

THE CATEGORY OF FINITELY PRESENTED SMOOTH
MOD p REPRESENTATIONS OF $GL_2(F)$

JACK SHOTTON

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ABSTRACT. Let F be a finite extension of \mathbb{Q}_p . We prove that the category of finitely presented smooth Z -finite representations of $GL_2(F)$ over a finite extension of \mathbb{F}_p is an abelian subcategory of the category of all smooth representations. The proof uses amalgamated products of completed group rings.

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1 INTRODUCTION

Let \mathbb{F} be a finite field of characteristic p . If G is a locally profinite topological group, let $\mathcal{C}_{\mathbb{F}}(G)$ be the category of smooth representations of G over \mathbb{F} . Throughout this paper, if K is an open subgroup of such a group G then ind_K^G denotes induction with compact support modulo K .

DEFINITION 1.1. Let V be a smooth \mathbb{F} -representation of a locally profinite group G . Then V is:

1. FINITELY GENERATED if for some compact open subgroup K of G there is a surjection of $\mathbb{F}[G]$ -modules

$$\text{ind}_K^G W \rightarrow V$$

for a smooth finite-dimensional \mathbb{F} -representation W of K ;

2. FINITELY PRESENTED if for some compact open subgroups K_1, K_2 of G there is an exact sequence

$$\mathrm{ind}_{K_1}^G W_1 \rightarrow \mathrm{ind}_{K_2}^G W_2 \rightarrow V \rightarrow 0$$

for W_1 and W_2 smooth finite-dimensional \mathbb{F} -representations of K_1 and K_2 respectively.

Let F be a finite extension of \mathbb{Q}_p . The purpose of this article is to prove:

THEOREM 1.2. *The category of finitely presented smooth \mathbb{F} -representations of $SL_2(F)$ is an abelian subcategory of $\mathcal{C}_{\mathbb{F}}(SL_2(F))$.*

The same holds for the category of finitely presented smooth Z -finite representations of $GL_2(F)$.

This is Theorem 5.1 and Corollary 5.2 below. In fact, we prove the same result with F replaced by any finite dimensional division algebra over \mathbb{Q}_p .

The theorem is equivalent to the statement that the kernel¹ of any map between finitely presented smooth representations is itself finitely presented. If $\mathcal{C}_{\mathbb{F}}(SL_2(F))$ were the category of modules over a ring R , this would be the statement that R is a coherent ring. Indeed, we will prove the theorem by considering smooth \mathbb{F} -representations as modules over the amalgamated product

$$\mathbb{F}[[K]] *_{\mathbb{F}[[I]]} \mathbb{F}[[K']],$$

where $K = SL_2(\mathcal{O}_F)$, $K' = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix}$ for π a uniformising element of D , and $I = K \cap K'$. Then a result of Åberg [Å82] shows that, under certain conditions, an amalgamated product of coherent rings over a noetherian ring is itself coherent. *Throughout, unless otherwise stated, by ‘module’, ‘noetherian’ or ‘coherent’ we mean ‘left module’, ‘left noetherian’ or ‘left coherent’.*

Finitely presented representations of $GL_2(F)$ were previously studied by Hu [Hu12], Vigneras [Vig11], and Schraen [Sch15].² In particular, [Vig11] Theorem 6 shows that a smooth *admissible* finitely presented representation of $GL_2(F)$ has finite length, and that all of its subquotients are also admissible and finitely presented. On the other hand, the main result of [Sch15] says that, if F is a quadratic extension of \mathbb{Q}_p , then an irreducible supersingular representation of $GL_2(F)$ admitting a central character is never finitely presented.

We are motivated by the construction (see [CEG⁺16]) of a ‘patched module’ M_{∞} that has an action of $G = GL_n(F)$ and, hopefully, interpolates the hypothetical p -adic Langlands correspondence. It is (only?) possible to directly obtain information about M_{∞} by considering $\mathrm{Hom}_{GL_n(F)}(\mathrm{ind}_K^G(W), M_{\infty}^{\vee})$ for

¹and the cokernel, but this is automatic

²The definition of ‘finitely presented’ in these articles is slightly different to ours, and automatically entails Z -finiteness.

locally algebraic representations of $K = GL_n(\mathcal{O}_F)$ on finitely generated \mathbb{Z}_p -modules W . This leads us to consider the category of finitely presented representations of G ; it also motivates us to prove a version of Theorem 1.2 with coefficients.

I do not know whether Theorem 1.2 holds when $G = GL_n(F)$ (or any p -adic Lie group). The method of this paper does not apply, because G is not (up to centre) an amalgam of two compact open subgroups. I am not sure whether Theorem 1.2 holds when F has positive characteristic; the method of this paper fails because $GL_2(\mathcal{O}_F)$ is not p -adic analytic and its completed group ring is not noetherian. I thank Billy Woods for a helpful discussion about this case.

I am grateful to Matthew Emerton for asking me the question that this paper answers, and for several helpful and motivational conversations. I also thank Julien Hauseux and Stefano Morra for comments and corrections. I am indebted to the anonymous referee for suggesting that I relate the amalgamated product of rings considered here to the ring $\Lambda(G)$ considered in [Koh17], which greatly clarified and simplified the arguments of this paper.

2 FINITELY PRESENTED REPRESENTATIONS

For the rest of this article, let \mathbb{F} be a finite field of characteristic p . Let A be a complete local noetherian $W(\mathbb{F})$ -algebra with maximal ideal \mathfrak{m} and residue field \mathbb{F} . Let G be a locally profinite group. Recall ([Eme10] definition 2.2.5) that a *smooth* A -representation of G is a representation of G on a torsion A -module V such that every $v \in V$ is fixed by a compact open subgroup of G .

DEFINITION 2.1. If K is a profinite group, then a *finite rank* A -representation of K is a representation of K on a finitely generated A -module M such that, for every $n \geq 0$, $M/\mathfrak{m}^n M$ is a smooth representation of K .

Strictly speaking, we should call these finite rank continuous A -representations of K .

DEFINITION 2.2. A representation of G on an A -module V is *K -finite* if for some (equivalently, any) compact open subgroup $K \subset G$, and for every $v \in V$, the $A[K]$ -module generated by v is a finite rank A -representation of K .

We let $\mathcal{C}_A^{K\text{-fin}}(G)$ be the category of all K -finite A -representations of G , with morphisms being morphisms of $A[G]$ -modules. Note that a representation of G on a *torsion* A -module V is smooth if and only if it is K -finite.

In the introduction (Definition 1.1) we gave the definitions of ‘finitely generated’ and ‘finitely presented’ smooth \mathbb{F} -representations of G . We now extend those to K -finite A -representations. First, note that if M is a finite rank A -representation of a compact open subgroup $K \subset G$, then $\text{ind}_K^G M$ is certainly K -finite.

DEFINITION 2.3. Let V be a K -finite A -representation of G . Then V is:

1. *finitely generated* if there is a compact open subgroup $K \subset G$, a finite rank A -representation W of K , and a surjection of $A[G]$ -modules

$$\mathrm{ind}_K^G(W) \twoheadrightarrow V;$$

2. *finitely presented* if for some compact open subgroups K_1, K_2 of G there is an exact sequence of $A[G]$ -modules

$$\mathrm{ind}_{K_1}^G W_1 \rightarrow \mathrm{ind}_{K_2}^G W_2 \rightarrow V \rightarrow 0$$

for W_1 and W_2 finite rank A -representations of K_1 and K_2 .

We start by establish some straightforward properties of finitely presented K -finite representations. Many of the proofs follow those of the properties of finitely presented modules over a ring given in [Sta17, Tag 0519].

LEMMA 2.4. *A K -finite A -representation V of G is finitely generated if and only if it is finitely generated as an $A[G]$ -module.*

Proof. For any W and K , $\mathrm{ind}_K^G W$ is generated (as an $A[G]$ -module) by the finitely generated A -submodule of functions supported on K . The ‘only if’ direction follows.

For the ‘if’ direction, let V be a K -finite representation generated by v_1, \dots, v_n as an $A[G]$ -module. Choose a compact open subgroup K and let W be the finite rank A -representation of K generated by v_1, \dots, v_n . Then V is a quotient of $\mathrm{ind}_K^G W$. \square

REMARK 2.5. It is not true that a finitely presented K -finite A -representation of G will be finitely presented as an $A[G]$ -module; this is already false for the \mathbb{F} -representation $\mathrm{ind}_K^G \mathbb{F}$, as long as K is not finitely generated. This is the main technical problem that we have to overcome in the next section.

LEMMA 2.6. *Suppose that $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ is a short exact sequence of K -finite A -representations of G .*

If V_1 and V_3 are finitely generated, so is V_2 .

Proof. This is immediate from Lemma 2.4 and the fact that an extension of finitely generated modules over $A[G]$ is finitely generated. \square

LEMMA 2.7. *Suppose that $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ is a short exact sequence of K -finite A -representations of G .*

1. *If V_2 is finitely presented and V_1 is finitely generated, then V_3 is finitely presented.*
2. *If V_3 is finitely presented and V_2 is finitely generated, then V_1 is finitely generated.*
3. *If V_1 and V_3 are finitely presented, so is V_2 .*

Proof. We use K and L, M, N to denote a suitably chosen compact open subgroup of G and finite rank A -representations of K .

1. Choose a presentation $\text{ind}_K^G N \xrightarrow{\alpha} \text{ind}_K^G M \rightarrow V_2 \rightarrow 0$ and choose v_1, \dots, v_r generating the image of V_1 in V_2 as an $A[G]$ -module. For each i , let \tilde{v}_i be a lift of v_i to $\text{ind}_K^G M$, and let L be the finite rank A -representation of K generated by the \tilde{v}_i . Then we have a map $\gamma : \text{ind}_K^G L \rightarrow \text{ind}_K^G M$, and the kernel of the (surjective) composition $\text{ind}_K^G M \rightarrow V_2 \rightarrow V_3$ is the sum of the image of α and the image of γ , and so is finitely generated.
2. Choose a presentation $\text{ind}_K^G N \rightarrow \text{ind}_K^G M \xrightarrow{\alpha} V_3 \rightarrow 0$. We may replace M by its image in V_3 , so that we have $M \subset V_3$ and $\text{ind}_K^G M \rightarrow V_3$ is the natural map. Let m_1, \dots, m_r generate $M \subset V_3$ as an A -module, and for each i let $\tilde{m}_i \in V_2$ be a lift of m_i . Let \tilde{M} be the $A[K]$ -span of the \tilde{m}_i in V_2 . Then there is a surjective map of K representations $\tilde{M} \rightarrow M$, and we let L be the kernel. There is also a map $\beta : \text{ind}_K^G \tilde{M} \rightarrow V_2$ giving a commuting diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & \text{ind}_K^G N & & \\
 & & & & \downarrow & & \\
 \text{ind}_K^G L & \longrightarrow & \text{ind}_K^G \tilde{M} & \longrightarrow & \text{ind}_K^G M & \longrightarrow & 0 \\
 & & \beta \downarrow & & \downarrow \alpha & & \\
 & & V_2 & \longrightarrow & V_3 & \longrightarrow & 0.
 \end{array}$$

Repeating the same argument, we may replace N by an $A[K]$ -submodule of $\text{ind}_K^G \tilde{M}$ and find a K -submodule $\tilde{N} \subset \text{ind}_K^G \tilde{M}$, together with a surjection $\tilde{N} \rightarrow N$ of $A[K]$ -modules, such that

$$\begin{array}{ccc}
 \text{ind}_K^G \tilde{N} & \longrightarrow & \text{ind}_K^G N \\
 \downarrow & & \downarrow \\
 \text{ind}_K^G \tilde{M} & \longrightarrow & \text{ind}_K^G M.
 \end{array}$$

commutes and has surjective horizontal maps. The kernel of $\text{ind}_K^G \tilde{M} \rightarrow V_3$ is the image of $\text{ind}_K^G(\tilde{N} \oplus L)$. Write γ for the restriction of β to $\text{ind}_K^G(\tilde{N} \oplus L)$. We obtain a commutative diagram

$$\begin{array}{ccccccc}
 \text{ind}_K^G(\tilde{N} \oplus L) & \longrightarrow & \text{ind}_K^G \tilde{M} & \longrightarrow & V_3 & \longrightarrow & 0 \\
 \gamma \downarrow & & \beta \downarrow & & \parallel & & \\
 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 \longrightarrow 0
 \end{array}$$

with exact rows, from which we see that $\text{cok}(\gamma) \cong \text{cok}(\beta)$. As V_2 is finitely generated, so is $\text{cok}(\beta)$ and hence also $\text{cok}(\gamma)$. Since $\text{im}(\gamma)$ is also finitely generated, we see that V_1 is finitely generated by Lemma 2.6.

- Choose surjections $\alpha : \text{ind}_K^G M \rightarrow V_1$ and $\beta : \text{ind}_K^G N \rightarrow V_3$. As before, we may assume that $N \subset V_3$. Let n_1, \dots, n_r generate N as an A -module, lift them to $\tilde{n}_i \in V_2$, and let \tilde{N} be the $A[K]$ -module generated by the \tilde{n}_i . Let γ be the resulting map $\text{ind}_K^G \tilde{N} \rightarrow V_2$. If we let $L = \ker(\tilde{N} \rightarrow N)$, then γ restricts to a map $\gamma' : \text{ind}_K^G L \rightarrow V_1$. We obtain a commuting diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ind}_K^G(M \oplus L) & \longrightarrow & \text{ind}_K^G(M \oplus \tilde{N}) & \longrightarrow & \text{ind}_K^G N \longrightarrow 0 \\
 & & \alpha + \gamma' \downarrow & & \alpha + \gamma \downarrow & & \downarrow \beta \\
 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 \longrightarrow 0
 \end{array}$$

with exact rows and surjective vertical maps. By the snake lemma there is a short exact sequence

$$0 \rightarrow \ker(\alpha + \gamma') \rightarrow \ker(\alpha + \gamma) \rightarrow \ker(\beta) \rightarrow 0.$$

Since the outer two terms are finitely generated by (2), so is the inner term (by Lemma 2.6). Thus V_2 is finitely presented, as required. \square

LEMMA 2.8. *Suppose that $G' \subset G$ is a finite index open subgroup. Then a K -finite A -representation V of G is finitely generated/presented if and only if its restriction to G' is.*

Proof. 1. If V is finitely generated as a representation of G' then it certainly is as a representation of G . Conversely, for any compact open subgroup K of G and any finite rank A -representation W of K , we have the Mackey formula

$$\text{res}_{G'}^G \text{ind}_K^G W \cong \bigoplus_{g \in G' \backslash G/K} \text{ind}_{gKg^{-1} \cap G'}^{G'} W^g.$$

So $\text{ind}_K^G W$ is finitely generated — in fact finitely presented — as a representation of G' . It follows that any finitely generated representation of G is finitely generated as a representation of G' .

- We showed in (1) that $\text{ind}_K^G W$ is finitely presented as a representation of G' for any finite rank A -representation W of a compact open subgroup K . It follows from Lemma 2.7 (1) that any K -finite finitely presented representation of G is finitely presented as a representation of G' .

Conversely, suppose that V is finitely presented as a representation of G' . By the first part, it is finitely generated as a representation of G , so that there is a surjection $\text{ind}_K^G W \rightarrow V$. Since the first term is finitely generated as a representation of G' by part (1), by Lemma 2.7 (2) the kernel is finitely generated as a representation of G' , and hence also as a representation of G . Therefore V is finitely presented as a representation of G by Lemma 2.7 (1). \square

2.1 Z -FINITENESS

Suppose that G is a locally profinite group with centre Z . We say that *Hypothesis Z is satisfied* if, for some (equivalently, any) compact open subgroup K of G , $Z/K \cap Z$ is finitely generated. Recall from [Eme10] the definitions of Z -finite and locally Z -finite representations: a representation is Z -finite if the action of $A[Z]$ on V factors through a quotient $A[Z]/I$ that is a finitely generated A -module. It is locally Z -finite if the $A[Z]$ -module spanned by any $v \in V$ is a finitely generated A -module. By [Eme10] Lemma 2.3.3, a representation of G , finitely generated as an $A[G]$ -module, is Z -finite if and only if it is locally Z -finite.

LEMMA 2.9. *Let V be a locally Z -finite, K -finite, A -representation of G .*

1. *The representation V is finitely generated if and only if there is a surjection*

$$\text{ind}_{KZ}^G W \rightarrow V \rightarrow 0$$

for some compact open subgroup K of G and finite rank A -representation W of KZ .

2. *If the representation V is finitely presented then there is an exact sequence*

$$\text{ind}_{K_1Z}^G W_1 \rightarrow \text{ind}_{K_2Z}^G W_2 \rightarrow V \rightarrow 0$$

for some compact open subgroup K of G and finite rank A -representations W_1 and W_2 of K_1Z and K_2Z . If Hypothesis Z is satisfied, the converse holds.

Proof. 1. The backwards implication is clear. For the forwards implication, let W be the $A[KZ]$ -span of a finite set of generators of V . It is finite-rank since V is K -finite and locally Z -finite. We therefore get a surjection $\text{ind}_{KZ}^G W \rightarrow V \rightarrow 0$ as required.

2. Suppose that V is finitely presented. Then there is a surjection $\text{ind}_{KZ}^G W_2 \rightarrow V \rightarrow 0$, by the first part. The kernel is finitely generated by Lemma 2.7 (2), and $\text{ind}_{KZ}^G W_2$ is Z -finite. Applying the first part again, we get an exact sequence $\text{ind}_{KZ}^G W_1 \rightarrow \text{ind}_{KZ}^G W_2 \rightarrow V \rightarrow 0$ as required.

For the other direction, it is enough to show that (under Hypothesis Z) $\text{ind}_{KZ}^G W_2$ is finitely presented for any representation W_2 of KZ on a finitely generated A -module. If U is the kernel of the natural map $\text{ind}_K^{KZ} W_2 \rightarrow W_2$ then there is a short exact sequence

$$0 \rightarrow \text{ind}_{KZ}^G U \rightarrow \text{ind}_K^G W_2 \rightarrow \text{ind}_{KZ}^G W_2 \rightarrow 0.$$

We have to show that U is finitely generated as a KZ -representation. This follows from Hypothesis Z , since this implies that $A[KZ/K]$ is a noetherian ring. \square

Now suppose that H is an open subgroup of G such that HZ has finite index in G and $Z \cap H$ is compact.

PROPOSITION 2.10. *Let V be a locally Z -finite, K -finite, A -representation of G .*

1. *The representation V of G is finitely generated if and only if its restriction to H is finitely generated.*
2. *If the representation V of G is finitely presented then its restriction to H is finitely presented. If Hypothesis Z holds, then the converse is true.*

Proof. By Lemma 2.8 we may assume that $G = HZ$. Let V be a locally Z -finite K -finite representation of G .

1. If V is finitely generated as an representation of H , it certainly is as a representation of G . Conversely, suppose that V is finitely generated as an representation of G . If $W \subset V$ is a finitely generated A -module that generates V as an representation of G , then the Z -span ZW is a finitely generated A -module that generates V as a representation of H . So V is a finitely generated representation of H as required.
2. Suppose that V is finitely presented as a representation of G . By Lemma 2.9 (2) and Lemma 2.7 (1), it suffices to show that $\text{ind}_{KZ}^G W$ is a finitely presented representation of H for $K \subset H$. This follows from the identity of representations of H

$$\text{ind}_{KZ}^{HZ} W = \text{ind}_{K(Z \cap H)}^H W$$

and the assumption that $Z \cap H$ is compact.

Finally, suppose that V is finitely presented as an representation of H and that Hypothesis Z holds. Then V is finitely generated as a representation of G , so by Lemma 2.9 (1) there is a surjection $\text{ind}_{KZ}^G W \rightarrow V$. By (1) and Lemma 2.7 (2) the kernel of this map is a finitely generated representation of H . By (1) again, it is a finitely generated representation of G , and so by Lemma 2.9 (1) we have an exact sequence

$$\text{ind}_{KZ}^G U \rightarrow \text{ind}_{KZ}^G W \rightarrow V \rightarrow 0.$$

As G satisfies Hypothesis Z , by the converse direction of Lemma 2.9 (2), V is a finitely presented representation of G . \square

3 COMPLETED GROUP RINGS

If K is a profinite group, let

$$A[[K]] = \varprojlim_{J \triangleleft K_{\text{open}}} A[K/J]$$

be the completed group ring, a compact topological A -algebra.

LEMMA 3.1. *Suppose that M is a finite rank A -representation of K . Then there is a unique $A[[K]]$ -module structure on M extending the $A[K]$ -module structure.*

Proof. For each n , the action of $A[K]$ on $M \otimes_A A/\mathfrak{m}_A^n$ factors through $A[K/J_n]$ for some open subgroup $J_n \subset K$ and so extends uniquely to an action of $A[[K]]$. Since M is finitely generated as an A -module, $M = \varprojlim M \otimes_A A/\mathfrak{m}_A^n$ and the lemma follows. \square

Kohlhaase [Koh17] has extended the notion of completed group ring beyond the compact case. Let G be a locally profinite group.

PROPOSITION 3.2 (Kohlhaase). *If $K \subset G$ is a compact open subgroup, then there is a unique A -algebra structure on*

$$A \langle G \rangle = A[G] \otimes_{A[K]} A[[K]]$$

such that the natural maps $A[G] \rightarrow A \langle G \rangle$ and $A[[K]] \rightarrow A \langle G \rangle$ are A -algebra homomorphisms. This A -algebra is independent of the choice of K up to canonical isomorphism.

Proof. This is shown in section 1 of [Koh17] when A is a field — where what we call $A \langle G \rangle$ is denoted $\Lambda(G)$ — but the proof works verbatim for general rings A . We recall the construction for the reader’s convenience. Firstly, if K' is an open subgroup of K , then the natural map of $(A[G], A[[K']])$ -bimodules

$$\rho_{K,K'} : A[G] \otimes_{A[K']} A[[K']] \rightarrow A[G] \otimes_{A[K]} A[[K]]$$

is an isomorphism.³ If $K'' \subset K'$ then we have $\rho_{K,K''} = \rho_{K',K''} \circ \rho_{K,K'}$, and so we may construct the direct limit

$$A \langle G \rangle = \varinjlim_K (A[G] \otimes_{A[K]} A[[K]])$$

which is (canonically) isomorphic to any one of its terms. Now, if $g \in G$ then there is an isomorphism of direct systems

$$\cdot g : A[G] \otimes_{A[K]} A[[K]] \rightarrow A[G] \otimes_{A[g^{-1}Kg]} A[[g^{-1}Kg]]$$

taking $h \otimes \kappa$ to $hg \otimes g^{-1}\kappa g$. This defines a right action of G on $A \langle G \rangle$ by left $A[G]$ -module isomorphisms, which suffices to define the required ring structure on $A \langle G \rangle$. Precisely, if $h \otimes \kappa \in A[G] \otimes_{A[K]} A[[K]]$ and $h' \otimes \kappa' \in A[G] \otimes_{A[K']} A[[K']]$ are representatives of elements of $A \langle G \rangle$, we may assume that $K \subset h^{-1}K'h$ and define

$$(h' \otimes \kappa')(h \otimes \kappa) = h'h \otimes h^{-1}\kappa'h\kappa \in A[G] \otimes_{A[h^{-1}K'h]} A[[h^{-1}K'h]]. \quad \square$$

³In [Koh17] this is stated for K' normal in K , but it is true for any K' and moreover this is necessary for the construction of the ring structure.

For later use, we record a flatness result:

LEMMA 3.3. *The A -algebra $A\langle G \rangle$ is flat as a right $A[[K]]$ -module for any compact open subgroup K of G .*

Proof. As in [Koh17], $A\langle G \rangle = A[G] \otimes_{A[K]} A[[K]] \cong \bigoplus_{h \in G/K} A[[K]]$ as right $A[[K]]$ -modules, so that $A\langle G \rangle$ is even a free right $A[[K]]$ -module. \square

REMARK 3.4. In the same way we could put an A -algebra structure on $A[[K]] \otimes_{A[K]} A[G]$ (for any compact open subgroup K) and the A -module map $A[[K]] \otimes_{A[K]} A[G] \rightarrow A\langle G \rangle$ defined by

$$\kappa \otimes h \mapsto h \otimes h^{-1}\kappa h \in A[G] \otimes_{A[h^{-1}Kh]} A[[h^{-1}Kh]]$$

is an isomorphism of A -algebras. Thus Lemma 3.3 holds with ‘right’ replaced by ‘left’.

LEMMA 3.5. *Suppose that V is a K -finite A -representation of G . Then there is a unique $A\langle G \rangle$ -module structure on V extending the $A[G]$ -module structure.*

Proof. Since V is K -finite, for any compact open subgroup K the action of $A[K]$ extends uniquely to an action of $A[[K]]$ by Lemma 3.1. By the unicity, we have that, for any $h \in G$ and $\kappa \in A[[K]]$, the two actions of $h^{-1}\kappa h$ defined on the one hand by the actions of G and $A[[K]]$, and on the other hand by the action of $A[[h^{-1}Kh]]$, agree. From the formula for multiplication in $A\langle G \rangle$ given in Proposition 3.2, it follows that we can define an action of $A\langle G \rangle$ on V by fixing K and setting

$$(h \otimes \kappa)(v) = h(\kappa(v))$$

for any $h \in A[G]$ and $\kappa \in A[[K]]$, which is clearly the unique action extending those of $A[G]$ and $A[[K]]$. \square

LEMMA 3.6. *Suppose that V is a K -finite A -representation of G . Then V is finitely generated if and only if it is finitely generated as an $A\langle G \rangle$ -module.*

Proof. Suppose that V is finitely generated. By Lemma 2.4, V is finitely generated as an $A[G]$ -module, and hence as a $A\langle G \rangle$ -module.

Conversely, let V be a K -finite A -representation of G that is finitely generated as a $A\langle G \rangle$ -module. Then, if v_1, \dots, v_r generate V and if M is their $A[K]$ -span, then M is also preserved by $A[[K]]$ and so

$$V = A\langle G \rangle \cdot M = (A[G] \otimes_{A[K]} A[[K]])M = A[G]M.$$

Therefore V is finitely generated, as required. \square

The key technical reason for us to introduce the ring $A\langle G \rangle$ is that it is true that a finitely presented K -finite A -representation of G is a finitely presented $A\langle G \rangle$ -module — see Remark 2.5. The starting point is the following result of Lazard (see [Eme10] Theorem 2.1.1).

THEOREM 3.7. *If G is a p -adic analytic group, then $A[[K]]$ is noetherian for every compact open subgroup K of G . \square*

PROPOSITION 3.8. *Suppose that G is a p -adic analytic group. Let V be a K -finite A -representation of G . Then V is finitely presented if and only if it is finitely presented as an $A\langle G \rangle$ -module.*

Proof. The backwards implication follows from Lemma 3.6. Suppose that V is finitely presented as an $A\langle G \rangle$ -module. Then by Lemma 3.6 there is a surjection

$$\alpha : \text{ind}_K^G W \rightarrow V \rightarrow 0$$

for some finite rank A -representation W of a compact open subgroup $K \subset G$. The kernel of α is a K -finite representation of G that is finitely generated as an $A\langle G \rangle$ -module, by [Sta17, Tag 0519] (5).⁴ Therefore it is finitely generated as an A -representation of G , by Lemma 3.6.

Suppose now that V is finitely presented. Then by Lemma 2.7 (2), there is a compact open subgroup K , a finite rank A -representation M of K , and a surjection $\text{ind}_K^G M \rightarrow V \rightarrow 0$ with finitely generated kernel.

By [Sta17, Tag 0519] (4) and Lemma 3.6, it is enough to show that $\text{ind}_K^G(M)$ is a finitely presented $A\langle G \rangle$ -module for the $A\langle G \rangle$ -module structure provided by Lemma 3.5. We may think of this instead as the tensor product

$$\text{ind}_K^G(M) \cong A[G] \otimes_{A[K]} M$$

via the isomorphism sending an element $f : G \rightarrow M$ of $\text{ind}_K^G(M)$ to $\sum_{g \in G/K} g \otimes f(g^{-1})$. By Lemma 3.1 the action of $A[K]$ on M extends uniquely to one of $A[[K]]$ and we have isomorphisms

$$A\langle G \rangle \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} M$$

of $A[G]$ -modules, and hence of $A\langle G \rangle$ -modules (by Lemma 3.5).

Since $A[[K]]$ is noetherian by Theorem 3.7, the finitely generated $A[[K]]$ -module M is finitely presented; let $A[[K]]^m \rightarrow A[[K]]^n \rightarrow M \rightarrow 0$ be a presentation. Applying $A\langle G \rangle \otimes_{A[[K]]} -$, we obtain an exact sequence

$$A\langle G \rangle^n \rightarrow A\langle G \rangle^m \rightarrow A\langle G \rangle \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} M \rightarrow 0$$

so that $\text{ind}_K^G M = A[G] \otimes_{A[K]} M$ is a finitely presented $A\langle G \rangle$ -module, as required. \square

4 AMALGAMATIONS AND COHERENCE

Let K_1, K_2 and I be profinite groups equipped with inclusions $f_i : I \hookrightarrow K_i$ of I as a common open subgroup of K_1 and K_2 . Then there are maps $f_i : A[[I]] \rightarrow A[[K_i]]$ of topological augmented A -algebras.

⁴Strictly speaking, [Sta17, Tag 0519] is only stated for modules over commutative rings. However, it is still true, with an identical proof, in the non-commutative case.

Let $H = K_1 *_I K_2$ be the amalgamation of K_1 and K_2 along I . By [Ser77], Théorème 1, the natural map $I \rightarrow H$ is injective. The following proposition shows that H is naturally a locally profinite topological group:

PROPOSITION 4.1. *With the colimit topology,⁵ H is a locally profinite group with a basis of open neighbourhoods of the identity being given by open neighbourhoods of I .*

Proof. Let H and H' respectively denote H with the colimit topology and the topology for which translates of open subgroups of I are a basis of open sets. Let $i : H \rightarrow H'$ and $j : H' \rightarrow H$ be the identity maps; we have to show that they are both continuous. But i is continuous by the universal property of H , and j is continuous because the map $I \rightarrow H$ is continuous. \square

We now consider the amalgamated product of rings, $A[[K_1]] *_A[[I]] A[[K_2]]$. Note first that $A[K_1] *_A[[I]] A[K_2]$ is simply the group ring of H over A . This is because the functor $G \mapsto A[G]$ from groups to A -algebras is a left-adjoint, and so commutes with the colimit $*$.

In general, we have A -algebra maps $A[[K_1]] \rightarrow A\langle H \rangle$ and $A[[K_2]] \rightarrow A\langle H \rangle$ which agree on $A[[I]]$, and so (by the universal property) an A -algebra map $\alpha : A[[K_1]] *_A[[I]] A[[K_2]] \rightarrow A\langle H \rangle$.

PROPOSITION 4.2. *The map*

$$\alpha : A[[K_1]] *_A[[I]] A[[K_2]] \rightarrow A\langle H \rangle$$

is an isomorphism of A -algebras.

Proof. Let $R = A[[K_1]] *_A[[I]] A[[K_2]]$. The composite map

$$A[H] = A[K_1] *_A[[I]] A[K_2] \rightarrow R \rightarrow A\langle H \rangle$$

is easily seen to be the natural map $A[H] \rightarrow A\langle H \rangle$. It follows that the image of R in $A\langle H \rangle$ contains $A[H]$ and $A[[K_i]]$ and so in fact is all of $A\langle H \rangle$, whence α is surjective.

Moreover, from the universal property of \otimes , for each i we have a map of $(A[H], A[[K_i]])$ -bimodules

$$A[H] \otimes_{A[[K_i]]} A[[K_i]] \rightarrow R$$

and these define the *same* map $\beta : A\langle H \rangle \rightarrow R$. Since this is a map of right $A[[K_1]]$ - and $A[[K_2]]$ -modules, we see that $\beta \circ \alpha$ is the identity — it is enough to check that it takes 1 to 1. Therefore α is injective and so an isomorphism. \square

⁵The coarsest topology on H such that for every topological group G equipped with continuous maps $K_i \rightarrow G$ agreeing on I , there is a continuous map $H \rightarrow G$ extending these.

4.1 COHERENCE

Recall that a ring R is (left) coherent if any of the following equivalent definitions hold:

1. every finitely generated left ideal of R is finitely presented;
2. if $f : M \rightarrow N$ is a map of finitely presented left R -modules, then $\ker(f)$ is finitely presented;
3. the category of finitely presented left R -modules is an abelian subcategory of the category of left R -modules.

PROPOSITION 4.3. *If the rings $A[[K_i]]$ are coherent and $A[[I]]$ is noetherian, then $A\langle H \rangle$ is coherent.*

Proof. This follows immediately from [Å82] Theorem 12; the hypotheses of that theorem are satisfied, by Lemma 3.3. For the convenience of the reader, we summarise the argument of [Å82] in the case of interest to us. It uses the characterisation — due to Chase [Cha60] — of left coherent rings as those for which arbitrary products of right flat modules are flat. Let R , S and T be rings such that S and T are R -algebras, and $Q = S *_R T$ is flat as a right R , S or T -module; we will take $R = A[[I]]$ and $S = A[[K_1]]$, $T = A[[K_2]]$. Then there is a Mayer–Vietoris sequence for Tor^Q in terms of Tor^S , Tor^R and Tor^T . If R is left noetherian and S and T are left coherent, then take a set $(F_i)_{i \in I}$ of right flat Q -modules and compare the Mayer–Vietoris sequence for $\mathrm{Tor}(\prod F_i, M)$ with the product of those for $\mathrm{Tor}(F_i, M)$, for an arbitrary left Q -module M . This gives $\mathrm{Tor}_i^Q(\prod F_i, M) = 0$ for $i > 1$. Since S and T are left coherent and that, as R is left noetherian and the F_i are right flat R -modules, $(\prod F_i) \otimes_R M \rightarrow \prod (F_i \otimes_R M)$ is injective by [Å82] Lemma 6. It follows that $\mathrm{Tor}_1^Q(\prod F_i, M)$ also vanishes, so that $\mathrm{Tor}_i^Q(\prod F_i, M) = 0$ for all $i > 0$ as required. \square

Combining with Theorem 3.7 we get:

COROLLARY 4.4. *Suppose that H is a p -adic analytic group that is an amalgamated product of two compact open subgroups. Then $A\langle H \rangle$ is coherent.* \square

THEOREM 4.5. *Suppose that H is a p -adic analytic group that is an amalgamated product of two compact open subgroups. Then the category of finitely presented K -finite A -representations of H is an abelian subcategory of the category of A -representations of H .*

Proof. It suffices to show that the kernel or cokernel of a map of finitely presented K -finite A -representations of H is also a finitely presented K -finite A -representation. This is straightforward for cokernels, and does not require the ring $A\langle H \rangle$. For kernels, suppose that $f : V \rightarrow W$ is a map of finitely presented K -finite A -representations of H . Then $\ker(f)$ is a K -finite A -representation

of H , and by Proposition 3.8 and Corollary 4.4 it is finitely presented as a left $A\langle H \rangle$ -module. By Proposition 3.8 again, it is a finitely presented A -representation of H . \square

5 APPLICATIONS

Let F be a local field of characteristic 0 with ring of integers \mathcal{O}_F and residue field k of characteristic p , and let D be a division algebra over F with ring of integers \mathcal{O}_D . Choose a uniformiser π of D . Let $G = GL_2(D)$ and let $G' = SL_2(D)$ be the subgroup of elements of reduced norm 1. Let $K_1 = GL_2(\mathcal{O}_D)$ and let $K'_1 = SL_2(\mathcal{O}_D) = K' \cap SL_2(D)$. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in G$, and let $K_2 = \alpha K_1 \alpha^{-1}$ and $K'_2 = K_2 \cap G'$. Let

$$I = K_1 \cap K_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1 : c \equiv 0 \pmod{\pi} \right\}$$

and $I' = I \cap G' = K'_1 \cap K'_2$.

THEOREM 5.1. *The category of finitely presented K -finite A -representations of G' is an abelian subcategory of $\mathcal{C}_A^{K\text{-fin}}(G')$.*

Proof. By a theorem of Ihara (Serre [Ser77] Chapter II Corollary 1) we know that $G' = K'_1 *_I K'_2$. The theorem follows from Theorem 4.5. \square

COROLLARY 5.2. *The category of finitely presented, K -finite, (locally) Z -finite A -representations of G is an abelian subcategory of $\mathcal{C}_A^{K\text{-fin}}(G)$.*

Proof. Let G^0 be the subgroup of G of elements whose reduced norm is in \mathcal{O}_F^\times and let Z be the centre of G . Then ZG^0 has finite index in G , $Z \cap G^0$ is compact, and $Z/Z \cap K$ is finitely generated for any compact open subgroup K of G . Let $f : V_1 \rightarrow V_2$ be a map of K -finite Z -finite finitely presented representations of G . By Proposition 2.10 they are finitely presented representations of G^0 . By [Ser77] Chapter II Theorem 3, $G^0 = K_1 *_I K_2$, and so Theorem 4.5 the kernel $\ker(f)$ is finitely presented as a representation of G^0 . By Proposition 2.10 again, it is a finitely presented representation of G . \square

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Jack Shotton
Department of Mathematics
Durham University
Lower Mountjoy
Stockton Road
Durham DH1 3LE
United Kingdom
jack.g.shotton@durham.ac.uk

