On the Classification of Lie Bialgebras by Cohomological Means

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Received: November 7, 2018
Revised: October 25, 2019

Communicated by Nikita Karpenko

Abstract. We approach the classification of Lie bialgebra structures on simple Lie algebras from the viewpoint of descent and non-abelian cohomology. We achieve a description of the problem in terms of faithfully flat cohomology over an arbitrary ring over $\mathbb{Q}$, and solve it for Drinfeld–Jimbo Lie bialgebras over fields of characteristic zero. We consider the classification up to isomorphism, as opposed to equivalence, and treat split and non-split Lie algebras alike. We moreover give a new interpretation of scalar multiples of Lie bialgebras hitherto studied using twisted Belavin–Drinfeld cohomology.

2010 Mathematics Subject Classification: 17B62, 17B37, 20G10
Keywords and Phrases: Lie bialgebra, quantum group, faithfully flat descent, Galois cohomology

1 Introduction

The "linearization problem" for quantum groups, outlined in spirit by Drinfeld [D], and solved in the celebrated work of Etingof and Kazhdan [EK1] and [EK2], naturally leads to the study of Lie bialgebras over the power series ring $R = \mathbb{C}[[t]]$. If $\mathfrak{g}$ is a finite dimensional (necessarily) split simple complex Lie algebra one can try to understand all possible Lie bialgebra structures that can be put on the R–Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} R$. This is exactly the program started by

[1] S. Alsaody wishes to thank the Knut and Alice Wallenberg Foundation for the grant KAW 2015.0367, by means of which he was supported as a postdoctoral researcher at Institut Camille Jordan, Lyon, and PIMS, for supporting him as a postdoctoral fellow at the University of Alberta.
[2] A. Pianzola wishes to thank NSERC and Conicet for their continuing support.

Kadets, Karolinsky, Pop and Stolin and pursued in [KKPS] and other papers, where this is done by considering the (algebraic) Laurent series field $\mathbb{K} = \mathbb{C}((t))$, and introducing (twisted and untwisted) Belavin–Drinfeld cohomologies. These cohomologies parametrized the possible Lie bialgebra structures on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$, and they were computed on a case-by-case fashion for the classical types.
We recall that the Belavin–Drinfeld theorem from [BD] gives a complete list (up to equivalence) of all possible Lie bialgebra structures on $\mathfrak{g} \otimes_{\mathbb{C}} \overline{\mathbb{K}}$. It is thus natural to approach the problem at hand by means of usual Galois cohomology. This is done in [PS], where the Belavin–Drinfeld cohomologies are shown to be usual Galois cohomology with values in the algebraic group $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ (the centralizer in the adjoint group $\mathbf{G}$ of $\mathfrak{g}$ of the given Belavin–Drinfeld r-matrix r_{BD}). The main results of [PS] state that:

(a) untwisted Belavin–Drinfeld cohomologies are the usual Galois cohomologies $H^1_{\mathrm{Gal}}(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}}))$;

(b) for the Drinfeld–Jimbo r-matrix r_{DJ} the twisted Belavin–Drinfeld cohomologies can be expressed in terms of the Galois cohomology set $H^1_{\mathrm{Gal}}(\mathbb{K}, \widetilde{\mathbf{C}}(\mathbf{G}, r_{\mathrm{DJ}}))$ for a twisted form $\widetilde{\mathbf{C}}(\mathbf{G}, r_{\mathrm{DJ}})$ of the $\mathbb{K}$-algebraic group $\mathbf{C}(\mathbf{G}, r_{\mathrm{DJ}})$. This form is split by the quadratic extension $\mathbb{L} = \mathbb{K}(\sqrt{t})$ of $\mathbb{K}$;

(c) $H^1_{\mathrm{Gal}}(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{\mathrm{DJ}}))$ is trivial by Hilbert 90, and $H^1(\mathbb{K}, \widetilde{\mathbf{C}}(\mathbf{G}, r_{\mathrm{DJ}}))$ is trivial by a theorem of Steinberg that is also used to establish the correspondence in (b). As a consequence, there are unique (up to Belavin–Drinfeld equivalence with gauge group $\mathbf{G}$) corresponding Lie bialgebra structures on $\mathfrak{g}$ with prescribed doubles (namely $\mathfrak{g} \times \mathfrak{g}$ in the untwisted case, and $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{L}$ in the twisted case).

The main objective of the present paper is to develop the theory of faithfully flat descent for Lie bialgebras over rings, with emphasis on what this theory is best suited for: the classification of twisted forms of a given Lie bialgebra *up to isomorphism* and *without the restriction that the underlying Lie algebras be split.* This is the main difference between our work and the recent paper [KPS], where the authors use Galois descent to obtain far-reaching results about Lie bialgebra structures on split Lie algebras up to equivalence. The Belavin–Drinfeld classification is up to equivalence, in the sense that two coboundary Lie bialgebra structures ∂r and $\partial r'$ on a split Lie algebra $\mathfrak{g}$ are considered equivalent if

$$r' = \alpha(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r)$$

for some invertible scalar α and some X in a suitably chosen group with corresponding simple Lie algebra $\mathfrak{g}$. This relation is not comparable to isomorphism. On the one hand, scalar multiples of r-matrices in general lead to non-isomorphic Lie bialgebras. On the other hand, non-equivalent Lie bialgebra

structures may still be isomorphic if the Lie algebra admits outer automorphisms. Nevertheless, the flexibility of the point of view that we take allows us to recover and in some cases explain all the results up to equivalence known heretofore. As we will see, it is well suited for understanding Lie bialgebras whose underlying Lie algebra is not necessarily split over arbitrary fields of characteristic zero, a topic that has been little investigated in the literature.
In the appendix of [KPS], the authors classify Lie bialgebra structures on $\mathfrak{sl}(2, R)$ and $\mathfrak{sl}(3, R)$ for a discrete valuation ring R using orders and lattices. It would therefore be interesting to find a cohomological interpretation and generalization of these results. Another instance where Lie bialgebras over rings are discussed is [BFS], where solutions to the quantum Yang–Baxter equation that arise from Frobenius algebras over rings are treated.
After the necessary definitions in this section, the paper proceeds with the statement of the formalism of faithfully flat descent for Lie bialgebras in Section 2. In Section 3 we give a description of the automorphism groups of Belavin–Drinfeld Lie bialgebras defined over the base ring. Our major result is in Section 4, where we solve the classification problem for standard (i.e. Drinfeld–Jimbo) Lie bialgebras over arbitrary fields of characteristic zero. In Section 5 we turn our attention to Lie bialgebras that are locally scalar multiples of Belavin–Drinfeld structures. This includes and provides context to previous results on twisted Belavin–Drinfeld bialgebras. In the final Section 6 we review some known classification results in the light of the results of the previous sections.

Acknowledgement

We are grateful to Alexander Stolin for fruitful discussions, and to the referees for their careful reading of our manuscript, and for their feedback.

1.1 Lie Bialgebras over Rings

The importance of considering Lie bialgebras over rings that are not fields was explained in the introduction. Throughout, we fix a unital, commutative ring R. All unadorned tensor products are understood to be over R. By an *R-ring* we understand a unital, commutative R-algebra. We will further always assume that Spec R, as a scheme, has characteristic 0; this amounts to saying that R is a $\mathbb{Q}$-ring. For any R-module M we will always write κ for the linear map $M \otimes M \to M \otimes M$ defined by the transposition $x \otimes y \mapsto y \otimes x$ of tensor factors. Let M be an R-module. A *Lie cobracket* on M is an R-linear map $\delta : M \to M \otimes M$ that is *anti-symmetric* in the sense that

$$\kappa \circ \delta = -\delta,$$

and satisfies the *co-Jacobi identity*

$$(\delta \otimes \mathrm{Id}_M) \circ \delta = (\mathrm{Id}_M \otimes \delta) \circ \delta + (\mathrm{Id}_M \otimes \kappa) \circ (\delta \otimes \mathrm{Id}_M) \circ \delta.$$

The pair (M, δ) is called a *Lie coalgebra.* From the definition it follows that the composition

$$M^* \otimes M^* \xrightarrow{\text{can}} (M \otimes M)^* \xrightarrow{\delta^*} M^*$$

is a Lie bracket on M^*.

Remark 1.1. If M is a finitely generated projective module, then the canonical map can is an isomorphism. In that case, identifying $M^* \otimes M^*$ with $(M \otimes M)^*$, it can be verified that δ is a Lie cobracket if and only if δ^* is a Lie bracket.

If $\mathfrak{g} = (\mathfrak{g}, [,])$ is a Lie algebra, and δ is a Lie cobracket on $\mathfrak{g}$ satisfying the *cocycle condition*

$$\delta([a, b]) = (\mathrm{ad}_a \otimes 1 + 1 \otimes \mathrm{ad}_a)\delta(b) - (\mathrm{ad}_b \otimes 1 + 1 \otimes \mathrm{ad}_b)\delta(a)$$

for any $a, b \in \mathfrak{g}$, then $(\mathfrak{g}, \delta)$ is called a *Lie bialgebra.* If $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}', \delta')$ are Lie bialgebras, then a map $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie bialgebras $(\mathfrak{g}, \delta) \to (\mathfrak{g}', \delta')$ if it is a Lie algebra morphism that in addition satisfies

$$(\phi \otimes \phi) \circ \delta = \delta' \circ \phi. \tag{1.1}$$

If ϕ is invertible, then ϕ^{-1} is a morphism of Lie bialgebras $(\mathfrak{g}', \delta') \to (\mathfrak{g}, \delta)$, and ϕ is called an isomorphism of Lie bialgebras. Thus R-Lie bialgebras form a category, which we denote by $\widehat{\mathfrak{LBi}}_R$. We denote by $\mathfrak{LBi}_R$ the full subcategory of $\widehat{\mathfrak{LBi}}_R$ whose objects are those Lie bialgebras whose underlying module is finitely generated and projective.

2 Descent for Lie Bialgebras

We will first establish the desired correspondence between twisted forms of bialgebras and certain cohomology classes. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra over R, and let S be a R-ring. On the S-algebra $\mathfrak{g}_S = \mathfrak{g} \otimes S$ one has a unique Lie bialgebra structure δ_S that satisfies

$$\delta_S(x \otimes 1) = \sum (y_i \otimes 1) \otimes_S (z_i \otimes 1)$$

for all $x \in \mathfrak{g}$, where $\sum y_i \otimes z_i = \delta(x) \in \mathfrak{g} \otimes \mathfrak{g}$. An R-Lie bialgebra $(\mathfrak{g}', \delta')$ is said to be an *S/R-twisted form of* $(\mathfrak{g}, \delta)$ if $(\mathfrak{g}'_S, \delta'_S) \simeq (\mathfrak{g}_S, \delta_S)$ as S-bialgebras. We will mainly be interested in the case where S is faithfully flat over R. This includes the special case where R is a field of characteristic zero and S is any field extension.

Let $(\mathfrak{g}, \delta)$ now be an S-Lie bialgebra, and let $\kappa : S \otimes S \to S \otimes S$ be the (R-linear) flip $\alpha \otimes \beta \mapsto \beta \otimes \alpha$. There are two ways to endow $\mathfrak{g} \otimes S$ with an $S \otimes S$-module structure; the $S \otimes S$-action being component-wise in the first, and twisted by κ in the second. We denote the two modules (algebras, bialgebras) by $\mathfrak{g} \otimes_{12} S$ and $\mathfrak{g} \otimes_{21} S$, respectively. Both modules (algebras, bialgebras) are seen as having the same underlying R-module (R-algebra, R-bialgebra) structure which we continue to denote by $\mathfrak{g} \otimes S$.

Definition 2.1. A *descent datum* on $\mathfrak{g}$ is an isomorphism $\theta : \mathfrak{g}\otimes_{12} S \to \mathfrak{g}\otimes_{21} S$ of $S \otimes S$- Lie bialgebras, satisfying the equality

$$(\mathrm{Id} \otimes \kappa)(\theta \otimes \mathrm{Id})(\mathrm{Id} \otimes \kappa) = (\theta \otimes \mathrm{Id})(\mathrm{Id} \otimes \kappa)(\theta \otimes \mathrm{Id})$$

of maps on $\mathfrak{g} \otimes S \otimes S$. The triple $(\mathfrak{g}, \delta, \theta)$ is called a *Lie bialgebra with a descent datum.* A *morphism of Lie bialgebras with descent data* $(\mathfrak{g}, \delta, \theta) \to (\mathfrak{g}', \delta', \theta')$ is a morphism of Lie bialgebras $f : (\mathfrak{g}, \delta) \to (\mathfrak{g}', \delta')$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} \otimes_{12} S & \longrightarrow & \mathfrak{g}' \otimes_{12} S \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} \\ \mathfrak{g} \otimes_{21} S & \longrightarrow & \mathfrak{g}' \otimes_{21} S \end{array}$$

commutes, where the horizontal arrows are given by $f \otimes \mathrm{Id}$.

Remark 2.2. Some authors set $\theta^0 = (\mathrm{Id}\otimes\kappa)(\theta\otimes\mathrm{Id})(\mathrm{Id}\otimes\kappa)$, $\theta^1 = (\theta\otimes\mathrm{Id})(\mathrm{Id}\otimes\kappa)$ and $\theta^2 = (\theta \otimes \mathrm{Id})$, and write the equality in the definition in the equivalent form $\theta^1 = \theta^0\theta^2$.

We recall the following fundamental fact of faithfully flat descent (see [KO, II.2.5]). Here and in what follows, if N is an R-module and T an R-ring, we shall often abbreviate $N \otimes T$ by N_T.

Lemma 2.3. *Let S be a faithfully flat R-ring and let N and N' be R-modules. There is an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_R(N, N') \longrightarrow \mathrm{Hom}_S(N_S, N'_S) \xrightarrow{\ \Delta\ } \mathrm{Hom}_{S\otimes S}(N_{S\otimes S}, N'_{S\otimes S})$$

of R-modules. Here Δ is defined by $f \mapsto (f \otimes \mathrm{Id}) - (\mathrm{Id} \otimes \kappa)(f \otimes \mathrm{Id})(\mathrm{Id} \otimes \kappa)$.

For an R-ring S, we write $\widehat{\mathfrak{LBi}}_R^S$ for the category of S-Lie bialgebras with descent data, and $\mathfrak{LBi}_R^S$ for the full subcategory formed by the objects whose underlying modules are finitely generated projective. If $(\mathfrak{g}, \delta) \in \widehat{\mathfrak{LBi}}_R$, then the *standard descent datum* on $\mathfrak{g} \otimes S$ is the map

$$\mathrm{Id}_{\mathfrak{g}} \otimes \kappa : (\mathfrak{g} \otimes S) \otimes_{12} S \to (\mathfrak{g} \otimes S) \otimes_{21} S.$$

It is straightforward to verify that this is indeed a descent datum, and that we thus get a functor $\mathfrak{D} = \mathfrak{D}_R^S : \widehat{\mathfrak{LBi}}_R \to \widehat{\mathfrak{LBi}}_R^S$, defined on objects by $\mathfrak{g} \mapsto (\mathfrak{g} \otimes S, \mathrm{Id}_{\mathfrak{g}} \otimes \kappa)$, and on morphisms by $f \mapsto f \otimes \mathrm{Id}_S$.
Faithfully flat descent for Lie bialgebras is then as follows.

Proposition 2.4. *If the R-ring S is faithfully flat, then $\mathfrak{D}$ is an equivalence of categories $\widehat{\mathfrak{LBi}}_R \to \widehat{\mathfrak{LBi}}_R^S$, and induces an equivalence $\mathfrak{LBi}_R \to \mathfrak{LBi}_R^S$.*

This result is well-known for modules (where the descent data are linear maps). We will use this in the proof below.

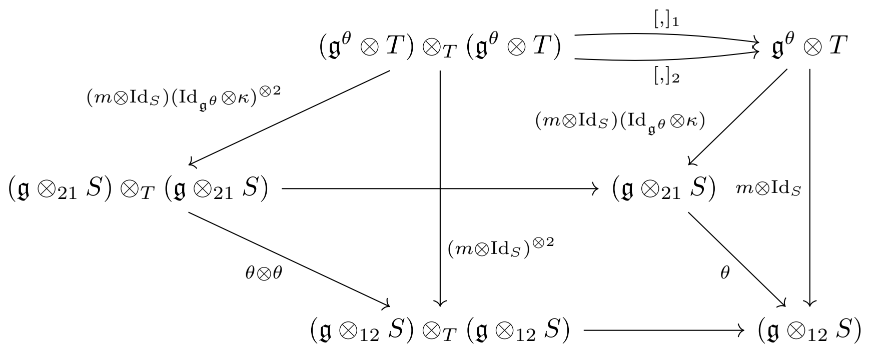
Proof. It is clear that $\mathfrak{D}(\mathfrak{g})$ is finitely generated projective if $\mathfrak{g}$ is. We claim that a quasi-inverse to $\mathfrak{D}$ is given by $\mathfrak{D}' : \widehat{\mathfrak{LBi}}_R^S \to \widehat{\mathfrak{LBi}}_R$, defined on objects by mapping $(\mathfrak{g}, \theta) \in \mathfrak{LBi}_R^S$ to

$$\mathfrak{g}^\theta := \{x \in \mathfrak{g} | \theta(x \otimes 1) = x \otimes 1\},$$

and on morphisms by restriction. It is known that $\mathfrak{g}^\theta$ is an R-submodule of $\mathfrak{g}$, which is finitely generated projective if $\mathfrak{g}$ is, and by [W, 17.2] $\mathfrak{g}^\theta \otimes S \simeq \mathfrak{g}$ as S-modules, via the isomorphism $m : x \otimes \alpha \mapsto \alpha x$.
The module $\mathfrak{g}^\theta$ is an algebra by [KO, II.3.]. We repeat their argument here. The Lie bracket on $\mathfrak{g}$ induces, via the map m, a Lie bracket $[,]_\theta$ on $\mathfrak{g}^\theta \otimes S$, thus making m into an algebra homomorphism fitting into the commutative diagram

$$\begin{array}{ccc} (\mathfrak{g}^\theta \otimes S) \otimes_S (\mathfrak{g}^\theta \otimes S) & \xrightarrow{[,]_\theta} & \mathfrak{g}^\theta \otimes S \\ \downarrow{\scriptstyle m \otimes_S m} & & \downarrow{\scriptstyle m} \\ \mathfrak{g} \otimes_S \mathfrak{g} & \longrightarrow & \mathfrak{g} \end{array} . \tag{2.1}$$

Consider the following diagram of $T := S \otimes S$-modules



(2.2)

where the middle and bottom horizontal arrows are the Lie brackets on the respective modules, and the maps $[,]_1$ and $[,]_2$ are defined by

$$[x \otimes \alpha \otimes \beta, y \otimes 1 \otimes 1]_1 = [x \otimes \alpha, y \otimes 1]_\theta \otimes \beta,$$
$$[x \otimes \beta \otimes \alpha, y \otimes 1 \otimes 1]_2 = (\mathrm{Id}_{\mathfrak{g}^\theta} \otimes \kappa)([x \otimes \alpha, y \otimes 1]_\theta \otimes \beta).$$

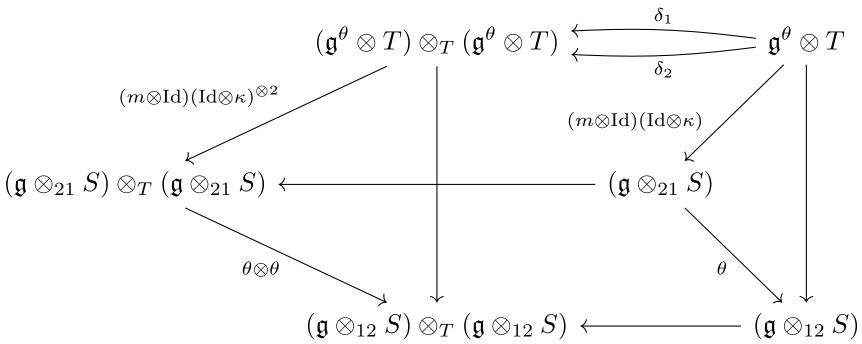
Note that the vertical arrows are isomorphisms: this holds for $m \otimes \mathrm{Id}_S$ since S is flat, and for $(m \otimes \mathrm{Id}_S)^{\otimes 2}$ since tensoring is right exact, hence preserves isomorphisms. Recall next that we have an isomorphism $f_U : (\mathfrak{g}^\theta \otimes U) \otimes_U (\mathfrak{g}^\theta \otimes U) \to \mathfrak{g}^\theta \otimes \mathfrak{g}^\theta \otimes U$ for any R-ring U. Given the isomorphism f_T and the construction of $[,]_1$ and $[,]_2$ it remains, by Lemma 2.3, to show that these two brackets are equal. The lemma then implies that $[,]_\theta$ is induced by a unique bracket on $\mathfrak{g}^\theta$, which satisfies the axioms of a Lie bracket since so does $[,]_\theta$ (see [KO, II.3.]).

Since the rightmost vertical arrow is an isomorphism, it suffices to show that the rear square commutes with each of $[,]_i$ as top arrow. For $[,]_1$, this follows from the commutativity of (2.1) by applying f_S, tensoring with S on the right, and applying f_T. Tensoring with S in the middle instead implies that the top parallelogram is commutative for $[,]_2$. The bottom parallelogram is commutative since θ is a Lie algebra homomorphism, and the two triangles commute by definition of a descent datum. This altogether implies that the square commutes with $[,]_2$ as well, as desired.

To show that $\mathfrak{g}^\theta$ is a Lie coalgebra, we use the same argument up to reversing some arrows. More specifically, we have a Lie cobracket δ_θ on $\mathfrak{g}^\theta \otimes S$, making m into a coalgebra homomorphism such that the diagram

$$\begin{array}{ccc} (\mathfrak{g}^\theta \otimes S) \otimes_S (\mathfrak{g}^\theta \otimes S) & \xleftarrow{\delta_\theta} & \mathfrak{g}^\theta \otimes S \\ \downarrow{\scriptstyle m \otimes_S m} & & \downarrow{\scriptstyle m} \\ \mathfrak{g} \otimes_S \mathfrak{g} & \longleftarrow & \mathfrak{g} \end{array} \tag{2.3}$$

commutes. Consider the diagram



(2.4)

which only differs from (2.2) at the horizontal arrows: the middle and bottom arrows are the Lie cobrackets on the respective modules, and the maps δ_1 and δ_2 are defined by

$$\begin{aligned} \delta_1(x \otimes \alpha \otimes \beta) &= \textstyle\sum (x_i \otimes \alpha_i \otimes \beta) \otimes_T (x_i' \otimes \alpha_i' \otimes 1) \\ \delta_2(x \otimes \beta \otimes \alpha) &= \textstyle\sum (x_i \otimes \beta \otimes \alpha_i) \otimes_T (x_i' \otimes 1 \otimes \alpha_i') \end{aligned}$$

with $\delta_\theta(x \otimes \alpha) = \sum (x_i \otimes \alpha_i) \otimes_S (x_i' \otimes \alpha_i')$. As above it suffices to check that the square commutes for each δ_i. For δ_1 this follows from the commutativity of (2.3) via f_S, tensoring with S on the right, and f_T. Again, tensoring with S in the middle implies that the top parallelogram is commutative for δ_2. The bottom parallelogram is commutative since θ is a Lie coalgebra homomorphism, and the triangles are those from (2.2). Thus δ_θ is induced by a unique cobracket on $\mathfrak{g}^\theta$, which satisfies the axioms of a Lie cobracket since δ_θ does.

The cocycle condition further follows, by faithful flatness, from that on $\mathfrak{g}^\theta \otimes S$, which holds in view of the isomorphism m. Thus $\mathfrak{g}^\theta$ is a Lie bialgebra.

To show that $\mathfrak{D}'$ is well-defined on morphisms, let $f:(\mathfrak{g},\theta)\to(\mathfrak{g}',\theta')$ be a morphism of S-Lie bialgebras with descent data. Then for all $x\in\mathfrak{g}^\theta$,

$$\theta'(f(x)\otimes 1)=\theta'(f\otimes\mathrm{Id})(x\otimes 1)=(f\otimes\mathrm{Id})\theta(x\otimes 1)=(f\otimes\mathrm{Id})(x\otimes 1)=f(x)\otimes 1,$$

where the second equality holds by definition of a map of Lie bialgebras with descent data, and the third by construction of $\mathfrak{g}^\theta$. Thus $f(x)\in(\mathfrak{g}')^{\theta'}$, and the functor is well-defined morphisms.

It remains to be shown that $\mathfrak{D}'$ is a quasi-inverse to $\mathfrak{D}$. Firstly, for each $\mathfrak{g}\in\widehat{\mathfrak{LBi}}_R$,

$$\mathfrak{D}'\mathfrak{D}(\mathfrak{g})=\{x\in\mathfrak{g}\otimes S|(\mathrm{Id}_\mathfrak{g}\otimes\kappa)(x\otimes 1)=x\otimes 1\}.$$

Applying Lemma 2.3 with $N=R$ and $N'=\mathfrak{g}$, we see that the assignment $x\mapsto x\otimes 1$ yields an isomorphism $\mathfrak{g}\to\mathfrak{D}'\mathfrak{D}(\mathfrak{g})$. Since this map is clearly a natural morphism of Lie bialgebras, this implies $\mathfrak{D}'\mathfrak{D}\simeq\mathrm{Id}_{\widehat{\mathfrak{LBi}}_R}$.

Secondly, if $(\mathfrak{g},\theta)\in\widehat{\mathfrak{LBi}}_R^S$, then

$$\mathfrak{D}\mathfrak{D}'(\mathfrak{g},\theta)=(\mathfrak{g}^\theta\otimes S,\mathrm{Id}_{\mathfrak{g}^\theta}\otimes\kappa).$$

From the above we know that the map $m:\mathfrak{g}^\theta\otimes S\to\mathfrak{g}$ is an isomorphism of Lie bialgebras. It is a map of Lie bialgebras with descent data, since for all $x\in\mathfrak{g}^\theta$ and $\alpha,\beta\in S$,

$$\theta(m\otimes\mathrm{Id}_S)(x\otimes\alpha\otimes\beta)=\theta(\alpha x\otimes\beta)=(\beta\otimes\alpha)\theta(x\otimes 1)=(\beta\otimes\alpha)(x\otimes 1)$$

which is equal to

$$\beta x\otimes\alpha=(m\otimes\mathrm{Id}_S)(x\otimes\beta\otimes\alpha)=(m\otimes\mathrm{Id}_S)(\mathrm{Id}_{\mathfrak{g}^\theta}\otimes\kappa)(x\otimes\alpha\otimes\beta).$$

Naturality is clear, and the equivalence of categories thus obtained induces an equivalence $\mathfrak{LBi}_R\to\mathfrak{LBi}_R^S$ by restriction. The proof is complete. □

For any R-Lie bialgebra $(\mathfrak{g},\delta)$ and any faithfully flat R-ring S, we wish to classify all *S/R-twisted forms of* $\mathfrak{g}$, i.e. all R-Lie bialgebras $(\mathfrak{g}',\delta')$ such that $(\mathfrak{g}'_S,\delta'_S)\simeq(\mathfrak{g}_S,\delta_S)$. Let $\mathbf{A}=\mathbf{Aut}((\mathfrak{g},\delta))$ be the automorphism group functor of $(\mathfrak{g},\delta)$. As one does for modules and algebras, we consider, for each faithfully flat R-ring S the cohomology set $H^1(S/R,\mathbf{A}):=H^1_{\mathrm{fppf}}(S/R,\mathbf{A})$, consisting of cohomology classes of 1-cocycles. Here a 1-cocycle is an element $\phi\in\mathbf{A}(S\otimes S)$ satisfying the cocycle condition

$$d^1\phi=(d^0\phi)(d^2\phi)$$

where $d^i\phi$ is the R-linear extension of ϕ to $\mathfrak{g}\otimes S\otimes S\otimes S$ obtained by applying Id_S to the i^{th} copy of S; two cocycles ϕ and ϕ' are defined to be cohomologous if

$$\phi'=(\mathrm{Id}_\mathfrak{g}\otimes\kappa)(\rho\otimes\mathrm{Id}_S)(\mathrm{Id}_\mathfrak{g}\otimes\kappa)\phi(\rho^{-1}\otimes\mathrm{Id}_S)$$

for some $\rho\in\mathbf{A}(S)$. The following is then a consequence of the above proposition, and the proof is analogous to that for descent of modules.

Corollary 2.5. *Let $(\mathfrak{g},\delta)$ be a Lie bialgebra over R with automorphism group scheme $\mathbf{A}$. Let S be a faithfully flat R-ring. Then there is a $1{-}1$-correspondence between $H^1(S/R,\mathbf{A})$ and R-isomorphism classes of S/R-twisted forms of $(\mathfrak{g},\delta)$.*

3 Belavin–Drinfeld Lie Bialgebras and Their Automorphisms

3.1 Coboundary Lie Bialgebras and r-matrices

A Lie bialgebra $(\mathfrak{g}, \delta)$ is said to be a *coboundary Lie bialgebra* if $\delta = \partial r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. if

$$\delta(a) = (\mathrm{ad}_a \otimes 1 + 1 \otimes \mathrm{ad}_a)(r) \tag{3.1}$$

for all $a \in \mathfrak{g}$. Using classical notation, alluding to the universal enveloping algebra, this is written as

$$\delta(a) = [a \otimes 1 + 1 \otimes a, r]$$

In general, not every $r \in \mathfrak{g} \otimes \mathfrak{g}$ gives rise to a Lie bialgebra structure via the above formula. We will come back to this point below.
Let $\mathbf{G}$ be a split simple adjoint group over R with $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. By Chevalley uniqueness [SGA3, XXIII.5] and the fact that R is a $\mathbb{Q}$-ring, up to isomorphism we may and will assume that $\mathbf{G}$ is defined over $\mathbb{Q}$; thus $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{Q}} R$ for a split simple $\mathbb{Q}$-Lie algebra $\mathfrak{g}_0$. Let $\mathbf{E}$ be a pinning of $\mathbf{G}$. The pinning provides a split maximal torus $\mathbf{H}$, a splitting Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(\mathbf{H})$ of $\mathfrak{g}$, a base Γ of the corresponding root system Δ, a set of positive roots $\Delta^+ \subset \Delta$ and, for each $\alpha \in \Delta$, a Chevalley generator $X_\alpha \neq 0$ of $\mathfrak{g}_\alpha$. This moreover provides a Casimir element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$, and we write $\Omega_{\mathfrak{h}}$ for its Cartan part. (More precisely, writing $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{Q}} R$, where $\mathfrak{h}_0$ is the Cartan subalgebra of $\mathfrak{g}_0$ corresponding to the above data, and taking an orthonormal basis (h_i) of $\mathfrak{h}_0$, $\Omega_{\mathfrak{h}}$ is the image of $\sum_i h_i \otimes h_i$ under the base change $\mathbb{Q} \to R$.)

Remark 3.1. It is known that in the above setting

$$\mathbb{Q}\Omega = \{s \in \mathfrak{g}_0 \otimes \mathfrak{g}_0 | \forall a \in \mathfrak{g}_0 : (\mathrm{ad}_a \otimes 1 + 1 \otimes \mathrm{ad}_a)(s) = 0\}, \tag{3.2}$$

and that any automorphism of $\mathfrak{g}_0$ fixes Ω. From this we will deduce that

$$R\Omega = \{s \in \mathfrak{g} \otimes \mathfrak{g} | \forall a \in \mathfrak{g} : (\mathrm{ad}_a \otimes 1 + 1 \otimes \mathrm{ad}_a)(s) = 0\}, \tag{3.3}$$

and that any automorphism of $\mathfrak{g}$ fixes Ω. Indeed, consider the $\mathbb{Q}$-linear map

$$F_0 : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{g}_0, \mathfrak{g}_0 \otimes \mathfrak{g}_0), s \mapsto \partial_s$$

where $\partial_s(a) = [1 \otimes a + a \otimes 1, s]$. The kernel of F_0 is the right hand side of (3.2), which thus is $\mathbb{Q}\Omega$. Base change gives a map

$$F_0 \otimes_{\mathbb{Q}} \mathrm{Id}_R : (\mathfrak{g}_0 \otimes_{\mathbb{Q}} \mathfrak{g}_0) \otimes_{\mathbb{Q}} R \to \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{g}_0, \mathfrak{g}_0 \otimes_{\mathbb{Q}} \mathfrak{g}_0) \otimes_{\mathbb{Q}} R,$$

where the right hand side is canonically isomorphic to $\mathrm{Hom}_R(\mathfrak{g}, (\mathfrak{g}_0 \otimes_{\mathbb{Q}} \mathfrak{g}_0) \otimes_{\mathbb{Q}} R)$ since $\mathfrak{g}_0$ is finite-dimensional. Since R is flat over $\mathbb{Q}$, the kernel of this map is

$\mathbb{Q}\Omega \otimes_{\mathbb{Q}} R$. Identifying $(\mathfrak{g}_0 \otimes_{\mathbb{Q}} \mathfrak{g}_0) \otimes_{\mathbb{Q}} R$ with $\mathfrak{g} \otimes_R \mathfrak{g}$ we thus deduce that the kernel of the map

$$F : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Hom}_R(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}), s \mapsto \partial_s,$$

is $R\Omega$. Furthermore $\mathbf{Aut}(\mathfrak{g}) = \mathbf{Aut}(\mathfrak{g}_0)_R$, whence for any R-ring S and any $\phi \in \mathbf{Aut}(\mathfrak{g})(S)$,

$$(\phi \otimes_S \phi)(\Omega \otimes_{\mathbb{Q}} 1_S) = \Omega \otimes_{\mathbb{Q}} 1_S$$

Note further that since $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{Q}} R$ and Ω is defined over $\mathbb{Q}$ and non-zero in $\mathfrak{g}_0 \otimes \mathfrak{g}_0$, it follows that if $\lambda \in R$ satisfies $\lambda\Omega = 0$ in $\mathfrak{g} \otimes \mathfrak{g}$, then $\lambda = 0$.

By an *r-matrix on* $\mathfrak{g}$ we understand an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying $\mathrm{CYB}(r) = 0$ and $r + \kappa(r) = \lambda\Omega$ for some $\lambda \in R$. Here the classical Yang–Baxter operator CYB is defined by

$$\mathrm{CYB}(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}],$$

which, writing $r = \sum_i s_i \otimes t_i$, is shorthand for[3]

$$\sum_{i,j} ([s_i, s_j] \otimes t_i \otimes t_j + s_i \otimes [t_i, s_j] \otimes t_j + s_i \otimes s_j \otimes [t_i, t_j]) \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}.$$

It is straightforward to check that if r is an r-matrix, then ∂r is a Lie bialgebra structure on $\mathfrak{g}$. Conversely, if $R = K$ is a field and δ is a Lie bialgebra structure on $\mathfrak{g}$, then by Whitehead's Lemma, $\delta = \partial r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. If moreover K is algebraically closed, then one may take r to be an r-matrix (see e.g. [ES]).

Remark 3.2. When R is a field, r-matrices r satisfying $r + \kappa(r) = 0$ are called skew-symmetric. These are excluded from the Belavin–Drinfeld classification. In the sequel we will only consider r-matrices satisfying $r + \kappa(r) = \lambda\Omega$ with $\lambda \in R^\times$, i.e. those that remain non-skew symmetric under any base change $R \to K$ with K a field.

Below we list a few properties of r-matrices and their coboundary structures for later use.

Lemma 3.3. *Let r_1 and r_2 be two r-matrices over $\mathfrak{g}$, and let ϕ be a surjective endomorphism of the Lie algebra $\mathfrak{g}$. Then ϕ is a morphism of Lie bialgebras $(\mathfrak{g}, \partial r_1) \to (\mathfrak{g}, \partial r_2)$ if and only if $(\phi \otimes \phi)(r_1) - r_2 \in R\Omega$.*

Proof. Set $s = (\phi \otimes \phi)(r_1) - r_2$. Combining (1.1) and (3.1), one sees that ϕ is a Lie bialgebra morphism if and only if $(\mathrm{ad}_{\phi(a)} \otimes 1 + 1 \otimes \mathrm{ad}_{\phi(a)})(s) = 0$ for all $a \in \mathfrak{g}$. Remark 3.1 then implies that $s \in R\Omega$, since ϕ is surjective. ☐

[3]The notation is motivated by the classical situation where one passes to the universal enveloping algebra $U(\mathfrak{g})$ and sets e.g. $(s \otimes t)_{13} = s \otimes 1 \otimes t$. The bracket then denotes the commutator in $U(\mathfrak{g})^{\otimes 3}$. This is however merely a convenient notation and there is no need to resort to $U(\mathfrak{g})$.

Lemma 3.4. *Let $\phi \in \mathrm{Aut}(\mathfrak{g})$ and let r be an r-matrix on $\mathfrak{g}$. Then ϕ is an automorphism of $(\mathfrak{g}, \partial r)$ if and only if $(\phi \otimes \phi)(r) = r$.*

Proof. From the previous lemma we know that $(\phi \otimes \phi)(r) = r + \mu\Omega$ for some $\mu \in R$. Moreover, r satisfies $r + \kappa(r) = \lambda\Omega$ for some $\lambda \in R$. Thus

$$(\phi \otimes \phi)(r + \kappa(r)) = (\phi \otimes \phi)(r) + (\phi \otimes \phi)\kappa(r) = r + \mu\Omega + \kappa(r) + \mu\Omega = (\lambda + 2\mu)\Omega$$

while the left hand side equals $(\phi \otimes \phi)(\lambda\Omega) = \lambda\Omega$. Thus $2\mu\Omega = 0$, whence $\mu = 0$ by Remark 3.1. □

Lemma 3.5. *Assume that R is an integral domain. Let, for $i = 1, 2$, r_1 and r_2 be two r-matrices with $r_i + \kappa(r_i) = \lambda_i\Omega$ with $\lambda_i \in R^\times$. If $r_2 = r_1 - \mu\Omega$ for some $\mu \in R$, then either $\mu = 0$ or $\mu = \lambda_1$.*

The result and its proof over fields have been communicated to us by A. Stolin. The proof over integral domains given here is almost identical.

Proof. Inserting $r_2 = r_1 - \mu\Omega$ into the equation $\mathrm{CYB}(r_2) = 0$, and simplifying the expression using the fact that $\mathrm{CYB}(r_1) = 0$ and $\kappa(r_1) = \lambda_1\Omega - r_1$, one sees that the equation is equivalent to $\mu(\mu - \lambda_1)[\Omega_{12}, \Omega_{13}] = 0$, where the subscripts are as in the definition of the classical Yang–Baxter operator. Now $[\Omega_{12}, \Omega_{13}]$ is defined over $\mathbb{Q}$ and non-zero in $\mathfrak{g}_0^{\otimes 3}$, whence it is free over R. Thus $\mu(\mu - \lambda_1) = 0$ and we conclude with the assumption on R. □

Lemma 3.6. *Assume that R is an integral domain, let r be an r-matrix on $\mathfrak{g}$ with $r + \kappa(r) = \lambda\Omega$, $\lambda \in R^\times$, and let $\alpha, \beta \in R^\times$. If $(\mathfrak{g}, \partial\alpha r) \simeq (\mathfrak{g}, \partial\beta r)$, then $\beta = \pm\alpha$.*

Proof. Assume that $\phi : (\mathfrak{g}, \partial\alpha r) \simeq (\mathfrak{g}, \partial\beta r)$ is an automorphism. By Lemma 3.3, $(\phi \otimes \phi)(\alpha r) = \beta r + \mu\Omega$ for some $\mu \in R$. As in the proof of Lemma 3.4 we get

$$(\phi \otimes \phi)(\alpha r + \kappa(\alpha r)) = (\beta\lambda + 2\mu)\Omega,$$

while the left hand side equals $(\phi \otimes \phi)(\alpha\lambda\Omega) = \alpha\lambda\Omega$. Thus Remark 3.1 gives $\beta\lambda = \alpha\lambda - 2\mu$. On the other hand, $r' = (\phi \otimes \phi)(\alpha r)$ is an r-matrix with $r' + \kappa(r') = \alpha\lambda\Omega$. Thus Lemma 3.5 implies that $\mu = 0$ or $\mu = \alpha\lambda$. Inserting these cases into $\beta\lambda = \alpha\lambda - 2\mu$ gives $\beta = \pm\alpha$, since λ is invertible. □

Finally, for later use, we recall the split exact sequence

$$\mathbf{1} \longrightarrow \mathbf{G} \xrightarrow{\mathrm{Ad}} \mathbf{Aut}(\mathfrak{g}) \xrightarrow{f} \mathbf{A}_\Gamma \longrightarrow \mathbf{1} \tag{3.4}$$

of affine R-group schemes, where Ad denotes the adjoint representation and $\mathbf{A}_\Gamma$ is the constant group scheme corresponding to the finite abstract group $\mathrm{Aut}(\Gamma)$.

Remark 3.7. This split exact sequence is obtained by base change from a similar split exact sequence of $\mathbb{Z}$-group schemes.

3.2 Belavin–Drinfeld Structures

Let $\mathbf{E}$ be a pinning of $\mathfrak{g}$. By an *admissible quadruple* we mean a quadruple $Q = (\Gamma_1, \Gamma_2, \tau, r_{\mathfrak{h}})$, where $(\Gamma_1, \Gamma_2, \tau)$ is an admissible triple, and $r_{\mathfrak{h}} \in \mathfrak{h} \otimes \mathfrak{h}$ satisfies $r_{\mathfrak{h}} + \kappa(r_{\mathfrak{h}}) = \Omega_{\mathfrak{h}}$ and

$$\forall \alpha \in \Gamma_1 : (\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_{\mathfrak{h}}) = 0.$$

Recall that $(\Gamma_1, \Gamma_2, \tau)$ being admissible means that $\Gamma_1, \Gamma_2 \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is an isometry such that for every $\alpha \in \Gamma_1$ there exists a positive integer k such that $\tau^k(\alpha) \notin \Gamma_1$. The map τ extends to a map $\mathrm{Span}_{\mathbb{Z}}(\Gamma_1) \to \mathrm{Span}_{\mathbb{Z}}(\Gamma_2)$, which is also denoted by τ. Associated to these data is the *Belavin–Drinfeld* r-matrix

$$r_{\mathrm{BD}} = r_{\mathrm{BD}}(\mathbf{E}, Q) = r_{\mathfrak{h}} + \sum_{\alpha \in \Delta^+} X_\alpha \otimes X_{-\alpha} + \sum_{\substack{\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} X_\alpha \wedge X_{-\tau^k(\alpha)}.$$

Remark 3.8. The requirement on $r_{\mathfrak{h}}$ above implies that r_{BD} is defined over R.

We denote by $\mathbf{A}_{\mathrm{BD}}$ the automorphism group of $(\mathfrak{g}, \partial r_{\mathrm{BD}})$ and by $\mathbf{C}_{\mathrm{BD}} \subseteq \mathbf{H}$ the centralizer of r_{BD}; since the action of $\mathbf{H}$ on $\mathfrak{g} \otimes \mathfrak{g}$ is linear we have

$$\mathbf{C}_{\mathrm{BD}}(S) = \{h \in \mathbf{H}(S) | \mathrm{Ad}_h \otimes \mathrm{Ad}_h(r_{\mathrm{BD}}) = r_{\mathrm{BD}}\}$$

for any R-ring S, which, since the induced action of $\mathbf{H}$ on $\mathfrak{h} \otimes \mathfrak{h}$ is trivial, equals

$$\{h \in \mathbf{H}(S) | \mathrm{Ad}_h \otimes \mathrm{Ad}_h(r'_{\mathrm{BD}}) = r'_{\mathrm{BD}}\},$$

where $r'_{\mathrm{BD}} = r_{\mathrm{BD}} - r_{\mathfrak{h}}$ is defined over $\mathbb{Q}$. Thus $\mathbf{C}_{\mathrm{BD}}$ is obtained from an affine $\mathbb{Q}$-group scheme by base change. Since any $\mathbb{Q}$-group scheme is smooth as $\mathbb{Q}$ is a field of characteristic zero, and since smoothness is preserved by base change, this proves the following.

Lemma 3.9. *The group $\mathbf{C}_{\mathrm{BD}}$ is smooth.*

Given an admissible quadruple $Q = (\Gamma_1, \Gamma_2, \tau, r_{\mathfrak{h}})$, we denote by $\mathbf{A}_\Gamma^Q$ the closed subgroup of $\mathbf{A}_\Gamma$ defined by the equations

$$\pi(\Gamma_1) = \Gamma_1, \quad \pi\tau = \tau\pi, \quad \text{and} \quad (\widehat{\pi} \otimes \widehat{\pi})(r_{\mathfrak{h}}) = r_{\mathfrak{h}}.$$

Here $\widehat{\pi}$ is the image of π under the (unique) splitting $s : \mathbf{A}_\Gamma \to \mathbf{Aut}(\mathbf{G})$ leaving invariant each of $\mathbf{H}$ and $\mathbf{E}$; this splitting exists by [SGA3, XXIII.5.5].

Theorem 3.10. *Let $\mathbf{G}$ be a split simple adjoint R-group with $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Let $\mathbf{E} = (\mathbf{H}, \Gamma, (X_\alpha)_{\alpha \in \Gamma})$ be a pinning of $\mathbf{G}$. Fix an admissible quadruple $Q = (\Gamma_1, \Gamma_2, \tau, r_{\mathfrak{h}})$ and consider $r_{\mathrm{BD}} = r_{\mathrm{BD}}(\mathbf{E}, Q)$. Then the sequence (3.4) induces a split exact sequence*

$$\mathbf{1} \longrightarrow \mathbf{C}_{\mathrm{BD}} \xrightarrow{\mathrm{Ad}} \mathbf{A}_{\mathrm{BD}} \xrightarrow{f} \mathbf{A}_\Gamma^Q \longrightarrow \mathbf{1}$$

of affine group schemes.

Proof. Via the above splitting $s : \mathbf{A}_\Gamma \to \mathbf{Aut}(\mathbf{G}), \pi \mapsto \widehat{\pi}$, $\mathbf{A}_\Gamma$ acts on $\mathbf{H}$ in such a way that for any R-ring S and any $h \in \mathbf{H}(S)$ and $\pi \in \mathbf{A}_\Gamma(S)$,

$$\mathrm{Ad}_{\pi\cdot h} = \widehat{\pi}\mathrm{Ad}_h\widehat{\pi}^{-1}.$$

If $\pi \in \mathbf{A}_\Gamma^Q(S)$, then by definition of $\mathbf{A}_\Gamma^Q$, $\widehat{\pi} \in \mathbf{A}_{\mathrm{BD}}$, whence for any $h \in \mathbf{C}_{\mathrm{BD}}(S)$ we have $\pi \cdot h \in \mathbf{C}_{\mathrm{BD}}(S)$. Hence the action induces an action of $\mathbf{A}_\Gamma^Q$ on $\mathbf{C}_{\mathrm{BD}}$ and we may form the semi-direct product $\mathbf{C}_{\mathrm{BD}} \cdot \mathbf{A}_\Gamma^Q$ with respect to this action, and we have a group morphism

$$\mathrm{Ad} \times s : \mathbf{C}_{\mathrm{BD}} \cdot \mathbf{A}_\Gamma^Q \to \mathbf{A}_{\mathrm{BD}}.$$

To prove the theorem it suffices to show that $\mathrm{Ad} \times s$ is an isomorphism. The group $\mathbf{C}_{\mathrm{BD}}$ is smooth (hence flat), and $\mathbf{A}_\Gamma^Q$ is a closed subgroup of a finite constant group, hence flat as well. Thus their semi-direct product is flat, and the fiber-wise isomorphism criterion [EGAIV, $_4$17.9.5] reduces the problem to the case where $R = K$ is a field of characteristic zero. In that case the above group schemes are smooth, and it suffices, by [KMRT, 22.5], to show that $(\mathrm{Ad} \times s)(\overline{K}) : \mathbf{C}_{\mathrm{BD}}(\overline{K}) \cdot \mathbf{A}_\Gamma^Q(\overline{K}) \to \mathbf{A}_{\mathrm{BD}}(\overline{K})$ is an isomorphism of abstract groups. This latter statement holds by the following lemma, which completes the proof. □

Lemma 3.11. *Let K be an algebraically closed field of characteristic zero. The map* $\mathrm{Ad} \times s$ *defines an isomorphism of groups* $\mathbf{C}_{\mathrm{BD}}(K) \cdot \mathbf{A}_\Gamma^Q(K) \to \mathbf{A}_{\mathrm{BD}}(K)$.

Proof. The map in question being the restriction of an injective homomorphism, it only remains to be shown that it is surjective. Let $\phi \in \mathbf{A}_{\mathrm{BD}}(K)$. Then $\phi = \widehat{\pi}\mathrm{Ad}_g$ for some $g \in \mathbf{G}(K)$ and $\pi \in \mathbf{A}_\Gamma(K)$. As a first step we will show that necessarily, $g \in \mathbf{H}(K)$, building on an argument from [KKPS]. Indeed, consider the isomorphism

$$\Xi : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}_K(g)$$

defined by sending $a \otimes b$ to the linear map $u \mapsto \langle a, u\rangle b$, where $\langle,\rangle$ is the Killing form on $\mathfrak{g}$. Now $\Xi((\phi \otimes \phi)(r_{\mathrm{BD}})) = \phi\Xi(r_{\mathrm{BD}})\phi^{-1}$. Hence, by Lemma 3.4, ϕ is an automorphism of $(\mathfrak{g}, \partial r_{\mathrm{BD}})$ if and only if ϕ commutes with $\Xi(r_{\mathrm{BD}})$, which holds if and only if ϕ commutes with its semisimple and nilpotent parts. Denote by D the semisimple part of $\Xi(r_{\mathrm{BD}})$. It is shown in [KKPS, Proof of Theorem 1] that the Borel subalgebra $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is the normalizer of the eigenspace of D corresponding to the eigenvalue 0 (resp. 1) (recall that we have fixed a pinning). Thus if $\phi \in \mathbf{A}_{\mathrm{BD}}$, then ϕ must preserve $\mathfrak{b}^+$ and $\mathfrak{b}^-$. Since this is true for $\widehat{\pi}$, it follows that Ad_g must preserve $\mathfrak{b}^+$ and $\mathfrak{b}^-$. The end of the proof of Theorem 1 of [KKPS] now applies to yield that this implies that $g \in \mathbf{H}(K)$.
Next we will show that $\pi \in \mathbf{A}_\Gamma^Q$. Once this is done, it follows from $(\phi \otimes \phi)(r_{\mathrm{BD}}) = r_{\mathrm{BD}}$ that $\mathrm{Ad}_g \otimes \mathrm{Ad}_g(r_{\mathrm{BD}}) = r_{\mathrm{BD}}$, i.e. that $g \in \mathbf{C}_{\mathrm{BD}}(K)$, and the

proof becomes complete. Now, $(\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(r_{\mathfrak{h}}) = r_{\mathfrak{h}}$, while $\mathrm{Ad}_g(X_\alpha) = \alpha(g)X_\alpha$ and $\mathrm{Ad}_g(X_{-\alpha}) = \alpha(g)^{-1}X_\alpha$. Thus $(\phi \otimes \phi)$ maps r_{BD} to

$$(\widehat{\pi} \otimes \widehat{\pi})(r_{\mathfrak{h}}) + \sum_{\alpha \in \Delta^+} X_\alpha \otimes X_{-\alpha} + \sum_{\substack{\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} \lambda_{\alpha,k}(g) X_{\pi(\alpha)} \wedge X_{-\pi\tau^k(\alpha)},$$

where $\lambda_{\alpha,k}(g) = \alpha(g)(\tau^k(\alpha)(g))^{-1} \in K^\times$. Thus $(\widehat{\pi} \otimes \widehat{\pi})(r_{\mathfrak{h}}) = r_{\mathfrak{h}}$, and

$$\sum_{\substack{\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} \lambda_{\alpha,k}(g) X_{\pi(\alpha)} \wedge X_{-\pi\tau^k(\alpha)} = \sum_{\substack{\beta \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ l>0}} X_\beta \wedge X_{-\tau^l(\beta)}.$$

Now π and τ map positive roots to positive roots, and the set $\{X_\alpha\}_{\alpha \in \Delta}$ is linearly independent in $\mathfrak{g}$. Thus for the above equality to hold, π must preserve the intersection of $\mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+$ with Δ and thus it preserves $\mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \cap \Gamma = \Gamma_1$. Thus for any $\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma)$, the sum of all terms with first component $X_{\pi(\alpha)}$ in the left-hand sum equals the sum of all terms with first component X_β in the right-hand sum, for $\beta = \pi(\alpha)$, whence

$$\sum_{k>0} \lambda_{\alpha,k}(g) X_{-\pi\tau^k(\alpha)} = \sum_{l>0} X_{-\tau^l\pi(\alpha)}$$

for all $\alpha \in \mathrm{Span}_{\mathbb{Z}}\Gamma_1^+$. We want to prove that this implies that $\pi\tau = \tau\pi$. For this it is enough to show that $\pi\tau(\alpha) = \tau\pi(\alpha)$ for all $\alpha \in \Gamma_1$. If $\Gamma_1 = \emptyset$, there is nothing to prove. Assume thus that $\Gamma_1 \neq \emptyset$. For any $\alpha \in \Gamma_1$, the definition of admissibility implies that there exists a unique integer $l_\alpha \geq 1$ such that $\tau^{l_\alpha}(\alpha) \notin \Gamma_1$ and $\tau^k(\alpha) \in \Gamma_1$ for $k < l_\alpha$. Since $\pi(\Gamma_1) = \Gamma_1$, the above equality and the construction of a Chevalley basis implies that all $\lambda_{\alpha,k} = 1$, that both sums have l_α terms, and that

$$\{\pi\tau(\alpha), \pi\tau^2(\alpha), \ldots, \pi\tau^{l_\alpha}(\alpha)\} = \{\tau\pi(\alpha), \tau^2\pi(\alpha), \ldots, \tau^{l_\alpha}\pi(\alpha)\}.$$

Since $l_{\tau(\alpha)} = l_\alpha - 1$, the set of all l_α is a segment of integers starting at 1. We may thus proceed by induction on l_α. If $l_\alpha = 1$, the above sets are singletons, giving the base of the induction. For $l_\alpha > 1$, by the induction hypothesis the left hand side is equal to

$$\{\pi\tau(\alpha), \tau\pi\tau(\alpha), \ldots, \tau\pi\tau^{l_\alpha - 1}(\alpha)\}.$$

Thus for the above equality of sets to hold, either $\tau\pi(\alpha) = \pi\tau(\alpha)$, or $\tau\pi(\alpha) = \tau\pi\tau^j(\alpha)$ for some $j > 0$. This second case is however not possible, since by injectivity of $\tau\pi$ it implies $\alpha = \tau^j(\alpha)$ for some $j > 0$, which violates admissibility. Thus $\pi\tau(\alpha) = \tau\pi(\alpha)$, and the result follows by induction. This completes the proof. □

3.3 Standard Structure

Associated to a pinning $\mathbf{E}$ is the *Drinfeld–Jimbo* r-matrix

$$r_{\mathrm{DJ}} = r_{\mathrm{DJ}}(\mathbf{E}) = \frac{1}{2}\Omega_{\mathfrak{h}} + \sum_{\alpha\in\Delta^+} X_\alpha \otimes X_{-\alpha}.$$

In other words $r_{\mathrm{DJ}} = r_{\mathrm{BD}}(\mathbf{E}, Q)$ for the trivial quadruple $Q = (\emptyset, \emptyset, \mathrm{Id}_\emptyset, \frac{1}{2}\Omega_{\mathfrak{h}})$.

Lemma 3.12. *If* $\mathbf{E}$ *and* $\mathbf{E}'$ *are pinnings of* $\mathbf{G}$*, then the Lie bialgebras* $(\mathfrak{g}, \partial r_{\mathrm{DJ}}(\mathbf{E}))$ *and* $(\mathfrak{g}, \partial r_{\mathrm{DJ}}(\mathbf{E}'))$ *are isomorphic.*

Proof. If $\mathbf{E}$ and $\mathbf{E}'$ are two pinnings of $\mathbf{G}$, then there is an automorphism of $\mathbf{G}$ mapping $\mathbf{E}$ to $\mathbf{E}'$ by [SGA3, XXIII.5.1]. As $\Omega_{\mathfrak{h}}$ is determined by $\mathfrak{h}$ and Ω, there is thus an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ with $(\phi \otimes \phi)(r_{\mathrm{DJ}}(\mathbf{E})) = r_{\mathrm{DJ}}(\mathbf{E}')$, and hence ϕ is an isomorphism of Lie bialgebras. □

Following [ES], we use the term *standard* or *Drinfeld–Jimbo* for the Lie bialgebra structure $\partial r_{\mathrm{DJ}}(\mathbf{E})$ on $\mathfrak{g}$. Up to isomorphism, the above lemma shows that it is independent of the choice of $\mathbf{E}$, and is thus determined by $\mathfrak{g}$. Classifying twisted forms of standard Lie bialgebras, i.e. Lie bialgebras $(\mathfrak{g}', \delta')$ such that $(\mathfrak{g}'_S, \delta'_S) \simeq (\mathfrak{g}_S, (\partial r_{\mathrm{DJ}})_S)$ for a split Lie algebra $\mathfrak{g}$ with standard Lie bialgebra structure ∂r_{DJ} and a faithfully flat R-ring S amounts, by Corollary 2.5, to describing the set $H^1(S/R, \mathbf{A})$, where $\mathbf{A}$ is the automorphism R-group scheme of $(\mathfrak{g}, \partial r_{\mathrm{DJ}})$. For this group scheme, Theorem 3.10 has the following consequence.

Corollary 3.13. *Let* $\mathbf{G}$ *be a split simple adjoint* R*-group with* $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$*, and let* $\mathbf{E} = (\mathbf{H}, \Gamma, (X_\alpha)_{\alpha\in\Gamma})$ *be a pinning of* $\mathbf{G}$*. Then the sequence* (3.4) *induces a split exact sequence*

$$\mathbf{1} \longrightarrow \mathbf{H} \xrightarrow{\mathrm{Ad}} \mathbf{A} \xrightarrow{f} \mathbf{A}_\Gamma \longrightarrow \mathbf{1} \tag{3.5}$$

of affine group schemes.

Proof. Any $h \in \mathbf{H}(S)$, with S an R-ring, maps $X_\alpha \otimes X_{-\alpha}$ to itself. Thus in the terminology of Theorem 3.10, $\mathbf{C}_{\mathrm{BD}} = \mathbf{H}$, and by construction $\mathbf{A}_\Gamma^Q = \mathbf{A}_\Gamma$. We conclude with Theorem 3.10. □

4 Standard Lie Bialgebras over Fields

We now specialize to the case where $R = K$ is a field (of characteristic zero), and consider $\overline{K}/K$-twisted forms of standard Lie bialgebras. Recall that over K, two (bi)algebras whose underlying modules are finitely generated are locally isomorphic with respect to the fppf topology if and only if they become isomorphic after scalar extension to $\overline{K}$. If $\mathbf{A}$ is an algebraic group over K, then $H^1_{\mathrm{Gal}}(K, \mathbf{A})$ stands in bijection to K-isomorphism classes of $\mathbf{A}$-torsors that become trivial over $\overline{K}$. In order to classify twisted forms of standard Lie bialgebras, we thus wish to compute $H^1_{\mathrm{Gal}}(K, \mathbf{A})$, where $\mathbf{A}$ is the automorphism group scheme of the standard Lie bialgebra $(\mathfrak{g}, \partial r_{\mathrm{DJ}})$. We can in fact prove the following.

Theorem 4.1. *The map f of (3.5) induces an isomorphism of pointed sets*

$$f^* : H^1_{\mathrm{Gal}}(K, \mathbf{A}) \xrightarrow{\sim} H^1_{\mathrm{Gal}}(K, \mathbf{A}_\Gamma).$$

Proof. The split exact sequence (3.5) induces an exact sequence

$$H^1_{\mathrm{Gal}}(K, \mathbf{H}) \xrightarrow{\mathrm{Ad}^*} H^1_{\mathrm{Gal}}(K, \mathbf{A}) \xrightarrow{f^*} H^1_{\mathrm{Gal}}(K, \mathbf{A}_\Gamma)$$

of pointed sets. The map f^* is surjective since, as a consequence of Remark 3.7, the sequence (3.5) is split by a Galois equivariant section. To prove injectivity of f^*, let $c \in H^1_{\mathrm{Gal}}(K, \mathbf{A})$. The set of all $c' \in H^1_{\mathrm{Gal}}(K, \mathbf{A})$ with $f^*(c) = f^*(c')$ is in bijection with a quotient of the set $H^1_{\mathrm{Gal}}(K, {}_{f^*(c)}\mathbf{H})$ by a certain equivalence relation, where ${}_{f^*(c)}$ indicates a twisted $\mathrm{Gal}(K)$-action. As $\mathbf{A}_\Gamma$ acts on $\mathbf{H}$ via permutation of the roots, ${}_{f^*(c)}\mathbf{H}$ is a permutation torus (i.e. the Weil restriction of a split torus $\mathbf{T}$ over some finite field extension $K \subset L$). By Shapiro's Lemma, $H^1_{\mathrm{Gal}}(K, {}_{f^*(c)}\mathbf{H}) = H^1_{\mathrm{Gal}}(L, \mathbf{T})$, which is trivial by Hilbert's Ninetieth Theorem. □

An immediate consequence is the following.

Corollary 4.2. *Assume that $\mathfrak{g}$ is of type A_1, B_n or C_n for any n, E_7, E_8, F_4 or G_2. Then all twisted forms of $(\mathfrak{g}, \partial r_{DJ})$ are isomorphic.*

Proof. The assumption implies that $\mathbf{A}_\Gamma$ is the trivial group, whence the triviality of $H^1_{\mathrm{Gal}}(K, \mathbf{A}_\Gamma)$ and, by the above theorem, of $H^1_{\mathrm{Gal}}(K, \mathbf{A})$. □

5 Scalar Multiples

The Belavin–Drinfeld classification is up to equivalence, which, as explained in the introduction, groups together scalar multiples of Lie bialgebra structures on a given Lie algebra. Therefore, it is reasonable to consider the set $B_\alpha(K)$ of K-Lie bialgebra structures on $\mathfrak{g}$ that, after scalar extension to $\overline{K}$, become isomorphic to $\alpha \partial r_{\mathrm{BD}}$ for some $\alpha \in \overline{K}^\times$ and some Belavin–Drinfeld r-matrix r_{BD}.

We start in a more general setting. Let $\mathfrak{g}$ be a split simple Lie algebra over an integral domain R (as always assumed to be a $\mathbb{Q}$-ring), let δ be an R-Lie bialgebra structure on $\mathfrak{g}$, and let $\alpha \in S^\times$ with S a faithfully flat R-ring. Consider the Lie bialgebra structure $\alpha\delta_S$ on $\mathfrak{g}_S$. It is worth noting that a priori, there are three possible scenarios for the descent properties of $\alpha\delta_S$. Recall that we wish to consider Lie bialgebra structures δ' on S/R-twisted forms $\mathfrak{g}'$ of $\mathfrak{g}$ such that $(\mathfrak{g}'_S, \delta'_S) \simeq (\mathfrak{g}_S, \alpha\delta_S)$. We phrase this by saying that $\alpha\delta_S$ *descends* to $\mathfrak{g}'$. Then either $\alpha\delta_S$ descends to $\mathfrak{g}$, or it does not descend to $\mathfrak{g}$, but descends to a (non-split) twisted form of $\mathfrak{g}$, or $\alpha\delta_S$ does not descend to any form of $\mathfrak{g}$.

Remark 5.1. Let δ_e be an S-Lie bialgebra structure on $\mathfrak{g}_S$. Then any R-Lie bialgebra $(\mathfrak{g}', \delta')$ with $(\mathfrak{g}'_S, \delta'_S) \simeq (\mathfrak{g}_S, \delta_e)$ is R-isomorphic to the restriction δ_e^θ of δ_e to

$$\mathfrak{g}_S^\theta = \{x \in \mathfrak{g} \otimes S | \theta(x \otimes 1) = x \otimes 1\}$$

for some descent datum θ of Lie algebras. Indeed, if θ' denotes the standard descent datum on $(\mathfrak{g}'_S, \delta'_S)$ and $\phi : (\mathfrak{g}'_S, \delta'_S) \to (\mathfrak{g}_S, \delta_e)$ is an S-isomorphism, then $\theta = (\phi \otimes \mathrm{Id}_S)\theta'(\phi^{-1} \otimes \mathrm{Id}_S)$ is a descent datum on $(\mathfrak{g}_S, \delta_e)$. Thus δ_e restricts to an R-bialgebra structure on $\mathfrak{g}_S^\theta$. This is R-isomorphic to $(\mathfrak{g}', \delta')$ since by construction of θ, the map ϕ is an isomorphism of S-Lie bialgebras with descent data between $(\mathfrak{g}'_S, \delta'_S)$ with datum θ' and $(\mathfrak{g}_S, \delta_e)$ with datum θ. Thus to classify R-Lie bialgebras that become isomorphic to $(\mathfrak{g}_S, \delta_e)$ over S, it suffices to consider restrictions of δ_e itself to twisted forms of $\mathfrak{g}$. This will be used throughout for the case $\delta_e = \alpha\delta_S$

5.1 Twisted Cohomology

The following proposition sheds some light on the occurrence of the possible cases discussed in the opening of this section. Our approach extends that used for so called twisted Belavin–Drinfeld cohomologies, see e.g. [KKPS, Section 7]. We begin with the case when $R = K$ is a field, where Galois cohomology gives a rather precise insight, and treat the more general case below.

Proposition 5.2. *Let $\mathfrak{g}$ be a split simple Lie algebra over K and let $\delta = \partial r$ for an r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ with $r + \kappa(r) = \lambda\Omega$ for some $\lambda \in K^\times$. Let $\alpha \in \overline{K}^\times$. Finally let $\mathfrak{g}'$ be a twisted form of $\mathfrak{g}$ and (u_γ) the corresponding Γ-cocycle, where $\Gamma = \mathrm{Gal}(K)$. Then $\alpha\overline{\delta}$ descends to $\mathfrak{g}'$ if and only if one of the following mutually exclusive conditions holds.*

1. *$\alpha \in K^\times$ and $(u_\gamma \otimes u_\gamma)(r) = r$ for each $\gamma \in \Gamma$.*
2. *$\alpha^2 \in K^\times \setminus (K^\times)^2$ and*

$$(u_\gamma \otimes u_\gamma)(r) = \begin{cases} r & \text{if } \gamma \in \mathrm{Gal}(K(\alpha)) \\ \kappa(r) & \text{if } \gamma(\alpha) = -\alpha. \end{cases} \tag{5.1}$$

Here we are using the notation $\overline{\mathfrak{g}} = \mathfrak{g}_{\overline{K}}$ and $\overline{\delta} = \delta_{\overline{K}}$.

Proof. Assume that $\alpha\overline{\delta}$ descends to $\mathfrak{g}'$. Then $\overline{\delta}(x) \in \mathfrak{g}' \otimes \mathfrak{g}'$ for any $x \in \mathfrak{g}'$, which in terms of cocycles implies

$$(u_\gamma \otimes u_\gamma)^{\gamma\otimes\gamma}(\alpha\overline{\delta}(x)) = \alpha\overline{\delta}(x)$$

for each $\gamma \in \Gamma$. Using the fact that $\delta = \partial r$, that ${}^{\gamma\otimes\gamma}r = r$ since $r \in \mathfrak{g} \otimes \mathfrak{g}$, and that $u_\gamma({}^\gamma x) = x$ since $x \in \mathfrak{g}'$, this is equivalent to

$$[1 \otimes x + x \otimes 1, \gamma(\alpha)(u_\gamma \otimes u_\gamma)(r)] = [1 \otimes x + x \otimes 1, \alpha r]$$

for each $\gamma \in \Gamma$. This is in turn equivalent to

$$\gamma(\alpha)(u_\gamma \otimes u_\gamma)(r) = \alpha r - \mu_\gamma \Omega \tag{5.2}$$

for some $\mu_\gamma \in K$. Now $\mathrm{CYB}(\alpha r) = \alpha^2 \mathrm{CYB}(r) = 0$ and

$$\mathrm{CYB}(\gamma(\alpha)(u_\gamma \otimes u_\gamma)(r)) = \gamma(\alpha)^2 u_\gamma^{\otimes 3} \mathrm{CYB}(r) = 0.$$

Since

$$(u_\gamma \otimes u_\gamma)(r) + \kappa(u_\gamma \otimes u_\gamma)(r) = (u_\gamma \otimes u_\gamma)(r + \kappa(r)) = \Omega,$$

we may, after extending scalars to $\overline{K}$, apply Lemma 3.5 with $r_1 = \alpha r$, $r_2 = \gamma(\alpha)(u_\gamma \otimes u_\gamma)(r)$, $\lambda_1 = \alpha\lambda$, $\lambda_2 = \gamma(\alpha)\lambda$ and $\mu = \mu_\gamma$. Thus we get $\mu_\gamma = j_\gamma \alpha \lambda$ for some $j_\gamma = 0, 1$. Since $r + \kappa(r) = \lambda\Omega$, (5.2) is then equivalent to

$$\gamma(\alpha)(u_\gamma \otimes u_\gamma)(r) = \alpha(-\kappa)^{j_\gamma}(r).$$

Applying κ to both sides gives

$$\gamma(\alpha)(u_\gamma \otimes u_\gamma)\kappa(r) = \alpha(-1)^{j_\gamma}\kappa^{j_\gamma+1}(r),$$

and adding the two equations, using $r + \kappa(r) = \lambda\Omega$ and $(u_\gamma \otimes u_\gamma)(\Omega) = \Omega$, yields

$$\gamma(\alpha)\Omega = (-1)^{j_\gamma}\alpha\Omega.$$

Thus $\gamma(\alpha^2) = \alpha^2$ for each $\gamma \in \Gamma$, whence $\alpha^2 \in K^\times$. For a given $\gamma \in \Gamma$, two cases are possible: either $j_\gamma = 0$, i.e. equivalently $\gamma(\alpha) = \alpha$, and then (5.2) gives $(u_\gamma \otimes u_\gamma)(r) = r$; or $j_\gamma = 1$, i.e. equivalently $\gamma(\alpha) = -\alpha$, and then (5.2) gives $(u_\gamma \otimes u_\gamma)(r) = \kappa(r)$. It follows that if $\alpha\overline{\delta}$ descends to $\mathfrak{g}'$, then (1) or (2) holds.

Assume, conversely, that (1) or (2) holds. It is straightforward to check that (5.2) is satisfied for each $\gamma \in \Gamma$, with $\mu_\gamma = 0$ if $\gamma(\alpha) = \alpha$, and $\mu_\gamma = \alpha\lambda$ if $\gamma(\alpha) = -\alpha$. By the chain of equivalences in the above argument, this implies that

$$(u_\gamma \otimes u_\gamma)^{\gamma\otimes\gamma}(\alpha\overline{\delta}(x)) = \alpha\overline{\delta}(x).$$

We may then conclude with the lemma below. □

Lemma 5.3. *Let $\mathfrak{g}$ be a finite-dimensional vector space over a field K, (u_γ) a $\Gamma = \mathrm{Gal}(K)$-cocycle in $\mathbf{GL}(\mathfrak{g})$ with corresponding twisted form $\mathfrak{g}'$, and δ a Lie coalgebra structure on $\overline{\mathfrak{g}}$. Then δ descends to $\mathfrak{g}'$ if and only if*

$$(u_\gamma \otimes u_\gamma)^{\gamma\otimes\gamma}(\delta(x)) = \delta(x)$$

for each $x \in \mathfrak{g}'$.

Remark 5.4. Of course $\mathfrak{g} \simeq \mathfrak{g}'$ as K-vector spaces. What the lemma achieves is to establish an easily checked condition for when the coalgebra structure δ is compatible with the Galois action.

Proof. What needs to be shown is that the inclusion $\mathfrak{g}' \otimes \mathfrak{g}' \subseteq (\overline{\mathfrak{g}} \otimes \overline{\mathfrak{g}})^{\Gamma'}$ is an equality, where the right-hand side denotes the fixed points of $\overline{\mathfrak{g}} \otimes \overline{\mathfrak{g}}$ under the component-wise twisted Γ-action

$$\gamma \cdot_u (y \otimes z) = u_\gamma({}^\gamma y) \otimes u_\gamma({}^\gamma z).$$

Let (e_i) be a K-basis of $\mathfrak{g}'$. It is then finite by assumption, and a $\overline{K}$-basis of $\overline{\mathfrak{g}}$. Thus if $x \in \overline{\mathfrak{g}} \otimes \overline{\mathfrak{g}}$, then

$$x = \sum_{i,j} \alpha_{ij} e_i \otimes e_j$$

for some $\alpha_{ij} \in \overline{K}$. Then for each $\gamma \in \Gamma$,

$$\gamma \cdot_u x = \sum_{i,j} \gamma(\alpha_{ij}) u_\gamma({}^\gamma e_i) \otimes u_\gamma({}^\gamma e_j) = \sum_{i,j} \gamma(\alpha_{ij}) e_i \otimes e_j,$$

where the last equality holds since $e_k \in \mathfrak{g}'$ implies $u_\gamma({}^\gamma e_k) = e_k$ for each k. If $x \in (\overline{\mathfrak{g}} \otimes \overline{\mathfrak{g}})^{\Gamma'}$, then for each $\gamma \in \Gamma$, $\gamma \cdot_u x = x$, which, together with the above, by linear independence implies that $\gamma(\alpha_{ij}) = \alpha_{ij}$, i.e. $\alpha_{ij} \in K$, for each i and j. Thus $x \in \mathfrak{g}' \otimes \mathfrak{g}'$, as desired. □

Remark 5.5. Note that this result encompasses those Lie bialgebras that in [KKPS] and [PS] are treated by means of twisted Belavin–Drinfeld cohomologies. Indeed, when they exist, these are obtained by constructing the cocycle (u_γ) as follows: one takes $\alpha \in \overline{K}^\times$ with $\alpha^2 \in K^\times \setminus (K^\times)^2$, and sets $u_\gamma = \mathrm{Id}$ for any $\gamma \in \mathrm{Gal}(K(\alpha))$, and $u_{\gamma_\alpha} = \mathrm{Ad}_X^{-1\gamma_\alpha}\mathrm{Ad}_X$ for $X \in \mathbf{G}(\overline{K})$ satisfying certain conditions. (Here $\mathbf{G}$ is an adjoint group with $\mathfrak{g} = \mathrm{Lie}\mathbf{G}$, and γ_α is the non-trivial element of $\mathrm{Gal}(K(\alpha)/K)$.) Note that, as the authors remark in [PS], this cocycle is trivial as a Lie algebra cocycle, i.e. the fixed locus is the split Lie algebra $\mathfrak{g}$. However, the descended Lie bialgebra is, by Lemma 3.6, not isomorphic to $(\mathfrak{g}, \beta\partial r)$ for any $\beta \in K^\times$.

Led by the above, we define twisted cohomologies as follows. For each $\alpha \in \overline{K}^\times$ with $\alpha^2 \in K^\times \setminus (K^\times)^2$ we write $\overline{Z}^1_\alpha = \overline{Z}^1(K, \mathbf{Aut}(\mathfrak{g}, \partial r), \alpha)$ for the set of all $\mathrm{Gal}(K)$-cocycles (u_γ) in $\mathbf{Aut}(\mathfrak{g})$ that satisfy (5.1). Thus $\overline{Z}^1_\alpha \subset \mathbf{Aut}(\mathfrak{g})(\overline{K})$, and we define an equivalence relation $\sim$ on $\overline{Z}^1_\alpha$ by

$$(v_\gamma) \sim (u_\gamma) \iff \exists \rho \in \mathbf{Aut}(\mathfrak{g}, \partial r)(\overline{K}) : \forall \gamma : v_\gamma = \rho^{-1} u_\gamma {}^\gamma \rho \tag{5.3}$$

and write $\overline{H}^1_{\mathrm{Gal}}(K, \mathbf{Aut}(\mathfrak{g}, \partial r), \alpha)$ for the set of equivalence classes. The purpose of this set is explained by the following result.

Proposition 5.6. *Let $\alpha \in \overline{K}^\times$. The set of all K-Lie bialgebras that become isomorphic to $(\overline{\mathfrak{g}}, \alpha\partial r)$ over $\overline{K}$ is in bijection with*

1. $H^1_{\mathrm{Gal}}(K, \mathbf{Aut}(\mathfrak{g}, \partial r))$, *if* $\alpha \in K^\times$,

2. *$\overline{H}^1_{\mathrm{Gal}}(K, \mathbf{Aut}(\mathfrak{g}, \partial r), \alpha)$, if $\alpha^2 \in K^\times \setminus (K^\times)^2$, and*

3. *the empty set, otherwise.*

Proof. Corollary 2.5 and the fact that $\mathbf{Aut}(\mathfrak{g}, \alpha\partial r)) \simeq \mathbf{Aut}(\mathfrak{g}, \partial r))$ whenever $\alpha \in K^\times$ together imply (1), and (3) follows from Proposition 5.2. For (2), the proposition provides a surjective map from the set of all K-Lie bialgebras that become isomorphic to $(\overline{\mathfrak{g}}, \alpha\partial r)$ to $\overline{Z}^1$. To show that it induces a bijection with the twisted cohomology, let first $\mathfrak{g}'$ and $\mathfrak{g}''$ be twisted forms of $\mathfrak{g}$ to which $\alpha\overline{\delta}$ descend (where $\delta = \partial r$), and let (u_γ) and (v_γ) be the cocycles corresponding to $\mathfrak{g}'$ and $\mathfrak{g}''$, respectively. Denote by δ' and δ'' the respective descended Lie bialgebra structures.
If $(\mathfrak{g}', \delta') \simeq (\mathfrak{g}'', \delta'')$, then there is a $\overline{K}$-automorphism ρ of $(\overline{\mathfrak{g}}, \alpha\partial r)$ that maps $\overline{\mathfrak{g}}^{\Gamma_u}$ to $\overline{\mathfrak{g}}^{\Gamma_v}$. (We denote by $\overline{\mathfrak{g}}^{\Gamma_u}$ the set of all $x \in \overline{\mathfrak{g}}$ that are fixed under the twisted action $\gamma \cdot x = u_\gamma {}^\gamma x$ of $\Gamma = \mathrm{Gal}(K)$, and likewise for Γ_v.) Then for any $\gamma \in \Gamma$ and $x \in \overline{\mathfrak{g}}^{\Gamma_u}$,

$$\rho(u_\gamma {}^\gamma x) = \rho(x) = v_\gamma {}^\gamma \rho(x).$$

Since $\overline{\mathfrak{g}}$ is generated as a $\overline{K}$-vector space by $\overline{\mathfrak{g}}^{\Gamma_u}$, this implies that $u_\gamma = \rho^{-1} v_\gamma ({}^\gamma\rho)$, i.e. $(u_\gamma) \sim (v_\gamma)$, since the $\overline{K}$-automorphisms of $(\overline{\mathfrak{g}}, \alpha\partial r)$ coincide with those of $(\overline{\mathfrak{g}}, \partial r)$. If conversely $(u_\gamma) \sim (v_\gamma)$ with ρ satisfying $u_\gamma = \rho^{-1} v_\gamma ({}^\gamma\rho)$, then it follows that ρ maps $\overline{\mathfrak{g}}^{\Gamma_u}$ to $\overline{\mathfrak{g}}^{\Gamma_v}$ and thus induces an isomorphism $(\mathfrak{g}', \delta') \to (\mathfrak{g}'', \delta'')$. ☐

In the more general case where R is an integral domain, Proposition 5.2 admits the following generalization.

Proposition 5.7. *Assume that R is an integral domain. Let $\mathfrak{g}$ be a split simple Lie algebra over R, and let $\mathfrak{g}' = \mathfrak{g}^\theta_S$ be the S/R-twisted form of $\mathfrak{g}$ corresponding to the descent datum θ and cocycle ϕ. Let $\delta = \partial r$ be a coboundary R-Lie bialgebra structure on $\mathfrak{g}$, with $r \in \mathfrak{g} \otimes \mathfrak{g}$ an r-matrix satisfying $r + \kappa(r) = \lambda\Omega$ for some $\lambda \in R^\times$. Finally let S be a faithfully flat R ring and $\alpha \in S^\times$. Then $\alpha\delta_S$ descends to $\mathfrak{g}'$ if and only if one of the following mutually exclusive conditions holds.*

1. *$\alpha \in R^\times$ and $(\phi \otimes \phi)(r \otimes 1 \otimes 1) = r \otimes 1 \otimes 1$,*

2. *$\alpha^2 \in R^\times \setminus (R^\times)^2$ and $(\phi \otimes \phi)(r \otimes 1 \otimes 1) = \kappa(r) \otimes 1 \otimes 1$.*

Let us give some explanation of the terminology of the proposition. If θ is a descent datum on $\mathfrak{g}_S$, then $\theta = (\mathrm{Id}_\mathfrak{g} \otimes \kappa)\phi$ for some $\phi \in \mathrm{Aut}(\mathfrak{g} \otimes S \otimes S)$ (recall that $(\mathrm{Id}_\mathfrak{g} \otimes \kappa)$ is the standard descent datum, corresponding to the S/R-form $\mathfrak{g}$). This automorphism is the *cocycle corresponding to* θ. We thus have

$$\mathfrak{g}' \simeq \{x \in \mathfrak{g} \otimes S | \phi(x \otimes 1) = (\mathrm{Id}_\mathfrak{g} \otimes \kappa)(x \otimes 1)\}.$$

We will also identify, for any ring T, T-ring U, and T-module M, $(M_U) \otimes_U (M_U)$ with $(M \otimes_T M) \otimes_T U$.

Proof. Write $T = S\otimes S$. Assume that $\alpha\delta_S$ descends to $\mathfrak{g}'$. Then $\alpha\delta_S(x) \in \mathfrak{g}'\otimes\mathfrak{g}'$ for any $x \in \mathfrak{g}'$, which in terms of descent data implies

$$(\theta \otimes_T \theta)(\alpha\delta_S(x) \otimes 1) = \alpha\delta_S(x) \otimes 1.$$

Using the fact that $\delta = \partial r$, that $(\theta \otimes_T \theta)(r \otimes 1 \otimes 1) = (\phi \otimes_T \phi)(r \otimes 1 \otimes 1)$ since $r \in \mathfrak{g} \otimes \mathfrak{g}$, and that $\theta(x \otimes 1) = x \otimes 1$ since $x \in \mathfrak{g}'$, this is equivalent to

$$[1\otimes_T(x\otimes 1)+(x\otimes 1)\otimes_T 1, (\phi\otimes_T\phi)(r\otimes 1\otimes\alpha)] = [1\otimes_T(x\otimes 1)+(x\otimes 1)\otimes_T 1, r\otimes\alpha\otimes 1].$$

Since the module $\mathfrak{g}_T$ is generated over T by elements of $\mathfrak{g}'$, this is, by Remark 3.1, equivalent to

$$(\phi \otimes \phi)(r \otimes 1 \otimes \alpha) = (r \otimes \alpha \otimes 1) - \mu(\Omega \otimes 1 \otimes 1). \tag{5.4}$$

Now for any $\beta \in T$, $\mathrm{CYB}(r \otimes \beta) = \beta^2\mathrm{CYB}(r \otimes 1 \otimes 1) = 0$ and $\mathrm{CYB}((\phi \otimes_T \phi)(r \otimes \beta)) = \beta^2\phi^{\otimes_T 3}\mathrm{CYB}(r \otimes 1 \otimes 1) = 0$. We may thus apply Lemma 3.5 to $\mathfrak{g}_T$ with $r_1 = r \otimes \alpha \otimes 1$, $r_2 = (\phi \otimes \phi)(r \otimes 1 \otimes \alpha)$, $\lambda_1 = \lambda\alpha \otimes 1$ and $\lambda_2 = \lambda \otimes \alpha$. Thus we get $\mu = j(\lambda\alpha \otimes 1)$ for some $j \in \{0, 1\}$. Since $r + \kappa(r) = \lambda\Omega$, (5.4) is then equivalent to

$$(\phi \otimes \phi)(r \otimes 1 \otimes \alpha) = (-\kappa)^j(r) \otimes \alpha \otimes 1.$$

Applying κ to both sides gives

$$(\phi \otimes \phi)(\kappa(r) \otimes 1 \otimes \alpha) = (-1)^j\kappa^{j+1}(r) \otimes \alpha \otimes 1.$$

and adding the two equations, using $r + \kappa(r) = \lambda\Omega$ and the fact that $\Omega \otimes 1 \otimes 1$ is fixed by automorphisms, we get

$$\Omega \otimes 1 \otimes \alpha = (-1)^j\Omega \otimes \alpha \otimes 1.$$

Thus $1 \otimes \alpha^2 = \alpha^2 \otimes 1$, whence by faithful flatness, $\alpha^2 \in R^\times$. Two cases are then possible: either $j = 0$, whence $\alpha \in R^\times$, and then $\mu = 0$ and (5.4) gives $(\phi \otimes \phi)(r \otimes 1 \otimes 1) = r \otimes 1 \otimes 1$; or $j = 1$, whence $1 \otimes \alpha = -\alpha \otimes 1$, and then $\mu = \lambda\alpha \otimes 1$ and (5.4) gives $(\phi \otimes \phi)(r \otimes 1 \otimes 1) = \kappa(r) \otimes 1 \otimes 1$. It follows that if $\alpha\delta_S$ descends to $\mathfrak{g}'$, then (1) or (2) holds.
Assume, conversely, that (1) or (2) holds. It is straight-forward to check that (5.4) is satisfied with $\mu = 0$ in case (1) and $\mu = \alpha\lambda \otimes 1$ in case (2). By the chain of equivalences in the above argument, this implies that

$$(\theta \otimes_T \theta)(\alpha\delta_S(x) \otimes 1) = \alpha\delta_S(x) \otimes 1.$$

We may then conclude with the lemma below. ☐

Lemma 5.8. *Let $\mathfrak{g}$ be a finitely generated projective R-module, let S be a faithfully flat R-ring and θ a descent datum on $\mathfrak{g}_S$ with corresponding twisted form $\mathfrak{g}'$, and δ a Lie coalgebra structure on $\mathfrak{g}_S$. Then δ descends to $\mathfrak{g}'$ if and only if*

$$\theta \otimes_{S\otimes S} \theta(\delta(x) \otimes 1) = \delta(x) \otimes 1$$

for each $x \in \mathfrak{g}'$.

Proof. Again we write $T = S \otimes S$. What needs to be shown is that the inclusion $\mathfrak{g}' \otimes \mathfrak{g}' \subseteq (\mathfrak{g}_S \otimes_S \mathfrak{g}_S)^\theta$ is an equality, where the right hand side denotes the set of all $w = \sum_i y_i \otimes z_i \in \mathfrak{g}_S \otimes_S \mathfrak{g}_S$ that satisfy

$$(\theta \otimes_T \theta)\left(\sum_i (y_i \otimes 1) \otimes_T (z_i \otimes 1)\right) = \sum_i (y_i \otimes 1) \otimes_T (z_i \otimes 1)$$

By descent it is enough to show that for some faithfully flat R-ring $\widehat{R}$, the inclusion $(\mathfrak{g}' \otimes \mathfrak{g}') \otimes \widehat{R} \subseteq (\mathfrak{g}_S \otimes_S \mathfrak{g}_S)^\theta \otimes \widehat{R}$ is an equality. The left hand side is isomorphic to $\mathfrak{g}'_{\widehat{R}} \otimes_{\widehat{R}} \mathfrak{g}'_{\widehat{R}}$, and the right hand side is isomorphic to

$$\left((\mathfrak{g}_{\widehat{R}})_{S\otimes\widehat{R}} \otimes_{S\otimes\widehat{R}} (\mathfrak{g}_{\widehat{R}})_{S\otimes\widehat{R}}\right)^{\widehat{\theta}},$$

where $\widehat{\theta}$ is obtained from θ by the base change induced by $R \to \widehat{R}$; this follows from faithful flatness and the canonical isomorphism

$$(\mathfrak{g}_{\widehat{R}})_{S\otimes\widehat{R}} \otimes_{S\otimes\widehat{R}} (\mathfrak{g}_{\widehat{R}})_{S\otimes\widehat{R}} \simeq (\mathfrak{g}_S \otimes_S \mathfrak{g}_S) \otimes \widehat{R}.$$

Thus, replacing R by a faithfully flat R-ring $\widehat{R}$ so that $\mathfrak{g}' \otimes \widehat{R}$ is free, we may assume that $\mathfrak{g}'$ is a free R-module. Let (e_i) be an R-basis of $\mathfrak{g}'$. It is then finite by assumption, and an $(e_i \otimes 1)$ is an S-basis of $\mathfrak{g}_S$. Thus if $w \in \mathfrak{g}_S \otimes \mathfrak{g}_S$, then

$$w = \sum_{i,j} (e_i \otimes \alpha_{ij}) \otimes_S (e_j \otimes 1)$$

for some $\alpha_{ij} \in S$. Then

$$\theta(w \otimes 1) = (\theta \otimes_T \theta)\left(\sum_{i,j} (e_i \otimes \alpha_{ij} \otimes 1) \otimes_T (e_j \otimes 1 \otimes 1)\right),$$

which equals $\sum_{i,j} (e_i \otimes 1 \otimes \alpha_{ij}) \otimes_T (e_j \otimes 1 \otimes 1)$ since $e_k \in \mathfrak{g}'$ implies that θ fixes $e_k \otimes 1 \otimes 1$ for each k. If $w \in (\mathfrak{g}_S \otimes \mathfrak{g}_S)^\theta$, then $\theta \otimes_T \theta$ fixes $w \otimes 1$, which by linear independence implies that $(\alpha_{ij} \otimes 1) = (1 \otimes \alpha_{ij})$ for each i and j, whence $\alpha_{ij} \in R$ for each i and j by faithful flatness. Thus $x \in \mathfrak{g}' \otimes \mathfrak{g}'$, as desired. □

As in the field case, we can encode this in terms of twisted cohomologies. Given a split simple Lie algebra g over an integral domain R, an r-matrix r on $\mathfrak{g}$, a faithfully flat R-ring S, and $\alpha \in S^\times$, we set

$$\overline{Z}^1 := \overline{Z}^1(S/R, \mathbf{Aut}(\mathfrak{g})) = \{\phi \in Z^1(S/R, \mathbf{Aut}(\mathfrak{g})) | (\phi \otimes \phi)(r) = \kappa(r)\},$$

where $Z^1(S/R, \mathbf{Aut}(\mathfrak{g}))$ is the set of 1-cocycles on $\mathbf{Aut}(\mathfrak{g})$ (using the conventions of [W, 17.6]). Thus $\overline{Z}^1 \subset \mathbf{Aut}(\mathfrak{g})(S \otimes S)$. We then define an equivalence relation $\sim$ on $\overline{Z}^1$ by

$$\psi \sim \phi \Longleftrightarrow \exists \rho \in \mathbf{Aut}(\mathfrak{g}, \partial r)(S) : \psi = (\mathrm{Id}_\mathfrak{g} \otimes \kappa)(\rho \otimes \mathrm{Id}_S)(\mathrm{Id}_\mathfrak{g} \otimes \kappa)\phi(\rho \otimes \mathrm{Id}_S)^{-1}$$

and write $\overline{H}^1(S/R, \mathbf{Aut}(\mathfrak{g}, \partial r))$ for the set of equivalence classes.

Corollary 5.9. *The set of all R-Lie bialgebras that become isomorphic to $(\mathfrak{g}_S, \alpha\partial r)$ over S is in bijection with*

1. $H^1(S/R, \mathbf{Aut}(\mathfrak{g}, \partial r))$, *if* $\alpha \in R^\times$,
2. $\overline{H}^1(S/R, \mathbf{Aut}(\mathfrak{g}, \partial r))$, *if* $\alpha^2 \in R^\times \setminus (R^\times)^2$, *and*
3. *the empty set, otherwise.*

Proof. Part (1) follows from Corollary 2.5 since $\mathbf{Aut}(\mathfrak{g}, \alpha\partial r)) \simeq \mathbf{Aut}(\mathfrak{g}, \partial r))$, and part (3) is immediate from Proposition 5.7. For part (2), in view of the proposition, we have a surjective map from the set of all R-Lie bialgebras that become isomorphic to $(\mathfrak{g}_S, \alpha\partial r)$ to $\overline{Z}^1$. To show that it induces a bijection with the twisted cohomology, assume first that $\mathfrak{g}'$ and $\mathfrak{g}''$ are twisted forms of $\mathfrak{g}$ to which $\alpha\delta_S$ descend, where $\delta = \partial r$, and let θ' and θ'' be the descent data corresponding to $\mathfrak{g}'$ and $\mathfrak{g}''$, respectively. Denote by δ' and δ'' the respective descended Lie bialgebra structures.

If now $(\mathfrak{g}', \delta') \simeq (\mathfrak{g}'', \delta'')$, then there is an S-automorphism ρ of $(\mathfrak{g}_S, \alpha\partial r)$ that maps $\mathfrak{g}_S^{\theta'}$ to $\mathfrak{g}_S^{\theta''}$. Thus for all $x \in \mathfrak{g}_S^{\theta'}$ we have

$$\theta''(\rho \otimes \mathrm{Id}_S)(x \otimes 1) = (\rho \otimes \mathrm{Id}_S)(x \otimes 1) = (\rho \otimes \mathrm{Id}_S)\theta'(x \otimes 1).$$

Now $\mathfrak{g} \otimes S \otimes S$ is generated as an $S \otimes S$-module by $\mathfrak{g}_S^{\theta'} \otimes 1$, and, for all $\lambda, \mu \in S$,

$$\theta''(\rho \otimes \mathrm{Id}_S)((\lambda \otimes \mu)(x \otimes 1)) = (\mu \otimes \lambda)\theta''(\rho \otimes \mathrm{Id}_S)(x \otimes 1)$$

which equals

$$(\mu \otimes \lambda)(\rho \otimes \mathrm{Id}_S)\theta'(x \otimes 1) = (\rho \otimes \mathrm{Id}_S)\theta'((\lambda \otimes \mu)(x \otimes 1)),$$

whence $\theta''(\rho \otimes \mathrm{Id}_S) = (\rho \otimes \mathrm{Id}_S)\theta'$. Expressed in terms of the corresponding cocycles ϕ for θ' and ψ for θ'' this precisely gives $\psi \sim \phi$, since the S-automorphisms of $(\mathfrak{g}_S, \alpha\partial r)$ coincide with the S-automorphisms of $(\mathfrak{g}_S, \partial r)$.

Assume conversely that $\psi \sim \phi$, with ρ such that

$$\psi = (\mathrm{Id}_{\mathfrak{g}} \otimes \kappa)(\rho \otimes \mathrm{Id}_S)(\mathrm{Id}_{\mathfrak{g}} \otimes \kappa)\phi(\rho \otimes \mathrm{Id}_S)^{-1},$$

or equivalently $\theta''(\rho \otimes \mathrm{Id}_S) = (\rho \otimes \mathrm{Id}_S)\theta'$ in terms of the corresponding descent data. Then the automorphism ρ of $(\mathfrak{g}_S, \partial r)$ maps $\mathfrak{g}_S^{\theta'}$ to $\mathfrak{g}_S^{\theta''}$ and thus induces an isomorphism of Lie bialgebras $(\mathfrak{g}', \delta') \to (\mathfrak{g}'', \delta'')$. This completes the proof. □

5.2 Interpreting Twisted Cohomologies

The twisted cohomologies defined above can be interpreted as (ordinary) cohomologies of twisted groups. To begin with, we need to determine when the twisted cohomology sets are non-empty. This is the content of the following. Throughout, we work over a field K.

Proposition 5.10. *Let r_{BD} be a Belavin–Drinfeld r-matrix with associated admissible quadruple $(\Gamma_1, \Gamma_2, \tau, r_{\mathfrak{h}})$. Then $\overline{H}^1_{\mathrm{Gal}}(K, \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}}))$ is non-empty if and only if there exists a diagram automorphism π of Γ satisfying*

$$\pi(\Gamma_1) = \Gamma_2, \quad \pi(\Gamma_2) = \Gamma_1, \quad \pi\tau\pi^{-1} = \tau^{-1} \quad \textit{and} \quad (\widehat{\pi} \otimes \widehat{\pi})(r_{\mathfrak{h}}) = r_{\mathfrak{h}}^{21}. \tag{5.5}$$

The following lemma will be helpful.

Lemma 5.11. *Let r_{BD} be the Belavin–Drinfeld r-matrix with associated admissible quadruple $(\Gamma_1, \Gamma_2, \tau, r_{\mathfrak{h}})$. If there exists a diagram automorphism π satisfying (5.5), then there exists a diagram automorphism π' of order 1 or 2 satisfying (5.5). More specifically, either π itself is of order 1 or 2, or $\mathfrak{g}$ is of type D_4 and $r_{\mathrm{BD}} = r_{\mathrm{DJ}}$, in which case* Id *satisfies (5.5).*

Proof. This is immediate if $\mathfrak{g}$ is not of type D_4. If $\mathfrak{g}$ is of type D_4 and $\Gamma_1 \neq \emptyset$, then it is straightforward to check, case by case, that for all admissible Γ_1, Γ_2 and τ, a diagram automorphism satisfying (5.5) must satisfy $\pi^2 = \mathrm{Id}$. If $\Gamma_1 = \Gamma_2 = \emptyset$ and $\pi^2 \neq \mathrm{Id}$, then $\pi^3 = \mathrm{Id}$. But then the condition $(\widehat{\pi} \otimes \widehat{\pi})(r_{\mathfrak{h}}) = r_{\mathfrak{h}}^{21}$ implies that $(\widehat{\pi}^2 \otimes \widehat{\pi}^2)(r_{\mathfrak{h}}) = r_{\mathfrak{h}}$, whence

$$r_{\mathfrak{h}} = (\widehat{\pi}^3 \otimes \widehat{\pi}^3)(r_{\mathfrak{h}}) = (\widehat{\pi} \otimes \widehat{\pi})(r_{\mathfrak{h}}) = r_{\mathfrak{h}}^{21}.$$

Thus since $r_{\mathfrak{h}} + r_{\mathfrak{h}}^{21} = \Omega_{\mathfrak{h}}$, this implies that $r_{\mathfrak{h}} = \frac{1}{2}\Omega_{\mathfrak{h}}$, whence $r_{\mathrm{BD}} = r_{\mathrm{DJ}}$. In that case Id satisfies (5.5), and the proof is complete. ☐

Proof of Proposition 5.10. Assume such an element π exists. By the above lemma, we may assume $\pi^2 = \mathrm{Id}$. Let χ be the Chevalley automorphism of $\mathfrak{g}$ and set $\phi = \chi\pi$, which we view as an element of $\mathbf{Aut}(\mathfrak{g})(\overline{K})$. We claim that

$$\phi \in \overline{Z}^1(K, \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}}), \alpha).$$

We have $\chi(h) = -h$ for any $h \in \mathfrak{h}$, whence $(\chi \otimes \chi)(r_{\mathfrak{h}}) = r_{\mathfrak{h}}$. Moreover π permutes all $\alpha \in \Delta^+$, and $\pi\tau^k = \tau^{-k}\pi$, whence

$$(\phi \otimes \phi)(r_{\mathrm{BD}}) = r_{\mathfrak{h}}^{21} + \sum_{\alpha \in \Delta^+} X_{-\alpha} \otimes X_\alpha + \sum_{\substack{\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} X_{-\pi(\alpha)} \wedge X_{\tau^{-k}\pi(\alpha)}.$$

For fixed α and k, setting $\alpha' = \tau^{-k}\pi(\alpha)$, the term $X_{-\pi(\alpha)} \wedge X_{\tau^{-k}\pi(\alpha)}$ becomes $X_{-\tau^k(\alpha')} \wedge X_{\alpha'}$. Summing over α' and k, we thus get

$$(\phi \otimes \phi)(r_{\mathrm{BD}}) = r_{\mathfrak{h}}^{21} + \sum_{\alpha \in \Delta^+} X_{-\alpha} \otimes X_\alpha + \sum_{\substack{\alpha' \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} X_{-\tau^k(\alpha')} \wedge X_{\alpha'},$$

which is equal to r_{BD}^{21}. Since χ and π commute and are of order at most two, we have $\phi^2 = 1$. Since ϕ is stable under the $\mathrm{Gal}(K)$-action, we get an element u_γ of $Z^1(K, \mathbf{Aut}(\mathfrak{g}))$ by setting

$$u_\gamma = \begin{cases} \mathrm{Id} & \text{if } \gamma \in \mathrm{Gal}(K(\alpha)) \\ \phi & \text{if } \gamma(\alpha) = -\alpha. \end{cases}$$

By construction, this element belongs to $\overline{Z}^1(K, \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}}), \alpha)$.
Conversely, assume that $\overline{Z}^1(K, \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}}), \alpha)$ is non-empty. Then in particular there exists $\phi \in \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}})(\overline{K})$ satisfying $(\phi \otimes \phi)(r_{\mathrm{BD}}) = r_{\mathrm{BD}}^{21}$. Arguing as in 3.11, we conclude that any such ϕ must map the semisimple part of $\Xi(r_{\mathrm{BD}})$ to the semisimple part of $\Xi(r_{\mathrm{BD}}^{21})$. By [KKPS], the semisimple part of $\Xi(r_{\mathrm{BD}})$ differs from $\Xi(r_{\mathrm{DJ}})$ only by an element in $\Xi(\mathfrak{h} \otimes \mathfrak{h})$, and thus likewise for $\Xi(r_{\mathrm{BD}}^{21})$. Thus we have $(\phi \otimes \phi)(r_{\mathrm{DJ}}) = r_{\mathrm{DJ}}^{21}$, and thence $\phi = \chi \mathrm{Ad}_h \pi$ for some $h \in \mathbf{H}(\overline{K})$ and a diagram automorphism π. Thus

$$(\pi \otimes \pi)(r_{\mathrm{BD}}) = \mathrm{Ad}_{h^{-1}} \chi(r_{\mathrm{BD}}^{21}).$$

This implies $(\pi \otimes \pi)(r_{\mathfrak{h}}) = r_{\mathfrak{h}}^{21}$ (since both Ad_h and χ leave $\mathfrak{h} \otimes \mathfrak{h}$ fixed), and

$$\sum_{\substack{\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} X_{\pi(\alpha)} \wedge X_{-\pi\tau^k(\alpha)} = \sum_{\substack{\beta \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ l>0}} \lambda_{\beta,l} X_{\tau^l(\beta)} \wedge X_{-\beta},$$

for non-zero scalars $\lambda_{\beta,l}$, which then must equal 1. Thus $(\pi \otimes \pi)(r_{\mathrm{BD}}) = \chi(r_{\mathrm{BD}}^{21})$, and $h^{-1} \in \mathbf{C}(\chi(r_{\mathrm{BD}}^{21}))$, which implies that $h \in \mathbf{C}(r_{\mathrm{BD}})$. Similarly to the case in the proof of Lemma 3.11 this implies that $\pi(\Gamma_1) = \Gamma_2$ and $\pi(\Gamma_2) = \Gamma_1$. Relabeling the terms, the equality becomes

$$\sum_{\substack{\alpha \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ k>0}} X_{\pi(\alpha)} \wedge X_{-\pi\tau^k(\alpha)} = \sum_{\substack{\beta \in \mathrm{Span}_{\mathbb{Z}}(\Gamma_1)^+ \\ l>0}} X_{\beta} \wedge X_{-\tau^{-l}(\beta)},$$

Proceeding again as in the proof of Lemma 3.11, this implies that for each $\alpha \in \Gamma_1$ we have the equality of sets

$$\{\pi\tau(\alpha), \dots, \pi\tau^{l_\alpha}(\alpha)\} = \{\tau^{-1}\pi(\alpha), \dots, \tau^{-l_\alpha}\pi(\alpha)\},$$

where l_α is defined as in the proof of Lemma 3.11. As there, we proceed by induction on l_α, the case $l_\alpha = 1$ being clear. For $l_\alpha > 1$, since $l_{\tau(\alpha)} = l_\alpha - 1$, by the induction hypothesis the left hand side is equal to

$$\{\pi\tau(\alpha), \tau^{-1}\pi\tau(\alpha), \dots, \tau^{-1}\pi\tau^{l_\alpha - 1}(\alpha)\}.$$

Thus $\tau^{-1}\pi(\alpha) = \pi\tau(\alpha)$, since by admissibility $\tau^{-1}\pi(\alpha) \neq \tau^{-1}\pi\tau^j(\alpha)$ for any $j > 0$. This completes the proof. □

We are now ready to re-interpret $\overline{H}^1(K, \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}}))$ whenever it is non-empty. Recall that in these cases there exists a diagram automorphism π of order at most two satisfying (5.5). Recall also that we are in the situation where $K \subset K(\alpha) \subseteq \overline{K}$, with $\alpha^2 \in K$.
Set $\mathbf{A}_{\mathrm{BD}} = \mathbf{Aut}(\mathfrak{g}, \partial r_{\mathrm{BD}})$. We have a map $v = v_\pi : \mathrm{Gal}(K) \to \mathbf{Aut}(\mathbf{A}_{\mathrm{BD}})(\overline{K})$ that is the identity on $\mathrm{Gal}(K(\alpha))$ and maps γ_α to $\rho \mapsto \chi\pi\rho(\chi\pi)^{-1} = \chi\pi\rho\pi\chi$; this is well defined since by the above, $\chi\pi(r_{\mathrm{BD}}) = r_{\mathrm{BD}}^{21}$, so

$$(\chi\pi\rho(\chi\pi)^{-1} \otimes \chi\pi\rho(\chi\pi)^{-1})(r_{\mathrm{BD}}) = r_{\mathrm{BD}}.$$

Since $\mathrm{Gal}(K)$ acts trivially on $\rho \mapsto \chi\pi\rho(\chi\pi)^{-1}$, and since the map is of order 2, it is a cocycle, and, following [KPS] we may consider the twisted group $(\mathbf{A}_{\mathrm{BD}})_v$. The $\mathrm{Gal}(K)$-action defining the cocycle set $Z^1_{\mathrm{Gal}}(K,(\mathbf{A}_{\mathrm{BD}})_v)$ and the cohomology $H^1_{\mathrm{Gal}}(K,(\mathbf{A}_{\mathrm{BD}})_v)$ is given by

$$\gamma \cdot \rho = v(\gamma)({}^\gamma\rho).$$

Theorem 5.12. *Let r_{BD} be a Belavin–Drinfeld r-matrix with associated admissible quadruple $(\Gamma_1, \Gamma_2, \tau, r_{\mathfrak{h}})$ such that $\overline{H}^1(K, \mathbf{A}_{\mathrm{BD}})$ is non-empty, and let π be any diagram automorphism of Γ of order at most two satisfying (5.5). Then the map*

$$\overline{Z}^1(K, \mathbf{A}_{\mathrm{BD}}, \alpha) \to Z^1_{\mathrm{Gal}}(K, (\mathbf{A}_{\mathrm{BD}})_v)$$

mapping the cocycle (u_γ) to the cocycle $\widehat{u}_\gamma$ defined by

$$\widehat{u}_\gamma = \begin{cases} u_\gamma & \text{if } \gamma \in \mathrm{Gal}(K(\alpha)) \\ u_\gamma\pi\chi & \text{if } \gamma(\alpha) = -\alpha, \end{cases}$$

induces an injective map

$$\overline{H}^1(K, \mathbf{A}_{\mathrm{BD}}) \to H^1_{\mathrm{Gal}}(K, (\mathbf{A}_{\mathrm{BD}})_v),$$

where $v = v_\pi$ is constructed as above.

Proof. First we show that the map is well-defined. If $(u_\gamma) \in \overline{Z}^1(K, \mathbf{A}_{\mathrm{BD}}, \alpha)$ we need to show that $(\widehat{u}_\gamma)$ is in $\mathbf{A}_{\mathrm{BD}}(\overline{K})$ and satisfies the twisted cocycle condition

$$\widehat{u}_{\gamma_1\gamma_2} = \widehat{u}_{\gamma_1} v(\gamma_1)({}^{\gamma_1}\widehat{u}_{\gamma_2}).$$

To show that each $\widehat{u}_\gamma$ is an automorphism, we must show that $(\widehat{u}_\gamma \otimes \widehat{u}_\gamma)(r_{\mathrm{BD}})$ is equal to r_{BD}. This is automatic whenever $\gamma(\alpha) = \alpha$, and holds when $\gamma(\alpha) = -\alpha$ by the proof of Proposition 5.10. The twisted cocycle condition follows from the cocycle condition on (u_γ) by a direct computation in each of the four cases $(\gamma_1(\alpha), \gamma_2(\alpha)) = (\pm\alpha, \pm\alpha)$. Thus the map is well defined. To show that it induces an injective map on cohomology, we need show that

$$(u_\gamma) \sim (w_\gamma) \Longleftrightarrow (\widehat{u}_\gamma) \sim (\widehat{w}_\gamma),$$

where the equivalence relations are in the respective cohomology sets. For $\gamma \in \mathrm{Gal}(K(\alpha))$, this holds by definition. For those $\gamma \in \mathrm{Gal}(K)$ with $\gamma(\alpha) = -\alpha$, the right hand equivalence amounts to

$$w_\gamma\pi\chi = \rho^{-1}u_\gamma\pi\chi\chi\pi{}^\gamma\rho\pi\chi$$

which is equivalent to

$$w_\gamma = \rho^{-1}u_\gamma{}^\gamma\rho,$$

which is precisely what the left hand equivalence amounts to. This completes the proof. □

6 Previous Results Revisited

In the light of the above, we will now review those results obtained in [PS] (for split Lie algebras) and [AS] (for a class of non-split Lie algebras) which are concerned with Drinfeld–Jimbo (i.e. standard) Lie bialgebra structures. Throughout, we work over a field K of characteristic zero and consider$(\overline{K}/K)$-twisted forms. We begin by the following consequence of Theorem 4.1.

Proposition 6.1. *Let $\mathfrak{g}$ be a split simple Lie algebra over K, and let $\mathfrak{g}'$ be a twisted form of $\mathfrak{g}$. Then there is, up to K-isomorphism, at most one Lie bialgebra structure δ' on $\mathfrak{g}'$ such that $(\mathfrak{g}', \delta')$ is a twisted form of the standard Lie bialgebra structure on $\mathfrak{g}$.*

Proof. The inclusion $i : \mathbf{A} \to \mathbf{Aut}(\mathfrak{g})$ induces a map

$$i^* : H^1_{\mathrm{Gal}}(K, \mathbf{A}) \to H^1_{\mathrm{Gal}}(K, \mathbf{Aut}(\mathfrak{g})).$$

Now the first (resp. second) of these cohomology sets classifies those Lie bialgebras (resp. Lie algebras) that are twisted forms of $(\mathfrak{g}, \partial r_{\mathrm{DJ}})$ (resp. of $\mathfrak{g}$), and the map i^* corresponds to sending the isomorphism class of a Lie bialgebra to the isomorphism class of the underlying Lie algebra. We thus need show that i^* is injective. But by construction, the isomorphism $f^* : H^1_{\mathrm{Gal}}(K, \mathbf{A}) \xrightarrow{\sim} H^1_{\mathrm{Gal}}(K, \mathbf{A}_\Gamma)$ of Theorem 4.1 factors through i^*, which therefore is injective. □

Remark 6.2. Note that in general, the map i^* is not surjective, meaning that there exist twisted forms $\mathfrak{g}'$ of $\mathfrak{g}$ which admit no Lie bialgebra structure that is a twisted form of the standard structure on $\mathfrak{g}$. By Corollary 4.2, this is in particular the case whenever $\mathfrak{g}'$ is non-split of type A_1, B_n, C_n, E_7, E_8, F_4 or G_2.

Remark 6.3. In [PS], this was proved in the special case $\mathfrak{g}' = \mathfrak{g}$, by formulating the problem in terms of Galois cohomology of split tori and using Steinberg's theorem. This essentially corresponds to considering Lie bialgebra structures on $\mathfrak{g}$ up to those isomorphisms that are inner automorphisms of $\mathfrak{g}$.

Remark 6.4. In [AS], the authors studied the classification of Lie bialgebra structures on certain non-split Lie algebras of type A_n, up to equivalence by certain natural gauge groups. More precisely, the authors considered special unitary Lie algebras under the action of unitary groups with respect to a non-square $d \in K^\times$. Curiously, for twisted forms of the standard structure, they showed that if such twisted forms exist, there exists a unique equivalence class if n is odd, but if n is even, the equivalence classes are parametrized by $K^\times/N(K(\sqrt{d})^\times)$. This does not contradict the above result, since the unitary group is not adjoint. It is rather straightforward to check that if one uses the corresponding adjoint group in the calculations of [AS], one obtains uniqueness in all cases.

The main part of [AS] is concerned with Lie bialgebras that upon extension to $\overline{K}$ become equivalent to $\alpha\partial r_{\mathrm{DJ}}$ for some $\alpha \in \overline{K}^\times \setminus K^\times$ with $d := \alpha^2 \in K^\times$.
To review these results, let K be a field of characteristic zero admitting a quadratic extension $L = K(\sqrt{d}) \supsetneq K$ with $d \in K^\times$. Set $\mathfrak{g} = \mathfrak{sl}_n(K)$ and consider the Lie algebra

$$\mathfrak{g}' = \mathfrak{su}_n(K,d) = \{x \in \mathfrak{sl}_n(L) | \overline{x}^t = -x\},$$

where $x \mapsto \overline{x}$ is the linear map induced by the non-trivial Galois automorphism of L/K.
Note that $\mathfrak{g}'_L \simeq \mathfrak{g}_L = \mathfrak{sl}_n(L)$. In the sequel we fix an algebraic closure $\overline{K}$ of K containing L, so that $\mathfrak{g}'_{\overline{K}} \simeq \mathfrak{sl}_n(\overline{K})$. As a direct consequence of Proposition 5.2, we obtain the following result from [AS].

Corollary 6.5. *Let $\alpha \in \overline{K}^\times$. If the Lie bialgebra structure $\alpha\partial r_{\mathrm{DJ}}$ on $\mathfrak{sl}_n(\overline{K})$ descends to $\mathfrak{g}'$, then $\alpha^2 \in K^\times$.*

One thus distinguishes two cases: $\alpha^2 \in (K^\times)^2 d$, in which case $K(\alpha) = L$, and $\alpha^2 \notin (K^\times)^2 d$, in which case $K(\alpha) \cap L = K$. The former case is the one that is thoroughly studied in [AS], and in this case, we obtain the following result.

Corollary 6.6. *Let $\alpha \in \overline{K}^\times$ with $\alpha^2 \in (K^\times)^2 d$. Then $\alpha\partial r_{\mathrm{DJ}}$ descends to $\mathfrak{g}'$.*

Proof. Note that the twisted form $\mathfrak{g}'$ of $\mathfrak{g}$ corresponds to the $\mathrm{Gal}(K)$-cocycle (u_γ) defined by $u_{\gamma_\alpha}(x) = -x^t$, where γ_α is the non-identity element of $\mathrm{Gal}(L/K)$, and by $u_\gamma = \mathrm{Id}$ whenever $\gamma \in \mathrm{Gal}(L)$. Since $r_{\mathrm{DJ}}^{t\otimes t} = \kappa(r_{\mathrm{DJ}})$, Proposition 5.2 implies that $\alpha\partial r_{\mathrm{DJ}}$ descends to $\mathfrak{g}'$. □

Remark 6.7. In [AS], the authors obtain, over e.g. fields of cohomological dimension at most 2, a classification parametrized by $(K^\times/N(L^\times))^m$ for a certain power m. Although we are here considering *isomorphism classes* of Lie bialgebras that become a unique *isomorphism* class after scalar extension, whereas in [AS] the authors consider *equivalence* classes that become a unique *equivalence* class upon extension, our results above can be used to explain the appearance of these norm classes, namely by restricting ourselves to inner automorphisms in Proposition 5.6. Let us first clarify what we mean. The embedding $\mathfrak{g}' \to \mathfrak{g}_L$ defines an L-isomorphism of Lie algebras $\mathfrak{g}'_L \to \mathfrak{g}_L$. A Lie bialgebra structure δ' on $\mathfrak{g}'$ induces, by means of this isomorphism, an L-Lie bialgebra structure δ'_L on $\mathfrak{g}_L$. We shall consider those δ' where $\delta'_L \simeq \sqrt{d}\partial r_{\mathrm{DJ}}$ via an inner automorphism of $\mathfrak{g}_L$, i.e. an element of the form Ad_X for $X \in \mathbf{GL}(L)$. Two such structures on $\mathfrak{g}'$ are considered (gauge) equivalent if they are isomorphic via an inner automorphism of $\mathfrak{g}'$. Recall further that an inner automorphism of $\mathfrak{g}_L$ is an automorphism of $(\mathfrak{g}_L, \partial r_{\mathrm{DJ}})$ if and only if it is of the form Ad_D with $D \in \mathbf{H}_n(L)$, where $\mathbf{H}_n$ is the split torus of $\mathbf{GL}_n$ fixed by the choice of a pinning.
This corresponds to considering, in $\overline{Z}^1_\alpha$, those cocycles (u_γ) with $u_{\gamma_\alpha} = \mathrm{Ad}_{D_u}\chi$ for the generator γ_α of $\mathrm{Gal}(L/K)$, with $D_u \in \mathbf{H}_n(L)$ and where χ is the

Chevalley automorphism. (Note that in general, not every D satisfies $\mathrm{Ad}_D\chi \in \overline{Z}^1_\alpha(L)$, as the map in Theorem 5.12 may not be surjective. We will not go into details, but refer to [AS].) Two cocycles $\mathrm{Ad}_{D_u}\chi$ and $\mathrm{Ad}_{D_v}\chi$ are then equivalent if they satisfy (5.3) with $\rho = \mathrm{Ad}_D$ for some $D \in \mathbf{H}(L)$. This equivalence condition translates as

$$\mathrm{Ad}_{D_v}\chi = \mathrm{Ad}_D^{-1}\mathrm{Ad}_{D_u}\chi\mathrm{Ad}_{\overline{D}}.$$

By definition of χ, and the fact that $D^t = D$, this is equivalent to

$$\mathrm{Ad}_{D_v}\chi = \mathrm{Ad}_D^{-1}\mathrm{Ad}_{D_u}\mathrm{Ad}_{\overline{D}}^{-1}\chi,$$

i.e. $\mathrm{Ad}_{D_u} = \mathrm{Ad}_{D\overline{D}D_v}$, which amounts to saying that each entry of D_v differs from the corresponding entry of D_u only by a factor in $N_{L/K}(L^\times)$. We thus retrieve the result obtained in [AS]. (Technically, $\overline{Z}^1_\alpha$ was defined with respect to the extension $\overline{K}/K$, but one can similarly consider any finite Galois extension.)

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