On the Whistle Cobordism Operation in String Topology of Classifying Spaces

Katsuhiko Kuribayashi

Received: July 27, 2019
Revised: December 3, 2019

Communicated by Max Karoubi

Abstract. In this manuscript, we consider cobordism operations in the 2-dimensional labeled open-closed topological quantum field theory for the classifying space of a connected compact Lie group in the sense of Guldberg. In particular, it is proved that the whistle cobordism operation is non-trivial in general provided the labels are in the set of maximal closed subgroups of the given Lie group. The non-triviality of cobordism operations induced by gluing the whistle with the opposite and other labeled cobordisms is also discussed.

2020 Mathematics Subject Classification: 55P50, 81T40, 55R35
Keywords and Phrases: String topology, classifying space, topological quantum field theory, Eilenberg–Moore spectral sequence

1 Introduction

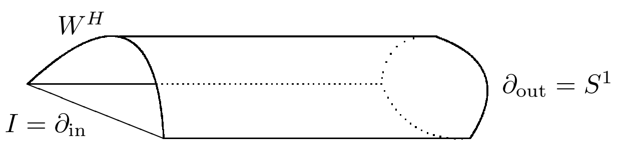
String topology initiated by Chas and Sullivan [6] until now provides many impressive and fruitful algebraic structures for the homology of the free loop spaces of orientable manifolds [9, 8, 12], orbifolds [19], the classifying spaces of Lie groups [7, 14, 15], Gorenstein spaces [11, 16] and differentiable stacks [3]. In this manuscript, we deal with string topology for classifying spaces, which is enriched with a 2-dimensional labeled open-closed *topological quantum field theory* (TQFT).

In [13], Guldberg has developed such a labeled TQFT for the classifying space BG of a connected compact Lie group G. In consequence, the homology groups of double coset spaces associated to G and of the free loop space LBG are simultaneously incorporated into the open-closed TQFT with labels in a set of closed connected subgroups of G. The structure is indeed induced by an open-closed *homological conformal field theory* (HCFT) which is an extended version of a closed HCFT for classifying spaces due to Chataur and Menichi [7]; see [13, Theorem 1.2.3 and Lemma 2.4.1]. The aim of this manuscript is to investigate the non-triviality of an important cobordism operation which connects the open and closed theories in the labeled TQFT.

To describe a labeled open-closed TQFT in general, we need to introduce the category $\mathsf{oc\text{-}Cobor}(\mathcal{S})$ of open-closed strings. Its objects are finite disjoint unions of oriented circle and intervals with ends labeled by elements of a fixed set $\mathcal{S}$. A morphism in the category is the diffeomorphism class of cobordisms from Y_0 to Y_1 labeled in $\mathcal{S}$, where such a cobordism is indeed a 2-dimensional oriented manifold Σ whose boundary consists of three parts $Y_0 \cup Y_1 \cup \partial_{\text{free}}\Sigma$. Here the part $\partial_{\text{free}}\Sigma$ called the *free boundary* is a cobordism between ∂Y_0 and ∂Y_1. Moreover, it is required that the connected component of $\partial_{\text{free}}\Sigma$ is labeled by elements of $\mathcal{S}$ compatible with the labeling of ∂Y_0 and ∂Y_1; see [21, Section 2] and [17, Section 3] for the precise definition and [18, Section 2] for a physical point of view. The compositions are given by gluing cobordisms provided the labelings of the boundaries are compatible. By definition, a 2-*dimensional labeled open-closed topological quantum field theory* is a monoidal functor μ from $(\mathsf{oc\text{-}Cobor}(\mathcal{S}), \coprod)$ to $(\mathbb{K}\text{-}\mathsf{Vect}, \otimes)$ the category of graded vector space over $\mathbb{K}$, where the monoidal structure $\coprod$ of $\mathsf{oc\text{-}Cobor}(\mathcal{S})$ is given by the disjoint union of cobordisms. We may write μ_Σ for the linear map assigned by a cobordism Σ. Moreover, we denote by $(\Sigma, \{\Sigma^H\}_{H\in\mathcal{S}'})$ a one or two dimensional labeled cobordism whose free boundary has conected components $\{\Sigma^H\}_{H\in\mathcal{S}'}$, where $\mathcal{S}'$ is a subset of $\mathcal{S}$.

While the labeled open-closed TQFT for a classifying space is investigated in this article, we refer the reader to the results due to Blumberg, Cohen and Teleman [4] for an open-closed TQFT labeled by submanifolds of a given manifold; see also [12] for an HCFT based on the free loop space of a manifold.

Let G be a compact connected Lie group and $\mathcal{B}$ a set consisting of connected closed subgroups of G. Let $W = (W, \{W^H\})$ denote the whistle cobrodism from the interval I to the circle S^1 whose incoming boundary $\partial_{\text{in}} := I$ is connected with an arc W^H labeled by a subgroup $H \in \mathcal{B}$ at the each endpoint; see the figure below for the whistle cobordism.



Observe that the arc W^H is the only free boundary of the whistle. In [13], the non-triviality of a cobordism operation of open strings, namely intervals, is revealed with a computational example; see Section 4 for more computations in an open TQFT. Since the whistle cobrodism connects open and closed strings, it is anticipated that the operation associated with the whistle plays a key role in the open-closed TQFT. In fact, an open-closed TQFT splits into the open theory and the closed one if all whistle cobordism operations are trivial; see [17, Propositions 3.8 and 3.9] for generators of morphisms in oc-Cobor($\mathcal{B}$) for example. In this article, we focus on the whistle cobordism operation and the non-triviality is discussed.

In what follows, the homology and cohomology are with coefficients in a field $\mathbb{K}$. Our main theorem is described as follows.

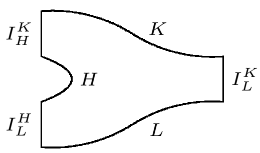
Theorem 1.1. *Let G be a connected compact Lie group and H a connected closed subgroup of maximal rank. Suppose that the integral homology groups of G and H are p-torsion free, where p is the characteristic of $\mathbb{K}$. Then the operations μ_W and $\mu_{W^{\rm op}}$ associated to the whistle cobordisms $(W, \{W^H\})$ and $(W^{\rm op}, \{(W^{\rm op})^H\})$ are non-trivial. Moreover, the composite operation $\mu_W \circ \mu_{W^{\rm op}} = \mu_{W\circ W^{\rm op}}$ is also non-trivial if $(\deg(B\iota)^*(x_i), p) = 1$ for any $i = 1, ..., l$, where $B\iota : BH \to BG$ stands for the map between classifying spaces induced by the inclusion $\iota : H \to G$ and $x_1, ..., x_l$ are generators of $H^*(BG; \mathbb{K})$.*

This enables us to conclude that the open theory and closed theory are inseparable in general. In consequence, we can explicitly determine every labeled open-closed cobordism operation over the rational $\mathbb{Q}$ under an appropriate assumption on the set of labels; see Assertion 4.1.

We observe that the cobordism $W \circ W^{\rm op}$ in Theorem 1.1 is the cylinder with a hole labeled by H a subgroup of maximal rank. Operations associated with other composites of the whistle cobordisms are discussed in Remark 3.3 below.

We also discuss a cobordism operation in the labelled open TQFT for classifying spaces. Let G be a connected compact Lie group whose cohomology with coefficients in $\mathbb{K}$ is a polynomial algebra over generators with even degree. Let K, H and L be connected closed subgroup of G of maximal rank, whose cohomology algebras satisfy the same condition as that of G. Then we have

Theorem 1.2. *Let Υ be the basic cobordism from two labeled intervals I_H^K and I_L^H to one labeled interval I_L^K, which is pictured below. Then the cobordism operation μ_Υ is trivial but not $\mu_{\Upsilon^{\rm op}}$ in general. More precisely, the operation $\mu_{\Upsilon^{\rm op}}$ is injective.*

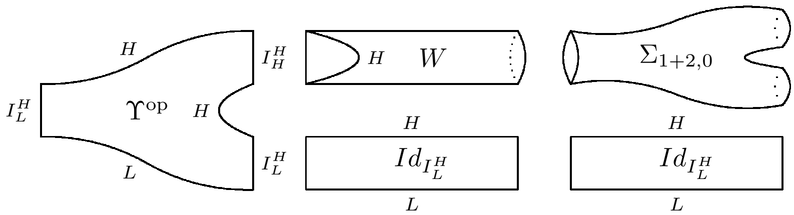


As an advantage of the explicit form of the operation $\mu_{\Upsilon^{op}}$ described in the proof of Theorem 1.2 in Section 4, we have a computational example of a more complicated operation associated with the cobordism

$$\Sigma_{\rm OC} := (\Sigma_{1+2,0} \coprod Id_{I_L^H}) \circ (W \coprod Id_{I_L^H}) \circ \Upsilon^{\rm op}$$

in the open-closed TQFT for the classifying space of a Lie group G; see Section 4, where each labelled cobordism is pictured below.

(1.1)



More precisely, we choose the unitary group $U(m+1)$ as the given Lie group G and $U(m) \times U(1)$ as the labeling H. Then we have

Proposition 1.3. *For arbitrary connected closed subgroup L of $U(m+1)$ whose integral homology is p-torsion free, the cobordism operation $\mu_{\Sigma_{\mathrm{OC}}}$ is injective.*

This gives the first nontrivial calculation in the 2-dimensional labeled open-closed TQFT for the classifying space.
The rest of the article is organized as follows. In Section 2, we briefly review the construction of a cobordism operation in the labeled TQFT for classifying spaces due to Guldberg. After describing our strategy for proving the main theorem, we consider the Eilenberg-Moore spectral sequences for appropriate pullback diagrams and give the proof in Section 3. Section 4 proves Theorem 1.2 and Proposition 1.3. In Appendix A, we discuss rational homotopy theoretical methods for computing the whistle cobordism operation.

2 A brief review of the labeled TQFT for classifying spaces

We recall the cobordism operation introduced in [13, Section 2.3]. In what follows, we denote by $\mathrm{map}(X, Y)$ the mapping space of maps from X to Y with compact-open topology. Observe that $\mathrm{map}(S^1, BG) = LBG$ by definition. For a two dimensional labeled cobordism $\Sigma := (\Sigma, \{\Sigma^H\}_{H\in\mathcal{B}})$ with in-coming boundary ∂_{in} and outgoing boundary ∂_{out}, we define a space $\mathcal{M}(\Sigma)$ by the pullback diagram

$$\begin{array}{ccc} \mathcal{M}(\Sigma) & \longrightarrow & \mathrm{map}(\Sigma, BG) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ \prod_H \mathrm{map}(\Sigma^H, BH) & \xrightarrow[B\iota_*]{} & \prod_H \mathrm{map}(\Sigma^H, BG), \end{array}$$

where $\iota : H \to G$ is the inclusion and $i : \coprod_H \Sigma^H = \partial_{\mathrm{free}}\Sigma \to \Sigma$ denotes the embedding. By applying the same pullback construction as above to a one dimensional cobordism of the form $\partial_{in} = (\partial_{in}, \{\Sigma^H \cap \partial_{in}\}_{H\in\mathcal{B}})$, we define a space $\mathcal{M}(\partial_{in})$. The naturality of the construction enables us to obtain a map $in^* : \mathcal{M}(\Sigma) \to \mathcal{M}(\partial_{in})$ from the inclusion $in : \partial_{in} \to \Sigma$ of the in-coming boundary.

Proposition 2.1. *[13, Proposition 2.3.9]* (i) *The map in^* induced the inclusion gives rise to a fibration $\mathcal{M}(\Sigma)_c \to \mathcal{M}(\Sigma) \stackrel{in^*}{\to} \mathcal{M}(\partial_{\rm in})$ whose fibre $\mathcal{M}(\Sigma)_c$ is the product of $\Omega BH \simeq H$, G/H and the total space E of a fibration of the form $\Omega BH \to E \to G/H$ in which H's are the labels of the cobordism Σ.*
(ii) *The fibration in* (i) *is orientable; that is, the action of the fundamental group of the base on the homology of the fibre is trivial.*

Thus for the fibration $h := in^* : \mathcal{M}(\Sigma) \to \mathcal{M}(\partial\Sigma)$, we can define the integration along the fibre $h_! : H_*(\mathcal{M}(\partial\Sigma)) \to H_{*+i}(\Sigma)$ with degree i the top degree of $H_*(\mathcal{M}(\Sigma)_c)$. The main result [13, Theorem 1.2.3] asserts that an operation μ_Σ defined by the composite

$$\mu_\Sigma : H_*(\mathcal{M}(\partial_{\rm in})) \xrightarrow{h_!} H_*(\mathcal{M}(\Sigma)) \xrightarrow{(out^*)_*} H_*(\mathcal{M}(\partial_{\rm out}))$$

for each labeled cobordism Σ gives rise to a labeled open-closed TQFT structure for the classifying space of G. We here and henceforth omit the action of *determinants* in the TQFT for classifying spaces; see [7, 13] for the action. This means that our computation of the cobordism operations below is made up to multiplication by non-zero scalar.
For a labeled whistle cobordism $W = (W, W^H)$, let a and b be the two endpoints of the arc W^H and hence they are also endpoints of the in-boundary $\partial_{\rm in}$ of W. In what follows, we may write ${}_a\cap_b$ and ${}_a—_b$ for the arc W^H and the in-boundary $\partial_{\rm in}$, respectively.
Remark 2.2. As for the whistle $W = (W, W^H)$, the fibration $h = in^* : \mathcal{M}(W) \to \mathcal{M}(\partial_{\rm in})$ induced by the embedding $\partial_{\rm in} \to W$ is the homotopy pullback of

$$in^* : \mathcal{M}(W^H) = \mathrm{map}(W^H, BG) \to \mathcal{M}(\{a, b\}) = \mathrm{map}(\{a, b\}, BH)$$

along the map $(in')^* : \mathcal{M}(\partial_{\rm in}) \to \mathcal{M}(\{a, b\})$. Thus the map in_* is regarded as the evaluation map $BH^I \to BH \times BH$ at 0 and 1. We see that the fibre of h has the homotopy type of ΩBH and then of H. Moreover, the fibration $k = out_* : \mathcal{M}(W) \to \mathcal{M}(\partial_{\rm out})$ is the homotopy pullback of the fibration $B\iota : BH \to BG$ along the evaluation map $ev_0 : \mathcal{M}(\partial_{\rm out}) = LBG \to BG$. This implies that the fibre of k has the homotopy type of the homogeneous space G/H. These results follow from the proof of [13, Proposition 2.3.9]. We observe that $\deg \mu_W = \dim\, H$ and $\deg \mu_{W^{\rm op}} = (\dim\, G/H) = (\dim\, G - \dim\, H)$.
If labels are in the set of subgroups of maximal rank, then Grassmann manifolds and flag manifolds can appear as the fibres of the fibrations $k : \mathcal{M}(W) \to \mathcal{M}(\partial_{\rm out})$.

3 Proof of the main result

We begin by describing our strategy for proving Theorem 1.1.

(i) We deal with the dual operation $D\mu_W = h^! \circ k^*$ on the cohomology.

(ii) Determine explicitly the cohomology algebras of $\mathcal{M}(\partial W)$ and $\mathcal{M}(W)$ with the Eilenberg–Moore spectral sequences, and investigate the behavior of the maps h^* and k^* for generators of the cohomology comparing the spectral sequences.

(iii) Consider the Leary–Serre spectral sequence (LSSS) for the fibration in Proposition 2.1 in order to compute the integration $h^!$ along the fibre.

(iv) Determine the value of the composite $h^! \circ k^*$ at an appropriate element of $H^*(\mathcal{M}(\partial_{\text{out}}))$.

(v) As for the latter half of the assertions, we also consider the dual operation $D\mu_{W^{\text{op}}} = k^! \circ h^*$ with the same strategy as above.

(vi) Reveal the nontriviality of the composite $D\mu_{W^{\text{op}}} \circ D\mu_W = D(\mu_{W \circ W^{\text{op}}})$ with the description of the fundamental class of the homogeneous space G/H due to Smith [24].

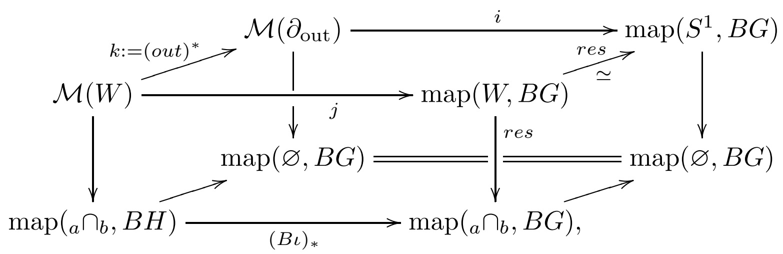
We recall the Eilenberg–Moore spectral sequence (EMSS) in a general setting. Let $p : E \to B$ be a fibration over a simply-connected base B and $\pi : E_\varphi \to X$ the pullback along a map $\varphi : X \to B$. We then have the Eilenberg–Moore spectral sequence [22] converging to the cohomology $H^*(E_\varphi)$ as an algebra with

$$E_2^{*,*} \cong \mathrm{Tot}^{*,*}_{H^*(B)}(H^*(E), H^*(X))$$

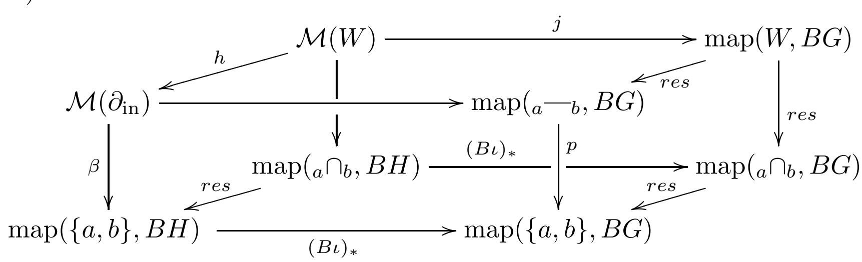
as a bigraded algebra. Observe that each term of the spectral sequence appears in the second quadrant.

In order to compute the cohomology concerning the whistle cobordism operation by using the EMSS, we consider commutative diagrams

(3.1)

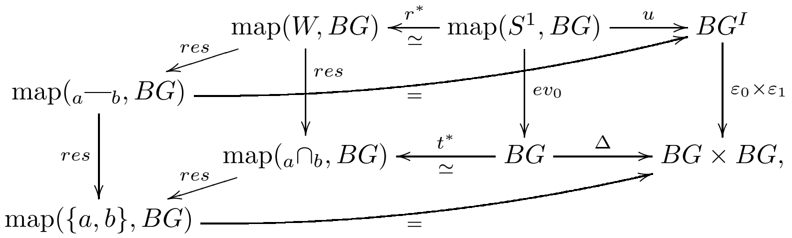


(3.2)



in which the front and back squares are pullback diagrams, and res and p denote the maps induced by the embeddings. Moreover, using a deformation retraction $r : W \to \partial_{\rm out}$ with $r(a) = 0 = r(b)$ and $r({}_a\!\!-\!\!{}_b) = \partial_{\rm out}$, which is a homotopy inverse of the embedding $out : \partial_{out} \to W$, we have commutative diagrams

(3.3)



in which the back right square is a pullback diagram, where $t : {}_a\cap_b \to \{a\}$ denotes a deformation retraction.

For morphisms $f : A \to M$ and $g : A \to N$ of algebras, we can regard M and N as A-modules via the morphisms. Then we write $\mathrm{Tor}_A(M,N)_{f,g}$ for the torsion product of M and N. By the assumption of the theorem, we see that the cohomology algebras of BG and BH are polynomial, say $H^*(BG) \cong \mathbb{K}[x_i, ..., x_l]$ and $H^*(BH) \cong \mathbb{K}[u_i, ..., u_l]$; see [20]. Applying the EMSS to the pullback diagrams in (3.1) and (3.2), we have commutative diagrams

(3.4)

$$\begin{array}{ccc}
H^*(\mathcal{M}(\partial_{\rm out})) & \xleftarrow[EM]{\cong} & \mathrm{Tor}_{\mathbb{K}}(\mathbb{K}, H^*(BG^{S^1})) =: A \\
\downarrow k^* & & \downarrow \mathrm{Tor}_\eta(\eta, (out^*)^*) \\
H^*(\mathcal{M}(W)) & \xleftarrow[EM]{\cong} & \mathrm{Tor}_{H^*(BG^{\cap})}(H^*(BH^{\cap}), H^*(BG^W)))_{(B\iota_*)^*, res^*} =: B \\
\uparrow h^* & & \uparrow \mathrm{Tor}_{res^*}(res^*, res^*) \\
H^*(\mathcal{M}(\partial_{\rm in})) & \xleftarrow[EM]{\cong} & \mathrm{Tor}_{H^*(BG^{\{a,b\}})}(H^*(BH^{\{a,b\}}), H^*(BG^{a-b})))_{(B\iota_*)^*, p^*} =: C,
\end{array}$$

where η is the unit, X^K is the mapping space $\mathrm{map}(K,X)$ and $\cap$ denotes the arc ${}_a\cap_b = W^H$. In fact, the argument with the *two sided Koszul resolution* implies that each spectral sequence collapses at the E_2-term. We observe that the resolution is of the form

$$(H^*(BG) \otimes H^*(BG) \otimes \wedge(y_1, ..., y_l), D) \xrightarrow{\mathsf{m}} H^*(BG) \cong H^*(BG^I) \longrightarrow 0,$$

where $D(y_i) = x_i \otimes 1 - 1 \otimes x_i$, bideg $y_i = (-1, \deg x_i)$ and m stands for the multiplication on $H^*(BG)$; see [2]. There exists no extension problem in the spectral sequences; see the diagram (3.6) below. Thus the naturality of each isomorphism $EM : \mathrm{Tot}E_2^{*,*} \to \mathrm{Tot}E_\infty^{*,*} \cong$ (the target cohomology), which is induced by the Eilenberg–Moore map [22], allows us to obtain the commutative diagram. Moreover, by using the retraction r mentioned above, we have commutative diagrams

$$
(3.5)\quad
\begin{array}{ccc}
A & \xrightarrow[=]{\qquad\qquad\qquad} & H^*(BG^{S^1}) =: A' \\
\downarrow & & \downarrow{\scriptstyle \iota\otimes 1} \\
B & \xrightarrow[\cong]{\mathrm{Tor}(1,(r^*)^*)} & \mathrm{Tor}_{H^*(BG^{\cap})}(H^*(BH^{\cap}), H^*(BG^{S^1})))_{(B\iota_*)^*,(r^*)^*\circ res^*} =: B' \\
\uparrow & & \uparrow{\scriptstyle \mathrm{Tor}(res^*,(r^*)^*\circ res^*)} \\
C & \xrightarrow[\cong]{} & \mathrm{Tor}_{H^*(BG^{\times 2})}(H^*(BH^{\times 2}), H^*(BG^{a—b}))_{((B\iota_*)^*)^{\otimes 2},res^*} =: C',
\end{array}
$$

where the left vertical arrows are the same ones as in (3.4). Since the back left diagram in (3.3) is commutative, it follows that $(r^*)^* \circ res^* = (ev_0)^* \circ (t^*)^*$. Moreover, the diagram (3.3) enables us to deduce the commutativity of the left-hand side two squares in the diagram

(3.6)

$$
\begin{array}{ccccc}
A' & \xleftarrow[EM]{\cong} & \mathrm{Tor}_{H^*(BG^{\times 2})}(H^*(BG), H^*(BG^{I})))_{\Delta^*,(\varepsilon_0\times e_1)^*} & \xleftarrow{\cong} & H^*(BG)\otimes\wedge(y_1,...,y_l) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle (B\iota)^*\otimes 1} \\
B' & \xrightarrow[\mathrm{Tor}_{t^*}(t^*,1)]{\cong} & \mathrm{Tor}_{H^*(BG)}(H^*(BH), H^*(BG^{S^1})))_{(B\iota_*)^*,(ev_0)^*} & \xleftarrow{\cong} & H^*(BH)\otimes\wedge(y_1,...,y_l) \\
\uparrow & & \uparrow{\scriptstyle \mathrm{Tor}_{\Delta^*}(\Delta^*,u^*)} & & \uparrow{\scriptstyle \mathsf{m}} \\
C' & \xrightarrow[\cong]{} & \mathrm{Tor}_{H^*(BG^{\times 2})}(H^*(BH^{\times 2}), H^*(BG^{I})_{((B\iota_*)^*)^{\otimes 2},res^*} & \xleftarrow{\cong} & \frac{H^*(BH)\otimes H^*(BH)}{((B\iota)^*x_i\otimes 1-1\otimes(B\iota)^*x_i)}.
\end{array}
$$

Explicit calculations of the EMSS's with the Koszul resolutions above give the commutative diagrams in the right-hand side in (3.6), where m denotes the map induced by the multiplication of the algebra $H^*(BH)$. We observe that

$$(B\iota)^*x_1 \otimes 1 - 1 \otimes (B\iota)^*x_1, ..., (B\iota)^*x_l \otimes 1 - 1 \otimes (B\iota)^*x_l$$

give a regular sequence since H is of maximal rank.

We are ready to prove main theorem.

Proof of the non-triviality of μ_W. In order to compute the integration along the fibre associated with the fibration $h := in^* : \mathcal{M}(W) \to \mathcal{M}(\partial_{\rm in})$, we consider the LSSS $\{_{LS}E_r^{*,*}, d_r\}$ for the fibration. As mentioned in Remark 2.2, the fibration h fits into the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{M}(W) & \longrightarrow & BH^I & \longrightarrow & K(\mathbb{Z}/p, \deg u_i)^I \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \varepsilon_0\times\varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_0\times\varepsilon_1} \\
\mathcal{M}(\partial_{\rm in}) & \xrightarrow[\beta]{} & BH \times BH & \xrightarrow[f_i\times f_i]{} & K(\mathbb{Z}/p, \deg u_i) \times K(\mathbb{Z}/p, \deg u_i)
\end{array}
$$

in which the left-hand side square is a pullback, where β is the map in (3.2) and the map f_i represents the element u_i. The argument with the EMSS yields that $H^*(\Omega BH) \cong H^*(H) \cong \wedge(z_1, ..., z_l)$ as algebras, where $\deg z_i = \deg u_i - 1$. Comparing the LSSS of the right two fibrations, we see that z_i is transgressive to $u_i \otimes 1 - 1 \otimes u_i$ for $i = 1, ..., l$ in the LSSS of the middle fibration. Therefore, we have

Lemma 3.1. *In $\{{}_{LS}E_r^{*,*}, d_r\}$, the element z_i is transgressive to an element of the form $u_i \otimes 1 - 1 \otimes u_i \in H^*(\mathcal{M}(\partial_{in}))$ for $i = 1, ..., l$, where $\beta^* u_i$ is identified with $u_i \in H^*(\mathcal{M}(\partial_{in}))$ via isomorphisms in (3.4), (3.5) and (3.6) for $i = 1, ..., l$.*

By [23, Lemma 3.4], we can write

$$(3.7) \qquad (B\iota)^* x_i \otimes 1 - 1 \otimes (B\iota)^* x_i = \sum_{j=1}^{l} \zeta_{ij}(u_j \otimes 1 - 1 \otimes u_j)$$

in $H^*(BH) \otimes H^*(BH)$ for $i = 1, ..., l$ with elements ζ_{ij} in $H^*(BH) \otimes H^*(BH)$ which satisfy the condition that $\mathsf{m}(\zeta_{ij}) = \frac{\partial (B\iota)^* x_i}{\partial u_j}$, where m denotes the multiplication on $H^*(BH)$. Then it follows from Lemma 3.1 that an element of the form $w_i := \sum_j^l \zeta_{ij} z_j$ is a permanent cycle for $i = 1, ..., l$. Moreover, we have

Lemma 3.2. *The elements $w_1, ..., w_l$ are linearly independent in the vector space $(Q\mathrm{Tot}_{LS}E_\infty^{*,*})^{\mathrm{odd}}$ of the indecomposable elements of $\mathrm{Tot}_{LS}E_\infty^{*,*}$ with odd degree.*

The computation in (3.4), (3.5) and (3.6) implies that

$$\mathbb{K}\{w_1, ..., w_l\} \cong (Q\mathrm{Tot}_{LS}E_\infty^{*,*})^{\mathrm{odd}} \cong (QH^*(\mathcal{M}(W)))^{\mathrm{odd}} \cong \mathbb{K}\{y_1, ..., y_l\}.$$

As for the first isomorphism, it follows from Lemma 3.2 that there exists an injective linear map $\mathbb{K}\{w_1, ..., w_l\} \to (Q\mathrm{Tot}_{LS}E_\infty^{*,*})^{\mathrm{odd}}$. The second and third isomorphisms yield that $\dim(Q\mathrm{Tot}_{LS}E_\infty^{*,*})^{\mathrm{odd}} = l$. We have the first isomorphism. It turns our that $0 \neq y_1 \cdots y_l = w_1 \cdots w_l = \det(\zeta_{ij}) z_1 \cdots z_l$ in ${}_{LS}E_\infty^{*,\dim H}$ changing the generators $y_1, ..., y_l$ if it is necessary. This implies that $\mu_W(1 \otimes y_1 \cdots y_l) = h^! \circ k^*(1 \otimes y_1 \cdots y_l) = h^!(1 \otimes y_1 \cdots y_l) = h^!(\det(\zeta_{ij}) z_1 \cdots z_l) = \det(\zeta_{ik})$. The last equality follows from the definition of the integration. □

Proof of Lemma 3.2. Let $\mathcal{K}_1$ be the graded algebra $\frac{H^*(BH) \otimes H^*(BH)}{((B\iota)^* x_i \otimes 1 - 1 \otimes (B\iota)^* x_i)}$. Let $\mathcal{K}$ denote a cochain algebra of the form $\wedge(z_1, ..., z_l) \otimes \mathcal{K}_1$ with $d(z_i) = u_1 \otimes 1 - 1 \otimes u_i$. We define the decreasing filtration of $\mathcal{K}$ by

$$F^p\mathcal{K} := \{\sum \alpha_i \otimes \beta_i \in \mathcal{K}_1 \mid \alpha_i \in \wedge(z_1, .., z_l), \beta_i \in \mathcal{K}_1, \deg z_i \geq p\}$$

for $p \geq 0$. We have the spectral sequence $\{\overline{E}_r^{*,*}, \overline{d}_r\}$ associated by the filtration, which converges to the homology $H(\mathcal{K})$. Lemma 3.1 enables us to obtain a morphism $\{f_r\} : \{\overline{E}_r^{*,*}, \overline{d}_r\} \to \{{}_{LS}E_r^{*,*}, d_r\}$ of spectral sequences for which f_2 is an isomorphism of cochain algebras. Then we see that f_∞ induces an isomorphism $\widehat{f_\infty} : \mathrm{Tot}\overline{E}_\infty^{*,*} \xrightarrow{\cong} \mathrm{Tot}_{LS}E_\infty^{*,*}$ which assigns w_i to $w_i = \sum_j^l \zeta_{ij} z_j$.

In order to prove the lemma, it suffices to show that $w_1, ..., w_l$ are linearly independent in the vector space $(QH(\mathcal{K}))^{\mathrm{odd}}$. We consider the surjective map of cochain algebra $\rho : \mathcal{K} \to (\wedge(z_1, ..., z_l) \otimes \Lambda \otimes \Lambda, d) =: \mathcal{K}'$ defined by $\rho(z_i) = z_i$ and $\rho([v]) = [v]$ for $v \in H^*(BG) \otimes H^*(BG)$, where $\Lambda = (\frac{H^*(BH)}{((B\iota)^* x_i)}, 0)$ and $d(z_i) = u_i \otimes 1 - 1 \otimes u_i$. The result [23, Lemma 3.3] yields that there exists an isomorphism $\Theta : H(\mathcal{K}') \xrightarrow{\cong} \wedge(w_1, ..., w_l) \otimes \Lambda$ of graded algebras with $\Theta(w_i) = w_i$. Thus we have a sequence

$$H(\mathcal{K}) \xrightarrow{H(\rho)} QH(\mathcal{K}') \xrightarrow[\cong]{\Theta} Q(\wedge(w_1, ..., w_l) \otimes \Lambda) = \mathbb{K}\{w_1, ..., w_l\} \oplus Q\Lambda$$

for which $\Theta H(\rho)(w_i) = w_i$. We have the result. ∎

Proof of the non-triviality of $\mu_{W^{op}}$. We consider the integration $k^!$ along the fibre associated with the fibration

$$(*) : G/H \xrightarrow{i} \mathcal{M}(W^{\mathrm{op}}) = \mathcal{M}(W) \xrightarrow{k} \mathcal{M}(\partial_{\mathrm{out}}) = \mathcal{M}((\partial^{\mathrm{op}})_{\mathrm{in}})$$

which is mentioned in Remark 2.2. By virtue of [1, 3.5 Proposition], we see that $H^*(BH) \cong \mathbb{K}[(B\iota)^* x_1, ..., (B\iota)^* x_l] \otimes M$ as an $H^*(BG)$-module for some graded vector space M. In particular, $H^*(\mathcal{M}(W))$ is a free $H^*(\mathcal{M}(\partial_{\mathrm{out}}))$-module. Thus the argument of the EMSS for the fibration $(*)$ enables us to deduce that $i^* : M \xrightarrow{\cong} H^*(G/H)$ is an isomorphism; see [22, Proposition 4.2]. It follows from the definition of $k^!$ that $k^!(\Lambda_W \cdot \alpha \otimes y^{i_1} \cdots y^{i_l}) = \alpha \otimes y^{i_1} \cdots y^{i_l}$, where $\alpha \in \mathbb{K}[(B\iota)^* x_i]$, $i_k = 0$ or 1 and $i^*(\Lambda_W)$ denotes the fundamental class of the homogeneous space G/H. Therefore, we see that $D\mu_{W^{\mathrm{op}}}(1 \otimes \Lambda_W \cdot \alpha) = k^! \circ h^*(1 \otimes \Lambda_W \cdot \alpha) = k^!(\Lambda_W \cdot \alpha) = \alpha$ for an element of the form

$$1 \otimes \Lambda_W \cdot \alpha \in \frac{H^*(BH) \otimes H^*(BH)}{((B\iota)^* x_i \otimes 1 - 1 \otimes (B\iota)^* x_i \mid 1 \le i \le l)} \cong H^*(\mathcal{M}(\partial_{\mathrm{in}})).$$

Thus $D\mu_{W^{\mathrm{op}}}$ is non-trivial in general. ∎

Proof of the latter half of Theorem 1.1. It remains to show that the composite

$$D\mu_{W^{\mathrm{op}}} \circ D\mu_W = D(\mu_W \circ \mu_{W^{\mathrm{op}}}) = D(\mu_{W \circ W^{\mathrm{op}}})$$

is non-trivial. By the assumption of the degree of $(B\iota)^* x_i$ for i and the result [24, Proposition 3], we see that $\det(\frac{\partial (B\iota)^* x_i}{\partial u_j})$ is the fundamental class Λ_W of G/H. The computation above and the choice of elements ζ_{ij} allow us to conclude that

$$\begin{aligned} D\mu_{W^{\mathrm{op}}} \circ D\mu_W(y_1 \cdots y_l) &= D\mu_{W^{\mathrm{op}}}(\det(\zeta_{ij})) = k^!(\mathsf{m}(\det(\zeta_{ij}))) \\ &= k^!(\det(\mathsf{m}(\zeta_{ij})) = k^!\Big(\det\Big(\frac{\partial (B\iota)^* x_i}{\partial u_j}\Big)\Big) = 1. \end{aligned}$$

This completes the proof. ∎

We conclude this section with remarks on other operations obtained by a composite with the whistle cobordism. The results show a fruitful structure of the labeled TQFT for classifying spaces.

Remark 3.3. (i) The integration $h^!$ along the fibre is a morphism of $H^*(\mathcal{M}(\partial_{\rm in}))$-modules via h^*. Thus the computation of μ_W above yields that for $\gamma \in H^*(BG)$,

$$D\mu_W(\gamma \otimes y_1^{i_1}\cdots y_l^{i_l}) = h^!((B\iota)^*\gamma \otimes y_1^{i_1}\cdots y_l^{i_l}) = (1\otimes (B\iota)^*\gamma)h^!(1\otimes y_1^{i_1}\cdots y_l^{i_l})$$
$$= \begin{cases} (1\otimes (B\iota)^*\gamma)\det(\zeta_{ij}) & \text{if } i_1\cdots i_l = 1 \\ 0 & \text{otherwise.}\end{cases}$$

(ii) The computation in the proof of Theorem 1.1 implies that the image of $D\mu_{W^{\rm op}}$ is in $H^*(BG)\otimes 1$. Then, it follows from (i) that $D(\mu_{W^{\rm op}\circ W_1}) = D(\mu_{W^{\rm op}} \circ \mu_{W_1}) = D\mu_{W_1} \circ D\mu_{W^{\rm op}} = 0$, where W_1 denotes a whistle cobordism whose label is not necessarily the same as that of W. This yields that in the labeled TQFT for the classifying space, the operation for a cobordism with n holes labeled by connected closed subgroups of maximal rank is trivial provided $n \geq 2$ and the characteristic of the underlying field is sufficiently large. As a consequence, under the same assumption, the Cardy condition [17, (2.14)] implies that the cobordism operation associated with the double-twist diagram [21, Fig. 11] in the open TQFT is trivial. This result also follows from Theorem 1.2 below in which the open theory is clarified in our setting.
(iii) Let Σ be the pair of pants with one incoming boundary. The result [15, Theorem 4.1], in which each y_j is replaced with the notation x_j, enables us to deduce that the operator $\mu_{\Sigma\circ W} = \mu_\Sigma \circ \mu_W$ is non-trivial in general; see an explicit calculation in the end of Section 4.
(iv) We can consider the whistle cobordism operation in a homological conformal field theory (HCFT). Let $\bigoplus H_*(BDiff^+(\Sigma;\partial))$ be the prop parameterized by the homology of mapping class groups. By using the prop, we have a HCFT structure for classifying spaces; see [7, 13]. Let C be the cylinder $S^1\times[0,1]$ and

$$\circ : H_*(BDiff^+(C;\partial)) \otimes H_*(BDiff^+(W;\partial)) \to H_*(BDiff^+(C\circ W;\partial))$$

the prop structure coming from the gluing of bordisms. The Dehn twist gives rise to the element Δ in $H_1(BDiff^+(C\circ W;\partial))$ via the Hurewicz map. In fact, the element Δ induces the Batalin-Vilkovisky (B–V) operator on $H^*(LBG)$ by the HCFT structure; see [7, Proposition 60]. Observe that μ_W is regarded as an element in $H_0(BDiff^+(W;\partial))$. Then under the same assumption as in (ii) and with the notation in the proof of Theorem 1.1, we see that $D(\Delta\circ \mu_W)(x_1y_2\cdots y_l) = (D\mu_W\circ D\Delta)(x_1y_2\cdots y_l) = D\mu_W(y_1\cdots y_l) \neq 0$. Observe that the B–V operator is a derivation with respect to the cup product on the cohomology. Then the second equality follows from [15, Theorem 3.1].

Under the same assumption as in Remark 3.3 (ii), the cobordism operation associated to the pair of pants with two incoming boundaries is trivial on

$H_*(LBG)$; see [15, Theorems 7.1 and 7.3]. Thus we can exactly understand the closed TQFT structure for classifying spaces. Thanks to the main theorem, we can also compute every cobordism operation in the open-closed TQFT labeled in $\mathcal{B}$ the set of connected closed subgroups of maximal rank if the open TQFT is clarified. The consideration of the open theory is the topic in the next section.

4 A labelled open TQFT for classifying spaces

This section is devoted to proving Theorem 1.2. We consider fibrations below which define the cobordism operations μ_Υ and $\mu_{\Upsilon^{\mathrm{op}}}$. Moreover, we investigate the cohomology algebras of total and base spaces. The results are described with the commutative diagram

(4.1)

$$\begin{array}{ccccc}
\mathcal{M}(I_L^K) & & H^*(\mathcal{M}(I_L^K)) & \xleftarrow{\ \cong\ } & \frac{H^*(BK)\otimes H^*(BL)}{((Bk)^*x_i\otimes 1-1\otimes(B\ell)^*x_i)} \\
\uparrow{\scriptstyle (out)^*} & & \downarrow{\scriptstyle ((out)^*)^*} & & \downarrow{\scriptstyle \varphi} \\
\mathcal{M}(\Upsilon) & & H^*(\mathcal{M}(\Upsilon)) & \xleftarrow{\ \cong\ } & \frac{H^*(BK)\otimes H^*(BH)\otimes H^*(BL)}{((Bk)^*x_i\otimes 1-1\otimes(B\iota)^*x_i,(B\iota)^*x_i\otimes 1-1\otimes(B\ell)^*x_i)} \\
\downarrow{\scriptstyle (in)^*} & & \uparrow{\scriptstyle ((in)^*)^*} & & \uparrow{\scriptstyle 1\otimes \mathsf{m}\otimes 1} \\
\mathcal{M}(I_H^K\coprod I_L^H) & & H^*(\mathcal{M}(I_H^K\coprod I_L^H)) & \xleftarrow{\ \cong\ } & \frac{H^*(BK)\otimes H^*(BH)}{((Bk)^*x_i\otimes 1-1\otimes(B\iota)^*x_i)}\otimes\frac{H^*(BH)\otimes H^*(BL)}{((B\iota)^*x_i\otimes 1-1\otimes(B\ell)^*x_i)}
\end{array}$$

where φ is the map induced by the natural map

$$H^*(BK)\otimes H^*(BL)\to H^*(BK)\otimes 1\otimes H^*(BL)\to H^*(BK)\otimes H^*(BH)\otimes H^*(BL),$$

and $k : K\to G$, $\iota : H\to G$ and $\ell : L\to G$ denote the inclusions. Observe that the fibres of the fibrations $(out)^*$ and $(in)^*$ are G/H and $\Omega BH\simeq H$, respectively; see the proof of [13, Lemma 2.3.11]. The cohomology algebras are computed with the Eilenberg–Moore spectral sequence as made in Section 3 associated with pullback diagrams defining the spaces $\mathcal{M}(I_L^K)$, $\mathcal{M}(\Upsilon)$ and $\mathcal{M}(I_H^K\coprod I_L^H)$.

The computation of $H^*(\mathcal{M}(\Upsilon))$ is here given. We choose a homotopy equivalence

$$r : \partial_{in} = a_1 \text{---}\bullet\frown a_2\text{---}a_3 \xrightarrow{\ \simeq\ } a_1\text{---}a_2\text{---}a_3 = I\cup_{a_2} I$$

with a homotopy inverse i which satisfies the condition that $i(a_s) = a_s$ for $s = 1, 2$ and 3. We regard Υ as a labeled cobordism $(\Upsilon, \{\Upsilon^{H_i}\}_{i=1,2,3})$, where $H_1 = K$, $H_2 = H$, $H_3 = L$. Observe that the in-boundaries of the free boundaries Υ^{H_1}, Υ^{H_2} and Υ^{H_3} are the sets $\{a_1\}$, $\{\bullet, a_2\}$ and $\{a_3\}$, respectively. By the definition of the space $\mathcal{M}(\Upsilon)$, we have commutative diagrams

$$\begin{array}{ccccccc}
\mathcal{M}(\Upsilon) & \longrightarrow & \mathrm{map}(\Upsilon, BG) & \underset{\simeq}{\xrightarrow{\ i^*\ }} & BG^{I\cup_{a_2}I} & \longrightarrow & BG^I\times BG^I \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle (ev_0, ev_{\frac{1}{2}}, ev_1)} & & \downarrow{\scriptstyle (ev_0\times ev_1)^{\times 2}} \\
\prod_{i=1}^3 \mathrm{map}(\Upsilon^{H_i}, BH_i) & \underset{\psi}{\longrightarrow} & \prod_{i=1}^3 \mathrm{map}(\Upsilon^{H_i}, BG) & \underset{\gamma}{\xrightarrow{\ \simeq\ }} & BG^{\times 3} & \underset{1\times\Delta\times 1}{\longrightarrow} & BG^{\times 4}
\end{array}$$

in which the right-hand side and left-hand side squares are pullback diagrams, where the map γ is defined by the embeddings which are homotopy inverses of deformation retractions $\Upsilon^{H_i} \to \{a_i\}$ and ψ is the map induced by the inclusions k, ι and ℓ mentioned above. Thus applying the EMSS to the big square which is a homotopy pullback, we can compute the cohomology algebra $H^*(\mathcal{M}(\Upsilon))$ in (4.1) with the two sided Koszul resolution mentioned in Section 3.

Proof of Theorem 1.2. By definition, we see that $D\mu_\Upsilon = ((in)^*)^! \circ ((out)^*)^*$. The image of $((out)^*)^*$ is in the image of $((in)^*)^*$. This follows from the commutativity of the diagram (4.1). Then the definition of the integration along the fibre yields that $D\mu_\Upsilon$ is trivial.
We investigate the Leray–Serre spectral sequence $\{E_r^{*,*}, d_r\}$ for the fibration $G/H \xrightarrow{j} \mathcal{M}(\Upsilon) \xrightarrow{(out)^*} \mathcal{M}(I_L^K)$. Since the subgroup H is of maximal rank, it follows that the E_2 term is generated by elements with even degree and hence the map j^* induced by the inclusion j is an epimorphism. Therefore, there exists an element Λ_Υ of the form $\sum_i q_i \cdot (1 \otimes b_i \otimes 1)$ in $H^*(\mathcal{M}(\Upsilon))$ with $q_i \in H^*(BK) \otimes H^*(BL)$ and $b_i \in H^*(BH)$ such that $j^*(\Lambda_\Upsilon)$ is the fundamental class of G/H. We can assume that $\deg q_i = 0$ for any i because q_i is in the image of φ in (4.1). Thus we see that $D\mu_{\Upsilon^{\mathrm{op}}}(\sum_i q_i \cdot (1 \otimes b_i \otimes 1 \otimes 1)) = ((out)^*)^! \circ ((in)^*)^*(\sum_i q_i \cdot (1 \otimes b_i \otimes 1 \otimes 1)) = ((out)^*)^!(\sum_i q_i \cdot (1 \otimes b_i \otimes 1)) = 1_{H^*((\mathcal{M}(I_L^K)))}$. The computations of the cohomology ring $H^*(\mathcal{M}(\Upsilon))$ and $H^*(\mathcal{M}(I_H^K \coprod I_L^H))$ via the EMSS show that the map $((in)^*)^*$ is a morphism of $H^*(BK) \otimes H^*(BH) \otimes H^*(BL)$-modules. Moreover, the construction of the shriek map $((out)^*)^!$ enables us to conclude that the map is a morphism of $H^*(BK) \otimes H^*(BL)$-modules. Since each element $x \in H^*(\mathcal{M}(I_L^K))$ is of the form $x = \beta 1_{H^*((\mathcal{M}(I_L^K)))}$ for some $\beta \in H^*(BK) \otimes H^*(BL)$, it follows that $D\mu_{\Upsilon^{\mathrm{op}}}(\beta \sum_i q_i \cdot (1 \otimes b_i \otimes 1 \otimes 1)) = \beta 1_{H^*((\mathcal{M}(I_L^K)))} = x$. This implies that the dual operator $D\mu_{\Upsilon^{\mathrm{op}}}$ is an epimorphism. We have the result. $\square$

The results [15, Theorems 4.1 and 7.1] asserts that the closed TQFT for the classifying space BG is completely determined if the cohomology $H^*(BG, \mathbb{K})$ is a polynomial algebra. Therefore, by virtue of Theorems 1.1, 1.2, and the result [17, Proposition 3.9], we have

Assertion 4.1. *Let $\mathcal{B}$ be the set of connected closed subgroup of G of maximal rank. Then one can make a calculation of each of the dual operations for the labeled TQFT $\mu : (\mathsf{oc\text{-}Cobor}(\mathcal{B}), \coprod) \to (\mathbb{Q}\text{-}\mathsf{Vect}, \otimes)$ introduced by Guldberg up to multiplication by non-zero scalar with the cohomology algebras and their generators described in (3.6) and (4.1), and moreover, with representatives Λ_W and Λ_Υ of the fundamental classes of the homogeneous spaces G/H in $H^*(\mathcal{M}(W))$ and $H^*(\mathcal{M}(\Upsilon))$; see the proof of Theorems 1.1 and 1.2 for Λ_W and Λ_Υ.*

In particular, we see that $\mu_\Sigma \equiv 0$ for a cobordism Σ which has two holes, or contains at least either one of Υ and the pair of pants with two in-boundaries as a component constructing the cobordism with gluing.

We conclude this section with a computational example which proves Proposition 1.3. Let G be the Lie group $U(m+1)$ and H the subgroup $U(m)\times U(1)$ of G. Let L be a maximal rank sub group of $U(m+1)$ whose homology is p-torsion free, where p is the characteristic of the underlying field $\mathbb{K}$. Observe that the homogeneous space G/H is nothing but the projective space $\mathbb{C}P^{m+1}$ and that

$$\begin{aligned} H^*(G/H;\mathbb{K}) &\cong \mathbb{K}[c_1, c_1', c_2', ..., c_m']/\big(\sum_{i+j=k} c_i c_j' \mid 0\le i\le 1, 0\le j\le m, k\ge 1\big) \\ &\cong \mathbb{K}[c_1]/\big(c_1^{m+1}\big). \end{aligned}$$

In what follows, we write $H^*(X)$ for $H^*(X;\mathbb{K})$. We show that the string operation $D\mu_{\Sigma_{\mathrm{OP}}}$ for the labeled cobordism $\Sigma_{\mathrm{OP}} = (\Sigma_{1+2,0}\coprod Id_{I_L^H})\circ(W\coprod Id_{I_L^H})\circ \Upsilon^{\mathrm{op}}$ is non trivial with an explicit calculation, where each cobordism is pictured in (1.1). We observe that the domain of the cobordism operation $D\mu_\Sigma$ is the cohomology $H^*(LBG)^{\otimes 2}\otimes H^*(\mathcal{M}(I_L^H)) \cong (H^*(BG)\otimes \wedge(y_1,...,y_{m+1}))^{\otimes 2}\otimes H^*(\mathcal{M}(I_L^H))$. The result [15, Theorem 4.1] gives an explicit formula of the cobordism operation $D\mu_{\Sigma_{1+2,0}}$. In fact the dual to the loop coproduct $\odot$ in the loop cohomology is induced by $a\odot b = (-1)^{d(d-\deg a)}D\mu_{\Sigma_{1+2,0}}(a\otimes b)$, where d denotes the dimension of G. As a consequence, we have (**) : $D\mu_{\Sigma_{1+2,0}}(y_1\cdots y_l\otimes y_1\cdots y_l) = y_1\cdots y_l$.
Recall the formula (3.7) to compute $D\mu_W(1\otimes y_1\cdots y_l)$. Then we see that the matrix $\big(\zeta_{ij}\big)$ is of the form

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ c_1'\otimes 1 & 1\otimes c_1 & 1 & 0 & \cdots & \cdots & 0 \\ c_2'\otimes 1 & 0 & 1\otimes c_1 & 1 & \cdots & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & & & & 1 \\ c_m'\otimes 1 & 0 & 0 & \cdots & & & 1\otimes c_1 \end{pmatrix}.$$

The direct calculation enables us to deduce that

$$\det\big(\zeta_{ij}\big) = 1\otimes c_1^m - c_1'\otimes c_1^{m-1} + \cdots (-1)^{l-1}c_{l-1}'\otimes c_1^{m-l+1} + \cdots + (-1)^m c_m'\otimes 1.$$

Let j be the inclusion of the fibration $G/H \xrightarrow{j} \mathcal{M}(\Upsilon) \xrightarrow{(out)^*} \mathcal{M}(I_L^H)$. One of important properties of the EMSS [22] allows us to conclude that the map $j^*: H^*(\mathcal{M}(\Upsilon)) \to H^*(G/H)$ induced by j coincides with the composite of maps

$$H^*(\mathcal{M}(\Upsilon)) \twoheadrightarrow \mathrm{Tor}^{0,*}_{H^*(\mathcal{M}(I_L^H))}(\mathbb{K}, H^*(\mathcal{M}(\Upsilon))) = E_2^{0,*} \twoheadrightarrow E_\infty^{0,*} \hookrightarrow H^*(G/H).$$

The proof of Theorem 1.2 implies that j^* is an epimorphism. Therefore, we can choose $j^*(1\otimes c^m\otimes 1_{H^*(BL)}) = j^*(\det(\zeta_{ij})\otimes 1_{H^*(BL)})$ as the fundamental class of $H^*(G/H) = H^*(\mathbb{C}P^m)$. The calculation in the proof of Theorem 1.2 allows us to deduce that $D\mu_{\Upsilon^{\rm op}}(\det(\zeta_{ij}) \otimes 1_{H^*(BL)}) = 1$. Combining it with (**), we have

$$\begin{aligned} & D\mu_{\Sigma_{\rm OP}}(y_1 \cdots y_l \otimes y_1 \cdots y_l \otimes 1) \\ = & D\mu_{\Upsilon^{\rm op}} \circ D\mu_{W \coprod Id_{I_L^H}} \circ D\mu_{\Sigma_{1+2,0} \coprod Id_{I_L^H}}(y_1 \cdots y_l \otimes y_1 \cdots y_l \otimes 1) \\ = & D\mu_{\Upsilon^{\rm op}} \circ D\mu_{W \coprod Id_{I_L^H}}(y_1 \cdots y_l \otimes 1) \\ = & D\mu_{\Upsilon^{\rm op}}(\det(\zeta_{ij}) \otimes 1) = 1. \end{aligned}$$

Proof of Proposition 1.3. The computation above shows that the unit of $H^*(\mathcal{M}(I_L^H))$ is in the image of the operation $D\mu_{\Sigma_{\rm OP}}$. Then the same argument as in the proof of Theorem 1.2 yields the result. □

Acknowledgments

The author is grateful to Anssi Lahtinen for inviting him to University of Copenhagen, and also thanks Jesper Grodal. The inspired discussions with them on string topology have enabled the author to refine the computations in this manuscript. The author would like to thank the referee for his/her suggestions in revising the paper.

5 Appendix A: An algebraic model for the whistle cobordism operation

We give an algebraic model for μ_W in cochain level over the rational. In this section, the cochain algebra $A^*_{PL}(X)$ for a space X is regarded as the DG algebra of PL differential forms on X thought the same notation as that of the singular cochain algebra is used; see [5] and [10, Section 10] for PL differential forms. In what follows, we assume that H is an arbitrary connected closed subgroup of a connected compact Lie group G.

We recall the commutative diagram in (3.2) and first consider the fibration $res : \mathrm{map}({}_a\cap_b, BH) \to \mathrm{map}(\{a,b\}, BH)$ in the left square. The results [10, Theorems 14.12 and 15.3] allow us to obtain a minimal relative Sullivan model for res of the form

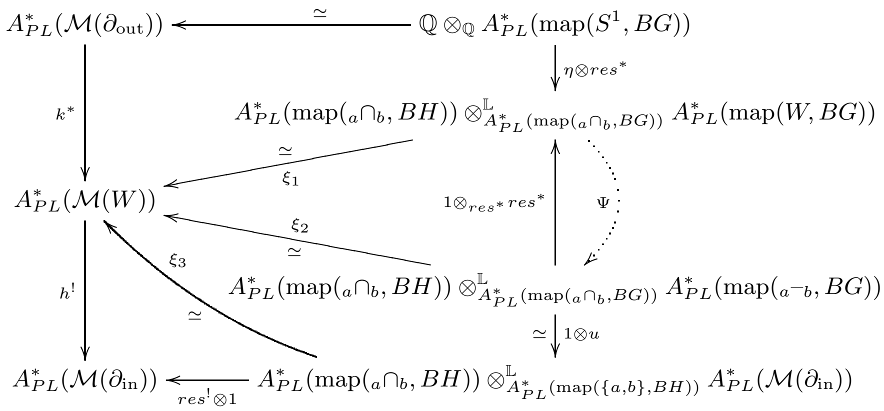
$$\zeta : A^*_{PL}(\mathrm{map}(\{a,b\}, BH)) \otimes H^*(\Omega BH) \xrightarrow{\simeq} A^*_{PL}(\mathrm{map}({}_a\cap_b, BH)).$$

Observe that the source is the tensor product as a vector space, but not as a DGA. By using the model ζ, we have a model $res^! : A^*_{PL}(\mathrm{map}(\{a,b\}, BH)) \otimes H^*(\Omega BH) \to A^*_{PL}(\mathrm{map}(\{a,b\}, BH))$ for the integration along the fibre of the fibration res mentioned above; see [11, Theorem 5]. The left-hand side diagram in (3.2) is the pullback described in Remark 2.2. Then the proof of [11, Theorem 6] yields that $res^! \otimes 1$ in (5.1) below is a model for the integration $h^!$. Moreover, a quasi-isomorphism

$$u : A^*_{PL}(\mathrm{map}(\{a,b\}, BH)) \otimes^{\mathbb{L}}_{A^*_{PL}(\mathrm{map}({}_a\cap_b, BG))} A^*_{PL}(\mathrm{map}({}_a-_b, BG)) \xrightarrow[\simeq]{} A^*_{PL}(\mathcal{M}(\partial_{\mathrm{in}}))$$

is induced by the front pullback in (3.2). Thus we have commutative diagrams with solid arrows

(5.1)



in which ξ_1 and ξ_3 are quasi-isomorphisms induced by the back pullback diagram and the left-hand side pulback diagram in (3.2), respectively, and ξ_2 is a quasi-isomorphism induced by the big pullback which the left-hand side and the front pullback diagrams give. By the lifting lemma [10, Proposition 12.9] enables us to obtain a right inverse Ψ of $1 \otimes_{res^*} res^*$ in the derived category of $A^*_{PL}(\mathrm{map}({}_a\cap_b, BG))$-modules. The last step in constructing the model for $\mu_W = h^! \circ k^*$ is not explicit. In fact, as expected from the proof of Theorem 1.1, it seems that the construction of the lift is complicated in general. Under appropriate assumptions on G and the subgroup H, it is anticipated that Sullivan models serve the explicit calculation. However, we do not pursue the topic in this article.

References

[1] P.F. Baum, On the cohomology of homogeneous spaces, Topology 7 (1968), 15–38.

[2] P.F. Baum and L. Smith, The real cohomology of differentiable fibre bundles, Comm. Math. Helv. 42 (1967), 171–179.

[3] K. Behrend, G. Ginot, B. Noohi, and P. Xu, String topology for stacks, Astérisque (2012), no. 343, xiv+169.

[4] A.J. Blumberg, R.L. Cohen, and C. Teleman, Open-closed field theories, string topology, and Hochschild homology, Alpine perspectives on algebraic topology, edited by C. Ausoni, K. Hess, and J. Scherer, Contemp. Math., AMS, 502 (2009), 53–76.

[5] A.K. Bousfield and V.K.A.M. Gugenheim, On PL de Rham theory and rational homotopy type, Memoirs of AMS 179 (1976).

[6] M. Chas and D. Sullivan, String topology, preprint `arXiv:math/9911159`.

[7] D. Chataur and L. Menichi, String topology of classifying spaces, J. Reine Angew. Math. 669 (2012), 1–45.

[8] R.L. Cohen and V. Godin, A polarized view of string topology, Topology, geometry and quantum field theory, 127–154, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.

[9] R.L. Cohen and J.D.S. Jones, A homotopy theoretic realization of string topology, Math. Ann. 324 (2002), 773–798.

[10] Y. Félix, S. Halperin and J.-C. Thomas, Rational Homotopy Theory, Graduate Texts in Mathematics 205, Springer-Verlag, 2001.

[11] Y. Félix and J.-C. Thomas, String topology on Gorenstein spaces, Math. Ann. 345 (2009), 417–452.

[12] V. Godin, Higher string topology operations, preprint `arXiv:0711.4859`.

[13] C. Guldberg, Labelled String Topology for Classifying Spaces of Compact Lie Groups: A 2-dimensional Homological Field Theory with D-branes, Thesis for the Master degree in Mathematics Department of Mathematical Sciences, University of Copenhagen, 2011.

[14] R. Hepworth and A. Lahtinen, On string topology of classifying spaces, Adv. Math. 281 (2015), 394–507.

[15] K. Kuribayashi and L. Menichi, The Batalin-Vilkovisky algebra in the string topology of classifying spaces, Canad. J. Math. 71 (2019), 843–889.

[16] K. Kuribayashi, L. Menichi, and T. Naito, Derived string topology and the Eilenberg–Moore spectral sequence, Isr. J. Math. 209 (2015), 745–802.

[17] A.D. Lauda and H. Pfeiffer, Open-closed string: Two-dimensional extended TQFTs and Frobenius algebras, Topology and its Applications, 155 (2008), 623–666.

[18] C.I. Lazaroiu, On the structure open-closed topological field theory in two dimensions, Nucl. Phys. B 603 (2001), 497–530.

[19] E. Lupercio, B. Uribe, and M.A. Xicoténcatl, Orbifold string topology, Geom. Topol. 12 (2008), 2203–2247.

[20] M. Mimura and H. Toda, Topology of Lie groups II, Translations of Math. Monographs 91, Amer. Math. Soc. 1991.

[21] G.W. Moore and G. Segal, D-branes and K-theory in 2D topological field theory, preprint `arXiv:hep-th/0609042`.

[22] L. Smith, Homological algebra and the Eilenberg–Moore spectral sequence. Trans. Amer. Math. Soc. 129 (1967), 58–93.

[23] L. Smith, On the characteristic zero cohomology of the free loop space, Amer. J. Math. 103 (1981), 887–910.

[24] L. Smith, A note on the realization of graded complete intersection algebras by the cohomology of a space, Quart. J. Math. Oxford (2), 33 (1982), 379–384.

[25] H. Tamanoi, Loop coproducts in string topology and triviality of higher genus TQFT operations. J. Pure Appl. Algebra 214 (2010), 605–615.

Katsuhiko Kuribayashi
Department of Mathematical Sciences
Faculty of Science
Shinshu University
Matsumoto, Nagano
390-8621, Japan
kuri@math.shinshu-u.ac.jp