Subtle Characteristic Classes
and Hermitian Forms

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Abstract. Following [14], we compute the motivic cohomology ring of the Nisnevich classifying space of the unitary group associated to the standard split hermitian form of a quadratic extension. This provides us with subtle characteristic classes which take value in the motivic cohomology of the Čech simplicial scheme associated to a hermitian form. Comparing these new classes with subtle Stiefel-Whitney classes arising in the orthogonal case, we obtain relations among the latter ones holding in the motivic cohomology of the Čech simplicial scheme associated to a quadratic form divisible by a 1-fold Pfister form. Moreover, we present a description of the motive of the torsor corresponding to a hermitian form in terms of its subtle characteristic classes.

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1 Introduction

The study of homotopy theory in the algebro-geometric world, which was initiated by Morel and Voevodsky in [9], has led to very deep results such as the affirmation of *Milnor Conjecture* ([17]). As a result, much attention has been devoted in the last years to transferring topological techniques into algebraic geometry.

For example, the study of classifying spaces BG and their respective characteristic classes in different cohomology theories have been extremely useful in

topology to approach the classification of principal G-bundles. In the same way, it is possible to study G-torsors in algebraic geometry by focusing on classifying spaces and characteristic classes in the motivic homotopic environment. We notice that here there are two different, but highly related, classifying spaces, namely the *Nisnevich* and the *étale*. For non special algebraic groups they have in general different cohomology rings and, consequently, they produce different characteristic classes. Although G-torsors are classified by the étale classifying space, it is undoubted that the Nisnevich version provides its own advantages. Good evidence of this is provided by [14], where torsors, Nisnevich classifying spaces and a general homotopic framework to deal with them have been deeply studied by the authors. In particular, they focus on BO_n, the Nisnevich classifying space of the orthogonal group associated to the standard split quadratic form, which allows to study O_n-torsors over the point that are in one-to-one correspondence with quadratic forms. They prove that the motivic cohomology ring with $\mathbb{Z}/2$-coefficients of BO_n is a polynomial algebra over the motivic cohomology of the base field generated by some elements that they call *subtle Stiefel-Whitney classes*. These classes are very informative, for example they are able to see if a quadratic form is in a power of the fundamental ideal of the Witt ring or not. Moreover, they are related to the J-invariant of quadrics introduced in [15].

In this work we will focus instead on the unitary group $U_n(E/k)$ associated to the standard split hermitian form of a quadratic extension E/k. In particular, we will compute the motivic cohomology with $\mathbb{Z}/2$-coefficients of its Nisnevich classifying space. As in the orthogonal case, this will provide us with subtle characteristic classes which allow to approach the classification of $U_n(E/k)$-torsors over the point that are nothing else but n-dimensional hermitian forms of E/k, which in turn are in one-to-one correspondence with $2n$-dimensional quadratic forms over k divisible by the norm form of the quadratic extension considered. In [14], the computation of the motivic cohomology of BO_n is conducted inductively by using fibrations with motivically Tate fibers. In our situation, new features will appear. In particular, the fibrations in the unitary case, similar to those considered in the orthogonal one, will have reduced fibers which (depending on parity) are not motivically Tate but, anyway, invertible, which will still allow the computation. These invertible motives are, not surprisingly, closely related to the *Rost motive* of our quadratic extension. As a consequence, we obtain that, unlike the orthogonal case, the classifying space of the unitary group is not cellular, but it becomes one once tensored with the Čech simplicial scheme of the Pfister form of the quadratic extension. Related to this, we observe an interesting interaction between invertible objects and idempotents in Voevodsky category. It is manifested, in particular, by the fact that the cohomology of the tensor product of $BU_n(E/k)$ with the Čech simplicial scheme above mentioned happens to be a direct limit of the cohomology of $BU_n(E/k)$ tensored with powers of an invertible motive. We also note that, although studying hermitian forms is the same as studying quadratic forms divisible by a Pfister form, the understanding of the unitary case allows to trace

back information from the hermitian world to the quadratic one. In particular, from the computation of the motivic cohomology of $BU_n(E/k)$ we get relations among subtle Stiefel-Whitney classes in the cohomology of the Čech simplicial scheme of the respective quadratic form divisible by a binary Pfister form. In this sense, for this special class of quadratic forms, the cohomology of $BU_n(E/k)$ is much closer to that of the Čech simplicial scheme of the torsor than the cohomology of BO_{2n}.

We will now summarise the content of the sections of this text. First of all, in section 2 we present a few notations which will be followed throughout the paper. Section 3 is devoted to recalling some preliminary definitions and results about the category of motives over a simplicial base studied in [16]. Moreover, following [14], we will prove some statements regarding fibrations with motivically invertible reduced fibers. In section 4 we will deal with Nisnevich classifying spaces and some of their main features. A study of the Čech simplicial scheme, the Rost motive of a quadratic extension and, especially, of some closely related invertible motives is presented in section 5. The main result of the paper, namely the computation of the motivic cohomology ring of $BU_n(E/k)$ is object of section 6. In section 7, we will compare the classifying space of the unitary group of the split hermitian form with that of the orthogonal group of the corresponding quadratic form. As a consequence, we will present the cohomology of the first as a quotient of the second. In particular, we will relate our subtle classes to subtle Stiefel-Whitney classes arising in the orthogonal case. Finally, in section 8, in the same fashion of [14], we find some applications to hermitian forms. For example, we deduce relations among subtle classes in the motivic cohomology of the respective Čech simplicial scheme, see that these subtle classes distinguish the triviality of the torsor and find an expression of the motive of the torsor associated to a hermitian form.

2 NOTATION

Throughout this paper we will work over a field k of characteristic different from 2.

The main categories we will consider are the category of motivic spaces $Spc(k) = \Delta^{op}Shv_{Nis}(Sm/k)$, the simplicial homotopy category $\mathcal{H}_s(k)$ constructed by Morel and Voevodsky in [9] and the triangulated category of effective motives $\mathcal{DM}^-_{eff}(k)$ constructed by Voevodsky in [18].

All motivic cohomology will be with $\mathbb{Z}/2$-coefficients. Moreover, we will denote by H the motivic cohomology of $Spc(k)$. From a result by Voevodsky ([17]),

we know that $H = K^M(k)/2[\tau]$, where τ is the generator of $H^{0,1} = \mathbb{Z}/2$.
Given a quadratic extension $E = k(\sqrt{\alpha})$ and an n-dimensional E-vector space V, an n-dimensional hermitian form is a map $h : V \times V \to E$ which is E-linear in the first factor and such that $h(v,w) = \sigma(h(w,v))$ (where σ is the generator of $Gal(E/k)$). It follows immediately from the definition that the diagonal part of a hermitian form takes values in k and is a quadratic form. We will denote by $\widetilde{h}$ this $2n$-dimensional quadratic form over k defined by $\widetilde{h}(v) = h(v,v)$ for any $v \in V$ considered as a $2n$-dimensional k-vector space. Moreover, notice that the quadratic form $\widetilde{h}$ just defined is divisible by $\langle\!\langle \alpha \rangle\!\rangle$, the 1-fold Pfister form associated to α. Indeed, more is true, namely any quadratic form over k divisible by $\langle\!\langle \alpha \rangle\!\rangle$ is associated to some hermitian form, and the correspondence is bijective. In fact, given two n-dimensional hermitian forms h and h', we have that $h \cong h'$ if and only if $\widetilde{h} \cong \widetilde{h'}$ ([6, Corollary 9.2]).
We will express by q_n the standard split quadratic form $\perp_{i=1}^{n} \langle (-1)^{i-1} \rangle$ and by $\mathbb{H}$ the hyperbolic form $\langle 1, -1 \rangle$. Similarly, we will denote by h_n the standard split hermitian form $\perp_{i=1}^{n} \langle (-1)^{i-1} \rangle$. Notice, in particular, that $\widetilde{h}_n = \langle\!\langle \alpha \rangle\!\rangle \otimes q_n$. By $U_n(E/k)$ we will mean the unitary group of invertible $n \times n$-matrices over E that preserve the standard split hermitian form h_n. Notice that this is a linear algebraic group over k.

3 Motives over a simplicial base

We start this section by recalling a few definitions and some results about the category of motives over a simplicial base studied by Voevodsky in [16].
Let $Y_\bullet$ be a smooth simplicial scheme over k and R a commutative ring with unity. As in [16], let $Sm/Y_\bullet$ be the category whose objects are pairs (U,j), where j is a non-negative integer and U is a smooth scheme over Y_j, and whose morphisms from (U,j) to (V,i) are pairs (f,θ), where $\theta : [i] \to [j]$ is a simplicial map, such that the following diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\;f\;} & V \\
\downarrow & & \downarrow \\
Y_j & \xrightarrow{\;Y_\theta\;} & Y_i
\end{array}
$$

commutes.
Then, we will denote by $Spc(Y_\bullet) = \Delta^{op}Shv_{Nis}(Sm/Y_\bullet)$ the category of motivic spaces over $Y_\bullet$, obtained by considering simplicial Nisnevich sheaves on $Sm/Y_\bullet$. In [16] the category of motives over $Y_\bullet$ with R coefficients was constructed. We will denote this category by $\mathcal{DM}_{eff}^{-}(Y_\bullet, R)$.
This category comes endowed with a sequence of functors

$$
r_i^* : \mathcal{DM}_{eff}^{-}(Y_\bullet, R) \to \mathcal{DM}_{eff}^{-}(Y_i, R)
$$

For simplicity we will write N_i for $r_i^*(N)$. We recall that, for any morphism $p : Y_\bullet \to Y'_\bullet$ of smooth simplicial schemes, there is an adjoint pair

$$\mathcal{DM}_{eff}^-(Y_\bullet, R)$$
$$Lp^* \uparrow \downarrow Rp_*$$
$$\mathcal{DM}_{eff}^-(Y'_\bullet, R)$$

If moreover p is smooth, then we also have the adjoint pair

$$\mathcal{DM}_{eff}^-(Y_\bullet, R)$$
$$Lp_\# \downarrow \uparrow p^*$$
$$\mathcal{DM}_{eff}^-(Y'_\bullet, R)$$

Besides, we will denote by $CC(Y_\bullet)$ the simplicial set obtained by applying to $Y_\bullet$ the functor CC which commutes with coproducts and sends any connected scheme to the point.

For any smooth simplicial scheme $Y_\bullet$ over k we can consider the projection to the base $Y_\bullet \to Spec(k)$. This morphism induces the triangulated functor

$$c^* : \mathcal{DM}_{eff}^-(k, R) \to \mathcal{DM}_{eff}^-(Y_\bullet, R)$$

We will denote by $M_{Y_\bullet}$ the motive $c^*(M)$ for any $M \in \mathcal{DM}_{eff}^-(k, R)$.

We report below two results about the category of motives over a simplicial base which will be essential throughout this paper in order to deal with fibrations with motivically invertible reduced fibers. First of all, in [14] it is proven the following proposition.

PROPOSITION 3.1 *[14, Proposition 3.1.5] Suppose that $H^1(CC(Y_\bullet), R^\times) = 0$. Let T be the unit in $\mathcal{DM}_{eff}^-(k, R)$ and $N \in \mathcal{DM}_{eff}^-(Y_\bullet, R)$ be such a motive that its graded components $N_i \in \mathcal{DM}_{eff}^-(Y_i, R)$ are isomorphic to T_{Y_i} and all the structure maps $N_\theta : LY_\theta^*(N_i) \to N_j$ are isomorphisms for any simplicial map $\theta : [i] \to [j]$. Then N is isomorphic to $T_{Y_\bullet}$.*

From the previous proposition we immediately deduce the following corollary which is a generalisation for all invertible motives.

COROLLARY 3.2 *Suppose that $H^1(CC(Y_\bullet), R^\times) = 0$. Let M be an invertible motive in $\mathcal{DM}_{eff}^-(k, R)$ and $N \in \mathcal{DM}_{eff}^-(Y_\bullet, R)$ be such a motive that its graded components $N_i \in \mathcal{DM}_{eff}^-(Y_i, R)$ are isomorphic to M_{Y_i} and all the structure maps $N_\theta : LY_\theta^*(N_i) \to N_j$ are isomorphisms for any simplicial map $\theta : [i] \to [j]$. Then N is isomorphic to $M_{Y_\bullet}$.*

Proof. Consider the motive $N \otimes M_{Y_\bullet}^{-1}$ in $\mathcal{DM}_{eff}^-(Y_\bullet, R)$. We notice that

$$(N \otimes M_{Y_\bullet}^{-1})_i \cong N_i \otimes M_{Y_i}^{-1} \cong M_{Y_i} \otimes M_{Y_i}^{-1} \cong T_{Y_i}$$
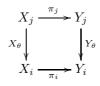
and, for any simplicial map $\theta : [i] \to [j]$, the morphisms

$$LY_\theta^*((N \otimes M_{Y_\bullet}^{-1})_i) \to (N \otimes M_{Y_\bullet}^{-1})_j$$

are nothing else but the isomorphisms $(LY_\theta^*(N_i) \to N_j) \otimes M_{Y_j}^{-1}$. Then, it follows from Proposition 3.1 that $N \otimes M_{Y_\bullet}^{-1} \cong T_{Y_\bullet}$, which completes the proof. $\square$

Notice that the condition $H^1(CC(Y_\bullet), R^\times) = 0$ is automatically satisfied if $R = \mathbb{Z}/2$, which is the case we will be interested in.

Before proceeding with the next results, we recall some definitions about coherence. A smooth morphism $\pi : X_\bullet \to Y_\bullet$ of smooth simplicial schemes is called smooth coherent if for any simplicial map $\theta : [i] \to [j]$ the following diagram

$$
\begin{array}{ccc}
X_j & \xrightarrow{\pi_j} & Y_j \\
{\scriptstyle X_\theta}\downarrow & & \downarrow{\scriptstyle Y_\theta} \\
X_i & \xrightarrow{\pi_i} & Y_i
\end{array}
$$

is cartesian and all the π_j are smooth. An object N in $\mathcal{DM}_{eff}^-(Y_\bullet, R)$ is called coherent if, for any simplicial map $\theta : [i] \to [j]$, the structural map $N_\theta : LY_\theta^*(N_i) \to N_j$ is an isomorphism. Let $\mathcal{DM}_{coh}^-(Y_\bullet, R)$ be the full subcategory of $\mathcal{DM}_{eff}^-(Y_\bullet, R)$ consisting of coherent objects. We notice that $\mathcal{DM}_{coh}^-(Y_\bullet, R)$ is closed under taking cones and arbitrary direct sums, since LY_θ^* is a triangulated functor. It immediately follows from these definitions that, if π is smooth coherent, then $L\pi_\#$ maps coherent objects to coherent ones and, in particular, $M(X_\bullet \xrightarrow{\pi} Y_\bullet)$ belongs to $\mathcal{DM}_{coh}^-(Y_\bullet, R)$, where $M(X_\bullet \xrightarrow{\pi} Y_\bullet)$ is nothing else but the image $L\pi_\#(T_{X_\bullet})$ of the trivial Tate motive.

We present now the main technique taken from [14] we will use in our computation. This result allows to generate long exact sequences (of the same nature of Gysin sequences for sphere bundles in topology) in motivic cohomology associated to fibrations with reduced fibers which are motivically invertible.

PROPOSITION 3.3 *Let $\pi : X_\bullet \to Y_\bullet$ be a smooth coherent morphism of smooth simplicial schemes over k and A a smooth k-scheme such that:*
1) over the 0 simplicial component π is isomorphic to the map $Y_0 \times A \to Y_0$;
2) $H^1(CC(Y_\bullet), R^\times) = 0$;
3) $\widetilde{M}(A)$ is an invertible motive in $\mathcal{DM}_{eff}^-(k, R)$, where by $\widetilde{M}(A)$ we mean $Cone(M(A) \to T)[-1]$.
Then, $Cone(\pi) \cong \widetilde{M}(A)_{Y_\bullet}[1] \in \mathcal{DM}_{eff}^-(Y_\bullet, R)$.

Proof. In the motivic category $\mathcal{DM}_{eff}^-(Y_\bullet, R)$ we have a distinguished triangle

$$M(X_\bullet \xrightarrow{\pi} Y_\bullet) \to T_{Y_\bullet} \to Cone(\pi) \to M(X_\bullet \xrightarrow{\pi} Y_\bullet)[1]$$

By condition 1 and from the fact that our morphism is smooth coherent it follows that it is the projection over any simplicial component. So, we obtain that the morphism $\pi_i : Y_i \times A \cong X_i \to Y_i$ induces in $\mathcal{DM}^-_{eff}(Y_i, R)$ the map $M(A)_{Y_i} \to T_{Y_i}$ for any i. Thus, $Cone(\pi)_i \cong \widetilde{M}(A)_{Y_i}[1]$ is an invertible motive in $\mathcal{DM}^-_{eff}(Y_i, R)$. Since $M(X_\bullet \xrightarrow{\pi} Y_\bullet)$ and $T_{Y_\bullet}$ belong to $\mathcal{DM}^-_{coh}(Y_\bullet, R)$, we have that $Cone(\pi)$ is in $\mathcal{DM}^-_{coh}(Y_\bullet, R)$ and, by Corollary 3.2, we get that $Cone(\pi) \cong \widetilde{M}(A)_{Y_\bullet}[1]$ in $\mathcal{DM}^-_{eff}(Y_\bullet, R)$, as we aimed to show. $\square$
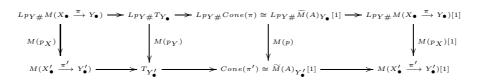
Later, we will also need the following result about functoriality of the isomorphism found in the previous proposition.

PROPOSITION 3.4 *Let $\pi : X_\bullet \to Y_\bullet$ and $\pi' : X'_\bullet \to Y'_\bullet$ be smooth coherent morphisms of smooth simplicial schemes over k and A a smooth k-scheme that satisfies all conditions from the previous proposition with respect to π' and such that the following diagram is cartesian with all morphisms smooth*
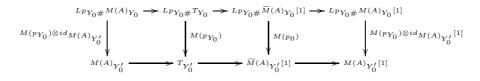
$$
\begin{array}{ccc}
X_\bullet & \xrightarrow{\pi} & Y_\bullet \\
{\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\
X'_\bullet & \xrightarrow{\pi'} & Y'_\bullet
\end{array}
$$

Then, the induced square of motives in the category $\mathcal{DM}^-_{eff}(Y'_\bullet, R)$ extends uniquely to a morphism of triangles where $Lp_{Y\#}Cone(\pi) \to Cone(\pi')$ is given by $M(p_Y) \otimes id_{\widetilde{M}(A)_{Y'_\bullet}[1]}$.

Proof. First of all, we notice that in $\mathcal{DM}^-_{eff}(Y'_\bullet, R)$ there is the following morphism of distinguished triangles

$$
\begin{array}{ccccccc}
Lp_{Y\#}M(X_\bullet \xrightarrow{\pi} Y_\bullet) & \longrightarrow & Lp_{Y\#}T_{Y_\bullet} & \longrightarrow & Lp_{Y\#}Cone(\pi) \cong Lp_{Y\#}\widetilde{M}(A)_{Y_\bullet}[1] & \longrightarrow & Lp_{Y\#}M(X_\bullet \xrightarrow{\pi} Y_\bullet)[1] \\
{\scriptstyle M(p_X)}\downarrow & & {\scriptstyle M(p_Y)}\downarrow & & {\scriptstyle M(p)}\downarrow & & {\scriptstyle M(p_X)[1]}\downarrow \\
M(X'_\bullet \xrightarrow{\pi'} Y'_\bullet) & \longrightarrow & T_{Y'_\bullet} & \longrightarrow & Cone(\pi') \cong \widetilde{M}(A)_{Y'_\bullet}[1] & \longrightarrow & M(X'_\bullet \xrightarrow{\pi'} Y'_\bullet)[1]
\end{array}
$$

where the isomorphisms in the diagram are due to Proposition 3.3. Once restricted to the 0 simplicial component the previous diagram becomes in $\mathcal{DM}^-_{eff}(Y'_0, R)$

$$
\begin{array}{ccccccc}
Lp_{Y_0\#}M(A)_{Y_0} & \longrightarrow & Lp_{Y_0\#}T_{Y_0} & \longrightarrow & Lp_{Y_0\#}\widetilde{M}(A)_{Y_0}[1] & \longrightarrow & Lp_{Y_0\#}M(A)_{Y_0}[1] \\
{\scriptstyle M(p_{Y_0})\otimes id_{M(A)_{Y'_0}}}\downarrow & & {\scriptstyle M(p_{Y_0})}\downarrow & & {\scriptstyle M(p_0)}\downarrow & & {\scriptstyle M(p_{Y_0})\otimes id_{M(A)_{Y'_0}}[1]}\downarrow \\
M(A)_{Y'_0} & \longrightarrow & T_{Y'_0} & \longrightarrow & \widetilde{M}(A)_{Y'_0}[1] & \longrightarrow & M(A)_{Y'_0}[1]
\end{array}
$$

Note that

$$Hom_{\mathcal{DM}^-_{eff}(Y'_0,R)}(Lp_{Y_0\#}\widetilde{M}(A)_{Y_0}[1], T_{Y'_0}) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(\widetilde{M}(A)_{Y_0}[1], p^*_{Y_0}T_{Y'_0}) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(\widetilde{M}(A)_{Y_0}[1], T_{Y_0}) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(k,R)}(\widetilde{M}(Y_0 \times A)[1], T) \cong 0$$

since $Y_0 \times A$ is a smooth scheme over k and, so, has no cohomology in bidegree $(0)[-1]$. From this we deduce that $M(p_0)$ must be $M(p_{Y_0}) \otimes id_{\widetilde{M}(A)_{Y'_0}[1]}$.

At this point we notice that both $M(p)$ and $M(p_Y) \otimes id_{\widetilde{M}(A)_{Y'_\bullet}[1]}$ belong to

$$Hom_{\mathcal{DM}^-_{eff}(Y'_\bullet,R)}(Lp_{Y\#}\widetilde{M}(A)_{Y_\bullet}[1], \widetilde{M}(A)_{Y'_\bullet}[1]) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(\widetilde{M}(A)_{Y_\bullet}, p^*_Y\widetilde{M}(A)_{Y'_\bullet}) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_\bullet,R)}(\widetilde{M}(A)_{Y_\bullet}, \widetilde{M}(A)_{Y_\bullet}) \cong H^{0,0}(Y_\bullet, R)$$

since $\widetilde{M}(A)_{Y_\bullet}$ is an invertible motive. Similarly $M(p_0) = M(p_{Y_0}) \otimes id_{\widetilde{M}(A)_{Y'_0}[1]}$ belongs to

$$Hom_{\mathcal{DM}^-_{eff}(Y'_0,R)}(Lp_{Y_0\#}\widetilde{M}(A)_{Y_0}[1], \widetilde{M}(A)_{Y'_0}[1]) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(\widetilde{M}(A)_{Y_0}, p^*_{Y_0}\widetilde{M}(A)_{Y'_0}) \cong$$

$$Hom_{\mathcal{DM}^-_{eff}(Y_0,R)}(\widetilde{M}(A)_{Y_0}, \widetilde{M}(A)_{Y_0}) \cong H^{0,0}(Y_0, R)$$

Now, since the complex $R(0)$ is quasi isomorphic to the constant sheaf R concentrated in degree 0, we have that $H^{0,0}(Y_\bullet, R)$ is just the sheaf cohomology group $H^0(Y_\bullet, R)$. From [3, sections 5.1 and 5.2], one knows how to compute the sheaf cohomology of a simplicial scheme in terms of the sheaf cohomology of its simplicial components. In particular, the group of global sections $\Gamma(Y_\bullet, R) = H^0(Y_\bullet, R)$ is given by the kernel of the morphism $\Gamma(Y_0, R) \to \Gamma(Y_1, R)$ induced by the simplicial data. This means that

$$H^{0,0}(Y_\bullet, R) = Ker(H^{0,0}(Y_0, R) \to H^{0,0}(Y_1, R))$$

In other words, $H^{0,0}(Y_\bullet, R)$ is the free R-module with rank equal to the number of connected components of $Y_\bullet$, where the set of connected components of $Y_\bullet$ is obtained from the set of connected components of Y_0 by identifying all the couples of components of Y_0 linked by a connected component of Y_1 via the face maps. On the other hand, $H^{0,0}(Y_0, R)$ is the free R-module with rank equal to the number of connected components of Y_0. It follows that the restriction

$$r^*_0 : H^{0,0}(Y_\bullet, R) \to H^{0,0}(Y_0, R)$$

is injective, hence $M(p) = M(p_Y) \otimes id_{\widetilde{M}(A)_{Y'_\bullet}}[1]$, which is what we aimed to prove. $\qquad\square$

4 The Nisnevich classifying space

Let us recall at this point some facts about Nisnevich and étale classifying spaces of a linear algebraic group G over $Spec(k)$.

Denote by EG the simplicial scheme defined by $(EG)_n = G^{n+1}$ with face and degeneracy maps given by partial projections and partial diagonals respectively. There is an obvious action of G on EG induced by the operation in G, then the Nisnevich classifying space BG is the simplicial scheme defined by $BG = EG/G$. In other words, BG is the simplicial Nisnevich sheaf with simplicial component $(BG)_n$ given by the Nisnevich sheaf $U \mapsto Hom_{Sm/k}(U, G^n)$ for any $n \geq 0$ and standard face and degeneracy maps of the bar construction.

Now, consider the morphism of sites $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$. This induces a pair of adjoint functors

$$\mathcal{H}_s((Sm/k)_{et})$$
$$\pi^* \uparrow \downarrow R\pi_*$$
$$\mathcal{H}_s((Sm/k)_{Nis})$$

where π_* is the restriction to Nisnevich topology and π^* is étale sheafification. Then, the étale classifying space is defined by $B_{et}G = R\pi_*\pi^*BG$. Furthermore, we recall that in [9] it is constructed, starting from a faithful representation $\rho : G \hookrightarrow GL(V)$, a geometric model BG_{gm} for the A^1-homotopy type of $B_{et}G$, obtained from an infinite-dimensional affine space $\oplus_{i=1}^\infty V$ by removing a closed subscheme in order to let the diagonal action of G be free and then taking the quotient.

In this paper we will be mainly interested in Nisnevich classifying spaces. We finish this section by showing some of their features.

Let G be a linear algebraic group over k and H an algebraic subgroup of G. Denote by $\widetilde{B}H$ the bisimplicial scheme $(EH \times EG)/H$, where H acts on $EH \times EG$ diagonally, i.e. $(h_1, \ldots, h_m, g_1, \ldots, g_n)h = (h_1h, \ldots, h_mh, g_1h, \ldots, g_nh)$ for any $h_1, \ldots, h_m, h$ in H and $g_1, \ldots, g_n$ in G, and by $\widehat{B}H$ the simplicial scheme EG/H. We observe that the natural fibration $\pi : \widehat{B}H \to BG$ is trivial over simplicial components and has fiber G/H. There are two natural maps $\phi : \widetilde{B}H \to BH$ and $\psi : \widetilde{B}H \to \widehat{B}H$. We notice that ϕ is an isomorphism in $\mathcal{H}_s(k)$ since over each simplicial component it is a trivial fibration with contractible fiber EG. On the other hand, ψ is not in general an isomorphism in $\mathcal{H}_s(k)$. However, we have the following statement.

Proposition 4.1 *If the map*

$$Hom_{\mathcal{H}_s(k)}(Spec(R), B_{et}H) \to Hom_{\mathcal{H}_s(k)}(Spec(R), B_{et}G)$$

is injective for any Henselian local ring R over k, then ψ is an isomorphism in $\mathcal{H}_s(k)$. In particular, $BH \cong \widehat{B}H$ in $\mathcal{H}_s(k)$.

Proof. We start by noticing that the restriction of ψ over any simplicial component is given by the morphism $(EH \times G^{n+1})/H \to G^{n+1}/H$. The simplicial scheme $(EH \times G^{n+1})/H$ is nothing else but the Čech simplicial scheme $\check{C}(G^{n+1} \to G^{n+1}/H)$ associated to the H-torsor $G^{n+1} \to G^{n+1}/H$ which becomes split once extended to G. In order to check that

$$\check{C}(G^{n+1} \to G^{n+1}/H) \to G^{n+1}/H$$

is a simplicial weak equivalence it is enough, by [9, Lemma 1.11], to evaluate on henselian local rings. Therefore, we only need to prove that the H-torsor $G^{n+1} \to G^{n+1}/H$ is Nisnevich locally split. Now, the fiber of $G^{n+1} \to G^{n+1}/H$ over any $Spec(R)$ of G^{n+1}/H, where R is henselian local, is given by a H-torsor $P \to Spec(R)$ whose extension to G is split, so split itself by hypothesis. Hence, $G^{n+1} \to G^{n+1}/H$ is Nisnevich locally split. This implies that ψ is an isomorphism in $\mathcal{H}_s(k)$. $\square$

In practice, in the unitary group case (as in many other cases), it will be enough to check the hypothesis of the previous proposition only for field extensions of k. The reason resides in the fact that rationally trivial hermitian forms are locally trivial (see [10, Theorem 9.2]).

There are obvious morphisms $j : BH \to \widehat{B}H$ and $g : BH \to BG$ induced by the embedding $H \hookrightarrow G$.

PROPOSITION 4.2 *Under the hypothesis of Proposition 4.1, j is an isomorphism in $\mathcal{H}_s(k)$.*

Proof. Under the hypothesis of Proposition 4.1 both ϕ and ψ are morphisms of bisimplicial schemes which are weak equivalences over simplicial components, hence the induced morphisms on the associated diagonal simplicial schemes $\phi : \Delta(\widetilde{B}H) \to BH$ and $\psi : \Delta(\widetilde{B}H) \to \widehat{B}H$ are weak equivalences. In order to complete the proof we only need to notice that the morphisms $j\phi$ and ψ are simplicial homotopic. A simplicial homotopy between them $F_i^{(n)} : (H^{n+1} \times G^{n+1})/H \to G^{n+2}/H$ is defined for any n and any $0 \leq i \leq n$ by

$$F_i^{(n)}(h_0, \ldots, h_n, g_0, \ldots, g_n) = (h_0, \ldots, h_i, g_i, \ldots, g_n)$$

$\square$

It immediately follows from the previous proposition and by noticing that $g = \pi j$ that the morphism $j^* : H(\widehat{B}H) \to H(BH)$ is an isomorphism of $H(BG)$-modules.

Propositions 4.1 and 4.2 apply in particular to the case when G and H are respectively O_n and O_{n-1}. We recall that

$$A_{q_n} \cong O_n/O_{n-1}$$

where A_{q_n} is the affine quadric defined by the equation $q_n = 1$. Moreover, we know that

$$M(A_{q_n}) = \mathbb{Z} \oplus \mathbb{Z}([n/2])[n-1] \in \mathcal{DM}_{eff}^-(k)$$

by [14, Proposition 3.1.3]. Therefore, Proposition 3.3 applies to the fibration $\widehat{BO}_{n-1} \to BO_n$.

By previous considerations and by an induction argument, in [14] it is proven the following theorem.

THEOREM 4.3 *[14, Theorem 3.1.1] There is a unique set $u_1, \ldots, u_n$ of classes in the motivic $\mathbb{Z}/2$-cohomology of BO_n such that $deg(u_i) = ([i/2])[i]$, u_i vanishes when restricted to $H(BO_{i-1})$ for any $2 \leq i \leq n$ and*

$$H(BO_n) = H[u_1, \ldots, u_n]$$

These new cohomology classes u_i are called *subtle Stiefel-Whitney classes*.

5   Čech simplicial scheme and Rost motive of a quadratic extension

Let $E = k(\sqrt{\alpha})$ be a quadratic extension of k. Then, the motive of $Spec(E)$ in $\mathcal{DM}_{eff}^-(k, \mathbb{Z}/2)$ is the Rost motive M_α of the Pfister form $\langle\!\langle \alpha \rangle\!\rangle$. It is proven in [12] that this motive comes endowed with two morphisms $M_\alpha \to T$ and $T \to M_\alpha$ such that the composition $T \to M_\alpha \to T$ is the 0 morphism and becomes a split distinguished triangle in $\mathcal{DM}_{eff}^-(E, \mathbb{Z}/2)$.

Moreover, in [17, Theorem 4.4] it is shown that M_α can be presented as an extension of two motives of Čech simplicial schemes. More precisely, in $\mathcal{DM}_{eff}^-(k, \mathbb{Z}/2)$ there is the following distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1] \qquad (*)$$

where $\mathfrak{X}_\alpha$ is the motive of the Čech simplicial scheme of the Pfister quadric associated to the Pfister form $\langle\!\langle \alpha \rangle\!\rangle$.

Let $N_>$ be $Cone(T \to M_\alpha)$ and $N_<$ be $Cone(M_\alpha \to T)[-1]$. Since $Hom(T, T[j]) = 0$ for $j \neq 0$ and we are working with $\mathbb{Z}/2$-coefficients, the morphism $T \to M_\alpha$ is uniquely liftable to $N_<$ while the morphism $M_\alpha \to T$ is uniquely extendable to $N_>$. It immediately follows from the octahedron axiom that $Cone(N_> \to T)[-1] \cong Cone(T \to N_<)$. We will denote this motive by $\widehat{M_\alpha}$.

In this section, we will study the above mentioned motives and their motivic cohomology. We start by establishing relations among them.

PROPOSITION 5.1 *The following isomorphisms hold in* $\mathcal{D}M^-_{eff}(k,\mathbb{Z}/2)$:
1) $M_\alpha \otimes \mathfrak{X}_\alpha \cong M_\alpha$ *via* $M_\alpha \otimes (\mathfrak{X}_\alpha \to T)$;
2) $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$ *via* $(N_> \to T) \otimes \mathfrak{X}_\alpha$;
3) $M_\alpha \otimes N_> \cong M_\alpha$ *via* $M_\alpha \otimes (N_> \to T)$;
4) $N_< \otimes N_> \cong T$;
5) $\widehat{M_\alpha} \otimes N_> \cong \widehat{M_\alpha}[1]$ *via* $\widehat{M_\alpha} \otimes (N_> \to T[1])$.

Proof. 1) Since $\mathfrak{X}_\alpha$ is a projector in $\mathcal{D}M^-_{eff}(k,\mathbb{Z}/2)$ we have that $\mathfrak{X}_\alpha \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$. Hence, by tensoring with $\mathfrak{X}_\alpha$ the distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1]$$

we obtain that $M_\alpha \otimes \mathfrak{X}_\alpha \cong M_\alpha$.
2) Therefore, by tensoring with $\mathfrak{X}_\alpha$ the distinguished triangle

$$T \to M_\alpha \to N_> \to T[1]$$

and by recalling that $\mathfrak{X}_\alpha \to M_\alpha$ from $(*)$ factors through T we get that $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$.
3) It follows formally from 1) and 2).
4) On the other hand, by tensoring with $N_>$ the distinguished triangle

$$N_< \to M_\alpha \to T \to N_<[1]$$

and by noticing that $(M_\alpha \to T) \otimes N_>$ coincides with $M_\alpha \to N_>$ we obtain that $N_< \otimes N_> \cong T$.
5) Finally, by tensoring with $N_>$ the distinguished triangle

$$T \to N_< \to \widehat{M_\alpha} \to T[1]$$

and by noticing that $(T \to N_<) \otimes N_>$ coincides with $N_> \to T$ we have that $\widehat{M_\alpha} \otimes N_> \cong \widehat{M_\alpha}[1]$. $\square$

From the previous proposition we immediately deduce the following lemma.

LEMMA 5.2 *In* $\mathcal{D}M^-_{eff}(k,\mathbb{Z}/2)$ *for any* $n \in \mathbb{N}$ *there are the following distinguished triangles:*
1) $N_>^{\otimes n-1} \to M_\alpha \to N_>^{\otimes n} \to N_>^{\otimes n-1}[1]$;
2) $\widehat{M_\alpha}[n-1] \to N_>^{\otimes n} \to N_>^{\otimes n-1} \to \widehat{M_\alpha}[n]$.
Here, $M_\alpha \to N_>^{\otimes n}$ *and* $N_>^{\otimes n} \to N_>^{\otimes n-1}$ *are the unique non-zero morphisms between the respective objects.*

Proof. 1) It follows immediately from 3) of Proposition 5.1 by tensoring the distinguished triangle

$$T \to M_\alpha \to N_> \to T[1]$$

with the appropriate power of $N_>$.

2) It follows immediately from 5) of Proposition 5.1 by tensoring the distinguished triangle

$$\widehat{M_\alpha} \to N_> \to T \to \widehat{M_\alpha}[1]$$

with the appropriate power of $N_>$. $\qquad\square$

At this point, we present the motivic cohomology of $\widehat{M_\alpha}$, which will be used in the main result of this section, namely the computation of the motivic cohomology of tensor powers of $N_>$.

LEMMA 5.3 *There exists a cohomology class μ of bidegree $(0)[1]$ such that the motivic cohomology of $\widehat{M_\alpha}$ is given by*

$$H(\widehat{M_\alpha}) = \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu$$

So, the motivic cohomology of $\widehat{M_\alpha}$ is concentrated on a single diagonal.

Proof. After applying the octahedron axiom twice to the distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1]$$

we get the distinguished triangle

$$\widehat{M_\alpha}[-1] \to \widetilde{\mathfrak{X}_\alpha}[1] \to \widetilde{\mathfrak{X}_\alpha} \to \widehat{M_\alpha}$$

where $\widetilde{\mathfrak{X}_\alpha}$ is $Cone(\mathfrak{X}_\alpha \to T)[-1]$.
The motivic cohomology of $\widetilde{\mathfrak{X}_\alpha}$ has been computed in the original version of [11] and [19]. It is described by

$$H(\widetilde{\mathfrak{X}_\alpha}) = \mathbb{Z}/2[\mu] \cdot \mu \otimes \frac{K^M(k)/2}{Ann(\{\alpha\})}$$

Therefore, by the long exact sequence in motivic cohomology induced by the previous distinguished triangle and by recalling that $H^{*,*'}(\widetilde{\mathfrak{X}_\alpha}) \to H^{*-1,*'}(\widetilde{\mathfrak{X}_\alpha})$ sends μ^j to μ^{j-1} we get the description of $H(\widehat{M_\alpha})$. $\qquad\square$

We are now ready to compute the motivic cohomology of any tensor power of $N_>$. This result will be essential in the next section for the proof of the main result.

PROPOSITION 5.4 *For any $n \in \mathbb{N}$ there exist cohomology classes μ_i of bidegree $(0)[i]$ for $1 \le i \le n$ such that the motivic cohomology of the nth tensor power of $N_>$ as an H-module is given by*

$$H(N_>^{\otimes n}) = H \oplus \bigoplus_{i=1}^{n} \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu_i$$

where the H-module structure is described by the relations $\tau\mu_i = \{\alpha\}\mu_{i-1}$ ($\mu_0 = 1$ by convention).

Proof. We will proceed by induction on n. For $n = 1$ the distinguished triangle

$$\widehat{M_\alpha} \to N_> \to T \to \widehat{M_\alpha}[1]$$

induces the following long exact sequence in motivic cohomology

$$\ldots \to H^{*-1,*'}(\widehat{M_\alpha}) \to H^{*,*'} \to H^{*,*'}(N_>) \to H^{*,*'}(\widehat{M_\alpha}) \to \ldots$$

From Lemma 5.3 it follows that

$$H^{*,*'}(N_>) = \begin{cases} H^{*,*'}(\widehat{M_\alpha}), & * > *' \\ H^{*,*'}, & * \leq *' \end{cases}$$

which implies that

$$H(N_>) = H \oplus \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu$$

On the other hand, after tensoring with $\mathfrak{X}_\alpha$ the distinguished triangle

$$T \to M_\alpha \to N_> \to T[1]$$

we get a morphism of long exact sequences in motivic cohomology

$$\ldots \longrightarrow H^{*-1,*'}(M_\alpha) \longrightarrow H^{*-1,*'} \longrightarrow H^{*,*'}(N_>) \longrightarrow H^{*,*'}(M_\alpha) \longrightarrow \ldots$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$
$$\ldots \longrightarrow H^{*-1,*'}(M_\alpha) \longrightarrow H^{*-1,*'}(\mathfrak{X}_\alpha) \longrightarrow H^{*,*'}(\mathfrak{X}_\alpha) \longrightarrow H^{*,*'}(M_\alpha) \longrightarrow \ldots$$

By a four lemma argument we deduce that $H(N_>) \to H(\mathfrak{X}_\alpha)$ is injective. Therefore, $\tau\mu = \{\alpha\}$ in $H(N_>)$, since the same relation holds in $H(\mathfrak{X}_\alpha)$. That completes the induction basis.

Now, suppose the statement holds for $n - 1$. Then, by 2) of Lemma 5.2 we have the following long exact sequence in motivic cohomology

$$\ldots \to H^{*-n,*'}(\widehat{M_\alpha}) \to H^{*,*'}(N_>^{\otimes n\text{-}1}) \to H^{*,*'}(N_>^{\otimes n}) \to H^{*-n+1,*'}(\widehat{M_\alpha}) \to \ldots$$

From Lemma 5.3 and by induction hypothesis we have

$$H^{*,*'}(N_>^{\otimes n}) = \begin{cases} H^{*-n+1,*'}(\widehat{M_\alpha}), & * > *' + n - 1 \\ H^{*,*'}(N_>^{\otimes n\text{-}1}), & * \leq *' + n - 1 \end{cases}$$

which implies that there exists μ_n in bidegree $(0)[n]$ such that

$$H(N_>^{\otimes n}) = H(N_>^{\otimes n\text{-}1}) \oplus \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu_n = H \oplus \bigoplus_{i=1}^{n} \frac{K^M(k)/2}{Ann(\{\alpha\})} \cdot \mu_i$$

From 1) of Lemma 5.2 we have the following long exact sequence in motivic cohomology

$$\ldots \to H^{*-1,*'}(N_>^{\otimes n-1}) \to H^{*,*'}(N_>^{\otimes n}) \to H^{*,*'}(M_\alpha) \to H^{*,*'}(N_>^{\otimes n-1}) \to \ldots$$

that maps $\mu_{i-1} \in H^{i-1,0}(N_>^{\otimes n-1})$ to $\mu_i \in H^{i,0}(N_>^{\otimes n})$ since $H^{i,0}(M_\alpha) = 0$ for $i > 0$. Hence, by induction hypothesis, $\tau\mu_i = \{\alpha\}\mu_{i-1}$ in $H(N_>^{\otimes n})$ and the proof is complete. $\qquad\square$

By 2) of Lemma 5.1 there is a chain of morphisms

$$\mathfrak{X}_\alpha \to \ldots \to N_>^{\otimes n} \to N_>^{\otimes n-1} \to \ldots \to N_> \to T$$

that induces in cohomology the chain of homomorphisms

$$H \to H(N_>) \to \ldots \to H(N_>^{\otimes n-1}) \to H(N_>^{\otimes n}) \to \ldots \to H(\mathfrak{X}_\alpha)$$

which sends $\mu \in H(N_>)$ to $\mu \in H(\mathfrak{X}_\alpha)$.
We now highlight an interesting relation between the invertible motive $N_>$ and the projector $\mathfrak{X}_\alpha$.

PROPOSITION 5.5 *The homomorphisms $H(N_>^{\otimes n}) \to H(\mathfrak{X}_\alpha)$ are injective for all $n \in \mathbb{N}$. Moreover, $H(\mathfrak{X}_\alpha) = \varinjlim H(N_>^{\otimes n})$.*

Proof. We have already noticed that $H(N_>) \to H(\mathfrak{X}_\alpha)$ is injective and maps $\mu = \mu_1$ to μ. Now, suppose by induction that the homomorphism $H(N_>^{\otimes n-1}) \to H(\mathfrak{X}_\alpha)$ is injective. Notice that there is a commutative diagram

$$
\begin{array}{ccc}
H(N_>^{\otimes n-1}) \otimes H(N_>) & \longrightarrow & H(N_>^{\otimes n}) \\
\downarrow & & \downarrow \\
H(\mathfrak{X}_\alpha) \otimes H(\mathfrak{X}_\alpha) & \overset{\smile}{\longrightarrow} & H(\mathfrak{X}_\alpha)
\end{array}
$$

where the bottom horizontal map is the usual cup product in $H(\mathfrak{X}_\alpha)$. It follows that the right vertical map sends μ_i to μ^i for any $i \le n$. This completes the proof. $\qquad\square$

Later on we will need also the following description of the motivic cohomology of $N_<$.

LEMMA 5.6 *The motivic cohomology of $N_<$ is given by*

$$H(N_<) = Ann(\{\alpha\}) \oplus H \cdot \tau$$

Proof. After applying the octahedron axiom to the distinguished triangle

$$M_\alpha \to \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha[1] \to M_\alpha[1]$$

we obtain

$$\mathfrak{X}_\alpha \to N_< \to \widetilde{\mathfrak{X}_\alpha} \to \mathfrak{X}_\alpha[1]$$

which induces in motivic cohomology the following long exact sequence

$$\ldots \to H^{*-1,*'}(\mathfrak{X}_\alpha) \to H^{*,*'}(\widetilde{\mathfrak{X}_\alpha}) \to H^{*,*'}(N_<) \to H^{*,*'}(\mathfrak{X}_\alpha) \to \ldots$$

Hence, the result follows by noticing that $H^{*,*'}(\widetilde{\mathfrak{X}_\alpha})$ is the $* > *'$ part of $H^{*,*'}(\mathfrak{X}_\alpha)$, while $H^{*,*'}$ is the $* \leq *'$ part of it, and that $H^{*-1,*'}(\mathfrak{X}_\alpha) \to H^{*,*'}(\widetilde{\mathfrak{X}_\alpha})$ sends μ^{i-1} to μ^i. $\square$

6   The motivic cohomology ring of $BU_n(E/k)$

Our goal in this section is to compute by using the techniques presented in section 3 and 4 the motivic cohomology of the Nisnevich classifying space of $U_n(E/k)$, the unitary group associated to the standard split hermitian form h_n of the extension E/k.
At first, let us show some preliminary results which will be useful in the proof of the main theorem.

PROPOSITION 6.1 *The homogeneous variety $U_n(E/k)/U_{n-1}(E/k)$ is isomorphic to the affine quadric A_{h_n} defined by the equation $\widetilde{h}_n = 1$.*

Proof. Let V be an n-dimensional E-vector space and let A_{h_n} be the subset of V defined by the equation $h_n = 1$. Then, V can be considered as a $2n$-dimensional k-vector space in which A_{h_n} is the affine quadric defined by the equation $\widetilde{h}_n = 1$. The action of $U_n(E/k)$ on A_{h_n} is transitive since for any two vectors of A_{h_n} there exists the product of at most two reflections which sends one to the other (see the proof of [13, Theorem 9.5]). Moreover, the isotropy group of the vector $(1, 0, \ldots, 0)$ is isomorphic to $U_{n-1}(E/k)$. This implies the desired result. $\square$

At this point, in order to apply Proposition 3.3 to the unitary case, we need to study the motive of the affine quadric A_{h_n}.

PROPOSITION 6.2 *The motive in $\mathcal{DM}_{eff}^-(k, \mathbb{Z}/2)$ of the affine quadric A_{h_n} is given by*

$$M(A_{h_n}) = \begin{cases} T \oplus N_>(n)[2n-1], & n \text{ odd} \\ T \oplus T(n)[2n-1], & n \text{ even} \end{cases}$$

Proof. We start by noticing that the quadratic form

$$\widetilde{h}_n = \begin{cases} \langle\!\langle\alpha\rangle\!\rangle \perp (n-1)\,\mathbb{H}, & n \text{ odd} \\ n\,\mathbb{H}, & n \text{ even} \end{cases}$$

For a quadratic form q let us denote by Q the projective quadric defined by $q = 0$, by Q' the projective quadric defined by $q = z^2$ and by A the affine quadric defined by $q = 1$. Then, we have in $\mathcal{DM}_{eff}^{-}(k, \mathbb{Z}/2)$ the following Gysin triangle

$$M(A) \to M(Q') \to M(Q)(1)[2] \to M(A)[1]$$

In the case $q = n\,\mathbb{H}$ the previous triangle becomes

$$M(A) \to \bigoplus_{i=0}^{2n-1} T(i)[2i] \to \bigoplus_{i=1}^{2n-1} T(i)[2i] \oplus T(n)[2n] \to M(A)[1]$$

which implies that, for n even, $M(A_{h_n}) = M(A) = T \oplus T(n)[2n-1]$.
In the case $q = \langle\!\langle\alpha\rangle\!\rangle$ we have

$$M(A) \to T \oplus T(1)[2] \to M_\alpha(1)[2] \to M(A)[1]$$

from which it follows that $M(A_{h_1}) = M(A) = T \oplus N_>(1)[1]$.
The general case n odd follows from [1, Lemma 34] . Namely, we have

$$\widetilde{M}(A_{h_n}) = \widetilde{M}(A_{h_1})(n-1)[2n-2] = N_>(n)[2n-1]$$

that implies $M(A_{h_n}) = T \oplus N_>(n)[2n-1]$. $\qquad\square$

Before going ahead with the main theorem of this section, we notice that $U_n(E/k)$-torsors over $Spec(k)$ are in one-to-one correspondence with hermitian forms associated to the quadratic extension E/k or, which is the same, with quadratic forms over k divisible by $\langle\!\langle\alpha\rangle\!\rangle$. The map from the set of $U_{n-1}(E/k)$-torsors to the set of $U_n(E/k)$-torsors sends a hermitian form h to $h \perp \langle(-1)^{n-1}\rangle$ or, analogously, a quadratic form q divisible by $\langle\!\langle\alpha\rangle\!\rangle$ to $q \perp (-1)^{n-1}\langle\!\langle\alpha\rangle\!\rangle$. Since Witt cancellation holds for quadratic forms, the previous remark assures that $U_{n-1}(E/k)$-torsors inject in $U_n(E/k)$-torsors over any field extension of k, which allows us to use Propositions 4.1 and 4.2 in the unitary case.

THEOREM 6.3 *For any $m, n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes c_i of bidegree $(i)[2i]$ for $1 \leq i \leq n$ such that the motivic cohomology of $\mathfrak{X}_\alpha \otimes BU_n(E/k)$ and $N_>^{\otimes m} \otimes BU_n(E/k)$ is described respectively by*

$$H(\mathfrak{X}_\alpha \otimes BU_n(E/k)) = H(\mathfrak{X}_\alpha)[c_1, \ldots, c_n]$$

and

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) = \bigoplus_{i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m + \sum_{l \text{ odd}} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

where the obvious homomorphisms of $H(BU_n(E/k))$-modules

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) \to H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$$

are injective. Moreover, $H(\mathfrak{X}_\alpha \otimes BU_n(E/k)) = \varinjlim H(N_>^{\otimes m} \otimes BU_n(E/k))$.

Proof. We will proceed by induction on n. The induction basis follows immediately from noticing that $BU_0(E/k) \cong Spec(k)$ and by Proposition 5.5.

Now, suppose the result holds for $n-1$. Then, since $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$ by 2) of Proposition 5.1 and applying Propositions 3.3, 4.1, 4.2 and 6.2 to the coherent morphism $\mathfrak{X}_\alpha \otimes \widehat{B}U_{n-1}(E/k) \to \mathfrak{X}_\alpha \otimes BU_n(E/k)$, we obtain the following long exact sequence in motivic cohomology

$$\ldots \to H^{*-1,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) \xrightarrow{h^*} H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \xrightarrow{f^*}$$

$$H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \xrightarrow{g^*} H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) \to \ldots$$

Note that, even after having replaced $H(\mathfrak{X}_\alpha \otimes \widehat{B}U_{n-1}(E/k))$ with $H(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k))$, this stays a sequence of $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$-modules by the remark just after Proposition 4.2. By induction hypothesis, $H(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k))$ is freely generated as an $H(\mathfrak{X}_\alpha)$-algebra by $c_1, \ldots, c_{n-1}$ which are all uniquely liftable to $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$, since $\mathfrak{X}_\alpha \otimes BU_n(E/k)$ is the motive of a smooth simplicial scheme and, so, has no cohomology in negative round degrees. Hence, g^* is an epimorphism as it is a ring homomorphism, h^* is trivial and f^* is a monomorphism. Denoting by c_n the element $f^*(1)$ we obtain the result

$$H(\mathfrak{X}_\alpha \otimes BU_n(E/k)) = H(\mathfrak{X}_\alpha)[c_1, \ldots, c_n]$$

For the rest of the induction step we will consider separately two cases.

1) *n even*: for any $m \in \mathbb{N}$ we have the following long exact sequence in motivic cohomology of $H(BU_n(E/k))$-modules

$$\ldots \to H^{*-1,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \xrightarrow{h^*} H^{*-2n,*'-n}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{f^*}$$

$$H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{g^*} H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \to \ldots$$

For $m = 0$, by induction hypothesis, $H(BU_{n-1}(E/k))$ is generated as an H-algebra by $c_1, \ldots, c_{n-1}$ and μc_l for any odd $l < n$. By degree reasons these cohomology classes are all uniquely liftable to $H(BU_n(E/k))$. Therefore, g^* is an epimorphism since it is a ring homomorphism. This assures that, for any m, $H(N_>^{\otimes m} \otimes BU_{n-1}(E/k))$ is generated as an $H(BU_n(E/k))$-module by μ_i for all $i \leq m$. By degree reasons the μ_i are all uniquely liftable to $H(N_>^{\otimes m} \otimes BU_n(E/k))$. Now g^* happens to be surjective since it is a homomorphism of $H(BU_n(E/k))$-modules. Hence, h^* is the 0 homomorphism and f^* is

a monomorphism. Then, denoting by c_n the cohomology class $f^*(1)$ we have, for any m, the following morphism of short exact sequences of $H(BU_n(E/k))$-modules

$$0 \to H^{*-2n,*'-n}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{\cdot c_n} H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) \to H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \to 0$$

$$0 \to H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \xrightarrow{\cdot c_n} H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \to H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) \to 0$$

By induction on square degree and by a standard four lemma argument, the central vertical morphism is injective. Moreover, by an induction argument on square degree and looking at the previous upper short exact sequence we get that

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \cdot c_n^i$$

$$= \bigoplus_{i_1,\dots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m + \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

as an $H(BU_n(E/k))$-submodule of $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$.

2) n odd: as before for any m we have the following long exact sequence in motivic cohomology of $H(BU_n(E/k))$-modules

$$\dots \to H^{*-1,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \xrightarrow{h^*} H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BU_n(E/k)) \xrightarrow{f^*}$$

$$H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) \xrightarrow{g^*} H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \to \dots$$

As in the previous case, for $m = 0$ the induction hypothesis implies that $H(BU_{n-1}(E/k))$ is generated as an H-algebra by $c_1,\dots,c_{n-1}$ and μc_l for any odd $l < n$. By the same degree reasons they are all uniquely liftable to $H(BU_n(E/k))$. Thus, g^* is an epimorphism since it is a ring homomorphism. This is enough to show that, for any m, $H(N_>^{\otimes m} \otimes BU_{n-1}(E/k))$ is generated as an $H(BU_n(E/k))$-module by μ_i for all $i \leq m$. Again the μ_i are uniquely liftable to $H(N_>^{\otimes m} \otimes BU_n(E/k))$. It follows that g^* is surjective, h^* is trivial and f^* is injective. Then, denoting by c_n the cohomology class $f^*(1)$ we have, for any m, the following morphism of short exact sequences of $H(BU_n(E/k))$-modules

$$0 \to H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BU_n(E/k)) \xrightarrow{\cdot c_n} H^{*,*'}(N_>^{\otimes m} \otimes BU_n(E/k)) \to H^{*,*'}(N_>^{\otimes m} \otimes BU_{n-1}(E/k)) \to 0$$

$$0 \to H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \xrightarrow{\cdot c_n} H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_n(E/k)) \to H^{*,*'}(\mathfrak{X}_\alpha \otimes BU_{n-1}(E/k)) \to 0$$

By the very same arguments of the previous case, the central vertical morphism

is injective and

$$H(N_>^{\otimes m} \otimes BU_n(E/k)) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+i} \otimes BU_{n-1}(E/k)) \cdot c_n^i$$

$$= \bigoplus_{i_1,\ldots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m + \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

as an $H(BU_n(E/k))$-submodule of $H(\mathfrak{X}_\alpha \otimes BU_n(E/k))$, which completes the proof. $\square$

As a corollary of the previous theorem we obtain the description of the motivic cohomology ring of $BU_n(E/k)$ as an H-algebra.

THEOREM 6.4 *For any $n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes c_i of bidegree $(i)[2i]$ for $1 \leq i \leq n$ and d_j of bidegree $(j)[2j+1]$ for $1 \leq j$ odd $\leq n$ such that the motivic cohomology ring of $BU_n(E/k)$ is given by*

$$H(BU_n(E/k)) = \frac{H[c_i, d_j]_{1 \leq i \leq n, 1 \leq j \, odd \leq n}}{R}$$

where R is the ideal generated by $\tau d_j + \{\alpha\} c_j$, $Ann(\{\alpha\}) \cdot d_j$ and $c_{j'} d_j + c_j d_{j'}$ for any $1 \leq j, j'$ odd $\leq n$.

Proof. By Theorem 6.3 we have a monomorphism of rings

$$H(BU_n(E/k)) = \bigoplus_{i_1,\ldots,i_n \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n} \to H(\mathfrak{X}_\alpha)[c_1,\ldots,c_n]$$

from which we deduce that $H(BU_n(E/k))$ is generated as an H-algebra by the c_i and the μc_j for j odd. Let us denote by d_j these elements. Then, the relations among c_i and d_j that generate R follow immediately by Proposition 5.4 and by noticing that $\mu c_j \cdot c_{j'} = c_j \cdot \mu c_{j'}$ for any $1 \leq j, j'$ odd $\leq n$. This amounts to say that there is an epimorphism

$$p : \frac{H[c_i, d_j]_{1 \leq i \leq n, 1 \leq j \, odd \leq n}}{R} \to H(BU_n(E/k))$$

We can check its injectivity by looking separately at each restriction

$$p : p^{-1}(H(N_>^{\otimes \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}) \to H(N_>^{\otimes \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$$

Notice that $H(N_>^{\otimes \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n}$ is generated as a $K^M(k)/2$-module by $\mu^m c_1^{i_1} \cdots c_n^{i_n}$ for any $0 < m \leq \sum_{l \, odd} i_l$ and $\tau^{m'} c_1^{i_1} \cdots c_n^{i_n}$ for any $m' \geq 0$. Moreover, the elements in $p^{-1}(H(N_>^{\otimes \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n})$ that map to these generators through p are unique. Then, injectivity follows by looking at the restriction of p on each diagonal of $p^{-1}(H(N_>^{\otimes \sum_{l \, odd} i_l}) \cdot c_1^{i_1} \cdots c_n^{i_n})$ which is an isomorphism to $\frac{K^M(k)/2}{Ann(\{\alpha\})}$ on positive diagonals and to $K^M(k)/2$ on the others. $\square$

7 COMPARISON BETWEEN $BU_n(E/k)$ AND $BO(\widetilde{h}_n)$

Since there exists an obvious homomorphism of groups $U_n(E/k) \to O(\widetilde{h}_n)$, it is reasonable to compare the classifying spaces $BU_n(E/k)$ and $BO(\widetilde{h}_n)$ and, in particular, the characteristic classes arising from both.

Before proceeding, we highlight that, given a quadratic form q, there is the following isomorphism

$$O(q \perp \langle b \rangle)/O(q) \cong A_{q \perp \langle b \rangle = b}$$

where by $A_{q \perp \langle b \rangle = b}$ we mean the affine quadric defined by the equation $q \perp \langle b \rangle = b$.

For sake of simplicity, we will express by p_n the quadratic form $\langle\!\langle \alpha \rangle\!\rangle \perp (n-1)\,\mathbb{H}$ and by $p_{n-\frac{1}{2}}$ the quadratic form $\langle -\alpha \rangle \perp (n-1)\,\mathbb{H}$.

In the following theorem we compute the motivic cohomology ring of $BO(p_n)$.

THEOREM 7.1 *For any $n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes u_i of bidegree $([i/2])[i]$ for $1 \leq i \leq 2n$ and a class v_{2n+1} of bidegree $(n)[2n+1]$ such that the motivic cohomology ring of $BO(p_n)$ is given by*

$$H(BO(p_n)) = \frac{H[u_1, \ldots, u_{2n}, v_{2n+1}]}{(\tau v_{2n+1} + \{\alpha\} u_{2n}, \mathrm{Ann}(\{\alpha\}) \cdot v_{2n+1})}$$

Proof. We start by noticing that

$$O(p_n)/O(p_{n-\frac{1}{2}}) \cong A_{p_n = 1}$$

From the fact that $O(p_{n-\frac{1}{2}}) \cong O_{2n-1}$ we obtain by Theorem 4.3 that

$$H(BO(p_{n-\frac{1}{2}})) = H[u_1, \ldots, u_{2n-1}]$$

Then, from [14, Proposition 3.2.4] it follows that

$$H(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) = H(N_>^{\otimes m}) \otimes_H H[u_1, \ldots, u_{2n-1}]$$

and

$$H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) = H(\mathfrak{X}_\alpha)[u_1, \ldots, u_{2n-1}]$$

Now, by recalling that $M(A_{p_n=1}) = T \oplus N_>(n)[2n-1]$ and $N_> \otimes \mathfrak{X}_\alpha \cong \mathfrak{X}_\alpha$ and using Propositions 3.3, 4.1 and 4.2, we obtain a long exact sequence in motivic cohomology

$$\ldots \to H^{*-1,*'}(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \xrightarrow{h^*} H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BO(p_n)) \xrightarrow{f^*}$$

$$H^{*,*'}(\mathfrak{X}_\alpha \otimes BO(p_n)) \xrightarrow{g^*} H^{*,*'}(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \to \ldots$$

Hence, by the same arguments of Theorem 6.3 and denoting by u_{2n} the class $f^*(1)$ we get that

$$H(\mathfrak{X}_\alpha \otimes BO(p_n)) = H(\mathfrak{X}_\alpha)[u_1, \ldots, u_{2n}]$$

As in the odd case of Theorem 6.3, for any m we get a long exact sequence of $H(BO(p_n))$-modules

$$\ldots \to H^{*-1,*'}(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) \xrightarrow{h^*} H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BO(p_n)) \xrightarrow{f^*}$$

$$H^{*,*'}(N_>^{\otimes m} \otimes BO(p_n)) \xrightarrow{g^*} H^{*,*'}(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) \to \ldots$$

Hence, by exactly the same arguments of Theorem 6.3 and denoting by u_{2n} the class $f^*(1)$ we obtain, for any m, a morphism of short exact sequences of $H(BO(p_n))$-modules

$$
\begin{array}{ccccccccc}
0 & \to & H^{*-2n,*'-n}(N_>^{\otimes m+1} \otimes BO(p_n)) & \xrightarrow{\cdot u_{2n}} & H^{*,*'}(N_>^{\otimes m} \otimes BO(p_n)) & \to & H^{*,*'}(N_>^{\otimes m} \otimes BO(p_{n-\frac{1}{2}})) & \to & 0 \\
 & & \downarrow & & \downarrow & & \uparrow & & \\
0 & \to & H^{*-2n,*'-n}(\mathfrak{X}_\alpha \otimes BO(p_n)) & \xrightarrow{\cdot u_{2n}} & H^{*,*'}(\mathfrak{X}_\alpha \otimes BO(p_n)) & \to & H^{*,*'}(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) & \to & 0
\end{array}
$$

From this it follows that

$$H(N_>^{\otimes m} \otimes BO(p_n)) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+i} \otimes BO(p_{n-\frac{1}{2}})) \cdot u_{2n}^i$$

$$= \bigoplus_{i_1,\ldots,i_{2n} \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes m+i_{2n}}) \cdot u_1^{i_1} \cdots u_{2n}^{i_{2n}}$$

and, setting $m = 0$, we obtain

$$H(BO(p_n)) = \bigoplus_{i_1,\ldots,i_{2n} \in \mathbb{Z}_{\geq 0}} H(N_>^{\otimes i_{2n}}) \cdot u_1^{i_1} \cdots u_{2n}^{i_{2n}}$$

Moreover, we have a monomorphism of H-algebras

$$H(BO(p_n)) \to H(\mathfrak{X}_\alpha \otimes BO(p_n))$$

from which we deduce, as in Theorem 6.4, that

$$H(BO(p_n)) = \frac{H[u_1, \ldots, u_{2n}, v_{2n+1}]}{(\tau v_{2n+1} + \{\alpha\} u_{2n}, Ann(\{\alpha\}) \cdot v_{2n+1})}$$

where v_{2n+1} is nothing else but the element that maps to μu_{2n} under the monomorphism $H(BO(p_n)) \to H(\mathfrak{X}_\alpha \otimes BO(p_n))$. $\square$

At this point, we recall that $\mathfrak{X}_\alpha \otimes BO(p_n) \cong BO_{2n} \otimes \mathfrak{X}_\alpha$ by [14, Proposition 2.6.1]. Moreover, note that this isomorphism is functorial just by the way it is

constructed in the proof of [14, Proposition 2.6.1]. In few words, for any torsor triple (G, X, H) the isomorphism is obtained by considering the bisimplicial scheme $EG \times X \times EH$, then taking the quotient with respect to the left action of G and the righ action of H in different orders. The claimed isomorphism in $H_s(k)$ is

$$\check{C}(X) \times BH \cong (G \backslash (EG \times X \times EH))/H = G \backslash ((EG \times X \times EH)/H) \cong BG \times \check{C}(X)$$

where $\check{C}(X)$ is the Čech simplicial scheme of X. So, a morphism of torsor triples $(G, X, H) \to (G', X', H')$ induces a commutative diagram in $H_s(k)$

$$
\begin{array}{ccc}
\check{C}(X) \times BH & \longleftarrow & BG \times \check{C}(X) \\
\downarrow & & \downarrow \\
\check{C}(X') \times BH' & \longleftarrow & BG' \times \check{C}(X')
\end{array}
$$

where the horizontal maps are isomorphisms.

PROPOSITION 7.2 *The isomorphism in motivic cohomology*

$$H(\mathfrak{X}_\alpha \otimes BO(p_n)) \longleftrightarrow H(BO_{2n} \otimes \mathfrak{X}_\alpha)$$

induced by the isomorphism $\mathfrak{X}_\alpha \otimes BO(p_n) \cong BO_{2n} \otimes \mathfrak{X}_\alpha$ *maps* u_{2i} *to* u_{2i} *and* u_{2i-1} *to* $u_{2i-1} + \mu u_{2i-2}$ *for any* $1 \leq i \leq n$.

Proof. We proceed by induction on n. For $n = 1$, by applying the argument just before this proposition to the morphism of torsor triples $(O(\langle -1 \rangle), Iso(\langle -1 \rangle \leftrightarrow \langle -\alpha \rangle), O(\langle -\alpha \rangle)) \to (O_2, Iso(q_2 \leftrightarrow \langle\!\langle \alpha \rangle\!\rangle), O(\langle\!\langle \alpha \rangle\!\rangle))$, we have the following commutative diagram

$$
\begin{array}{ccc}
H(\mathfrak{X}_\alpha \otimes BO(\langle\!\langle \alpha \rangle\!\rangle)) & \longleftarrow & H(BO_2 \otimes \mathfrak{X}_\alpha) \\
\downarrow & & \downarrow \\
H(\mathfrak{X}_\alpha \otimes BO(\langle -\alpha \rangle)) & \longleftarrow & H(BO(\langle -1 \rangle) \otimes \mathfrak{X}_\alpha)
\end{array}
$$

where the bottom horizontal isomorphism maps u_1 to $u_1 + \mu$ by [14, Proposition 2.6.1 and Lemma 3.2.6]. Then, the result follows from the fact that u_1 and u_2 are uniquely determined both in $H(\mathfrak{X}_\alpha \otimes BO(\langle\!\langle \alpha \rangle\!\rangle))$ and in $H(BO_2 \otimes \mathfrak{X}_\alpha)$ by the fact that u_1 restricts to u_1 and u_2 vanishes respectively in $H(\mathfrak{X}_\alpha \otimes BO(\langle -\alpha \rangle))$ and in $H(BO(\langle -1 \rangle) \otimes \mathfrak{X}_\alpha)$.

Now, suppose the statement is true for $n - 1$. Then, the chain of morphisms of torsor triples $(O_{2n-2}, Iso(q_{2n-2} \leftrightarrow p_{n-1}), O(p_{n-1})) \to (O(-q_{2n-1}), Iso(-q_{2n-1} \leftrightarrow p_{n-\frac{1}{2}}), O(p_{n-\frac{1}{2}})) \to (O_{2n}, Iso(q_{2n} \leftrightarrow p_n), O(p_n))$

induces the following commutative diagram

$$
\begin{CD}
H(\mathfrak{X}_\alpha \otimes BO(p_n)) @<<< H(BO_{2n} \otimes \mathfrak{X}_\alpha) \\
@VVV @VVV \\
H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) @<<< H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha) \\
@VVV @VVV \\
H(\mathfrak{X}_\alpha \otimes BO(p_{n-1})) @<<< H(BO_{2n-2} \otimes \mathfrak{X}_\alpha)
\end{CD}
$$

In this case we need to understand first the homomorphism

$$
H(BO(p_{n-\frac{1}{2}})) \to H(BO(p_{n-1}))
$$

In order to do so, we notice that

$$
O(p_{n-\frac{1}{2}})/O(p_{n-1}) \cong A_{p_{n-\frac{1}{2}}=-1} \cong A_{\langle\alpha\rangle \perp (n-1)\,\mathbb{H}=1}
$$

From $\widetilde{M}(A_{\alpha x^2=1}) = N_<$ we deduce that

$$
\widetilde{M}(A_{\langle\alpha\rangle \perp (n-1)\,\mathbb{H}=1}) = N_<(n-1)[2n-2]
$$

Therefore, by Proposition 3.3 we have a long exact sequence in motivic cohomology

$$
\ldots \to H^{*-1,*'}(BO(p_{n-1})) \xrightarrow{h^*} H^{*-2n+1,*'-n+1}(N_< \otimes BO(p_{n-\frac{1}{2}})) \xrightarrow{f^*}
$$

$$
H^{*,*'}(BO(p_{n-\frac{1}{2}})) \xrightarrow{g^*} H^{*,*'}(BO(p_{n-1})) \to \ldots
$$

At this point, notice that $H(N_< \otimes BO(p_{n-\frac{1}{2}})) = H(N_<) \otimes_H H[u_1, \ldots, u_{2n-1}]$ which implies that u_i and v_{2n-1} are all uniquely liftable to $H(BO(p_{n-\frac{1}{2}}))$ by degree reasons, since $H^{*,*'}(N_< \otimes BO(p_{n-\frac{1}{2}}))$ is 0 for $*' < 0$ and for $(*')[*] = (0)[0]$ and $(0)[1]$, and by Lemma 5.6. Hence, g^* is an epimorphism since it is a ring homomorphism. Moreover, $g^*(u_i) = u_i$ for $i \leq 2n-2$ since the natural restriction $H(BO(p_{n-\frac{1}{2}})) \to H(BO(p_{n-\frac{3}{2}}))$ factors through $H(BO(p_{n-1}))$ and the classes u_i are uniquely determined, both in $H(BO(p_{n-\frac{1}{2}}))$ and in $H(BO(p_{n-1}))$, by the fact that they restrict to the respective u_i or vanish for $i = 2n-2$ in $H(BO(p_{n-\frac{3}{2}}))$. For the same reason, since v_{2n-1} vanishes in $H(BO(p_{n-\frac{3}{2}}))$, the element that covers v_{2n-1} through g^* has the shape $u_{2n-1} + \epsilon u_1 u_{2n-2}$, where ϵ is 0 or 1. Suppose $\epsilon = 1$, then by [14, Proposition 3.1.12] we have that $Sq^1(u_{2n-1} + u_1 u_{2n-2}) = Sq^1 Sq^1 u_{2n-2} = 0$, so $Sq^1 v_{2n-1} = 0$ as well. But, $Sq^1 v_{2n-1}$ maps to $Sq^1(\mu u_{2n-2})$ in $H(\mathfrak{X}_\alpha \otimes BO(p_{n-1}))$, which again maps to $Sq^1(\mu u_{2n-2}) = \mu^2 u_{2n-2} + \mu u_1 u_{2n-2} \neq 0$ in $H(BO_{2n-2} \otimes \mathfrak{X}_\alpha) = H(\mathfrak{X}_\alpha)[u_1, \ldots, u_{2n-2}]$, and we get a contradiction. Hence, ϵ must be 0 and $g^*(u_{2n-1}) = v_{2n-1}$.

Therefore, we have that the isomorphism $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \leftrightarrow H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha)$ maps u_{2i} to u_{2i} and u_{2i-1} to $u_{2i-1} + \mu u_{2i-2}$ for any $1 \leq i \leq 2n-2$. Moreover, since $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \to H(\mathfrak{X}_\alpha \otimes BO(p_{n-1}))$ maps $u_{2n-1} + \mu u_{2n-2}$ to 0, we have that $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}})) \leftrightarrow H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha)$ maps u_{2n-1} to $u_{2n-1} + \mu u_{2n-2}$.

Now, the result follows from the fact that the u_i are uniquely determined both in $H(\mathfrak{X}_\alpha \otimes BO(p_n))$ and in $H(BO_{2n} \otimes \mathfrak{X}_\alpha)$ by the fact that they restrict to u_i for $i \leq 2n-1$ and vanishes for $i = 2n$ respectively in $H(\mathfrak{X}_\alpha \otimes BO(p_{n-\frac{1}{2}}))$ and in $H(BO(-q_{2n-1}) \otimes \mathfrak{X}_\alpha)$. $\qquad\square$

From Theorem 7.1 we get immediately the following result which provides the motivic cohomology ring of $BO(\widetilde{h}_n)$.

THEOREM 7.3 *For any $n \in \mathbb{Z}_{\geq 0}$ there exist cohomology classes u_i of bidegree $([i/2])[i]$ for $1 \leq i \leq 2n$ and a class v_{2n+1} of bidegree $(n)[2n+1]$ only for n odd such that the motivic cohomology ring of $BO(\widetilde{h}_n)$ is given by*

$$H(BO(\widetilde{h}_n)) = \begin{cases} \dfrac{H[u_1,\ldots,u_{2n},v_{2n+1}]}{(\tau v_{2n+1} + \{\alpha\}u_{2n}, Ann(\{\alpha\})\cdot v_{2n+1})}, & n \text{ odd} \\ H[u_1,\ldots,u_{2n}], & n \text{ even} \end{cases}$$

Proof. It follows from the fact that $\widetilde{h}_n$ is split for n even and is isomorphic to p_n for n odd. $\qquad\square$

Once we know both the motivic cohomology of $BO(\widetilde{h}_n)$ and $BU_n(E/k)$, we can relate the subtle classes arising from the orthogonal group and those arising from the unitary group. In particular, we have the following result.
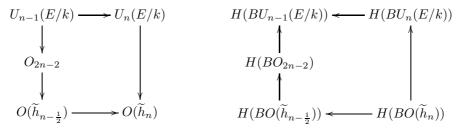
PROPOSITION 7.4 *For any $n \in \mathbb{Z}_{\geq 0}$ the natural embedding $U_n(E/k) \hookrightarrow O(\widetilde{h}_n)$ induces an epimorphism*

$$H(BO(\widetilde{h}_n)) \to H(BU_n(E/k))$$

sending u_{2i} to c_i for any $1 \leq i \leq n$, u_{2l+1} to 0 for any $0 \leq l$ even $< n$, u_{2j+1} to d_j for any $1 \leq j$ odd $< n$ and v_{2n+1} to d_n only for n odd.

Proof. We will proceed by induction. Notice that the induction basis is provided by the isomorphism $U_0 \cong O_0 \cong Spec(k)$.

For n odd we have the following commutative diagrams

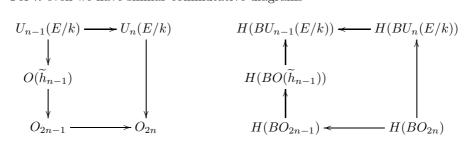where by $\widetilde{h}_{n-\frac{1}{2}}$ here we mean the quadratic form $\langle -\alpha \rangle \perp (n-1)\,\mathbb{H}$.

By induction hypothesis we have that u_{2i} goes to c_i for any $1 \le i \le n-1$, u_{2l+1} to 0 for any $0 \le l\,even < n-1$ and u_{2j+1} to d_j for any $1 \le j\,odd < n-1$. The class u_{2n-1} goes to 0 via the map $H(BO(\widetilde{h}_n)) \to H(BU_{n-1}(E/k))$ since this factors through $H(BO_{2n-2})$. Hence, u_{2n-1} maps to 0 in $H(BU_n(E/k))$ since the morphism $H(BU_n(E/k)) \to H(BU_{n-1}(E/k))$ is injective in bidegree $(n-1)[2n-1]$. Moreover, noticing that

$$U_n(E/k)/U_{n-1}(E/k) \cong O(\widetilde{h}_n)/O(\widetilde{h}_{n-\frac{1}{2}})$$

and by Proposition 3.4, we obtain that u_{2n} goes to c_n and v_{2n+1} goes to d_n.

For n even we have similar commutative diagrams

$$
\begin{array}{ccc}
U_{n-1}(E/k) \longrightarrow U_n(E/k) & \qquad & H(BU_{n-1}(E/k)) \longleftarrow H(BU_n(E/k)) \\
\downarrow \qquad\qquad\quad \downarrow & & \uparrow \qquad\qquad\qquad\qquad \uparrow \\
O(\widetilde{h}_{n-1}) & & H(BO(\widetilde{h}_{n-1})) \\
\downarrow & & \uparrow \\
O_{2n-1} \longrightarrow O_{2n} & & H(BO_{2n-1}) \longleftarrow H(BO_{2n})
\end{array}
$$

In this case we need to study the homomorphism

$$H(BO_{2n-1}) \to H(BO(\widetilde{h}_{n-1}))$$

In order to do so, we notice that

$$O_{2n-1}/O(\widetilde{h}_{n-1}) \cong A_{\widetilde{h}_{n-1}\perp\langle\alpha\rangle=\alpha} \cong A_{\alpha^{-1}\widetilde{h}_{n-1}\perp\langle 1\rangle=1}$$

From $\widetilde{M}(A_{x^2=\alpha}) = N_<$ and since $\alpha^{-1}\widetilde{h}_{n-1} \perp \langle 1 \rangle$ is isomorphic to $\langle \alpha^{-1} \rangle \perp (n-1)\,\mathbb{H}$, we deduce that

$$\widetilde{M}(A_{\alpha^{-1}\widetilde{h}_{n-1}\perp\langle 1\rangle=1}) = N_<(n-1)[2n-2]$$

Hence, by Proposition 3.3 we have a long exact sequence in motivic cohomology

$$\ldots \to H^{*-1,*'}(BO(\widetilde{h}_{n-1})) \xrightarrow{h^*} H^{*-2n+1,*'-n+1}(N_< \otimes BO_{2n-1}) \xrightarrow{f^*}$$

$$H^{*,*'}(BO_{2n-1}) \xrightarrow{g^*} H^{*,*'}(BO(\widetilde{h}_{n-1})) \to \ldots$$

Then, by repeating exactly the same arguments that appear in Proposition 7.2 we get that $g^*(u_i) = u_i$ for $i \le 2n-2$ and $g^*(u_{2n-1}) = v_{2n-1}$.

Therefore, by induction hypothesis we have that u_{2i} goes to c_i for any $1 \leq i \leq n-1$, u_{2l+1} to 0 for any $0 \leq l\ even \leq n-1$ and u_{2j+1} to d_j for any $1 \leq j\ odd \leq n-1$. Moreover, recalling that

$$U_n(E/k)/U_{n-1}(E/k) \cong O_{2n}/O_{2n-1}$$

and by Proposition 3.4, we obtain that u_{2n} goes to c_n, as we aimed to show. $\square$

As a corollary of the previous proposition and of Theorem 6.4 we get a description of $H(BU_n(E/k))$ as a quotient of $H(BO(\widetilde{h}_n))$.

COROLLARY 7.5 *For any $n \in \mathbb{Z}_{\geq 0}$ there is an isomorphism*

$$H(BU_n(E/k)) \cong \frac{H(BO(\widetilde{h}_n))}{R}$$

where R is the ideal generated by u_{4j+1}, $u_{4i+3}u_{4j+2} + u_{4j+3}u_{4i+2}$, $\tau u_{4j+3} + \{\alpha\}u_{4j+2}$ and $Ann(\{\alpha\}) \cdot u_{4j+3}$ for any $0 \leq i, j \leq [\frac{n-1}{2}]$, where u_{2n+1} is substituted by v_{2n+1} for n odd.

8 APPLICATIONS TO HERMITIAN FORMS

Throughout this section we exploit previous results to study subtle Stiefel-Whitney classes of quadratic forms divisible by $\langle\langle\alpha\rangle\rangle$. The general idea is that $H(BU_n(E/k))$ is closer to the cohomology of the Čech simplicial scheme of a quadratic form associated to a hermitian form than $H(BO_{2n})$.

From [14], we know that for every hermitian form h of the quadratic extension E/k there exists a commutative diagram

$$
\begin{array}{ccccc}
\check{C}(X_h) & \longrightarrow & BU_n(E/k) & \longrightarrow & BO(\widetilde{h}_n) \\
\downarrow & & \downarrow & & \downarrow \\
Spec(k) & \stackrel{h}{\longrightarrow} & B_{et}U_n(E/k) & \longrightarrow & B_{et}O(\widetilde{h}_n)
\end{array}
$$

where $\check{C}(X_h)$ is the Čech simplicial scheme of the torsor $X_h = Iso(h \leftrightarrow h_n)$. Hence, the computation of the motivic cohomology of $BU_n(E/k)$ provides us with subtle characteristic classes for hermitian forms and relations among them. More precisely, we have the following proposition.

PROPOSITION 8.1 *For any n-dimensional hermitian form h, in $H(\check{C}(X_h))$ the following relations hold for any $1 \leq j, j' odd \leq n$:*
1) $c_{j'}(h)d_j(h) + c_j(h)d_{j'}(h) = 0$;
2) $\tau d_j(h) + \{\alpha\}c_j(h) = 0$;
3) $Ann(\{\alpha\}) \cdot d_j(h) = 0$.

Proof. It follows immediately from Theorem 6.4.   □

We now move to consider quadratic forms associated to hermitian ones and their subtle Stiefel-Whitney classes.

Recall that two hermitian forms are isomorphic if and only if the corresponding quadratic forms over k are isomorphic. In particular, for even dimensional hermitian forms we have that they split if and only if the respective quadratic forms split. It follows that $\check{C}(X_h) \cong \check{C}(X_{\widetilde{h}})$, for even dimensional hermitian forms.
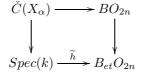
PROPOSITION 8.2 *For n even, in $H(\check{C}(X_{\widetilde{h}}))$ the following relations hold for any $0 \leq i, j \leq \frac{n}{2} - 1$:*

1) $u_{4j+1}(\widetilde{h}) = 0$;

2) $u_{4i+3}(\widetilde{h})u_{4j+2}(\widetilde{h}) = u_{4j+3}(\widetilde{h})u_{4i+2}(\widetilde{h})$;

3) $\tau u_{4j+3}(\widetilde{h}) = \{\alpha\}u_{4j+2}(\widetilde{h})$;

4) $Ann(\{\alpha\}) \cdot u_{4j+3}(\widetilde{h}) = 0$.

Proof. It follows immediately from Corollary 7.5.   □

On the other hand, if q is an odd dimensional quadratic form, then $\langle\langle\alpha\rangle\rangle \otimes q$ is split over a field extension of k if and only if $\langle\langle\alpha\rangle\rangle$ is split over the same field extension. It follows from this remark that, for odd dimensional hermitian forms, $\check{C}(X_{\widetilde{h}}) \cong \check{C}(X_\alpha)$, where $\check{C}(X_\alpha)$ stands for the Čech simplicial scheme associated to the Pfister form $\langle\langle\alpha\rangle\rangle$.

PROPOSITION 8.3 *For n odd, in $H(\check{C}(X_{\widetilde{h}})) = H(\check{C}(X_\alpha))$ the following relations hold for any $0 \leq j \leq \frac{n-1}{2}$:*

1) $u_{4j+1}(\widetilde{h}) = \mu u_{4j}(\widetilde{h})$;

2) $u_{4j-1}(\widetilde{h}) = 0$.

Proof. Together with the commutative diagram at the beginning of this section, we have the following one

$$\begin{array}{ccc} \check{C}(X_\alpha) & \longrightarrow & BO_{2n} \\ \downarrow & & \downarrow \\ Spec(k) & \xrightarrow{\widetilde{h}} & B_{et}O_{2n} \end{array}$$

By [14, Proposition 2.6.1], we know that after tensoring both with $\check{C}(X_\alpha)$ they coincide. Therefore, our restriction morphism $H(BO_{2n}) \to H(\check{C}(X_\alpha))$ factors as

$$H(BO_{2n}) \to H(BO_{2n} \times \check{C}(X_\alpha)) \leftrightarrow H(\check{C}(X_\alpha) \times BO(\widetilde{h}_n)) \to$$
$$H(\check{C}(X_\alpha) \times BU_n(E/k)) \to H(\check{C}(X_\alpha))$$

which implies the result by Theorem 6.3, Proposition 7.2 and Proposition 7.4.
□

We now show that the subtle classes arising in the unitary case see the triviality of the torsor of a hermitian form in the same way subtle Stiefel-Whitney classes do for quadratic forms ([14, Corollary 3.2.32]).

PROPOSITION 8.4 $h \cong h_n$ *if and only if* $c_{2^r}(h) = 0$ *for any* r.

Proof. Let us start from the case n even. Then, we have already noticed that h splits if and only if $\widetilde{h}$ splits. This is equivalent to say that $u_{2^{r+1}}(\widetilde{h})$ vanishes in $H(\check{C}(X_{\widetilde{h}}))$ for any r, which is the same of vanishing of $c_{2^r}(h)$ in $H(\check{C}(X_h))$, since in this case $\check{C}(X_h) \cong \check{C}(X_{\widetilde{h}})$ and by Proposition 7.4.

For n odd, we have that h splits if and only if $h \perp \langle -1 \rangle$ (which is even dimensional) splits. This amounts to say that $c_{2^r}(h \perp \langle -1 \rangle) = 0$ in $H(\check{C}(X_{h \perp \langle -1 \rangle}))$ for any r, which is equivalent to say that $c_{2^r}(h) = 0$ in $H(\check{C}(X_h))$ for any r. □

We conclude by presenting an expression of the motive of the torsor associated to a hermitian form. Indeed, by the very same arguments of [14, Propositions 3.1.11 and 3.2.2] one obtains the description of the motive of the torsor X_h in terms of motives of Čech simplicial schemes and subtle characteristic classes, where h is any hermitian form.
Before stating the results, let us denote by $\widetilde{c}_j$ a morphism $T \to N_>(j)[2j]$ in $\mathcal{DM}_{eff}^-(BU_n(E/k))$ which composed with the only non-zero morphism $N_>(j)[2j] \to T(j)[2j]$ gives c_j for any j odd. It is actually the unique cohomology class in $H(N_< \otimes BU_n(E/k))$ that maps to c_j under the homomorphism induced by the only non-zero morphism $T \to N_<$. Then, we have the following two propositions.

PROPOSITION 8.5 *In* $\mathcal{DM}_{eff}^-(BU_n(E/k))$ *we have*

$$M(EU_n(E/k) \to BU_n(E/k)) = \bigotimes_{1 \leq i\ even \leq n} Cone[-1](T \xrightarrow{c_i} T(i)[2i]) \otimes$$

$$\bigotimes_{1 \leq j\ odd \leq n} Cone[-1](T \xrightarrow{\widetilde{c}_j} N_>(j)[2j])$$

PROPOSITION 8.6 *In* $\mathcal{DM}_{eff}^-(k)$ *we have*

$$M(X_h) = \bigotimes_{1 \leq i\ even \leq n} Cone[-1](\mathfrak{X}_h \xrightarrow{c_i(h)} \mathfrak{X}_h(i)[2i]) \otimes$$

$$\bigotimes_{1 \leq j\ odd \leq n} Cone[-1](\mathfrak{X}_h \xrightarrow{\widetilde{c}_j(h)} N_> \otimes \mathfrak{X}_h(j)[2j]).$$

References

[1] T. Bachmann, *On the invertibility of motives of affine quadrics*, Doc. Math. 22 (2017), 363-395.

[2] T. Bachmann, A. Vishik, *Motivic equivalence of affine quadrics*, Math. Ann. 371 (2018), no. 1-2, 741-751.

[3] P. Deligne, *Théorie de Hodge. III*, Publications Mathématiques de l'IHÉS, no. 44 (1974), 5-77.

[4] S. I. Gelfand, Y. I. Manin, *Methods of homological algebra*, Translated from the 1988 Russian original. Springer-Verlag, Berlin, 1996.

[5] P. G. Goerss, J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, 174. Birkhäuser Verlag, Basel, 1999.

[6] N. A. Karpenko, *Unitary Grassmannians*, J. Pure Appl. Algebra 216 (2012), no. 12, 2586-2600.

[7] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI, 1998.

[8] C. Mazza, V. Voevodsky, C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, 2. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[9] F. Morel, V. Voevodsky, A^1-*homotopy theory of schemes*, Publications Mathématiques de l'IHÉS, no. 90 (1999), 45-143.

[10] M. Ojanguren, I. Panin, *Rationally trivial Hermitian spaces are locally trivial*, Math. Z. 237 (2001), no. 1, 181-198.

[11] D. Orlov, A. Vishik, V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Ann. Math. 165 (2007), 1-13.

[12] M. Rost, *The motive of a Pfister form*, Preprint, 1998, available at: https://www.math.uni-bielefeld.de/∼rost/motive.html.

[13] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 270. Springer-Verlag, Berlin, 1985.

[14] A. Smirnov, A. Vishik, *Subtle Characteristic Classes*, arXiv:1401.6661v1 (2014).

[15] A. Vishik, *On the Chow groups of quadratic Grassmannians*, Doc. Math. 10 (2005), 111-130.

[16] V. Voevodsky, *Motives over simplicial schemes*, J. K-Theory 5 (2010), no. 1, 1-38.

[17] V. Voevodsky, *Motivic cohomology with Z/2-coefficients*, Publications Mathématiques de l'IHÉS, no. 98, pp. 59-104, 2003.

[18] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, 188-238, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[19] N. Yagita, *Applications of Atiah-Hirzebruch spectral sequences for motivic cobordisms*, Proc. London Math. Soc. 90 (2005), 783-816.

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