langle f_E^D, f_E^D \rangle_{\widehat{R}^\times_{\Pi^D}} = \sum_{j=1}^r \int_{\Gamma_j \backslash \mathfrak{H}^3} |f^D_{E,x_j}(z)|^2 \mathrm{Im}(z)^{\underline{k}} d\mu(z).$$

Here $z = (z_1, z_2, z_3) \in \mathfrak{H}^3$, $\mathrm{Im}(z)^{\underline{k}} = \prod_{\ell=1}^3 \mathrm{Im}(z_\ell)^{k_\ell}$, and the measure $d\mu(z)$ on $\mathfrak{H}^3$ is given by

$$d\mu(z) = \prod_{\ell=1}^3 y_\ell^{-2} dx_\ell dy_\ell \qquad (z_\ell = x_\ell + i y_\ell, \quad 1 \le \ell \le 3),$$

where dx_ℓ and dy_ℓ are the usual Lebesgue measures on $\mathbf{R}$. Clearly, $\langle f_E^D, f_E^D \rangle_{\widehat{R}^\times_{\Pi^D}}$ is independent of the choice of the set $\{x_1, \cdots, x_r\}$. Similarly, we can define the norm $\langle f_E, f_E \rangle_{K_0(\widehat{\mathfrak{n}})}$ of f_E.
On the other hand, the Petersson norms of $\mathbf{f}_E$ and $\mathbf{f}_E^D$ are given by

$$\int_{\mathbf{A}_E^\times \mathrm{GL}_2(E) \backslash \mathrm{GL}_2(\mathbf{A}_E)} |\mathbf{f}_E(h)|^2 dh \quad \text{and} \quad \int_{\mathbf{A}_E^\times D^\times(E) \backslash D^\times(\mathbf{A}_E)} |\mathbf{f}_E^D(h)|^2 dh,$$

where dh are the Tamagawa measures on $\mathbf{A}_E^\times \backslash \mathrm{GL}_2(\mathbf{A}_E)$ and $\mathbf{A}_E^\times \backslash D^\times(\mathbf{A}_E)$, respectively. By [IP18, Lemma 6.1 and Lemma 6.3], we have

$$\begin{aligned} \langle f_E, f_E \rangle_{K_0(\widehat{\mathfrak{n}})} &= h_E \left[\mathrm{GL}_2(\widehat{\mathcal{O}}_E) : K_0(\widehat{\mathfrak{n}}) \right] \mathcal{D}_E^{3/2} \zeta_E(2) \\ &\quad \times \int_{\mathbf{A}_E^\times \mathrm{GL}_2^\times(E) \backslash \mathrm{GL}_2^\times(\mathbf{A}_E)} |\mathbf{f}_E(h)|^2 dh, \\ \langle f_E^D, f_E^D \rangle_{\widehat{R}^\times_{\Pi^D}} &= h_E \left[\widehat{\mathcal{O}}^\times_{D(E)} : \widehat{R}^\times_{\Pi^D} \right] \mathcal{D}_E^{3/2} \zeta_E(2) \\ &\quad \times \prod_{p \mid N^-} (p-1) \prod_{p \in \Sigma_1 \cap \Sigma_D} (p-1)^2 \prod_{p \in \Sigma_3 \cap \Sigma_D, p^3 \| M} (p^2+p+1) \\ &\quad \times \int_{\mathbf{A}_E^\times D^\times(E) \backslash D^\times(\mathbf{A}_E)} |\mathbf{f}_E^D(h)|^2 dh. \end{aligned} \tag{6.11}$$

Here $h_E := \sharp(E^\times \backslash \mathbf{A}_E^\times / E_\infty^\times \widehat{\mathcal{O}}_E^\times)$ is the class number of E. We mention that

$$\begin{aligned} dh &= \mathcal{D}_E^{-3/2} \zeta_E(2)^{-1} \\ &\quad \times \prod_{p \mid N^-} (p-1)^{-1} \prod_{p \in \Sigma_3 \cap \Sigma_D} (p-1)^{-2} \prod_{p \in \Sigma_1 \cap \Sigma_D, p^3 \| M} (p^2+p+1)^{-1} \\ &\quad \times \prod_v dh_v, \end{aligned}$$

where dh is the Tamagawa measures on $\mathbf{A}_E^\times\backslash D^\times(\mathbf{A}_E)$ and dh_v is the Haar measure on $E_v^\times\backslash D^\times(E_v)$ defined in §4.1 and §5.1 for each place v of $\mathbf{Q}$.

LEMMA 6.1. *We have*

$$\langle f_E, f_E\rangle_{K_0(\widehat{\mathfrak{n}})} = 2^{-k_1-k_2-k_3+c-3} h_E \mathcal{D}_E N_{\mathbf{Q}}^E(\mathfrak{n}) \cdot L(1, \Pi, \mathrm{Ad}),$$

where c is given by (6.3).

Proof. By specializing the formula in [Wal85, Proposition 6], we have

$$\int_{\mathbf{A}_E^\times \mathrm{GL}_2(E)\backslash \mathrm{GL}_2(\mathbf{A}_E)} |\mathbf{f}_E(h)|^2 dh$$
$$= 2^{-k_1-k_2-k_3+c-3}\zeta_E(2)^{-1}\mathcal{D}_E^{-1/2}N_{\mathbf{Q}}^E(\mathfrak{n})\left[\mathrm{GL}_2(\widehat{\mathcal{O}}_E) : K_0(\widehat{\mathfrak{n}})\right]^{-1} L(1,\Pi,\mathrm{Ad}).$$

The lemma follows form combining this with the equation (6.11). □

For each place v, let $\mathbf{t}_v \in D^\times(E_v)$ be the element defined in §2.3 for Π_v^D and put $\mathbf{t} = \otimes_v \mathbf{t}_v$, $\hat{\mathbf{t}} = \otimes_p \mathbf{t}_p$. Recall $N^- = \prod_{p\in\Sigma_D} p$ and $M = \prod_{p\mid N_{\mathbf{Q}}^E(\mathfrak{n})} p$. Let

$$\Gamma_{M/N^-}^D = D^\times(\mathbf{Q}) \cap \left(D^\times(\mathbf{R})^+ \times \widehat{R}_{M/N^-}^\times\right) \subset \mathrm{SL}_2(\mathbf{R}), \tag{6.12}$$

which is a Fuchsian group of the first kind. Remember that $k_3 \geq k_1 + k_2$. Set

$$2m = k_3 - k_1 - k_2.$$

Recall that $\nu(\Pi)$ is the non-negative integer defined in the last paragraph of §6.3.

Let

$$(\mathbf{f}_E^D)^*(h) = \overline{\mathbf{f}_E^D(h\tau_\infty)}$$

for $h \in D^\times(\mathbf{A}_E)$, where

$$\tau_\infty = \left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right) \in D^\times(E_\infty).$$

Since Π^D is unitary and self-contragredient, we have $\bar{\Pi}^D \cong \tilde{\Pi}^D \cong \Pi^D$, where $\bar{\Pi}^D$ is the conjugate representation of Π^D. The multiplicity one theorem then implies $(\mathbf{f}_E^D)^* \in \Pi^D$. By the theory of newform, there exists a non-zero constant α such that $\mathbf{f}_E^D = \alpha \cdot (\mathbf{f}_E^D)^*$ for all $h \in D^\times(\mathbf{A}_E)$. Since $((\mathbf{f}_E^D)^*)^* = \mathbf{f}_E^D$, we see that $\alpha\bar{\alpha} = 1$. It follows that we can always normalize $\mathbf{f}_E^D$ such that $\mathbf{f}_E^D = (\mathbf{f}_E^D)^*$.

THEOREM 6.2.

(1) *Suppose $\mathbf{f}_E^D$ is normalized so that $\mathbf{f}_E^D = (\mathbf{f}_E^D)^*$. We have*

$$\left(\int_{\Gamma_{M/N^-}^D\backslash\mathfrak{H}} (1\otimes\delta_{k_2}^m\otimes 1) f_E^D((z,z,-\overline{z}),\hat{\mathbf{t}}) y^{k_3-2} dxdy\right)^2$$
$$= 2^{-2k_3+\nu(\Pi)-2} M\mathcal{D}_E^{-1/2}\frac{\langle f_E^D, f_E^D\rangle_{\widehat{R}_{\Pi^D}^\times}}{\langle f_E, f_E\rangle_{K_0(\widehat{\mathfrak{n}})}} L\left(\frac{1}{2},\Pi,r\right).$$

(2) The central value is non-negative, that is

$$L\left(\frac{1}{2}, \Pi, r\right) \geq 0.$$

Proof. We first prove that the central values are non-negative. By our normalization, we have

$$\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)} \mathbf{f}_E^D(h)\mathbf{f}_E^D(h\tau_\infty)dh = \int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)} \left|\mathbf{f}_E^D(h)\right|^2 dh.$$

On the other hand, since $\mathrm{Ad}\tau_{\mathbf{R}}(V_+) = V_+ - 2\sqrt{-1}I_2$ and Π^D has trivial central character, we also have

$$\begin{aligned}\int_{\mathbf{A}^\times D^\times\backslash D^\times(\mathbf{A})} \Pi_\infty^D(\mathbf{t}_\infty)\mathbf{f}_E^D(h\hat{\mathbf{t}})dh &= \int_{\mathbf{A}^\times D^\times\backslash D^\times(\mathbf{A})} \Pi_\infty^D(\mathbf{t}_\infty\tau_\infty)\overline{\mathbf{f}_E^D(h\hat{\mathbf{t}})}dh \\ &= \overline{\int_{\mathbf{A}^\times D^\times\backslash D^\times(\mathbf{A})} \Pi_\infty^D(\mathbf{t}_\infty)\mathbf{f}_E^D(h\hat{\mathbf{t}})dh}.\end{aligned}$$

On the other hand, by Ichino's formula [Ich08, Theorem 1.1 and Remark 1.3]) and the choices of the Haar measures in §4.1 and §5.1, we find that

$$\frac{\left(\int_{\mathbf{A}^\times D^\times\backslash D^\times(\mathbf{A})} \Pi^D(\mathbf{t})\mathbf{f}_E^D(h)dh\right)^2}{\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)} \mathbf{f}_E^D(h)\mathbf{f}_E^D(h\tau_\infty)dh} \tag{6.13}$$

$$= 2^{-c}\prod_{p|N^-}(p-1)^{-1}\cdot\frac{\zeta_E(2)}{\zeta_{\mathbf{Q}}(2)^2}\cdot\frac{L(1/2,\Pi,r)}{L(1,\Pi,\mathrm{Ad})}\cdot\prod_v I^*(\Pi_v^D,\mathbf{t}_v), \tag{6.14}$$

where c is given by (6.3). Since $L(1,\Pi,\mathrm{Ad}) > 0$ by Lemma 6.1 and $I^*(\Pi_v,\mathbf{t}_v) > 0$ for all v by our results in the previous sections, we see immediately that assertion (2) holds.

To drive our formula, we note that from (6.2) and the definition of $\mathbf{t}_\infty$, the function $\Pi^D(\mathbf{t})\mathbf{f}_E^D$ is the adelic lift of the automorphic function

$$((z_1,z_2,z_3),h) \mapsto (1\otimes\delta_{k_2}^m\otimes 1)f_E^D((z_1,z_2,-\overline{z_3}),h\hat{\mathbf{t}})$$

for $((z_1,z_2,z_3),h) \in \mathfrak{H}^3\times\mathrm{GL}_2(\widehat{E})$. Applying lemmas 6.1 and 6.3 in [IP18], we obtain

$$\begin{aligned}&\left(\int_{\mathbf{A}^\times D^\times\backslash D^\times(\mathbf{A})} \Pi^D(\mathbf{t})\mathbf{f}_E^D(h)dh\right)^2 \\ &= \zeta_{\mathbf{Q}}(2)^{-2}\prod_{p|M/N^-}(1+p)^{-2}\prod_{p|N^-}(p-1)^{-2} \\ &\times\left(\int_{\Gamma_{M/N^-}^D\backslash\mathfrak{H}}(1\otimes\delta_{k_2}^m\otimes 1)f_E^D((z,z,-\bar{z}),\hat{\mathbf{t}})y^{k_3-2}dxdy\right)^2.\end{aligned}$$

The formula then follows from combining this with Lemma 6.1 and our results for $I^*(\Pi_v^D,\mathbf{t}_v)$ in §4 and §5. ∎

6.5 Balanced case

Assume $\epsilon(\Pi) = 1$ and $\epsilon(\Pi_\infty) = -1$ in this section. We have

$$D^\times(E_\infty) = \mathbf{H}^\times \times \mathbf{H}^\times \times \mathbf{H}^\times \quad \text{and} \quad (\Pi_\infty^D, V_{\Pi_\infty^D}) = (\rho_{\underline{k}}, \mathcal{L}_{\underline{k}}(\mathbf{C})).$$

Let $\mathcal{A}(D^\times(\mathbf{A}_E))_{\Pi^D}$ be the underlying space of Π^D in $\mathcal{A}(D^\times(\mathbf{A}_E))$ and put

$$\mathcal{A}_{\underline{k}}(D, E; \widehat{R}_{\Pi^D}^\times)[\Pi^D] = \mathcal{A}_{\underline{k}}(D, E; \widehat{R}_{\Pi^D}^\times) \cap \mathcal{A}(D^\times(\mathbf{A}_E))_{\Pi^D}.$$

By the multiplicity one theorem and the theory of newform, there exists a unique (up to constants) non-zero element $f_E^D \in \mathcal{M}_{\underline{k}}(D, E; \widehat{R}_{\Pi^D}^\times)$ such that the map $\mathbf{v} \mapsto \Phi(\mathbf{v} \otimes f_E^D)$ defines a $D^\times(E_\infty)$-isomorphism form $\mathcal{L}_{\underline{k}}(\mathbf{C})$ onto $\mathcal{A}_{\underline{k}}(D, E; \widehat{R}_{\Pi^D}^\times)[\Pi^D]$. Let $\mathbf{P}_{\underline{k}} \in \mathcal{L}_{\underline{k}}(\mathbf{C})$ be the $\mathbf{H}^\times$-fixed element given by (5.3). We put $\mathbf{f}_E^D = \Phi(\mathbf{P}_{\underline{k}} \otimes f_E^D)$. Then its immediately form the definition that $\mathbf{f}_E^D$ is right $\mathbf{H}^\times$-invariant.

To state our central value formula for the balanced case, we need some notations. Let $\mathrm{Cl}(R_{\Pi^D})$ and $\mathrm{Cl}(R_{M/N^-})$ be sets of representatives of

$$\widehat{E}^\times D^\times(E) \backslash D^\times(\widehat{E}) / \widehat{R}_{\Pi^D}^\times \quad \text{and} \quad \widehat{\mathbf{Q}}^\times D^\times(\mathbf{Q}) \backslash D^\times(\widehat{\mathbf{Q}}) / \widehat{R}_{M/N^-}^\times,$$

respectively. Let Γ_α be finite sets defined by

$$\left(D^\times(E) \cap \widehat{E}^\times \alpha\, \widehat{R}_{\Pi^D}^\times\, \alpha^{-1}\right) / E^\times \quad \text{or} \quad \left(D^\times(\mathbf{Q}) \cap \widehat{\mathbf{Q}}^\times \alpha\, \widehat{R}_{M/N^-}^\times\, \alpha^{-1}\right) / \mathbf{Q}^\times,$$

according to $\alpha \in \mathrm{Cl}(R_{\Pi^D})$ or $\alpha \in \mathrm{Cl}(R_{M/N^-})$, respectively. We put

$$\langle f_E^D, f_E^D \rangle_{\widehat{R}_{\Pi^D}^\times} = \sum_{\alpha \in \mathrm{Cl}(R_{\Pi^D})} \frac{1}{\sharp \Gamma_\alpha} \langle f_E^D(\alpha), f_E^D(\alpha) \rangle_{\underline{k}}.$$

For each place v, let $\mathbf{t}_v \in D^\times(E_v)$ be the element defined in §2.3 for Π_v^D and put $\mathbf{t} = \otimes_v \mathbf{t}_v$. Recall that $M = \prod_{p | N_{\mathbf{Q}}^E(\mathfrak{n})} p$ and that $\nu(\Pi)$ is the non-negative integer defined in the last paragraph of §6.3.

Theorem 6.3.

(1) We have

$$\left(\sum_{\alpha \in \mathrm{Cl}(R_{M/N^-})} \frac{1}{\sharp \Gamma_\alpha} \langle f_E^D(\alpha \mathbf{t}), \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} \right)^2$$
$$= 2^{-(k_1+k_2+k_3+1)+\nu(\Pi)} M \mathcal{D}_E^{-1/2} \frac{\langle f_E^D, f_E^D \rangle_{\widehat{R}_{\Pi^D}^\times}}{\langle f_E, f_E \rangle_{K_0(\widehat{\mathfrak{n}})}} L\left(\frac{1}{2}, \Pi, r\right).$$

(2) The central value is non-negative, that is

$$L\left(\frac{1}{2}, \Pi, r\right) \geq 0.$$

Proof. (1) By Lemmas 6.1 and 6.3 in [IP18], we have

$$\left(\sum_{\alpha\in \mathrm{Cl}(R_{M/N^-})}\frac{1}{\sharp\Gamma_\alpha}\langle f_E^D(\alpha\mathbf{t}),\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}\right)=\frac{1}{24}\prod_{p\mid M/N^-}(1+p)\prod_{p\mid N^-}(p-1)\times\int_{\mathbf{A}^\times D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})}\mathbf{f}_E^D(h\mathbf{t})dh,$$

where dh is the Tamagawa measure on $\mathbf{A}^\times\backslash D^\times(\mathbf{A})$. On the other hand, applying same lemmas, we obtain

$$\begin{aligned}\langle f_E^D,f_E^D\rangle_{\widehat{R}^\times_{\Pi^D}}&=2^{-6}\pi^{-3}h_E\mathcal{D}_E^{3/2}\left[\widehat{\mathcal{O}}^\times_{D(E)}:\widehat{R}^\times_{\Pi^D}\right]\\&\times\prod_{p\mid N^-}(p-1)\prod_{p\in\Sigma_1\cap\Sigma_D}(p-1)^2\prod_{p\in\Sigma_3\cap\Sigma_D,p^3\parallel M}(p^2+p+1)\\&\times\zeta_E(2)\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)}\langle f_E^D(h),f_E^D(h)\rangle_{\underline{k}}dh,\end{aligned}$$

where dh is the Tamagawa measure on $\mathbf{A}_E^\times\backslash D^\times(\mathbf{A}_E)$. Schur's orthogonal relation implies

$$\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)}\mathbf{f}_E^D(h)\mathbf{f}_E^D(h)dh=\frac{\langle\mathbf{P}_{\underline{k}},\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}}{(k_1-1)(k_2-1)(k_3-1)}\times\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)}\langle f_E^D(h),f_E^D(h)\rangle_{\underline{k}}dh.$$

The measure dh on the RHS of the equation above is also the Tamagawa measure on $\mathbf{A}_E^\times\backslash D^\times(\mathbf{A}_E)$. By Ichino's formula [Ich08, Theorem 1.1 and Remark 1.3] and the choices of Haar measures in §4.1 and §5.1, we find that

$$\frac{\left(\int_{\mathbf{A}^\times D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})}\mathbf{f}_E^D(h\mathbf{t})dh\right)^2}{\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)}\mathbf{f}_E^D(h)\mathbf{f}_E^D(h)dh}=2^{3-c}3^{-1}\prod_{p\mid N^-}(p-1)^{-1}\cdot\frac{\zeta_E(2)}{\zeta_{\mathbf{Q}}(2)}\cdot\frac{L(1/2,\Pi,r)}{L(1,\Pi,\mathrm{Ad})}\cdot\prod_v I^*(\Pi_v^D,\mathbf{t}_v).$$

Here the constant c is given by (6.3). The central value formula follows from the equations above together with Lemma 6.1 and the results for $I^*(\Pi_v^D,\mathbf{t}_v)$ in §4 and §5.

To prove (2), it suffices to show that the ratio

$$\frac{\left(\int_{\mathbf{A}^\times D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})}\mathbf{f}_E^D(h\mathbf{t})dh\right)^2}{\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)}\langle f_E^D(h),f_E^D(h)\rangle_{\underline{k}}dh}$$

is non-negative. To do this we consider $(f_E^D)^*(h) = \overline{f_E^D(h\tau_\infty)}$ for $h \in D^\times(\mathbf{A}_E)$, where

$$\tau_\infty = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in D^\times(E_\infty).$$

The function $(f_E^D)^*$ satisfy the same conditions as f_E^D. By the uniqueness, there exists a non-zero constant α such that $f_E^D(h) = \alpha \cdot \overline{f_E^D(h\tau_\infty)}$ for all $h \in D^\times(\mathbf{A}_E)$. On one hand, we have

$$\begin{aligned}\int_{\mathbf{A}^\times D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})} \langle f_E^D(h\mathbf{t}), \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} dh &= \alpha \int_{\mathbf{A}^\times D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})} \langle \overline{f_E^D(h\mathbf{t}\tau)}, \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} dh \\ &= \alpha \cdot \overline{\int_{\mathbf{A}^\times D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})} \langle f_E^D(h\mathbf{t}), \mathbf{P}_{\underline{k}} \rangle_{\underline{k}} dh}.\end{aligned}$$

On the other hand, recall that

$$\mathcal{H}_{\underline{k}}(v,w) = \langle v, \Pi_\infty^D(\tau_\infty)\bar{w} \rangle_{\underline{k}} \quad v,w \in \mathcal{L}(\mathbf{C}),$$

defines an $D^\times(E_\infty)$-invariant *Hermitian* pairing on $V_{\Pi_\infty^D}$. We have

$$\begin{aligned}&\int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)} \langle f_E^D(h), f_E^D(h) \rangle_{\underline{k}} dh \\ &= \alpha \int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)} \langle f_E^D(h), \overline{f_E^D(h\tau_\infty)} \rangle_{\underline{k}} dh \\ &= \alpha \int_{\mathbf{A}_E^\times D^\times(E)\backslash D^\times(\mathbf{A}_E)} \mathcal{H}_{\underline{k}}(f_E^D(h), f_E^D(h)) dh.\end{aligned}$$

This finishes the proof. □

6.6 Algebraicity of the central critical value

Now we apply Theorems 6.2 and 6.3 to prove the algebraicity of the central critical values of the triple product L-functions. We keep the notations in §6.3. When $\epsilon(\Pi) = 1$ and $\epsilon(\Pi_\infty) = 1$ (resp. $\epsilon(\Pi_\infty) = -1$), we will follow the setting in §6.4 (resp. §6.5). We define the motivic triple product L-function and its associated completed L-function for f_E by

$$L(s, f_E, r) = \prod_p L\left(s - \frac{w}{2}, \Pi_p, r_p\right) \quad \text{and} \quad \Lambda(s, f_E, r) = L\left(s - \frac{w}{2}, \Pi, r\right).$$

Here w is given by (6.5).

When $\epsilon(\Pi) = 1$ and $\epsilon(\Pi_\infty) = 1$, we assume that $k_3 \geq k_1 + k_2$ and $E = K \times \mathbf{Q}$ with $K = \mathbf{Q} \times \mathbf{Q}$ or K is a real quadratic extension of $\mathbf{Q}$. Then $\mathfrak{n}_K = (N_1\mathbf{Z}, N_2\mathbf{Z})$, $N_3 = N$ and $g_K = f_1 \otimes f_2$, $f_3 = f$ when $K = \mathbf{Q} \times \mathbf{Q}$. In any case, we have

$$f_E = g_K \otimes f \quad \text{and} \quad \mathfrak{n} = (\mathfrak{n}_K, N\mathbf{Z}).$$

Define the Petersson norm of f by

$$\langle f, f\rangle_{\Gamma_0(N)} = \int_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z)|^2 y^{k_3-2} dxdy, \quad (z = x+iy).$$

Here dx, dy are the usual Lebesgue measures on $\mathbf{R}$.
The first corollary to Theorems 6.2 and 6.3 is the Galois-equivariant property of the central L-value.

Corollary 6.4. *Let $\sigma \in \mathrm{Aut}(\mathbf{C})$.*

(1) Assume $\epsilon(\mathit{\Pi}_v) = 1$ for all places v. We have

$$\left(\frac{L((w+1)/2, f_E, r)}{\mathcal{D}_E^{1/2}\pi^{2k_3}\langle f, f\rangle^2_{\Gamma_0(N)}}\right)^\sigma = \frac{L((w+1)/2, f_E^\sigma)}{\mathcal{D}_E^{1/2}\pi^{2k_3}\langle f^\sigma, f^\sigma\rangle^2_{\Gamma_0(N)}}.$$

(2) Assume $\epsilon(\mathit{\Pi}) = 1$ and $\epsilon(\mathit{\Pi}_\infty) = -1$. We have

$$\left(\frac{L\left((w+1)/2, f_E, r\right)}{\mathcal{D}_E^{1/2}\pi^{w+2}\langle f_E, f_E\rangle_{K_0(\widehat{\mathfrak{n}})}}\right)^\sigma = \frac{L\left((w+1)/2, f_E^\sigma, r\right)}{\mathcal{D}_E^{1/2}\pi^{w+2}\langle f_E^\sigma, f_E^\sigma\rangle_{K_0(\widehat{\mathfrak{n}})}}.$$

(3) Assume $\epsilon(\mathit{\Pi}) = -1$. We have

$$L\left(\frac{w+1}{2}, f_E, r\right) = 0.$$

Proof. By the Galois equivariance property for the local Langlands correspondence [Hen01, §7] and $\mathit{\Pi}_\infty^\sigma \cong \mathit{\Pi}_\infty$, we have $\epsilon(\mathit{\Pi}_v) = \epsilon(\mathit{\Pi}_v^\sigma)$ for all $\sigma \in \mathrm{Aut}(\mathbf{C})$. First we assume $\epsilon(\mathit{\Pi}_v) = 1$ for all v. Then $D = \mathrm{M}_2$. Let $\iota : \mathfrak{H} \to \mathfrak{H}^2$ be the diagonal embedding $z \mapsto (z,z)$. The $\mathrm{GL}_2(\mathbf{Q}_p)$ component of $\mathbf{t}_p \in \mathrm{GL}_2(K_p) \times \mathrm{GL}_2(\mathbf{Q}_p)$ is equal to 1 for all p. Thus we may view $\hat{\mathbf{t}}$ as an element in $\mathrm{GL}_2(\widehat{K})$. Note that $(1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K$ is a nearly holomorphic Hilbert modular form over K of weight (k_1, k_2+2m), where ρ denote the right translation of $\mathrm{GL}_2(\widehat{K})$. Let

$$\iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K)(z) = (1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K((z,z),1)$$

be its pullback along ι at the identity cusp. Then it is a nearly holomorphic modular form of weight k_3 and level $\Gamma_0(M)$. We consider the period integral $\langle \iota^*((1 \otimes \delta_{k_2}^m)g_K), f\rangle$ defined by

$$\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K), f\rangle = \int_{\Gamma_0(M)\backslash\mathfrak{H}} \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K)(z)\overline{f(z)}y^{k_3-2}dxdy,$$

where $z = x+iy$ and dx, dy are the usual Lebesgue measures on $\mathbf{R}$. Let $\sigma \in \mathrm{Aut}(\mathbf{C})$. By our normalization of g_K, we have

$$(\iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K))^\sigma = \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K^\sigma).$$

Since $\iota^*((1\otimes\delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K)$ is nearly holomorphic and f is a newform, by [Stu80, Theorem 4] and [Shi76], we have

$$\left(\frac{\langle \iota^*((1\otimes\delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K),f\rangle}{\langle f,f\rangle_{\Gamma_0(N)}}\right)^\sigma=\frac{\langle(\iota^*((1\otimes\delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K^\sigma)),f^\sigma\rangle}{\langle f^\sigma,f^\sigma\rangle_{\Gamma_0(N)}}.$$

In particular, we have $\langle \iota^*((1\otimes\delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K),f\rangle\in\mathbf{R}$. Note that

$$(1\otimes\delta_{k_2}^m\otimes 1)\rho(\hat{\mathbf{t}})f_E((z,z,-\overline{z}),1)=\iota^*((1\otimes\delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K)(\tau)\overline{f(z)}.$$

By Theorem 6.2, we have

$$\frac{\langle \iota^*((1\otimes\delta_{k_2}^m)\rho(\hat{\mathbf{t}})g_K),f\rangle^2}{\langle f,f\rangle_{\Gamma_0(N)}^2}=2^{-2k_3-1+\nu(\Pi)}M\frac{\Lambda((w+1)/2,f_E,r)}{\mathcal{D}_K^{1/2}\langle f,f\rangle_{\Gamma_0(N)}^2}.$$

The assertion then follows from applying σ to both sides and applying our central value formula to the left hand side again.
Assume $\epsilon(\mathit{\Pi})=1$ and $\epsilon(\mathit{\Pi}_\infty)=-1$. Put

$$\langle f_E^D,\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}=\left(\sum_{\alpha\in\mathrm{Cl}(R_{M/N^-})}\frac{1}{\sharp\Gamma_\alpha}\langle f_E^D(\alpha\mathbf{t}),\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}\right)^2.$$

Since the equation above only evaluate at the finite adeles and we are considering the ratios, we have

$$\left(\frac{\langle f_E^D,\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}}{\langle f_E^D,f_E^D\rangle_{\widehat{R}_{\mathit{\Pi}D}^\times}}\right)^\sigma=\frac{\langle (f_E^\sigma)^D,\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}}{\langle (f_E^\sigma)^D,(f_E^\sigma)^D\rangle_{\widehat{R}_{\mathit{\Pi}D}^\times}}$$

for all $\sigma\in\mathrm{Aut}(\mathbf{C})$. The assertion then follows from Theorem 6.3.
Finally, assume $\epsilon(\mathit{\Pi})=-1$. By the results of [HK04] and [PSP08], we have

$$L_{\mathrm{PSR}}\left(\frac{w+1}{2},f_E,r\right)=0,$$

where $L_{\mathrm{PSR}}(s,f_E,r)$ is the triple product L-function associated with f_E defined by the integral representation in [PSR87] and [Ike89]. On the other hand, by Theorem D in [CCI19], we have

$$L(s,f_E,r)=L_{\mathrm{PSR}}(s,f_E,r).$$

This completes the proof. ☐

Remark 6.5. We can also prove the algebraicity of the central value when $\epsilon(\mathit{\Pi})=1$ and $\epsilon(\mathit{\Pi}_\infty)=1$ (see Corollary 6.6). However, to prove the Galois-equivariant property, one needs to refine the results of Harris in [Har93, Lemma 2.5.5] and [Har94, Theorem 1].

The following corollary is a refinement of the results of Harris-Kudla in [HK91, Theorems 11.6 and 12.4]. We prove that the ratio between the central L-value and the Petersson norms is essentially a square in the Hecke field $\mathbf{Q}(\mathit{\Pi})$ of $\mathit{\Pi}$.

Corollary 6.6. *Assume $\epsilon(\mathit{\Pi})=1$ and f_E^D is $\mathbf{Q}(\mathit{\Pi})$-arithmetic in the sense described in [HK91] and [Har93]. Let $\mathbf{Q}(\mathit{\Pi})$ be the Hecke field of $\mathit{\Pi}$ and $\Omega_{f_E}\in\mathbf{C}^\times$ defined by*

$$\Omega_{f_E}=\begin{cases}2^{\nu(\mathit{\Pi})}M^{-1}\mathcal{D}_E^{1/2}\dfrac{\langle f_E,f_E\rangle_{K_0(\widehat{\mathfrak{n}})}}{\langle f_E^D,f_E^D\rangle_{\widehat{R}^\times_{\mathit{\Pi}^D}}}\langle f,f\rangle^2_{\Gamma_0(N)} & \text{if } \epsilon(\mathit{\Pi}_\infty)=1,\\ 2^{1+\nu(\mathit{\Pi})}M^{-1}\mathcal{D}_E^{1/2}\dfrac{\langle f_E,f_E\rangle_{K_0(\widehat{\mathfrak{n}})}}{\langle f_E^D,f_E^D\rangle_{\widehat{R}^\times_{\mathit{\Pi}^D}}} & \text{if } \epsilon(\mathit{\Pi}_\infty)=-1.\end{cases}$$

We have

$$\frac{L((w+1)/2,f_E,r)}{\Omega_{f_E}}\in\mathbf{Q}(\mathit{\Pi})^2.$$

Proof. First we assume $\epsilon(\mathit{\Pi})=1$ and $\epsilon(\mathit{\Pi}_\infty)=1$. Since f_E^D is $\mathbf{Q}(\mathit{\Pi})$-arithmetic, one can show that $(f_E^D)^*$ is also $\mathbf{Q}(\mathit{\Pi})$-arithmetic by the arithmeticity criterion [HK91, Theorem 14.7]. As $\mathbf{Q}(\mathit{\Pi})$ is totally real, we deduce that $f_E^D=\pm(f_E^D)^*$. If $f_E^D=-(f_E^D)^*$, then it follows from Theorem 6.2-(2) and (6.13) that $L((w+1)/2,f_E,r)=0$. Therefore, we assume $f_E^D=(f_E^D)^*$. Let τ be the irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by the adelic lift of f and τ^D its Jacquet-Langlands lift to $D^\times(\mathbf{A})$. Let $f^D\in\tau^D$ be non-zero holomorphic $\mathbf{Q}(\mathit{\Pi})$-arithmetic cusp form. Recall $\Gamma^D_{M/N^-}\subset\mathrm{SL}_2(\mathbf{R})$ is the Fuchsian group of the first kind defined in (6.12). By [HK91, Theorem 12.3], we have

$$\frac{\langle f,f\rangle_{\Gamma_0(N)}}{\langle f^D,f^D\rangle_{\Gamma^D_{M/N^-}}}\in\mathbf{Q}(\mathit{\Pi}).$$

Since f_E^D is assumed to be $\mathbf{Q}(\mathit{\Pi})$-arithmetic, by [Har90, Corollary 7.7.1] and [Har94, Theorem 1] (see also [HK91, Lemma 15.1]), we have

$$\frac{\displaystyle\int_{\Gamma^D_{M/N^-}\backslash\mathfrak{H}}(1\otimes\delta^m_{k_2}\otimes 1)f_E^D((z,z,-\overline{z}),\hat{\mathbf{t}})y^{k_3-2}dxdy}{\langle f^D,f^D\rangle_{\Gamma^D_{M/N^-}}}\in\mathbf{Q}(\mathit{\Pi}).$$

The assertion then follows from Theorem 6.2-(1).
Assume $\epsilon(\mathit{\Pi})=1$ and $\epsilon(\mathit{\Pi}_\infty)=-1$. Since f_E^D is assumed to be $\mathbf{Q}(\mathit{\Pi})$-arithmetic, we have

$$\sum_{\alpha\in\mathrm{Cl}(R_{M/N^-})}\frac{1}{\sharp\Gamma_\alpha}\langle f_E^D(\alpha\mathbf{t}),\mathbf{P}_{\underline{k}}\rangle_{\underline{k}}\in\mathbf{Q}(\mathit{\Pi}).$$

The assertion then follows from Theorem 6.3. This completes the proof. □

7 Applications

In this section, we prove our main results of this paper. Let N_1, N_2 be positive square-free integers, and κ', κ be positive even integers. Put $w = 2\kappa + \kappa' - 3$. Let $N = \gcd(N_1, N_2)$ and $M = \mathrm{lcm}(N_1, N_2)$. Let $f \in S_{\kappa'}(\Gamma_0(N_1))$ and $g \in S_\kappa(\Gamma_0(N_2))$ be normalized elliptic newforms and $\mathbf{f}$ and $\mathbf{g}$ be the adelic lifts of f and g, respectively. Let $\tau = \otimes'_v \tau_v$ and $\pi = \otimes'_v \pi_v$ be the irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A})$ generated by $\mathbf{f}$ and $\mathbf{g}$, respectively.

If F' is a cyclic extension of $\mathbf{Q}$ with prime degree, we let $\pi_{F'}$ be the base change lift of π to $\mathrm{GL}_2(\mathbf{A}_{F'})$. We note that $\pi_{F'}$ is a unitary irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{F'})$ whose central character is trivial. Indeed, if $\eta = \otimes'_v \eta_v$ is a character of $\mathbf{A}^\times$ associated to $F'/\mathbf{Q}$ by the global class field theory, then $\pi \not\cong \pi \otimes \eta$. Otherwise, since N_2 is square-free, the condition $\pi_p \cong \pi_p \otimes \eta_p$ for all p implies η_p is unramified for all p, which is impossible since the ground field is $\mathbf{Q}$. Now we can apply [AC89, Theorem 4.2 (a) and Theorem 5.1].

We define the motivic L-function and its associated completed L-function for $\mathrm{Sym}^2(g) \otimes f$ by

$$L(s, \mathrm{Sym}^2(g) \otimes f) = \prod_p L\left(s - \frac{w}{2}, \mathrm{Sym}^2(\pi_p) \otimes \tau_p\right),$$

$$\Lambda(s, \mathrm{Sym}^2(g) \otimes f) = \prod_v L\left(s - \frac{w}{2}, \mathrm{Sym}^2(\pi_v) \otimes \tau_v\right).$$

Note that $L(s, \mathrm{Sym}^2(g) \otimes f)$ is holomorphic at $s = (w+1)/2$. We have the functional equation

$$\begin{aligned} \Lambda(s, \mathrm{Sym}^2(g) \otimes f) &= \epsilon(\mathrm{Sym}^2(g) \otimes f) \left(M^4 N N_1^{-1}\right)^{-s+(w+1)/2} \\ &\quad \times \Lambda(w+1-s, \mathrm{Sym}^2(g) \otimes f), \end{aligned} \tag{7.1}$$

where $\epsilon(\mathrm{Sym}^2(g) \otimes f) \in \{\pm 1\}$ is given by

$$\epsilon(\mathrm{Sym}^2(g) \otimes f) = (-1)^{\delta(\kappa, \kappa') + \kappa'/2} \prod_{p \mid N_1/N} w_f(p),$$

with

$$\delta(\kappa, \kappa') = \begin{cases} 1 & \text{if } 2\kappa \le \kappa', \\ -1 & \text{if } 2\kappa > \kappa', \end{cases}$$

and $w_f(p) \in \{\pm 1\}$ is the eigenvalue of the Atkin-Lehner involution of f at p. Recall the Deligne's period $\Omega_{f,g} \in \mathbf{C}^\times$ of the tensor motive associated to $\mathrm{Sym}^2(g) \otimes f$ with sign $\epsilon = (-1)^{\kappa'/2 - 1}$ defined in (1.1).

Corollary 7.1. *For $\sigma \in \mathrm{Aut}(\mathbf{C})$, we have*

$$\left(\frac{L((w+1)/2, \mathrm{Sym}^2(g) \otimes f)}{(2\pi\sqrt{-1})^{3(w+1)/2} \Omega_{f,g}}\right)^\sigma = \frac{L((w+1)/2, \mathrm{Sym}^2(g^\sigma) \otimes f^\sigma)}{(2\pi\sqrt{-1})^{3(w+1)/2} \Omega_{f^\sigma, g^\sigma}}.$$

Proof. First we assume $\kappa' \geq 2\kappa$. Define Ξ to be the set of real quadratic extensions $K/\mathbf{Q}$ such that

- $N \mid D_K$.
- $\gcd(D_K, M/N) = 1$.
- $\left(\frac{D_K}{p}\right) = 1$ for $p \mid M/N_2$.

Here D_K is the discriminant of $K/\mathbf{Q}$. Certainly Ξ contains infinitely many elements. Let $K \in \Xi$ and $\chi_K = \otimes_v \chi_{K,v} : K^\times \backslash \mathbf{A}_K^\times \to \mathbf{C}$ be the idele class character associated to K by class field theory. Put

$$\mathit{\Pi} = \pi_K \boxtimes \tau.$$

One checks that $\epsilon(\mathit{\Pi}_v) = 1$ for all v. For example, if $p \mid N$, then K_p is a ramified quadratic extension over $\mathbf{Q}_p$, and π_{K_p} (resp. τ_p) is a unramified special representation of $\mathrm{GL}_2(K_p)$ (resp. $\mathrm{GL}_2(\mathbf{Q}_p)$). Then Proposition 4.8 (2) implies $\epsilon(\mathit{\Pi}_p) = 1$. On the other hand, by the results of [Pra90] and [Pra92], we have

$$\epsilon(\mathit{\Pi}_v) = \epsilon\left(\frac{1}{2}, \mathit{\Pi}_v, r_v, \psi_v\right)\chi_{K,v}(-1),$$

for all place v. In particular, $\epsilon\left(1/2, \mathit{\Pi}, r\right) = 1$ and the matrix algebra M_2 is the unique quaternion algebra over $\mathbf{Q}$ satisfying (6.6). We see from the factorization $\epsilon(s, \mathit{\Pi}, r) = \epsilon(s, \mathrm{Sym}^2(\pi) \otimes \tau)\epsilon(s, \tau \otimes \chi_K)$ that

$$\epsilon\left(\frac{1}{2}, \mathrm{Sym}^2(\pi) \otimes \tau\right) = \epsilon\left(\frac{1}{2}, \tau \otimes \chi_K\right).$$

If $\epsilon\left(1/2, \tau \otimes \chi_K\right) = -1$, then $\epsilon\left(1/2, \mathrm{Sym}^2(\pi) \otimes \tau\right) = -1$. On the other hand, by the Galois equivariance property for the local Langlands correspondence described in [Hen01, §7], we also have $\epsilon\left(1/2, \mathrm{Sym}^2(\pi^\sigma) \otimes \tau^\sigma\right) = -1$. Therefore

$$L\left(\frac{w+1}{2}, \mathrm{Sym}^2(g^\sigma) \otimes f^\sigma\right) = L\left(\frac{w+1}{2}, \mathrm{Sym}^2(g) \otimes f\right) = 0$$

for all $\sigma \in \mathrm{Aut}(\mathbf{C})$ by the functional equation. Otherwise, by the nonvanishing theorem of [FH95], there exists $K' \in \Xi$ such that $L\left(\kappa'/2, f \otimes \chi_{K'}\right) \neq 0$. Let $\sigma \in \mathrm{Aut}(\mathbf{C})$. By [Shi77], we have

$$\left(\frac{L(\kappa'/2, f \otimes \chi_{K'})}{D_{K'}^{1/2}\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_f^{-\epsilon}}\right)^\sigma = \frac{L(\kappa'/2, f^\sigma \otimes \chi_{K'})}{D_{K'}^{1/2}\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_{f^\sigma}^{-\epsilon}},$$

$$\left(\frac{\langle f, f\rangle}{(\sqrt{-1})^{\kappa'-1}\Omega_f^+\Omega_f^-}\right)^\sigma = \frac{\langle f^\sigma, f^\sigma\rangle}{(\sqrt{-1})^{\kappa'-1}\Omega_{f^\sigma}^+\Omega_{f^\sigma}^-}.$$

Let $g_{K'}$ be the normalized Hilbert modular newform associated to $\pi_{K'}$, the base change lift of π to $\mathrm{GL}_2(\mathbf{A}_{K'})$. By Corollary 6.4-(1), we have

$$\left(\frac{L((w+1)/2, g_{K'} \otimes f, r)}{D_{K'}^{1/2}\pi^{2\kappa'}\langle f, f\rangle^2_{\Gamma_0(N)}}\right)^\sigma = \frac{L((w+1)/2, g^\sigma_{K'} \otimes f^\sigma, r)}{D_{K'}^{1/2}\pi^{2\kappa'}\langle f^\sigma, f^\sigma\rangle^2_{\Gamma_0(N)}}.$$

Note that $g^\sigma_{K'} = (g^\sigma)_{K'}$. Now the corollary then follows from combining these equations with the following factorization

$$L\left(\frac{w+1}{2}, g_{K'} \otimes f, r\right) = L\left(\frac{w+1}{2}, \mathrm{Sym}^2(g) \otimes f\right) L\left(\frac{\kappa'}{2}, f \otimes \chi_{K'}\right).$$

This completes the proof in case $\kappa' \geq 2\kappa$.
Assume $2\kappa > \kappa'$. The case when $N_1 = 1$ is proved in [Che19]. We assume $N_1 > 1$. By the non-vanishing results of [FH95], the assumption $N_1 > 1$ enable us to choose a real quadratic field K with fundamental discriminant $\mathcal{D} > 0$ such that $L\left(\frac{\kappa'}{2}, f \otimes \chi_{\mathcal{D}}\right) \neq 0$, where $\chi_{\mathcal{D}}$ is the Dirichlet character associated to $K/\mathbf{Q}$ by class field theory. Let g_K be the normalized Hilbert modular newform associated to π_K and $\mathbf{g}_K \in \pi_K$ be its adelic lift. By equation (6.11), the Petersson norm of g_K is given by

$$\begin{aligned}\langle g_K, g_K\rangle = h_K &\left[\mathrm{GL}_2(\widehat{\mathcal{O}}_K) : K_0(N_2\mathcal{O}_K)\right] \mathcal{D}_K^{3/2}\zeta_K(2) \\ &\times \int_{\mathbf{A}_K^\times \mathrm{GL}_2^\times(K)\backslash \mathrm{GL}_2^\times(\mathbf{A}_K)} |\mathbf{g}_K(h)|^2 dh,\end{aligned}$$

where h_K is the class number of K and dh is the Tamagawa measure on $\mathbf{A}_K^\times\backslash\mathrm{GL}_2(\mathbf{A}_K)$. Define the Petersson norm of g by

$$\langle g, g\rangle = \int_{\Gamma_0(N_2)\backslash\mathfrak{H}} |g(\tau)|^2 y^{\kappa-2} d\tau.$$

We have

$$\left(\frac{\langle g_K, g_K\rangle}{\langle g, g\rangle^2}\right)^\sigma = \frac{\langle (g^\sigma)_K, (g^\sigma)_K\rangle}{\langle g^\sigma, g^\sigma\rangle^2}.$$

This equality follows from combining the factorization

$$L(1, \pi_K, \mathrm{Ad}) = L(1, \pi, \mathrm{Ad})L(1, \pi, \mathrm{Ad}, \chi),$$

and a result of Sturm [Stu89]. Put

$$\mathit{\Pi} = \pi_K \boxtimes \tau.$$

Since $L\left(\frac{\kappa'}{2}, f \otimes \chi_{\mathcal{D}}\right) \neq 0$, we deduce that if $\epsilon(\mathit{\Pi}) = -1$, then $L(\frac{w+1}{2}, \mathrm{Sym}^2(g) \otimes f) = 0$. Therefore, we may assume $\epsilon(\mathit{\Pi}) = 1$. The rest of the proof is similar to the above case except we use Corollary 6.4-(2) here instead. This completes the proof. □

Combining with the result of Januszewski in [Jan18], we obtain a conditional result on Deligne's conjecture for arbitrary critical values with abelian twists.

Corollary 7.2. *Assume* $2\kappa > \kappa'$ *and* $L(\frac{w+1}{2}, \mathrm{Sym}^2(g) \otimes f) \neq 0$. *Let* $n \in \mathbf{Z}$ *be a critical integer for* $L(s, \mathrm{Sym}^2(g) \otimes f)$ *and* χ *be a Dirichlet character such that* $(-1)^n\chi(-1) = \epsilon$. *For* $\sigma \in \mathrm{Aut}(\mathbf{C})$, *we have*

$$\left(\frac{L(n, \mathrm{Sym}^2(g) \otimes f \otimes \chi)}{G(\chi)^3(2\pi\sqrt{-1})^{3n}\Omega_{f,g}}\right)^\sigma = \frac{L(n, \mathrm{Sym}^2(g^\sigma) \otimes f^\sigma \otimes \chi^\sigma)}{G(\chi^\sigma)^3(2\pi\sqrt{-1})^{3n}\Omega_{f^\sigma,g^\sigma}}.$$

Proof. Since N_2 is square-free, the functorial lift of $\mathrm{Sym}^2(\pi)$ to $\mathrm{GL}_3(\mathbf{A})$ is a cuspidal automorphic representation by Theorem 9.3 in [GJ78]. By Theorem A in [Jan18], there exists cohomological periods $\Omega_\pm(f, g) \in \mathbf{C}^\times$ such that

$$\begin{aligned}&\left(\frac{L(n, \mathrm{Sym}^2(g) \otimes f \otimes \chi)}{G(\chi)^3(2\pi\sqrt{-1})^{3n}\Omega_{(-1)^n\chi(-1)}(f, g)}\right)^\sigma \\ &= \frac{L(n, \mathrm{Sym}^2(g^\sigma) \otimes f^\sigma \otimes \chi^\sigma)}{G(\chi^\sigma)^3(2\pi\sqrt{-1})^{3n}\Omega_{(-1)^n\chi(-1)}(f^\sigma, g^\sigma)}\end{aligned} \tag{7.2}$$

for $\sigma \in \mathrm{Aut}(\mathbf{C})$. Note that the condition $2\kappa > \kappa'$ is equivalent to the balanced condition in [Jan18]. For $n = \frac{w+1}{2}$ and $\chi = 1$, by Corollary 7.1, (7.2), and the assumption $L\left(\frac{w+1}{2}, \mathrm{Sym}^2(g) \otimes f\right) \neq 0$, we have

$$\left(\frac{\Omega_{f,g}}{\Omega_\epsilon(f, g)}\right)^\sigma = \frac{\Omega_{f^\sigma,g^\sigma}}{\Omega_\epsilon(f^\sigma, g^\sigma)} \tag{7.3}$$

for $\sigma \in \mathrm{Aut}(\mathbf{C})$. The assertion follows from (7.2) and (7.3). This completes the proof. ☐

We consider the case when E is a cubic Galois extension over $\mathbf{Q}$. Put $w = 3\kappa' - 3$. We define the motivic L-function for $\mathrm{Sym}^3(f)$ by

$$L(s, \mathrm{Sym}^3(f)) = \prod_p L\left(s - \frac{w}{2}, \mathrm{Sym}^3(\tau_p)\right).$$

Note that $L(s, \mathrm{Sym}^3(f))$ is holomorphic at $s = \frac{w+1}{2}$. Denote $\Omega_{f,\mathrm{Sym}^3} \in \mathbf{C}^\times$ be the period defined as in (1.2).

Corollary 7.3. *Assume* $N_1 > 1$ *and there exist a cubic Dirichlet character* χ *such that* $L\left(\frac{\kappa'}{2}, f \otimes \chi\right) \neq 0$. *For* $\sigma \in \mathrm{Aut}(\mathbf{C})$, *we have*

$$\left(\frac{L((w+1)/2, \mathrm{Sym}^3(f))}{(2\pi\sqrt{-1})^{w+1}\Omega_{f,\mathrm{Sym}^3}}\right)^\sigma = \frac{L((w+1)/2, \mathrm{Sym}^3(f^\sigma))}{(2\pi\sqrt{-1})^{w+1}\Omega_{f^\sigma,\mathrm{Sym}^3}}.$$

Proof. The argument is similar to that of Corollary 7.1. Let E be the cubic Galois extension of $\mathbf{Q}$ associated to χ by global class filed theory, and χ_E be a idele class character associated to $E/\mathbf{Q}$. Let f_E be the normalized Hilbert modular newform associated to π_E and $\langle f_E, f_E\rangle_{K_0(N_1\mathcal{O}_E)}$ be the Petersson norm of f_E defined in §6.4. The factorization

$$L(1,\pi_E,\mathrm{Ad}) = L(1,\pi,\mathrm{Ad})L(1,\pi,\mathrm{Ad},\chi_E)L(1,\pi,\mathrm{Ad},\bar{\chi}_E),$$

toghther with Lemma 6.1 and Sturm's result [Stu80] yield

$$\left(\frac{\langle f,f\rangle^3}{\langle f_E,f_E\rangle_{K_0(N_1\mathcal{O}_E)}}\right)^\sigma = \frac{\langle f^\sigma,f^\sigma\rangle^3}{\langle (f^\sigma)_E,(f^\sigma)_E\rangle_{K_0(N_1\mathcal{O}_E)}}.$$

Using again [Shi77], we have

$$\left(\frac{L(\kappa'/2,f\otimes\chi)}{G(\chi)\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_f^{-\epsilon}}\right)^\sigma = \frac{L(\kappa'/2,f^\sigma\otimes\chi)}{G(\chi^\sigma)\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_{f^\sigma}^{-\epsilon}},$$

$$\left(\frac{L(\kappa'/2,f\otimes\bar{\chi})}{G(\bar{\chi})\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_f^{-\epsilon}}\right)^\sigma = \frac{L(\kappa'/2,f^\sigma\otimes\bar{\chi})}{G(\bar{\chi}^\sigma)\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_{f^\sigma}^{-\epsilon}},$$

$$\left(\frac{\langle f,f\rangle}{(\sqrt{-1})^{\kappa'-1}\Omega_f^+\Omega_f^-}\right)^\sigma = \frac{\langle f^\sigma,f^\sigma\rangle}{(\sqrt{-1})^{\kappa'-1}\Omega_{f^\sigma}^+\Omega_{f^\sigma}^-}.$$

Here $G(\chi)$ (resp. $G(\overline{\chi})$) is the Gauss sum associated to χ (resp. $\overline{\chi}$) defined in [Shi77]. Notice that since the Hecke field of f is totally real, we have

$$L\left(\frac{\kappa'}{2},f\otimes\bar{\chi}\right) = \overline{L\left(\frac{\kappa'}{2},f\otimes\chi\right)} \neq 0.$$

Also, as $E/\mathbf{Q}$ is Galois, $\mathcal{D}_E$ is a square. The corollary then follows from these equations together with Corollary 6.4-(2) and the factorization

$$L\left(\frac{w+1}{2},f_E,r\right) = L\left(\frac{w+1}{2},\mathrm{Sym}^3(f)\right)L\left(\frac{\kappa'}{2},f\otimes\chi\right)L\left(\frac{\kappa'}{2},f\otimes\bar{\chi}\right).$$

This finishes the proof. □

Combining with the result of Januszewski in [Jan16] and Jiang-Sun-Tian in [JST19], we obtain a conditional result on Deligne's conjecture for arbitrary critical values with abelian twists.

Corollary 7.4. *Suppose that $N_1 > 1$, $L(\frac{w+1}{2},\mathrm{Sym}^3(f)) \neq 0$, and there exist a cubic Dirichlet character χ such that $L\left(\frac{\kappa'}{2},f\otimes\chi\right) \neq 0$. Let $n \in \mathbf{Z}$ be a critical integer for $L(s,\mathrm{Sym}^3(f))$ and μ be a Dirichlet character such that $(-1)^n\mu(-1) = \epsilon$. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, we have*

$$\left(\frac{L(n,\mathrm{Sym}^3(f)\otimes\mu)}{G(\mu)^2(2\pi\sqrt{-1})^{2n}\Omega_{f,\mathrm{Sym}^3}}\right)^\sigma = \frac{L(n,\mathrm{Sym}^3(f^\sigma)\otimes\mu^\sigma)}{G(\mu^\sigma)^2(2\pi\sqrt{-1})^{2n}\Omega_{f^\sigma,\mathrm{Sym}^3}}.$$

Proof. Since N_1 is square-free, the functorial lift of $\mathrm{Sym}^3(\tau)$ to $\mathrm{GL}_4(\mathbf{A})$ is a cuspidal automorphic representation by Theorem B in [KS02]. The rest of the proof is similar to that of Corollary 7.2 except in this case we use Theorem A in [Jan16] and Theorem 1.1 in [JST19]. □

References

[AC89] J. Arthur and L. Clozel. *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula.* Number 120. Annals of Mathematics Studies, 1989. Princeton University Press, Princeton, NJ.

[BFH05] B. Brubaker, S. Friedberg, and J. Hoffstein. Cubic twists of GL(2) automorphic L-functions. *Inventiones mathematicae*, 160:31–58, 2005.

[BJ79] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations, and L-functions*, volume 33 of *Proceedings of Symposia in Pure Mathematics*, pages 189–202, 1979. part 1.

[Bla87] D. Blasius. Appendix to Orloff, Critical values of certain tensor product L-functions. *Inventiones mathematicae*, 90:181–188, 1987.

[BSP96] S. Bocherer and R. Schulze-Pillot. On the central critical value of the triple product L-function. In *Number theory*, volume 235 of *London Mathematical Society Lecture Note Series*, pages 1–46. Cambridge Univ. Press, Cambridge, 1996.

[Bum98] D. Bump. *Automorphic Forms and Representations.* Cambridge Studies in Advanced Mathematics 55. Cambridge University Press, Cambridge, first edition, 1998.

[Cas73] W. Casselman. On some results of Atkin and Lehner. *Mathematische Annalen*, 201:301–314, 1973.

[CCI19] S.-Y. Chen, Y. Cheng, and I. Ishikawa. Gamma factors for Asai representations of GL(2). *J. Number Theory*, 2019. DOI: 10.1016/j.jnt.2019.08.008.

[Che19] S.-Y. Chen. Pullback formulas for nearly holomorphic Saito-Kurokawa lifts. *Manuscripta Math.*, 2019. DOI: 10.1007/s00229-019-01111-2.

[FH95] S. Friedberg and J. Hoffstein. Nonvanishing theorems for automorphic L-functions on GL(2). *Annals of Mathematics*, 142(2):385–423, 1995.

[Fla79] D. Flath. Decomposition of representations into tensor products. In *Automorphic forms, representations, and L-functions*, volume 33 of *Proc. Symp. Pure Math*, Part 1, pages 179–183, 1979.

[FM14] M. Furusawa and K. Morimoto. On special values of certain L-functions. *American Journal of Mathematics*, 136(5):1385–1407, 2014.

[FM16] M. Furusawa and K. Morimoto. On special values for certain L-functions. II. *American Journal of Mathematics*, 138(4):1117–1166, 2016.

[Gar87] P. Garrett. Decomposition of Eisenstein series: Rankin triple products. *Annals of Mathematics*, 125(2):209–235, 1987.

[GH93] P. Garrett and M. Harris. Special values of triple product L-functions. *American Journal of Mathematics*, 115(1):161–240, 1993.

[GJ72] R. Godement and H. Jacquet. *Zeta functions of simple algebras*, volume 260 of *Lecture Notes in Mathematics*. Springer, 1972.

[GJ78] S. Gelbart and H. Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Scient. Ecole Norm. Sup.*, 11(4):471–542, 1978.

[GK92] B. Gross and S. Kudla. Heights and the central critical values of triple product L-functions. *Compositio Mathematica*, 81(2):143–209, 1992.

[Har90] M. Harris. Automorphic forms of $\overline{\partial}$-cohomology type as coherent cohomology classes. *Journal of Differential Geometry*, 32:1–63, 1990.

[Har93] M. Harris. L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms. *Journal of American Mathematical Society*, 6(3):637–719, 1993.

[Har94] M. Harris. Period invariants of Hilbert modular forms. II. *Compositio Mathematica*, 94:201–226, 1994.

[Hen01] G. Henniart. Sur la conjecture de Langlands locale pour GL_n. *Journal de Théorie des Nombres de Bordeaux*, 13(1):167–187, 2001.

[Hid93] H. Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[HK91] M. Harris and S. Kudla. The central critical value of a triple product L-function. *Annals of Mathematics*, 133(3):605–672, 1991.

[HK04] M. Harris and S. Kudla. *On a conjecture of Jacquet*, pages 355–371. Johns Hopkins Univ. Press, Baltimore, 2004.

[Ich05] A. Ichino. Pullbacks of Saito-Kurokawa lifts. *Inventiones mathematicae*, 162:551–647, 2005.

[Ich08] A. Ichino. Trilinear forms and the central values of triple product L-functions. *Duke Mathematical Journal*, 145(2):281–307, 2008.

[II10] A. Ichino and T. Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. *Goemetric and functional analysis*, 19:1378–1425, 2010.

[Ike89] T. Ikeda. On the functional equations of the triple L-functions. *Kyoto Journal of Mathematics*, 29:175–219, 1989.

[Ike98] T. Ikeda. On the gamma factor of the triple L-function. II. *Journal fur die reine und angewandte Mathematik*, 499:199–223, 1998.

[IP18] A. Ichino and K. Prasanna. Period of quaternionic Shimura varieties. I. *Mem. Amer. Math. Soc.*, 2018. to appear.

[Jan16] F. Januszewski. On period relations for automorphic L-functions. II. 2016. arXiv:1604.04253.

[Jan18] F. Januszewski. On period relations for automorphic L-functions. I. *Trans. Am. Math. Soc.*, 2018. DOI: 10.1090/tran/7527.

[JL70] H. Jacquet and R. Langlands. *Automorphic Forms on* GL(2), volume 114 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin and New York, 1970.

[JST19] D. Jiang, B. Sun, and F. Tian. Period relations for standard L-functions of symplectic type. 2019. arXiv:1909.03476.

[KS02] H. H. Kim and F. Shahidi. Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. *Annals of Mathematics*, 155:837–893, 2002.

[Nel11] P. Nelson. Equidistribution of cusp forms in the level aspect. *Duke Mathematical Journal*, 160(3):467–501, 2011.

[Orl87] T. Orloff. Special values and mixed weight triple products. *Inventiones mathematicae*, 90:169–180, 1987. with appendix by Don Blasius.

[Pra90] D. Prasad. Trilinear forms for representations of GL(2) and local ϵ-factors. *Compositio Mathematica*, 75(1):1–46, 1990.

[Pra92] D. Prasad. Invariant linear forms for representations of GL(2) over a local field. *American Journal of Mathematics*, 114:1317–1363, 1992.

[PSP08] D. Prasad and R. Schulze-Pillot. Generalised form of a conjecture of Jacquet and a local consequence. *Journal fur die reine und angewandte Mathematik*, (616):219–236, 2008.

[PSR87] I. Piatetski-Shapiro and S. Rallis. Rankin triple L-functions. *Compositio Mathematica*, 64(1):31–115, 1987. MR 911357.

[Rag09] A. Raghuram. On the special values of certain Rankin-Selberg L-functions and applications to odd symmetric power L-functions of modular forms. *International Mathematics Research Notices*, 2010(2):334–372, 2009.

[Sat87] T. Satoh. Some remarks on triple L-functions. *Mathematische Annalen*, 276:687–698, 1987.

[Shi76] G. Shimura. The special values of the zeta functions associated with cusp forms. *Communications on Pure and Applied Mathematics*, pages 783–804, 1976.

[Shi77] G. Shimura. On the periods of modular forms. *Mathematische Annalen*, 229:211–221, 1977.

[Shi78] G. Shimura. The special values of the zeta functions associated with Hilbert modular forms. *Duke Mathematical Journal*, 45(3):637–679, 1978.

[Shi81] G. Shimura. On certain Zeta functions attached to two Hilbert modular forms. II. The case of automorphic forms on a quaternion algebra. *Annals of Mathematics*, 114(3):569–607, 1981.

[Stu80] J. Sturm. Special values of zeta functions, and Eisenstein series of half Integral weight. *American Journal of Mathematics*, 102(2):219–240, 1980.

[Stu89] J. Sturm. Evaluation of the symmetric square at the near center point. *American Journal of Mathematics*, 111(4):585–598, 1989.

[Wal85] J-L. Waldspurger. Sur les valeurs de certaines fonctions L automorphes en leur centre de symetrie. *Compositio Mathematica*, 54(2):173–242, 1985.

[Xue19] H. Xue. Central values of degree six L-functions. *J. Number Theory*, 2019. DOI: 10.1016/j.jnt.2019.03.007.

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