Rational Models for Automorphisms of Fiber Bundles

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Abstract. Given a fiber bundle, we construct a differential graded Lie algebra model, in the sense of Quillen's rational homotopy theory, for the classifying space of the monoid of homotopy equivalences of the base covered by a fiberwise isomorphism of the total space.

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1 Introduction

Consider a fiber bundle ξ with projection $p\colon E \to X$ and structure group G over a simply connected finite CW-complex X and let $aut_\circ(\xi)$ denote the topological monoid of bundle maps

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X, \end{array}$$

such that f is homotopic to the identity map of X and φ is a fiberwise isomorphism. The goal of this paper is to construct a dg Lie algebra model for the classifying space $Baut_\circ(\xi)$ in the sense of Quillen's rational homotopy theory [14]. In particular, this yields tractable models for the computation of the rational homotopy and cohomology groups of $Baut_\circ(\xi)$. We assume that BG is a nilpotent space, i.e., that the group $\pi_0(G)$ is nilpotent and acts nilpotently on $\pi_k(G)$ for all $k \geq 1$.

Theorem 1.1. *Let L be the minimal Quillen model for X and let Π be a dg Lie algebra model for BG. Furthermore, let $\tau\colon \mathcal{C}L \to \Pi$ be a twisting function that models the classifying map of the bundle $\nu\colon X \to BG$. Then $Baut_\circ(\xi)$ is rationally homotopy equivalent to the geometric realization of the dg Lie algebra*

$$\mathrm{Hom}^\tau(\mathcal{C}L, \Pi)\langle 0\rangle \rtimes_{\tau_*} \big(\mathrm{Der}\, L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle. \tag{1}$$

Here, $\mathcal{C}L$ is the Chevalley-Eilenberg complex of L, we use $\langle n\rangle$ to indicate the n-connected cover, and the decorations τ and τ_* indicate that we take a twisted semi-direct product (see §3.5 and §3.6).

In many cases of interest, there are simplifications of this model. For example, we have the following if G is a compact connected Lie group.

Theorem 1.2. *If G is a compact connected Lie group, then $Baut_\circ(\xi)$ is rationally homotopy equivalent to the geometric realization of the dg Lie algebra*

$$\big(H^*(X;\mathbb{Q}) \otimes \pi_*(G)\big)\langle 0\rangle \rtimes_{\tau_*} \big(\mathrm{Der}\, L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle.$$

The twisting function is given explicitly by

$$\tau = \sum_{i=1}^n u_i(\xi) \otimes \gamma_i,$$

where $u_i(\xi) \in H^*(X;\mathbb{Q})$ are the characteristic classes of ξ associated to generators $u_1, \ldots, u_n$ of the cohomology ring $H^*(BG;\mathbb{Q})$ and $\gamma_i \in \pi_*(G)\otimes\mathbb{Q} = \pi_{*+1}(BG)\otimes\mathbb{Q}$ is dual to u_i.

Similar simplifications are possible whenever $H^*(BG;\mathbb{Q})$ is a free graded commutative algebra, see §4. An immediate consequence is the following.

Corollary 1.3. *If G is a compact connected Lie group, then the rational homotopy type of $Baut_\circ(\xi)$ depends only on the rational homotopy type of X and the rational characteristic classes of ξ.*

The main application which motivated this work is the construction of a dg Lie algebra model for the classifying space of the block diffeomorphism group of a simply connected smooth n-manifold M with boundary $\partial M = S^{n-1}$ ($n \geq 5$). The construction is carried out in [3, §4], using the results of this paper. The key point is roughly speaking that, rationally, the block diffeomorphism group may be replaced by the automorphisms of the stable tangent bundle (or stable normal bundle), see [3] for more precise statements.

We now turn to some more direct applications. Earlier work on the rational homotopy theory of automorphisms of fibrations has focused on the submonoid $aut_X(\xi) \subseteq aut_\circ(\xi)$ where f is *equal* to the identity on X, see e.g. [6]. Our results yield models not only for $Baut_X(\xi)$, but for the whole homotopy fiber sequence

$$Baut_X(\xi) \to Baut_\circ(\xi) \to Baut_\circ(X), \tag{2}$$

where $aut_\circ(X)$ is the monoid of self-maps of X homotopic to the identity. We show that (2) is modeled by the short exact sequence of dg Lie algebras

$$\mathrm{Hom}^\tau(\mathcal{C}L,\Pi)\langle 0\rangle \to \mathrm{Hom}^\tau(\mathcal{C}L,\Pi)\langle 0\rangle \rtimes_{\tau_*} \big(\mathrm{Der}\, L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle \to \big(\mathrm{Der}\, L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle$$

associated to the twisted semi-direct product (1).

As an application, the following gives an interesting class of spaces X for which the fibration (2) is always split. Recall that a simply connected space X is called *elliptic* if both $H^*(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ are finite dimensional. The *Halperin conjecture* asserts that $aut_\circ(X)$ is rationally homotopy equivalent to a product of odd dimensional spheres if X is an elliptic space with positive Euler characteristic, see e.g. [8, §9.7.2]. Examples of spaces for which the Halperin conjecture is known to hold are homogeneous spaces G/H, for $H\subseteq G$ a closed subgroup of maximal rank, or elliptic Kähler manifolds with positive Euler characteristic.

Theorem 1.4. *Let X be an elliptic space with positive Euler characteristic and let ξ be a fiber bundle over X with structure group a compact connected Lie group G. If the Halperin conjecture is valid for X, then*

1. *There is a rational splitting,*

$$Baut_\circ(\xi)\sim_{\mathbb{Q}} Baut_X(\xi)\times Baut_\circ(X).$$

2. *The cohomology ring $H^*(Baut_\circ(\xi);\mathbb{Q})$ is a polynomial algebra on finitely many generators of even degree.*

The above result applies in particular to bundles over even dimensional spheres. Bundles over odd dimensional spheres exhibit a different behavior; these provide examples where the fibration (2) does not split rationally, though it does so after looping once. Also, the cohomology ring has an interesting structure in this case.

Theorem 1.5. *Let $n\geq 3$ be odd and let ξ be a fiber bundle over S^n with structure group a compact connected Lie group G.*

1. *There is a rational splitting,*

$$aut_\circ(\xi)\sim_{\mathbb{Q}} aut_{S^n}(\xi)\times aut_\circ(S^n),$$

but the fibration

$$Baut_{S^n}(\xi)\to Baut_\circ(\xi)\to Baut_\circ(S^n)$$

does not split, in fact the map $H^*(Baut_\circ(\xi);\mathbb{Q})\to H^*(Baut_{S^n}(\xi);\mathbb{Q})$ is not surjective.

2. *Write* $H^*(BG;\mathbb{Q}) = \mathbb{Q}[u_1,\dots,u_k,v_1,\dots,v_\ell]$, *where* $|u_i| < n$ *and* $|v_j| > n$. *There is an algebra isomorphism*

$$H^*(Baut_\circ(\xi);\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_k] \otimes \big(\mathbb{Q}[z] \oplus d(\Omega^*_{\mathbb{Q}[v_1,\dots,v_\ell]|\mathbb{Q}})\big),$$

where z *is a generator of degree* $n+1$ *and the right factor is the (nilpotent) algebra of exact Kähler differential forms on the polynomial ring* $\mathbb{Q}[v_1,\dots,v_\ell]$.

For an example where the fibration (2) does not split rationally even after looping, see Remark 4.10.

We end this introduction with some observations about $\mathrm{Diff}(S^n)$, the topological group of orientation preserving diffeomorphisms of the sphere, as a corollary of the above results. It is well known that the map $SO(n+1) \to \mathrm{Diff}(S^n)$ admits a left homotopy inverse, in fact there is a homotopy equivalence

$$SO(n+1) \times \mathrm{Diff}(S^n, D^n) \to \mathrm{Diff}(S^n), \tag{3}$$

where $\mathrm{Diff}(S^n, D^n) \subseteq \mathrm{Diff}(S^n)$ is the subgroup of diffeomorphisms that fix a smoothly embedded disk $D^n \subseteq S^n$ pointwise. The usual argument (see e.g. [1, Lemma 1.1.5]) goes as follows. Let ξ denote the oriented frame bundle, $p\colon F(S^n) \to S^n$, and consider the map

$$\mathrm{Diff}(S^n) \to F(S^n), \tag{4}$$

given by evaluating the differential on a standard frame over the basepoint. Its fiber is homotopy equivalent to $\mathrm{Diff}(S^n, D^n)$ and the composite $SO(n+1) \to \mathrm{Diff}(S^n) \to F(S^n)$ is a homotopy equivalence. This implies the homotopy equivalence (3). One might ask to what extent the splitting deloops. The map (4) does not deloop, at least not in any evident way, so the above argument does not let us say whether $BSO(n+1) \to B\,\mathrm{Diff}(S^n)$ splits. However, the map (4) factors as

$$\mathrm{Diff}(S^n) \to aut_\circ(\xi) \to F(S^n),$$

and the same argument shows that $SO(n+1) \to aut_\circ(\xi)$ is split. In particular,

$$BSO(n+1) \to Baut_\circ(\xi) \tag{5}$$

induces and injection on rational homotopy groups. If n is even, then the cohomology ring $H^*(Baut_\circ(\xi);\mathbb{Q})$ is free by Theorem 1.4, and this implies that (5) admits a left homotopy inverse in the rational homotopy category. *A fortiori*, we get

Corollary 1.6. *For* n *even,* $BSO(n+1) \to B\,\mathrm{Diff}(S^n)$ *admits a left homotopy inverse, rationally.*

For n odd, say $n = 2r-1$, the cohomology calculation in Theorem 1.5 shows that the map (5) is not surjective in rational cohomology (e.g., $p_{r-1} \in$

$H^*(BSO(2r);\mathbb{Q})$ is not hit), so it does not split rationally. However, this yields no information about $BSO(n+1) \to B\,\mathrm{Diff}(S^n)$, except that an argument for a splitting would need to use methods that do not apply to $Baut_\circ(\xi)$. (In fact, $SO(4) \to \mathrm{Diff}(S^3)$ is known to be a homotopy equivalence [9].)

The paper is structured as follows. In Section 2 we interpret $Baut_\circ(\xi)$ as a classifying space for certain fibrations and relate it to the geometric bar construction $B\big(map(X, BG), aut(X), *\big)$ in the sense of [13, §7]. Section 3 contains general results about dg Lie algebra models and $\mathbb{Q}$-localizations of geometric bar constructions. These are used in the proof of Theorem 1.1, which is given at the end of Section 3. A new technical tool is the construction of a simplicial nilpotent group $\exp_\bullet(\mathfrak{g})$ associated to any degree-wise nilpotent dg Lie algebra $\mathfrak{g}$. It generalizes the Malcev group associated to a nilpotent Lie algebra over $\mathbb{Q}$, and provides a functorial simplicial group model of the loop space $\Omega\mathsf{MC}_\bullet(\mathfrak{g})$ of the nerve of the dg Lie algebra $\mathfrak{g}$, see §3.2. Section 4 contains the proofs of Theorem 1.2, Theorem 1.4 and Theorem 1.5.

2 Moduli spaces of $\mathcal{F}$-fibrations

We will utilize the general framework for the classification of fibrations provided by May [13]. Let $(\mathcal{F}, F)$ be a category of fibers in the sense of [13, Definition 4.1] and assume it satisfies the hypotheses of the classification theorem [13, Theorem 9.2]. Also recall the notions of $\mathcal{F}$-spaces and $\mathcal{F}$-maps from [5].

For a suitable choice of $\mathcal{F}$, fiber bundles with structure group G are examples of $\mathcal{F}$-fibrations, see [13, Example 6.11]. Another special case is when $\mathcal{F}$ is the category of all spaces weakly equivalent to a given CW-complex X, with morphisms all weak equivalences between such spaces. In this case, the 'structure group' $G = aut(X)$ is the group-like monoid of homotopy automorphisms of X, and an $\mathcal{F}$-fibration is the same thing as a fibration with fiber weakly homotopy equivalent to X. We will refer to such fibrations as *X-fibrations*.

Returning to the general situation, we let G denote the group-like topological monoid $\mathcal{F}(F, F)$, to be thought of as the structure group for $\mathcal{F}$-fibrations. Let $p_\infty\colon E_\infty \to B_\infty$ denote the universal $\mathcal{F}$-fibration, the existence of which is ensured by May's classification theorem, and define

$$Fib(X, \mathcal{F}) = B\big(map(X, B_\infty), aut(X), *\big),$$

where the right hand side denotes the geometric bar construction, in the sense of [13, §7], of the group-like monoid $aut(X)$ acting on the space $map(X, B_\infty)$ from the right by precomposition. It is a consequence of May's 'Classification of Y-structures' [13, §11] that $Fib(X, \mathcal{F})$ may be thought of as a moduli space of $\mathcal{F}$-fibrations with base weakly equivalent to X. More precisely, we have the following:

Theorem 2.1. *For a CW-complex A, there is a bijective correspondence between homotopy classes of maps*

$$A \to Fib(X, \mathcal{F})$$

and equivalence classes of X-fibrations $p\colon E \to A$ with a B_∞-structure $\theta\colon E \to B_\infty$.

Proof. This follows readily from [13, Theorem 11.1]. □

In particular, since an X-fibration over a point is just a space weakly equivalent to X, we see that the set of path components,

$$\pi_0 Fib(X, \mathcal{F}),$$

is in bijective correspondence with the set of equivalence classes of $\mathcal{F}$-fibrations with base weakly homotopy equivalent to X.

Definition 2.2. Given an $\mathcal{F}$-fibration $p\colon E \to B$, let $aut^{\mathcal{F}}(p)$ denote the space of $\mathcal{F}$-self equivalences of p, i.e., the topological monoid consisting of commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X, \end{array}$$

such that f is a weak homotopy equivalence and φ is a fiberwise $\mathcal{F}$-map, topologized as a subset of $map(B,B) \times map(E,E)$. Let $aut_\circ^{\mathcal{F}}(p) \subseteq aut^{\mathcal{F}}(p)$ denote the submonoid consisting of those pairs (f, φ) such that f is homotopic to the identity map on X. If $D \subseteq C \subseteq X$ are subsets, then let $aut_C^{\mathcal{F}}(p)$ denote the submonoid consisting of pairs as above such that f restricts to the identity map on C, and write $aut_C^{D,\mathcal{F}}(p)$, or simply $aut_C^D(p)$, for the submonoid of $aut_C^{\mathcal{F}}(p)$ where φ restricts to the identity isomorphism on the fibers over points in D. Finally, let $aut_{C,\circ}^D(p)$ denote $aut_C^D(p) \cap aut_\circ^{\mathcal{F}}(p)$.

By using standard properties of the geometric bar construction from [13, §7], we can obtain information about the homotopy types of the components of $Fib(X, \mathcal{F})$.

Theorem 2.3. *1. There is a bijection*

$$\pi_0 Fib(X, \mathcal{F}) \cong [X, B_\infty]/\pi_0 aut(X).$$

2. There is a weak equivalence of spaces over $Baut(X)$,

$$Fib(X, \mathcal{F}) \sim \coprod_{[p]} Baut^{\mathcal{F}}(p),$$

where the union is over all equivalence classes of $\mathcal{F}$-fibrations $p\colon E \to B$, with B weakly equivalent to X.

Proof. As follows from [13, Proposition 7.9], there is a homotopy fiber sequence

$$aut(X) \to map(X, B_\infty) \to Fib(X, \mathcal{F}) \to Baut(X).$$

The first statement follows by looking at the induced long exact sequence of homotopy groups.
The space of $\mathcal{F}$-maps $map^{\mathcal{F}}(p, p_\infty)$ is weakly contractible for every $\mathcal{F}$-fibration $p\colon E \to X$ by [5, Proposition 3.1]. Consider the diagram

$$\begin{array}{ccccccc} aut^{\mathcal{F}}(p) & \longrightarrow & map^{\mathcal{F}}(p, p_\infty) & \longrightarrow & B\big(map^{\mathcal{F}}(p, p_\infty), aut^{\mathcal{F}}(p), *\big) & \longrightarrow & Baut^{\mathcal{F}}(p) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ aut(X) & \longrightarrow & map(X, B_\infty) & \longrightarrow & B\big(map(X, B_\infty), aut(X), *\big) & \longrightarrow & Baut(X). \end{array}$$

According to [13, Proposition 7.9] the rows are quasifibration sequences. The leftmost square is homotopy cartesian. It follows that the third vertical map from the left induces a weak equivalence between the connected components containing ν. Since $map^{\mathcal{F}}(p, p_\infty)$ is weakly contractible, the rightmost map in the top row is a weak homotopy equivalence. The rightmost square yields a zig-zag of weak homotopy equivalences showing $Baut^{\mathcal{F}}(p) \sim B\big(map(X, B_\infty), aut(X), *\big)_\nu$ as spaces over $Baut(X)$, where ν indicates the component containing (the class of) ν. □

COROLLARY 2.4. *There are weak homotopy equivalences*

$$Baut^{\mathcal{F}}(p) \sim B\big(map(X, B_\infty)_\nu, aut(X)_{[\nu]}, *\big),$$

$$Baut^{\mathcal{F}}_\circ(p) \sim B\big(map(X, B_\infty)_\nu, aut_\circ(X), *\big),$$

where $aut(X)_{[\nu]}$ denotes the monoid of homotopy equivalences $\varphi\colon X \to X$ such that $\nu \circ \varphi \simeq \nu$ and $map(X, B_\infty)_\nu$ denotes the component of ν.

Proof. We have just seen that $Baut^{\mathcal{F}}(p) \sim B\big(map(X, B_\infty), aut(X), *\big)_\nu$. The latter is easily seen to be weakly equivalent to $B\big(map(X, B_\infty)_\nu, aut(X)_{[\nu]}, *\big)$. The second claim is proved similarly. □

3 RATIONAL MODELS

This section contains the proof of the main theorem. We begin by examining the effect of $\mathbb{Q}$-localization on the geometric bar construction of [13, §7]. Then we will construct a dg Lie model for the $\mathbb{Q}$-localized bar construction, by combining Schlessinger-Stasheff's [15] and Tanré's [16] theory of fibrations of dg Lie algebras with Quillen's theory of principal dg coalgebra bundles [14].

3.1 Rationalization

Lemma 3.1. *Let X be a connected nilpotent finite CW-complex, let Z be a connected nilpotent space, and fix a map $\nu\colon X \to Z$. Then the space $B\big(map(X,Z)_\nu, aut_\circ(X), *\big)$ is rationally homotopy equivalent to*

$$B\big(map(X_{\mathbb{Q}}, Z_{\mathbb{Q}})_{\nu_{\mathbb{Q}}}, aut_\circ(X_{\mathbb{Q}}), *\big).$$

Proof. By using a functorial $\mathbb{Q}$-localization for nilpotent spaces, e.g., the Bousfield-Kan $\mathbb{Q}$-completion, we can construct a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\nu} & Z \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ X_{\mathbb{Q}} & \xrightarrow{\nu_{\mathbb{Q}}} & Z_{\mathbb{Q}}, \end{array}$$

where the vertical maps are $\mathbb{Q}$-localizations. We may also assume that r is a cofibration. Define the monoid $aut_\circ(r)$ as the pullback

$$\begin{array}{ccc} aut_\circ(r) & \overset{\sim}{\twoheadrightarrow} & aut_\circ(X) \\ \downarrow{\scriptstyle \sim_{\mathbb{Q}}} & & {\scriptstyle \sim_{\mathbb{Q}}}\downarrow{\scriptstyle r_*} \\ aut_\circ(X_{\mathbb{Q}}) & \underset{r^*}{\overset{\sim}{\twoheadrightarrow}} & map(X, X_{\mathbb{Q}})_r. \end{array}$$

Thus, the monoid $aut_\circ(r)$ consists of pairs (f,g) where f and g are self-maps homotopic to the identity of X and $X_{\mathbb{Q}}$, respectively, such that $r \circ f = g \circ r$. Since r is a cofibration, the map r^* is a fibration. It is also a weak equivalence by standard properties of $\mathbb{Q}$-localization. The map r_* is a rational homotopy equivalence by [10, Theorem II.3.11]. It follows that the projections from $aut_\circ(r)$ to $aut_\circ(X)$ and $aut_\circ(X_{\mathbb{Q}})$ are a weak equivalence and a rational homotopy equivalence, respectively.

There are right actions of the monoid $aut_\circ(r)$ on $map(X,Z)$ and $map(X_{\mathbb{Q}}, Z_{\mathbb{Q}})$ through the projections to $aut_\circ(X)$ and $aut_\circ(X_{\mathbb{Q}})$, respectively. We get a zig-zag of rational homotopy equivalences of right $aut_\circ(r)$-spaces

$$map(X,Z)_\nu \xrightarrow{q_*} map(X, Z_{\mathbb{Q}})_{q\nu} \xleftarrow{r^*} map(X_{\mathbb{Q}}, Z_{\mathbb{Q}})_{\nu_{\mathbb{Q}}}.$$

This accounts for the top horizontal zig-zag in the following diagram, where we write $\bullet$ instead of $B\big(map(X, Z_{\mathbb{Q}})_{q\nu}, aut_\circ(r), *\big)$ to save space,

$$\begin{array}{ccccc} B\big(map(X,Z)_\nu, aut_\circ(r), *\big) & \xrightarrow{\sim_{\mathbb{Q}}} & \bullet & \xleftarrow{\sim_{\mathbb{Q}}} & B\big(map(X_{\mathbb{Q}}, Z_{\mathbb{Q}})_{\nu_{\mathbb{Q}}}, aut_\circ(r), *\big) \\ \downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim_{\mathbb{Q}}} \\ B\big(map(X,Z)_\nu, aut_\circ(X), *\big) & & & & B\big(map(X_{\mathbb{Q}}, Z_{\mathbb{Q}})_{\nu_{\mathbb{Q}}}, aut_\circ(X_{\mathbb{Q}}), *\big). \end{array}$$

□

3.2 Geometric realization of dg Lie algebras

Let $\mathfrak{g}$ be a dg Lie algebra over $\mathbb{Q}$, possibly unbounded as a chain complex. For $n \geq 0$, the *n-connected cover* is the dg Lie subalgebra $\mathfrak{g}\langle n\rangle \subseteq \mathfrak{g}$ defined by

$$\mathfrak{g}\langle n\rangle_i = \begin{cases} \mathfrak{g}_i, & i > n, \\ \ker(\mathfrak{g}_n \xrightarrow{d} \mathfrak{g}_{n-1}), & i = n, \\ 0, & i < n. \end{cases}$$

We call $\mathfrak{g}$ *connected* if $\mathfrak{g} = \mathfrak{g}\langle 0\rangle$ and *simply connected* if $\mathfrak{g} = \mathfrak{g}\langle 1\rangle$.
The lower central series of $\mathfrak{g}$ is the descending filtration

$$\mathfrak{g} = \Gamma^1\mathfrak{g} \supseteq \Gamma^2\mathfrak{g} \supseteq \cdots$$

characterized by $\Gamma^1\mathfrak{g} = \mathfrak{g}$ and $[\Gamma^k\mathfrak{g}, \mathfrak{g}] = \Gamma^{k+1}\mathfrak{g}$. We call $\mathfrak{g}$ *nilpotent* if the lower central series terminates *degree-wise*, meaning that for every n, there is a k such that $(\Gamma^k\mathfrak{g})_n = 0$. This definition of nilpotence mirrors the notion of nilpotence for topological spaces. Indeed, a connected dg Lie algebra $\mathfrak{g}$ is nilpotent if and only if the Lie algebra $\mathfrak{g}_0$ is nilpotent and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_n$ is nilpotent for all n. And, clearly, every simply connected dg Lie algebra is nilpotent.
If $\mathfrak{g}$ is an ordinary nilpotent Lie algebra, then $\exp(\mathfrak{g})$ denotes the nilpotent group whose underlying set is $\mathfrak{g}$ and where the group operation is given by the Baker-Campbell-Hausdorff formula, see e.g. [14]. The following generalizes this to dg Lie algebras. Let $\mathfrak{g}$ be a connected nilpotent dg Lie algebra. If Ω is a commutative cochain algebra, then the chain complex $\mathfrak{g} \otimes \Omega$ becomes a dg Lie algebra with

$$[x \otimes \alpha, y \otimes \beta] = (-1)^{|\alpha||y|}[x, y] \otimes \alpha\beta$$

for $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \Omega$. If $\Omega^k = 0$ unless $0 \leq k \leq n$ for some n, then the degree 0 component of $\mathfrak{g} \otimes \Omega$ decomposes as

$$(\mathfrak{g} \otimes \Omega)_0 = (\mathfrak{g}_0 \otimes \Omega^0) \oplus (\mathfrak{g}_1 \otimes \Omega^1) \oplus \cdots \oplus (\mathfrak{g}_n \otimes \Omega^n).$$

From the fact that $[\mathfrak{g}_i \otimes \Omega^i, \mathfrak{g}_j \otimes \Omega^j] \subseteq \mathfrak{g}_{i+j} \otimes \Omega^{i+j}$ and that $\mathfrak{g}_0$ acts nilpotently on $\mathfrak{g}_k$ for all k, one sees that $(\mathfrak{g} \otimes \Omega)_0$ is a nilpotent Lie algebra. Hence, so is the Lie subalgebra of zero-cycles $Z_0(\mathfrak{g} \otimes \Omega)$.
Let $\Omega_\bullet$ be the simplicial commutative differential graded algebra where Ω_n is the Sullivan-de Rham algebra of polynomial differential forms on the n-simplex, see [7]. Since $\Omega_n^k = 0$ unless $0 \leq k \leq n$, the above construction may be applied levelwise to the simplicial dg Lie algebra $\mathfrak{g} \otimes \Omega_\bullet$.

Definition 3.2. Let $\mathfrak{g}$ be a connected nilpotent dg Lie algebra. We define $\exp_\bullet(\mathfrak{g})$ to be the simplicial nilpotent group

$$\exp_\bullet(\mathfrak{g}) = \exp Z_0(\mathfrak{g} \otimes \Omega_\bullet).$$

Next, we recall the definition of the nerve $\mathsf{MC}_\bullet(\mathfrak{g})$ of a dg Lie algebra $\mathfrak{g}$. As we will see below, the nerve $\mathsf{MC}_\bullet(\mathfrak{g})$ is a delooping of the simplicial group $\exp_\bullet(\mathfrak{g})$.

Definition 3.3. A *Maurer-Cartan element* in $\mathfrak{g}$ is an element τ of degree -1 such that

$$d(\tau) + \frac{1}{2}[\tau, \tau] = 0.$$

The set of Maurer-Cartan elements is denoted $\mathsf{MC}(\mathfrak{g})$. The *nerve* of $\mathfrak{g}$ is the simplicial set

$$\mathsf{MC}_\bullet(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g} \otimes \Omega_\bullet).$$

Define the *geometric realization* of a dg Lie algebra to be the geometric realization of its nerve,

$$|\mathfrak{g}| = |\mathsf{MC}_\bullet(\mathfrak{g})|.$$

3.3 Geometric realization of dg coalgebras

Let Ω be a commutative cochain algebra over $\mathbb{Q}$. A dg coalgebra over Ω is a coalgebra in the symmetric monoidal category of Ω-modules, i.e., a dg Ω-module C together with a coproduct and a counit,

$$\Delta\colon C \to C \otimes_\Omega C, \quad \epsilon\colon C \to \Omega,$$

such that the appropriate diagrams commute. We let $dgc(\Omega)$ denote the category of dg coalgebras over Ω. If C is a dg coalgebra over Ω, we let

$$\mathcal{G}(C)$$

denote the set of group-like elements, i.e., elements $\xi \in C$ of degree 0 such that

$$\Delta(\xi) = \xi \otimes \xi, \quad d(\xi) = 0, \quad \epsilon(\xi) = 1.$$

Given a dg coalgebra C over $\mathbb{Q}$, the free Ω-module $C \otimes \Omega$ is a dg coalgebra over Ω. Clearly,

$$\Omega \mapsto \mathcal{G}(C \otimes \Omega)$$

defines a functor from commutative cochain algebras to sets.

Definition 3.4. Let C be a dg coalgebra. We defined the *spatial realization* of C to be the simplicial set

$$\langle C \rangle = \mathcal{G}(C \otimes \Omega_\bullet).$$

A dg Lie algebra over Ω is a dg Ω-module L together with a Lie bracket $\ell\colon L \otimes_\Omega L \to L$ satisfying the usual anti-symmetry and Jacobi relations. Quillen's generalization of the Chevalley-Eilenberg construction can be extended to dg Lie algebras over Ω, yielding a functor

$$\mathcal{C}_\Omega\colon dgl(\Omega) \to dgc(\Omega).$$

The underlying coalgebra $\mathcal{C}_\Omega(L)$ is the symmetric coalgebra $S_\Omega(sL)$, where

$$S_\Omega(V) = \bigoplus_{k \geq 0} (V^{\otimes_\Omega k})_{\Sigma_n},$$

for an Ω-module V. The differential is defined as usual, see e.g., [7, p.301]. If L is a dg Lie algebra over $\mathbb{Q}$, then $L \otimes \Omega$ is a dg Lie algebra over Ω and there is a natural isomorphism of dg coalgebras over Ω,

$$\mathcal{C}_\Omega(L \otimes \Omega) \cong \mathcal{C}(L) \otimes \Omega.$$

PROPOSITION 3.5. *Let L be a connected dg Lie algebra and let Ω be a bounded commutative cochain algebra. There is a natural bijection*

$$e\colon \mathsf{MC}(L \otimes \Omega) \cong \mathcal{G}\big(\mathcal{C}_\Omega(L \otimes \Omega)\big),$$

$$e(\tau) = \sum_{k \geq 0} \frac{1}{k!} s\tau^{\wedge_\Omega k} \in \mathcal{C}_\Omega(L \otimes \Omega).$$

Proof. The crucial observation is that this series converges since Ω is bounded and L is connected. Say $\Omega^k =$ unless $0 \leq k \leq n$. Then

$$\big(L \otimes \Omega\big)_{-1} = L_0 \otimes \Omega^1 \oplus \cdots \oplus L_{n-1} \otimes \Omega^n,$$

whence $\tau \in L \otimes \Omega^+$ for every element τ of degree -1. Since $(\Omega^+)^k = 0$ for $k > n$, this implies that

$$s\tau \wedge_\Omega \cdots \wedge_\Omega s\tau = 0$$

whenever there are more than n factors. Clearly, $\Delta(e(\tau)) = e(\tau) \otimes e(\tau)$ and $\epsilon(e(\tau)) = 0$. As the reader may check, the equation $d(e(\tau)) = 0$ is equivalent to the Maurer-Cartan equation for τ. □

COROLLARY 3.6. *There is a natural isomorphism of simplicial sets,*

$$\mathsf{MC}_\bullet(L) \cong \langle \mathcal{C}(L) \rangle,$$

for every connected dg Lie algebra L.

Proof. Indeed, $\mathsf{MC}_\bullet(L) = \mathsf{MC}(L \otimes \Omega_\bullet) \cong \mathcal{G}\big(\mathcal{C}_\Omega(L \otimes \Omega_\bullet)\big) \cong \mathcal{G}\big(\mathcal{C}(L) \otimes \Omega_\bullet\big)$. □

Recall that for a commutative dg algebra A, the spatial realization is defined by

$$\langle A \rangle = \mathrm{Hom}_{dga}(A, \Omega_\bullet),$$

see e.g., [2]. We use the same notation as for the coalgebra realization, but it should be clear from the context which one is used.

PROPOSITION 3.7. *Let A be a commutative cochain algebra of finite type with dual dg coalgebra $A^\vee$. Then there is a natural isomorphism*

$$\langle A^\vee \rangle \cong \langle A \rangle.$$

Proof. For a bounded commutative cochain algebra Ω and a finite type dg algebra A, there is a natural isomorphism of chain complexes

$$A^\vee \otimes \Omega \cong \mathrm{Hom}(A, \Omega).$$

Under this isomorphism, group-like elements in the dg coalgebra $A^\vee \otimes \Omega$ correspond to morphisms of dg algebras $A \to \Omega$. ☐

Note that the spatial realization of dg coalgebras preserves products, $\langle C \otimes D \rangle \cong \langle C \rangle \times \langle D \rangle$. In particular, since the universal enveloping algebra $U\mathfrak{g}$ of a dg Lie algebra $\mathfrak{g}$ is a dg Hopf algebra, i.e., a group object in the category of dg coalgebras, its spatial realization $\langle U\mathfrak{g} \rangle$ is a simplicial group. We also remark that for every commutative cochain algebra Ω, the forgetful functor $dga(\Omega) \to dgl(\Omega)$ admits a left adjoint $U_\Omega \colon dgl(\Omega) \to dga(\Omega)$.

Proposition 3.8. *Let $\mathfrak{g}$ be a simply connected dg Lie algebra. There is a natural isomorphism of simplicial groups*

$$\exp_\bullet(\mathfrak{g}) \cong \langle U\mathfrak{g} \rangle.$$

Proof. Let Ω be a bounded commutative cochain algebra, say $\Omega^k = 0$ unless $0 \leq k \leq n$. Observe that there is a canonical isomorphism $U\mathfrak{g} \otimes \Omega \cong U_\Omega(\mathfrak{g} \otimes \Omega)$. The isomorphism is effected by the exponential map

$$\exp\colon Z_0(\mathfrak{g} \otimes \Omega) \to \mathcal{G}U_\Omega(\mathfrak{g} \otimes \Omega),$$

$$\exp(x) = \sum_{k \geq 0} \frac{1}{k!} x^k,$$

where the product x^k is taken in $U_\Omega(\mathfrak{g} \otimes \Omega)$. The crucial point is that the sum converges. Indeed, since $\mathfrak{g}$ is simply connected,

$$\big(\mathfrak{g} \otimes \Omega\big)_0 = \mathfrak{g}_1 \otimes \Omega^1 + \cdots + \mathfrak{g}_n \otimes \Omega^n,$$

so $x \in \mathfrak{g} \otimes \Omega^+$, whence $x^k = 0$ for $k > n$, whenever x is an element of degree 0. The fact that exp respects the group structure is essentially by design of the Baker-Campbell-Hausdorff group structure. ☐

3.4 Principal dg coalgebra bundles

Next, recall Quillen's theory of principal dg coalgebra bundles [14, Appendix B, §5]. In particular, recall that $\mathcal{C}(\mathfrak{g})$ serves as a classifying space for principal $\mathfrak{g}$-bundles. Quillen's universal principal $\mathfrak{g}$-bundle may be identified with

$$U\mathfrak{g} \to \mathcal{C}(U\mathfrak{g}; \mathfrak{g}) \to \mathcal{C}(\mathfrak{g}),$$

where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{C}(U\mathfrak{g}; \mathfrak{g})$ is the Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in the right $\mathfrak{g}$-module $U\mathfrak{g}$.

Theorem 3.9. *Let $\mathfrak{g}$ be a simply connected dg Lie algebra of finite type. The realization of the universal principal $\mathfrak{g}$-bundle,*

$$\langle U\mathfrak{g}\rangle \to \langle \mathcal{C}(U\mathfrak{g};\mathfrak{g})\rangle \to \langle \mathcal{C}(\mathfrak{g})\rangle,$$

is a universal principal $\langle U\mathfrak{g}\rangle$-bundle.

Proof. This is proved in [7, Chapter 25]. Indeed, when $\mathfrak{g}$ is simply connected and of finite type, the coalgebra realization of $U\mathfrak{g}$ is the same as the algebra realization of the dual dg algebra $U\mathfrak{g}^{\vee}$. □

Corollary 3.10. *Let $\mathfrak{g}$ be a simply connected dg Lie algebra of finite type. The nerve $\mathsf{MC}_\bullet(\mathfrak{g})$ is a delooping of the simplicial group $\exp_\bullet(\mathfrak{g})$.*

Proof. We have the isomorphisms $\exp_\bullet(\mathfrak{g}) \cong \langle U\mathfrak{g}\rangle$ and $\mathsf{MC}_\bullet(\mathfrak{g}) \cong \langle \mathcal{C}(\mathfrak{g})\rangle$. □

Remark 3.11. Since we work with coalgebras, the finite type hypothesis on $\mathfrak{g}$ can be dropped in Theorem 3.9 and Corollary 3.10. However, we will not repeat the lengthy argument here since $\mathfrak{g}$ will be of finite type in our applications.

3.5 Twisted semi-direct products and Borel constructions

We begin by recalling certain aspects of Tanré's classification of fibrations in the category of dg Lie algebras [16, Chapitre VII].

Definition 3.12. Let $\mathfrak{g}$ and L be dg Lie algebras. An *outer action* of $\mathfrak{g}$ on L consists of two maps

$$\alpha\colon L\otimes \mathfrak{g} \to L, \quad \xi\colon \mathfrak{g}\to L,$$

satisfying the following conditions for all $x, y\in\mathfrak{g}$ and $a, b\in L$, where we write

$$a.\, x = \alpha(a\otimes x), \quad x.\, a = -(-1)^{|a||x|}a.\, x.$$

Firstly, the map α defines an action of $\mathfrak{g}$ on L by derivations, i.e.,

$$[x,y].\, a = x.\,(y.\, a) - (-1)^{|x||y|}y.\,(x.\, a),$$

$$x.\,[a,b] = [x.\, a, b] + (-1)^{|x||a|}[a, x.\, b].$$

Secondly, the map ξ is a chain map of degree -1 and a derivation, i.e.,

$$d\xi(x) = -\xi(dx),$$

$$\xi[x,y] = \xi(x).\, y + (-1)^{|x|}x.\,\xi(y).$$

Finally, the action and ξ are connected by the equation

$$d(x.\, a) = d(x).\, a + (-1)^{|x|}x.\, d(a) + [\xi(x), a].$$

Definition 3.13. Given an outer action of $\mathfrak{g}$ on L, the *twisted semi-direct product* $L \rtimes_\xi \mathfrak{g}$ is the dg Lie algebra whose underlying graded Lie algebra is the semi-direct product of $\mathfrak{g}$ acting on L,

$$\big[(a,x),(b,y)\big] = \big([a,b] + x.\, b + a.\, y, [x,y]\big),$$

and whose differential is twisted by ξ in the sense that

$$\partial^\xi(a,x) = (da + \xi(x), dx).$$

The twisted semi-direct product is the total space in a short exact sequence (i.e. fibration sequence) of dg Lie algebras,

$$0 \to L \to L \rtimes_\xi \mathfrak{g} \to \mathfrak{g} \to 0. \tag{6}$$

The section $\mathfrak{g} \to L \rtimes_\xi \mathfrak{g}$, $x \mapsto (0,x)$, is a morphism of graded Lie algebras, but it commutes with differentials if and only if $\xi = 0$.
Outer actions on L are classified by the dg Lie algebra

$$\operatorname{Der} L \ltimes_{\mathrm{ad}} sL,$$

whose underlying graded Lie algebra is the semi-direct product of $\operatorname{Der} L$ acting on the abelian dg Lie algebra sL from the left by

$$\theta.\, sx = (-1)^{|\theta|} s\theta(x),$$

and whose differential is given by

$$\partial\big(\theta, sx\big) = \big(\partial(\theta) + \mathrm{ad}_x, -sd(x)\big),$$

where $\mathrm{ad}_x \in \operatorname{Der} L$ is given by $\mathrm{ad}_x(y) = [x,y]$.

Proposition 3.14. *Specifying an outer action of $\mathfrak{g}$ on L is tantamount to specifying a morphism of dg Lie algebras*

$$\phi\colon \mathfrak{g} \to \operatorname{Der} L \ltimes_{\mathrm{ad}} sL.$$

The correspondence is given by

$$\phi(x) = \big(\theta_x, -s\xi(x)\big),$$

where $\theta_x(a) = x.\, a$.

Proof. The proof is a straightforward calculation. □

An outer action of $\mathfrak{g}$ on L defines an action of $\mathfrak{g}$ on $\mathcal{C}(L)$ by coderivations by the following formula:

$$\begin{aligned}(sa_1 \wedge \cdots \wedge sa_n).\, x = {} & sa_1 \wedge \cdots \wedge sa_n \wedge s\xi(x) \\ & + \sum_{i=1}^n \pm sa_1 \wedge \cdots \wedge s(a_i.\, x) \wedge \cdots \wedge sa_n.\end{aligned}$$

Equivalently, $\mathcal{C}(L)$ becomes a $U\mathfrak{g}$-module coalgebra, i.e., a right $U\mathfrak{g}$-module such that the structure map $\mathcal{C}(L)\otimes U\mathfrak{g} \to \mathcal{C}(L)$ is a morphism of dg coalgebras. The action of $\operatorname{Der} L \ltimes_{\mathrm{ad}} sL$ by coderivations on $\mathcal{C}(L)$, derived from the tautological outer action on L, gives rise to a morphism of dg Lie algebras that we will denote

$$\chi\colon \operatorname{Der} L \ltimes_{\mathrm{ad}} sL \to \operatorname{Coder} \mathcal{C}(L). \tag{7}$$

Theorem 3.15. *Let $\mathfrak{g}$ be a simply connected dg Lie algebra of finite type with an outer action on a connected dg Lie algebra L. There is a right action of the simplicial group $G = \exp_\bullet(\mathfrak{g})$ on $\mathsf{MC}_\bullet(L)$ and a weak equivalence of simplicial sets over $\mathsf{MC}_\bullet(\mathfrak{g})$,*

$$\mathsf{MC}_\bullet(L \rtimes_\xi \mathfrak{g}) \sim \mathsf{MC}_\bullet(L) \times_G EG.$$

Proof. The action of $\mathfrak{g}$ on $\mathcal{C}(L)$ makes $\mathcal{C}(L)$ into a right $U\mathfrak{g}$-module coalgebra. This yields a right action of $\exp_\bullet(\mathfrak{g}) \cong \langle U\mathfrak{g}\rangle$ on $\mathsf{MC}_\bullet(L) \cong \langle \mathcal{C}(L)\rangle$. The key observation, which may be checked by hand, is that there is an isomorphism of dg coalgebras

$$\mathcal{C}(L \rtimes_\xi \mathfrak{g}) \cong \mathcal{C}(\mathcal{C}(L);\mathfrak{g}).$$

Secondly, we have the standard isomorphism

$$\mathcal{C}(\mathcal{C}(L);\mathfrak{g}) \cong \mathcal{C}(L) \otimes_{U\mathfrak{g}} \mathcal{C}(U\mathfrak{g};\mathfrak{g}).$$

By combining these isomorphisms and taking realizations, we get isomorphisms of simplicial sets

$$\mathsf{MC}_\bullet(L \rtimes_\xi \mathfrak{g}) \cong \langle \mathcal{C}(L \rtimes_\xi \mathfrak{g})\rangle \cong \langle \mathcal{C}(L) \otimes_{U\mathfrak{g}} \mathcal{C}(U\mathfrak{g};\mathfrak{g})\rangle \cong \langle \mathcal{C}(L)\rangle \times_{\langle U\mathfrak{g}\rangle} \langle \mathcal{C}(U\mathfrak{g};\mathfrak{g})\rangle.$$

By Theorem 3.9, the simplicial set $\langle \mathcal{C}(U\mathfrak{g};\mathfrak{g})\rangle$ is a model for EG. This finishes the proof. □

Let L be a simply connected cofibrant dg Lie algebra of finite type with geometric realization

$$X = |\mathsf{MC}_\bullet(L)|,$$

and consider the simply connected dg Lie algebra

$$\mathfrak{g} = \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle,$$

with associated topological group

$$G = |\exp_\bullet(\mathfrak{g})|.$$

There is an evident outer action of $\mathfrak{g}$ on L, whence an action of the simplicial group $\exp_\bullet(\mathfrak{g})$ on the nerve $\mathsf{MC}_\bullet(L)$, cf. Theorem 3.15, whence an action of G on X. Since $\mathfrak{g}$ is simply connected, the simplicial group $\exp_\bullet(\mathfrak{g})$ is reduced, i.e., has only one vertex. In particular, the topological group G is connected. Therefore, the action yields a map of group-like monoids

$$G \to aut_\circ(X). \tag{8}$$

This map is a weak homotopy equivalence, as follows from, e.g., Tanré's theory [16, Chapitre VII].

3.6 Twisting functions and mapping spaces

Let C be a dg coalgebra with coproduct $\Delta\colon C \to C \otimes C$ and let L be a dg Lie algebra with Lie bracket $\ell\colon L \otimes L \to L$. Recall that a *twisting function* $\tau\colon C \to L$ is a Maurer-Cartan element in the dg Lie algebra $\mathrm{Hom}(C, L)$, whose differential and Lie bracket are given by

$$\partial(f) = d_L \circ f - (-1)^{|f|} f \circ d_C,$$

$$[f, g] = \ell \circ (f \otimes g) \circ \Delta.$$

If τ is a twisting function, then $\mathrm{Hom}^\tau(C, L)$ denotes the dg Lie algebra with the same underlying graded Lie algebra but twisted differential

$$\partial^\tau(f) = \partial(f) + [\tau, f].$$

Furthermore, there is an outer action of $\mathrm{Coder}\, C$ on $\mathrm{Hom}^\tau(C, L)$ given by

$$f.\theta = f \circ \theta, \quad \xi(\theta) = \tau_*(\theta) = \tau \circ \theta,$$

for $f \in \mathrm{Hom}^\tau(C, L)$ and $\theta \in \mathrm{Coder}\, C$. We note for future reference that we may make the identification

$$\big(\mathrm{Hom}(C, L) \rtimes \mathrm{Coder}\, C\big)^\tau = \mathrm{Hom}^\tau(C, L) \rtimes_{\tau_*} \mathrm{Coder}\, C \tag{9}$$

for every twisting function $\tau\colon C \to L$.

Theorem 3.16. *Let L and Π be connected dg Lie algebras and suppose Π is nilpotent and of finite type. There is a natural weak homotopy equivalence of simplicial sets*

$$\mathsf{MC}\big(\mathrm{Hom}(\mathcal{C}L, \Pi \otimes \Omega_\bullet)\big) \xrightarrow{\sim} map\big(\mathsf{MC}_\bullet(L), \mathsf{MC}_\bullet(\Pi)\big).$$

Proof. Let Ω be a bounded commutative cochain algebra. We define a natural map

$$\mathsf{MC}\,\mathrm{Hom}(\mathcal{C}(L), \Pi \otimes \Omega) \times \mathsf{MC}(L \otimes \Omega) \to \mathsf{MC}(\Pi \otimes \Omega) \tag{10}$$

as follows. First, make the identifications

$$\mathrm{Hom}(\mathcal{C}(L), \Pi \otimes \Omega) = \mathrm{Hom}_\Omega(\mathcal{C}_\Omega(L \otimes \Omega), \Pi \otimes \Omega),$$
$$\mathsf{MC}(L \otimes \Omega) = \mathcal{G}\big(\mathcal{C}_\Omega(L \otimes \Omega)\big),$$

the second of which is justified by Proposition 3.5, and then define

$$\epsilon\colon \mathsf{MC}\,\mathrm{Hom}_\Omega(\mathcal{C}_\Omega(L \otimes \Omega), \Pi \otimes \Omega) \times \mathcal{G}\big(\mathcal{C}_\Omega(L \otimes \Omega)\big) \to \mathsf{MC}(\Pi \otimes \Omega),$$

simply by evaluation,

$$\epsilon(\tau, \xi) = \tau(\xi).$$

We need to verify that $\tau(\xi)$ satisfies the Maurer-Cartan equation. Since τ is a twisting function, it satisfies the equation

$$0 = \partial(\tau) + \frac{1}{2}[\tau, \tau].$$

Evaluating both sides at the group-like element ξ yields

$$0 = d\tau(\xi) + \tau d(\xi) + \frac{1}{2}\ell \circ (\tau \otimes \tau) \circ \Delta(\xi) = d\tau(\xi) + \frac{1}{2}[\tau(\xi), \tau(\xi)],$$

showing that $\tau(\xi)$ satisfies the Maurer-Cartan equation.
The map is clearly natural in Ω and yields a simplicial map

$$\mathsf{MC}\,\mathrm{Hom}(\mathcal{C}(L), \Pi \otimes \Omega_\bullet) \times \mathsf{MC}(L \otimes \Omega_\bullet) \to \mathsf{MC}(\Pi \otimes \Omega_\bullet).$$

The map in the theorem is defined to be the adjoint of this map.
To show it is a weak homotopy equivalence, one argues as in [2, Theorem 6.6] by induction on a suitable complete filtration of Π. The proof is entirely analogous so we omit the details. □

Remark 3.17. The dg Lie algebra $\mathrm{Hom}(\mathcal{C}L, \Pi)$ with the descending filtration

$$F^{r+1} = \mathrm{Hom}(\mathcal{C}L, \Pi\langle r\rangle), \quad r \geq 0,$$

is a complete dg Lie algebra in the sense of [2, Definition 5.1]. By [2, Theorem 6.3] (see also Definition 5.3 and Remark 6.4 in *loc.cit.*), the Kan complex

$$\widehat{\mathsf{MC}}_\bullet(\mathrm{Hom}(\mathcal{C}L, \Pi)) = \varprojlim \mathsf{MC}_\bullet\,\mathrm{Hom}(\mathcal{C}L, \Pi/\Pi\langle r\rangle)$$

is homotopy equivalent to $map(\mathsf{MC}_\bullet(L), \mathsf{MC}_\bullet(\Pi))$. We would like to remark how this relates to the statement in Theorem 3.16.
Since $\Pi/\Pi\langle r\rangle$ is finite dimensional for all r, we have

$$\mathrm{Hom}(\mathcal{C}L, \Pi/\Pi\langle r\rangle) \otimes \Omega_\bullet \cong \mathrm{Hom}(\mathcal{C}L, \Pi/\Pi\langle r\rangle \otimes \Omega_\bullet)$$

Upon taking the inverse limit, we get an isomorphism of simplicial sets

$$\widehat{\mathsf{MC}}_\bullet(\mathrm{Hom}(\mathcal{C}L, \Pi)) \cong \mathsf{MC}(\mathrm{Hom}(\mathcal{C}L, \Pi \otimes \Omega_\bullet)).$$

Thus, Theorem 3.16 and Theorem 6.3 in [2] say the same thing. The advantage of Theorem 3.16 is that the explicit formula for the map gives us control over equivariance properties, as we will see next.

Let L be a simply connected cofibrant dg Lie algebra of finite type. Precomposition defines a right action of the dg Lie algebra $\mathrm{Coder}\,\mathcal{C}(L)$ on the complete dg Lie algebra $\mathrm{Hom}(\mathcal{C}(L), \Pi)$. By composing with (7), and restricting to the simply connected cover, we get an action of the dg Lie algebra

$$\mathfrak{g} = \big(\mathrm{Der}\,L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle$$

on $\operatorname{Hom}(\mathcal{C}(L),\Pi)$. By Theorem 3.15, this induces an action of the simplicial group $\exp_\bullet(\mathfrak{g})$ on the simplicial set $\mathsf{MC}\operatorname{Hom}(\mathcal{C}(L),\Pi\otimes\Omega_\bullet)\cong\widehat{\mathsf{MC}}_\bullet(\operatorname{Hom}(\mathcal{C}(L),\Pi))$. On the other hand, $\exp_\bullet(\mathfrak{g})$ acts on $\mathsf{MC}_\bullet(L)$ and hence also on $map(\mathsf{MC}_\bullet(L),\mathsf{MC}_\bullet(\Pi))$. The following is an important addendum to Theorem 3.16.

Proposition 3.18. *The weak equivalence of Theorem 3.16,*

$$\mathsf{MC}\operatorname{Hom}(\mathcal{C}L,\Pi\otimes\Omega_\bullet)\xrightarrow{\sim} map(\mathsf{MC}_\bullet(L),\mathsf{MC}_\bullet(\Pi)),$$

is equivariant with respect to the action of the simplicial group $\exp_\bullet(\mathfrak{g})$.

Proof. The proof boils down to the easily checked fact that the map ϵ in the proof of Theorem 3.16 satisfies

$$\epsilon(\theta.\,f,\xi)=\epsilon(f,\xi.\,\theta),$$

for $\theta\in\mathcal{G}U_{\Omega_\bullet}(\mathfrak{g}\otimes\Omega_\bullet)$, $f\in\mathsf{MC}\operatorname{Hom}_\Omega(\mathcal{C}_\Omega(L\otimes\Omega),\Pi\otimes\Omega)$ and $\xi\in\mathcal{G}\big(\mathcal{C}_\Omega(L\otimes\Omega)\big)$. □

Proposition 3.19. *Let $X_\mathbb{Q}$ and $Z_\mathbb{Q}$ be $\mathbb{Q}$-local connected nilpotent spaces of finite $\mathbb{Q}$-type. Let L be a finite type cofibrant dg Lie algebra model for $X_\mathbb{Q}$ and let Π be any dg Lie model for $Z_\mathbb{Q}$. The geometric bar construction,*

$$B\big(map(X_\mathbb{Q},Z_\mathbb{Q}),aut_\circ(X_\mathbb{Q}),*\big),$$

is weakly homotopy equivalent to the geometric realization of the dg Lie algebra

$$\operatorname{Hom}(\mathcal{C}L,\Pi)\rtimes\big(\operatorname{Der}L\ltimes_{\mathrm{ad}}sL\big)\langle 1\rangle.$$

Proof. We may as well assume $X_\mathbb{Q}=\mathsf{MC}_\bullet(L)$ and $Z_\mathbb{Q}=\mathsf{MC}_\bullet(\Pi)$. By Theorem 3.15, there is a weak homotopy equivalence

$$\widehat{\mathsf{MC}}_\bullet\big(\operatorname{Hom}(\mathcal{C}L,\Pi)\rtimes\mathfrak{g}\big)\sim B\big(\widehat{\mathsf{MC}}_\bullet\big(\operatorname{Hom}(\mathcal{C}L,\Pi)\big),\exp_\bullet(\mathfrak{g}),*\big).$$

The weak equivalence $\exp_\bullet(\mathfrak{g})\to aut_\circ(X)$ of group-like simplicial monoids and the weak equivalence of $\exp_\bullet(\mathfrak{g})$-spaces of Proposition 3.18 combine to give a weak homotopy equivalence

$$B\big(\widehat{\mathsf{MC}}_\bullet\big(\operatorname{Hom}(\mathcal{C}L,\Pi)\big),\exp_\bullet(\mathfrak{g}),*\big)\xrightarrow{\sim} B\big(map(X_\mathbb{Q},Z_\mathbb{Q}),aut_\circ(X_\mathbb{Q}),*\big).$$

□

3.7 Proof of the main result

Theorem 3.20. *Suppose that $\mathcal{F}$ is a category of fibers such that the classifying space B_∞ is connected and nilpotent. Let $p\colon E\to X$ be an $\mathcal{F}$-fibration over a simply connected finite CW-complex X.*

Let L be a simply connected cofibrant dg Lie algebra model for X and let Π be a connected nilpotent dg Lie algebra model for B_∞. Let $\tau\colon \mathcal{C}L \to \Pi$ be a twisting function that models the map $\nu\colon X \to B_\infty$ that classifies p.
Then the classifying space $Baut_\circ^{\mathcal{F}}(p)$ is rationally homotopy equivalent to the geometric realization of the dg Lie algebra

$$\operatorname{Hom}^\tau(\mathcal{C}L,\Pi)\langle 0\rangle \rtimes_{\tau_*} \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle.$$

Proof. For notational convenience, let $Z = B_\infty$. As before, let

$$\mathfrak{g} = \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle.$$

That the dg Lie algebras L and Π are models for X and Z means that we may use their geometric realizations as models for the $\mathbb{Q}$-localizations of X and Z;

$$X_{\mathbb{Q}} = |\mathsf{MC}_\bullet(L)|, \quad Z_{\mathbb{Q}} = |\mathsf{MC}_\bullet(\Pi)|.$$

By Corollary 2.4 and Lemma 3.1 we have

$$Baut_\circ^{\mathcal{F}}(p) \sim B\big(map(X,Z)_\nu, aut_\circ(X), *\big) \sim_{\mathbb{Q}} B\big(map(X_{\mathbb{Q}},Z_{\mathbb{Q}})_{\nu_{\mathbb{Q}}}, aut_\circ(X_{\mathbb{Q}}), *\big).$$

The latter space is weakly homotopy equivalent to the component

$$B\big(map(X_{\mathbb{Q}},Z_{\mathbb{Q}}), aut_\circ(X_{\mathbb{Q}}), *\big)_{\nu_{\mathbb{Q}}}.$$

By Proposition 3.19,

$$B\big(map(X_{\mathbb{Q}},Z_{\mathbb{Q}}), aut_\circ(X_{\mathbb{Q}}), *\big) \sim \widehat{\mathsf{MC}}_\bullet\big(\operatorname{Hom}(\mathcal{C}L,\Pi) \rtimes \mathfrak{g}\big).$$

Let $\tau\colon \mathcal{C}L \to \Pi$ be a twisting function that corresponds to $\nu_{\mathbb{Q}}$. It follows from [2, Corollary 1.3] that the component

$$\widehat{\mathsf{MC}}_\bullet\big(\operatorname{Hom}(\mathcal{C}L,\Pi) \rtimes \mathfrak{g}\big)_\tau \sim \widehat{\mathsf{MC}}_\bullet\big((\operatorname{Hom}(\mathcal{C}L,\Pi) \rtimes \mathfrak{g})^\tau\langle 0\rangle\big).$$

Finally, as in (9) one checks that there is an isomorphism of dg Lie algebras

$$(\operatorname{Hom}(\mathcal{C}L,\Pi) \rtimes \mathfrak{g})^\tau\langle 0\rangle \cong \operatorname{Hom}^\tau(\mathcal{C}L,\Pi)\langle 0\rangle \rtimes_{\tau_*} \mathfrak{g}.$$

This finishes the proof. ☐

REMARK 3.21. It is straightforward to derive the following variants of the main result: If $A \subseteq X$ is a subspace such that the inclusion of A into X is a cofibration, then we may consider the submonoid $aut_{A,\circ}^{\mathcal{F}}(p) \subseteq aut_\circ^{\mathcal{F}}(p)$ where the homotopy automorphism of the base restricts to the identity map on A. If

$$\mathfrak{g}_A \to \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle$$

is a dg Lie algebra morphism that models the map $Baut_{A,\circ}(X) \to Baut_\circ(X)$, then

$$\operatorname{Hom}^\tau(\mathcal{C}L,\Pi)\langle 0\rangle \rtimes_{\tau_*} \mathfrak{g}_A$$

is a dg Lie algebra model for the space $Baut_{A,\circ}^{\mathcal{F}}(p)$. Similarly, if we pick a base-point $* \in A \subseteq X$, and let $aut_{A,\circ}^*(p)$ denote the submonoid of $aut_{A,\circ}^{\mathcal{F}}(p)$ where the automorphism of the total space restricts to the identity over the base-point, then one gets a model for $Baut_{A,\circ}^*(p)$ by replacing $\mathcal{C}(L)$ with the *reduced* Chevalley-Eilenberg chains.

4 Examples and applications

Many classifying spaces of interest in geometry have simple rational homotopy types:

- If G is a compact connected Lie group, then $H^*(BG;\mathbb{Q})$ is a polynomial algebra on finitely many generators of even degree (see, e.g., [8, Theorem 1.81]).
- The stable classifying spaces BO, $BTop$, BPL are infinite loop spaces and have rational cohomology rings of the form $\mathbb{Q}[p_1, p_2, \ldots]$, where p_i is a generator of degree $4i$ (see, e.g., [12]).
- When the 'structure group' is $G = aut_\circ(X)$ for a finite simply connected CW-complex X, Halperin's conjecture is equivalent to the statement that $H^*(BG;\mathbb{Q})$ is a polynomial algebra whenever X is an elliptic space with non-zero Euler characteristic.

In this section, we will provide a simplification of the model in Theorem 3.20 in the case when $H^*(B_\infty;\mathbb{Q})$ is a polynomial algebra or, more generally, a free graded commutative algebra $\Lambda(u_1, u_2, \ldots)$. We allow infinitely many generators, but we assume there are only finitely many generators in each degree. Let X be a simply connected finite CW-complex together with an $\mathcal{F}$-bundle ξ classified by a map

$$f\colon X \to B_\infty.$$

The characteristic classes of the bundle are defined by pulling back the universal classes u_i along f;

$$u_i(\xi) = f^*(u_i) \in H^*(X;\mathbb{Q}).$$

We first need a lemma about cochain algebra models for the map f. Note that a space with free cohomology ring is intrinsically formal; the minimal Sullivan model for B_∞ is the cohomology ring with zero differential, $(\Lambda(u_1, u_2, \ldots), 0)$.

Lemma 4.1. *Let (A, d) be any cochain algebra model for X. For any choice of cocycles α_i in (A, d) that represent $u_i(\xi)$, the cdga map $\varphi\colon \mathbb{Q}[u_1, u_2, \ldots] \to (A, d)$, defined by $\varphi(u_i) = \alpha_i$, is a model for the map $f\colon X \to B_\infty$.*

Proof. We know from the general theory of Sullivan models that there exists a cdga model $\psi\colon \mathbb{Q}[u_1, u_2, \ldots] \to (A, d)$ for the map f. In particular, $\beta_i = \psi(u_i)$ is then a cocycle representative of the class $u_i(\xi)$, but this is possibly different from our chosen α_i. However, $\beta_i = \alpha_i + d\omega_i$ for some $\omega_i \in A$ and the cdga map $h\colon \mathbb{Q}[u_1, u_2, \ldots] \to \Lambda(t, dt) \otimes A$ defined by

$$h(u_i) = (1 - t)\alpha_i + t\beta_i + dt\omega_i$$

shows that ψ and φ are cdga homotopic, which means that φ is a model for f as well. □

We now turn to dg Lie algebra models. Consider the graded vector space

$$\Pi = \pi_*(\Omega B_\infty) \otimes \mathbb{Q},$$

and let $\gamma_i \in \pi_{|u_i|-1}(\Omega B_\infty) \otimes \mathbb{Q} = \pi_{|u_i|}(B_\infty) \otimes \mathbb{Q}$ be dual to u_i under the Hurewicz pairing between cohomology and homotopy. Equipped with trivial differential and Lie bracket, Π is a dg Lie algebra model for B_∞, because the Chevalley-Eilenberg cochains $C^*(\Pi)$ is isomorphic to the minimal model $(\Lambda(u_1, u_2, \ldots), 0)$.

Let L denote the minimal Quillen model of X. It has the form

$$L = \big(\mathbb{L}(V), \delta\big),$$

where $V = s^{-1}\widetilde{H}_*(X;\mathbb{Q})$ and the differential δ is decomposable in the sense that $\delta(L) \subseteq [L, L]$. Thus, we may identify

$$H_*(X;\mathbb{Q}) \cong \mathbb{Q} \oplus sL/[L, L]. \tag{11}$$

There is a quasi-isomorphism of chain complexes,

$$g\colon \mathcal{C}L \to \mathbb{Q} \oplus sL/[L, L], \tag{12}$$

defined by $g(1) = 1$, $g(sx) = s[x]$ and $g(sx_1 \wedge \cdots \wedge sx_n) = 0$ for $n \geq 2$, see [7, Proposition 22.8]. For future reference, we note that g factors as

$$\mathcal{C}L \xrightarrow{1\oplus s\tau_L} \mathbb{Q} \oplus sL \xrightarrow{1\oplus sa} \mathbb{Q} \oplus sL/[L, L], \tag{13}$$

where $\tau_L\colon \mathcal{C}L \to L$ is the universal twisting morphism, and $a\colon L \to L/[L, L]$ is the canonical projection.

Consider the degree -1 map of graded vector spaces

$$\rho\colon \widetilde{H}_*(X;\mathbb{Q}) \to \pi_*(\Omega B_\infty) \otimes \mathbb{Q},$$
$$e \mapsto \sum_i \langle u_i(\xi), e\rangle \gamma_i,$$

where $\langle -, -\rangle$ denotes the standard pairing between cohomology and homology (and $\langle u, e\rangle = 0$ unless u and e have the same degree). Note that we may interpret ρ as a morphism of dg Lie algebras $L/[L, L] \to \Pi$.

Proposition 4.2. *The composite morphism of dg Lie algebras $L \xrightarrow{a} L/[L, L] \xrightarrow{\rho} \Pi$ is a model for the map $f\colon X \to B_\infty$.*

Proof. The quasi-isomorphism g in (12) dualizes to a quasi-isomorphism of cochain complexes

$$g^*\colon H^*(X;\mathbb{Q}) = \mathbb{Q} \oplus (sL/[L, L])^\vee \to C^*(L).$$

In particular, we may take $\alpha_i = g^*(u_i(\xi))$ as a cocycle representative of $u_i(\xi)$ in $C^*(L)$. By Lemma 4.1, the map $\varphi\colon \Lambda(u_1, u_2, \ldots) \to C^*(L)$ defined by $\varphi(u_i) = \alpha_i$ is then a cdga model for f. Using the factorization (13), one checks that φ agrees with the map $(\rho a)^*\colon C^*(\Pi) \to C^*(L)$ induced by the dg Lie algebra morphism $\rho a\colon L \to \Pi$. This means that ρa is a dg Lie algebra model for f. □

Corollary 4.3. *The composite map*

$$\tau\colon \mathcal{C}(L) \xrightarrow{g} \mathbb{Q} \oplus sL/[L,L] = H_*(X;\mathbb{Q}) \xrightarrow{\rho} \pi_*(\Omega B_\infty) \otimes \mathbb{Q}$$

is a twisting function that models the map $\nu\colon X \to B_\infty$.

Proof. Using the factorization (13), one sees that $g\rho$ is the same as $\tau_L^*(\rho a)$. □

There is an action of $\operatorname{Der} L \ltimes_{\mathrm{ad}} sL$ on $\mathbb{Q} \oplus sL/[L,L]$ given by

$$\theta.\, 1 = 0, \qquad sx.\, 1 = s[x],$$
$$\theta.\, s[a] = (-1)^{|\theta|} s[\theta(a)], \qquad sx.\, s[a] = 0,$$

for $x, a \in L$ and $\theta \in \operatorname{Der} L$. The following is an easy but important observation.

Proposition 4.4. *The quasi-isomorphism of chain complexes*

$$g\colon \mathcal{C}L \to \mathbb{Q} \oplus sL/[L,L]$$

is a morphism of $\operatorname{Der} L \ltimes_{\mathrm{ad}} sL$*-modules.*

Proof. The verification is direct and left to the reader. □

Remark 4.5. The result may be interpreted as a formality result; it says that $\mathcal{C}L$ is formal as a dg $\operatorname{Der} L \ltimes_{\mathrm{ad}} sL$-module. Note however that g is in general far from being a morphism of dg coalgebras.

Theorem 4.6. *If* $H^*(B_\infty;\mathbb{Q})$ *is free graded commutative, then the classifying space* $Baut_\circ(\xi)$ *is rationally homotopy equivalent to the geometric realization of the dg Lie algebra*

$$\operatorname{Hom}(H_*(X;\mathbb{Q}), \pi_*(\Omega B_\infty) \otimes \mathbb{Q})\langle 0 \rangle \rtimes_{\rho_*} \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1 \rangle.$$

Proof. By Corollary 4.3, the map $\tau = \rho \circ g\colon \mathcal{C}L \to \Pi$ is a twisting function that models the map $f\colon X \to B_\infty$. By Theorem 3.20, the dg Lie algebra

$$\operatorname{Hom}^\tau\big(\mathcal{C}L, \Pi\big)\langle 0 \rangle \rtimes_{\tau_*} \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1 \rangle$$

is a model for $Baut_\circ(\xi)$. Since Π is abelian, twisting has no effect, i.e., we have $\operatorname{Hom}^\tau\big(\mathcal{C}L, \Pi\big) = \operatorname{Hom}\big(\mathcal{C}L, \Pi\big)$ as dg Lie algebras. Since $g^*(\rho) = \tau$ by definition of τ, we see that the quasi-isomorphism of $\operatorname{Der} L \ltimes_{\mathrm{ad}} sL$-modules from Proposition 4.4 induces a quasi-isomorphism of dg Lie algebras

$$\operatorname{Hom}(H_*(X;\mathbb{Q}), \Pi)\langle 0 \rangle \rtimes_{\rho_*} \mathfrak{g} \xrightarrow{g^* \rtimes 1} \operatorname{Hom}(\mathcal{C}L, \Pi)\langle 0 \rangle \rtimes_{\tau_*} \mathfrak{g},$$

where $\mathfrak{g} = \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1 \rangle$. □

REMARK 4.7. If $H_*(X;\mathbb{Q})$ is finite dimensional, we can rewrite the result in terms of cohomology,

$$\operatorname{Hom}(H_*(X;\mathbb{Q}),\pi_*(\Omega B_\infty)) \cong H^*(X;\mathbb{Q})\otimes \pi_*(\Omega B_\infty).$$

Under this identification, ρ assumes the form

$$\rho = \sum_i u_i(\xi)\otimes\gamma_i \in H^*(X;\mathbb{Q})\otimes\pi_*(\Omega B_\infty).$$

REMARK 4.8. There are variants of this result that can be proved in a similar fashion. If

$$\mathfrak{h} \subseteq \big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle$$

is a sub dg Lie algebra that models a connected submonoid $H \subseteq aut_\circ(X)$, then

$$\operatorname{Hom}\big(H_*(X;\mathbb{Q}),\pi_*(\Omega B_\infty)\otimes\mathbb{Q}\big)\langle 0\rangle \rtimes_{\rho_*} \mathfrak{h}$$

is a dg Lie algebra model for the submonoid $aut_H(\xi)\subseteq aut_\circ(\xi)$ of pairs (f,φ) such that $f\in H$.
For example, let $A\subseteq X$ be a subcomplex and let $aut_{A,\circ}(\xi)\subseteq aut_\circ(\xi)$ denote the submonoid where the homotopy automorphism of the base restricts to the identity map on A. If $L'\to L$ is a cofibration of dg Lie algebras that models the inclusion of A into X, then the dg Lie algebra $\operatorname{Der}(L;L')\langle 1\rangle$ of derivations on L that vanish on L' is a dg Lie algebra model for $Baut_A(X)$ (see [4]), and

$$\operatorname{Hom}\big(H_*(X;\mathbb{Q}),\pi_*(\Omega B_\infty)\otimes\mathbb{Q}\big)\langle 0\rangle \rtimes_{\rho_*} \operatorname{Der}(L;L')\langle 1\rangle$$

is a dg Lie algebra model for the space $Baut_{A,\circ}(\xi)$.

We finally turn to the proof of the results in the introduction.

Proof of Theorem 1.4. Writing $\mathfrak{g}=\big(\operatorname{Der} L\ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle$ for brevity, the fibration (2) is modeled by the short exact sequence of dg Lie algebras

$$0\to H^*(X;\mathbb{Q})\otimes\pi_*(G)\langle 0\rangle \to \big(H^*(X;\mathbb{Q})\otimes\pi_*(G)\big)\langle 0\rangle \rtimes_{\tau_*}\mathfrak{g}\to\mathfrak{g}\to 0.$$

But $H_*(\mathfrak{g})\cong\pi_*(aut_\circ(X))\otimes\mathbb{Q}$ is concentrated in odd degrees by Halperin's conjecture. On the other hand, it is known that the rational cohomology of an elliptic space with positive Euler characteristic is concentrated in even degrees, and that the rational homotopy of a compact connected Lie group is concentrated in odd degrees, so it follows that $H^*(X;\mathbb{Q})\otimes\pi_*(G)$ is concentrated in odd degrees as well. Hence, the long exact sequence in homology splits and

$$W=\pi_*(aut_\circ(\xi))\otimes\mathbb{Q}$$

is concentrated in odd degrees as well. The minimal Sullivan model of $Baut_\circ(\xi)$ has the form $(\Lambda V,d)$ where $V=(sW)^\vee$. Since V is concentrated in even degrees, the differential must be zero, showing $Baut_\circ(\xi)$ has cohomology ΛV, which is polynomial on $\dim V$ generators. Since the differentials in the minimal models are trivial, the splitting of homotopy groups carries over to a splitting of the minimal models, showing

$$Baut_\circ(\xi) \sim_{\mathbb{Q}} Baut_X(\xi) \times Baut_\circ(X). \qquad \square$$

Proof of Theorem 1.5. The minimal Quillen model for S^n is given by the free graded Lie algebra $L = \mathbb{L}(\alpha)$ on one generator α of degree $n-1$, with zero differential. The dg Lie algebra $\big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle$ is one-dimensional abelian with a generator $s\alpha$ in degree n. This generator acts as $\frac{\partial}{\partial x}$ on $H^*(S^n;\mathbb{Q}) = \Lambda x$. The characteristic classes of ξ are zero, because these live in even degrees. Writing $\pi_*(G)\otimes\mathbb{Q} = \langle \gamma_1,\ldots,\gamma_k,\nu_1,\ldots,\nu_\ell\rangle$, where γ_i is dual to u_i and ν_j to v_j, the dg Lie algebra model for $Baut_\circ(\xi)$ of Theorem 1.2 assumes the form

$$\langle \gamma_1,\ldots,\gamma_k,\nu_1,\ldots,\nu_\ell, x\nu_1,\ldots,x\nu_\ell, \frac{\partial}{\partial x}\rangle,$$

where the only non-zero Lie brackets are

$$[\frac{\partial}{\partial x}, x\nu_i] = \nu_i, \quad i = 1,2,\ldots,\ell,$$

and the differential is zero. The cohomology $H^*(Baut_\circ(\xi);\mathbb{Q})$ may be computed as the cohomology of the Chevalley-Eilenberg cochain complex, which has the form

$$\Lambda(u_1,\ldots,u_k,v_1,\ldots,v_\ell,w_1,\ldots,w_\ell,z),$$

where $|w_i| = |v_i| - n$, $|z| = n+1$, and the only non-trivial differentials are given by $dv_i = w_i z$. This cdga may be written as

$$\mathbb{Q}[u_1,\ldots,u_k]\otimes \Omega^*_{\mathbb{Q}[v_1,\ldots,v_\ell]|\mathbb{Q}}[z],$$

where

$$\Omega^*_{\mathbb{Q}[v_1,\ldots,v_\ell]|\mathbb{Q}} = \big(\mathbb{Q}[v_1,\ldots,v_\ell]\otimes\Lambda(w_1,\ldots,w_\ell),\ dv_i = w_i\big)$$

is the complex of Kähler differentials on $\mathbb{Q}[v_1,\ldots,v_\ell]$ and where $\Omega[z]$, for a cdga Ω, denotes the cdga $\Omega\otimes\mathbb{Q}[z]$ with differential zd. There is an exact sequence

$$0 \to d\Omega \to H^*(\Omega[z]) \to H^*(\Omega)[z] \to 0,$$

from which one deduces

$$H^*(\Omega[z]) \cong \mathbb{Q}[z]\oplus d\Omega,$$

when $H^*(\Omega)\cong\mathbb{Q}$. $\square$

Remark 4.9. As is well-known, the exact Kähler differentials agree with the negative cyclic homology of $\mathbb{Q}[v_1,\ldots,v_\ell]$ (appropriately regraded), and the result may be written

$$H^*(Baut_\circ(\xi);\mathbb{Q}) \cong \mathbb{Q}[u_1,\ldots,u_k]\otimes HC^-_*(\mathbb{Q}[v_1,\ldots,v_\ell]).$$

The appearance of cyclic homology in this example can be explained as follows: noting that $aut_\circ(S^n) \sim_{\mathbb{Q}} S^n$ for n odd, the fibration

$$Baut_{S^n}(\xi) \to Baut_\circ(\xi) \to Baut_\circ(S^n)$$

is similar to the fibration

$$LBG \to ES^1 \times_{S^1} LBG \to BS^1$$

associated to the S^1-action on the free loop space of BG.

Remark 4.10. It is easy to construct examples of bundles for which the fibration (2) does not split rationally even after looping. For example, consider the complex vector bundle ξ over $S^3 \times S^3$ classified by the composite

$$S^3 \times S^3 \to S^6 \to BU(3),$$

where the first map is the collapse map and the second a representative for a generator of $\pi_6 BU(3) = \pi_5 U(3) = \mathbb{Z}$. The Chern classes of ξ are $c_1(\xi) = 0$, $c_2(\xi) = 0$ and $c_3(\xi)$ is a generator for $H^6(S^3 \times S^3)$.
The minimal Quillen model for $S^3 \times S^3$ is given by $(\mathbb{L}(\alpha, \beta, \gamma), \delta)$ where $|\alpha| = |\beta| = 2$, $|\gamma| = 5$ and $\delta(\gamma) = [\alpha, \beta]$. A calculation shows that the projection

$$\big(\operatorname{Der} L \ltimes_{\mathrm{ad}} sL\big)\langle 1\rangle \to sL/[L,L] = \langle s\alpha, s\beta\rangle$$

is a quasi-isomorphism of dg Lie algebras, where the right hand side is given the trivial differential and Lie bracket. If $s\alpha$ and $s\beta$ are made to act on $H^*(S^3 \times S^3; \mathbb{Q}) = \Lambda(x, y)$ by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively, we obtain a quasi-isomorphism of dg Lie algebras from the model in Theorem 1.2 to

$$\mathfrak{g} = \big(H^*(S^3 \times S^3; \mathbb{Q}) \otimes \pi_* U(3)\big)\langle 0\rangle \rtimes_{\rho_*} \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle.$$

Writing $\pi_* U(3) \otimes \mathbb{Q} = \langle \gamma_1, \gamma_2, \gamma_3\rangle$, where γ_i is dual to c_i, we get the following explicit description.

$$\mathfrak{g} = \langle \gamma_1, \gamma_2, \gamma_3, x\gamma_2, y\gamma_2, x\gamma_3, y\gamma_3, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle,\ D.$$

The differential D is governed by $\rho = xy\gamma_3$. The only non-trivial differentials are

$$D(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}(xy\gamma_3) = y\gamma_3,$$
$$D(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}(xy\gamma_3) = -x\gamma_3.$$

In particular, this shows that the map $\pi_* aut_X(\xi) \otimes \mathbb{Q} \to \pi_* aut_\circ(\xi) \otimes \mathbb{Q}$ is not injective in this example.
The cohomology of be computed explicitly from the Chevalley-Eilenberg cochain complex. We omit the details of the computation. The result is

$$H^*(Baut_\circ(\xi); \mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, c_3] \otimes \Lambda(\alpha, \beta),$$

for certain classes α, β of degree 1.

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