Annihilating Wild Kernels

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Abstract. Let L/K be a finite Galois extension of number fields with Galois group G. Let p be an odd prime and $r > 1$ be an integer. Assuming a conjecture of Schneider, we formulate a conjecture that relates special values of equivariant Artin L-series at $s = r$ to the compact support cohomology of the étale p-adic sheaf $\mathbb{Z}_p(r)$. We show that our conjecture is essentially equivalent to the p-part of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}[G])$. We derive from this explicit constraints on the Galois module structure of Banaszak's p-adic wild kernels.

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1 Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. To each finite set S of places of K containing all archimedean places, one can associate a so-called 'Stickelberger element' θ_S in the center of the complex group algebra $\mathbb{C}[G]$. This Stickelberger element is defined via L-values at zero of S-truncated Artin L-functions attached to the (complex) characters of G. Let us denote the roots of unity of L by μ_L and the class group of L by cl_L. Assume that S contains all finite primes of K that ramify in L/K. Then it was independently shown in [Bar78], [CN79] and [DR80] that when G is abelian we have

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S \subseteq \mathbb{Z}[G], \tag{1.1}$$

where we denote by $\mathrm{Ann}_\Lambda(M)$ the annihilator ideal of M regarded as a module over the ring Λ. Now a conjecture of Brumer asserts that $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S$ annihilates cl_L.

Using L-values at integers $r < 0$, one can define higher Stickelberger elements $\theta_S(r)$. When G is abelian, Coates and Sinnott [CS74] conjectured that these elements can be used to construct annihilators of the higher K-groups $K_{-2r}(\mathcal{O}_{L,S})$, where we denote by $\mathcal{O}_{L,S}$ the ring of $S(L)$-integers in L for any finite set S of places of K; here, we write $S(L)$ for the set of places of L which lie above those in S. Coates and Sinnott essentially proved a p-adic étale cohomological version of their conjecture in the case $K = \mathbb{Q}$. First results on the K-theoretic version are due to Banaszak [Ban92, Ban93] and Nguyen Quang Do [NQD92]. However if, for example, L is totally real and r is even, these conjectures merely predict that zero annihilates $K_{-2r}(\mathcal{O}_{L,S})$ if $r < 0$ and cl_L if $r = 0$.

In the case $r = 0$, Burns [Bur11] presented a universal theory of refined Stark conjectures. In particular, the Galois group G may be non-abelian, and he uses leading terms rather than values of Artin L-functions to construct conjectural nontrivial annihilators of the class group. His conjecture thereby extends the aforementioned conjecture of Brumer (we point out that there are different generalizations of Brumer's conjecture due to the author [Nic11b] and Dejou and Roblot [DR14]). Similarly, in the case $r < 0$ the author [Nic11a] has formulated a conjecture on the annihilation of higher K-groups which generalises the Coates–Sinnott conjecture and a conjecture of Snaith [Sna06]. More precisely, using leading terms at negative integers a certain 'canonical fractional Galois ideal' $\mathcal{J}_r^S$ is defined. It is then conjectured that for every odd prime p and every $x \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(K_{1-2r}(\mathcal{O}_{L,S})_{\mathrm{tor}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ one has

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(x) \cdot \mathcal{H}_p(G) \cdot \mathcal{J}_r^S \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(K_{-2r}(\mathcal{O}_{L,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

Here, the subscript 'tor' refers to the torsion submodule of $K_{1-2r}(\mathcal{O}_{L,S})$, we denote the reduced norm of any $x \in \mathbb{Q}_p[G]$ by $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(x)$, and $\mathcal{H}_p(G)$ denotes a certain 'denominator ideal' (introduced in [Nic10]; see §7.2).

When G is abelian and $r = 1$, Solomon [Sol08] has defined a certain ideal which he conjectures to annihilate the p-part of the class group. This has recently been generalized to arbitrary (finite) Galois groups G by Castillo and Jones [CJ13]. All these annihilation conjectures are implied by appropriate special cases of the equivariant Tamagawa number conjecture (ETNC) formulated by Burns and Flach [BF01].

Now let $r > 1$ be a positive integer. When L/K is an abelian extension of totally real fields and r is even, Barrett [Bar09] has defined a 'Higher Solomon ideal' which he conjectures to annihilate the p-adic wild kernel $K^w_{2r-2}(\mathcal{O}_{L,S})_p$ of Banaszak [Ban93] (see also [NQD92]). There is an analogue on 'minus parts' when L/K is an abelian CM-extension and r is odd. Under the same conditions Barrett and Burns [BB13] have constructed conjectural annihilators of the p-adic wild kernel via integer values of p-adic Artin L-functions. This approach has been further generalized to the non-abelian situation by Burns and Macias Castillo [BMC14].

In this paper we consider the most general case, where L/K is an arbitrary (not necessarily abelian or totally real) Galois extension and $r > 1$ is an arbitrary integer. Let G_L be the absolute Galois group of L. Assuming conjectures of Gross [Gro05] and of Schneider [Sch79], we define a canonical fractional Galois ideal $\mathcal{J}_r^S$ and conjecture that for every $x \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_L})$ we have that

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(x) \cdot \mathcal{H}_p(G) \cdot \mathcal{J}_r^S \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(K_{2r-2}^w(\mathcal{O}_{L,S})_p).$$

Note that the conjectures of Gross and Schneider are known when L is totally real and r is even (see Theorem 5.2 and Theorem 3.9 below, respectively). When in addition L/K is abelian, we show that our conjecture is compatible with that of Barrett.

In order to show that our conjecture is implied by the appropriate special case of the ETNC, we reformulate the ETNC for the pair $h^0(\mathrm{Spec}(L)(r), \mathbb{Z}[G])$ in the spirit of the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss [GRW99] and the 'leading term conjectures' of Breuning and Burns [BB07]. Note that the leading term conjecture at $s = 1$ is equivalent to the ETNC for the pair $h^0(\mathrm{Spec}(L)(1), \mathbb{Z}[G])$ when Leopoldt's conjecture holds (see [BB10]), and that Schneider's conjecture is a natural analogue when $r > 1$. This reformulation is more explicit than the rather involved and general formulation of Burns and Flach [BF01]. This will actually occupy a large part of the paper and is interesting in its own right. Moreover, the relation to the ETNC will lead to a proof of our annihilation conjecture in several important cases.

In a little more detail, we modify the compact support cohomology of the étale p-adic sheaf $\mathbb{Z}_p(r)$ such that we obtain a complex which is acyclic outside degrees 2 and 3. We show that this complex is a perfect complex of $\mathbb{Z}_p[G]$-modules provided that Schneider's conjecture holds. Assuming Gross' conjecture we define a trivialization of this complex that involves Soulé's p-adic Chern class maps [Sou79] and the Bloch–Kato exponential map [BK90]. These data define a refined Euler characteristic which our conjecture relates to the special values of the equivariant Artin L-series at $s = r$ and determinants of a certain regulator map. This relation is expressed as an equality in a relative algebraic K-group.

This article is organized as follows. In §2 we review the higher Quillen K-theory of rings of integers in number fields. We discuss its relation to étale cohomology and introduce Banaszak's wild kernels. In §3 we prove basic properties of the compact support cohomology of the étale p-adic sheaf $\mathbb{Z}_p(r)$, where $r > 1$ is an integer. We recall Schneider's conjecture and provide a reformulation in terms of Tate–Shafarevich groups (which originates with Barrett [Bar09]). We then construct the aforementioned complex of $\mathbb{Z}_p[G]$-modules which is perfect when Schneider's conjecture holds. We recall some background on relative algebraic K-theory and in particular on refined Euler characteristics in §4. We state Gross' conjecture on leading terms of Artin L-functions at negative integers in §5 and give a reformulation at positive integers by means of the functional equation. In §6 we construct a trivialization of our conjecturally perfect complex and formulate a leading term conjecture at $s = r$ for every integer $r > 1$. We show that our conjecture is essentially equivalent to the ETNC for the pair $h^0(\mathrm{Spec}(L)(r), \mathbb{Z}[G])$. Finally, in §7 we define the canonical fractional Galois ideal and give a precise formulation of our conjecture on the annihilation of p-adic wild kernels. We show that this conjecture is implied by the leading term conjecture of §6. The relation to the ETNC then implies that our conjectures hold in several important cases. We also discuss the relation to a recent conjecture of Burns, Kurihara and Sano [BKS].

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Notation and conventions

All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. Unadorned tensor products will always denote tensor products over $\mathbb{Z}$. If K is a field, we denote its absolute Galois group by G_K. For a module M we write M_{tor} for its torsion submodule and set $M_{\mathrm{tf}} := M/M_{\mathrm{tor}}$ which we regard as embedded into $\mathbb{Q} \otimes M$. If R is a ring, we write $M_{m\times n}(R)$ for the set of all $m \times n$ matrices with entries in R. We denote the group of invertible matrices in $M_{n\times n}(R)$ by $\mathrm{GL}_n(R)$.

2 Higher K-theory of rings of integers

2.1 The setup

Let L/K be a finite Galois extension of number fields with Galois group G. We write S_∞ for the set of archimedean places of K and let S be a finite set of places of K containing S_∞. We let $\mathcal{O}_{L,S}$ be the ring of $S(L)$-integers in L, where $S(L)$ denotes the finite set of places of L that lie above a place in S; we will abbreviate $\mathcal{O}_{L,S_\infty}$ to $\mathcal{O}_L$.

For any place v of K we choose a place w of L above v and write G_w and I_w for the decomposition group and inertia subgroup of L/K at w, respectively. We denote the completions of L and K at w and v by L_w and K_v, respectively, and identify the Galois group of the extension L_w/K_v with G_w. We put $\overline{G_w} := G_w/I_w$ which we identify with the Galois group of the residue field extension which we denote by $L(w)/K(v)$. Finally, we let $\phi_w \in \overline{G_w}$ be the Frobenius automorphism, and we denote the cardinality of $K(v)$ by $N(v)$.

2.2 Higher K-theory

For an integer $n \geq 0$ and a ring R we write $K_n(R)$ for the Quillen K-theory of R. In the case $R = \mathcal{O}_{L,S}$ or $R = L$ the groups $K_n(\mathcal{O}_{L,S})$ and $K_n(L)$ are equipped with a natural G-action and for every integer $r > 1$ the inclusion $\mathcal{O}_{L,S} \subseteq L$ induces an isomorphism of $\mathbb{Z}[G]$-modules

$$K_{2r-1}(\mathcal{O}_{L,S}) \simeq K_{2r-1}(L). \tag{2.1}$$

Moreover, if S' is a second finite set of places of K containing S, then for every $r > 1$ there is a natural exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to K_{2r}(\mathcal{O}_{L,S}) \to K_{2r}(\mathcal{O}_{L,S'}) \to \bigoplus_{w \in S'(L) \setminus S(L)} K_{2r-1}(L(w)) \to 0. \tag{2.2}$$

Both results, (2.1) and (2.2), follow from work of Soulé [Sou79], see [Wei13, Chapter V, Theorem 6.8]. We also note that sequence (2.2) remains left-exact in the case $r = 1$. The structure of the finite $\mathbb{Z}[\overline{G_w}]$-modules $K_{2r-1}(L(w))$ has been determined by Quillen [Qui72] (see also [Wei13, Chapter IV, Theorem 1.12 and Corollary 1.13]) to be

$$K_{2r-1}(L(w)) \simeq \mathbb{Z}[\overline{G_w}]/(\phi_w - N(v)^r). \tag{2.3}$$

If S contains all places of K that ramify in L/K, we thus have an isomorphism of $\mathbb{Z}[G]$-modules

$$\bigoplus_{w \in S'(L) \setminus S(L)} K_{2r-1}(L(w)) \simeq \bigoplus_{v \in S' \setminus S} \mathrm{Ind}_{G_w}^{G} \mathbb{Z}[G_w]/(\phi_w - N(v)^r), \tag{2.4}$$

where we write $\mathrm{Ind}_U^G M := \mathbb{Z}[G] \otimes_{\mathbb{Z}[U]} M$ for any subgroup U of G and any $\mathbb{Z}[U]$-module M. We also note that the even K-groups $K_{2r}(\mathbb{F})$ of a finite field $\mathbb{F}$ all vanish.

2.3 The regulators of Borel and Beilinson

Let $\Sigma(L)$ be the set of embeddings of L into the complex numbers $\mathbb{C}$; we then have $|\Sigma(L)| = r_1 + 2r_2$, where r_1 and r_2 are the number of real embeddings and the number of pairs of complex embeddings of L, respectively. For an integer $k \in \mathbb{Z}$ we define

$$H_k(L) := \bigoplus_{\Sigma(L)} (2\pi i)^{-k} \mathbb{Z}$$

which is endowed with a natural $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action, diagonally on $\Sigma(L)$ and on $(2\pi i)^{-k}$. The invariants of $H_k(L)$ under this action will be denoted by $H_k^+(L)$, and it is easily seen that

$$d_k := \mathrm{rank}_{\mathbb{Z}}(H^+_{1-k}(L)) = \begin{cases} r_1 + r_2 & \text{if} \quad 2 \nmid k \\ r_2 & \text{if} \quad 2 \mid k. \end{cases} \tag{2.5}$$

Let $r > 1$ be an integer. Borel [Bor74] has shown that the even K-groups $K_{2r-2}(\mathcal{O}_L)$ (and thus $K_{2r-2}(\mathcal{O}_{L,S})$ for any S as above by (2.2) and (2.3)) are finite, and that the odd K-groups $K_{2r-1}(\mathcal{O}_L)$ are finitely generated abelian groups of rank d_r. More precisely, Borel constructed regulator maps

$$\rho_r : K_{2r-1}(\mathcal{O}_L) \to H^+_{1-r}(L) \otimes \mathbb{R} \tag{2.6}$$

with finite kernel. Its image is a full lattice in $H^+_{1-r}(L) \otimes \mathbb{R}$. The covolume of this lattice is called the Borel regulator and will be denoted by $R_r(L)$. Moreover, Borel showed that

$$\frac{\zeta^*_L(1-r)}{R_r(L)} \in \mathbb{Q}^\times, \tag{2.7}$$

where $\zeta^*_L(1-r)$ denotes the leading term at $s = 1-r$ of the Dedeking zeta function $\zeta_L(s)$ attached to L.

Remark 2.1. In the context of the ETNC it is often more natural to work with Beilinson's regulator map [Beĭ84]. However, by a result of Burgos Gil [BG02] Borel's regulator map is twice the regulator map of Beilinson. As we will eventually work prime by prime and exclude the prime 2, there will be no essential difference which regulator map we use.

2.4 Derived categories and Galois cohomology

Let Λ be a noetherian ring and $\mathrm{PMod}(\Lambda)$ be the category of all finitely generated projective Λ-modules. We write $\mathcal{D}(\Lambda)$ for the derived category of Λ-modules and $\mathcal{C}^b(\mathrm{PMod}(\Lambda))$ for the category of bounded complexes of finitely generated projective Λ-modules. Recall that a complex of Λ-modules is called perfect if it is isomorphic in $\mathcal{D}(\Lambda)$ to an element of $\mathcal{C}^b(\mathrm{PMod}(\Lambda))$. We denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ comprising perfect complexes by $\mathcal{D}^{\mathrm{perf}}(\Lambda)$.

If M is a Λ-module and n is an integer, we write $M[n]$ for the complex

$$\cdots \to 0 \to 0 \to M \to 0 \to 0 \to \cdots,$$

where M is placed in degree $-n$. We will also use the following convenient notation: When $t \geq 1$ and $n_1, \ldots, n_t$ are integers, we put

$$M\{n_1, \ldots, n_t\} := \bigoplus_{i=1}^{t} M[-n_i].$$

Note that $M\{n\} = M[-n]$ and $M\{n_1, \ldots, n_t\}[n] = M\{n_1 - n, \ldots, n_t - n\}$.

Recall the notation of §2.1. In particular, L/K is a Galois extension of number fields with Galois group G. For a finite set S of places of K containing S_∞ we let $G_{L,S}$ be the Galois group over L of the maximal extension of L that is unramified outside $S(L)$. For any topological $G_{L,S}$-module M we write $R\Gamma(\mathcal{O}_{L,S}, M)$ for the complex of continuous cochains of $G_{L,S}$ with coefficients in M. If F is a field and M is a topological G_F-module, we likewise define $R\Gamma(F, M)$ to be the complex of continuous cochains of G_F with coefficients in M.

If F is a global or a local field of characteristic zero, and M is a discrete or a compact G_F-module, then for $r \in \mathbb{Z}$ we denote the r-th Tate twist of M by $M(r)$. Now let p be a prime and suppose that S also contains all p-adic places of K. Then we will particularly be interested in the complexes $R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ in $\mathcal{D}(\mathbb{Z}_p[G])$. Note that for an integer i the cohomology group in degree i of $R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ naturally identifies with $H^i_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$, the i-th étale cohomology group of the affine scheme $\mathrm{Spec}(\mathcal{O}_{L,S})$ with coefficients in the étale p-adic sheaf $\mathbb{Z}_p(r)$.

2.5 p-adic Chern class maps

Fix an odd prime p and assume that S contains S_∞ and the set S_p of all p-adic places of K. Then for any integer $r > 1$ and $i = 1, 2$ Soulé [Sou79] has constructed canonical G-equivariant p-adic Chern class maps

$$ch^{(p)}_{r,i} : K_{2r-i}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p \to H^i_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)).$$

We need the following deep result.

Theorem 2.2 (Quillen–Lichtenbaum Conjecture). *Let p be an odd prime. Then for any integer $r > 1$ and $i = 1, 2$ the p-adic Chern class maps $ch^{(p)}_{r,i}$ are isomorphisms.*

Proof. Soulé [Sou79] proved surjectivity. Building on work of Rost and Voevodsky, Weibel [Wei09] completed the proof of the Quillen–Lichtenbaum Conjecture. □

Corollary 2.3. *Let $r > 1$ be an integer and let p be an odd prime. Then we have isomorphisms of $\mathbb{Z}_p[G]$-modules*

$$H^i R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq H^i_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq \begin{cases} K_{2r-1}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p & \text{if} \quad i = 1 \\ K_{2r-2}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p & \text{if} \quad i = 2 \\ 0 & \text{if} \quad i \neq 1, 2 \end{cases}$$

Proof. This follows from Theorem 2.2 and the fact that the Galois group $G_{L,S}$ has cohomological p-dimension 2 by [NSW08, Proposition 8.3.18]. □

2.6 K-theory of local fields

Let p be a prime. For an integer $n \geq 0$ and a ring R we write $K_n(R;\mathbb{Z}_p)$ for the K-theory of R with coefficients in $\mathbb{Z}_p$. Now let p be odd and let w be a finite place of L. We write $\mathcal{O}_w$ for the ring of integers in L_w. If w does not belong to $S_p(L)$, then for $r > 1$ and $i = 1, 2$ we have isomorphisms of $\mathbb{Z}_p[G_w]$-modules

$$K_{2r-i}(\mathcal{O}_w;\mathbb{Z}_p) \simeq K_{2r-i}(L(w);\mathbb{Z}_p) \simeq (\mathbb{Q}_p/\mathbb{Z}_p(r-i+1))^{G_{L_w}}.$$

Here, the first isomorphism is a special case of Gabber's Rigidity Theorem [Wei13, Chapter IV, Theorem 2.10]. As the even K-groups of a finite field vanish, the Universal Coefficient Theorem [Wei13, Chapter IV, Theorem 2.5] identifies $K_{2r-i}(L(w);\mathbb{Z}_p)$ with $K_{2r-1}(L(w)) \otimes \mathbb{Z}_p$ if $i = 1$ and with $K_{2r-3}(L(w)) \otimes \mathbb{Z}_p$ if $i = 2$. Now (2.3) gives the second isomorphism. Note that in particular $K_{2r-i}(\mathcal{O}_w;\mathbb{Z}_p)$ is a finite group. We likewise have

$$\begin{array}{ccccc} H^1_{\text{ét}}(L_w,\mathbb{Z}_p(r)) & \simeq & H^0_{\text{ét}}(L_w,\mathbb{Q}_p/\mathbb{Z}_p(r)) & = & (\mathbb{Q}_p/\mathbb{Z}_p(r))^{G_{L_w}}, \\ H^2_{\text{ét}}(L_w,\mathbb{Z}_p(r)) & \simeq & H^0_{\text{ét}}(L_w,\mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee & \simeq & (\mathbb{Q}_p/\mathbb{Z}_p(r-1))^{G_{L_w}}, \end{array}$$

where $(-)^\vee := \mathrm{Hom}(-,\mathbb{Q}_p/\mathbb{Z}_p)$ denotes the Pontryagin dual and we have used local Tate duality (see also [NSW08, Proposition 7.3.10] and the subsequent remark). This shows the case $w \notin S_p(L)$ of the following well-known theorem. The case $w \in S_p(L)$ is another instance of the Quillen–Lichtenbaum Conjecture and has been proven by Hesselholt and Madsen [HM03].

Theorem 2.4 (Gabber rigidity and Hesselholt-Madsen). *Let p be an odd prime and let w be a finite place of L. Then for any integer $r > 1$ and $i = 1, 2$ there are canonical isomorphisms of $\mathbb{Z}_p[G_w]$-modules*

$$K_{2r-i}(\mathcal{O}_w;\mathbb{Z}_p) \simeq H^i_{\text{ét}}(L_w,\mathbb{Z}_p(r)).$$

2.7 Wild Kernels

Let p be an odd prime and let S be a finite set of places of K containing all archimedean and all p-adic places. The following definition is due to Banaszak [Ban93] (a variant has been defined slightly earlier by Nguyen Quang Do [NQD92]).

Definition 2.5. *Let $r > 1$ be an integer. The kernel of the natural map*

$$K_{2r-2}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p \to \bigoplus_{w \in S(L)} H^2_{\text{ét}}(L_w,\mathbb{Z}_p(r))$$

is called the p-adic wild kernel *and will be denoted by $K^w_{2r-2}(\mathcal{O}_{L,S})_p$.*

Remark 2.6. This can be described in purely K-theoretic terms as follows. As p is odd, the cohomology groups $H^2_{\text{ét}}(L_w,\mathbb{Z}_p(r))$ vanish for archimedean w. Thus Theorem 2.4 implies that $K^w_{2r-2}(\mathcal{O}_{L,S})_p$ identifies with the kernel of the map

$$K_{2r-2}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p \to \bigoplus_{w \in S(L) \setminus S_\infty(L)} K_{2r-2}(\mathcal{O}_w;\mathbb{Z}_p).$$

Remark 2.7. Let S' be a second finite set of places of K such that $S \subseteq S'$. As we have observed in §2.6, we have isomorphisms

$$K_{2r-2}(\mathcal{O}_w; \mathbb{Z}_p) \simeq K_{2r-2}(L(w); \mathbb{Z}_p) \simeq K_{2r-3}(L(w)) \otimes \mathbb{Z}_p$$

for every $w \in S'(L) \setminus S(L)$. Taking sequence (2.2) into account, a diagram chase shows that the p-adic wild kernel $K^w_{2r-2}(\mathcal{O}_{L,S})_p$ does in fact not depend on the set S.

3 The conjectures of Leopoldt and Schneider

3.1 Local Galois cohomology

We keep the notation of §2.1. In particular, L/K is a finite Galois extension of number fields with Galois group G. Let p be an odd prime. We denote the (finite) set of places of K that ramify in L by S_{ram} and let S be a finite set of places of K containing S_{ram} and all archimedean and p-adic places (i.e. we have $S_\infty \cup S_p \cup S_{\mathrm{ram}} \subseteq S$).

Let M be a topological $G_{L,S}$-module. Then M becomes a topological G_{L_w}-module for every $w \in S(L)$ by restriction. For any $i \in \mathbb{Z}$ we put

$$P^i(\mathcal{O}_{L,S}, M) := \bigoplus_{w \in S(L)} H^i_{\text{ét}}(L_w, M).$$

We write S_f for the subset of S comprising all finite places in S.

Lemma 3.1. *Let $r > 1$ be an integer. Then we have isomorphisms of $\mathbb{Z}_p[G]$-modules*

$$P^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq \begin{cases} H^+_{-r}(L) \otimes \mathbb{Z}_p & \text{if } i = 0 \\ \bigoplus_{w \in S_f(L)} K_{2r-1}(\mathcal{O}_w; \mathbb{Z}_p) & \text{if } i = 1 \\ \bigoplus_{w \in S_f(L)} K_{2r-2}(\mathcal{O}_w; \mathbb{Z}_p) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first observe that $H^0_{\text{ét}}(L_w, \mathbb{Z}_p(r))$ vanishes unless w is a complex place or w is a real place and r is even, whereas in these cases we have $H^0_{\text{ét}}(L_w, \mathbb{Z}_p(r)) = \mathbb{Z}_p(r)$. Thus the isomorphism

$$\bigoplus_{\Sigma(L)} \mathbb{Z}_p(r) \simeq \left(\bigoplus_{\Sigma(L)} (2\pi i)^r \mathbb{Z} \right) \otimes \mathbb{Z}_p$$

that maps a generator of $\mathbb{Z}_p(r)$ to $(2\pi i)^r$ restricts to an isomorphism

$$P^0(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) = \bigoplus_{w \in S_\infty(L)} H^0_{\text{ét}}(L_w, \mathbb{Z}_p(r)) \simeq H^+_{-r}(L) \otimes \mathbb{Z}_p.$$

Now let $i > 0$. As p is odd, it is clear that $H^i_{\acute{e}t}(L_w, \mathbb{Z}_p(r))$ vanishes for all archimedean w. Now let w be a finite place of L. Since the cohomological dimension of G_{L_w} equals 2 by [NSW08, Theorem 7.1.8(i)], we have $H^i_{\acute{e}t}(L_w, \mathbb{Z}_p(r)) = 0$ for $i > 2$. The remaining cases now follow from Theorem 2.4. ☐

COROLLARY 3.2. *Let $r > 1$ be an integer. Then*

$$\operatorname{rank}_{\mathbb{Z}_p}\left(P^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))\right) = \begin{cases} d_{r+1} & \text{if } i = 0 \\ [L:\mathbb{Q}] & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In degree zero the result follows from Lemma 3.1 and the definition of d_{r+1}. We have already observed that the groups $K_{2r-i}(\mathcal{O}_w; \mathbb{Z}_p)$ are finite for $i = 1, 2$ and all finite places w of L which are not p-adic. If w belongs to $S_p(L)$, then $K_{2r-2}(\mathcal{O}_w; \mathbb{Z}_p)$ is finite, whereas $K_{2r-1}(\mathcal{O}_w; \mathbb{Z}_p)$ has $\mathbb{Z}_p$-rank $[L_w : \mathbb{Q}_p]$ by [Wei13, Chapter VI, Theorem 7.4]. The result for $i \neq 0$ now follows again from Lemma 3.1 and the formula $[L:\mathbb{Q}] = \sum_{w \in S_p(L)} [L_w : \mathbb{Q}_p]$. ☐

For any integers r and i we define $P^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))$ to be $P^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The following result is also proven in [Bar09, Lemma 5.2.4].

LEMMA 3.3. *Let $r > 1$ be an integer. Then we have isomorphisms of $\mathbb{Q}_p[G]$-modules*

$$P^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq \begin{cases} H^+_{-r}(L) \otimes \mathbb{Q}_p & \text{if } i = 0 \\ L \otimes_{\mathbb{Q}} \mathbb{Q}_p & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from Lemma 3.1 and Corollary 3.2 unless $i = 1$. To handle this case we let $w \in S_p(L)$ and put $D^{L_w}_{dR}(\mathbb{Q}_p(r)) := H^0(L_w, B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r))$, where B_{dR} denotes Fontaine's de Rham period ring. Then the Bloch–Kato exponential map

$$\exp^{BK}_r : L_w = D^{L_w}_{dR}(\mathbb{Q}_p(r)) \to H^1_{\acute{e}t}(L_w, \mathbb{Q}_p(r))$$

is an isomorphism for every $w \in S_p(L)$ as follows from [BK90, Corollary 3.8.4 and Example 3.9]. Thus we have isomorphisms of $\mathbb{Q}_p[G]$-modules

$$P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \simeq \bigoplus_{w \in S_p(L)} H^1_{\acute{e}t}(L_w, \mathbb{Q}_p(r)) \simeq \bigoplus_{w \in S_p(L)} L_w \simeq L \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

☐

By abuse of notation we write $\exp^{BK}_r$ also for the isomomrphism $L \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))$.

3.2 Schneider's conjecture

We recall the following conjecture of Schneider [Sch79, p. 192].

Conjecture 3.4 (Schneider). *Let $r \neq 0$ be an integer. Then the cohomology group $H^2_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ vanishes.*

Remark 3.5. It can be shown that Schneider's conjecture for $r = 1$ is equivalent to Leopoldt's conjecture (see [NSW08, Chapter X, §3]).

Remark 3.6. For a given number field L and a fixed prime p, Schneider's conjecture holds for almost all r. This follows from [Sch79, §5, Corollar 4] and [Sch79, §6, Satz 3].

Definition 3.7. *Let M be a topological $G_{L,S}$-module. For any integer i we denote the kernel of the natural localization map*

$$H^i_{\text{ét}}(\mathcal{O}_{L,S}, M) \to P^i(\mathcal{O}_{L,S}, M)$$

by $Ш^i(\mathcal{O}_{L,S}, M)$. We call $Ш^i(\mathcal{O}_{L,S}, M)$ the Tate–Shafarevich group *of M in degree i.*

The relation of Tate–Shafarevich groups to Schneider's conjecture is explained by the following result (see also [Bar09, Lemma 3.2.10]).

Proposition 3.8. *Let $r \neq 0$ be an integer and let p be an odd prime. Then the following holds.*

(i) The Tate–Shafarevich group $Ш^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is torsion-free.

(ii) Schneider's conjecture holds at r and p if and only if the Tate–Shafarevich group $Ш^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ vanishes.

Proof. We first claim that the group $H^2_{\text{ét}}(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ vanishes for every place w of L. This is clear when w is archimedean. If w is a finite place, then the Pontryagin dual of $H^2_{\text{ét}}(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ naturally identifes with $H^0_{\text{ét}}(L_w, \mathbb{Z}_p(r)) = 0$ by local Tate duality. Now by Poitou–Tate duality [NSW08, Theorem 8.6.9] and the claim we have

$$Ш^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq Ш^2(\mathcal{O}_{L,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee = H^2_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee.$$

This implies (ii) and also (i) as the groups $H^2_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))$ are divisible [Sch79, Lemma 2]. □

We record some cases, where Schneider's conjecture is known.

Theorem 3.9. *Let p be an odd prime.*

(i) If $r < 0$ is an integer, then Schneider's conjecture holds at r and p.

(ii) If $r > 0$ is even and L is a totally real field, then Schneider's conjecture holds at r and p.

Proof. Case (i) is due to Soulé [Sou79] (see also [NSW08, Theorem 10.3.27]). Now suppose that $r > 0$ is even and that L is totally real. Then the K-groups $K_{2r-1}(\mathcal{O}_{L,S})$ are finite by work of Borel (see §2.3). The Quillen–Lichtenbaum Conjecture (Theorem 2.2) implies that the groups $H^1_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ are finite as well. It follows that the Tate–Shafarevich group $\text{Ш}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite and thus vanishes by Proposition 3.8 (i). Now (ii) follows from Proposition 3.8 (ii). ☐

3.3 Compact support cohomology

Let M be a topological $G_{L,S}$-module. Following Burns and Flach [BF01] we define the compact support cohomology complex to be

$$R\Gamma_c(\mathcal{O}_{L,S}, M) := \text{cone}\left(R\Gamma(\mathcal{O}_{L,S}, M) \to \bigoplus_{w \in S(L)} R\Gamma(L_w, M)\right)[-1],$$

where the arrow is induced by the natural restriction maps. For any $i \in \mathbb{Z}$ we abbreviate $H^i R\Gamma_c(\mathcal{O}_{L,S}, M)$ to $H^i_c(\mathcal{O}_{L,S}, M)$. If r is an integer, we set $H^i_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) := H^i_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Lemma 3.10. *For every topological $G_{L,S}$-module M we have*

$$H^0_c(\mathcal{O}_{L,S}, M) = \text{Ш}^0(\mathcal{O}_{L,S}, M) = 0.$$

Proof. This is [Bar09, Lemma 3.1.6]. We repeat the short argument for the reader's convenience.
By definition, the groups $H^0_c(\mathcal{O}_{L,S}, M)$ and $\text{Ш}^0(\mathcal{O}_{L,S}, M)$ both identify with the kernel of the map

$$H^0_{\text{ét}}(\mathcal{O}_{L,S}, M) \to P^0(\mathcal{O}_{L,S}, M)$$

which is just the diagonal embedding $M^{G_{L,S}} \hookrightarrow \bigoplus_{w \in S(L)} M^{G_{L_w}}$. ☐

Proposition 3.11. *Let r be an integer. Then the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ belongs to $\mathcal{D}^{\text{perf}}(\mathbb{Z}_p[G])$.*

Proof. This is a special case of [BF96, Proposition 1.20], for example. ☐

Proposition 3.12. *Let $r > 1$ be an integer and let p be an odd prime. Then the following holds.*

(i) We have an exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to H^+_{-r}(L) \otimes \mathbb{Z}_p \to H^1_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to \text{Ш}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to 0.$$

In particular, we have $H^1_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq H^+_{-r}(L) \otimes \mathbb{Z}_p$ if and only if Schneider's conjecture 3.4 holds.

(ii) We have an isomorphism of $\mathbb{Z}_p[G]$-modules

$$H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq \mathbb{Z}_p(r-1)_{G_L}$$

(iii) We have an exact sequence of $\mathbb{Z}_p[G]$-modules

$$\begin{array}{rl} 0 & \to \text{Ш}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to H^2_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \\ & \to \bigoplus_{w\in S(L)} \mathbb{Z}_p(r-1)_{G_{L_w}} \to \mathbb{Z}_p(r-1)_{G_L} \to 0. \end{array}$$

(iv) We have an isomorphism of $\mathbb{Z}_p[G]$-modules

$$\text{Ш}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq K^w_{2r-2}(\mathcal{O}_{L,S})_p.$$

In particular, $\text{Ш}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite and does not depend on S.

(v) Schneider's conjecture 3.4 holds if and only if the $\mathbb{Z}_p$-rank of $H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ equals d_{r+1}.

Proof. We first observe that Artin–Verdier duality implies

$$\begin{array}{rcl} H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) & \simeq & H^0_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee \\ & = & (\mathbb{Q}_p/\mathbb{Z}_p(1-r)^{G_L})^\vee = \mathbb{Z}_p(r-1)_{G_L} \end{array}$$

giving (ii). For any $w \in S(L)$ local Tate duality likewise implies

$$\begin{array}{rcl} H^2_{\text{ét}}(L_w, \mathbb{Z}_p(r)) & \simeq & H^0_{\text{ét}}(L_w, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee \\ & = & (\mathbb{Q}_p/\mathbb{Z}_p(1-r)^{G_{L_w}})^\vee = \mathbb{Z}_p(r-1)_{G_{L_w}}. \end{array}$$

As $H^0_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ vanishes by Lemma 3.10, the long exact sequence in cohomology associated to the exact triangle

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to \bigoplus_{w\in S(L)} R\Gamma(L_w, \mathbb{Z}_p(r))$$

now gives the exact sequences in (i) and (iii) by Lemma 3.1 and the very definition of Tate–Shafarevich groups (in view of (iv) the sequence in (iii) then actually coincides with the sequence in [Sch79, Satz 8]). It is then also clear that Schneider's conjecture implies that we have an isomorphism $H^1_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq H^+_{-r}(L) \otimes \mathbb{Z}_p$. Conversely, if these two $\mathbb{Z}_p[G]$-modules are isomorphic, they are in particular finitely generated $\mathbb{Z}_p$-modules of the same rank. The short exact sequence in (i) then implies that the Tate–Shafarevich group $\text{Ш}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is torsion and thus vanishes by Proposition 3.8 (i). Hence Schneider's conjecture holds by Proposition 3.8 (ii). This completes the proof of (i). Claim (iv) is an easy consequence of Theorem 2.2 and Remark 2.7. Alternatively, it can be derived from [Ban13, Corollary 4.2 and Theorem 5.10(7)]. Finally, it follows from Theorem 2.2, Corollary 3.2 and the exact sequence

$$\begin{array}{rlll} 0 & \to \text{Ш}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) & \to H^1_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) & \to P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \\ & \to H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) & \to \text{Ш}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) & \to 0 \end{array}$$

that the $\mathbb{Z}_p$-rank of $H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ equals

$$[L:\mathbb{Q}] - d_r + \mathrm{rank}_{\mathbb{Z}_p}(\text{Ш}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))) = d_{r+1} + \mathrm{rank}_{\mathbb{Z}_p}(\text{Ш}^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))).$$

Thus (v) is a consequence of Proposition 3.8. $\square$

3.4 A conjecturally perfect complex

We keep the notation of the last subsection and also recall the notation of §2.4. Let $C_{L,S}(r) \in \mathcal{D}(\mathbb{Z}_p[G])$ be the cone of the map

$$H^1_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))\{1,4\} \to R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \oplus (H^+_{1-r}(L) \otimes \mathbb{Z}_p)\{2,3\}$$

which on cohomology induces the identity map in degree 1 and the zero map in all other degrees.

PROPOSITION 3.13. *Let $r > 1$ be an integer and let p be an odd prime. Then the following holds.*

(i) The complex $C_{L,S}(r)$ is acyclic outside degrees 2 and 3.

(ii) There is an isomorphism of $\mathbb{Z}_p[G]$-modules

$$H^2(C_{L,S}(r)) \simeq H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \oplus H^+_{1-r}(L) \otimes \mathbb{Z}_p.$$

In particular, there is a surjection $H^2(C_{L,S}(r)) \to \text{Ш}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$.

(iii) Assume that Schneider's conjecture 3.4 holds. Then the complex $C_{L,S}(r)$ belongs to $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ and we have an isomorphism of $\mathbb{Z}_p[G]$-modules

$$H^3(C_{L,S}(r)) \simeq H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \oplus \left(H^+_{-r}(L) \oplus H^+_{1-r}(L)\right) \otimes \mathbb{Z}_p.$$

Proof. This follows easily from Propositions 3.12 and 3.11 once we have observed that the $\mathbb{Z}_p[G]$-module $H^+_k(L) \otimes \mathbb{Z}_p$ is projective for every $k \in \mathbb{Z}$. Indeed, the $\mathbb{Z}[G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module $H_k(L)$ is free over $\mathbb{Z}[G]$ of rank $[K:\mathbb{Q}]$ and $H^+_k(L) \otimes \mathbb{Z}_p$ is a direct summand of $H_k(L) \otimes \mathbb{Z}_p$ as p is odd. $\square$

4 Relative algebraic K-theory

For further details and background on algebraic K-theory used in this section, we refer the reader to [CR87] and [Swa68].

4.1 Algebraic K-theory

Let R be a noetherian integral domain of characteristic 0 with field of fractions E. Let A be a finite-dimensional semisimple E-algebra and let Λ be an R-order in A. Recall that $\mathrm{PMod}(\Lambda)$ denotes the category of finitely generated projective left Λ-modules. Then $K_0(\Lambda)$ naturally identifies with the Grothendieck group of

PMod(Λ) (see [CR87, §38]) and $K_1(\Lambda)$ with the Whitehead group (see [CR87, §40]). For any field extension F of E we set $A_F := F \otimes_E A$. Let $K_0(\Lambda, F)$ denote the relative algebraic K-group associated to the ring homomorphism $\Lambda \hookrightarrow A_F$. We recall that $K_0(\Lambda, F)$ is an abelian group with generators $[X, g, Y]$ where X and Y are finitely generated projective Λ-modules and $g : F \otimes_R X \to F \otimes_R Y$ is an isomorphism of A_F-modules; for a full description in terms of generators and relations, we refer the reader to [Swa68, p. 215]. Furthermore, there is a long exact sequence of relative K-theory

$$K_1(\Lambda) \longrightarrow K_1(A_F) \xrightarrow{\partial_{\Lambda,F}} K_0(\Lambda, F) \longrightarrow K_0(\Lambda) \longrightarrow K_0(A_F) \tag{4.1}$$

(see [Swa68, Chapter 15]). We write $\zeta(A)$ for the center of (any ring) A. The reduced norm map

$$\mathrm{Nrd}_A : A \longrightarrow \zeta(A)$$

is defined componentwise (see [Rei03, §9]) and extends to matrix rings over A in the obvious way; hence this induces a map $K_1(A) \to \zeta(A)^\times$ which we also denote by Nrd_A.
Let P be a finitely generated projective A-module and let γ be an A-endomorphism of P. Choose a finitely generated projective A-module Q such that $P \oplus Q$ is free. Then the reduced norm of $\gamma \oplus \mathrm{id}_Q$ with respect to a chosen basis yields a well-defined element $\mathrm{Nrd}_A(\gamma) \in \zeta(A)$. In particular, if γ is invertible, then γ defines a class $[\gamma] \in K_1(A)$ and we have $\mathrm{Nrd}_A(\gamma) = \mathrm{Nrd}_A([\gamma])$.

4.2 Refined Euler characteristics

For any $C^\bullet \in \mathcal{C}^b(\mathrm{PMod}(\Lambda))$ we define Λ-modules

$$C^{ev} := \bigoplus_{i \in \mathbb{Z}} C^{2i}, \quad C^{odd} := \bigoplus_{i \in \mathbb{Z}} C^{2i+1}.$$

Similarly, we define $H^{ev}(C^\bullet)$ and $H^{odd}(C^\bullet)$ to be the direct sum over all even and odd degree cohomology groups of $C^\bullet$, respectively. A pair $(C^\bullet, t)$ consisting of a complex $C^\bullet \in \mathcal{D}^{\mathrm{perf}}(\Lambda)$ and an isomorphism $t : H^{odd}(C^\bullet_F) \to H^{ev}(C^\bullet_F)$ is called a trivialized complex, where we write $C^\bullet_F$ for $F \otimes^{\mathbb{L}}_R C^\bullet$. We refer to t as a trivialization of $C^\bullet$. One defines the refined Euler characteristic $\chi_{\Lambda,F}(C^\bullet, t) \in K_0(\Lambda, F)$ of a trivialized complex as follows: Choose a complex $P^\bullet \in \mathcal{C}^b(\mathrm{PMod}(\Lambda))$ which is quasi-isomorphic to $C^\bullet$. Let $B^i(P^\bullet_F)$ and $Z^i(P^\bullet_F)$ denote the i-th cobounderies and i-th cocycles of $P^\bullet_F$, respectively. For every $i \in \mathbb{Z}$ we have the obvious exact sequences

$$0 \to B^i(P^\bullet_F) \to Z^i(P^\bullet_F) \to H^i(P^\bullet_F) \to 0,$$
$$0 \to Z^i(P^\bullet_F) \to P^i_F \to B^{i+1}(P^\bullet_F) \to 0.$$

If we choose splittings of the above sequences, we get an isomorphism of A_F-modules

$$\phi_t : P^{odd}_F \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P^\bullet_F) \oplus H^{odd}(P^\bullet_F) \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P^\bullet_F) \oplus H^{ev}(P^\bullet_F) \simeq P^{ev}_F,$$

where the second map is induced by t. Then the refined Euler characteristic is defined to be
$$\chi_{\Lambda,F}(C^\bullet, t) := [P^{odd}, \phi_t, P^{ev}] \in K_0(\Lambda, F)$$
which indeed is independent of all choices made in the construction. For further information concerning refined Euler characteristics we refer the reader to [Bur04].

4.3 K-theory of group rings

Let p be a prime and let G be a finite group. By a well-known theorem of Swan (see [CR81, Theorem (32.1)]) the map $K_0(\mathbb{Z}_p[G]) \to K_0(\mathbb{Q}_p[G])$ induced by extension of scalars is injective. Thus from (4.1) we obtain an exact sequence

$$K_1(\mathbb{Z}_p[G]) \longrightarrow K_1(\mathbb{Q}_p[G]) \longrightarrow K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \longrightarrow 0. \tag{4.2}$$

The reduced norm map induces an isomorphism $K_1(\mathbb{Q}_p[G]) \longrightarrow \zeta(\mathbb{Q}_p[G])^\times$ (use [CR87, Theorem (45.3)]) and $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G])) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}((\mathbb{Z}_p[G])^\times)$ (this follows from [CR87, Theorem (40.31)]). Hence from (4.2) we obtain an exact sequence

$$(\mathbb{Z}_p[G])^\times \xrightarrow{\mathrm{Nrd}_{\mathbb{Q}_p[G]}} \zeta(\mathbb{Q}_p[G])^\times \xrightarrow{\partial_p} K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \longrightarrow 0, \tag{4.3}$$

where we write ∂_p for $\partial_{\mathbb{Z}_p[G],\mathbb{Q}_p}$. The canonical maps $K_0(\mathbb{Z}[G], \mathbb{Q}) \to K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ induce an isomorphism

$$K_0(\mathbb{Z}[G], \mathbb{Q}) \simeq \bigoplus_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \tag{4.4}$$

where the sum ranges over all primes (see the discussion following [CR87, (49.12)]). By abuse of notation we let

$$\partial_p : \zeta(\mathbb{Q}[G])^\times \to K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$$

also denote the composite map of the inclusion $\zeta(\mathbb{Q}[G])^\times \to \zeta(\mathbb{Q}_p[G])^\times$ and the surjection ∂_p in sequence (4.3). Finally, the reduced norm $\mathrm{Nrd}_{\mathbb{R}[G]}$: $K_1(\mathbb{R}[G]) \to \zeta(\mathbb{R}[G])^\times$ is injective and there is an extended boundary homomorphism
$$\hat{\partial} : \zeta(\mathbb{R}[G])^\times \longrightarrow K_0(\mathbb{Z}[G], \mathbb{R})$$
such that $\hat{\partial} \circ \mathrm{Nrd}_{\mathbb{R}[G]}$ coincides with the usual boundary homomorphism $\partial_{\mathbb{Z}[G],\mathbb{R}}$ in sequence (4.1) (see [BF01, §4.2]).

5 Rationality conjectures

5.1 Artin L-series

Let L/K be a finite Galois extension of number fields with Galois group G and let S be a finite set of places of K containing all archimedean places. For any

irreducible complex-valued character χ of G we denote the S-truncated Artin L-series by $L_S(s,\chi)$, and the leading coefficient of $L_S(s,\chi)$ at an integer r by $L_S^*(r,\chi)$. We will use this notion even if $L_S^*(r,\chi) = L_S(r,\chi)$ (which will happen frequently in the following).
There is a canonical isomorphism $\zeta(\mathbb{C}[G]) \simeq \prod_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)} \mathbb{C}$, where $\mathrm{Irr}_{\mathbb{C}}(G)$ denotes the set of irreducible complex characters of G. We define the equivariant S-truncated Artin L-series to be the meromorphic $\zeta(\mathbb{C}[G])$-valued function

$$L_S(s) := (L_S(s,\chi))_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)}.$$

For any $r \in \mathbb{Z}$ we also put

$$L_S^*(r) := (L_S^*(r,\chi))_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)} \in \zeta(\mathbb{R}[G])^\times.$$

Now let $v \in S_\infty$ be an archimedean place of K. Let χ be an irreducible complex character of G and let V_χ be a $\mathbb{C}[G]$-module with character χ. We set

$$n_\chi := \dim_{\mathbb{C}}(V_\chi) = \chi(1),\ n_{\chi,v}^+ := \dim_{\mathbb{C}}(V_\chi^{G_w}),\ n_{\chi,v}^- := n_\chi - n_{\chi,v}^+.$$

We write $S_{\mathbb{R}}$ and $S_{\mathbb{C}}$ for the subsets of S_∞ consisting of real and complex places, respectively, and define ϵ-factors

$$\epsilon_v(s,\chi) := \begin{cases} (2\cdot(2\pi)^{-s}\Gamma(s))^{n_\chi} & \text{if} \quad v\in S_{\mathbb{C}},\\ L_{\mathbb{R}}(s)^{n_{\chi,v}^+}\cdot L_{\mathbb{R}}(s+1)^{n_{\chi,v}^-} & \text{if} \quad v\in S_{\mathbb{R}}, \end{cases}$$

where $L_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma(s)$ denotes the usual Gamma function. The completed Artin L-series is then defined to be

$$\Lambda(s,\chi) := \left(\prod_{v\in S_\infty} \epsilon_v(s,\chi)\right) L_{S_\infty}(s,\chi) = \prod_v \epsilon_v(s,\chi),$$

where the second product runs over all places of K and for a finite place v of K we have

$$\epsilon_v(s,\chi) := \det(1-\phi_w N(v)^{-s} \mid V_\chi^{I_w})^{-1}.$$

We denote the contragradient of χ by $\check{\chi}$. Then the completed Artin L-series satisfies the functional equation

$$\Lambda(s,\chi) = \epsilon(s,\chi)\Lambda(1-s,\check{\chi}), \tag{5.1}$$

where the ϵ-factor $\epsilon(s,\chi)$ is defined as follows. Let d_K be the absolute discriminant of K. We write $W(\chi)$ and $\mathfrak{f}(\chi)$ for the Artin root number and the Artin conductor of χ, respectively. We then have

$$\begin{aligned} c(\chi) &:= |d_K|^{n_\chi} N(\mathfrak{f}(\chi)),\\ \epsilon(s,\chi) &:= W(\chi)c(\chi)^{1/2-s}. \end{aligned}$$

We also define equivariant ϵ-factors and the completed equivariant Artin L-series by

$$\epsilon_v(s) := (\epsilon_v(s,\chi))_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)}, \quad \epsilon(s) := (\epsilon(s,\check{\chi}))_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)},$$
$$\Lambda(s) := (\Lambda(s,\chi))_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)}.$$

The functional equations (5.1) for all irreducibe characters then combine to give an equality

$$\Lambda(s)^\sharp = \epsilon(s)\Lambda(1-s), \tag{5.2}$$

where $x \mapsto x^\sharp$ denotes the $\mathbb{C}$-linear anti-involution of $\mathbb{C}[G]$ which sends each $g \in G$ to its inverse.

5.2 A conjecture of Gross

Let $r > 1$ be an integer. Since the Borel regulator map ρ_r induces an isomorphism of $\mathbb{R}[G]$-modules, the Noether–Deuring theorem (see [NSW08, Lemma 8.7.1] for instance) implies the existence of $\mathbb{Q}[G]$-isomorphisms

$$\phi_{1-r} : H^+_{1-r}(L) \otimes \mathbb{Q} \xrightarrow{\simeq} K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Q}. \tag{5.3}$$

Let χ be a complex character of G and let V_χ be a $\mathbb{C}[G]$-module with character χ. Composition with $\rho_r \circ \phi_{1-r}$ induces an automorphism of $\mathrm{Hom}_G(V_{\check{\chi}}, H^+_{1-r}(L) \otimes \mathbb{C})$. Let $R_{\phi_{1-r}}(\chi) \in \mathbb{C}^\times$ be its determinant. If χ' is a second character, then clearly $R_{\phi_{1-r}}(\chi + \chi') = R_{\phi_{1-r}}(\chi) \cdot R_{\phi_{1-r}}(\chi')$ so that we obtain a map

$$\begin{array}{rcl} R_{\phi_{1-r}} : R(G) & \longrightarrow & \mathbb{C}^\times \\ \chi & \mapsto & \det(\rho_r \circ \phi_{1-r} \mid \mathrm{Hom}_G(V_{\check{\chi}}, H^+_{1-r}(L) \otimes \mathbb{C})), \end{array}$$

where $R(G)$ denotes the ring of virtual complex characters of G. We likewise define

$$\begin{array}{rcl} A^S_{\phi_{1-r}} : R(G) & \longrightarrow & \mathbb{C}^\times \\ \chi & \mapsto & R_{\phi_{1-r}}(\chi)/L^*_S(1-r,\chi). \end{array}$$

Gross [Gro05, Conjecture 3.11] conjectured the following higher analogue of Stark's conjecture.

CONJECTURE 5.1 (Gross). *We have $A^S_{\phi_{1-r}}(\chi^\sigma) = A^S_{\phi_{1-r}}(\chi)^\sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.*

It is straightforward to see that Gross' conjecture does not depend on S and the choice of ϕ_{1-r} (see also [Nic11a, Remark 6]). We briefly collect what is known about Conjecture 5.1. When L/K is a CM-extension, recall that χ is odd when $\chi(j) = -\chi(1)$, where $j \in G$ denotes complex conjugation.

THEOREM 5.2. *Conjecture 5.1 holds in each of the following cases:*

(i) χ *is the trivial character;*

(ii) χ *is absolutely abelian, i.e.* $L^{\ker(\chi)}/\mathbb{Q}$ *is abelian;*

(iii) $L^{\ker(\chi)}$ *is totally real and* r *is even;*

(iv) $L^{\ker(\chi)}/K$ *is a CM-extension,* χ *is an odd character and* r *is odd.*

Proof. (i) is Borel's result (2.7) above. In cases (iii) and (iv) the regulator map disappears, and Conjecture 5.1 boils down to the rationality of the L-values at negative integers which is a classical result of Siegel [Sie70]. Finally, Gross' conjecture for all characters χ of G is equivalent to the rationality statement of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(1-r), \mathbb{Z}[G])$ by [Bur10, Lemma 6.1.1 and Lemma 11.1.2] (see also [Nic11a, Proposition 2.15]). In fact, the full ETNC is known for absolutely abelian extensions by work of Burns and Greither [BG03] and of Flach [Fla11] (see also Huber and Kings [HK03]) which implies (ii). $\square$

Remark 5.3. Let $f : R(G) \to \mathbb{C}^\times$ be a homomorphism. Then we may view f as an element in $\zeta(\mathbb{C}[G])^\times \simeq \prod_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} \mathbb{C}^\times$ by declaring its χ-component to be $f(\chi)$, $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. Conversely, each $f = (f_\chi)_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)}$ in $\zeta(\mathbb{C}[G])^\times$ defines a unique homomorphism $f : R(G) \to \mathbb{C}^\times$ such that $f(\chi) = f_\chi$ for each $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. Under this identification Conjecture 5.1 asserts that $A^S_{\phi_{1-r}} \in \zeta(\mathbb{C}[G])^\times$ actually belongs to $\zeta(\mathbb{Q}[G])^\times$.

5.3 A reformulation of Gross' conjecture

In this subsection we give a reformulation of Gross' conjecture using the functional equation of Artin L-series. For any integer k we write

$$\iota_k : L \otimes_{\mathbb{Q}} \mathbb{C} \to \bigoplus_{\Sigma(L)} \mathbb{C} = \big(H^+_{1-k}(L) \oplus H^+_{-k}(L)\big) \otimes \mathbb{C} \tag{5.4}$$

for the canonical $\mathbb{C}[G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-equivariant isomorphism which is induced by mapping $l \otimes z$ to $(\sigma(l)z)_{\sigma \in \Sigma(L)}$ for $l \in L$ and $z \in \mathbb{C}$. Now fix an integer $r > 1$. We define an $\mathbb{R}[G]$-isomorphism

$$\begin{aligned} \lambda_r : \big(K_{2r-1}(\mathcal{O}_L) \oplus H^+_{-r}(L)\big) \otimes \mathbb{R} \quad &\simeq \quad \big(H^+_{1-r}(L) \oplus H^+_{-r}(L)\big) \otimes \mathbb{R} \\ &\simeq \quad (L \otimes_{\mathbb{Q}} \mathbb{C})^+ = L \otimes_{\mathbb{Q}} \mathbb{R}. \end{aligned}$$

Here, the first isomorphism is $\rho_r \oplus \mathrm{id}_{H^+_{-r}(L)}$ and the second isomorphism is induced by ι_r^{-1}. As above, there exist $\mathbb{Q}[G]$-isomorphisms

$$\phi_r : L \xrightarrow{\simeq} \big(K_{2r-1}(\mathcal{O}_L) \oplus H^+_{-r}(L)\big) \otimes \mathbb{Q}.$$

We now define maps

$$\begin{aligned} R_{\phi_r} : R(G) \quad &\longrightarrow \quad \mathbb{C}^\times \\ \chi \quad &\mapsto \quad \det\left(\lambda_r \circ \phi_r \mid \mathrm{Hom}_G(V_{\check{\chi}}, L \otimes_{\mathbb{Q}} \mathbb{C})\right) \end{aligned}$$

and

$$\begin{array}{rcl} A^S_{\phi_r} : R(G) & \longrightarrow & \mathbb{C}^\times \\ \chi & \mapsto & R_{\phi_r}(\chi)/L^*_S(r,\check{\chi}). \end{array}$$

CONJECTURE 5.4. *We have* $A^S_{\phi_r}(\chi^\sigma) = A^S_{\phi_r}(\chi)^\sigma$ *for all* $\sigma \in \mathrm{Aut}(\mathbb{C})$.

It is again easily seen that this conjecture does not depend on S and the choice of ϕ_r. In fact we have the following result.

PROPOSITION 5.5. *Fix an integer* $r > 1$ *and a character* χ. *Then Gross' Conjecture 5.1 holds if and only if Conjecture 5.4 holds.*

Proof. Let k be an integer. If k is even, multiplication by $(2\pi i)^k$ induces a $\mathbb{Q}[G]$-isomorphism $H_0^+(L) \otimes \mathbb{Q} \simeq H_{-k}^+(L) \otimes \mathbb{Q}$. Similarly, multiplication by $(2\pi i)^{k-1}$ induces a $\mathbb{Q}[G]$-isomorphism $H_0^-(L) \otimes \mathbb{Q} \simeq H_{1-k}^+(L) \otimes \mathbb{Q}$. When k is odd, we likewise have $\mathbb{Q}[G]$-isomorphisms $H_0^+(L) \otimes \mathbb{Q} \simeq H_{1-k}^+(L) \otimes \mathbb{Q}$ and $H_0^-(L) \otimes \mathbb{Q} \simeq H_{-k}^+(L) \otimes \mathbb{Q}$ induced by multiplication by $(2\pi i)^{k-1}$ and $(2\pi i)^k$, respectively. So for any k we obtain a $\mathbb{Q}[G]$-isomorphism

$$\mu_k : H_0(L) \otimes \mathbb{Q} \simeq \left(H_{1-k}^+(L) \oplus H_{-k}^+(L)\right) \otimes \mathbb{Q}. \tag{5.5}$$

Moreover, we define an $\mathbb{R}[G]$-isomorphism

$$\pi_L : L \otimes_{\mathbb{Q}} \mathbb{R} \simeq \left(H_0^+(L) \oplus H_{-1}^+(L)\right) \otimes \mathbb{R} \stackrel{(1,-i)}{\longrightarrow} H_0(L) \otimes \mathbb{R},$$

where the first isomorphism is induced by ι_1. It is clear that π_L agrees with the map π_L in [BB03, p. 554]. Bley and Burns define an explicit $\mathbb{Q}[G]$-isomorphism

$$\phi : L \stackrel{\simeq}{\longrightarrow} H_0(L) \otimes \mathbb{Q}. \tag{5.6}$$

Building on a result of Fröhlich [Frö89] on Galois Gauss sums, the authors [BB03, equation (12) and (13)] then show that

$$\mathrm{Nrd}_{\mathbb{R}[G]}((\phi \otimes 1) \circ \pi_L^{-1}) \cdot \epsilon(0) \in \zeta(\mathbb{Q}[G])^\times. \tag{5.7}$$

Now choose a $\mathbb{Q}[G]$-isomorphism ϕ_{1-r} as in (5.3). We define ϕ_r to be the composite map

$$\phi_r := (\phi_{1-r} \oplus \mathrm{id}_{H_{-r}^+(L) \otimes \mathbb{Q}}) \circ \mu_r \circ \phi.$$

Let $a, b \in \zeta(\mathbb{C}[G])^\times$. In the following we write $a \sim b$ if $ab^{-1} \in \zeta(\mathbb{Q}[G])^\times$. Under the identification in Remark 5.3 we thus have to show that $A^S_{\phi_r} \sim A^S_{\phi_{1-r}}$. We observe that

$$\lambda_r \circ \phi_r = \iota_r^{-1} \circ (\rho_r \oplus \mathrm{id}_{H_{-r}^+(L)}) \circ (\phi_{1-r} \oplus \mathrm{id}_{H_{-r}^+(L)}) \circ \mu_r \circ \phi,$$

where we view each map as a $\mathbb{C}[G]$-isomorphism by extending scalars. This implies that

$$R_{\phi_r} = R_{\phi_{1-r}} \cdot \mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \phi)^\sharp.$$

We now use (5.7), the fact that $c(\chi^\sigma) = c(\chi)^\sigma$ for all $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, the definition of ϵ and the functional equation (5.2) to compute

$$\begin{aligned}
R_{\phi_r} &\sim R_{\phi_{1-r}} \cdot \left(\mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \pi_L) \cdot \epsilon(0)^{-1}\right)^\sharp \\
&\sim R_{\phi_{1-r}} \cdot \left(\mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \pi_L) \cdot \epsilon(1-r)^{-1}\right)^\sharp \\
&\sim R_{\phi_{1-r}} \cdot \mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \pi_L)^\sharp \cdot \frac{\Lambda^*(r)^\sharp}{\Lambda^*(1-r)}
\end{aligned}$$

Now let v be an archimedean place of K. As $\Gamma(k)$ is a non-zero rational number for every positive integer k and $\Gamma(s)$ has simple poles with rational residues at $s = k$ for every non-positive integer k, an easy computation shows that for $v \in S_{\mathbb{C}}$ one has

$$\frac{\epsilon_v(r)^\sharp}{\epsilon_v^*(1-r)} \sim \left(\pi^{(1-2r)n_\chi}\right)_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)}. \tag{5.8}$$

Moreover, using $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(1/2) = \sqrt{\pi}$ we find that $\Gamma((2k+1)/2) \in \sqrt{\pi} \cdot \mathbb{Q}^\times$ for every integer k. Then a computation shows that for $v \in S_{\mathbb{R}}$ one has

$$\frac{\epsilon_v(r)^\sharp}{\epsilon_v^*(1-r)} \sim \begin{cases} \left(\pi^{(1-r)n_{\chi,v}^+ - rn_{\chi,v}^-}\right)_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} & \text{if} \quad 2 \nmid r \\ \left(\pi^{(1-r)n_{\chi,v}^- - rn_{\chi,v}^+}\right)_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} & \text{if} \quad 2 \mid r. \end{cases} \tag{5.9}$$

The automorphism $\mu_r \circ \pi_L \circ \iota_r^{-1}$ on $(H_{1-r}^+(L) \oplus H_{-r}^+(L)) \otimes \mathbb{C}$ is given up to sign by multiplication by $(2\pi)^{r-1}$ and $(2\pi)^r$ on the first and second direct summand, respectively. It follows that

$$\mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \pi_L)^\sharp = \mathrm{Nrd}_{\mathbb{C}[G]}(\mu_r \circ \pi_L \circ \iota_r^{-1})^\sharp$$

$$\sim \begin{cases} \left(\pi^{(r-1)(|S_{\mathbb{C}}|n_\chi + \sum_{v \in S_{\mathbb{R}}} n_{\chi,v}^+) + r(|S_{\mathbb{C}}|n_\chi + \sum_{v\in S_{\mathbb{R}}} n_{\chi,v}^-)}\right)_\chi & \text{if} \quad 2 \nmid r \\ \left(\pi^{(r-1)(|S_{\mathbb{C}}|n_\chi + \sum_{v \in S_{\mathbb{R}}} n_{\chi,v}^-) + r(|S_{\mathbb{C}}|n_\chi + \sum_{v\in S_{\mathbb{R}}} n_{\chi,v}^+)}\right)_\chi & \text{if} \quad 2 \mid r. \end{cases}$$

If we compare this to (5.8) and (5.9) we find that

$$\mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \pi_L)^\sharp \sim \left(\prod_{v \in S_\infty} \frac{\epsilon_v(r)^\sharp}{\epsilon_v^*(1-r)}\right)^{-1}.$$

Finally, by the very definition of $\Lambda(s)$ we have $\Lambda(s) = \left(\prod_{v \in S_\infty} \epsilon_v(s)\right) \cdot L_{S_\infty}(s)$. We obtain

$$\begin{aligned}
R_{\phi_r} &\sim R_{\phi_{1-r}} \cdot \mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r^{-1} \circ \mu_r \circ \pi_L)^\sharp \cdot \left(\prod_{v \in S_\infty} \frac{\epsilon_v(r)^\sharp}{\epsilon_v^*(1-r)}\right) \cdot \frac{L_{S_\infty}^*(r)^\sharp}{L_{S_\infty}^*(1-r)} \\
&\sim R_{\phi_{1-r}} \cdot \frac{L_{S_\infty}^*(r)^\sharp}{L_{S_\infty}^*(1-r)}
\end{aligned}$$

which exactly means that

$$A_{\phi_r}^{S_\infty} = \frac{R_{\phi_r}}{L^*_{S_\infty}(r)^\sharp} \sim \frac{R_{\phi_{1-r}}}{L^*_{S_\infty}(1-r)} = A_{\phi_{1-r}}^{S_\infty}.$$

As both conjectures do not depend on the choice of S we are done. $\square$

6 Equivariant leading term conjectures

We fix a finite Galois extension L/K with Galois group G and an odd prime p. Let $r > 1$ be an integer. In this section we assume throughout that *Schneider's conjecture 3.4 holds*. In particular, if S is a sufficiently large finite set of places of K as in §3, then the complex $C_{L,S}(r) \in \mathcal{D}(\mathbb{Z}_p[G])$ constructed in §3.4 is perfect by Proposition 3.13.

6.1 Choosing a trivialization

In this subsection we construct a trivialization of $C_{L,S}(r)$. We first choose a $\mathbb{Q}[G]$-isomorphism

$$\alpha_r : L \to \big(H^+_{1-r}(L) \oplus H^+_{-r}(L)\big) \otimes \mathbb{Q}. \tag{6.1}$$

For instance, we may take $\alpha_r = \mu_r \circ \phi$, where μ_r and ϕ are the isomorphisms (5.5) and (5.6), respectively. Moreover, we choose a $\mathbb{Q}[G]$-isomorphism

$$\phi_{1-r} : H^+_{1-r}(L) \otimes \mathbb{Q} \xrightarrow{\simeq} K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Q}$$

as in (5.3). We let $\psi_r := (\phi_{1-r}, \alpha_r)$ be the corresponding pair of $\mathbb{Q}[G]$-isomorphisms. As $Ш^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ vanishes by Proposition 3.8 and $Ш^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite by Proposition 3.12 (iv), we have an exact sequence of $\mathbb{Q}_p[G]$-modules

$$0 \to H^1_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \to P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \to H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \to 0. \tag{6.2}$$

Since $\mathbb{Q}_p[G]$ is semisimple, we may choose a $\mathbb{Q}_p[G]$-equivariant splitting σ_r of this sequence. We now define a trivialization $t(\psi_r, \sigma_r, S)$ of $C_{L,S}(r)$ to be the composite of the following $\mathbb{Q}_p[G]$-isomorphisms (note that we have $H^{ev}(C_{L,S}(r)) = H^2(C_{L,S}(r))$ and $H^{odd}(C_{L,S}(r)) = H^3(C_{L,S}(r))$ by Proposition 3.13):

$$\begin{array}{rcl}
H^3(C_{L,S}(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \longrightarrow & \big(H^+_{1-r}(L) \oplus H^+_{-r}(L)\big) \otimes \mathbb{Q}_p \\
& \xrightarrow{\alpha_r^{-1}} & L \otimes_{\mathbb{Q}} \mathbb{Q}_p \\
& \xrightarrow{\exp_r^{BK}} & P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \\
& \xrightarrow{\sigma_r} & H^1_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \\
& \xrightarrow{(ch^{(p)}_{r,1})^{-1} \oplus \mathrm{id}} & (K_{2r-1}(\mathcal{O}_{L,S}) \otimes \mathbb{Q}_p) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \\
& \longrightarrow & (K_{2r-1}(\mathcal{O}_{L}) \otimes \mathbb{Q}_p) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \\
& \xrightarrow{\phi_{1-r}^{-1} \oplus \mathrm{id}} & (H^+_{1-r}(L) \otimes \mathbb{Q}_p) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \\
& \longrightarrow & H^2(C_{L,S}(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\end{array}$$

Here, the unlabelled isomorphisms come from Propositions 3.13 and 3.12 (ii) and (2.1). We now define

$$\Omega_{\psi_r,S} := \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(C_{L,S}(r), t(\psi_r, \sigma_r, S)) \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$$

which is easily seen to be independent of the splitting σ_r (see §4.1 and §4.2).

6.2 The leading term conjecture at $s = r$

We are now in a position to formulate the central conjectures of this article. We keep the notation of the last subsection. In particular, recall the pair $\psi_r = (\phi_{1-r}, \alpha_r)$. Define a $\mathbb{Q}[G]$-isomorphism

$$\phi_r : L \xrightarrow{\simeq} \big(K_{2r-1}(\mathcal{O}_L) \oplus H^+_{-r}(L)\big) \otimes \mathbb{Q}$$

by $\phi_r := (\phi_{1-r} \oplus \mathrm{id}_{H^+_{-r}(L)}) \circ \alpha_r$.

Conjecture 6.1. *Let L/K be a finite Galois extension of number fields with Galois group G and let $r > 1$ be an integer. Let p be an odd prime.*

(i) The Tate–Shafarevich group $Ш^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ vanishes.

(ii) We have that $A^S_{\phi_r}$ belongs to $\zeta(\mathbb{Q}[G])^\times$.

(iii) We have an equality $\partial_p(A^S_{\phi_r})^\sharp = -\Omega_{\psi_r,S}$.

Remark 6.2. Part (i) and (ii) of Conjecture 6.1 are equivalent to Schneider's conjecture 3.4 and Gross' conjecture 5.1 by Propositions 3.8 and 5.5, respectively.

Proposition 6.3. *Suppose that part (i) and part (ii) of Conjecture 6.1 both hold. Then part (iii) does not depend on any of the choices made in the construction.*

Proof. Let S' be a second sufficiently large finite set of places of K. By embedding S and S' into the union $S \cup S'$ we may and do assume that $S \subseteq S'$. By induction we may additionally assume that $S' = S \cup \{v\}$, where v is not in S. In particular, v is unramified in L/K and $v \nmid p$. We compute

$$A_{\phi_r}^{S'}(\chi)/A_{\phi_r}^{S}(\chi) = L_S^*(r,\check{\chi})/L_{S'}^*(r,\check{\chi}) = \epsilon_v(r,\check{\chi}). \tag{6.3}$$

On the other hand, by [BF01, (30)] we have an exact triangle

$$\mathrm{Ind}_{G_w}^G R\Gamma_f(L_w,\mathbb{Z}_p(r))[-1] \longrightarrow R\Gamma_c(\mathcal{O}_{L,S'},\mathbb{Z}_p(r)) \longrightarrow R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(r)),$$

where $R\Gamma_f(L_w,\mathbb{Z}_p(r))$ is a perfect complex of $\mathbb{Z}_p[G_w]$-modules which is naturally quasi-isomorphic to

$$\mathbb{Z}_p[G_w] \xrightarrow{1-\phi_w N(v)^{-r}} \mathbb{Z}_p[G_w]$$

with terms in degree 0 and 1. As Schneider's conjecture holds by assumption, the cohomology group $H_c^1(\mathcal{O}_{L,S},\mathbb{Z}_p(r))$ does not depend on S by Proposition 3.12. Thus by the definition of $C_{L,S}(r)$ we likewise have an exact triangle

$$\mathrm{Ind}_{G_w}^G R\Gamma_f(L_w,\mathbb{Z}_p(r))[-1] \longrightarrow C_{L,S'}(r) \longrightarrow C_{L,S}(r).$$

We therefore may compute

$$\begin{aligned}
\Omega_{\psi_r,S} - \Omega_{\psi_r,S'} &= \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}\left(\mathrm{Ind}_{G_w}^G R\Gamma_f(L_w,\mathbb{Z}_p(r)),0\right)\\
&= \partial_p(\epsilon_v(r))\\
&= \partial_p(A_{\phi_r}^{S'})^\sharp - \partial_p(A_{\phi_r}^{S})^\sharp,
\end{aligned}$$

where the last equality follows from (6.3). This shows that Conjecture 6.1 (iii) does not depend on S. Now suppose that α_r' is a second choice of $\mathbb{Q}[G]$-isomorphism as in (6.1). Let $\phi_r' := (\phi_{1-r} \oplus \mathrm{id}_{H_{-r}^+(L)}) \circ \alpha_r'$. Then we have

$$\begin{aligned}
\left(A_{\phi_r}^S \cdot \left(A_{\phi_r'}^S\right)^{-1}\right)^\sharp &= \left(R_{\phi_r} \cdot R_{\phi_r'}^{-1}\right)^\sharp \qquad (6.4)\\
&= \mathrm{Nrd}_{\mathbb{Q}[G]}((\phi_r')^{-1}\phi_r)\\
&= \mathrm{Nrd}_{\mathbb{Q}[G]}((\alpha_r')^{-1}\alpha_r).
\end{aligned}$$

Letting $\psi_r' := (\phi_{1-r},\alpha_r')$ we likewise compute

$$\begin{aligned}
\Omega_{\psi_r,S} - \Omega_{\psi_r',S} &= \partial_p\left(\mathrm{Nrd}_{\mathbb{Q}[G]}(\alpha_r'\alpha_r^{-1})\right)\\
&= -\partial_p\left(\mathrm{Nrd}_{\mathbb{Q}[G]}((\alpha_r')^{-1}\alpha_r)\right).
\end{aligned}$$

Finally, a similar computation shows that the conjecture does not depend on the choice of ϕ_{1-r}. $\square$

It is therefore convenient to put

$$T\Omega(L/K,r)_p := -\left(\partial_p(A^S_{\phi_r})^\sharp + \Omega_{\psi_r,S}\right) \in K_0(\mathbb{Z}_p[G],\mathbb{Q}_p).$$

Then Conjecture 6.1 (iii) simply asserts that $T\Omega(L/K,r)_p$ vanishes. The reason for the minus sign will become apparent in the next subsection (see Theorem 6.5).

Now choose an isomorphism $j:\mathbb{C}\simeq\mathbb{C}_p$. By functoriality, this induces a map

$$j_*: K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}_p[G],\mathbb{C}_p).$$

We define a trivialization $t(r,S,j)$ of the complex $C_{L,S}(r)$ as in §6.1, but we tensor with $\mathbb{C}_p$ and replace the isomorphisms α_r and ϕ_{1-r} by $\iota_r \otimes_j \mathbb{C}_p$ and $\rho_r^{-1}\otimes_j \mathbb{C}_p$ (see (2.6) and (5.4)). Thus we obtain an object

$$\Omega^j_{r,S} := \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(C_{L,S}(r), t(r,S,j)) \in K_0(\mathbb{Z}_p[G],\mathbb{C}_p).$$

Then the argument in the proof of Proposition 6.3 shows the following result.

Proposition 6.4. *Let $j:\mathbb{C}\simeq\mathbb{C}_p$ be an isomorphism. Suppose that part (i) and (ii) of Conjecture 6.1 both hold. Then we have an equality*

$$T\Omega(L/K,r)_p = j_*\left(\hat{\partial}(L^*_S(r))\right) - \Omega^j_{r,S}$$

in $K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$.

6.3 The relation to the equivariant Tamagawa number conjecture

We now compare our invariant $T\Omega(L/K,r)_p$ to the equivariant Tamagawa number conjecture (ETNC) as formulated by Burns and Flach [BF01].

For an arbitrary integer r we set $\mathbb{Q}(r)_L := h^0(\mathrm{Spec}(L))(r)$ which we regard as a motive defined over K and with coefficients in the semisimple algebra $\mathbb{Q}[G]$. The ETNC [BF01, Conjecture 4(iv)] for the pair $(\mathbb{Q}(r)_L,\mathbb{Z}[G])$ asserts that a certain canonical element $T\Omega(\mathbb{Q}(r)_L,\mathbb{Z}[G])$ in $K_0(\mathbb{Z}[G],\mathbb{R})$ vanishes. Note that in this case the element $T\Omega(\mathbb{Q}(r)_L,\mathbb{Z}[G])$ is indeed well-defined as observed in [BF03, §1]. If $T\Omega(\mathbb{Q}(r)_L,\mathbb{Z}[G])$ is rational, i.e. belongs to $K_0(\mathbb{Z}[G],\mathbb{Q})$, then by means of (4.4) we obtain elements $T\Omega(\mathbb{Q}(r)_L,\mathbb{Z}[G])_p$ in $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p)$. We say that the '$p$-part' of the ETNC for the pair $(\mathbb{Q}(r)_L,\mathbb{Z}[G])$ holds if $T\Omega(\mathbb{Q}(r)_L,\mathbb{Z}[G])_p$ vanishes.

Theorem 6.5. *Let L/K be a finite Galois extension of number fields with Galois group G and let $r>1$ be an integer. Then the following holds.*

(i) Conjecture 6.1 (ii) holds if and only if $T\Omega(\mathbb{Q}(r)_L,\mathbb{Z}[G])$ belongs to $K_0(\mathbb{Z}[G],\mathbb{Q})$.

(ii) *Suppose that part (i) and (ii) of Conjecture 6.1 both hold. Then*

$$T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])_p = T\Omega(L/K, r)_p.$$

In particular, Conjecture 6.1 (iii) and the p-part of the ETNC for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ are equivalent.

Proof. Conjecture 6.1(ii) is equivalent to Gross' conjecture 5.1 by Proposition 5.5. The latter conjecture is equivalent to the rationality of $T\Omega(\mathbb{Q}(1-r)_L, \mathbb{Z}[G])$ by [Bur10, Lemma 6.1.1 and Lemma 11.1.2]. Finally, $T\Omega(\mathbb{Q}(1-r)_L, \mathbb{Z}[G])$ is rational if and only if $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ is rational by [BF01, Theorem 5.2]. This proves (i).

For (ii) we briefly recall some basic facts on virtual objects. If Λ is a noetherian ring, we write $V(\Lambda)$ for the Picard category of virtual objects associated to the category $\mathrm{PMod}(\Lambda)$. We fix a unit object $\mathbb{1}_\Lambda$ and write $\boxtimes$ for the bifunctor in $V(\Lambda)$. For each object M there is an object M^{-1}, unique up to unique isomorphism, with an isomorphism $\tau_M : M \boxtimes M^{-1} \xrightarrow{\sim} \mathbb{1}_\Lambda$. If N is an object in $\mathrm{PMod}(\Lambda)$, we write $[N]$ for the associated object in $V(\Lambda)$. More generally, if $C^\bullet$ belongs to $\mathcal{D}^{\mathrm{perf}}(\Lambda)$, we write $[C^\bullet] \in V(\Lambda)$ for the associated object (see [BF01, Proposition 2.1]). We let $V(\mathbb{Z}_p[G], \mathbb{C}_p)$ be the Picard category associated to the ring homomorphism $\mathbb{Z}_p[G] \hookrightarrow \mathbb{C}_p[G]$ as defined in [BB05, §5]. We recall that objects in $V(\mathbb{Z}_p[G], \mathbb{C}_p)$ are pairs (M, t), where M is an object in $V(\mathbb{Z}_p[G])$ and t is an isomorphism $\mathbb{C}_p \otimes_{\mathbb{Z}_p} M \simeq \mathbb{1}_{\mathbb{C}_p[G]}$ in $V(\mathbb{C}_p[G])$. By [BB05, Lemma 5.1] one has an isomorphism

$$\pi_0(V(\mathbb{Z}_p[G], \mathbb{C}_p)) \simeq K_0(\mathbb{Z}_p[G], \mathbb{C}_p), \tag{6.5}$$

where $\pi_0(\mathcal{P})$ denotes the group of isomorphism classes of objects of a Picard category $\mathcal{P}$.

For any motive M which is defined over K and admits an action of a finite dimensional $\mathbb{Q}$-algebra A, Burns and Flach [BF01, (29)] define an element $\Xi(M)$ of $V(A)$. In the case $M = \mathbb{Q}(r)_L$ and $A = \mathbb{Q}[G]$ one has

$$\Xi(\mathbb{Q}(r)_L) = [K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Q}]^{-1} \boxtimes [H^+_{-r}(L) \otimes \mathbb{Q}]^{-1} \boxtimes [L] \in V(\mathbb{Q}[G]).$$

The regulator map (2.6) and (5.4) then induce an isomorphism in $V(\mathbb{R}[G])$:

$$\vartheta_\infty : \mathbb{R} \otimes_{\mathbb{Q}} \Xi(\mathbb{Q}(r)_L) \simeq \mathbb{1}_{\mathbb{R}[G]}.$$

Moreover, Burns and Flach construct for each prime p an isomorphism

$$\vartheta_p : \mathbb{Q}_p \otimes_{\mathbb{Q}} \Xi(\mathbb{Q}(r)_L) \simeq [R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))]$$

in $V(\mathbb{Q}_p[G])$ (see [BF01, p. 526]). These data determine an element $R\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ in $K_0(\mathbb{Z}[G], \mathbb{R})$ and one has $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G]) = \hat{\partial}(L^*_{S_\infty}(r)) + R\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ by definition.

Now suppose that part (i) and (ii) of Conjecture 6.1 both hold. Recall the definition of $C_{L,S}(r)$. The isomorphisms $\tau_{[N]}$, where $N = H^1_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ and $N = H^+_{1-r}(L) \otimes \mathbb{Z}_p$, yield an isomorphism

$$[C_{L,S}(r)] \simeq [R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))] \tag{6.6}$$

in $V(\mathbb{Z}_p[G])$. Now let $j : \mathbb{C} \simeq \mathbb{C}_p$ be an isomorphism. Then the trivialization $t(r, S, j)$ induces an isomorphism

$$\vartheta_{p,j} : [\mathbb{C}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p} C_{L,S}(r)] \simeq \mathbb{1}_{\mathbb{C}_p[G]}$$

in $V(\mathbb{C}_p[G])$. After extending scalars to $\mathbb{C}_p$, the isomorphisms (6.6), ϑ_p^{-1} and ϑ_∞ likewise induce an isomorphism

$$\vartheta'_{p,j} : [\mathbb{C}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p} C_{L,S}(r)] \simeq \mathbb{1}_{\mathbb{C}_p[G]}$$

in $V(\mathbb{C}_p[G])$. Taking [BF01, Remark 4] into account, we see that the class of the pair $([C_{L,S}(r)], \vartheta_{p,j})$ in $\pi_0(V(\mathbb{Z}_p[G], \mathbb{C}_p))$ maps to $-\Omega^j_{r,S}$ under the isomorphism (6.5), whereas $([C_{L,S}(r)], \vartheta'_{p,j})$ corresponds to $j_*(R\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G]))$. Unwinding the definitions of $\vartheta_{p,j}$ and $\vartheta'_{p,j}$ one sees that both isomorphisms almost coincide. The only difference rests on the following.

Let Λ be a noetherian ring and let $\phi : P \to P$ be an automorphism of a finitely generated projective Λ-module P. Consider the complex $C : P \xrightarrow{\phi} P$, where P is placed in degree 0 and 1. Then there a two isomorphisms $[C] \simeq \mathbb{1}_\Lambda$ induced by $\tau_{[P]}$ and the acyclicity of C, respectively. Now for every finite place $v \in S$, there appears such an acyclic complex of $\mathbb{Q}_p[G_w]$-modules in the construction of $R\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])$. Namely, if $v \nmid p$ this is the complex $R\Gamma_f(L_w, \mathbb{Q}_p(r))$ which is canonically quasi-isomorphic to

$$\mathbb{Q}_p[\overline{G_w}] \xrightarrow{1-\phi_w N(v)^{-r}} \mathbb{Q}_p[\overline{G_w}]$$

with terms in degree 0 and 1 (see [BF01, (19)]). If v divides p, this complex appears as the rightmost complex in [BF01, (22)] and is given by

$$D^{L_w}_{cris}(\mathbb{Q}_p(r)) \xrightarrow{1-\phi_{cris}} D^{L_w}_{cris}(\mathbb{Q}_p(r)),$$

where $D^{L_w}_{cris}(\mathbb{Q}_p(r)) := H^0(L_w, B_{cris} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r))$ naturally identifies with the maximal unramified subextension of L_w and ϕ_{cris} denotes the Frobenius on the crystalline period ring B_{cris}. Burns and Flach choose the isomorphisms induced by the corresponding τ's, whereas we have implicitly used the acyclicity of these complexes. For each such v this gives rise to an Euler factor $\epsilon_v(r)$ (for more details we refer the reader to [BF98, §2]; though the authors consider a slightly different situation, the arguments naturally carry over to the case at hand). This discussion gives an equality

$$j_*\left(R\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])\right) = -\Omega^j_{r,S} + j_*\left(\hat{\partial}\left(\prod_{v \in S} \varepsilon_v(r)\right)\right).$$

Thus $T\Omega(\mathbb{Q}(r)_L, \mathbb{Z}[G])_p$ and $T\Omega(L/K,r)_p$ have the same image under the injective map $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p) \to K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$. $\square$

7 Annihilating wild kernels

7.1 Generalised adjoint matrices

Let G be a finite group and let p be a prime. Let $\mathfrak{M}_p(G)$ be a maximal $\mathbb{Z}_p$-order such that $\mathbb{Z}_p[G] \subseteq \mathfrak{M}_p(G) \subseteq \mathbb{Q}_p[G]$. Let $e_1,\dots,e_t$ be the central primitive idempotents of $\mathbb{Q}_p[G]$. Then each Wedderburn component $\mathbb{Q}_p[G]e_i$ is isomorphic to an algebra of $m_i \times m_i$ matrices over a skewfield D_i and $F_i := \zeta(D_i)$ is a finite field extension of $\mathbb{Q}_p$. We denote the Schur index of D_i by s_i so that $[D_i : F_i] = s_i^2$ and put $n_i := m_i \cdot s_i$. We let $\mathcal{O}_i$ be the ring of intgers in F_i.
Choose $n \in \mathbb{N}$ and let $H \in M_{n\times n}(\mathfrak{M}_p[G])$. Then we may decompose H into

$$H = \sum_{i=1}^t H_i \in M_{n\times n}(\mathfrak{M}_p(G)) = \bigoplus_{i=1}^t M_{n\times n}(\mathfrak{M}_p(G)e_i),$$

where $H_i := He_i$. The reduced characteristic polynomial $f_i(X) = \sum_{j=0}^{n_i n} \alpha_{ij} X^j$ of H_i has coefficients in $\mathcal{O}_i$. Moreover, the constant term α_{i0} is equal to $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(H_i)\cdot(-1)^{n_i n}$. We put

$$H_i^* := (-1)^{n_i n+1}\cdot \sum_{j=1}^{n_i n} \alpha_{ij} H_i^{j-1}, \quad H^* := \sum_{i=1}^t H_i^*.$$

Note that this definition of H^* differs slightly from the definition in [Nic10, §4], but follows the conventions in [JN13]. Let $1_{n\times n}$ denote the $n\times n$ identity matrix.

Lemma 7.1. *We have that $H^* \in M_{n\times n}(\mathfrak{M}_p(G))$ and $H^*H = HH^* = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(H)\cdot 1_{n\times n}$.*

Proof. The first assertion is clear by the above considerations. Since $f_i(H_i)=0$, we find that

$$H_i^*\cdot H_i = H_i \cdot H_i^* = (-1)^{n_i n+1}(-\alpha_{i0}) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(H_i),$$

as desired (see also [JN13, Lemma 3.4]). $\square$

7.2 Denominator ideals

We define

$$\begin{aligned}
\mathcal{H}_p(G) &:= \{x \in \zeta(\mathbb{Z}_p[G]) \mid \\
&\qquad xH^* \in M_{n\times n}(\mathbb{Z}_p[G])\, \forall H \in M_{n\times n}(\mathbb{Z}_p[G])\, \forall n \in \mathbb{N}\},\\
\mathcal{I}_p(G) &:= \langle \mathrm{Nrd}_{\mathbb{Q}_p[G]}(H) \mid H \in M_{n\times n}(\mathbb{Z}_p[G]),\, n \in \mathbb{N}\rangle_{\zeta(\mathbb{Z}_p[G])}.
\end{aligned}$$

Since $x \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}(H) = xH^*H \in \zeta(\mathbb{Z}_p[G])$ by Lemma 7.1, in particular we have

$$\mathcal{H}_p(G) \cdot \mathcal{I}_p(G) = \mathcal{H}_p(G) \subseteq \zeta(\mathbb{Z}_p[G]). \tag{7.1}$$

Hence $\mathcal{H}_p(G)$ is an ideal in the commutative $\mathbb{Z}_p$-order $\mathcal{I}_p(G)$. We will refer to $\mathcal{H}_p(G)$ as the *denominator ideal* of the group ring $\mathbb{Z}_p[G]$. The following result determines the primes p for which the denominator ideal $\mathcal{H}_p(G)$ is best possible.

Proposition 7.2. *We have $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p[G])$ if and only if p does not divide the order of the commutator subgroup G' of G. Furthermore, when this is the case we have that $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$.*

Proof. The first assertion is a special case of [JN13, Proposition 4.8]. The second assertion follows from (7.1). □

7.3 A canonical fractional Galois ideal

Let L/K be a finite Galois extension of number fields with Galois group G and let $r > 1$ be an integer. Let $p \neq 2$ be a prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty \cup S_p$. Recall the notation of §6.1. As p is odd, the $\mathbb{Z}_p[G]$-module

$$Y_r := \left(H^+_{1-r}(L) \oplus H^+_{-r}(L)\right) \otimes \mathbb{Z}_p$$

is projective. We also observe that $P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tf}}$ does not depend on S by Lemma 3.1 and the fact that $K_{2r-1}(\mathcal{O}_w; \mathbb{Z}_p)$ is finite for each $w \notin S_p(L)$. We let

$$\begin{aligned} E(\alpha_r) &:= \{\gamma \in \mathrm{End}_{\mathbb{Q}_p[G]}(Y_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \mid \\ &\qquad \exp_r^{BK} \alpha_r^{-1}\gamma(Y_r) \subseteq P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tf}}\}, \\ \mathcal{E}(\alpha_r) &:= \langle \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma) \mid \gamma \in E(\alpha_r)\rangle_{\zeta(\mathbb{Z}_p[G])} \subseteq \zeta(\mathbb{Q}_p[G]). \end{aligned}$$

Now suppose that Schneider's conjecture 3.4 holds. Then we have the short exact sequence (6.2) and we may choose a $\mathbb{Q}_p[G]$-equivariant splitting σ_r of this sequence:

$$\sigma_r : P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \xrightarrow{\simeq} H^1_{\acute{e}t}(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)).$$

We let

$$\sigma_r^1 : P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \longrightarrow H^1_{\acute{e}t}(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))$$

be the composite map of σ_r and the projection onto the first component. We put

$$\begin{aligned} F(\phi_{1-r}, \sigma_r) &:= \{\delta \in \mathrm{End}_{\mathbb{Q}_p[G]}(H^+_{1-r}(L) \otimes \mathbb{Q}_p) \mid \\ &\qquad \delta\phi_{1-r}^{-1}(ch_{r,1}^{(p)})^{-1}\sigma_r^1\left(P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tf}}\right) \subseteq H^+_{1-r}(L) \otimes \mathbb{Z}_p\}, \\ \mathcal{F}(\phi_{1-r}) &:= \langle \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\delta) \mid \delta \in F(\phi_{1-r}, \sigma_r) \text{ for some } \sigma_r\rangle_{\zeta(\mathbb{Z}_p[G])}. \end{aligned}$$

Recall that $\phi_r = (\phi_{1-r} \oplus \mathrm{id}_{H^+_{-r}(L)}) \circ \alpha_r$.

PROPOSITION 7.3. *Let L/K be a finite Galois extension of number fields with Galois group G and let $r > 1$ be an integer. Let $p \neq 2$ be a prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty \cup S_p$. Suppose that Schneider's Conjecture 3.4 and Gross' Conjecture (Conjecture 5.4) both hold. Then with the notation above*

$$\mathcal{J}_r^S = \mathcal{J}_r^S(L/K,p) := \mathcal{E}(\alpha_r)\mathcal{F}(\phi_{1-r}) \cdot \left((A_{\phi_r}^S)^{-1}\right)^\sharp \subseteq \zeta(\mathbb{Q}_p[G])$$

only depends upon L/K, p, r and S. We call $\mathcal{J}_r^S$ the CANONICAL FRACTIONAL GALOIS IDEAL.

Proof. Suppose that α_r' is a second choice of $\mathbb{Q}[G]$-isomorphism as in (6.1). Let $\phi_r' := (\phi_{1-r} \oplus \mathrm{id}_{H_{-r}^+(L)}) \circ \alpha_r'$. Then we have a bijection

$$\begin{array}{ccc} E(\alpha_r) & \longrightarrow & E(\alpha_r') \\ \gamma & \mapsto & \alpha_r'\alpha_r^{-1}\gamma \end{array}$$

which implies $\mathcal{E}(\alpha_r) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\alpha_r(\alpha_r')^{-1})\mathcal{E}(\alpha_r')$. Now (6.4) implies that $\mathcal{J}_r^S$ does not depend on the choice of α_r. The argument for ϕ_{1-r} is similar. ☐

Example 7.4. Suppose that L/K is an extension of totally real fields and that r is even. Then the conjectures of Schneider and Gross both hold by Theorem 3.9 and Theorem 5.2, respectively. We have that $H_{1-r}^+(L)$ vanishes by (2.5) and thus $\mathcal{F}(\phi_{1-r}) = \zeta(\mathbb{Z}_p[G])$. Moreover, we have $Y_r = H_{-r}^+(L) \otimes \mathbb{Z}_p$ and $\alpha_r = \phi_r$. We conclude that we have

$$\mathcal{J}_r^S = \mathcal{E}(\phi_r) \cdot \left((A_{\phi_r}^S)^{-1}\right)^\sharp \subseteq \zeta(\mathbb{Q}_p[G])$$

unconditionally. We also have

$$\left((A_{\phi_r}^S)^{-1}\right)^\sharp = L_S^*(r) \cdot \mathrm{Nrd}_{\mathbb{C}[G]}(\iota_r\phi_r^{-1}).$$

Put $d := [K : \mathbb{Q}]$ and fix an isomorphism $j : \mathbb{C} \simeq \mathbb{C}_p$. We observe that $\iota_r = (2\pi i)^{-r}\mu_r \circ \iota_0$ and that $\mu_r(H_0(L) \otimes \mathbb{Z}_p) = Y_r$. We let

$$\begin{array}{rcl} E' & := & \{\gamma' \in \mathrm{End}_{\mathbb{Q}_p[G]}(H_0(L) \otimes \mathbb{Q}_p) \mid \\ & & \exp_r^{BK} \iota_0^{-1}\gamma'(H_0(L) \otimes \mathbb{Z}_p) \subseteq P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tf}}\} \end{array}$$

and obtain (substitute γ' by $\mu_r^{-1}\iota_r\phi_r^{-1}\gamma\mu_r$)

$$\mathcal{J}_r^S = \mathrm{Nrd}_{\mathbb{C}_p[G]}(j(2\pi i)^{-r})^d \cdot \langle \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma') \mid \gamma' \in E' \rangle_{\zeta(\mathbb{Z}_p[G])} \cdot j(L_S^*(r)).$$

Now suppose in addition that L/K is abelian. The inverse of the Bloch–Kato exponential map and $\iota_0 \otimes_j \mathbb{C}_p$ induce a map

$$P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \longrightarrow H_0(L) \otimes \mathbb{C}_p$$

which in turn induces a regulator map

$$\mathfrak{s}_{S,r}^{(j)} : \bigwedge_{\mathbb{Z}_p[G]}^{d} P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \longrightarrow \bigwedge_{\mathbb{C}_p[G]}^{d} (H_0(L) \otimes \mathbb{C}_p) \simeq \mathbb{C}_p[G].$$

It is then not hard to show that

$$\begin{aligned}\mathcal{J}_r^S &= j(2\pi i)^{-rd} \cdot \mathrm{Im}(\mathfrak{s}_{S,r}^{(j)}) \cdot j(L_S^*(r)) \\ &= j\left(\frac{i}{\pi}\right)^{rd} \cdot \mathrm{Im}(\mathfrak{s}_{S,r}^{(j)}) \cdot j(L_S^*(r)),\end{aligned}$$

where the second equality holds, since p is odd and r is even. This shows that in this case the canonical fractional Galois ideal $\mathcal{J}_r^S$ coincides with the 'Higher Solomon ideal' of Barrett [Bar09, Definition 5.3.1]. When L/K is a CM-extension and r is odd, similar observations hold on minus parts.

Example 7.5. Let L/K be a Galois extension of totally real fields, but now we assume that r is odd. Then (2.5) implies that $H_{-r}^+(L)$ vanishes and that we have $Y_r = H_{1-r}^+(L) \otimes \mathbb{Z}_p$. We assume that Schneider's conjecture holds so that the natural localization maps induce an isomorphism of $\mathbb{Q}_p[G]$-modules

$$H^1_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Q}_p(r)) \xrightarrow{\simeq} P^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))$$

by Propositions 3.8 and 3.12(v). We let $\sigma_r = \sigma_r^1$ be the inverse of this isomorphism. We set $\tau_r := (ch_{r,1}^{(p)})^{-1} \sigma_r^1 \exp_r^{BK}$, which is an isomorphism $L \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq K_{2r-1}(\mathcal{O}_{L,S}) \otimes \mathbb{Q}_p$, and define

$$\begin{aligned}G(\phi_{1-r}, \alpha_r) &:= \{\gamma \in \mathrm{End}_{\mathbb{Q}_p[G]}(Y_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \mid \phi_{1-r}^{-1} \tau_r \alpha_r^{-1} \gamma\,(Y_r) \subseteq Y_r\}, \\ \mathcal{G}(\phi_{1-r}, \alpha_r) &:= \langle \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma) \mid \gamma \in G(\phi_{1-r}, \alpha_r) \rangle_{\zeta(\mathbb{Z}_p[G])} \\ &= \mathcal{E}(\alpha_r) \cdot \mathcal{F}(\phi_{1-r}),\end{aligned}$$

where the last equality follows easily from the definitions. Clearly, the set $G(\phi_{1-r}, \alpha_r)$ contains $\gamma_r := \alpha_r \tau_r^{-1} \phi_{1-r}$ and hence $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma_r) \in \mathcal{G}(\phi_{1-r}, \alpha_r)$. Conversely, for every $\gamma \in G(\phi_{1-r}, \alpha_r)$ we have that $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma_r^{-1}\gamma) \in \mathcal{I}_p(G)$. In other words, we have an equality

$$\mathcal{G}(\phi_{1-r}, \alpha_r) \cdot \mathcal{I}_p(G) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma_r) \cdot \mathcal{I}_p(G).$$

Define a $\mathbb{C}_p[G]$-automorphism of $H_{1-r}^+(L) \otimes \mathbb{C}_p$ by $\vartheta_r^{(j)} := \rho_r \tau_r \iota_r^{-1}$, where we extend scalars via the isomorphism $j : \mathbb{C} \simeq \mathbb{C}_p$ on the right hand side. Noting that $\mathcal{H}_p(G)$ is an ideal in $\mathcal{I}_p(G)$, we compute

$$\begin{aligned}\mathcal{H}_p(G) \cdot \mathcal{J}_r^S &= \mathcal{H}_p(G) \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\gamma_r) \cdot \left((A_{\phi_r}^S)^{-1}\right)^\sharp \\ &= \mathcal{H}_p(G) \cdot \mathrm{Nrd}_{\mathbb{C}_p[G]}(\vartheta_r^{(j)}) \cdot j(L_S^*(r)).\end{aligned}$$

If L/K is a CM-extension and r is even, similar observations again hold on minus parts.

7.4 The annihilation conjecture

Let L/K be a finite Galois extension of number fields with Galois group G and let $r > 1$ be an integer. Let $p \neq 2$ be a prime and let S be a finite set of places of K containing $S_{\mathrm{ram}} \cup S_\infty \cup S_p$. Suppose that Schneider's Conjecture 3.4 and Gross' Conjecture (Conjecture 5.4) both hold.

Conjecture 7.6. *For every $x \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_L})$ we have that*

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(x) \cdot \mathcal{H}_p(G) \cdot \mathcal{J}_r^S \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(K_{2r-2}^w(\mathcal{O}_{L,S})_p).$$

Remark 7.7. The $\mathbb{Z}_p[G]$-annihilator of $\mathbb{Z}_p(r-1)_{G_L} \simeq (\mathbb{Q}_p/\mathbb{Z}_p(1-r)^{G_L})^\vee$ is generated by the elements $1 - \phi_w N(v)^{1-r}$, where v runs through the finite places of K with $v \notin S_{\mathrm{ram}} \cup S_p$ (cf. [Coa77]). Moreover, if L/K is totally real and r is even, then $\mathbb{Z}_p(r-1)_{G_L}$ vanishes.

Remark 7.8. If p does not divide the order of the commutator subgroup of G, then we have $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p[G])$ by Proposition 7.2. In particular, if G is abelian, then Conjecture 7.6 simplifies to the assertion

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_L}) \cdot \mathcal{J}_r^S \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(K_{2r-2}^w(\mathcal{O}_{L,S})_p).$$

Taking Example 7.4 and Remark 7.7 into account, we see that our conjecture is compatible with [Bar09, Conjecture 5.3.4].

Remark 7.9. The author also expects that for every $x \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_L})$ we have that

$$\mathrm{Nrd}_{\mathbb{Q}_p[G]}(x) \cdot \mathcal{J}_r^S \subseteq \mathcal{I}_p(G). \tag{7.2}$$

Then (7.1) implies that the left hand side in Conjecture 7.6 belongs to $\zeta(\mathbb{Z}_p[G])$.

Lemma 7.10. *Let S' be a second finite set of places of K such that $S \subseteq S'$.*

(i) If Conjecture 7.6 holds for S, then it holds for S' as well.

(ii) If (7.2) holds for S, then it holds for S' as well.

Proof. Recall from Remark 2.7 that the p-adic wild kernel does not depend on S. Thus (i) follows once we have shown that

$$\mathcal{H}_p(G) \cdot \mathcal{J}_r^{S'} \subseteq \mathcal{H}_p(G) \cdot \mathcal{J}_r^S. \tag{7.3}$$

By definition we have

$$\mathcal{J}_r^{S'} = \mathcal{J}_r^S \cdot \left(\prod_{v \in S' \setminus S} \epsilon_v(r)^{-1} \right)^\sharp.$$

However, each $\epsilon_v(r)^{-1} = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(1 - \phi_w N(v)^{-r})$ belongs to $\mathcal{I}_p(G)$ as for $v \notin S$ we have $v \nmid p$ and thus $N(v) \in \mathbb{Z}_p^\times$. This implies (ii) and also (7.3) by (7.1). $\square$

7.5 Noncommutative Fitting invariants

We briefly recall the definition and some basic properties of noncommutative Fitting invariants introduced in [Nic10] and further developed in [JN13].
Let G be a finite group and let p be a prime. Let N and M be two $\zeta(\mathbb{Z}_p[G])$-submodules of a $\mathbb{Z}_p$-torsion-free $\zeta(\mathbb{Z}_p[G])$-module. Then N and M are called Nrd-equivalent if there exists an integer n and a matrix $U \in \mathrm{GL}_n(\mathbb{Z}_p[G])$ such that $N = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(U) \cdot M$. We denote the corresponding equivalence class by $[N]$. We say that N is Nrd-contained in M (and write $[N] \subseteq [M]$) if for all $N' \in [N]$ there exists $M' \in [M]$ such that $N' \subseteq M'$. Note that it suffices to check this property for one $N_0 \in [N]$. We will say that x is contained in $[N]$ (and write $x \in [N]$) if there is $N_0 \in [N]$ such that $x \in N_0$.
Now let M be a finitely presented $\mathbb{Z}_p[G]$-module and let

$$\mathbb{Z}_p[G]^a \xrightarrow{h} \mathbb{Z}_p[G]^b \longrightarrow M \longrightarrow 0 \tag{7.4}$$

be a finite presentation of M. We identify the homomorphism h with the corresponding matrix in $M_{a\times b}(\mathbb{Z}_p[G])$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geq b$. In the case $a = b$ we call (7.4) a *quadratic presentation*. The Fitting invariant of h over $\mathbb{Z}_p[G]$ is defined to be

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]}(h) = \begin{cases} [0] & \text{if} \quad a < b \\ \left[\langle \mathrm{Nrd}_{\mathbb{Q}_p[G]}(H) | H \in S(h)\rangle_{\zeta(\mathbb{Z}_p[G])}\right] & \text{if} \quad a \geq b. \end{cases}$$

We call $\mathrm{Fitt}_{\mathbb{Z}_p[G]}(h)$ a *(noncommutative) Fitting invariant* of M over $\mathbb{Z}_p[G]$. One defines $\mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}(M)$ to be the unique Fitting invariant of M over $\mathbb{Z}_p[G]$ which is maximal among all Fitting invariants of M with respect to the partial order "$\subseteq$". If M admits a quadratic presentation h, one also puts $\mathrm{Fitt}_{\mathbb{Z}_p[G]}(M) := \mathrm{Fitt}_{\mathbb{Z}_p[G]}(h)$ which is independent of the chosen quadratic presentation. The following result is [Nic10, Theorem 4.2].

Theorem 7.11. *If M is a finitely presented $\mathbb{Z}_p[G]$-module, then*

$$\mathcal{H}_p(G) \cdot \mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(M).$$

Lemma 7.12. *Let $C^\bullet \in \mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ be a perfect complex such that $H^i(C^\bullet)$ is finite for all $i \in \mathbb{Z}$ and vanishes if $i \neq 2,3$. Choose $\mathcal{L} \in \zeta(\mathbb{Q}_p[G])^\times$ such that $\partial_p(\mathcal{L}) = \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(C^\bullet, 0)$. Then we have an equality*

$$\mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}((H^2(C^\bullet))^\vee)^\sharp = \mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}(H^3(C^\bullet)) \cdot \mathcal{L}.$$

Proof. This is an obvious reformulation of [Nic11a, Lemma 4.4] (with a shift by 2). □

7.6 The relation to the leading term conjecture

The aim of this subsection is to prove the following theorem which describes the relation of Conjecture 7.6 to the leading term conjecture at $s = r$ and thus also to the ETNC for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ by Theorem 6.5.

Theorem 7.13. *Let L/K be a finite Galois extension of number fields with Galois group G. Let $r > 1$ be an integer and let p be an odd prime. Suppose that the leading term conjecture at $s = r$ (Conjecture 6.1) holds for L/K at p. Then Conjecture 7.6 is also true.*

Corollary 7.14. *Fix an odd prime p and suppose that L is abelian over $\mathbb{Q}$. Then the leading term conjecture at $s = r$ and Conjecture 7.6 both hold for almost all $r > 1$ (and all even r if L is totally real).*

Proof. As $L/\mathbb{Q}$ is abelian, the ETNC for the pair $(\mathbb{Q}(r)_L, \mathbb{Z}[G])$ holds for all $r \in \mathbb{Z}$ by work of Burns and Flach [BF06]. Now fix an odd prime p. Then Schneider's conjecture holds for almost all r by Remark 3.6 and for all even $r > 1$ if L is totally real by Theorem 3.9. Thus the result follows from Theorem 6.5 and Theorem 7.13. ☐

Proof of Theorem 7.13. Recall the notation from §7.3. Let $\gamma \in E(\alpha_r)$, $\delta \in F(\phi_{1-r}, \sigma_r)$ and $x \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p(r-1)_{G_L})$. We have to show that

$$\mathcal{H}_p(G) \cdot \mathrm{Nrd}(x) \cdot \mathrm{Nrd}(\gamma) \cdot \mathrm{Nrd}(\delta) \cdot \left((A^S_{\phi_r})^{-1}\right)^\sharp \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(K^w_{2r-2}(\mathcal{O}_{L,S})_p), \quad (7.5)$$

where we denote $\mathrm{Nrd}_{\mathbb{Q}_p[G]}$ by Nrd in this proof for simplicity. As the reduced norm is continuous for the p-adic topology, we may and do assume that γ and δ are $\mathbb{Q}_p[G]$-automorphisms (and not just endomorphisms). By the definition of $E(\alpha_r)$ we therefore get an injection

$$\exp_r^{BK} \alpha_r^{-1}\gamma : Y_r \longrightarrow P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tf}} \quad (7.6)$$

which we may lift to an injection

$$\eta_r : Y_r \longrightarrow P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)),$$

since Y_r is a projective $\mathbb{Z}_p[G]$-module. Likewise, by the definition of $F(\phi_{1-r}, \sigma_r)$ we obtain a map

$$\delta\phi_{1-r}^{-1}(ch_{r,1}^{(p)})^{-1}\sigma_r^1 : P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tf}} \longrightarrow H^+_{1-r}(L) \otimes \mathbb{Z}_p. \quad (7.7)$$

We may therefore define a $\mathbb{Z}_p[G]$-homomorphism

$$\xi_r : Y_r \longrightarrow (H^+_{1-r}(L) \otimes \mathbb{Z}_p) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$$

such that the projection onto $H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is the composition of η_r and the natural map $P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$, whereas the projection onto $H^+_{1-r}(L) \otimes \mathbb{Z}_p$ is given by the composite map of (7.6) and (7.7). We then have an equality

$$\xi_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = (\delta \oplus \mathrm{id}_{H^2_c(\mathcal{O}_{L,S}, \mathbb{Q}_p(r))}) t(\psi_r, \sigma_r, S)\gamma \quad (7.8)$$

which implies that ξ_r is injective.
The perfect complex $C_{L,S}(r)$ is isomorphic in $\mathcal{D}(\mathbb{Z}_p[G])$ to a complex $A \to B$ of $\mathbb{Z}_p[G]$-modules of finite projective dimension, where A is placed in degree 2.

Choose $n \in \mathbb{N}$ such that $p^n\gamma(Y_r) \subseteq Y_r$. As Y_r is projective, we may construct the following commutative diagram of $\mathbb{Z}_p[G]$-modules with exact rows and columns.

$$\begin{tikzcd}
Y_r \arrow[r, equal] \arrow[d, hook, "\xi_r"] & Y_r \arrow[r, "0"] \arrow[d, hook] & Y_r \arrow[r, equal] \arrow[d, hook] & Y_r \arrow[d, hook, "p^n\gamma"] \\
(H^+_{1-r}(L) \otimes \mathbb{Z}_p) \oplus H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \arrow[r, hook] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r, two heads] \arrow[d] & H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \oplus Y_r \arrow[d] \\
\mathrm{cok}(\xi_r) \arrow[r, hook] & A' \arrow[r] & B' \arrow[r, two heads] & H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \oplus \mathrm{cok}(p^n\gamma)
\end{tikzcd}$$

The arrow $A' \to B'$ defines a complex C' in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ (where we place A' in degree 2; note that C' depends on a lot of choices which we suppress in the notation). The cohomology groups of this complex are finite and vanish outside degrees 2 and 3. Thus the zero map is the unique trivialization of this complex. Likewise the arrow $Y_r \xrightarrow{0} Y_r$ defines the complex $Y_r\{2,3\}$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ and we choose $t_{\delta,n} := p^n(\delta^{-1} \oplus \mathrm{id}_{H^+_{-r}(L)\otimes\mathbb{Q}_p})$ as a trivialization. Using equation (7.8) we compute

$$\begin{aligned}
-\partial_p(A^S_{\phi_r})^\sharp &= \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(C_{L,S}(r), t(\psi_r, \sigma_r, S)) \\
&= \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(C', 0) + \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(Y_r\{2,3\}, t_{\delta,n}) \\
&= \chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(C', 0) + \partial_p(\mathrm{Nrd}(t_{\delta,n})),
\end{aligned}$$

where the first equality is Conjecture 6.1. Now Lemma 7.12 implies the first equality in the following computation.

$$\begin{aligned}
\mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}(\mathrm{cok}(\xi_r)^\vee)^\sharp &= \mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}(H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \oplus \mathrm{cok}(p^n\gamma)) \cdot \\
&\qquad \left((A^S_{\phi_r})^\sharp \mathrm{Nrd}(t_{\delta,n})\right)^{-1} \\
&\supseteq \mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}\left(H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))\right) \cdot \mathrm{Fitt}_{\mathbb{Z}_p[G]}(\mathrm{cok}(p^n\gamma)) \cdot \\
&\qquad \left((A^S_{\phi_r})^\sharp \mathrm{Nrd}(t_{\delta,n})\right)^{-1} \\
&= \mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}\left(H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))\right) \cdot \mathrm{Nrd}(p^n\gamma) \cdot \\
&\qquad \left((A^S_{\phi_r})^\sharp \mathrm{Nrd}(t_{\delta,n})\right)^{-1} \\
&= \mathrm{Fitt}^{\max}_{\mathbb{Z}_p[G]}\left(H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))\right) \cdot \\
&\qquad \mathrm{Nrd}(\gamma) \cdot \mathrm{Nrd}(\delta) \cdot \left((A^S_{\phi_r})^\sharp\right)^{-1} \\
&\ni \mathrm{Nrd}(x) \cdot \mathrm{Nrd}(\gamma) \cdot \mathrm{Nrd}(\delta) \cdot \left((A^S_{\phi_r})^\sharp\right)^{-1}.
\end{aligned}$$

The inclusion follows from [Nic10, Proposition 3.5]. The second equality holds, since $p^n\gamma : Y_r \to Y_r$ is a quadratic presentation of $\mathrm{cok}(p^n\gamma)$. The definition of $t_{\delta,n}$ gives the third equality. Finally, the $\mathbb{Z}_p[G]$-module $H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is cyclic by Proposition 3.12 (ii) and thus $\mathrm{Nrd}(x)$ belongs to its maximal Fitting invariant by [JN13, Theorem 3.1(i) and Theorem 5.1(i)]. As $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{cok}(\xi_r)^\vee)^\sharp$ equals $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{cok}(\xi_r))$, Theorem 7.11 implies that

$$\mathcal{H}_p(G) \cdot \mathrm{Nrd}(x) \cdot \mathrm{Nrd}(\gamma) \cdot \mathrm{Nrd}(\delta) \cdot \left((A_{\phi_r}^S)^{-1}\right)^\sharp \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{cok}(\xi_r)). \tag{7.9}$$

However, the composition of ξ_r and the projection onto $H_c^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ factors through $P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ via η_r and thus there is a surjection of $\mathrm{cok}(\xi_r)$ onto

$$\begin{aligned}\mathrm{cok}\left(P^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \to H_c^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))\right) &\simeq Ш^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \\ &\simeq K_{2r-2}^w(\mathcal{O}_{L,S})_p,\end{aligned}$$

where the last isomorphism is Proposition 3.12 (iv). This and (7.9) now imply (7.5). ☐

Remark 7.15. The proof also shows that Conjecture 6.1 implies the containment (7.2).

7.7 The relation to a conjecture of Burns, Kurihara and Sano

Let L/K be an *abelian* extension of number fields with Galois group G and let r be an integer. In [BKS] the authors define a certain ideal in terms of 'generalized Stark elements of weight $-2r$' (in particular, this involves the equivariant L-value $L_S^*(r)$) and conjecture that this ideal coincides with the initial Fitting ideal of $H_{\text{ét}}^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$. In this final subsection, we will explain the relation of their conjecture to our Conjecture 7.6 if $r > 1$.
So let us henceforth assume that $r > 1$. Fix a second finite set T of places of K, which is disjoint from S. Following [BKS, §3.2] we define $R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$ to be a complex that lies in an exact triangle in the derived category $\mathcal{D}(\mathbb{Z}_p[G])$ of the form

$$R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)) \to R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)) \to \bigoplus_{w \in T(L)} R\Gamma(L(w), \mathbb{Z}_p(1-r)), \tag{7.10}$$

where the second arrow is induced by the natural morphism. For each $i \in \mathbb{Z}$ we abbreviate $H^i R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$ by $H_T^i(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$.
The conjecture of Burns, Kurihara and Sano [BKS, Conjecture 3.5] concerns the initial Fitting ideal and thus also the annihilator ideal of the finite cohomology group $H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$. In order to relate their conjecture to ours, we have to determine the relation between this cohomolgy group and the wild kernel $K_{2r-2}^w(\mathcal{O}_{L,S})_p$. Artin-Verdier duality and the triangle (7.10) give an exact triangle in $\mathcal{D}(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w \in T(L)} R\Gamma(L(w), \mathbb{Z}_p(1-r)) \to C_{S,T}^\bullet(r) \to D_S^\bullet(r) \tag{7.11}$$

(see [BF03, (6)] or [BKS, §4.1]), where we have set

$$\begin{aligned} C^\bullet_{S,T}(r) &:= R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))[1] \oplus (H_r^+(L) \otimes \mathbb{Z}_p)[-1]; \\ D^\bullet_S(r) &:= R\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)), \mathbb{Z}_p)[-2]. \end{aligned}$$

For any $\mathbb{Z}_p$-module M we write M^* for its $\mathbb{Z}_p$-linear dual. We henceforth assume that Schneider's conjecture holds. Then Proposition 3.12 implies that $H^1_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \simeq H^+_{-r}(L) \otimes \mathbb{Z}_p$ is $\mathbb{Z}_p[G]$-projective. Thus the complex $D^\bullet_S(r)$ is acyclic outside degrees 0 and 1 and we have canonical isomorphisms of $\mathbb{Z}_p[G]$-modules

$$\begin{aligned} H^0(D^\bullet_S(r))_{\mathrm{tor}} &\simeq H^3_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^\vee, \\ H^0(D^\bullet_S(r))_{\mathrm{tf}} &\simeq H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))^*, \\ H^1(D^\bullet_S(r)) &\simeq (H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tor}})^\vee \oplus (H^+_r(L) \otimes \mathbb{Z}_p). \end{aligned}$$

In particular, the triangle (7.11) yields a right exact sequence of $\mathbb{Z}_p[G]$-modules

$$\bigoplus_{w \in T(L)} \mathbb{Z}_p(r-1)_{G_w} \to H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)) \to (H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))_{\mathrm{tor}})^\vee \to 0.$$

Moreover, we have a surjection

$$H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r)) \twoheadrightarrow K^w_{2r-2}(\mathcal{O}_{L,S})_p$$

by Proposition 3.12(iv). Thus [BKS, Conjecture 3.5] and our conjecture predict annihilators of the torsion subgroup and a finite quotient of $H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$, respectively. In order to compare the two conjectures we will hence assume that $H^2_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite so that we have an inclusion

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r)))^\sharp \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(K^w_{2r-2}(\mathcal{O}_{L,S})_p). \tag{7.12}$$

By Proposition 3.12(v) and (2.5) this implies that L is totally real and that r is odd. Since $H^+_{-r}(L)$ vanishes in this case, the wedge product which occurs in [BKS, Conjecture 3.5] is empty (see [BKS, Hypothesis 2.2]) so that this conjecture predicts that the initial Fitting ideal of $H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-r))$ is generated by an element $\eta_{L/K,S,T}(r)$ as defined in [BKS, §2.2]. By its very definition (and taking [BKS, Remark 2.5] into account) this element is given by

$$\eta_{L/K,S,T}(r)^\sharp = \left(\prod_{v \in T} (1 - \phi_w N(v)^{1-r}) \right) \cdot \mathrm{Nrd}_{\mathbb{C}_p[G]}(\vartheta_r^{(j)}) \cdot j(L^*_S(r)).$$

Now the inclusion (7.12), Remark 7.7 and Example 7.5 imply the following result.

Proposition 7.16. *Let L/K be an abelian extension of totally real fields and let $r > 1$ be an odd integer. Assume that Schneider's Conjecture 3.4 and Gross' Conjecture 5.1 both hold. Then [BKS, Conjecture 3.5] for all T implies Conjecture 7.6.*

Remark 7.17. The conjecture of Burns, Kurihara and Sano indeed involves the choice of a certain idempotent ε of $\mathbb{Z}_p[G]$. Under the hypotheses of Proposition 7.16 it suffices to consider their conjecture for $\varepsilon = 1$ (which implies their conjecture for all admissible idempotents). However, we point out that in general 1 is not an admissible idempotent. For instance, this happens if L/K is a CM-extension. If we further assume that r is even, then $e^- := (1-c)/2$ is admissible, where $c \in G$ denotes complex conjugation. In this case $e^- H_c^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(r))$ is finite and one can formulate an analogue of Proposition 7.16 on minus parts.

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