Heat Kernel of Anisotropic Nonlocal Operators

Krzysztof Bogdan, Paweł Sztonyk,
and Victoria Knopova

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Abstract. We construct and estimate the fundamental solution of highly anisotropic space-inhomogeneous integro-differential operators. We use the Levy method. We give applications to the Cauchy problem for such operators.

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1 Introduction and main results

Semigroups of operators are at the core of mathematical analysis. They describe evolutionary phenomena, resolve parabolic differential equations and have many connections to spectral theory and integro-differential calculus. In this paper we focus on Markovian semigroups – that is, probability kernels satisfying the Chapman-Kolmogorov equation. We construct the semigroups from the integral kernel $\nu(x, A)$, called the Lévy kernel and interpreted as the intensity of occurrence of dislocations of mass, or jumps, from the position $x \in \mathbb{R}^d$ to the set $x + A \subset \mathbb{R}^d$.
The construction of the semigroup from the Lévy kernel is intrinsically difficult when ν is rough, just like the construction of a flow from a non-Lipschitz direction field or a diffusion from a second order elliptic operator with merely bounded or degenerate coefficients. Below under appropriate assumptions on ν we obtain the semigroup and estimate its integral kernel $p_t(x, y)$, called the heat kernel or the fundamental solution or the transition probability density,

and we prove regularity and uniqueness of the kernel. Our results are analogues to the construction and estimates of the heat kernel for the second order elliptic operators with rough or degenerate coefficients.

A unique feature of our methodology is that we can deal with highly anisotropic Lévy kernels, meaning that $\nu(x, A)$ may vanish in certain jump directions. In fact ν may be concentrated on a set of directions of Lebesgue measure zero. We should note that despite recent rapid accumulation of estimates of heat kernels of nonlocal integro-differential Lévy-type generators with kernels $\nu(x, A)$, so far there were virtually none on generators with highly anisotropic kernels. A notable exception are the papers by Sztonyk et al. [7, 34, 33, 51] but they only concern translation invariant generators and convolution semigroups, for which the existence and many properties follow by Fourier methods. We also mention the estimates of anisotropic non-convolution heat kernels $p_t(x, y)$ given in [50] and [32], however these are obtained under the assumption that the heat kernel exists, without constructing it.

We consider jump kernels $\nu(x, dz)$ comparable to the Lévy measure $\nu_0(dz)$ of a symmetric anisotropic α-stable Lévy process in $\mathbb{R}^d$. Here and below we always assume that $0 < \alpha < 2$ and $d = 1, 2, \ldots$. For important technical reasons we also require Hölder continuity in x of the Radon-Nikodym derivative $\nu(x, dz)/\nu_0(dz)$. Recall that the Lévy measure ν_0 of the α-stable Lévy process has the form of a product measure in polar coordinates: $\nu_0(drd\theta) = r^{-1-\alpha}dr\mu_0(d\theta)$. The anisotropy mentioned above means that the spherical marginal μ_0 may even be singular with respect to the surface measure on the unit sphere. In fact we assume that ν_0 and ν have Hausdorff-type regularity outside of the origin. The order γ of this regularity is a fundamental factor in our development: we require $\alpha + \gamma > d$; the assumption is essentially optimal as we explain below.

To construct the heat kernel p from the Lévy kernel ν we use the parametrix method. It is a general approach which starts from an implicit equation and some first approximation p^0 for p. Iterating the equation produces an explicit (parametrix) series. The series formally solves the equation but the proof requires delicate analysis of the convergence, which critically depends on the choice of the first approximation p^0. The method was proposed by E. Levi [45] to solve an elliptic Cauchy problem. It was then extended by Dressel [13] to parabolic systems and by Feller [20] to parabolic operators perturbed by bounded non-local operator. Further developments were given in papers of Drin' [14], Eidelman and Drin' [16], Kochubeĭ [39] and Kolokoltsov [41]. We also refer to the monograph by Eidelman, Ivasyshin and Kochubeĭ [17] and to the classical monograph of Friedman [22] on the second-order parabolic differential operators. The parametrix method has a version called the perturbation (or Duhamel) formula. This version is appropriate for adding a "lower order" term to the generator of a given semigroup and the role of the first approximation is played by the "unperturbed" semigroup. This is, however, not the case in the present paper, because $\nu(x, dz) - \nu_0(dz)$ is not of "lower order" in comparison with $\nu_0(dz)$.

For recent developments in the parametrix and perturbation methods for nonlocal operators we refer the interested reader to Bogdan and Jakubowski [6], Knopova and Kulik [36], [37], Ganychenko, Knopova and Kulik [24], Kulik [43], Chen and Zhang [11], Kim, Song and Vondracek [35] and Kühn [42]. We should note again that the listed papers assume that $\nu(x, dz)/dz$ is locally comparable with a radial function. This is what we call the isotropic setting. The anisotropic setting has different methods and very few results. Here we show how to handle space-dependent anisotropic generators using suitable majorization and recent precise estimates for stable convolution semigroups.

A different Hilbert-space approach was developed in Jacob [29, 31], Hoh [27] and Böttcher [8, 9] and relies on the symbolic calculus, see also Tsutsumi [52, 28] and Kumano-go [44].

After verifying that the parametrix series representing $p_t(x, y)$ is convergent, one is challenged to prove that p is indeed the fundamental solution, in particular that it is Markovian and the *generator* of the semigroup coincides with the integro-diferential *operator* defined by ν for sufficiently large class of functions. This is a complicated task. The method described by Friedman [22] consists in (1) proving that $p_t(x, y)$ gives solutions to the respective Cauchy problem for the operator and (2) using the maximum principle for the operator. This approach is extended to rather isotropic nonlocal operators by Kochubeĭ [39] and further developed in the isotropic setting by Chen and Zhang [11] and by Kim, Song and Vondracek [35]. In our work we indeed profited a lot by following the outline of Kochubeĭ [39]. Another method, based on suitable approximations of the fundamental solutions was developed by Knopova and Kulik [37, 36]. A more probabilistic approach, based on the notion of the martingale problem, is given by Kulik [43]. We should note that the construction of semigroups generated by nonlocal integro-differential operators is related to the existence and uniqueness of solutions to stochastic differential equations with jumps. For an overview of the results and references in this direction, including the probabilistic interpretation of the parametrix method we refer the reader to [36]. The reader interested in probabilistic methods may consult further results and references in [12, 36, 37, 40, 43, 46].

Our development is purely analytic. We treat operators not manageable by the currently existing methods and give precise estimates for the heat kernels; our upper bounds of $p_t(x, y)$ are essentially optimal (see below). We thus give a framework for further investigations of the inhomogeneous Cauchy problem and of the regularity of solutions to nonlocal equations. The approach also gives guidelines for further developments of the parametrix method. In particular, extensions to anisotropic jump kernels $\nu(x, dz)$ with different radial decay profiles, cf. [33], should be possible along the same lines. Such extensions call for estimates and regularity of suitable convolution semigroup majorants, and they are certainly non-trivial.

Here are the main actors of our presentation. Let $d \in \{1, 2, \ldots\}$ and let

$\nu(z,du) \ge 0$ be an integral kernel on $\mathbb{R}^d$ satisfying

$$\sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{0\}} (1\wedge |u|^2)\,\nu(z,du) < \infty. \tag{1}$$

Let ν be symmetric in the second argument, meaning that for all $z \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$,

$$\nu(z,A) = \nu(z,-A). \tag{2}$$

We note that this is a different symmetry than the one used in the theory of Dirichlet forms [23]. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous functions vanishing at infinity, then we write $f \in C_0(\mathbb{R}^d)$ and for $x, z \in \mathbb{R}^d$ and $\delta > 0$, we let

$$L^{z,\delta} f(x) := \frac{1}{2}\int_{|u|>\delta} \big[f(x+u) + f(x-u) - 2f(x)\big]\,\nu(z,du),$$

and

$$L^z f(x) := \lim_{\delta\to 0} L^{z,\delta} f(x), \tag{3}$$

provided that this limit exits and is finite. We note L^z and $L^{z,\delta}$ satisfy the maximum principle: if $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x)$, then $L^{z,\delta} f(x_0) \le 0$ and $L^z f(x_0) \le 0$. If, say, $\nu(x_0, du)$ has unbounded support, then we even have $L^z f(x_0) < 0$ provided $f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) > 0$, because $f(x_0+u) + f(x_0-u)$ is close to zero on a set of positive measure $\nu(x_0, du)$. We let

$$L^\delta f(x) := L^{x,\delta} f(x), \qquad Lf(x) := L^x f(x),$$

and define the domain of L:

$$D(L) = \{f \in C_0(\mathbb{R}^d) : \text{ finite } Lf(x) \text{ exists for all } x \in \mathbb{R}^d\}. \tag{4}$$

We often write $L_x p_t(x,y)$, etc., meaning that L acts on the first spatial variable x of $p_t(x,y)$. By the Taylor expansion and (1), $D(L)$ contains $C_0^2(\mathbb{R}^d)$. Here, as usual, $f \in C_0^2(\mathbb{R}^d)$ means that f and all its derivatives of order up to 2 are continuous and converge to zero at infinity. We have

$$Lf(x) = \int_{\mathbb{R}^d} \frac{1}{2}\big[f(x+u) + f(x-u) - 2f(x)\big]\,\nu(x,du) \tag{5}$$

$$= \int_{\mathbb{R}^d} \big[f(x+u) - f(x) - u\cdot\nabla f(x)\mathbb{1}_{|u|\le 1}\big]\,\nu(x,du), \qquad f \in C_0^2(\mathbb{R}^d). \tag{6}$$

We now fully specify the properties of ν used in this paper. Let $\alpha \in (0,2)$. Let μ_0 be a finite measure concentrated on the unit sphere $\mathbb{S} := \{x \in \mathbb{R}^d : |x| = 1\}$. Define

$$\nu_0(A) = \int_{\mathbb{S}} \int_0^\infty \mathbb{1}_A(r\theta) r^{-1-\alpha}\, dr \mu_0(d\theta)\,, \quad A \subset \mathbb{R}^d\,, \tag{7}$$

where $\mathbb{1}_A$ is the indicator function of A. This is the standard form of the Lévy measure of α-stable distribution [48, Theorem 14.3]. We further assume that μ_0

is symmetric and non-degenerate, that is not concentrated on a proper linear subspace of $\mathbb{R}^d$. In particular, $0 < \mu_0(\mathbb{R}^d) < \infty$, ν_0 is infinite at the origin, and

$$\int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |y|^2)\, \nu_0(dy) < \infty. \tag{8}$$

Definition 1.1. *We say that ν_0 is a γ-measure at $\mathbb{S}$ if $\gamma \geq 0$ and*

$$\nu_0(B(x,r)) \leq m_0 r^\gamma\,, \quad x \in \mathbb{S},\ 0 < r < 1/2\,. \tag{9}$$

This is a Hausdorff-type condition on ν_0 outside of the origin. Since $\nu_0(drd\theta) = r^{-1-\alpha}dr\mu_0(d\theta)$, ν_0 is at least a 1-measure and at most a d-measure at $\mathbb{S}$. In fact, the spherical measure μ_0 is a $(\gamma-1)$-measure at $\mathbb{S}$ if and only if the Lévy measure ν_0 is a γ-measure at $\mathbb{S}$. For the rest of the paper we fix $\gamma \in [1, d]$ and make the following assumptions.

A1. ν_0 is given by (7) with non-degenerate finite symmetric spherical measure μ_0, ν_0 is a γ-measure at $\mathbb{S}$, and $\alpha + \gamma > d$.

A2. There exist constants $M_0 > 0$, $\eta \in (0, 1]$ such that

$$M_0^{-1}\nu_0(A) \leq \nu(z, A) \leq M_0\nu_0(A), \quad z \in \mathbb{R}^d,\ A \subset \mathbb{R}^d, \tag{10}$$

and

$$|\nu(z_1, A) - \nu(z_2, A)| \leq M_0\,(|z_1 - z_2|^\eta \wedge 1)\, \nu_0(A), \quad z_1, z_2 \in \mathbb{R}^d, \quad A \subset \mathbb{R}^d. \tag{11}$$

By A1, $\alpha + \gamma - d \in (0, \alpha)$. By the Radon-Nikodym theorem, A2 is equivalent to having $\nu(z, du) = h(z, u)\nu_0(du)$, where $M_0^{-1} \leq h(z, u) \leq M_0$ and $h(z, u)$ is η-Hölder continuous with respect to z. Note that (10) and (8) imply (1).

We now indicate how to define the heat kernel $p_t(x, y)$ corresponding to ν (details and justification are given in Section 3). Let $p_t^z(y-x)$ be the transition probability density corresponding to the Lévy measure $\nu(z, \cdot)$, with $z \in \mathbb{R}^d$ fixed, see (36). For $t > 0$, $x, y \in \mathbb{R}^d$ we define the "zero-order" approximation of $p_t(x, y)$:

$$p_t^0(x, y) = p_t^y(y - x), \quad t > 0, \quad x, y \in \mathbb{R}^d. \tag{12}$$

Note that it is the "target point" y that determines the Lévy measure $\nu(y, \cdot)$ used to define $p_t^0(x, y)$. This is important for regularity of $x \mapsto p_t^0(x, y)$. We let

$$\Phi_t(x, y) = \Big(L_x - \partial_t\Big)p_t^0(x, y), \tag{13}$$

and

$$\Psi_t(x, y) = \sum_{k=1}^{\infty} \Phi_t^{\boxtimes k}(x, y), \tag{14}$$

where we use the k-fold convolution (26). Then we let

$$p_t(x, y) = p_t^0(x, y) + \big(p^0 \boxtimes \Psi\big)_t(x, y). \tag{15}$$

The following three theorems reflect the main steps in our development.

Theorem 1.1. *We have*

$$(\partial_t - L_x)p_t(x,y) = 0, \qquad t>0, \quad x,y\in\mathbb{R}^d, \tag{16}$$

and for all $f\in C_0(\mathbb{R}^d)$, uniformly in $x\in\mathbb{R}^d$

$$\lim_{t\to 0}\int_{\mathbb{R}^d} f(y)p_t(x,y)\,dy = f(x). \tag{17}$$

To describe the growth and regularity of $p_t(x,y)$, for $\beta>0$ define

$$G^{(\beta)}(x) = (|x|\vee 1)^{-\beta}, \quad G_t^{(\beta)}(x) = \frac{1}{t^{d/\alpha}}G^{(\beta)}\left(\frac{x}{t^{1/\alpha}}\right), \quad t>0,\ x\in\mathbb{R}^d. \tag{18}$$

Of course, if $\beta>d$, then

$$\int_{\mathbb{R}^d} G_t^{(\beta)}(x)dx = \int_{\mathbb{R}^d} G^{(\beta)}(x)dx < \infty, \quad t>0. \tag{19}$$

Theorem 1.2. *There exist constants $C, c_1, c_2, t_0>0$ such that for all $x,y\in\mathbb{R}^d$,*

$$\left|\partial_t^k p_t(x,y)\right| \le Ct^{-k}e^{c_1 t}G_t^{(\alpha+\gamma)}(y-x), \quad k=0,1, \quad t>0, \tag{20}$$

and

$$p_t(x,y)\approx t^{-d/\alpha}, \quad |y-x|\le c_2 t^{1/\alpha}, \quad t\in(0,t_0], \tag{21}$$

and for all $t>0$, $x_1,x_2,y\in\mathbb{R}^d$,

$$\left|p_t(x_1,y)-p_t(x_2,y)\right| \le C\Big(\frac{|x_1-x_2|}{t^{1/\alpha}}\Big)^{\theta} e^{c_1 t}\Big(G_t^{(\alpha+\gamma)}(y-x_1)+G_t^{(\alpha+\gamma)}(y-x_2)\Big), \tag{22}$$

for some $\theta\in(0,\eta\wedge(\alpha+\gamma-d))$. Furthermore, $p_t(x,y)$ is continuous in y.

The correspondence of p and L is detailed as follows.

Theorem 1.3. *For $f\in C_0(\mathbb{R}^d)$, $t>0$ and $x\in\mathbb{R}^d$ define*

$$P_tf(x) = \int_{\mathbb{R}^d} p_t(x,y)f(y)dy. \tag{23}$$

Then (P_t) is a strongly continuous Markovian semigroup on $C_0(\mathbb{R}^d)$ and the function $u(t,x)=P_tf(x)$ defines the unique solution to the Cauchy problem

$$\begin{cases} \big(\partial_t - L_x\big)u(t,x) = 0, \quad t>0,\ x\in\mathbb{R}^d, \\ \qquad\qquad u(0,x) = f(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{24}$$

such that $e^{-\lambda t}u(t,x)\in C_0([0,\infty)\times\mathbb{R}^d)$ for some $\lambda\in\mathbb{R}$.

Before we go to the proofs we discuss typical applications, the sharpness and further questions related to our results.

There exist many measures ν_0 and ν satisfying the conditions A1 and A2. Recall that ν_0 is a γ-measure at $\mathbb{S}$ if and only if μ_0 is a $(\gamma-1)$-measure at $\mathbb{S}$. We also see that ν_0 is a d-measure if and only if it is absolutely continuous with respect to the Lebesgue measure and has a density function locally bounded on $\mathbb{R}^d \setminus \{0\}$. In this case the condition A1 holds trivially and Theorem 1.2 recovers well-known upper estimates of convolution semigroups – for detailed discussion of this case we refer the reader to [15] and [26]. One of possible ways of constructing more general ν_0 is the following. For every $\gamma \in [1, d]$ there exists a set $F \subset \mathbb{S}$ with positive finite Hausdorff measure of order $(\gamma-1)$ [2] and a set $E \subset F$ such that the Hausdorff measure restricted to E, say μ_0, is a nonzero $(\gamma-1)$-measure [19, Prop. 4.11]. Then ν_0 defined by (7) is a γ-measure at $\mathbb{S}$, and A1 holds provided that $\alpha > d-\gamma$. For instance, if $d=2$ and E is the usual ternary Cantor set on $\mathbb{S}$ and $\gamma - 1 = \log 2/\log 3$, then A1 is satisfies provided that $\alpha > 1 - \log 2/\log 3 \approx 0.3791$.

In the simplest case of Theorem 1.2, when the (symmetric non-degenerate) measure μ_0 is a finite sum of Dirac measures, then $\gamma = 1$ and (20) with $k=0$ cannot be improved. Indeed, it is optimal for α-stable convolution semigroups, when $\nu(x, dz) = \nu_0(dz)$, because in the directions of the support of ν_0, the corresponding convolution semigroup has a matching lower bound [53, Theorem 1.1].

In fact, our upper bound (20) with $k=0$ is also optimal for general γ, as follows by inspecting the translation invariant case [53, Theorem 1.1]; see also [34, Theorem 2] for estimates of convolution semigroups related to a wider class of Lévy processes.

It is safe to bet that one cannot expect simple and precise upper bounds for $p_t(x, y)$ in the anisotropic α-stable setting of Theorem 1.2. The difficulties with anisotropy are also seen in the more general setting of [37] – although [37] allows to handle anisotropic Lévy kernels, it is yet to be seen how to obtain precise upper bounds for $p_t(x, y)$ from the series representation given there. In the forthcoming paper [38] some examples are given for the α-stable-like case which specifically show that even finite pointwise upper bounds are not always possible.

In view toward further developments our results and [7, 47, 21] suggest further questions about more precise estimates of the semigroup in large time, regularity of the resolvent, Harnack inequality for harmonic functions, estimates of the Green function and Poisson kernel, etc. We also hope that our emphasis on the usage of the so-called sub-convolution property and auxiliary majorants based on kernels of convolution semigroups, see e.g. [5] for examples, will bring further progress and more synthetic approach to the Levi method.

The structure of the paper is as follows. In Section 2 we give the notation, definitions and preliminary results. The main results of this section are Lemmas 2.2 and 2.11. In Section 3 we prove the convergence of the series (14) and prove Theorem 1.1. In Section 4 we estimate the time derivative of $p_t(x, y)$ and prove Theorem 1.2. In Section 5 we prove Theorem 1.3. We also show that the generator $\mathcal{L}$ of (P_t) coincides with the operator L on $C_0^2(\mathbb{R}^d)$ and that the kernel $p_t(x,y)$ with the above properties is unique.

2 Auxiliary convolution semigroups

2.1 Notation and preliminaries

Let $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$ and $\mathbb{N}_0^d = (\mathbb{N}_0)^d$. For (multiindex) $\beta = (\beta_1,\ldots,\beta_d) \in \mathbb{N}_0^d$ we denote $|\beta| = \beta_1 + \ldots + \beta_d$. For $x = (x_1,\ldots,x_d)$, $y = (y_1,\ldots,y_d) \in \mathbb{R}^d$ and $r > 0$ we let $x \cdot y = \sum_{i=1}^d x_i y_i$ and $|x| = \sqrt{x \cdot x}$. We denote by $B(x,r) \subset \mathbb{R}^d$ the ball of radius r centered at $x \in \mathbb{R}^d$, so $\mathbb{S} = \partial B(0,1)$ is the unit sphere. All sets, functions and measures considered in this paper are assumed Borel. For measure λ we let $|\lambda|$ denote the total variation of λ. *Constants* mean positive real numbers and we denote them by c, C, c_i, etc. For nonnegative functions f,g we write $f \approx g$ to indicate that for some constant $c > 0$, $c^{-1} f \le g \le cf$. We write $c = c(p,q,\ldots,r)$ if the constant c can be obtained as a function of $p,q,\ldots,r$ only.

The convolution of measures is, as usual, $\lambda_1 * \lambda_2(A) = \int_{\mathbb{R}^d} \lambda_1(A-z)\lambda_2(dz)$, where $A \subset \mathbb{R}^d$. We also consider the following compositions of functions on space and space-time, respectively:

$$(\phi_1 \star \phi_2)(x,y) := \int_{\mathbb{R}^d} \phi_1(x,z)\phi_2(z,y)dz, \tag{25}$$

$$(f_1 \boxtimes f_2)(t,x,y) := \int_0^t \int_{\mathbb{R}^d} f_1(t-\tau,x,z) f_2(\tau,z,y)dzd\tau. \tag{26}$$

Here $x,y \in \mathbb{R}^d$, $t \in [0,\infty)$ and the integrands are assumed to be nonnegative or absolutely integrable.

We consider the Lévy measure ν_0 and the Lévy kernel ν introduced in Section 1. For clarity, it is always assumed that $A \mapsto \nu(x,A)$ is a Borel measure on $\mathbb{R}^d$ for every $x \in \mathbb{R}^d$ and $x \mapsto \nu(x,A)$ is Borel measurable for every Borel $A \subset \mathbb{R}^d$. By construction, ν_0 is symmetric, non-degenerate and homogeneous of order $-\alpha$:

$$\nu_0(rA) = r^{-\alpha}\nu_0(A), \quad 0 < r < \infty,\ A \subset \mathbb{R}^d.$$

The correspondence of ν_0 and μ_0 is a bijection [48, Remark 14.4]. We call μ_0 the *spherical measure* of ν_0. Since μ_0 is non-degenerate,

$$\inf_{\xi \in \mathbb{S}} \int_{\mathbb{S}} |\xi \cdot \theta|^\alpha \mu_0(d\theta) > 0. \tag{27}$$

The respective characteristic (Lévy-Khintchine) exponent q_{ν_0} is defined by

$$q_{\nu_0}(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{i\xi\cdot u} + i\xi\cdot u \mathbb{1}_{\{|u|\le 1\}}\right)\nu_0(du) \tag{28}$$

$$= \int_{\mathbb{R}^d\setminus\{0\}} \left(1-\cos(\xi\cdot u)\right)\nu_0(du)$$

$$= \frac{\pi}{2\sin\frac{\pi\alpha}{2}\Gamma(1+\alpha)} \int_{\mathbb{S}} |\xi\cdot\theta|^\alpha \,\mu_0(d\theta), \quad \xi\in\mathbb{R}^d. \tag{29}$$

By scaling and (27),

$$c_1|\xi|^\alpha \le q_{\nu_0}(\xi) \le c_2|\xi|^\alpha, \quad \xi\in\mathbb{R}^d. \tag{30}$$

By the Lévy-Khintchine formula and (30) there is a convolution semigroup of probability density functions whose Fourier transform is $\exp(-tq_{\nu_0}(\xi))$, see, e.g., [1, 48]. If $q_{\nu_0}(\xi) = |\xi|^\alpha$, then the corresponding convolution semigroup $g(t,x)$ satisfies

$$g(t,x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} = G_t^{(d+\alpha)}(x), \quad t>0,\ x\in\mathbb{R}^d. \tag{31}$$

The comparison was proved by Blumenthal and Getoor [3] (see [5, (29)] for explicit constants). In the next section we prove a version of the upper bound in (31) for the semigroups corresponding to ν. To this end we first learn how to bound integro-differential operators with kernel ν. In what follows, we denote, as usual,

$$\operatorname{diam}(A) = \sup\{|x-y| : x,y\in A\}.$$

We also denote

$$\delta(A) = \operatorname{dist}(A,0) := \inf\{|x| : x\in A\}.$$

The lemma below is an easy consequence of (7) and (9).

Lemma 2.1. *Let $m_1 = \max\{m_0, 2^\gamma|\mu_0|/\alpha\}$. For every $A\subset\mathbb{R}^d$ we have*

$$\nu_0(A) \le m_1\delta(A)^{-\alpha-\gamma}\operatorname{diam}(A)^\gamma. \tag{32}$$

Proof. If $\delta(A)=0$, then (32) is trivial, so we assume $\delta(A)>0$. By the homogeneity of ν_0, for every $x_0\in A$,

$$\nu_0(A) \le \nu_0(B(x_0,\operatorname{diam}(A))\cap B(0,\delta(A))^c)$$

$$= |x_0|^{-\alpha}\nu_0\left(B\left(\frac{x_0}{|x_0|},\frac{\operatorname{diam}(A)}{|x_0|}\right)\cap B\left(0,\frac{\delta(A)}{|x_0|}\right)^c\right).$$

If $\operatorname{diam}(A)/|x_0| \le \frac12$, then from (9) we get

$$\nu_0(A) \le |x_0|^{-\alpha}m_0\left(\frac{\operatorname{diam}(A)}{|x_0|}\right)^\gamma \le m_0\,\delta(A)^{-\alpha-\gamma}\operatorname{diam}(A)^\gamma,$$

and if $\mathrm{diam}(A)/|x_0| \geq \frac{1}{2}$, then

$$\begin{aligned}\nu_0(A) &\leq |x_0|^{-\alpha}\nu_0\left(B\left(0,\frac{\delta(A)}{|x_0|}\right)^c\right) = |x_0|^{-\alpha}\frac{|\mu_0|}{\alpha}\left(\frac{\delta(A)}{|x_0|}\right)^{-\alpha}\\ &= \frac{|\mu_0|}{\alpha}\delta(A)^{-\alpha} \leq \frac{2^\gamma|\mu_0|}{\alpha}\delta(A)^{-\alpha-\gamma}\,\mathrm{diam}(A)^\gamma.\end{aligned}$$

Thus, in either case we get (32). □

Let $C_b^2(\mathbb{R}^d)$ be the class of all the functions bounded together with their derivatives up to order 2. For $t>0$, $x\in\mathbb{R}^d$ and $f\in C_b^2(\mathbb{R}^d)$ we denote

$$\mathcal{A}_t^{\#}f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left|f(x+u)-f(x)-u\cdot\nabla_x f(x)\mathbb{1}_{\{|u|\leq t^{1/\alpha}\}}\right|\,\nu_0(du).$$

The lemma below is the main result of this subsection.

Lemma 2.2. *Let $f:(0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ be such that $f_t(\cdot):=f(t,\cdot)\in C_b^2(\mathbb{R}^d)$ for every $t>0$ and there are constants $K,\zeta>0,\kappa\in[0,\alpha+\gamma-d)$ such that*

$$|\partial_x^\beta f_t(x)| \leq Kt^{-(\zeta+|\beta|)/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa}, \quad x\in\mathbb{R}^d,\, t>0, \tag{33}$$

for every multiindex $\beta\in\mathbb{N}_0^d$ with $|\beta|=0$ or 2. Then there exists $c_{\mathcal{A}}>0$ such that

$$\mathcal{A}_t^{\#}f_t(x) \leq c_{\mathcal{A}}Kt^{-1-\zeta/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa}, \quad x\in\mathbb{R}^d,\, t>0.$$

Proof. We have $\mathcal{A}_t^{\#}f_t(x) = I_1+I_2$, where

$$\begin{aligned}I_1 &= \int_{|u|\leq t^{1/\alpha}} \left|f_t(x+u)-f_t(x)-u\cdot\nabla_x f_t(x)\mathbb{1}_{\{|u|\leq t^{1/\alpha}\}}\right|\,\nu_0(du),\\ I_2 &= \int_{|u|> t^{1/\alpha}} \left|f_t(x+u)-f_t(x)-u\cdot\nabla_x f_t(x)\mathbb{1}_{\{|u|\leq t^{1/\alpha}\}}\right|\,\nu_0(du).\end{aligned}$$

From the Taylor expansion and (33) we get

$$\begin{aligned}I_1 &= \int_{|u|\leq t^{1/\alpha}} |f_t(x+u)-f_t(x)-u\cdot\nabla_x f_t(x)|\,\nu_0(du)\\ &\leq Kd^2 2^{\alpha+\gamma-\kappa-1}\int_{|u|\leq t^{1/\alpha}} |u|^2 t^{-(\zeta+2)/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa}\,\nu_0(du)\\ &= Kd^2 2^{\alpha+\gamma-\kappa-1}t^{-(\zeta+2)/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa}\frac{|\mu_0|}{2-\alpha}t^{(2-\alpha)/\alpha}\\ &= Kc_1t^{-1-\zeta/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa}.\end{aligned}$$

We split I_2 in the following way,

$$
\begin{aligned}
I_2 &= \int_{|u|>t^{1/\alpha}} |f_t(x-u)-f_t(x)|\ \nu_0(du) \\
&\le \int_{|u|>t^{1/\alpha}} |f_t(x-u)|\,\nu_0(du) + |f_t(x)| \int_{|u|>t^{1/\alpha}} \nu_0(du) \\
&= \left(\int_{|u|>t^{1/\alpha},\,|x-u|>t^{1/\alpha}} + \int_{|u|>t^{1/\alpha},\,|x-u|\le t^{1/\alpha}} \right) |f_t(x-u)|\ \nu_0(du) \\
&\quad + |f_t(x)|\,\frac{|\mu_0|}{\alpha t} =: I_{21} + I_{22} + |f_t(x)|\frac{|\mu_0|}{\alpha t}.
\end{aligned}
$$

Using (32) and (33) we obtain

$$
\begin{aligned}
I_{22} &\le K \int_{|u|>t^{1/\alpha},\,|x-u|\le t^{1/\alpha}} t^{-\zeta/\alpha}\,\nu_0(du) \\
&= K t^{-\zeta/\alpha} \nu_0 \left(B(x,t^{1/\alpha}) \cap B(0,t^{1/\alpha})^c \right) \\
&\le K m_1 t^{-\zeta/\alpha} (2t^{1/\alpha})^\gamma \left(\max\{|x|-t^{1/\alpha}, t^{1/\alpha}\} \right)^{-\gamma-\alpha} \\
&\le K c_2 t^{-1-\zeta/\alpha} \left(1 + t^{-1/\alpha}|x| \right)^{-\gamma-\alpha}.
\end{aligned}
$$

In order to estimate I_{21} we define

$$
\begin{aligned}
J_1 &= \int_{|u|>t^{1/\alpha},\,\max\{|x|/4,t^{1/\alpha}\}>|x-u|>t^{1/\alpha}} |f_t(x-u)|\ \nu_0(du), \\
J_2 &= \int_{|u|>t^{1/\alpha},\,|x-u|\ge\max\{|x|/4,t^{1/\alpha}\}} |f_t(x-u)|\ \nu_0(du),
\end{aligned}
$$

and observe that $I_{21} = J_1 + J_2$. Using (33) we get

$$
\begin{aligned}
J_2 &\le \int_{|u|>t^{1/\alpha},\,|x-u|\ge|x|/4} K t^{-\zeta/\alpha} (1+t^{-1/\alpha}|x-u|)^{-\gamma-\alpha+\kappa}\,\nu_0(du) \\
&\le (K|\mu_0|/\alpha) t^{-1-\zeta/\alpha} (1+t^{-1/\alpha}|x|/4)^{-\gamma-\alpha+\kappa} \\
&\le K c_3 t^{-1-\zeta/\alpha} (1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa}.
\end{aligned}
$$

If $|x| < 4t^{1/\alpha}$, then $J_1 = 0$. If $|x| \ge 4t^{1/\alpha}$, then $L := \lfloor \log_2(t^{-1/\alpha}|x|/4) \rfloor \ge 0$, and

$$\begin{aligned}
J_1 &\le \int_{|u|>t^{1/\alpha},\,|x|/4>|x-u|>t^{1/\alpha}} Kt^{-\zeta/\alpha}(1+t^{-1/\alpha}|x-u|)^{-\gamma-\alpha+\kappa}\,\nu_0(du)\\
&\le \sum_{n=0}^{L}\int_{2^{n+1}t^{1/\alpha}\ge|x-u|>2^n t^{1/\alpha}} Kt^{-\zeta/\alpha}(1+t^{-1/\alpha}|x-u|)^{-\gamma-\alpha+\kappa}\,\nu_0(du)\\
&\le Kt^{-\zeta/\alpha}\sum_{n=0}^{L}2^{-n(\alpha+\gamma-\kappa)}\nu_0\left(B(x,2^{n+1}t^{1/\alpha})\right)\\
&\le Kt^{-\zeta/\alpha}\sum_{n=0}^{L}2^{-n(\alpha+\gamma-\kappa)}m_1 2^{\alpha+\gamma}|x|^{-\gamma-\alpha}(2^{n+2}t^{1/\alpha})^{\gamma}\\
&\le Kc_4 t^{(-\zeta+\gamma)/\alpha}|x|^{-\gamma-\alpha}\le Kc_5 t^{-1-\zeta/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha},
\end{aligned}$$

where in the fourth inequality we use (32) and the fact that $2^{n+1}t^{1/\alpha}\le|x|/2$ for $n\le L$. We obtain

$$I_2 = I_{21}+I_{22}+|f_t(x)|\frac{|\mu_0|}{\alpha t}\le Kc_6 t^{-1-\zeta/\alpha}(1+t^{-1/\alpha}|x|)^{-\gamma-\alpha+\kappa},$$

and the lemma follows. □

2.2 Estimates of $p_t^z(x)$

In this section we estimate the convolution semigroup corresponding to the Lévy measure $\nu(z,\cdot)$ with fixed but arbitrary $z\in\mathbb{R}^d$. We are interested in majorants which are integrable in space, like (19). We note that the results [7] cannot be directly used here because we also need Hölder continuity of $z\mapsto p_t^z$, which is crucial for the proof of Theorem 1.1. Recall that each $\nu(z,\cdot)$ is symmetric and comparable to ν_0, i.e. it satisfies (10). Therefore,

$$\begin{aligned}
q(z,\xi) &:= \int_{\mathbb{R}^d\setminus\{0\}}(1-e^{i\xi\cdot u}+i\xi\cdot u\mathbb{1}_{\{|u|\le1\}})\,\nu(z,du)\\
&= \int_{\mathbb{R}^d\setminus\{0\}}(1-\cos\xi\cdot u)\,\nu(z,du),
\end{aligned}\tag{34}$$

is real-valued and there exist constants $c,C>0$ such that

$$c|\xi|^\alpha\le q(z,\xi)\le C|\xi|^\alpha,\quad \xi,z\in\mathbb{R}^d.\tag{35}$$

By (35),

$$p_t^z(x):=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{-ix\cdot\xi-tq(z,\xi)}\,d\xi,\quad t>0,\,x\in\mathbb{R}^d,\tag{36}$$

is infinitely smooth in t and x. Note that for each z, $(p_t^z)_{t>0}$ is a convolution semigroup of probability densities. The operator L^z equals to the generator of the semigroup on $C_0^2(\mathbb{R}^d)$, see, e.g., [4]. Therefore,

$$\partial_t p_t^z(x)=L^z p_t^z(x),\quad t>0,\ x\in\mathbb{R}^d,\tag{37}$$

and so

$$\partial_t p_t^z(x) = \int_{\mathbb{R}^d} \left(p_t^z(x+u) - p_t^z(x) - u \cdot \nabla_x p_t^z(x) \mathbb{1}_{\{|u|\le 1\}}\right) \nu(z, du).$$

We recall the definition (18) and give approximation for convolutions of $G_t^{(\beta)}$.

Lemma 2.3. *For every $\beta \in (d, d+2)$,*

$$\int_{\mathbb{R}^d} G_{t-s}^{(\beta)}(x-z) G_s^{(\beta)}(z)\, dz \approx G_t^{(\beta)}(x), \quad x \in \mathbb{R}^d,\ 0 < s < t. \tag{38}$$

Proof. Denote $\delta = \beta - d$. Let $g(t,x)$ be the density function of the isotropic rotation invariant δ-stable Lévy process. We have $G_t^{(\beta)}(x) \approx g(t^{\delta/\alpha}, x)$, see, e.g., [3].

$$\begin{aligned} \int_{\mathbb{R}^d} G_{t-s}^{(\beta)}(x-z) G_s^{(\beta)}(z)\, dz &\approx \int_{\mathbb{R}^d} g((t-s)^{\delta/\alpha}, x-z) g(s^{\delta/\alpha}, z)\, dz \\ &= g((t-s)^{\delta/\alpha} + s^{\delta/\alpha}, x). \end{aligned}$$

Since an easy calculation gives

$$(1 \wedge 2^{1-\delta/\alpha}) t^{\delta/\alpha} \le (t-s)^{\delta/\alpha} + s^{\delta/\alpha} \le (1 \vee 2^{1-\delta/\alpha}) t^{\delta/\alpha},$$

from (31) we get

$$g((t-s)^{\delta/\alpha} + s^{\delta/\alpha}, x) \approx t^{-d/\alpha} \left(1 \vee \frac{|x|}{t^{1/\alpha}}\right)^{-\beta} = G_t^{(\beta)}(x),$$

and (38) follows. □

Lemma 2.4. *For every $\beta \in \mathbb{N}_0^d$ there is $c = c(\nu_0, \beta, M_0) > 0$ such that*

$$|\partial_x^\beta p_t^z(x)| \le c t^{-|\beta|/\alpha} G_t^{(\alpha+\gamma)}(x), \quad t > 0,\ x, z \in \mathbb{R}^d.$$

The proof of Lemma 2.4 relies on the auxiliary results which we give first. Fix an arbitrary $z \in \mathbb{R}^d$. Let $\vartheta(\cdot) = \nu(z, \cdot)$. For $\varepsilon > 0$ let $\bar\vartheta_\varepsilon = \mathbb{1}_{B(0,\varepsilon)^c} \vartheta$, $\tilde\vartheta_\varepsilon = \mathbb{1}_{B(0,\varepsilon)} \vartheta$, and

$$q_{\bar\vartheta_\varepsilon}(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\xi\cdot u}\right) \bar\vartheta_\varepsilon(du), \quad q_{\tilde\vartheta_\varepsilon}(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{i\xi\cdot u} + i\xi\cdot u \mathbb{1}_{\{|u|\le 1\}}\right) \tilde\vartheta_\varepsilon(du),$$

and

$$\tilde p_t^\varepsilon(x) = \left(\mathcal{F}^{-1} \exp(-t q_{\tilde\vartheta_\varepsilon}(\cdot))\right)(x), \quad t > 0,\ x \in \mathbb{R}^d,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. By (30) we see that $\tilde p_t^\varepsilon(x)$ is smooth. The probability measure with the characteristic function $\exp(-t q_{\bar\vartheta_\varepsilon}(\xi))$ is

$$\bar P_t^\varepsilon(dy) = e^{-t|\bar\vartheta_\varepsilon|} \sum_{n=0}^\infty \frac{t^n \bar\vartheta_\varepsilon^{*n}(dy)}{n!}, \quad t > 0. \tag{39}$$

We have

$$p_t^z = \tilde{p}_t^\varepsilon * \bar{P}_t^\varepsilon \,. \tag{40}$$

The first step in the proof of Lemma 2.4 is to estimate the terms in the series (39). The following Lemma is a version of [7, Lemma 1] and [34, Cor. 10].

Proposition 2.5. *There exists $m_3 > 0$ such that for $\varepsilon > 0$ and $n \geq 1$,*

$$\bar{\vartheta}_\varepsilon^{*n}(B(x,r)) \leq m_3^n \varepsilon^{-(n-1)\alpha} |x|^{-\alpha-\gamma} r^\gamma \,, \quad x \in \mathbb{R}^d \setminus \{0\},\, r < |x|/2.$$

Consequently,

$$\bar{P}_t^\varepsilon\,(B(x,r)) \leq \varepsilon^\alpha e^{m_3 \varepsilon^{-\alpha} t} |x|^{-\alpha-\gamma} r^\gamma.$$

Proof. The result follows from [34, Lemma 9]. Indeed, one can check that the conditions (23) from [34, Lemma 9] hold true with $f(s) = s^{-\alpha-\gamma}$, which gives

$$\bar{\vartheta}_\varepsilon^{*n}(A) \leq c^n \mathfrak{q}_{\nu_0}(1/\varepsilon)^{n-1} f\left(\delta(A)/2\right) \operatorname{diam}(A)^\gamma, \tag{41}$$

where $\mathfrak{q}_{\nu_0}(r) = \sup_{|\xi|\leq r} q_{\nu_0}(\xi)$, $r > 0$. Since $\mathfrak{q}_{\nu_0}(r) \approx r^\alpha$, we get the required estimates. ☐

We denote $\bar{P}_t = \bar{P}_t^{t^{1/\alpha}}$, $\tilde{p}_t = \tilde{p}_t^{t^{1/\alpha}}$ and $\tilde{\vartheta} = \tilde{\vartheta}_{t^{1/\alpha}}$.

Lemma 2.6. *For every $n \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^d$ there is $c > 0$ such that*

$$|\partial_x^\beta \tilde{p}_t(x)| \leq c t^{-|\beta|/\alpha} G_t^{(n)}(x), \quad t > 0, x \in \mathbb{R}^d. \tag{42}$$

Proof. Let $g_t(x) = t^{d/\alpha} \tilde{p}_t(t^{1/\alpha} x)$, $t > 0$, $x \in \mathbb{R}^d$. For each t, $g_t(x)$ is the density function of an infinitely divisible distribution. We denote by $\phi_t(\xi)$ and $\eta_t(du)$ the corresponding characteristic exponent and the Lévy measure, respectively. To prove (42) will apply [49, Prop. 2.1], for which it suffices to check that

$$\int_{\mathbb{R}^d} |\xi|^k e^{-\operatorname{Re}\phi_t(\xi)} d\xi < c, \quad \int_{\mathbb{R}^d} |u|^k \eta_t(du) < c \tag{43}$$

for every $k \geq 2$ with constant c independent of t. Indeed, a direct calculation gives $\eta_t(A) = t\tilde{\vartheta}_{t^{1/\alpha}}(t^{1/\alpha} A)$. Then by (10) we get

$$\int |y|^k \, \eta_t(dy) \leq M_0 t \int_{|y|<t^{1/\alpha}} \left(\frac{|y|}{t^{1/\alpha}}\right)^k \nu_0(dy) = \frac{M_0 |\mu_0|}{k-\alpha}.$$

Further,

$$\begin{aligned}
\operatorname{Re}\phi_t(\xi) &= \int (1 - \cos(\xi \cdot y))\, \eta_t(dy) \\
&\geq M_0^{-1} t \int_{|y|<t^{1/\alpha}} \left(1 - \cos\left(\xi \cdot \frac{y}{t^{1/\alpha}}\right)\right) \nu_0(dy) \\
&= M_0^{-1} t q_{\nu_0}(\xi/t^{1/\alpha}) - M_0^{-1} t \int_{|y|\geq t^{1/\alpha}} \left(1 - \cos\left(\xi \cdot \frac{y}{t^{1/\alpha}}\right)\right) \nu_0(dy) \\
&\geq M_0^{-1} t q_{\nu_0}(\xi/t^{1/\alpha}) - M_0^{-1} t \nu_0(B(0, t^{1/\alpha})^c) \geq c_1 |\xi|^\alpha - c_2.
\end{aligned}$$

Therefore,

$$\int e^{-\operatorname{Re}\phi_t(\xi)}|\xi|^k\,d\xi \le e^{c_2}\int e^{-c_1|\xi|^\alpha}|\xi|^k\,d\xi \le c_3 < \infty.$$

Thus, (43) holds true and applying the result from [49, Prop.2.1] we get

$$|\partial_x^\beta g_t(x)| \le c_4\,(1+|x|)^{-n}\,,\quad n\ge 0,\, t>0,\, x\in\mathbb{R}^d.$$

Coming back to $\tilde{p}_t$ we get the desired estimate. □

Proof of Lemma 2.4. We have

$$\begin{aligned} |\partial_x^\beta p_t^z(x)| &= |\,(2\pi)^{-d}\,(-i)^{|\beta|}\int \xi^\beta e^{-ix\cdot\xi}e^{-tq_\vartheta(\xi)}\,d\xi| \\ &\le (2\pi)^{-d}\int |\xi|^{|\beta|}e^{-tc_1|\xi|^\alpha}\,d\xi = c_2 t^{(-d-|\beta|)/\alpha},\quad t>0,\, x\in\mathbb{R}^d. \end{aligned}$$

Using Proposition 2.5 and Lemma 2.6 with $n\ge\alpha+\gamma$, for $|x|>2t^{1/\alpha}$ we obtain

$$\begin{aligned} &\left|\partial_x^\beta\left(\tilde{p}_t * \bar{P}_t\right)(x)\right| = \left|\int_{\mathbb{R}^d}\partial_x^\beta\tilde{p}_t(x-y)\bar{P}_t(dy)\right| \le \int_{\mathbb{R}^d}\left|\partial_x^\beta\tilde{p}_t(x-y)\right|\bar{P}_t(dy) \\ &\le c_3 t^{\frac{-d-|\beta|}{\alpha}}\int_{\mathbb{R}^d}(1+t^{-1/\alpha}|x-y|)^{-n}\bar{P}_t(dy) \\ &= c_3 t^{\frac{-d-|\beta|}{\alpha}}\int_{\mathbb{R}^d}\int_0^{(1+t^{-1/\alpha}|x-y|)^{-n}} ds\,\bar{P}_t(dy) \\ &= c_3 t^{\frac{-d-|\beta|}{\alpha}}\int_0^1\int_{\mathbb{R}^d}\mathbb{1}_{(1+t^{-1/\alpha}|x-y|)^{-n}>s}\,\bar{P}_t(dy)ds \\ &= c_3 t^{\frac{-d-|\beta|}{\alpha}}\int_0^1\bar{P}_t\left(B(x,t^{1/\alpha}(s^{-\frac{1}{n}}-1))\right)ds, \end{aligned}$$

thus

$$\begin{aligned} &\left|\partial_x^\beta\left(\tilde{p}_t * \bar{P}_t\right)(x)\right| \\ &\le c_4 t^{\frac{-d-|\beta|}{\alpha}}\Bigg(\int_{(1+\frac{|x|}{2t^{1/\alpha}})^{-n}}^1 t|x|^{-\alpha-\gamma}\left(t^{1/\alpha}(s^{-\frac{1}{n}}-1)\right)^\gamma ds \\ &\qquad + \int_0^{(1+\frac{|x|}{2t^{1/\alpha}})^{-n}} ds\Bigg) \\ &\le c_4 t^{\frac{-d-|\beta|}{\alpha}}\left(t^{1+\gamma/\alpha}|x|^{-\alpha-\gamma}\int_0^1 s^{-\gamma/n}\,ds + \left(1+\frac{|x|}{2t^{1/\alpha}}\right)^{-n}\right) \\ &= c_5 t^{\frac{-d-|\beta|}{\alpha}}\left(t^{1+\gamma/\alpha}|x|^{-\alpha-\gamma} + \left(1+\frac{|x|}{2t^{1/\alpha}}\right)^{-n}\right) \\ &\le c_6 t^{\frac{-d-|\beta|}{\alpha}}\left(1+\frac{|x|}{t^{1/\alpha}}\right)^{-\alpha-\gamma}. \end{aligned}$$

□

For the regularity in time we have another estimate. Here the spatial bound is satisfactory, cf. (19), and the temporal growth at $t=0$ will later be tempered by making use of cancellations.

Lemma 2.7. *For every $\beta \in \mathbb{N}_0^d$ there exists a constant $c>0$ such that*

$$|\partial_t \partial_x^\beta p_t^z(x)| \le c t^{-1-|\beta|/\alpha} G_t^{(\alpha+\gamma)}(x), \quad x, z \in \mathbb{R}^d, t>0. \tag{44}$$

Proof. It follows from (36) and (10) that

$$\partial_t \partial_x^\beta p_t^z(x) = \partial_x^\beta \partial_t p_t^z(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} q(z,\xi)(-1)^{|\beta|+1} \xi^\beta e^{-ix\cdot\xi - tq(z,\xi)}\, d\xi.$$

Recall that

$$\partial_t p_t^z(x) = \int_{\mathbb{R}^d} \left(p_t^z(x+u) - p_t^z(x) - u \cdot \nabla_x p_t^z(x) \mathbb{1}_{\{|u|\le 1\}} \right) \nu(z,du),$$

cf. (37). Differentiating with respect to x and using A2 we get

$$\begin{aligned} \left|\partial_t \partial_x^\beta p_t^z(x)\right| &= \left|\partial_x^\beta \partial_t p_t^z(x)\right| \\ &= \left| \int \left(\partial_x^\beta p_t^z(x+u) - \partial_x^\beta p_t^z(x) - \partial_x^\beta u \cdot \nabla_x p_t^z(x) \mathbb{1}_{\{|u|\le 1\}} \right) \nu(z,du) \right| \\ &\le C \int \left| \partial_x^\beta p_t^z(x+u) - \partial_x^\beta p_t^z(x) - u \cdot \nabla_x \partial_x^\beta p_t^z(x) \mathbb{1}_{\{|u|\le t^{1/\alpha}\}} \right| \nu_0(du). \end{aligned}$$

Using Proposition 2.4 and Lemma 2.2 we obtain (44). □

The Hölder continuity of $z \mapsto p_t^z$, will be proved in Lemma 2.11 after some auxiliary lemmas. In the first one we prove the symmetry of the operators L^w.

Lemma 2.8. *For every $w \in \mathbb{R}^d$ the operator L^w is symmetric, i.e.,*

$$\int_{\mathbb{R}^d} \varphi(x) L^w f(x)\, dx = \int_{\mathbb{R}^d} L^w \varphi(x) f(x)\, dx \tag{45}$$

for all $\varphi, f \in C_b^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Proof. For every $\delta>0$ we have

$$\int \int_{|u|>\delta} |f(x+u)\varphi(x)| \nu(w,du) dx \le \|f\|_\infty \nu(w, B(0,\delta)^c) \|\varphi\|_1 < \infty.$$

Hence, by Fubini's theorem, change of variables and symmetry of $\nu(w,\cdot)$ we get

$$\int_{\mathbb{R}^d} \varphi(x) \int_{|u|>\delta} f(x+u) \nu(w,du)\, dx \quad = \quad \int_{\mathbb{R}^d} f(y) \int_{|u|>\delta} \varphi(y+u)\, \nu(w,du)\, dy.$$

By subtracting $\int_{\mathbb{R}^d}\int_{|u|>\delta} f(x)\varphi(x)\nu(w,du)dx$, we obtain

$$\int_{\mathbb{R}^d} \varphi(x)L^{w,\delta}f(x)\,dx = \int_{\mathbb{R}^d} L^{w,\delta}\varphi(x)f(x)\,dx.$$

Let $\delta \to 0$. By dominated convergence we get (45), since for $g \in C_b^2(\mathbb{R}^d)$, $\delta \in (0,1)$,

$$\begin{aligned} |L^{w,\delta}g(x)| &= \left| \int_{|u|>\delta} \big(g(x+u) - g(x) - \mathbb{1}_{B(0,1)}(u)\nabla g(x)\cdot u\big)\ \nu(w,du)\right| \\ &\le \tfrac{1}{2}d^2\|g\|_2 \int_{|u|<1} |u|^2\,\nu(w,du) + 2\|g\|_\infty \int_{|u|>1} \nu(w,du) < \infty. \end{aligned}$$

Here, as usual, $\|g\|_2 = \sup\{|\partial^\beta g(x)| :\ x \in \mathbb{R}^d, \beta \in \mathbb{N}_0^d, |\beta| \le 2\}$. □

Corollary 2.9. *For every $f \in C_b^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $w \in \mathbb{R}^d$ such that $L^w f \in L^1(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} L^w f(x)\,dx = 0.$$

Proof. Let $\varphi_n \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \le \varphi_n(x) \le 1$ and $\varphi_n(x) = 1$ for every $x \in B(0,n)$ and $\|\varphi_n\|_2 \le c_0$ for every $n \in \mathbb{N}$. Note that

$$|L^w\varphi_n(x)| \le c_1 \int_{|u|<1} |u|^2\,\nu(w,du) + 2\int_{|u|\ge 1} \nu(w,du) < \infty,$$

and for $n > |x|$ we have

$$|L^w\varphi_n(x)| = \left|\int_{|x+u|>n} (\varphi_n(x+u) - 1)\ \nu(w,du)\right| \le 2\nu(w, B(-x,n)^c),$$

which yields $\lim_{n\to\infty} L^w\varphi_n(x) = 0$ for every $x \in \mathbb{R}^d$. By the symmetry of L^w,

$$\int_{\mathbb{R}^d} \varphi_n(x)L^w f(x)\,dx = \int_{\mathbb{R}^d} L^w\varphi_n(x)f(x)\,dx,$$

and the corollary follows by the dominated convergence theorem. □

Lemma 2.10. *Let $t > 0$, $x, w_1, w_2 \in \mathbb{R}^d$ and*

$$\phi(s) = \begin{cases} p_{t-s}^{w_1} * p_s^{w_2}(x) & \text{if} \quad s \in (0,t), \\ p_t^{w_1}(x) & \text{if} \quad s = 0, \\ p_t^{w_2}(x) & \text{if} \quad s = t. \end{cases}$$

Then ϕ is continuous on $[0,t]$, $\partial_s\phi(s)$ exists on $(0,t)$ and

$$\partial_s\phi(s) = \int_{\mathbb{R}^d} \big(p_{t-s}^{w_1}(z-x)L^{w_2}p_s^{w_2}(z) - p_s^{w_2}(z)L^{w_1}p_{t-s}^{w_1}(z-x)\big)\ dz.$$

Proof. We have

$$\begin{aligned}|\phi(s)-\phi(0)| &= \left|\int_{\mathbb{R}^d} p^{w_1}_{t-s}(x-z)p^{w_2}_s(z)\,dz - p^{w_1}_t(x)\right|\\ &\le \left|\int_{\mathbb{R}^d} p^{w_1}_{t}(x-z)p^{w_2}_s(z)\,dz - p^{w_1}_t(x)\right|\\ &\quad + \int_{\mathbb{R}^d} |p^{w_1}_{t-s}(x-z)-p^{w_1}_t(x-z)|p^{w_2}_s(z)\,dz\\ &= I_1(s)+I_2(s),\end{aligned}$$

and $\lim_{s\to 0} I_1(s)=0$, since the semigroup $P^{w_2}_s f(x)=\int_{\mathbb{R}^d} f(x-z)p^{w_2}_s(z)\,dz$ is strongly continuous and $p^{w_1}_t(x-\cdot)\in C_0(\mathbb{R}^d)$. For $s\in(0,t/2)$ from Lemma 2.7,

$$|p^{w_1}_{t-s}(x-z)-p^{w_1}_t(x-z)| \le s \sup_{u\in(t-s,t)} |\partial_u p^{w_1}_u(x-z)|$$
$$\le cs \sup_{u\in(t-s,t)} \left\{u^{-1}G^{(\alpha+\gamma)}_u(x-z)\right\} \le cst^{-1-d/\alpha}\left(\frac{|x-z|}{t^{1/\alpha}}\vee 1\right)^{-\alpha-\gamma}.$$

From the strong continuity of $s\mapsto P^{w_2}_s$ we have

$$\lim_{s\to 0}\int_{\mathbb{R}^d}\left(\frac{|x-z|}{t^{1/\alpha}}\vee 1\right)^{-\alpha-\gamma} p^{w_2}_s(z)\,dz = \left(\frac{|x|}{t^{1/\alpha}}\vee 1\right)^{-\alpha-\gamma},$$

therefore $\lim_{s\to 0} I_2(s)=0$. This yields the continuity of ϕ at $s=0$. The proof of the continuity at $s=t$ is analogous.
For every $z\in\mathbb{R}^d$ and $s\in(0,t)$ from (37) we obtain

$$\partial_s\left(p^{w_1}_{t-s}(x-z)p^{w_2}_s(z)\right) = p^{w_1}_{t-s}(z-x)L^{w_2}p^{w_2}_s(z) - p^{w_2}_s(z)L^{w_1}p^{w_1}_{t-s}(z-x).$$

From Lemmas 2.4 and 2.2 we get

$$|\partial_s\left(p^{w_1}_{t-s}(x-z)p^{w_2}_s(z)\right)| \le cG^{(\alpha+\gamma)}_s(z)G^{(\alpha+\gamma)}_{t-s}(z-x)\left(s^{-1}+(t-s)^{-1}\right).$$

Hence for every $\delta\in(0,t/2)$ and $s\in(\delta,t-\delta)$ we obtain

$$|\partial_s\left(p^{w_1}_{t-s}(x-z)p^{w_2}_s(z)\right)| \le 2c\delta^{-1-2d/\alpha}\left(\frac{|z|}{t^{1/\alpha}}\vee 1\right)^{-\alpha-\gamma}\left(\frac{|z-x|}{t^{1/\alpha}}\vee 1\right)^{-\alpha-\gamma}, \quad (46)$$

and since $\int(\frac{|z|}{t^{1/\alpha}}\vee 1)^{-\alpha-\gamma}(\frac{|z-x|}{t^{1/\alpha}}\vee 1)^{-\alpha-\gamma}\,dz<\infty$, this yields

$$\partial_s\phi(s) = \int_{\mathbb{R}^d}\left(p^{w_1}_{t-s}(z-x)L^{w_2}p^{w_2}_s(z) - p^{w_2}_s(z)L^{w_1}p^{w_1}_{t-s}(z-x)\right)dz, \quad s\in(0,t).$$

□

Lemma 2.11. *For any $\beta\in\mathbb{N}^d_0$ and $\theta\in(0,\eta\wedge(\alpha+\gamma-d))$ there exists $c>0$ such that for all $x,w_1,w_2\in\mathbb{R}^d$, $t>0$,*

$$\left|\partial^\beta_x p^{w_1}_t(x) - \partial^\beta_x p^{w_2}_t(x)\right| \le c(|w_1-w_2|^\eta\wedge 1)t^{-|\beta|/\alpha}G^{(\alpha+\gamma-\theta)}_t(x). \quad (47)$$

Proof. Let us prove first the statement with $\beta = 0$. Since for $a, b > 0$ we have $|e^{-a} - e^{-b}| \le |a-b|e^{-(a\wedge b)}$, by Hölder continuity of $q(z,\xi)$ and $q(z,\xi) \approx |\xi|^\alpha$,

$$\begin{aligned}\left|p_t^{w_1}(x) - p_t^{w_2}(x)\right| &= (2\pi)^{-d}\Big|\int_{\mathbb{R}^d} e^{-i\xi x}\Big(e^{-tq(w_1,\xi)} - e^{-tq(w_2,\xi)}\Big)d\xi\Big| \\ &\le c_1(|w_1-w_2|^\eta \wedge 1)\Big|\int_{\mathbb{R}^d} t|\xi|^\alpha e^{-ct|\xi|^\alpha}d\xi\Big| \\ &\le c_2(|w_1-w_2|^\eta\wedge 1)t^{-d/\alpha}, \quad t>0,\ w_1,w_2,x\in\mathbb{R}^d.\end{aligned}$$

Since for $|x| \le t^{1/\alpha}$ we have $G_t^{(\alpha+\gamma-\theta)}(x) = t^{-d/\alpha}$, we get (47) for such t and x. Suppose now that $|x| \ge t^{1/\alpha}$. We note that $p_t^z \in C_b^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ for every $t>0$ and $z\in\mathbb{R}^d$. By Lemma 2.10, (46) (which yields integrability of $\partial_s\phi(s)$ on $[\delta, t-\delta]$ for every $\delta\in(0,t/2)$), Lemma 2.8 and the symmetry of $p_t^w(x)$ in x,

$$\begin{aligned}p_t^{w_2}(x) - p_t^{w_1}(x) &= \int_0^t \partial_s \int_{\mathbb{R}^d} p_{t-s}^{w_1}(x-z)p_s^{w_2}(z)\,dzds \\ &= \int_0^t\int_{\mathbb{R}^d}\big[p_{t-s}^{w_1}(z-x)L^{w_2}p_s^{w_2}(z) - p_s^{w_2}(z)L^{w_1}p_{t-s}^{w_1}(z-x)\big]\,dzds \\ &= \int_0^t\int_{\mathbb{R}^d} p_s^{w_2}(z)\big[L^{w_2}-L^{w_1}\big]p_{t-s}^{w_1}(z-x)\,dzds \\ &= \int_0^t\int_{\mathbb{R}^d}\big(p_s^{w_2}(z)-p_s^{w_2}(x)\big)\big[L^{w_2}-L^{w_1}\big]p_{t-s}^{w_1}(z-x)\,dzds \\ &\quad + \int_0^t\int_{\mathbb{R}^d} p_s^{w_2}(x)\big[L^{w_2}-L^{w_1}\big]p_{t-s}^{w_1}(z-x)\,dzds = I_1 + I_2.\end{aligned}$$

From Lemma 2.2 and Corollary 2.9 we derive

$$\int_{\mathbb{R}^d}\big[L^{w_2}-L^{w_1}\big]p_{t-s}^{w_1}(z-x)dz = 0,$$

hence, $I_2 = 0$. Next we observe that for all $w,x,y\in\mathbb{R}^d$, $t>0$,

$$\left|p_t^w(x) - p_t^w(y)\right| \le c\left(\frac{|x-y|}{t^{1/\alpha}}\wedge 1\right)\left(G_t^{(\alpha+\gamma)}(x) + G_t^{(\alpha+\gamma)}(y)\right), \tag{48}$$

which follows from the Taylor expansion of $p_t^w(x)$. Indeed, if $|x-y|\ge t^{1/\alpha}$, then (48) is straightforward: we just estimate the difference of functions by their sum and use Lemma 2.4. If $|x-y|\le t^{1/\alpha}$, then using the Taylor expansion and Lemma 2.4 with $|\beta|=1$ we get

$$\begin{aligned}|p_t^w(x)-p_t^w(y)| &\le |x-y|\cdot\sup_{\zeta\in[0,1]}|\nabla_x p_t^w(x+\zeta(y-x))| \\ &\le c_1|x-y|t^{-1/\alpha}\sup_{\zeta\in[0,1]}G_t^{(\alpha+\gamma)}(x+\zeta(y-x)) \\ &\le c_2\frac{|x-y|}{t^{1/\alpha}}\left(G_t^{(\alpha+\gamma)}(x)+G_t^{(\alpha+\gamma)}(y)\right),\end{aligned}$$

since for $|x| \le 2t^{1/\alpha}$ and every $\zeta \in (0,1)$ we have

$$G_t^{(\alpha+\gamma)}(x+\zeta(y-x)) = t^{-d/\alpha}\Big(\frac{|x+\zeta(y-x)|}{t^{1/\alpha}} \vee 1\Big)^{-\alpha-\gamma} \le t^{-d/\alpha} \le 2^{\alpha+\gamma} G_t^{(\alpha+\gamma)}(x),$$

and for $|x| > 2t^{1/\alpha}$ we have $|x+\zeta(y-x)| \ge |x| - |y-x| \ge |x|/2$, which yields

$$G_t^{(\alpha+\gamma)}(x+\zeta(y-x)) = t^{1-(d-\gamma)/\alpha}|x+\zeta(y-x)|^{-\alpha-\gamma} \le 2^{\alpha+\gamma} t^{1-(d-\gamma)/\alpha}|x|^{-\alpha-\gamma} = 2^{\alpha+\gamma} G_t^{(\alpha+\gamma)}(x).$$

Further, using (48), A2, Lemma 2.4 and Lemma 2.2 with $\zeta = d$ we get

$$\begin{aligned} |I_1| &\le c_1(|w_1-w_2|^\eta \wedge 1)\int_0^t \int_{\mathbb{R}^d} \Big(\frac{|x-z|}{s^{1/\alpha}} \wedge 1\Big)\Big(G_s^{(\alpha+\gamma)}(x) + G_s^{(\alpha+\gamma)}(z)\Big) \\ &\quad \cdot (t-s)^{-1} G_{t-s}^{(\alpha+\gamma)}(x-z)\, dzds \\ &\le c_2(|w_1-w_2|^\eta \wedge 1)\int_0^t s^{-\theta/\alpha}(t-s)^{-1+\theta/\alpha} \\ &\quad \cdot \int_{\mathbb{R}^d} G_{t-s}^{(\alpha+\gamma-\theta)}(x-z)\Big(G_s^{(\alpha+\gamma)}(x) + G_s^{(\alpha+\gamma)}(z)\Big) dzds, \end{aligned}$$

where in the second inequality above we use the fact that

$$\Big(\frac{|x-z|}{s^{1/\alpha}} \wedge 1\Big) \le \Big(\frac{t-s}{s}\Big)^{\theta/\alpha}\Big(\frac{|x-z|}{(t-s)^{1/\alpha}} \vee 1\Big)^{\theta}, \quad x,z \in \mathbb{R}^d,\ t > s > 0,\ \theta \in (0,1].$$

By Lemma 2.3 we obtain

$$|I_1| \le c_3(|w_1-w_2|^\eta \wedge 1)\int_0^t s^{-\theta/\alpha}(t-s)^{-1+\theta/\alpha}\Big(G_s^{(\alpha+\gamma)}(x) + G_t^{(\alpha+\gamma-\theta)}(x)\Big)\, ds.$$

Note that for $|x|^\alpha \ge t \ge s$ we have $G_s^{(\alpha+\gamma)}(x) = s^{(\alpha+\gamma-d)/\alpha}/|x|^{\alpha+\gamma}$. Therefore,

$$\begin{aligned} &\int_0^t s^{-\theta/\alpha}(t-s)^{-1+\theta/\alpha}\Big(G_s^{(\alpha+\gamma)}(x) + G_t^{(\alpha+\gamma-\theta)}(x)\Big)\, ds \\ &\qquad = B\left(\tfrac{2\alpha+\gamma-d-\theta}{\alpha}, \tfrac{\theta}{\alpha}\right)\frac{t^{(\alpha+\gamma-d)/\alpha}}{|x|^{\alpha+\gamma}} + \frac{\pi}{\sin(\pi\theta/\alpha)} G_t^{(\alpha+\gamma-\theta)}(x) \\ &\qquad \le c\Big(G_t^{(\alpha+\gamma)}(x) + G_t^{(\alpha+\gamma-\theta)}(x)\Big) \le 2c\, G_t^{(\alpha+\gamma-\theta)}(x). \end{aligned}$$

Thus we have (47) also for $|x| \ge t^{1/\alpha}$.

To prove the statement for $|\beta| \ge 1$, denote by $h_t(x)$ the convolution semigroup corresponding to the Lévy measure $(2M_0)^{-1}\nu_0(du)$, and denote by $\tilde{h}_t^z(x)$ the convolution semigroup with the Lévy measure $\nu^{\#}(z,du) = \nu(z,du) - (2M_0)^{-1}\nu_0(du)$. We note that $p_t^z(x) = h_t * \tilde{h}_t^z(x)$ and $\nu^{\#}$ satisfies A2 with constant $2M_0$ instead of M_0 and therefore (47) holds also for $\tilde{h}_t^z(x)$ with $\beta = 0$ (and perhaps a different constant c). Hence, using Lemma 2.4 for h_t and Lemma 2.3, we get for $\beta \in \mathbb{N}_0^d$

$$\begin{aligned}
\big|\partial_x^\beta p_t^{w_1}(x) - \partial_x^\beta p_t^{w_2}(x)\big| &= \big|\partial_x^\beta \int_{\mathbb{R}^d} h_t(x-y)\big(\tilde{h}_t^{w_1}(y) - \tilde{h}_t^{w_2}(y)\big)\,dy\big| \\
&= \big|\int_{\mathbb{R}^d} \partial_x^\beta h_t(x-y)\big(\tilde{h}_t^{w_1}(y) - \tilde{h}_t^{w_2}(y)\big)\,dy\big| \\
&\le c_1(|w_1-w_2|^\eta \wedge 1)t^{-|\beta|/\alpha}\int_{\mathbb{R}^d} G_t^{(\alpha+\gamma)}(x-y)G_t^{(\alpha+\gamma-\theta)}(y)\,dy \\
&\le c_2(|w_1-w_2|^\eta \wedge 1)t^{-|\beta|/\alpha}G_t^{(\alpha+\gamma-\theta)}(x).
\end{aligned}$$

This finishes the proof. □

Below we establish a similar continuity property.

Lemma 2.12. *For all $t>0$ and $y \in \mathbb{R}^d$ we have*

$$\lim_{z\to y}\sup_{x\in\mathbb{R}^d} |p_t^z(z-x) - p_t^y(y-x)| = 0. \tag{49}$$

Proof. From Lemma 2.11 and (48), we get

$$\begin{aligned}
|p_t^z(z-x)-p_t^y(y-x)| &\le |p_t^z(z-x) - p_t^y(z-x)| + |p_t^y(z-x) - p_t^y(y-x)| \\
&\le c_1(|z-y|^\eta \wedge 1)G_t^{(\alpha+\gamma-\theta)}(z-x) \\
&\quad + c_2\left(\frac{|z-y|}{t^{1/\alpha}} \wedge 1\right)\left(G_t^{(\alpha+\gamma)}(z-x) + G_t^{(\alpha+\gamma)}(y-x)\right) \\
&\le c_1 t^{-d/\alpha}(|z-y|^\eta \wedge 1) + 2c_2\left(\frac{|z-y|}{t^{1/\alpha}} \wedge 1\right)t^{-d/\alpha},
\end{aligned}$$

and (49) follows. □

Lemma 2.11 also yields Lemma 2.13 and 2.14 below. We have the following result on strong continuity of $p_t^0(x,y) = p_t^y(y-x)$.

Lemma 2.13. *For $f \in C_0(\mathbb{R}^d)$, $\lim_{t\to 0}\sup_x \left|\int_{\mathbb{R}^d} p_t^y(y-x)f(y)\,dy - f(x)\right| = 0$.*

Proof. We have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} p_t^y(y-x)f(y)\,dy - f(x)\Big| \\
&\le \Big|\int_{\mathbb{R}^d} p_t^x(y-x)f(y)\,dy - f(x)\Big| + \int_{\mathbb{R}^d} |p_t^y(y-x) - p_t^x(y-x)||f(y)|\,dy \\
&\le \int_{\mathbb{R}^d} |f(y)-f(x)|p_t^x(y-x)\,dy + \int_{\mathbb{R}^d} |p_t^y(y-x) - p_t^x(y-x)||f(y)|\,dy \\
&= I_1(t) + I_2(t).
\end{aligned}$$

Let $\delta > 0$. Using Lemma 2.4 for every $t \in (0, \delta^\alpha)$ we obtain

$$\begin{aligned} I_1(t) &\le c_1 \int_{\mathbb{R}^d} |f(y) - f(x)| G_t^{(\alpha+\gamma)}(y-x)\, dy \\ &= c_1 \int_{|y-x|\le t^{1/\alpha}} |f(y)-f(x)| t^{-d/\alpha}\, dy \\ &\quad + c_1 \int_{t^{1/\alpha}<|y-x|\le\delta} |f(y)-f(x)| t^{1-(d-\gamma)/\alpha} |y-x|^{-\alpha-\gamma}\, dy \\ &\quad + c_1 \int_{|y-x|>\delta} |f(y)-f(x)| t^{1-(d-\gamma)/\alpha} |y-x|^{-\alpha-\gamma}\, dy \\ &\le c_2 \sup_{|y-x|\le\delta} |f(y)-f(x)| + c_3 \|f\|_\infty t^{1-(d-\gamma)/\alpha} \delta^{-\alpha-\gamma+d}. \end{aligned}$$

Taking $\delta > 0$ such that $|f(y) - f(x)| \le \varepsilon/(2c_2)$ for $|y-x| \le \delta$, and t_0 such that

$$c_3 \|f\|_\infty t^{1-(d-\gamma)/\alpha} \delta^{-\alpha-\gamma+d} \le \varepsilon/2 \quad \text{for } t \in (0, t_0),$$

we get $\sup_{x\in\mathbb{R}^d} I_1(t,x) \le \varepsilon$, hence $\sup_{x\in\mathbb{R}^d} I_1(t,x) \to 0$, as $t \to 0$. To estimate $I_2(t,x)$ we take $\epsilon \in (\frac{d}{d+\eta}, 1)$. By Lemma 2.11 for $\theta \in (0, \eta \wedge (\alpha+\gamma-d))$ we get

$$\begin{aligned} I_2(t) &= \left(\int_{|y-x|\le t^{\epsilon/\alpha}} + \int_{|y-x|>t^{\epsilon/\alpha}} \right) |p_t^y(y-x) - p_t^x(y-x)| |f(y)|\, dy \qquad (50) \\ &\le c_4 \|f\|_\infty t^{\epsilon(d+\eta)/\alpha - d/\alpha} + c_4 \|f\|_\infty \int_{|y-x|>t^{\epsilon/\alpha}} G_t^{(\alpha+\gamma-\theta)}(y-x)\, dy \\ &= c_4 \|f\|_\infty t^{\epsilon(d+\eta)/\alpha - d/\alpha} + c_4 \|f\|_\infty \int_{|z|>t^{(\epsilon-1)/\alpha}} (|z| \vee 1)^{-\gamma-\alpha+\theta}\, dz. \end{aligned}$$

By our choice of ϵ, both terms tend to 0 as $t \to 0$. □

We now point out the impact of cancellations, cf. Lemma 2.7.

Lemma 2.14. *For every* $\theta \in \left(0, \eta \wedge \frac{\alpha+\gamma-d}{2}\right)$ *we have*

$$\left| \int_{\mathbb{R}^d} \partial_t p_t^y(y-x) dy \right| \le c t^{-1+\theta/\alpha}, \quad x \in \mathbb{R}^d,\ t > 0. \qquad (51)$$

Proof. Using the fact that $\partial_t p_t^z(x) = L^z p_t^z(x)$ we get

$$\begin{aligned} \int_{\mathbb{R}^d} \partial_t p_t^z(z-x) dz &= \int_{\mathbb{R}^d} L^z p_t^z(z-x)\, dz = \int_{\mathbb{R}^d} L^z \big(p_t^z(z-x) - p_t^x(z-x)\big)\, dz \\ &\quad + \int_{\mathbb{R}^d} \big(L^z - L^x\big) p_t^x(z-x)\, dz \quad + \int_{\mathbb{R}^d} L^x p_t^x(z-x)\, dz. \end{aligned}$$

Lemma 2.9 yields $\int_{\mathbb{R}^d} L^x p^x_t(z-x)\,dz = 0$. Further, by Lemma 2.11 and 2.2,

$$\begin{aligned}\int_{\mathbb{R}^d} &\big|L^z(p^z_t(z-x) - p^x_t(z-x))\big| dz \\ &\le c_1 \int_{\mathbb{R}^d} (|x-z|^\eta \wedge 1) t^{-1} G^{(\alpha+\gamma-\theta)}_t(z-x)\,dz \\ &= c_1 t^{-1-d/\alpha} \int_{\mathbb{R}^d} (|y|^\eta \wedge 1)(1 \vee (t^{-1/\alpha}|y|))^{-\gamma-\alpha+\theta}\,dy \\ &\le c_2 t^{-1+(\eta\wedge(\alpha+\gamma-\theta-d))/\alpha} \le c_2 t^{-1+\theta/\alpha}.\end{aligned}$$

Similarly, by A2, Lemma 2.4 and Lemma 2.2 we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \big|\big(L^z - L^x\big)p^x_t(z-x)\big|\,dz &\le c_1 t^{-1-d/\alpha} \int_{\mathbb{R}^d} (|z-x|^\eta \wedge 1) G^{(\alpha+\gamma)}_t(z-x)\,dz \\ &\le c_2 t^{-1+(\eta\wedge(\alpha+\gamma-d))/\alpha} \le c_2 t^{-1+\theta/\alpha},\end{aligned}$$

which finishes the proof of the lemma. □

3 Parametrix

3.1 Proof of convergence

In this section we prove that p_t given by (15) is well defined. To this end let

$$\Psi^\#_t(x,y) := \sum_{k=1}^{\infty} |\Phi|^{\boxtimes k}_t(x,y), \tag{52}$$

and

$$p^\#_t(x,y) = p^0_t(x,y) + \big(p^0 \boxtimes \Psi^\#\big)_t(x,y). \tag{53}$$

Our first result shows that the series (52) and the function $p^\#_t(x,y)$ are finite and possess nice estimates. Then $p_t(x,y)$ is well defined, with the same upper bounds.

Proposition 3.1. *The series* (52) *converges, the integral* $p^0 \boxtimes \Psi^\#$ *exists, and*

$$p^\#_t(x,y) \le Ce^{ct} G^{(\alpha+\gamma)}_t(y-x), \quad t>0,\ x,y \in \mathbb{R}^d. \tag{54}$$

The result depends on the auxiliary estimates of $|\Phi_t(x,y)|$ and its convolutions, which we now give. The proof of Proposition 3.1 will be given in the end of the next subsection.

Lemma 3.2. *Under condition* A2 *there exists* $C_\Phi > 0$ *such that*

$$\big|\Phi_t(x,y)\big| \le C_\Phi t^{-1}(1 \wedge |y-x|^\eta) G^{(\alpha+\gamma)}_t(y-x), \quad x,y \in \mathbb{R}^d,\ t>0, \tag{55}$$

the function $\partial_t \Phi_t(x,y)$ *exists for all* $t>0$, $x,y \in \mathbb{R}^d$, *is continuous in* t, *and*

$$\big|\partial_t \Phi_t(x,y)\big| \le C_\Phi t^{-2}(1 \wedge |y-x|^\eta) G^{(\alpha+\gamma)}_t(y-x), \quad x,y \in \mathbb{R}^d,\ t>0. \tag{56}$$

Proof. By the symmetry of $\nu(x,\cdot)$ for every $x\in\mathbb{R}^d$ and A2,

$$\begin{aligned}&\left|\Phi_t(x,y)\right|\\&=\left|\int\left(p_t^0(x+u,y)-p_t^0(x,y)-u\cdot\nabla_x p_t^0(x,y)\mathbb{1}_{\{|u|\le t^{1/\alpha}\}}\right)\right.\\&\qquad\cdot\left(\nu(x,du)-\nu(y,du)\right)\Big|\\&\le C\left(|y-x|^\eta\wedge 1\right)\mathcal{A}_t^{\#}p_t^0(x,y).\end{aligned}$$

By Lemma 2.4 and Lemma 2.2 with $\zeta=d$ and $\kappa=0$ we get (55). The estimate (56) follows from Lemma 2.2 and the fact that

$$\begin{aligned}&\partial_t\Phi_t(x,y)\\&=\int\partial_t\left(p_t^0(x+u,y)-p_t^0(x,y)-u\cdot\nabla_x p_t^0(x,y)\mathbb{1}_{|u|\le1}\right)\left(\nu(x,du)-\nu(y,du)\right).\end{aligned}$$

We can change the order of differentiation and integration, because by Lemma 2.7 for every $t>0$ and $\epsilon\in(-t/2,t/2)$ we have

$$\begin{aligned}&\left|\partial_t p_{t+\epsilon}^0(x+u,y)-\partial_t p_{t+\epsilon}^0(x,y)-u\cdot\nabla_x\partial_t p_{t+\epsilon}^0(x,y)\mathbb{1}_{\{|u|\le1\}}\right|\\&\quad\le c_1(t+\epsilon)^{-1-(d+2)/\alpha}|u|^2\,\mathbb{1}_{\{|u|\le1\}}+c_2(t+\epsilon)^{-1-d/\alpha}\mathbb{1}_{\{|u|>1\}}\\&\quad\le c_3t^{-1-(d+2)/\alpha}|u|^2\,\mathbb{1}_{\{|u|\le1\}}+c_4t^{-1-d/\alpha}\mathbb{1}_{\{|u|>1\}}\\&\quad=:g_t(u),\quad u\in\mathbb{R}^d,x,y\in\mathbb{R}^d,\end{aligned}$$

and $\int_{\mathbb{R}^d}g_t(u)\,\nu(w,du)<\infty$ for every $w\in\mathbb{R}^d$. This yields (56) and the continuity of $t\mapsto\partial_t\Phi_t(x,y)$. □

To estimate $\Phi^{\boxtimes k}$ we will use the following *sub-convolution* property.

Definition 3.1. *A non-negative kernel $H_t(x), t>0, x\in\mathbb{R}^d$, has the sub-convolution property if there is a constant $C_H>0$ such that*

$$(H_{t-s}*H_s)(x)\le C_H H_t(x),\quad 0<s<t,\quad x\in\mathbb{R}^d.\tag{57}$$

It follows from Lemma 2.3 that $G_t^{(\beta)}(x)$ has the sub-convolution property. On the other hand, the kernel $t^{-1}(1\wedge|x|^\eta)G_t^{(\alpha+\gamma)}(x)$ from Lemma 3.2 does not have it; take for instance $x=0$ in (57) or see [36] in the case when $d=\gamma$. To circumvent this problem, for $\zeta>0$ and $\kappa\in(d-\alpha,d]$ we define

$$H_t^{(\kappa,\zeta)}(x)=\left(t^{-\zeta/\alpha}\wedge\left(\frac{|x|}{t^{1/\alpha}}\vee1\right)^\zeta\right)G_t^{(\alpha+\kappa)}(x).\tag{58}$$

Proposition 3.3. *Assume that*

$$\alpha+\kappa-d>\zeta.\tag{59}$$

Then the kernels $H_t^{(\kappa,\zeta)}(x)$ satisfy the sub-convolution property with some constant $C_H>0$ and there exists a positive constant $C>0$ such that

$$\int_{\mathbb{R}^d}H_t^{(\kappa,\zeta)}(x)dx\le C,\quad t>0.\tag{60}$$

Proof. We follow the proof of [36, Proposition 3.3]. We have

$$H_t^{(\kappa,\zeta)}(x) \le (t^{-\zeta/\alpha} \wedge 1) G_t^{(\kappa+\alpha-\zeta)}(x), \quad x \in \mathbb{R}^d, t > 0, \tag{61}$$

and

$$H_t^{(\kappa,\zeta)}(x) = (t^{-\zeta/\alpha} \wedge 1)\, G_t^{(\kappa+\alpha-\zeta)}(x), \quad |x| \le 1 \vee t^{1/\alpha}, t > 0. \tag{62}$$

Clearly, (61) implies that

$$\int_{\mathbb{R}^d} H_t^{(\kappa,\zeta)}(x) dx \le C. \tag{63}$$

We notice that

$$((t-s)^{-\zeta/\alpha} \wedge 1)(s^{-\zeta/\alpha} \wedge 1) \le 2^{\zeta/\alpha}(t^{-\zeta/\alpha} \wedge 1), \quad 0 < s < t.$$

By this, Lemma 2.3, (61) and (62),

$$\left(H_{t-s}^{(\kappa,\zeta)} * H_s^{(\kappa,\zeta)}\right)(x) \le C H_t^{(\kappa,\zeta)}(x), \quad |x| \le 1 \vee t^{1/\alpha}, t > 0.$$

To complete the proof we assume that $|x| \ge 1 \vee t^{1/\alpha}$. We have

$$\left(H_{t-s}^{(\kappa,\zeta)} * H_s^{(\kappa,\zeta)}\right)(x) \le \left(\int_{|z|\ge|x|/2} + \int_{|x-z|\ge|x|/2}\right) H_{t-s}^{(\kappa,\zeta)}(z) H_s^{(\kappa,\zeta)}(x-z) dz.$$

By the structure of $H_t^{(\kappa,\zeta)}(x)$, for $|z| \ge |x|/2$ we obtain $H_{t-s}^{(\kappa,\zeta)}(z) \le c H_{t-s}^{(\kappa,\zeta)}(x)$. We have $H_t^{(\kappa,\zeta)}(x) = t^{1-(\zeta+d-\kappa)/\alpha}|x|^{-\kappa-\alpha}$, and by (59),

$$\begin{aligned} H_{t-s}^{(\kappa,\zeta)}(x) &= (t-s)^{1-(\zeta+d-\kappa)/\alpha}|x|^{-\kappa-\alpha} \\ &\le t^{1-(\zeta+d-\kappa)/\alpha}|x|^{-\kappa-\alpha} = H_t^{(\kappa,\zeta)}(x). \end{aligned}$$

Using (63) we get

$$\begin{aligned} \int_{|z|\ge|x|/2} H_{t-s}^{(\kappa,\zeta)}(z) H_s^{(\kappa,\zeta)}(x-z)\, dz &\le c H_t^{(\kappa,\zeta)}(x) \int_{|z|\ge|x|/2} H_t^{(\kappa,\zeta)}(x-z)\, dz \\ &\le c H_t^{(\kappa,\zeta)}(x) \int_{\mathbb{R}^d} H_t^{(\kappa,\zeta)}(z)\, dz \le C H_t^{(\kappa,\zeta)}(x). \end{aligned}$$

Similarly,

$$\int_{|x-z|\ge|x|/2} H_{t-s}^{(\kappa,\zeta)}(z) H_s^{(\kappa,\zeta)}(x-z) dz \le C H_t^{(\kappa,\zeta)}(x). \quad \square$$

Let us rewrite the upper estimate in (55). Since

$$1 \wedge |x|^\theta = t^{\theta/\alpha}\left(t^{-\theta/\alpha} \wedge \left(\frac{|x|}{t^{1/\alpha}}\right)^\theta\right) \le t^{\theta/\alpha}\left(t^{-\theta/\alpha} \wedge \left(\left(\frac{|x|}{t^{1/\alpha}}\right)^\theta \vee 1\right)\right),$$

we get

$$t^{-\theta/\alpha}(1 \wedge |x|^\theta) G_t^{(\alpha+\gamma)}(x) \le H_t^{(\gamma,\theta)}(x), \tag{64}$$

which implies for $\theta \le \eta$

$$\big|\Phi_t(x,y)\big| \le C_\Phi t^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, t>0. \tag{65}$$

Using the sub-convolution property of $H_t^{(\gamma,\theta)}(x)$, we can estimate $\Phi_t^{\boxtimes k}(x,y)$. Let

$$0 < \theta < \eta \wedge (\alpha+\gamma-d). \tag{66}$$

Lemma 3.4. *For every $k \ge 2$ and θ satisfying (66) we have*

$$|\Phi|_t^{\boxtimes k}(x,y) \le \frac{C_1 C_2^k}{\Gamma(k\theta/\alpha)} t^{-1+k\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0. \tag{67}$$

Proof. Let $C_1 = C_H^{-1}$, $C_2 = C_\Phi C_H \Gamma(\theta/\alpha)$, where C_Φ is from (65) and C_H is from Proposition 3.3. We use induction. For $k=1$ we already have (65). Suppose that (67) holds for k. By the sub-convolution property of $H_t^{(\gamma,\theta)}$,

$$\begin{aligned}
&\big|\Phi_t^{\boxtimes(k+1)}(x,y)\big| \\
&\le \frac{C_1 C_\Phi C_2^k}{\Gamma(k\theta/\alpha)} \int_0^t (t-s)^{-1+k\theta/\alpha} s^{-1+\theta/\alpha} \int_{\mathbb{R}^d} H_{t-s}^{(\gamma,\theta)}(x-z) H_s^{(\gamma,\theta)}(z-y)\, dzds \\
&\le \frac{C_1 C_\Phi C_H C_2^k}{\Gamma(k\theta/\alpha)} H_t^{(\gamma,\theta)}(y-x) \int_0^t (t-s)^{-1+k\theta/\alpha} s^{-1+\theta/\alpha}\, ds \\
&= \frac{C_1 C_\Phi C_H C_2^k}{\Gamma(k\theta/\alpha)} t^{-1+(k+1)\theta/\alpha} H_t^{(\gamma,\theta)}(y-x) \frac{\Gamma(k\theta/\alpha)\Gamma(\theta/\alpha)}{\Gamma((k+1)\theta/\alpha)} \\
&= \frac{C_1 C_2^{k+1}}{\Gamma((k+1)\theta/\alpha)} t^{-1+(k+1)\theta/\alpha} H_t^{(\gamma,\theta)}(y-x).
\end{aligned}$$

□

Corollary 3.5. *For $x,y \in \mathbb{R}^d$ and $k=1,2,...$, $t \to \Phi_t^{\boxtimes k}(x,y)$ is continuous.*

Proof. For every $h \in (0,t/2)$ we have

$$\begin{aligned}
&\left|\Phi_{t+h}^{\boxtimes(k+1)}(x,y) - \Phi_t^{\boxtimes(k+1)}(x,y)\right| \\
&\le \int_0^{t-h} \int_{\mathbb{R}^d} |\Phi_{t+h-s}(x,z) - \Phi_{t-s}(x,z)|\, \Phi_s^{\boxtimes k}(z,y)\, dzds \\
&\quad + \int_{t-h}^{t} \int_{\mathbb{R}^d} |\Phi_{t+h-s}(x,z) - \Phi_{t-s}(x,z)|\, \Phi_s^{\boxtimes k}(z,y)\, dzds \\
&\quad + \int_t^{t+h} \int_{\mathbb{R}^d} \Phi_{t+h-s}(x,z) \Phi_s^{\boxtimes k}(z,y)\, dzds = I_1(h) + I_2(h) + I_3(h).
\end{aligned}$$

Using Lemma 3.2, Lemma 3.4 and (64) we obtain

$$
\begin{aligned}
&I_1(h)\\
&\le c_1 h\int_0^{t-h}\int_{\mathbb{R}^d}(t-s)^{-2}(1\wedge|z-x|^\theta)G^{(\alpha+\gamma)}_{t-s}(z-x)\Phi^{\boxtimes k}_s(z,y)\,dzds\\
&\le c_2 h\int_0^{t-h}\int_{\mathbb{R}^d}(t-s)^{-2+\theta/\alpha}s^{-1+k\theta/\alpha}H^{(\gamma,\theta)}_{t-s}(z-x)H^{(\gamma,\theta)}_s(y-z)\,dzds\\
&\le c_3 hH^{(\gamma,\theta)}_t(y-x)\int_0^{t-h}(t-s)^{-2+\theta/\alpha}s^{-1+k\theta/\alpha}\,ds\\
&\le c_4 H^{(\gamma,\theta)}_t(y-x)t^{-1+k\theta/\alpha}h^{\theta/\alpha},
\end{aligned}
$$

and so $\lim_{h\to 0^+} I_1(h)=0$. Furthermore,

$$
\begin{aligned}
&I_2(h)\\
&\le c_1\int_{t-h}^{t}\int_{\mathbb{R}^d}(1\wedge|z-x|^\theta)(t+h-s)^{-1}G^{(\alpha+\gamma)}_{t+h-s}(z-x)\Phi^{\boxtimes k}_s(z,y)\,dzds\\
&\quad+\ c_2\int_{t-h}^{t}\int_{\mathbb{R}^d}(1\wedge|z-x|^\theta)(t-s)^{-1}G^{(\alpha+\gamma)}_{t-s}(z-x)\Phi^{\boxtimes k}_s(z,y)\,dzds\\
&\le c_3 H^{(\gamma,\theta)}_{t+h}(y-x)\int_{t-h}^{t}s^{-1+k\theta/\alpha}(t+h-s)^{-1+\theta/\alpha}\,ds\\
&\quad+\ c_4 H^{(\gamma,\theta)}_{t}(y-x)\int_{t-h}^{t}s^{-1+k\theta/\alpha}(t-s)^{-1+\theta/\alpha}\,ds\\
&\le c_5\left(H^{(\gamma,\theta)}_{t+h}(y-x)+H^{(\gamma,\theta)}_{t}(y-x)\right)t^{-1+k\theta/\alpha}h^{\theta/\alpha}.
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
&I_3(h)\\
&\le c_1\int_{t}^{t+h}\int_{\mathbb{R}^d}(1\wedge|z-x|^\theta)(t+h-s)^{-1}G^{(\alpha+\gamma)}_{t+h-s}(z-x)\Phi^{\boxtimes k}_s(z,y)\,dzds\\
&\le c_2\int_{t}^{t+h}\int_{\mathbb{R}^d}s^{-1+k\theta/\alpha}(t+h-s)^{-1+\theta/\alpha}H^{(\gamma,\theta)}_{t+h-s}(z-x)H^{(\gamma,\theta)}_s(y-z)\,dzds\\
&\le c_3 H^{(\gamma,\theta)}_{t+h}(y-x)t^{-1+k\theta/\alpha}h^{\theta/\alpha},
\end{aligned}
$$

so $\lim_{h\to 0^+} I_2(h)=\lim_{h\to 0^+} I_3(h)=0$ (and analogously for negative h). □

Proof of Proposition 3.1. By Lemma 3.4, the series

$$
\Psi^{\#}_t(x,y)=\sum_{m=1}^{\infty}|\Phi|^{(\boxtimes k)}_t(x,y)
$$

converges uniformly on compact subsets of $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. By the sub-convolution property of $H_t^{(\gamma,\theta)}(x)$, (65), (67), and the estimate

$$\sum_{k=0}^{\infty}\frac{C_2^k t^{\zeta k}}{\Gamma((k+1)\zeta)}\le c_1 e^{c_2 t},\quad \zeta>0, t>0, \tag{68}$$

for which see, e.g., [25], we get

$$|\Psi_t(x,y)|\le \Psi_t^{\#}(x,y)\le c_3 t^{-1+\theta/\alpha}e^{c_2 t}H_t^{(\gamma,\theta)}(y-x),\quad x,y\in\mathbb{R}^d,\quad t>0. \tag{69}$$

For every $t>0$ we have

$$G_s^{(\alpha+\gamma)}(x)\le \frac{1}{t^{-\theta/\alpha}\wedge 1}H_s^{(\gamma,\theta)}(x),\quad x\in\mathbb{R}^d, s\in(0,t]. \tag{70}$$

Then, for $x,y\in\mathbb{R}^d$, $t>0$,

$$\begin{aligned}
&\big|\big(p^0\boxtimes\Psi\big)_t(x,y)\big|\le\big(p^0\boxtimes\Psi^{\#}\big)_t(x,y) && (71)\\
&=\int_0^t\int_{\mathbb{R}^d}p^0_{t-s}(x,z)\Psi_s^{\#}(z,y)\,dzds\\
&\le c_4\int_0^t s^{-1+\theta/\alpha}e^{c_2 s}\int_{\mathbb{R}^d}G_{t-s}^{(\alpha+\gamma)}(z-x)H_s^{(\gamma,\theta)}(y-z)\,dzds\\
&\le c_4\int_0^t\frac{s^{-1+\theta/\alpha}e^{c_2 s}}{t^{-\theta/\alpha}\wedge 1}\int_{\mathbb{R}^d}H_{t-s}^{(\gamma,\theta)}(z-x)H_s^{(\gamma,\theta)}(y-z)\,dzds\\
&\le c_5 t^{\theta/\alpha}e^{c_2 t}H_t^{(\gamma,\theta)}(y-x)\le c_5 e^{c_2 t}G_t^{(\alpha+\gamma)}(y-x), && (72)
\end{aligned}$$

which follows from (61). This proves (54). □

From (72) we see in particular that $p_t(x,y)$ is well defined.

Lemma 3.6. *The following perturbation formula holds for all $t>0$, $x,y\in\mathbb{R}^d$,*

$$p_t(x,y)=p_t^0(x,y)+\int_0^t\int_{\mathbb{R}^d}p_s(x,z)\Phi_{t-s}(z,y)\,dzds. \tag{73}$$

Proof. The identity follows from (15), (14) and Proposition 3.1. □

3.2 Regularity of $\Psi_s(x,y)$ and $p_t(x,y)$

The statement of Proposition 3.1 implies the existence of the function $p_t(x,y)$. In this section we establish the Hölder continuity in x of the function Ψ and a few auxiliary results for the proof of Theorem 1.1.

Lemma 3.7. *For all $\epsilon\in(0,\theta)$, where θ satisfies (66), and $T>0$ there exists $C=C(T)>0$ such that for all $t\in(0,T]$, $x_1,x_2,y\in\mathbb{R}^d$,*

$$\begin{aligned}
&\big|\Psi_t(x_1,y)-\Psi_t(x_2,y)\big|\\
&\qquad\le C\big(|x_1-x_2|^{\theta-\epsilon}\wedge 1\big)t^{-1+\epsilon/\alpha}\Big(H_t^{(\gamma,\theta)}(y-x_1)+H_t^{(\gamma,\theta)}(y-x_2)\Big).
\end{aligned}$$

Proof. We begin with the proof of the following inequality for $t \in (0,T]$, $x_1, x_2, y \in \mathbb{R}^d$:

$$\begin{aligned}&\big|\Phi_t(x_1,y)-\Phi_t(x_2,y)\big|\\&\qquad\le C\big(|x_1-x_2|^{\theta-\epsilon}\wedge 1\big)t^{-1+\epsilon/\alpha}\Big(H_t^{(\gamma,\theta)}(y-x_1)+H_t^{(\gamma,\theta)}(y-x_2)\Big).\end{aligned}$$

For $|x_1-x_2|\ge 1$ the estimate simply follows from (65). Suppose now that $t^{1/\alpha}\le |x_1-x_2|\le 1$. Then,

$$\begin{aligned}\big|\Phi_t(x_1,y)-\Phi_t(x_2,y)\big| &\le c_1\big(\big|\Phi_t(x_1,y)\big|+\big|\Phi_t(x_2,y)\big|\big)\\&\le c_2 t^{-1+\theta/\alpha}\big(H_t^{(\gamma,\theta)}(y-x_1)+H_t^{(\gamma,\theta)}(y-x_2)\big)\\&\le c_3|x_1-x_2|^{\theta-\epsilon}t^{-1+\epsilon/\alpha}\big(H_t^{(\gamma,\theta)}(y-x_1)+H_t^{(\gamma,\theta)}(y-x_2)\big).\end{aligned}$$

Let $|x_1-x_2|\le t^{1/\alpha}\wedge 1$ and

$$g(x,y,u)=p_t^0(x+u,y)-p_t^0(x,y)-u\cdot\nabla_x p_t^0(x,y)\mathbb{1}_{\{|u|\le t^{1/\alpha}\}}.$$

We have

$$\begin{aligned}\Phi_t(x_1,y)-\Phi_t(x_2,y)&=\int_{\mathbb{R}^d} g(x_1,y,u)[\nu(x_1,du)-\nu(x_2,du)]\\&+\int_{\mathbb{R}^d}\big(g(x_1,y,u)-g(x_2,y,u)\big)[\nu(x_2,du)-\nu(y,du)]=I_1+I_2.\end{aligned}$$

For I_1 by A2, Lemma 2.4 and Lemma 2.2 with $\zeta=d$, $\kappa=0$, we get

$$|I_1|\le c_1(|x_1-x_2|^\theta\wedge 1)\mathcal{A}_t^{\#}p_t^0(x_1,y)\le c_2(|x_1-x_2|^\theta\wedge 1)t^{-1}G_t^{(\alpha+\gamma)}(y-x_1).$$

To estimate I_2 let

$$f_t(x)=p_t^y(x+x_2-x_1)-p_t^y(x).$$

Using the Taylor expansion, Lemma 2.4 and the fact that $|x_2-x_1|\le t^{1/\alpha}$, we get

$$\begin{aligned}|f_t(x)| &= |(x_2-x_1)\cdot\nabla_x p_t^y(x+\zeta(x_2-x_1))|\\&\le c_1|x_2-x_1|t^{-1/\alpha}G_t^{\alpha+\gamma}(x+\zeta(x_2-x_1))\\&\le c_2|x_2-x_1|t^{-1/\alpha}G_t^{\alpha+\gamma}(x),\quad x\in\mathbb{R}^d,\ t>0,\end{aligned}$$

where we used some $\zeta\in[0,1]$. Similarly, if $\beta\in\mathbb{N}_0^d$, $|\beta|=2$, then

$$|\partial_x^\beta f_t(x)|\le c|x_2-x_1|t^{-3/\alpha}G_t^{\alpha+\gamma}(x).$$

By A2 and Lemma 2.2 (applied with $\zeta=d+1$, $\kappa=0$) we get

$$\begin{aligned}|I_2|&\le M_0(|x_2-y|^\theta\wedge 1)\mathcal{A}_t^{\#}f_t(y-x_2)\\&\le c_2(|x_2-y|^\theta\wedge 1)|x_2-x_1|t^{-1-1/\alpha}G_t^{(\alpha+\gamma)}(y-x_2).\end{aligned}\tag{74}$$

Then for $|x_1 - x_2| \le t^{1/\alpha} \wedge 1$, $t \in (0, T]$, using the inequality

$$G_t^{(\alpha+\gamma)}(x) \le (T^{\theta/\alpha} \vee 1) H_t^{(\gamma,\theta)}(x), \tag{75}$$

we derive

$$|I_1| \le c_1 (T^{\theta/\alpha} \vee 1) |x_1 - x_2|^{\theta-\epsilon} t^{-1+\epsilon/\alpha} H_t^{(\gamma,\theta)}(y - x_1).$$

Furthermore, using (64) we obtain

$$\begin{aligned} |I_2| &\le c_1(|x_2 - y|^\theta \wedge 1)|x_2 - x_1| t^{-1-1/\alpha} G_t^{(\alpha+\gamma)}(y - x_2) \\ &\le c_1 |x_2 - x_1| t^{-1-1/\alpha+\theta/\alpha} H_t^{(\gamma,\theta)}(y - x_2) \\ &= c_1 |x_2 - x_1|^{\theta-\epsilon} |x_2 - x_1|^{1-\theta+\epsilon} t^{-1-1/\alpha+\theta/\alpha} H_t^{(\gamma,\theta)}(y - x_2) \\ &\le c_1 |x_2 - x_1|^{\theta-\epsilon} t^{-1+\epsilon/\alpha} H_t^{(\gamma,\theta)}(y - x_2). \end{aligned}$$

We now prove the inequality in the statement of the lemma. For $|x_1 - x_2| \ge 1$ the estimate follows from the bound (69) on $\Psi_t(x, y)$, so we let $|x_1 - x_2| \le 1$. Since

$$\Psi_t(x, y) = \Phi_t(x, y) + (\Phi \boxtimes \Psi)_t(x, y),$$

by Proposition 3.3 for $t \in (0, T]$ we get

$$\begin{aligned} &\Big| \Psi_t(x_1, y) - \Psi_t(x_2, y) \Big| \\ &\quad \le c_1 |x_1 - x_2|^{\theta-\epsilon} t^{-1+\epsilon/\alpha} \Big(H_t^{(\gamma,\theta)}(y - x_1) + H_t^{(\gamma,\theta)}(y - x_2) \Big) \\ &\qquad + c_2 |x_1 - x_2|^{\theta-\epsilon} \int_0^t \int_{\mathbb{R}^d} (t - s)^{-1+\epsilon/\alpha} \\ &\qquad \cdot \Big(H_{t-s}^{(\gamma,\theta)}(z - x_1) + H_{t-s}^{(\gamma,\theta)}(z - x_2) \Big) \cdot s^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(y - z)\, dz ds \\ &\quad \le c_3 |x_1 - x_2|^{\theta-\epsilon} t^{-1+\epsilon/\alpha} \Big(H_t^{(\gamma,\theta)}(y - x_1) + H_t^{(\gamma,\theta)}(y - x_2) \Big). \end{aligned}$$

□

We can finally apply the operator L to $p_t(x, y)$.

Lemma 3.8. *For all $y \in \mathbb{R}^d$ and $t > 0$ we have $p_t(\cdot, y) \in D(L)$, and*

$$L_x p_t(x, y) = L_x p_t^0(x, y) + \int_0^t \int_{\mathbb{R}^d} L_x p_{t-s}^0(x, z) \Psi_s(z, y)\, dz ds. \tag{76}$$

Proof. Since $p_t^0(\cdot, y) \in C_\infty^2(\mathbb{R}^d)$, the term $L_x p_t^0(x, y)$ is well defined. Using the representation of $L^{x,\delta}$, Lemma 2.4 and Lemma 2.2, for every $\delta > 0$ we get

$$\begin{aligned} &|L_x^\delta p_t^0(x, y)| \\ &\le \int_{|u|>\delta} \left| p_t^0(x + u, y) - p_t^0(x, y) - u \cdot \nabla_x p_t^0(x, y) \mathbb{1}_{\{|u| \le t^{1/\alpha}\}} \right| \nu(x, du) \\ &\le \mathcal{A}_t^{\#} p_t^0(x, y) \le c t^{-1} G_t^{(\alpha+\gamma)}(y - x). \end{aligned}$$

Let us show that the function

$$f_t^y(x) := \int_0^t \int_{\mathbb{R}^d} p^0_{t-s}(x,z)\Psi_s(z,y)\,dzds \tag{77}$$

belongs to $D(L)$ and

$$L_x f_t^y(x) = \int_0^t \int_{\mathbb{R}^d} L_x p^0_{t-s}(x,z)\Psi_s(z,y)\,dzds.$$

We use the definition (3) of L_x. By (71) for every $\delta > 0$ we get

$$\begin{aligned}
\int_{|u|>\delta} \int_0^t \int_{\mathbb{R}^d} &\left|\left(p^0_{t-s}(x+u,z) - p^0_{t-s}(x,z)\right)\Psi_s(z,y)\right|\,dzds\nu(x,du)\\
&\le c_1 \int_{|u|>\delta} (G_t^{(\alpha+\gamma)}(y-x-u) + G_t^{(\alpha+\gamma)}(y-x))\,\nu(x,du)\\
&< c_2 t^{-d/\alpha}\nu_0(B(0,\delta)^c).
\end{aligned}$$

By Fubini's theorem and the symmetry of ν we get

$$\begin{aligned}
L^\delta f_t^y(x) &= \int_0^t \int_{\mathbb{R}^d} L_x^\delta p^0_{t-s}(x,z)\Psi_s(z,y)\,dzds\\
&= \int_0^t \int_{\mathbb{R}^d} L_x^\delta p^0_{t-s}(x,z)\big[\Psi_s(z,y) - \Psi_s(x,y)\big]\,dzds\\
&\quad + \int_0^t \int_{\mathbb{R}^d} L_x^\delta p^0_{t-s}(x,z)\Psi_s(x,y)\,dzds\\
&= \int_0^t \int_{\mathbb{R}^d} L_x^\delta p^0_{t-s}(x,z)\big[\Psi_s(z,y) - \Psi_s(x,y)\big]\,dzds\\
&\quad + \int_0^t \int_{\mathbb{R}^d} L^{x,\delta}(p^z_{t-s} - p^x_{t-s})(z-x)\Psi_s(x,y)\,dzds\\
&\quad + \int_0^t \int_{\mathbb{R}^d} L^{x,\delta} p^x_{t-s}(z-x)\Psi_s(x,y)\,dzds = I_1(\delta) + I_2(\delta) + I_3(\delta).
\end{aligned}$$

Let us estimate the functions under the integrals $I_1(\delta)$ and $I_2(\delta)$. Using Lemma 3.7 and (64) for $T > 0$ and $0 < s < t \le T$ we get

$$\begin{aligned}
&\left|L_x^\delta p^0_{t-s}(x,z)\big[\Psi_s(z,y) - \Psi_s(x,y)\big]\right|\\
&\le c_1 (t-s)^{-1} G_{t-s}^{(\alpha+\gamma)}(z-x)\big(|x-z|^{\theta-\epsilon} \wedge 1\big) s^{-1+\epsilon/\alpha}\\
&\qquad \cdot \big[H_s^{(\gamma,\theta)}(y-z) + H_s^{(\gamma,\theta)}(y-x)\big]\\
&\le c_2 s^{-1+\epsilon/\alpha}(t-s)^{-1+(\theta-\epsilon)/\alpha} H_{t-s}^{(\gamma,\theta-\epsilon)}(z-x)\big[H_s^{(\gamma,\theta)}(y-z) + H_s^{(\gamma,\theta)}(y-x)\big]\\
&=: g_t^{(x,y)}(s,z),
\end{aligned}$$

with $c_1, c_2 > 0$ depending on T. Using Proposition 3.3 and the inequality

$$H_{t-s}^{(\gamma,\theta-\epsilon)}(z-x) \le (T^{\epsilon/\alpha} \vee 1) H_{t-s}^{(\gamma,\theta)}(z-x), \quad 0 < s < t \le T,$$

we obtain

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} g_t^{(x,y)}(s,z)\, dz ds \le c_1 \int_0^t s^{-1+\epsilon/\alpha}(t-s)^{-1+(\theta-\epsilon)/\alpha} \\ \cdot \left[H_t^{(\gamma,\theta)}(y-x) + H_s^{(\gamma,\theta)}(y-x)\right] ds \le c_2 t^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y-x) \quad (78) \\ + c_3 \int_0^t s^{-1+(\epsilon-d)/\alpha}(t-s)^{-1+(\theta-\epsilon)/\alpha} \left(1 \vee \frac{|y-x|}{s^{1/\alpha}}\right)^{-\gamma-\alpha+\theta} ds. \end{aligned}$$

We need to estimate carefully the integral. We split

$$\begin{aligned} \Big(\int_0^{|x-y|^\alpha \wedge t} + \int_{|x-y|^\alpha \wedge t}^t \Big) s^{-1+(\epsilon-d)/\alpha}(t-s)^{-1+(\theta-\epsilon)/\alpha} \\ \cdot \left(1 \vee \frac{|y-x|}{s^{1/\alpha}}\right)^{-\gamma-\alpha+\theta} ds = J_1 + J_2. \end{aligned}$$

For J_1 after changing variables we get

$$J_1 = t^{-1+\frac{\gamma+\alpha-d}{\alpha}} |x-y|^{-\alpha-\gamma+\theta} \int_0^{\frac{|x-y|^\alpha}{t} \wedge 1} \tau^{-1+\frac{\epsilon-d-\theta+\alpha+\gamma}{\alpha}} (1-\tau)^{-1+\frac{\theta-\epsilon}{\alpha}} d\tau.$$

Treating two cases $|x-y| \le (t/2)^{1/\alpha}$ and $|x-y| > (t/2)^{1/\alpha}$ separately, we get

$$\begin{aligned} J_1 &\le C t^{-1+\frac{\gamma+\alpha-d}{\alpha}} |x-y|^{-\alpha-\gamma+\theta} \\ &\quad \cdot \left(\left(\frac{|x-y|^\alpha}{t}\right)^{\frac{\epsilon-d-\theta+\alpha+\gamma}{\alpha}} \mathbb{1}_{\{|x-y| \le t^{1/\alpha}\}} + \mathbb{1}_{\{|x-y| > t^{1/\alpha}\}}\right) \\ &= C t^{-1+\frac{\theta}{\alpha}} \Big(\frac{t^{\frac{-\epsilon}{\alpha}}}{|x-y|^{d-\epsilon}} \mathbb{1}_{\{|x-y| \le t^{1/\alpha}\}} + \frac{t^{\frac{\gamma+\alpha-d-\theta}{\alpha}}}{|x-y|^{\alpha+\gamma-\theta}} \mathbb{1}_{\{|x-y| > t^{1/\alpha}\}} \Big) \\ &=: C t^{-1+\frac{\theta}{\alpha}} K_t^{(1)}(x,y). \end{aligned}$$

For J_2 we have

$$\begin{aligned} \int_{|x-y|^\alpha \wedge t}^t & s^{-1+(\epsilon-d)/\alpha}(t-s)^{-1+(\theta-\epsilon)/\alpha} ds \\ &= t^{-1+\frac{\theta-d}{\alpha}} \int_{\frac{|x-y|^\alpha}{t} \wedge 1}^1 \tau^{-1+\frac{\epsilon-d}{\alpha}} (1-\tau)^{-1+\frac{\theta-\epsilon}{\alpha}} d\tau \\ &\le C |y-x|^{\epsilon-d} t^{-1+\frac{\theta-\epsilon}{\alpha}} \mathbb{1}_{\{|x-y| \le t^{1/\alpha}\}} \\ &\le C t^{-1+\frac{\theta}{\alpha}} K_t^{(1)}(x,y). \end{aligned}$$

Thus,

$$I_1(\delta) \le Ct^{-1+\theta/\alpha}\big[H_t^{(\gamma,\theta)}(y-x) + K_t^{(1)}(x,y)\big].$$

For later convenience note that

$$\int_{\mathbb{R}^d} [H_t^{(\gamma,\theta)}(y-x) + K_t^{(1)}(x,y)]dy \le C, \quad x \in \mathbb{R}^d,\ t \in (0,T].$$

To estimate the integrand in $I_2(\delta)$ we use Lemma 2.11, 2.2, (69) and (64):

$$\begin{aligned}
&\Big|L^{x,\delta}(p_{t-s}^z - p_{t-s}^x)(z-x)\,|\Psi_s(x,y)|\Big| \le \mathcal{A}_{t-s}^{\#}(p_{t-s}^z - p_{t-s}^x)(z-x)\,\Psi_s(x,y)\\
&\quad \le c_1(|z-x|^\eta \wedge 1)(t-s)^{-1}G_{t-s}^{\alpha+\gamma-\theta}(z-x)s^{-1+\theta/\alpha}H_s^{(\gamma,\theta)}(y-x)\\
&\quad \le c_2 s^{-1+\theta/\alpha}(t-s)^{-1+\theta/\alpha}H_{t-s}^{(\gamma-\theta,\theta)}(z-x)H_s^{(\gamma,\theta)}(y-x) =: h_t^{(x,y)}(s,z).
\end{aligned}$$

Using Proposition 3.3 and the same argument as for estimating (78), for $\theta < (\alpha+\gamma-d)/2$ we get

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} h_t^{(x,y)}(s,z)\,dzds &\le c_1 \int_0^t s^{-1+\theta/\alpha}(t-s)^{-1+\theta/\alpha}H_s^{(\gamma,\theta)}(y-x)\,ds\\
&\le c_1 \int_0^t s^{-1+(\theta-d)/\alpha}(t-s)^{-1+\theta/\alpha}\left(1 \vee \frac{|y-x|}{s^{1/\alpha}}\right)^{-\gamma-\alpha+\theta} ds\\
&\le c_3 t^{-1+\theta/\alpha}K_t^{(2)}(x,y),
\end{aligned}$$

where

$$K_t^{(2)}(x,y) = \frac{1}{|x-y|^{d-\theta}}\mathbb{1}_{\{|x-y|\le t^{1/\alpha}\}} + \frac{t^{\frac{\gamma+\alpha-d}{\alpha}}}{|x-y|^{\alpha+\gamma-\theta}}\mathbb{1}_{\{|x-y|>t^{1/\alpha}\}}.$$

Observe that

$$\int_{\mathbb{R}^d} K_t^{(2)}(x,y)dy \le Ct^{\theta/\alpha}, \quad x \in \mathbb{R}^d,\ t \in (0,T].$$

We get

$$I_2(\delta) \le Ct^{-1+\theta/\alpha}K_t^{(2)}(x,y), \quad x,y \in \mathbb{R}^d,\ t \in (0,T].$$

Furthermore, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|u|>\delta} \left|p_{t-s}^x(z-x+u) - p_{t-s}^x(z-x)\right| \nu(x,du)dz\\
&\le \int_{|u|>\delta}\int_{\mathbb{R}^d}\Big(G_{t-s}^{(\alpha+\gamma)}(z-x+u) + G_{t-s}^{(\alpha+\gamma)}(z-x)\Big)\,dz\nu(x,du)\\
&\le c\nu_0(B(0,\delta)^c),
\end{aligned}$$

hence by Fubini's theorem we get for every $\delta > 0$,

$$\begin{aligned}
&I_3(\delta)\\
&= \int_0^t \int_{|u|>\delta} \left[\int_{\mathbb{R}^d} \left(p^x_{t-s}(z-x+u) - p^x_{t-s}(z-x)\right)\, dz\right] \nu(x,du)\Psi_s(x,y)\, ds\\
&= \int_0^t \int_{|u|>\delta} (1-1)\, \nu(x,du)\Psi_s(x,y)\, ds = 0.
\end{aligned}$$

Thus, by the dominated convergence theorem,

$$\begin{aligned}
\lim_{\delta\to 0^+} & L^\delta f^y_t(x)\\
&= \lim_{\delta\to 0^+} (I_1(\delta) + I_2(\delta) + I_3(\delta))\\
&= \int_0^t \int_{\mathbb{R}^d} L_x p^0_{t-s}(x,z)\big[\Psi_s(z,y) - \Psi_s(x,y)\big]\, dzds\\
&\quad + \int_0^t \int_{\mathbb{R}^d} L^x (p^z_{t-s} - p^x_{t-s})(z-x)\, dz \Psi_s(x,y)\, ds,\\
&= \int_0^t \int_{\mathbb{R}^d} \big[L_x p^0_{t-s}(x,z)\Psi_s(z,y) - L^x p^x_{t-s}(z-x)\Psi_s(x,y)\big]\, dzds.
\end{aligned}$$

By Corollary 2.9 we also have $\int_{\mathbb{R}^d} L^x p^x_{t-s}(z-x)\, dz = 0$, and (76) follows for every $t \in (0,T]$; since T is arbitrary it holds for every $t > 0$. ☐

Corollary 3.9. *We have*

$$|L^\delta_x p_t(x,y)| \le C t^{-1}\Big(G^{\alpha+\gamma}_t(y-x) + t^{\theta/\alpha} K_t(x,y)\Big), \tag{79}$$

for all $t \in (0,T]$, $x,y \in \mathbb{R}^d$, $\delta > 0$, $\theta \in (0, \frac{\alpha+\gamma-d}{2} \wedge \eta)$ *and some kernel* $K_t(x,y) \ge 0$ *such that*

$$\int_{\mathbb{R}^d} K_t(x,y)dy \le C, \quad t \in (0,T],\, x \in \mathbb{R}^d.$$

Furthermore,

$$\left|\int_{\mathbb{R}^d} L^\delta_x p_t(x,y)dy\right| \le C t^{-1+\theta/\alpha}, \quad t \in (0,T],\ x \in \mathbb{R}^d,\ \delta > 0. \tag{80}$$

Proof. Let $K = K^{(1)} + K^{(2)} + H^{(\gamma,\theta)}$, where the terms on the right-hand side are as in the proof of Lemma 3.8. This gives (79). To prove (80), consider

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d} L^\delta_x p^0_t(x,y)\, dy\Big| \le \Big|\int_{\mathbb{R}^d} L^\delta_x [p^y_t(y-x) - p^x_t(y-x)]dy\Big|\\
+ \Big|\int_{\mathbb{R}^d} L^\delta_x p^x_t(y-x)dy\Big|.
\end{aligned}$$

The last integral is 0. To estimate the first integral we use the Hölder continuity of $p_x^z(y-x)$ in z, i.e. (47) with $\beta = 0$ and $|\beta| = 2$:

$$\begin{aligned}\left|\partial_w^\beta p_t^y(y-x) - \partial_w^\beta p_t^x(w)\right| &\le c(|w|^\eta \wedge 1)t^{-|\beta|/\alpha} G_t^{(\alpha+\gamma-\theta)}(w)\\ &\le c|w|^\theta t^{-|\beta|/\alpha} G_t^{(\alpha+\gamma-\theta)}(w)\\ &\le ct^{(\theta-|\beta|)/\alpha} G_t^{(\alpha+\gamma-2\theta)}(w).\end{aligned}$$

Applying Lemma 2.2 with $\kappa = 2\theta$, $\zeta = d-\theta$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \left|L_x^\delta[p_t^y(y-x) - p_t^x(y-x)]\right| dy &\le Ct^{-1+\theta/\alpha}\int_{\mathbb{R}^d} G_t^{(\alpha+\gamma-2\theta)}(y-x)dy\\ &\le Ct^{-1+\theta/\alpha}.\end{aligned}$$

The proof is complete. □

Next we show how to differentiate $(p^0 \boxtimes \Psi)_t(x,y)$ in t.

Lemma 3.10. *For $x, y \in \mathbb{R}^d$, $0 < s < t$, $0 < t \le T$, we have*

$$\partial_t \int_{\mathbb{R}^d} p_{t-s}^0(x,z)\Psi_s(z,y)\,dz = \int_{\mathbb{R}^d} \partial_t p_{t-s}^0(x,z)\Psi_s(z,y)\,dz, \tag{81}$$

$$\int_s^t \left|\int_{\mathbb{R}^d} \partial_r p_{r-s}^0(x,z)\Psi_s(z,y)\,dz\right| dr < \infty, \tag{82}$$

$$\int_0^t \int_0^r \left|\int_{\mathbb{R}^d} \partial_r p_{r-s}^0(x,z)\Psi_s(z,y)\,dz\right| dsdr < \infty. \tag{83}$$

Proof. In order to prove (81) it suffices to show that for all fixed $t > s > 0$ and $x, y \in \mathbb{R}^d$ there is $\varepsilon_0 > 0$ and a function $g(z) \ge 0$ such that $\int_{\mathbb{R}^d} g(z)\,dz < \infty$, and

$$\left|\partial_t p_{t+\varepsilon-s}^0(x,z)\Psi_s(z,y)\right| \le g(z), \quad z \in \mathbb{R}^d, \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Using Lemma 2.7 and (69), for every $\varepsilon_0 < t-s$ we get

$$\begin{aligned}&\left|\partial_t p_{t+\varepsilon-s}^0(x,z)\Psi_s(z,y)\right|\\ &\le c_1(t+\varepsilon-s)^{-1} G_{t+\varepsilon-s}^{(\alpha+\gamma)}(z-x)s^{-1+\theta/\alpha}e^{c_2 s}H_s^{(\gamma,\theta)}(z-y)\\ &\le c_1(t-\varepsilon_0-s)^{-1-d/\alpha}s^{-1+\theta/\alpha}e^{c_2 s}H_s^{(\gamma,\theta)}(z-y) := g(z),\end{aligned}$$

and the finiteness of $\int_{\mathbb{R}^d} g(z)\,dz$ follows from Proposition 3.3.
The integral in (83) is not bigger than

$$\begin{aligned}&\int_0^t \int_0^r \left|\int_{\mathbb{R}^d} \partial_r p_{r-s}^0(x,z)(\Psi_s(z,y) - \Psi_s(x,y))\,dz\right| dsdr\\ &+ \int_0^t \int_0^r \left|\int_{\mathbb{R}^d} \partial_r p_{r-s}^0(x,z)\,dz\right| |\Psi_s(x,y)|\,dsdr = I_1 + I_2.\end{aligned}$$

Consider now I_1. By Lemma 2.7, Lemma 3.7, (64) and Proposition 3.3 we derive that for every $\epsilon \in (0,\theta)$ and $\theta \in (0, \eta \wedge (\gamma - d + \alpha))$,

$$
\begin{aligned}
I_1 \quad &\leq \quad c_1 \int_0^t \int_0^r \int_{\mathbb{R}^d} (r-s)^{-1} G_{r-s}^{(\alpha+\gamma)}(z-x)\big(|x-z|^{\theta-\epsilon} \wedge 1\big) s^{-1+\epsilon/\alpha} \\
&\qquad \cdot \big[H_s^{(\gamma,\theta)}(y-z) + H_s^{(\gamma,\theta)}(y-x)\big]\, dz\, ds\, dr \\
&\leq \quad c_1 \int_0^t \int_0^r \int_{\mathbb{R}^d} (r-s)^{-1+(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} H_{r-s}^{(\gamma,\theta-\epsilon)}(z-x) \\
&\qquad \cdot \big[H_s^{(\gamma,\theta)}(y-z) + H_s^{(\gamma,\theta)}(y-x)\big]\, dz\, ds\, dr \\
&\leq \quad c_2 \int_0^t \int_0^r \int_{\mathbb{R}^d} (r-s)^{-1+(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} H_{r-s}^{(\gamma,\theta)}(z-x) \\
&\qquad \cdot \big[H_s^{(\gamma,\theta)}(y-z) + H_s^{(\gamma,\theta)}(y-x)\big]\, dz\, ds\, dr.
\end{aligned}
$$

Further,

$$
\begin{aligned}
I_1 \quad &\leq \quad c_3 \int_0^t \int_0^r (r-s)^{-1+(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} \\
&\qquad \big[H_r^{(\gamma,\theta)}(y-x) + H_s^{(\gamma,\theta)}(y-x)\big]\, dsdr \\
&= \quad c_3 \int_0^t \int_s^t (r-s)^{-1+(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} \\
&\qquad \big[H_r^{(\gamma,\theta)}(y-x) + H_s^{(\gamma,\theta)}(y-x)\big]\, drds \\
&= \quad c_4 \Big[\int_0^t \int_s^t (r-s)^{-1+(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} H_r^{(\gamma,\theta)}(y-x)\, drds \\
&\qquad + \int_0^t (t-s)^{(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} H_s^{(\gamma,\theta)}(y-x)\, ds \Big].
\end{aligned}
$$

By the estimate $H_t^{(\gamma,\theta)}(x) \leq c|x|^{-\alpha-\gamma+\theta} t^{1+(\gamma-\theta-d)/\alpha}$ we obtain

$$
\begin{aligned}
I_1 \leq c_4 \Big[&\int_0^t \int_s^t (r-s)^{-1+(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha} r^{-d/\alpha} \left(\frac{|y-x|}{r^{1/\alpha}} \right)^{-\gamma-\alpha+\theta} drds \\
&+ \int_0^t (t-s)^{(\theta-\epsilon)/\alpha} s^{-1+(\epsilon-d)/\alpha} \left(\frac{|y-x|}{s^{1/\alpha}} \right)^{-\gamma-\alpha+\theta} ds \Big] \\
\leq c_5 |y-x|^{-\alpha-\gamma+\theta} &\Big[t^{1+(\gamma-d-\theta)/\alpha} \int_0^t (t-s)^{(\theta-\epsilon)/\alpha} s^{-1+\epsilon/\alpha}\, ds \\
&+ \int_0^t (t-s)^{(\theta-\epsilon)/\alpha} s^{(\gamma-\theta+\epsilon-d)/\alpha}\, ds \Big] = c_6 |y-x|^{-\alpha-\gamma+\theta} t^{1+(\gamma-d)/\alpha}.
\end{aligned}
$$

We note that the constants c_i in this proof may depend on T. From

Lemma 2.14, (69) and (64) with $\theta \in (0, \eta \wedge \frac{\alpha+\gamma-d}{2})$ we get similarly

$$\begin{aligned} I_2 &\le c_1 \int_0^t \int_0^r (r-s)^{-1+\theta/\alpha} |\Psi_s(x,y)| \, dsdr \\ &\le c_2 \int_0^t \int_0^r (r-s)^{-1+\theta/\alpha} s^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(y-x) \, dsdr \\ &= c_2 \int_0^t \int_s^t (r-s)^{-1+\theta/\alpha} s^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(y-x) \, drds \\ &\le c_3 |y-x|^{-\alpha-\gamma+\theta} \int_0^t (t-s)^{\theta/\alpha} s^{(\gamma-d)/\alpha} ds \\ &= c_4 |y-x|^{-\alpha-\gamma+\theta} t^{1+(\theta+\gamma-d)/\alpha}, \end{aligned}$$

because we assumed $\gamma - d + \alpha > 0$. This yields (82) and (83). □

3.3 Proof of Theorem 1.1

From (15), Lemma 2.4, (72), Lemma 2.13 and (63) we obtain (17). Next we verify (16). Using Lemmas 3.10, 3.7 and 2.13 we get

$$\int_s^t \Big[\partial_r \int_{\mathbb{R}^d} p^0_{r-s}(x,z)\Psi_s(z,y)\,dz\Big]\,dr = \int_{\mathbb{R}^d} p^0_{t-s}(x,z)\Psi_s(z,y)\,dz - \Psi_s(x,y).$$

Integrating the above equation from 0 to t and using Lemma 3.10 and Fubini's theorem we obtain

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} p^0_{t-s}(x,z)\Psi_s(z,y)\,dzds - \int_0^t \Psi_s(x,y)\,ds \\ = \int_0^t \int_0^r \int_{\mathbb{R}^d} \partial_r p^0_{r-s}(x,z)\Psi_s(z,y)\,dzdsdr. \end{aligned}$$

By Corollary 3.5 and the locally uniform convergence of the series defining Ψ_t the function $t \to \Psi_t(x,y)$ is continuous, implying

$$\partial_t p_t(x,y) = \partial_t p^0_t(x,y) + \Psi_t(x,y) + \int_0^t \int_{\mathbb{R}^d} \partial_t p^0_{t-s}(x,z)\Psi_s(z,y)\,dzds. \tag{84}$$

Subtracting $L_x p_t(x,y)$ from both sides and using Lemma 3.8 we get

$$\begin{aligned} \big(\partial_t - L_x\big)p_t(x,y) = -\Phi_t(x,y) + \Psi_t(x,y) \\ - \int_0^t \int_{\mathbb{R}^d} \Phi_{t-s}(x,z)\Psi_s(z,y)\,dzds = 0. \end{aligned}$$

The proof of Theorem 1.1 is complete. □

4 Further regularity

4.1 Time derivatives of $\Psi_t(x,y)$

We begin with an auxiliary estimate of the time derivative of $\Psi_t(x,y)$.

Lemma 4.1. *The function $\Psi_t(x,y)$ is differentiable in t and $\partial_t\Psi_t(x,y)$ is continuous on $(0,\infty)$. There are $C,c>0$ and $\theta\in(0,\eta\wedge(\alpha+\gamma-d))$ such that*

$$\left|\partial_t\Psi_t(x,y)\right|\le Ce^{ct}t^{-2+\theta/\alpha}H_t^{(\gamma,\theta)}(y-x),\quad x,y\in\mathbb{R}^d,\ t>0. \tag{85}$$

Proof. It follows from Lemma 3.2 and (64) that $\partial_t\Phi_t(x,y)$ is continuous and

$$\left|\partial_t\Phi_t(x,y)\right|\le C_\Phi t^{-2+\theta/\alpha}H_t^{(\gamma,\theta)}(y-x),\quad x,y\in\mathbb{R}^d,\ t>0. \tag{86}$$

We show by induction for all $k\ge1$ that $\partial_t\Phi_t^{\boxtimes k}=\partial_t(\Phi_t^{\boxtimes k})$ exists and

$$\left|\partial_t\Phi_t^{\boxtimes k}(x,y)\right|\le\frac{C_3C_4^k}{\Gamma(k\theta/\alpha)}t^{-2+\theta k/\alpha}H_t^{(\gamma,\theta)}(y-x),\quad x,y\in\mathbb{R}^d,\ t>0, \tag{87}$$

where

$$C_3=(1\vee(\Gamma(\theta/\alpha))^{-1})C_1,\quad C_4=8\big(1\vee(2-2\theta/\alpha)^{-\theta/\alpha}\big)C_2, \tag{88}$$

and C_1, C_2 come from (67). The case of $k=1$ is verified by (86).
Note that

$$\begin{aligned}\Phi_t^{\boxtimes(k+1)}(x,y)=&\int_0^{t/2}\int_{\mathbb{R}^d}\Phi_{t-s}^{\boxtimes k}(x,z)\Phi_s(z,y)\,dzds\\&+\int_0^{t/2}\int_{\mathbb{R}^d}\Phi_s^{\boxtimes k}(x,z)\Phi_{t-s}(z,y)\,dzds,\end{aligned}$$

for $k\in\mathbb{N}$. Accordingly, we claim that for $k\in\mathbb{N}$,

$$\begin{aligned}\partial_t\Phi_t^{\boxtimes(k+1)}(x,y)=&\int_0^{t/2}\int_{\mathbb{R}^d}\partial_t\Phi_{t-s}^{\boxtimes k}(x,z)\Phi_s(z,y)\,dzds\\&+\int_0^{t/2}\int_{\mathbb{R}^d}\Phi_s^{\boxtimes k}(x,z)\partial_t\Phi_{t-s}(z,y)\,dzds\quad+\int_{\mathbb{R}^d}\Phi_{t/2}^{\boxtimes k}(x,z)\Phi_{t/2}(z,y)\,dz.\end{aligned} \tag{89}$$

Indeed, we consider $\left(\Phi_{t+h}^{\boxtimes(k+1)}(x,y)-\Phi_t^{\boxtimes(k+1)}(x,y)\right)/h$ as $h\to0$. If for some $k\ge1$, continuous $\partial_t\Phi_t^{\boxtimes k}(x,y)$ exists for all $t>0$, $x,y\in\mathbb{R}^d$, and (87) holds for every $t>0$, then for $h\in(-t/4,t/4)$ we have

$$\begin{aligned}&\left|\partial_t\Phi_{t+h-s}^{\boxtimes k}(x,z)\Phi_s(z,y)\right|\\&\le c_1(t+h-s)^{-2+k\theta/\alpha}s^{-1+\theta/\alpha}H_{t+h-s}^{(\gamma,\theta)}(z-x)H_s^{(\gamma,\theta)}(y-z)\\&\le c_2(t-s)^{-2+k\theta/\alpha}s^{-1+\theta/\alpha}H_{t-s}^{(\gamma,\theta)}(z-x)H_s^{(\gamma,\theta)}(y-z)\\&=:g_t^{(x,y)}(s,z),\qquad s\in(0,t/2),x,y,z\in\mathbb{R}^d.\end{aligned}$$

It follows from Proposition 3.3 that $\int_0^{t/2}\int_{\mathbb{R}^d} g_t^{(x,y)}(s,z)\,dzds<\infty$. Estimating similarly $\left|\Phi_s^{\boxtimes k}(x,z)\partial_t\Phi_{t+h-s}(z,y)\right|$, by the continuity of $t\mapsto\Phi_t^{\boxtimes k}(x,z)$ we get (89). Denote by I_1, I_2, I_3 the integrals in (89), respectively. Using induction, Lemma 3.2 and Proposition 3.3 we get for the first term

$$\begin{aligned}
|I_1| &\leq \frac{2C_\Phi C_3 C_4^k t^{-1}}{\Gamma(k\theta/\alpha)}\int_0^t\int_{\mathbb{R}^d}(t-s)^{-1+k\theta/\alpha}s^{-1+\theta/\alpha}\\
&\qquad\cdot H_{t-s}^{(\gamma,\theta)}(z-x)H_s^{(\gamma,\theta)}(y-z)\,dzds\\
&\leq \frac{2C_3C_\Phi C_H C_4^k}{\Gamma(k\theta/\alpha)}B\big(k\theta/\alpha,\theta/\alpha\big)t^{-2+(k+1)\theta/\alpha}H_t^{(\gamma,\theta)}(y-x)\\
&= \frac{C_3}{\Gamma((k+1)\theta/\alpha)}\cdot\big(2C_2C_4^k\big)\cdot t^{-2+(k+1)\theta/\alpha}H_t^{(\gamma,\theta)}(y-x).
\end{aligned}$$

The same estimate holds for I_2, so let us estimate I_3. By (67),

$$\begin{aligned}
|I_3| &\leq \frac{C_1C_\Phi C_2^k}{\Gamma(k\theta/\alpha)}\Big(\frac{t}{2}\Big)^{-2+(k+1)\theta/\alpha}\int_{\mathbb{R}^d}H_{t/2}^{(\gamma,\theta)}(z-x)H_{t/2}^{(\gamma,\theta)}(y-z)\,dz\\
&\leq \frac{C_1C_\Phi C_H C_2^k}{\Gamma(k\theta/\alpha)}\Big(\frac{t}{2}\Big)^{-2+(k+1)\theta/\alpha}H_t^{(\gamma,\theta)}(y-x).
\end{aligned}$$

Using the inequality $u\leq e^u$, valid for all $u\in\mathbb{R}$, we get for $\zeta=\theta/\alpha$,

$$\Gamma((k+1)\zeta)=\int_0^\infty e^{-u}u^{(k+1)\zeta-1}\,du\leq(1-\zeta)^{-k\zeta}\Gamma(k\zeta).$$

Therefore,

$$\begin{aligned}
|I_3| &\leq \frac{C_1C_\Phi C_H C_2^k 2^{2-(k+1)\theta/\alpha}}{(1-\theta/\alpha)^{k\theta/\alpha}\Gamma((k+1)\theta/\alpha)}t^{-2+(k+1)\theta/\alpha}H_t^{(\gamma,\theta)}(y-x)\\
&\leq \frac{C_3}{\Gamma((k+1)\theta/\alpha)}4\Big(\frac{C_2}{(2-2\theta/\alpha)^{\theta/\alpha}}\Big)^{k+1}t^{-2+(k+1)\theta/\alpha}H_t^{(\gamma,\theta)}(y-x),
\end{aligned}$$

because $C_2=C_\Phi C_H\Gamma(\theta/\alpha)$. Observe that for C_4 given in (88) we have

$$4C_2C_4^k+4\Big(\frac{C_2}{(2-2\theta/\alpha)^{\theta/\alpha}}\Big)^{k+1}\leq C_4^{k+1},$$

and thus

$$I_1+I_2+I_3\leq\frac{C_3C_k^{k+1}}{\Gamma((k+1)\theta/\alpha)}t^{-2+(k+1)\theta/\alpha}H_t^{(\gamma,\theta)}(y-x),$$

proving (87). By (68) and (87) we get (85). □

4.2 Proof of Theorem 1.2

The proof of (20) for $k = 0$ easily follows from Proposition 3.1. Let us show (20) for $k = 1$. Our starting point is (15). Lemma 2.7 estimates $\partial_t p_t^0(x, y)$. We then use the estimate for $\partial_t \Psi_t(x, y)$ given in Lemma 4.1. The estimate of $\partial_t\big(p^0 \boxtimes \Psi\big)_t(x, y)$ can be obtained similarly as the estimates for $\partial_t \Phi_t(x, y)$ in Lemma 4.1. Indeed, as in the proof of (89), using (70) for every $h \in (-t/4, t/4)$ we get

$$\begin{aligned}\Big|\partial_t p^0_{t+h-s}(x,z)\Psi_s(z,y)\Big| &\le c_1 (t-s)^{-1} G^{(\alpha+\gamma)}_{t-s}(z-x) s^{-1+\theta/\alpha} e^{c_2 s} H^{(\gamma,\theta)}_s(y-z)\\ &\le c_1 \frac{(t-s)^{-1}}{t^{-\theta/\alpha}\wedge 1} H^{(\gamma,\theta)}_{t-s}(z-x) s^{-1+\theta/\alpha} e^{c_2 s} H^{(\gamma,\theta)}_s(y-z)\\ &=: g^{(x,y)}_t(s,z), \quad s \in (0, t/2),\end{aligned}$$

and it follows from Proposition 3.3 that the majorant satisfies

$$\int_0^{t/2}\int_{\mathbb{R}^d} g^{(x,y)}_t(s,z)\, dsdz < c t^{-1+\theta/\alpha} e^{c_3 t} H^{(\gamma,\theta)}_t(y-x) < \infty.$$

Similarly we estimate $|p^0_s(x,z)(\partial_t\Psi)_{t+h-s}(z,y)|$. These bounds and the continuity of $t \mapsto p^0_t(x,y)$ and $t \mapsto \Psi_t(x,y)$ allow us to write

$$\begin{aligned}\partial_t\big(p^0 \boxtimes \Psi\big)_t(x,y) &= \int_0^{t/2}\int_{\mathbb{R}^d} (\partial_t p^0)_{t-s}(x,z)\Psi_s(z,y)\, dzds\\ &+ \int_0^{t/2}\int_{\mathbb{R}^d} p^0_s(x,z)(\partial_t\Psi)_{t-s}(z,y)\, dzds + \int_{\mathbb{R}^d} p^0_{t/2}(x,z)\Psi_{t/2}(z,y)\, dz.\end{aligned}$$

We obtain

$$\big|\partial_t\big(p^0 \boxtimes \Psi\big)_t(x,y)\big| \le C t^{-1} e^{ct} G^{(\alpha+\gamma)}_t(y-x).$$

This finishes the proof of (20). To verify (21), we observe that

$$p^0_t(x,y) \approx t^{-d/\alpha}, \quad |y-x| \le c t^{1/\alpha},\ t > 0,$$

where the upper bound follows from (35) and (36) and the lower one from Lemma 7 in [34]. The second term in (15), which we denote by $R_t(x, y)$, can be estimated (cf. (71) and (72)) as follows:

$$|R_t(x,y)| \le C t^{\theta/\alpha} t^{-d/\alpha}, \quad |y-x| \le c t^{1/\alpha},\ t \in (0,1].$$

Combining these estimates we get (21) for $t \in (0, t_0]$, if $t_0 > 0$ is small enough. We finally prove (22). We observe that by (48),

$$\big|p^0_t(x_1,y) - p^0_t(x_2,y)\big| \le C\Big(\frac{|x_1-x_2|}{t^{1/\alpha}}\wedge 1\Big)\Big(G^{(\alpha+\gamma)}_t(y-x_1) + G^{(\alpha+\gamma)}_t(y-x_2)\Big).$$

Suppose first that $t \in (0,1]$. Using (69) for Ψ, (70) and the sub-convolution property of $H_t^{(\gamma,\theta)}(x)$ we obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} |p^0_{t-s}(x_1,z) - p^0_{t-s}(x_2,z)||\Psi_s(z,y)|\,dzds \\
&\quad \le c_1|x_2-x_1|^\theta \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\theta/\alpha}\Big(G^{(\alpha+\gamma)}_{t-s}(z-x_1) + G^{(\alpha+\gamma)}_{t-s}(z-x_2)\Big) \\
&\qquad \cdot s^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(y-z)\,dzds \\
&\quad \le c_2|x_2-x_1|^\theta \left(H_t^{(\gamma,\theta)}(y-x_1) + H_t^{(\gamma,\theta)}(y-x_2)\right) \\
&\quad \le c_2\Big(\frac{|x_2-x_1|}{t^{1/\alpha}}\Big)^\theta \left(G_t^{(\alpha+\gamma)}(y-x_1) + G_t^{(\alpha+\gamma)}(y-x_2)\right),
\end{aligned}$$

where in the last line we used that $t^{\theta/\alpha} H_t^{(\gamma,\theta)}(x) \le G_t^{(\alpha+\gamma)}(x)$ for $t \in (0,1]$. Next we assume that $t > 1$. Using the estimate for Ψ and (70) twice we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} |p^0_{t-s}(x_1,z) - p^0_{t-s}(x_2,z)||\Psi_s(z,y)|\,dzds \\
&\quad \le c_1|x_2-x_1|^\theta e^{ct} \int_0^t (t-s)^{-\theta/\alpha} \int_{\mathbb{R}^d} \Big(G^{(\alpha+\gamma)}_{t-s}(z-x_1) + G^{(\alpha+\gamma)}_{t-s}(z-x_2)\Big) \\
&\qquad \cdot s^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(y-z)\,dzds \\
&\quad \le c_1|x_2-x_1|^\theta e^{ct} \int_0^t (1 \vee (t-s)^{-\theta/\alpha}) \\
&\qquad \cdot \int_{\mathbb{R}^d} \Big(H^{(\gamma,\theta)}_{t-s}(z-x_1) + H^{(\gamma,\theta)}_{t-s}(z-x_2)\Big)\, s^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(y-z)\,dzds \\
&\quad \le c_2|x_2-x_1|^\theta t^{\theta/\alpha} e^{ct} \left(H_t^{(\gamma,\theta)}(y-x_1) + H_t^{(\gamma,\theta)}(y-x_2)\right) \\
&\quad \le c_2|x_2-x_1|^\theta e^{ct} \left(G_t^{(\alpha+\gamma)}(y-x_1) + G_t^{(\alpha+\gamma)}(y-x_2)\right) \\
&\quad \le c_2\left(\frac{|x_2-x_1|}{t^{1/\alpha}}\right)^\theta e^{c_3 t}\Big(G_t^{(\alpha+\gamma)}(y-x_1) + G_t^{(\alpha+\gamma)}(y-x_2)\Big).
\end{aligned}$$

This finishes the proof of (22) for $t > 0$.

According to our choice of the first approximation $p_t^0(x,y)$, the regularity of $y \mapsto p_t(x,y)$ is less obvious than that of $x \mapsto p_t(x,y)$. The next result gives a preparation for such regularity and may be confronted with Lemma 3.7.

LEMMA 4.2. *For all $t > 0$ and $y \in \mathbb{R}^d$ we have*

$$\lim_{z \to y} \sup_{x \in \mathbb{R}^d} |\Phi_t(x,z) - \Phi_t(x,y)| = 0. \tag{90}$$

Proof. Since $\partial_t p_t^z(x) = L^z p_t^z(x)$ we get

$$\begin{aligned}|\partial_t(p_t^z(z-x) - p_t^y(y-x))| &= |L^z p_t^z(z-x) - L^y p_t^y(y-x)| \\ &\le |L^z p_t^z(z-x) - L^y p_t^z(z-x)| + |L^y p_t^z(z-x) - L^y p_t^y(z-x)| \\ &\quad + |L^y p_t^y(z-x) - L^y p_t^y(y-x)| \\ &= I_1 + I_2 + I_3.\end{aligned}$$

From A2 and Lemma 2.2 we have

$$\begin{aligned} I_1 &\le \int_{\mathbb{R}^d} \left|p_t^z(z-x+u) - p_t^z(z-x) - u\cdot\nabla_x p_t^z(z-x)\mathbb{1}_{|u|\le t^{1/\alpha}}(u)\right| \\ &\quad \cdot |\nu(z,du) - \nu(y,du)| \\ &\le M_0(|z-y|^\eta \wedge 1)\mathcal{A}_t^{\#} p_t^z(z-x) \\ &\le c_1(|z-y|^\eta\wedge 1)t^{-1}G_t^{(\alpha+\gamma)}(z-x) \le c_1 t^{-1-d/\alpha}(|z-y|^\eta \wedge 1).\end{aligned}$$

From Lemma 2.11 and Lemma 2.2 we obtain

$$\begin{aligned} I_2 &= |L^y(p_t^z - p_t^y)(z-x)| \le \mathcal{A}_t^{\#}(p_t^z - p_t^y)(z-x) \\ &\le c_2(|z-y|^\eta\wedge 1)t^{-1}G_t^{(\alpha+\gamma-\theta)}(z-x) \le c_2 t^{-1-d/\alpha}(|z-y|^\eta\wedge 1).\end{aligned}$$

Finally, let $|y-z| < t^{1/\alpha}$ and $g_t(w) = p_t^y(w-(y-z)) - p_t^y(w)$. Using Taylor expansion and Lemma 2.4, for every $\beta \in \mathbb{N}_0^d$ such that $|\beta| \le 2$ we get

$$|\partial_w^\beta g_t(w)| \le c_3|z-y|t^{(-1-|\beta|)/\alpha}G_t^{(\alpha+\gamma)}(w).$$

This and Lemma 2.2 yield

$$\begin{aligned} I_3 &= |L^y g_t(y-x)| \le \mathcal{A}_t^{\#} g_t(y-x) \le c_4|z-y|t^{-1-1/\alpha}G_t^{(\alpha+\gamma)}(y-x) \\ &\le c_4 t^{-1-(1+d)/\alpha}|z-y|.\end{aligned}$$

Therefore,

$$\lim_{z\to y}\sup_{x\in\mathbb{R}^d} |\partial_t(p_t^z(z-x) - p_t^y(y-x))| = 0.$$

Similarly,

$$\begin{aligned} &|L^x p_t^z(z-x) - L^x p_t^y(y-x)| \\ &\qquad \le |L^x(p_t^z - p_t^y)(z-x)| + |L^x p_t^y(z-x) - L^x p_t^y(y-x)| \\ &\qquad \le \mathcal{A}_t^{\#}(p_t^z - p_t^y)(z-x) + \mathcal{A}_t^{\#} g_t(y-x) \\ &\qquad \le c_5 t^{-1-d/\alpha}(|z-y|^\eta\wedge 1) + c_6 t^{-1-(1+d)/\alpha}|z-y|,\end{aligned}$$

$\lim_{z\to y}\sup_{x\in\mathbb{R}^d} |L^x p_t^z(z-x) - L^x p_t^y(y-x)| = 0$, and (90) follows. □

Lemma 4.3. *For all $t > 0$ and $x \in \mathbb{R}^d$ the function $y \mapsto p_t(x,y)$ is continuous.*

Proof. For the proof we rely on (73). It is straightforward to see that

$$H_s^{(\gamma,\theta)}(y+h) \le c H_s^{(\gamma,\theta)}(y),$$

where $y \in \mathbb{R}^d$, $s>0$, $h \in \mathbb{R}^d$ and $|h| < s^{1/\alpha}$. Let $T \in (0,\infty)$ and $t \in (0,T]$. By Theorem 1.2, (70), (65) and Proposition 3.3 for every $\varepsilon \in (0,t)$ and $|h| < \varepsilon^{1/\alpha}$,

$$\begin{aligned}
&\int_0^{t-\varepsilon}\int_{\mathbb{R}^d} |p_s(x,z)\Phi_{t-s}(z,y+h)|\, dzds \\
\le\ & \frac{c_1 e^{c_2 t}}{t^{-\theta/\alpha}\wedge 1}\int_0^{t-\varepsilon}\int_{\mathbb{R}^d} H_s^{(\gamma,\theta)}(z-x)(t-s)^{-1+\theta/\alpha} H_{t-s}^{(\gamma,\theta)}(y+h-z)\, dzds \\
\le\ & c_3 \int_0^{t-\varepsilon}\int_{\mathbb{R}^d} (t-s)^{-1+\theta/\alpha} H_s^{(\gamma,\theta)}(z-x) H_{t-s}^{(\gamma,\theta)}(y-z)\, dzds \\
\le\ & c_4 \int_0^{t-\varepsilon} (t-s)^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y-x)\, ds \le c_5 H_t^{(\gamma,\theta)}(y-x),
\end{aligned}$$

with c_3, c_4, c_5 depending on T. By the dominated convergence and Lemma 4.2,

$$\lim_{h\to 0}\int_0^{t-\varepsilon}\int_{\mathbb{R}^d} p_s(x,z)\Phi_{t-s}(z,y+h)\, dzds = \int_0^{t-\varepsilon}\int_{\mathbb{R}^d} p_s(x,z)\Phi_{t-s}(z,y)\, dzds.$$

Furthermore, for every $|h| < t^{1/\alpha}$,

$$\begin{aligned}
&\left|\int_{t-\varepsilon}^{t}\int_{\mathbb{R}^d} p_s(x,z)\Phi_{t-s}(z,y+h)\, dzds\right| \\
\le\ & c_6 \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^d} H_s^{(\gamma,\theta)}(z-x)(t-s)^{-1+\theta/\alpha} H_{t-s}^{(\gamma,\theta)}(y+h-z)\, dzds \\
\le\ & c_7 \int_{t-\varepsilon}^{t} (t-s)^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y+h-x)\, ds \le c_8 \varepsilon^{\theta/\alpha} H_t^{(\gamma,\theta)}(y-x) < \infty.
\end{aligned}$$

This and Lemma 2.12 yield the continuity of $y \mapsto p_t(x,y)$. □

The proof of Theorem 1.2 is complete. □

Remark 4.1. *The lower bound in (21) extends to $t \in (0,T]$ for every finite $T>0$. The interested reader may use Lemma 5.4 below for a proof. By (20), the upper bound in (21) holds for all $x,y \in \mathbb{R}^d$ and $t \in (0,T]$, if $T<\infty$.*

5 The maximum principle

In this part of our development we follow Kochubeĭ's argument from [39, Section 6] with some modifications – we temper by $e^{-\lambda t}$ rather than restrict time. For $\lambda \in \mathbb{R}$ we let $\tilde{p}_t(x,y) = e^{-\lambda t} p_t(x,y)$, where $t>0$, $x,y \in \mathbb{R}^d$. By Theorem 1.2, $\tilde{p}_t(x,y) \le C e^{-(\lambda-c)t} G_t^{\alpha+\gamma}(y-x)$. We can give a solution to the Cauchy problem for $L-\lambda$.

Lemma 5.1. *If* $f \in C_0(\mathbb{R}^d)$, $u(t,x) = \int_{\mathbb{R}^d} \tilde{p}_t(x,y)f(y)\,dy$ *for* $t > 0$ *and* $u(0,x) = f(x)$, $x \in \mathbb{R}^d$, *then* u *is a continuous function on* $[0,\infty) \times \mathbb{R}^d$, *and*

$$\big(\partial_t - L_x + \lambda\big)u(t,x) = 0, \quad t > 0,\ x \in \mathbb{R}^d. \tag{91}$$

If $\lambda > c$, *where* c *is from* Theorem 1.2, *then* $u \in C_0([0,\infty) \times \mathbb{R}^d)$.

Proof. Let $(t_0,x_0) \in (0,\infty) \times \mathbb{R}^d$. We have

$$\begin{aligned}|u(t_0,x_0) - u(t,x)| &\le \int_{\mathbb{R}^d} |\tilde{p}_{t_0}(x_0,y) - \tilde{p}_t(x,y)||f(y)|\,dy \\ &\le e^{-\lambda t_0}\int_{\mathbb{R}^d} |p_{t_0}(x_0,y) - p_{t_0}(x,y)||f(y)|\,dy \\ &\quad + \int_{\mathbb{R}^d} |\tilde{p}_{t_0}(x,y) - \tilde{p}_t(x,y)||f(y)|\,dy \to 0,\end{aligned}$$

as $(t,x) \to (t_0,x_0)$. This follows from the dominated convergence, since Theorem 1.2 yields for $|x - x_0| < t_0^{1/\alpha}$

$$\begin{aligned}\big|p_{t_0}(x_0,y) &- p_{t_0}(x,y)\big| \\ &\le c_1\left(\frac{|x_0-x|}{t_0^{1/\alpha}}\right)^{\theta} e^{ct_0}\left(G_{t_0}^{(\alpha+\gamma)}(y-x_0) + G_{t_0}^{(\alpha+\gamma)}(y-x)\right) \\ &\le c_2\left(\frac{|x_0-x|}{t_0^{1/\alpha}}\right)^{\theta} e^{ct_0}G_{t_0}^{(\alpha+\gamma)}(y-x_0),\end{aligned}$$

and for $|t - t_0| \le t_0/2$ and some $s \in (t \wedge t_0, t \vee t_0)$ we have

$$\begin{aligned}|\tilde{p}_{t_0}(x,y) - \tilde{p}_t(x,y)| &= \left|\left(e^{-\lambda s}\partial_s p_s(x,y) - \lambda e^{-\lambda s}p_s(x,y)\right)(t_0 - t)\right| \\ &\le c_3(s^{-1}+\lambda)e^{(c-\lambda)s}G_s^{(\alpha+\gamma)}(y-x)|t_0-t| \\ &\le c_4(2/t_0+\lambda)e^{[(c-\lambda)\vee 0](3t_0/2)}G_{t_0}^{(\alpha+\gamma)}(y-x)|t_0-t|.\end{aligned}$$

If $(t,x) \to (0,x_0)$ for some $x_0 \in \mathbb{R}^d$, then by Theorem 1.1 and continuity of f,

$$|f(x_0) - u(t,x)| \le |f(x_0) - f(x)| + |f(x) - u(t,x)| \to 0,$$

This gives the continuity of u on $[0,\infty) \times \mathbb{R}^d$.
Let $\delta > 0$. Using the notation from Section 1, by Fubini's theorem we get

$$L^\delta u(t,x) = \int_{|u|>\delta} \left(u(t,x+u) - u(t,x)\right)\nu(x,du) = \int L_x^\delta \tilde{p}_t(x,y)f(y)dy.$$

By (79) and the dominated convergence theorem we get

$$Lu(t,x) = \int L_x\tilde{p}_t(x,y)f(y)\,dy. \tag{92}$$

In order to show that $\partial_t u(t,x) = \int \partial_t \tilde{p}_t(x,y) f(y)\,dy$, it suffices to estimate $|\partial_t \tilde{p}_t(x,y)|$ for every $t_0 > 0$ and all $t \in (t_0/2, 3t_0/2)$ by an integrable function depending only on t_0 and x, y. We obtain the estimate by using Theorem 1.2, which yields

$$\begin{aligned} |\partial_t \tilde{p}_t(x,y)| &\le \lambda e^{-\lambda t}|p_t(x,y)| + e^{-\lambda t}|\partial_t p_t(x,y)| \\ &\le c_1 e^{(c-\lambda)t} G_t^{(\alpha+\gamma)}(y-x)(\lambda + t^{-1}) \\ &\le \lambda c_2 e^{[(c-\lambda)\vee 0](3t_0/2)} G_{t_0}^{(\alpha+\gamma)}(y-x)(\lambda + (t_0/2)^{-1}). \end{aligned}$$

By the dominated convergence theorem,

$$\partial_t u(t,x) = \int \partial_t \tilde{p}_t(x,y) f(y)\,dy. \tag{93}$$

We note here that (92) and (93) hold in fact for every bounded function f. Now it easily follows from Theorem 1.1 that

$$\begin{aligned} (\partial_t - L_x)\tilde{p}_t(x,y) &= e^{-\lambda t}\partial_t p_t(x,y) - \lambda e^{-\lambda t} p_t(x,y) - e^{-\lambda t} L_x p_t(x,y) \\ &= -\lambda \tilde{p}_t(x,y), \end{aligned}$$

which, together with (93) and (92), yields (91). If $\lambda > c$, then we have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \tilde{p}_t(x,y) f(y)\,dy \right| &\le c_1 e^{-(\lambda-c)t} \int_{\mathbb{R}^d} G_t^{\alpha+\gamma}(y-x)|f(y)|\,dy \\ &= c_1 e^{-(\lambda-c)t} \int_{\mathbb{R}^d} G_1^{\alpha+\gamma}(y)|f(t^{1/\alpha}y + x)|\,dy \\ &\le c_1 e^{-(\lambda-c)t}\|f\|_\infty \int_{\mathbb{R}^d} G_1^{\alpha+\gamma}(y)\,dy \le c_2\|f\|_\infty. \end{aligned}$$

In fact, $e^{-(\lambda-c)t}|f(t^{1/\alpha}y + x)| \to 0$ as $|(t,x)| \to \infty$. By the dominated convergence theorem, $\lim_{|(t,x)|\to\infty} u(t,x) = 0$. ☐

Lemma 5.2. *If $u(t,x) \in C_0([0,\infty) \times \mathbb{R}^d)$, $\lambda \ge 0$ and $(\partial_t - L_x + \lambda)u(t,x) = 0$ on $(0,\infty) \times \mathbb{R}^d$, then*

$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} |u(t,x)| = \sup_{x\in\mathbb{R}^d} |u(0,x)|.$$

Proof. Let $m = \inf_{(t,x)\in[0,\infty)\times\mathbb{R}^d} u(t,x)$ and $M = \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} u(t,x)$. We have $-\infty < m \le 0 \le M < \infty$. If $M > 0$ and $u(t_0,x_0) = M$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$, then $\partial_t u(t_0,x_0) = 0$ and $L_x u(t_0,x_0) < 0$ by the maximum principle from Section 1. Hence $(\partial_t - L_x + \lambda)u(t_0,x_0) > 0$. This contradicts the assumptions of the lemma, hence $M = 0$ or the supremum of u is attained at some boundary point $(0,x_0)$. Similarly, if $m < 0$ and $u(t_0,x_0) = m$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$, then $\partial_t u(t_0,x_0) = 0$, $L_x u(t_0,x_0) > 0$, hence $(\partial_t - L_x + \lambda)u(t_0,x_0) < 0$. Again, we conclude that $m = 0$ or the infimum of u is attained at some point $(0,x_0)$. ☐

Corollary 5.3. *Let $\lambda \geq 0$. There is at most one solution $u \in C_0([0,\infty)\times\mathbb{R}^d)$ to the Cauchy problem for $L-\lambda$.*

Proof. By Lemma 5.2, the difference of two solutions is zero on $[0,\infty)\times\mathbb{R}^d$. □

Lemma 5.4. *p is nonnegative and satisfies the Chapman-Kolmogorov equation.*

Proof. Let, as usual, $\tilde{p}_t(x,y) = e^{-\lambda t}p_t(x,y)$ and pick $\lambda > c$, the constant in Theorem 1.2. Let $f \in C_0(\mathbb{R}^d)$. By Lemma 5.1, $u(t,x) := \int \tilde{p}_t(x,y)f(y)dy$ extends to a function of the class $C_0([0,\infty)\times\mathbb{R}^d)$. Recall that p is continuous (see Lemma 4.3), hence $\tilde{p}$ is continuous. Taking into account that all the nonnegative functions $f \in C_0(\mathbb{R}^d)$ are allowed here, by the proof of Lemma 5.2 we get that $\tilde{p} \geq 0$ and thus $p \geq 0$.
Next we consider $s > 0$ and $u(s,x)$ defined above. For $t \geq 0$, $x \in \mathbb{R}^d$, let $w(t,x)$ be the solution to the Cauchy problem for $L-\lambda$ with the initial condition $w(0,x) = u(s,x)$, $x \in \mathbb{R}^d$. By Lemma 5.1 and Corollary 5.3,

$$\int_{\mathbb{R}^d} \tilde{p}_{s+t}(x,y)f(y)dy = u(s+t,x) = w(t,x) = \int_{\mathbb{R}^d} \tilde{p}_t(x,y)\int_{\mathbb{R}^d} \tilde{p}_s(y,z)f(z)dzdy.$$

Since $f \in C_0(\mathbb{R}^d)$ is arbitrary, using Fubini's theorem we see that $\tilde{p}$ satisfies the Chapman-Kolmogorov equation and so does p. □

For $f \in C_0(\mathbb{R}^d)$, $t > 0$ and $x \in \mathbb{R}^d$, we let

$$\tilde{P}_t f(x) = \int_{\mathbb{R}^d} \tilde{p}_t(x,y)f(y)dy.$$

We conclude that $\{\tilde{P}_t\}$ and $\{P_t\}$ are strongly continuous semigroups on $C_0(\mathbb{R}^d)$.

Lemma 5.5. *If $\lambda > c_1$, the constant from* Theorem 1.2*, then $\{\tilde{P}_t\}$ is sub-Markovian.*

Proof. By Lemma 5.4, $\tilde{P}_t f \geq 0$ if $f \in C_0(\mathbb{R}^d)$ and $f \geq 0$. By Lemma 5.1 and Lemma 5.2, $\|\tilde{P}_t f\|_\infty \leq \|f\|_\infty$, as needed. □

In particular, if $\lambda > c_1$, then for all $t > 0$ and $x \in \mathbb{R}^d$ we have $\int_{\mathbb{R}^d} \tilde{p}_t(x,y)dy \leq 1$. We next verify that $p_t(x,y)$ is in fact a transition probability density. The result requires preparation. Let $\mathcal{L}$ be the generator of $\{P_t\}$. Then $\mathcal{L}-\lambda$ is the generator of $\{\tilde{P}_t\}$, with the same domain, say $D(\mathcal{L})$, a dense subset of $C_0(\mathbb{R}^d)$. We will make a connection between L and $\mathcal{L}$. Let $\phi \in C_0(\mathbb{R}^d)$, $0 < T < \infty$ and $f = \int_0^T P_s\phi ds$. By the general semigroup theory, $f \in D(\mathcal{L})$ and $\partial_t P_t f = \mathcal{L}P_t f \in C_0(\mathbb{R}^d)$ for every $t > 0$. By Lemma 5.1, $\partial_t P_t f(x) = LP_t f(x)$ for all $t > 0$ and $x \in \mathbb{R}^d$, hence $\mathcal{L}P_t f = LP_t f$ for all such t and f. Therefore $L = \mathcal{L}$ on a dense subset of $C_0(\mathbb{R}^d)$.
The following more explicit result is rather delicate.

Theorem 5.6. *$\mathcal{L}f = Lf$ for $f \in C_0^2(\mathbb{R}^d)$.*

Proof. We first prove that for Hölder continuous function $g \in C_0(\mathbb{R}^d)$ we have

$$L \int_0^t P_s g(x) ds = \int_0^t L P_s g(x) ds, \quad x \in \mathbb{R}^d. \tag{94}$$

Indeed, for $\delta > 0$ the operator L^δ is bounded and linear on $C_0(\mathbb{R}^d)$, hence

$$\begin{aligned} L \int_0^t P_s g(x) ds &= \lim_{\delta \to 0} L^\delta \int_0^t P_s g(x) ds = \lim_{\delta \to 0} \int_0^t L^\delta P_s g(x) ds \\ &= \lim_{\delta \to 0} \int_0^t \int_{\mathbb{R}^d} L_x^\delta p_s(x,y) g(y) dy ds = I + II, \end{aligned}$$

where

$$I = \lim_{\delta \to 0} \int_0^t \int_{\mathbb{R}^d} L_x^\delta p_s(x,y)[g(y) - g(x)] dy ds,$$

$$II = g(x) \lim_{\delta \to 0} \int_0^t \int_{\mathbb{R}^d} L_x^\delta p_s(x,y) dy ds$$

are finite, as we will shortly see. For II, by Corollary 3.9 and Lemma 3.8,

$$\lim_{\delta \to 0} \int_{\mathbb{R}^d} L_x^\delta p_s(x,y) dy = \int_{\mathbb{R}^d} \lim_{\delta \to 0} L_x^\delta p_s(x,y) dy = \int_{\mathbb{R}^d} L_x p_s(x,y) dy.$$

Therefore by (80) and the dominated convergence theorem,

$$II = g(x) \int_0^t \lim_{\delta \to 0} \int_{\mathbb{R}^d} L_x^\delta p_s(x,y) dy ds = g(x) \int_0^t \Big(\int_{\mathbb{R}^d} L_x p_s(x,y) dy \Big) ds.$$

It is important to notice that the last (outer) integral $\int_0^t (\ldots) ds$ converges absolutely. We now turn to I. Let $\epsilon > 0$ be such that $\alpha + \gamma - \epsilon > d$. Let g be Hölder continuous of order ϵ. Then for $x, y \in \mathbb{R}^d$ and $s \in (0,t)$, by (79) we get

$$\begin{aligned} \big| L_x^\delta p_s(x,y)[g(y) - g(x)] \big| &\le c |L_x^\delta p_s(x,y)| (1 \wedge |x-y|^\epsilon) \\ &\le c s^{-1} \big(G_s^{\alpha+\gamma}(y-x) + s^{\theta/\alpha} K_s(x,y) \big) (1 \wedge |x-y|^\epsilon) \\ &\le c s^{-1+\epsilon/\alpha} G_s^{(\alpha+\gamma-\epsilon)}(y-x) + s^{-1+\theta/\alpha} K_s(x,y). \end{aligned}$$

The above expression is integrable in $dyds$. Of course, $\lim_{\delta \to 0} L_x^\delta p_s(x,y) = L_x p_s(x,y)$. By the dominated convergence theorem,

$$I = \int_{(0,t) \times \mathbb{R}^d} L_x p_s(x,y)[g(y) - g(x)] dy ds,$$

which is finite. Adding I and II we obtain

$$L \int_0^t P_s g(x) ds = \int_0^t \Big(\int_{\mathbb{R}^d} L_x p_s(x,y) g(y) dy \Big) ds.$$

By (79) and the boundedness of L^δ we have that

$$\int_{\mathbb{R}^d} L_x p_s(x,y)g(y)dy = \int_{\mathbb{R}^d} \lim_{\delta\to 0} L_x^\delta p_s(x,y)g(y)dy = \lim_{\delta\to 0}\int_{\mathbb{R}^d} L_x^\delta p_s(x,y)g(y)dy$$
$$= \lim_{\delta\to 0} L^\delta \int_{\mathbb{R}^d} p_s(x,y)g(y)dy = \lim_{\delta\to 0} L^\delta P_s g(x).$$

Therefore, $\int_{\mathbb{R}^d} L_x p_s(x,y)g(y)dy = LP_s g(x)$, which gives (94).
We claim that for $f \in C_0^2(\mathbb{R}^d)$, $0<t<\infty$ and $x\in\mathbb{R}^d$,

$$P_t f(x) - f(x) = \int_0^t P_s L f(x) ds. \tag{95}$$

To prove (95) we let $\lambda > c > 0$, with c from Theorem 1.2, and we define

$$u(t,x) = e^{-\lambda t}\Big[P_t f(x) - f(x) - \int_0^t P_s L f(x) ds\Big].$$

We also let $u(0,x)=0$. By Lemma 5.1, $u \in C_0([0,\infty)\times\mathbb{R}^d)$. We can directly verify that Lf is Hölder continuous on $\mathbb{R}^d$, and then by (94) with $g = Lf$,

$$(\partial_t - L)u(t,x) = -\lambda u(t,x) + e^{-\lambda t}\Big[Lf(x) - P_t L f(x) + \int_0^t LP_s Lf(x)ds\Big].$$

From the discussion of (94) the last integral is absolutely convergent, implying that $\partial_s P_s L f(x) = LP_s Lf(x)$ is also absolutely integrable, cf. Lemma 5.1. Therefore,

$$(\partial_t - L)u(t,x) = -\lambda u(t,x) + e^{-\lambda t}\left[-\int_0^t \partial_s P_s L f(x) ds + \int_0^t LP_s Lf(x)ds\right]$$
$$= -\lambda u(t,x).$$

We now prove that $u\equiv 0$. Recall that $u\in C_0([0,\infty)\times\mathbb{R}^d)$. If u attains a strictly positive maximum at some point $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^d$, then $(\partial_t - L)u(t_0,x_0) = -\lambda u(t_0,x_0) < 0$, but the maximum principle for L contradicts this: $(\partial_t - L)u(t_0,x_0) = -Lu(t_0,x_0) > 0$. Therefore we must have $u\le 0$. Analogously we prove that $u \ge 0$ and so $u=0$ everywhere. Finally, we divide both sides of (95) by t and let $t\to 0$. We obtain $\mathcal{L}f(x) = Lf(x)$. The proof is complete: the operator L and the generator $\mathcal{L}$ coincide on $C_0^2(\mathbb{R}^d)$. □

5.1 Proof of Theorem 1.3

We only need to prove that for all $t>0$ and $x\in\mathbb{R}^d$ we have $\int_{\mathbb{R}^d} p_t(x,y)dy = 1$. We know that the operators $P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y)f(y)\,dy$, $t>0$, form a strongly continuous semigroup on $C_0(\mathbb{R}^d)$ with the generator $\mathcal{L}$. We fix $t>0$. If f is in the domain of $\mathcal{L}$, then from the general theory of semigroups (see [18, Ch. 2, Lemma 1.3] or [30, Lemma 4.1.14]),

$$P_t f(x) - f(x) = \int_0^t P_s \mathcal{L} f(x)\, ds.$$

In particular, let $f \in C_0^2(\mathbb{R}^d)$ be such that $|f(x)| \le 1$ for all $x \in \mathbb{R}^d$ and $f(x) = 1$ for $|x| < 1$. Let $f_n(x) = f(x/n)$, $n \ge 1$. We have $\lim_{n\to\infty} f_n(x) = 1$ and $\lim_{n\to\infty} P_t f_n(x) = \int_{\mathbb{R}^d} p_t(x,y)\, dy$, which easily follows from bounded convergence. Furthermore,

$$P_t f_n(x) - f_n(x) = \int_0^t P_s L f_n(x)\, ds. \tag{96}$$

If $x \in \mathbb{R}^d$ is fixed and $n > 2|x|$, then

$$\begin{aligned} |Lf_n(x)| &= \frac{1}{2}\left|\int_{\mathbb{R}^d} \left(f_n(x+u) + f_n(x-u) - 2f_n(x)\right) \nu(x,du)\right| \\ &= \frac{1}{2}\left|\int_{|u|>n/2} \left(f((x+u)/n) + f((x-u)/n) - 2\right) \nu(x,du)\right| \\ &\le \nu(x, B(0,n/2)^c) \le M_0 \nu_0(B(0,n/2)^c) \le c_1 n^{-\alpha}. \end{aligned}$$

This yields

$$\left|\int_0^t P_s L f_n(x)\, ds\right| \le \int_0^t |P_s L f_n(x)|\, ds \le c_2 t n^{-\alpha} \to 0,$$

as $n \to \infty$. By (96) and the above discussion we get $\int_{\mathbb{R}^d} p_t(x,y)\, dy = 1$. The proof of Theorem 1.3 is complete. □

We end the paper by pointing out in which sense $p_t(x,y)$ is unique. Plainly, if $\mathfrak{p}_t(x,y)$ has the properties listed in Theorem 1.1 and 1.2, then $\mathfrak{p}_t(x,y) = p_t(x,y)$ for all $t > 0$, $x,y \in \mathbb{R}^d$. Indeed, let $s > 0$ and $z \in \mathbb{R}^d$. By the proof of Lemma 5.1, $u(t,x) := e^{-\lambda t} \int_{\mathbb{R}^d} \mathfrak{p}_t(x,y) p_s(y,z) dy$ and $\mathfrak{u}(t,x) := e^{-\lambda t} \mathfrak{p}_{t+s}(x,z)$ give solutions to the same Cauchy problem for $L - \lambda$, and they are in $C_0([0,\infty) \times \mathbb{R}^d)$ for large $\lambda > 0$. By Corollary 5.3,

$$\int_{\mathbb{R}^d} \mathfrak{p}_t(x,y) p_s(y,z) dy = \mathfrak{p}_{t+s}(x,z), \quad s,t > 0,\ x,y \in \mathbb{R}^d.$$

We claim that for all $f \in C_0(\mathbb{R}^d)$, uniformly in $x \in \mathbb{R}^d$ we have

$$\lim_{t\to 0} \int_{\mathbb{R}^d} f(x) p_t(x,y)\, dx = f(y). \tag{97}$$

For clarity, this is different from (17). To prove (97) we note that

$$\int_{\mathbb{R}^d} p_t^0(x,y) dx = \int_{\mathbb{R}^d} p^y(y-x) dx = 1, \quad t > 0,\ y \in \mathbb{R}^d,$$

we recall (15), (72), (63), Lemma 2.4 with $\beta = 0$, the scaling of $G_t^{(\alpha+\gamma)}$ and the dominated convergence. By (97) we get $p_s(x,z) = \mathfrak{p}_s(x,z)$ for all $s > 0$, $x, z \in \mathbb{R}^d$.

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Krzysztof Bogdan
Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
Wyb. Wyspiańskiego 27
50-370 Wrocław
Poland
krzysztof.bogdan@pwr.edu.pl

Paweł Sztonyk
Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
Wyb. Wyspiańskiego 27
50-370 Wrocław
Poland
pawel.sztonyk@pwr.edu.pl

Victoria Knopova
TU Dresden
Zellescher Weg 12–14
Dresden
Germany
victoria.knopova@tu-dresden.de