

ESSENTIALLY FINITE VECTOR BUNDLES  
ON NORMAL PSEUDO-PROPER ALGEBRAIC STACKS

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ABSTRACT. Let  $X$  be a normal, connected and projective variety over an algebraically closed field  $k$ . In [BdS] and [AM] it is proved that a vector bundle  $V$  on  $X$  is essentially finite if and only if it is trivialized by a proper surjective morphism  $f : Y \rightarrow X$ . In this paper we introduce a different approach to this problem which allows to extend the results to normal, connected and strongly pseudo-proper algebraic stack of finite type over an arbitrary field  $k$ .

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## 1 INTRODUCTION

Let  $k$  be the base field, and let  $X$  be a connected and reduced scheme over  $k$  equipped with a rational point  $x \in X(k)$ . In [Nori] M. Nori constructed the *Nori fundamental group scheme*  $\pi^N(X, x)$  following two different paths:

1. He proved that the category  $N(X/k, x)$  of pointed torsors, that is, a triple  $(P, G, p)$  where  $P$  is a  $G$ -torsor over  $X$ ,  $G$  is a finite  $k$ -group scheme, and  $p$  is a  $k$ -rational point lying over  $x$ , is cofiltered, then he defined

$$\pi^N(X, x) := \varprojlim_{(P, G, p) \in N(X/k, x)} G$$

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This definition is very abstract, so it's not easy to understand the Nori fundamental group directly from the construction. However, this construction also gave birth to several generalizations, e.g. in [Zh] the second author studied this construction with  $x$  being replaced by a geometric point.

2. In the case when  $X$  is *proper* and the field  $k$  is perfect, Nori proved that the full subcategory of  $\mathbf{Vect}(\mathcal{X})$  consisting of vector bundles which are *essentially finite* (see 2.1) is Tannakian, and its Tannakian group is precisely  $\pi^N(X, x)$ . This Tannakian interpretation via essentially finite vector bundles makes the Nori fundamental group more accessible, but in order to follow this way, one has to understand better these essentially finite vector bundles.

In [BdS] and [BdS2], I. Biswas and J. P. Dos Santos gave a more geometric characterization of essentially finite vector bundles. If  $X$  is a smooth, connected and projective variety over an algebraically closed field they showed that a vector bundle  $V$  on  $X$  is essentially finite if and only if it is trivialized by a proper surjective morphism  $f : Y \rightarrow X$ , that is  $f^*V$  is a free vector bundle. This result has then been generalized to normal varieties in [AM]. In this paper we present a new proof of this result which applies to more general  $X$  and does not require the use of semistable sheaves. Let us introduce some notions before stating our results.

A category  $\mathcal{X}$  fibered in groupoid over a field  $k$  is *pseudo-proper* (resp. *strongly pseudo-proper*) if for all vector bundles (resp. finitely presented sheaves)  $E$  on  $\mathcal{X}$  the  $k$ -vector space  $H^0(\mathcal{X}, E)$  is finite dimensional. It is also required to satisfy a finiteness condition which is automatic for algebraic stacks of finite type over  $k$  (see 2.2). Proper algebraic stacks, finite stacks and affine gerbes are examples of strongly pseudo-proper fibred categories (see [BV, Example 7.2, pp. 20]). This is the generalization of [BdS] and [AM] we present.

**THEOREM I.** *Let  $\mathcal{X}$  be a connected, normal and strongly pseudo-proper algebraic stack of finite type over  $k$ . Then a vector bundle  $V$  on  $\mathcal{X}$  is essentially finite if and only if it is trivialized by a surjective morphism  $f : \mathcal{Y} \rightarrow \mathcal{X}$  between algebraic stacks such that  $f_*\mathcal{O}_{\mathcal{Y}}$  is a coherent sheaf.*

We stress that the proof of this result is stacky in nature and not only because we already start with stacks. The key example that enlightens the strategy is when  $X = \mathcal{X}$  is a normal variety and  $f : Y = \mathcal{Y} \rightarrow \mathcal{X}$  is the normalization of  $X$  in a Galois extension  $L/k(X)$  with group  $G$ . Then we have a splitting  $Y \rightarrow \mathcal{Z} = [Y/G] \xrightarrow{\pi} X$ . Since  $Y$  is a  $G$ -torsor over  $\mathcal{Z}$  and  $\pi^*V$  is trivialized by this torsor,  $\pi^*V$  is essentially finite. The conclusion that  $V$  is essentially finite then follows formally from the fact that  $\pi_*\mathcal{O}_{\mathcal{Z}} \simeq \mathcal{O}_X$ . Thus the key step here is to pass to a new space  $\mathcal{Z}$  which may not be a scheme. When  $f$  is general one can reduce the problem to the above situation. This is not possible for stacks and in this case we introduce a different kind of ‘‘Galois cover’’ where the group  $G$  is replaced by the symmetric group.

In [TZ2, Corollary I] we prove a variant of Theorem I without any regularity requirement on  $\mathcal{X}$  but assuming that  $f$  is proper and flat. The proofs of those results are independent.

For completeness we also deal with the analogous result of [BdS2] in our context:

**THEOREM II.** *Let  $\mathcal{X}$  be a connected, normal and strongly pseudo-proper algebraic stack of finite type over  $k$  and  $f : \mathcal{Y} \rightarrow \mathcal{X}$  be a surjective morphism of algebraic stacks such that  $f_*\mathcal{O}_{\mathcal{Y}}$  is a coherent sheaf. Consider the full subcategory of  $\mathbf{Vect}(\mathcal{X})$*

$$\mathbf{Vect}(\mathcal{X})_f := \{ V \in \mathbf{Vect}(\mathcal{X}) \mid f^*V \text{ is trivial} \}$$

and set  $L = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ , which is a finite field extension of  $k$ . We have

1. if  $\mathrm{Spec}(f_*\mathcal{O}_{\mathcal{Y}})$  is reduced (e.g. if  $\mathcal{Y}$  is reduced), then  $\mathbf{Vect}(\mathcal{X})_f$  is an  $L$ -Tannakian category;
2. if the generic fiber of  $\mathrm{Spec}(f_*\mathcal{O}_{\mathcal{Y}}) \rightarrow \mathcal{X}$  is étale then the affine gerbe over  $L$  corresponding to the full sub tannakian category generated by  $\mathbf{Vect}(\mathcal{X})_f \subseteq \mathbf{EFin}(\mathbf{Vect}(\mathcal{X}))$  is finite and étale.

Notice that in the above hypothesis the category  $\mathbf{EFin}(\mathbf{Vect}(\mathcal{X}))$  is indeed an  $L$ -Tannakian category (see 2.4). In situation (1) the gerbe associated with  $\mathbf{Vect}(\mathcal{X})_f$  is in general not finite, as already shown in [BdS2].

In [ABETZ] is studied the behaviour of essentially finite vector bundles under pushforward rather than pullback as in the present paper. We plan to investigate the relations among these results.

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2 PRELIMINARIES

In this section we fix a base field  $k$ . We will collect here some preliminary results and definitions needed for the next section. All fibered categories considered will be fibered in groupoids.

We will freely talk about affine gerbes over a field (often improperly called just gerbes) and Tannakian categories and use their properties. Please refer to [TZ, Appendix B] for details.

**DEFINITION 2.1.** [BV, Definition 7.7] Let  $\mathcal{C}$  be an additive and monoidal category. An object  $E \in \mathcal{C}$  is called finite if there exist  $f \neq g \in \mathbb{N}[X]$  polynomials with natural coefficients and an isomorphism  $f(E) \simeq g(E)$ , it is called *essentially finite* if it is a kernel of a map of finite objects of  $\mathcal{C}$ . We denote by  $\mathbf{EFin}(\mathcal{C})$  the full subcategory of  $\mathcal{C}$  consisting of essentially finite objects.

DEFINITION 2.2. [BV, Definition 5.3 and Definition 7.1] Let  $\mathcal{X}$  be a fibred category over  $k$ . It is called *inflexible* over  $k$  if any map from  $\mathcal{X}$  to a finite stack factors through a finite gerbe. It is called *pseudo-proper* (resp. *strongly pseudo-proper*) if

1. there exists a quasi-compact scheme  $U$  and a representable morphism  $U \rightarrow \mathcal{X}$  which is faithfully flat, quasi-compact and quasi-separated;
2. for any vector bundle (resp. finitely presented quasi-coherent sheaf)  $E$  on  $\mathcal{X}$  the  $k$ -vector space  $H^0(\mathcal{X}, E)$  is finite dimensional.

REMARK 2.3. The following are some general remarks about inflexible fibred categories.

1. Let  $\mathcal{X}$  be an algebraic stack of finite type over  $k$ . If  $\mathcal{X}$  is inflexible then it is geometrically connected. If  $\mathcal{X}$  is geometrically connected and geometrically reduced then it is inflexible. (See [BV, Proposition 5.5, p. 11])
2. Let  $\mathcal{X}$  be a fibred category over  $k$ . If  $\mathcal{X}$  is inflexible then  $k$  is integrally closed in  $H^0(\mathcal{O}_{\mathcal{X}})$ . The converse holds if  $\mathcal{X}$  is reduced, quasi-compact and quasi-separated. (See [TZ, Theorem 4.4, p. 13])

REMARK 2.4. By [BV, Theorem 7.9, p. 22], if  $\mathcal{X}$  is a pseudo-proper and inflexible fibred category then  $\text{EFin}(\text{Vect}(\mathcal{X}))$  is a  $k$ -Tannakian category, which corresponds to the so called Nori fundamental gerbe  $\Pi_{\mathcal{X}/k}^N$ .

Recall that a pseudo-proper fibred category  $\mathcal{X}$  which is inflexible satisfies  $H^0(\mathcal{O}_{\mathcal{X}}) = k$  and the converse is true if  $\mathcal{X}$  is reduced, quasi-compact and quasi-separated (see [TZ, Theorem 4.4]). In particular in Theorem I and II the algebraic stack  $\mathcal{X}$  is automatically inflexible over the field  $H^0(\mathcal{O}_{\mathcal{X}})$ .

Recall that if  $\mathcal{X}$  is an algebraic stack with a map  $\lambda: \mathcal{X} \rightarrow \Gamma$  to a finite gerbe and  $W \in \text{Vect}(\Gamma)$  then  $\lambda^*W$  is essentially finite. Indeed all vector bundles on  $\Gamma$  are essentially finite by [BV, Proposition 7.8] and  $\lambda^*$  is exact and monoidal because  $\lambda$  is flat.

We start by looking at a special case of Theorem I, that is the case of a torsor.

LEMMA 2.5. *Let  $\mathcal{X}$  be a pseudo-proper and inflexible fibred category over  $k$  and let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a torsor under some finite  $k$ -group scheme  $G$ . If  $V$  is a vector bundle on  $\mathcal{X}$  which is trivialized by  $f$  then  $V$  is an essentially finite vector bundle on  $\mathcal{X}$ . Moreover the full subcategory of  $\text{Vect}(\mathcal{X})$  of vector bundles trivialized by  $f$  is a  $k$ -Tannakian category whose associated gerbe  $\Gamma$  is the Nori reduction of  $\mathcal{X} \rightarrow \mathcal{B}_k G$ , that is, it is the image of the unique map  $\Pi_{\mathcal{X}/k}^N \rightarrow \mathcal{B}_k G$  which corresponds to  $\mathcal{X} \rightarrow \mathcal{B}_k G$ .*

*Proof.* Consider the following 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \text{Spec}(k) \\ \downarrow f & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{B}_k G \end{array}$$

By [BV, Lemma 5.12, Lemma 7.11] there is a finite gerbe  $\Gamma$  and a factorization of  $\mathcal{X} \rightarrow \mathcal{B}_k G$  as:  $\lambda: \mathcal{X} \rightarrow \Gamma$  such that  $\lambda_* \mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_{\Gamma}$ ;  $\Gamma \rightarrow \mathcal{B}_k G$  faithful. Notice that, since  $\Gamma$  is finite, the map  $\Gamma \rightarrow \mathcal{B}_k G$  is affine. See for instance [TZ, Remark B7]. In particular we also have a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\delta} & \text{Spec}(A) \\ \downarrow f & & \downarrow \\ \mathcal{X} & \xrightarrow{\lambda} & \Gamma \end{array}$$

By cohomology and base change along the flat map  $\text{Spec } A \rightarrow \Gamma$  we have  $\delta_* \mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\text{Spec}(A)}$ . Pulling back the adjunction  $\lambda^* \lambda_* V \rightarrow V$  along  $f$  we get the map  $\delta^* \delta_* f^* V \cong f^* \lambda^* \lambda_* V \rightarrow f^* V$ , which is an isomorphism as  $f^* V \cong \mathcal{O}_{\mathcal{Y}}^{\oplus m}$  and  $\delta_* \mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\text{Spec}(A)}$ . Since  $f$  is faithfully flat it follows that  $\lambda^* \lambda_* V \rightarrow V$  is an isomorphism. Thus  $V$  is the pull back of a vector bundle on a finite gerbe and hence it is essentially finite.

The map  $\text{Vect}(\Gamma) \rightarrow \text{Vect}(\mathcal{X})$  is fully faithful and embeds  $\text{Vect}(\Gamma)$  as a sub Tannakian category of  $\text{EFin}(\text{Vect}(\mathcal{X}))$  by [BV, Theorem 7.13, p. 24]. To conclude the proof we just have to show that if  $W \in \text{Vect}(\Gamma)$  then  $f^* \lambda^* W$  is free. Thus it is enough to show that  $A$  is a finite  $k$ -algebra and we can assume  $k$  algebraically closed. In this case  $\Gamma = \mathcal{B}_k H$ , where  $H$  is a closed subgroup of  $G$ , and a direct computation shows that  $\text{Spec } A \simeq G/H$  which is a finite  $k$ -scheme.  $\square$

The following result is also a variant of Theorem I, which is useful because it will allow us to replace an arbitrary finite map by a generically étale one. The same result is present in [TZ2, Lemma 2.3]. We include it here for completeness.

LEMMA 2.6. *Let  $\mathcal{X}$  be an inflexible and pseudo-proper fibred category over  $k$ ,  $V \in \text{Vect}(\mathcal{X})$  and denote by  $F: \mathcal{X} \rightarrow \mathcal{X}$  the absolute Frobenius. If there exists  $m \in \mathbb{N}$  such that  $F^{m*} V \in \text{Vect}(\mathcal{X})$  is essentially finite then  $V$  is essentially finite too.*

*Proof.* We can consider the case  $m = 1$  only. Set  $n = \text{rk } V$ . The vector bundle  $V$  is given by an  $\mathbb{F}_p$ -map  $v: \mathcal{X} \rightarrow \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p})$ : the stack  $\mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p})$  has a universal vector bundle  $\mathcal{E}$  of rank  $n$  such that  $v^* \mathcal{E} \simeq V$ . By 2.4  $V$  is essentially finite if and only if  $v$  factors as  $\mathcal{X} \xrightarrow{\phi} \Gamma \rightarrow \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p})$  where  $\Gamma$  is a finite  $k$ -gerbe and  $\phi$  is  $k$ -linear. The vector bundle  $F^* V$  corresponds to the composition  $\mathcal{X} \xrightarrow{v} \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p}) \xrightarrow{\overline{F}} \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p})$ , where  $\overline{F}$  is the absolute Frobenius of  $\mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p})$ . Thus we have a diagram

$$\begin{array}{ccccc} & & v & & \\ & & \curvearrowright & & \\ \mathcal{X} & \xrightarrow{\quad} & \Delta & \longrightarrow & \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p}) \\ & \searrow \phi & \downarrow & & \downarrow \overline{F} \\ & & \Gamma & \longrightarrow & \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n, \mathbb{F}_p}) \end{array}$$

where  $\Gamma$  is a finite  $k$ -gerbe and the square is 2-Cartesian. We conclude by showing that  $\Delta$  is a finite gerbe over  $k$ .

The map  $\overline{F}: \mathcal{B}_{\mathbb{F}_p}(\mathrm{GL}_{n,\mathbb{F}_p}) \rightarrow \mathcal{B}_{\mathbb{F}_p}(\mathrm{GL}_{n,\mathbb{F}_p})$  is induced by the Frobenius of  $\mathrm{GL}_{n,\mathbb{F}_p}$ . Since this last map is a surjective group homomorphism with finite kernel it follows that  $\overline{F}$  and therefore  $\Delta \rightarrow \Gamma$  is a finite relative gerbe. This plus the assumption that  $\Gamma$  is a finite gerbe implies that  $\Delta$  is a finite gerbe.  $\square$

We finish this section by explicitly showing how to associate with an étale degree  $n$  cover an  $S_n$ -torsor and conversely. Given a scheme  $U$  and an étale cover  $E \rightarrow U$  of degree  $n$  we define  $\mathcal{S}_{E/U}$  as the complement in  $E^n$  (the product of  $n$ -copies of  $E$  over  $U$ ) of the open and closed subsets given by the union of the diagonals  $E^{n-1} \subseteq E^n$ . The symmetric group  $S_n$  acts on  $\mathcal{S}_{E/U}$  over  $U$ , and by looking at the fibers of a geometric point of  $U$  we see that the natural map

$$\rho : \mathcal{S}_{E/U} \times S_n \rightarrow \mathcal{S}_{E/U} \times_U \mathcal{S}_{E/U}$$

sending  $(a, s)$  to  $(a, as)$  (with  $a \in \mathcal{S}_{E/U}$  and  $s \in S_n$ ) is a surjective map between two étale covers of  $U$  of the same degree. Thus  $\rho$  is an isomorphism and  $\mathcal{S}_{E/U}$  becomes an  $S_n$ -torsor over  $U$ .

PROPOSITION 2.7. *Let  $n > 0$  and denote by  $\mathrm{Et}_n$  the stack over  $\mathbb{Z}$  of étale covers of degree  $n$ . Then*

$$\mathcal{S} : \mathrm{Et}_n \rightarrow \mathcal{B}S_n, E/U \mapsto \mathcal{S}_{E/U}$$

*is an equivalence. Moreover  $\mathrm{pr}_1 : \mathcal{S}_{E/U} \rightarrow E$  is natural and the action of  $S_{n-1}$  on the last components makes it into a  $S_{n-1}$ -torsor.*

*Proof.* The last claim follows directly from the construction. We only have to prove that  $\mathcal{S}$  is an equivalence. Let  $I := \{1, \dots, n\}$ . There is a global object  $I_{\mathbb{Z}} = I \times \mathrm{Spec} \mathbb{Z} \in \mathrm{Et}_n(\mathbb{Z})$ . Since all étale covers of degree  $n$  of a scheme  $U$  became locally a disjoint union of copies of the base, it follows that  $\mathrm{Et}_n$  is a trivial gerbe, and more precisely, that there is an equivalence

$$\mathcal{B}G \rightarrow \mathrm{Et}_n \text{ where } G = \underline{\mathrm{Aut}}_{\mathrm{Et}_n}(I_{\mathbb{Z}})$$

mapping the trivial torsor to  $I_{\mathbb{Z}}$ . By the equivalence between neutral affine gerbes and affine group schemes, the composition  $\mathcal{B}G \rightarrow \mathrm{Et}_n \xrightarrow{\mathcal{S}} \mathcal{B}S_n$  is induced by a morphism of group schemes  $\phi : G = \underline{\mathrm{Aut}}_{\mathrm{Et}_n}(I_{\mathbb{Z}}) \rightarrow S_{n,\mathbb{Z}}$ . We must show it is an isomorphism. It is easy to see that  $G$  sends any scheme  $U$  to the set of continuous maps  $|U| \rightarrow S_n$ , where  $|U|$  is the underlying topological space of  $U$  and  $S_n$  is equipped with the discrete topology. Thus  $G$  is isomorphic to the constant group scheme  $S_{n,\mathbb{Z}}$ . The composition of the isomorphism  $S_{n,\mathbb{Z}} = G(\mathrm{Spec}(\mathbb{Z})) \times \mathrm{Spec}(\mathbb{Z}) \cong G$  defined by the inclusion of  $\mathbb{Z}$ -rational points with  $\phi : G \rightarrow S_{n,\mathbb{Z}}$  is the identity. Thus  $\phi$  is an isomorphism.  $\square$

3 THEOREM I AND II

*Proof of Theorem I and Theorem II.* Without loss of generality we can assume that  $H^0(\mathcal{O}_{\mathcal{X}})$  is  $k$ , so that, in particular,  $\mathcal{X}$  is inflexible (see 2.4). We split the proof in several steps.

3.1 THEOREM I, THE “ONLY IF” PART

If  $V \in \text{EFin}(\text{Vect}(\mathcal{X}))$  then by 2.4  $V$  is the pullback along a morphism  $\mathcal{X} \rightarrow \Gamma$  of a vector bundle on  $\Gamma$ , where  $\Gamma$  is a finite gerbe. Choose a point  $\text{Spec}(k') \rightarrow \Gamma$  where  $k'/k$  is a finite field extension. Then  $V$  is trivialized by the finite flat morphism  $\mathcal{X} \times_{\Gamma} \text{Spec}(k') \rightarrow \mathcal{X}$ .

3.2 REDUCTION TO  $f$  FINITE

Consider now a map  $f: \mathcal{Y} \rightarrow \mathcal{X}$  as in the statement,  $V \in \text{Vect}(\mathcal{X})$  and write  $f': \mathcal{Y}' = \text{Spec}(f_*\mathcal{O}_{\mathcal{Y}}) \rightarrow \mathcal{X}$  and  $r = \text{rk } V$ . Since  $\text{Iso}_{\mathcal{X}}(V, \mathcal{O}_{\mathcal{X}}^r) \rightarrow \mathcal{X}$  is affine we have that  $f^*V$  is free if and only if  $f'^*V$  is free. Since by hypothesis  $f_*\mathcal{O}_{\mathcal{Y}}$  is coherent, we can therefore assume that  $f: \mathcal{Y} \rightarrow \mathcal{X}$  is a finite map.

3.3 PROOF OF THEOREM II, (1) ASSUMING THEOREM I

We can assume that  $\mathcal{Y}$  is reduced. Since by Theorem I  $\text{Vect}(\mathcal{X})_f$  is contained in the tannakian category  $\text{EFin}(\text{Vect}(\mathcal{X}))$ , to prove that  $\text{Vect}(\mathcal{X})_f$  is tannakian it is enough to prove that  $\text{Vect}(\mathcal{X})_f$  is stable under taking tensor products, dual, kernels and cokernels. Tensor products and dual can be easily checked, while kernels and cokernels are reduced to check the following: given  $V \in \text{Vect}(\mathcal{X})_f$  and an embedding  $W \hookrightarrow V$  or a quotient  $V \twoheadrightarrow W$  in  $\text{EFin}(\text{Vect}(\mathcal{X}))$  we have  $W \in \text{Vect}(\mathcal{X})_f$ . Write  $\mathcal{Y} = \coprod_{i \in I} \mathcal{Y}_i$  for the decomposition into connected components. Each  $\mathcal{Y}_i$  is a reduced connected algebraic stack of finite type over  $k$  and, since  $\mathcal{X}$  is strongly pseudo-proper, it is pseudo-proper. Thus  $H^0(\mathcal{Y}_i, \mathcal{O}_{\mathcal{Y}_i}) = k'$  is a finite field extension of  $k$ , and by [TZ, Theorem 4.4, pp. 13]  $\mathcal{Y}_i$  is inflexible and pseudo-proper over  $k'$ . Thus  $\text{EFin}(\text{Vect}(\mathcal{Y}_i))$  is a tannakian category, so the pullback of  $W$  along  $\mathcal{Y}_i \rightarrow \mathcal{X}$  is a subobject or a quotient object of a trivial object, and consequently it is trivial itself. Thus  $f^*W$  is free on each component  $\mathcal{Y}_i$  and of the correct rank  $\text{rk } W$ , which means that  $f^*W$  is free on  $\mathcal{Y}$ .

3.4 PROOF OF THEOREM I AND THEOREM II, (2)

We first show how to reduce the proof of Theorem I to the case when the generic fiber of  $f$  is étale. Let  $h: U \rightarrow \mathcal{X}$  be a smooth atlas and  $\xi \in U$  be a generic point such that  $h(\xi)$  is the generic point of  $\mathcal{X}$ . Set  $K = k(\xi)$  and consider  $h(\xi): \text{Spec } K \rightarrow \mathcal{X}$  and  $\text{Spec } A = \text{Spec}(K) \times_{\mathcal{X}} \mathcal{Y}$ . Notice that  $A \neq 0$  because  $\mathcal{Y} \rightarrow \mathcal{X}$  is surjective. By [TZ, Lemma 2.3, pp. 8] there exists  $i \geq 0$  such that all residue fields of  $A^{(i,K)}$ ,

the  $i$ -th Frobenius twist of  $A$ , are separable over  $K$ . We have the following Cartesian diagrams

$$\begin{array}{ccccc}
 \mathrm{Spec} B & \longrightarrow & \mathcal{Y}_{\mathrm{red}}^{(i, \mathcal{X})} & & \\
 \downarrow & & \downarrow & & \\
 \mathrm{Spec} (A^{(i, K)}) & \longrightarrow & \mathcal{Y}^{(i, \mathcal{X})} & \longrightarrow & \mathcal{Y} \\
 \downarrow & & \downarrow & & \downarrow f \\
 \mathrm{Spec} K & \xrightarrow{h(\xi)} & \mathcal{X} & \xrightarrow{F^i} & \mathcal{X}
 \end{array}$$

where  $(-)\mathrm{red}$  denotes the reduction and  $F$  is the absolute Frobenius of  $\mathcal{X}$ . Set  $f' : \mathcal{Y}_{\mathrm{red}}^{(i, \mathcal{X})} \rightarrow \mathcal{X}$  for the composition. Since  $h(\xi)$  is the composition of a smooth map  $U \rightarrow \mathcal{X}$  and a generic point  $\mathrm{Spec} k(\xi) \rightarrow U$ , we can conclude that  $B$  is reduced. Since it is a quotient of  $A^{(i, K)}$  we can conclude that  $B/K$  is étale over  $K$ , that is, the generic fiber of  $f'$  is étale. Notice that if  $f^*V$  is free then  $f'^*(F^{i*}V)$  is free and, by 2.6,  $V$  is essentially finite if  $F^{i*}V$  is essentially finite. Thus we can replace  $\mathcal{Y} \rightarrow \mathcal{X}$  by  $\mathcal{Y}_{\mathrm{red}}^{(i, \mathcal{X})} \rightarrow \mathcal{X}$  and assume that the generic fiber of  $f$  is étale of degree say  $n \geq 0$ .

### 3.5 CONSTRUCTION OF $\Theta_{\mathcal{X}}$ AND ITS QUOTIENT

For each smooth map  $U \rightarrow \mathcal{X}$ , where  $U$  is a connected scheme, let  $Y_U := \mathcal{Y} \times_{\mathcal{X}} U$ . Since the generic point of  $U$  goes to the generic point of  $\mathcal{X}$ , the generic fibre  $L_U$  of  $Y_U \rightarrow U$  is finite étale of degree  $n$ . By 2.7  $L_U$  corresponds to an  $S_n$ -torsor  $\mathcal{S}_{L_U/K(U)}$  over  $\mathrm{Spec} (K(U))$ , where  $K(U)$  is the function field of  $U$ . Let  $\Theta_U$  be the normalization of  $U$  inside  $\mathcal{S}_{L_U/K(U)} = H^0(\mathcal{S}_{L_U/K(U)})$ . Then  $\Theta_U$  is equipped with an action of  $S_n$  and a morphism  $\Theta_U \rightarrow Y_U$  define by the projection  $\mathcal{S}_{L_U/K(U)} \rightarrow \mathrm{Spec} (L_U)$  (the last claim of 2.7). This construction is functorial. If  $V \rightarrow U$  is a smooth morphism then by 2.7 and [SP, 03GC] we have

$$\begin{aligned}
 s_{U, V} : \mathcal{S}_{L_U/K(U)} \otimes_{K(U)} K(V) &\cong \mathcal{S}_{L_V/K(V)} \\
 \theta_{U, V} : (\Theta_U \times_U V \rightarrow Y_U \times_U V) &\cong (\Theta_V \rightarrow Y_V)
 \end{aligned}$$

and also the action of  $S_n$  on  $\Theta_V$  is the same as the one obtained by the pull-back of the action on  $\Theta_U$ . Moreover, if there is a third map  $W \rightarrow V$ , then  $s_{U, V}, s_{V, W}, s_{U, W}$  and  $\theta_{U, V}, \theta_{V, W}, \theta_{U, W}$  are compatible in a natural way. This allow us to construct maps  $\Theta_{\mathcal{X}} \rightarrow \mathcal{Z} \rightarrow \mathcal{X}$  fitting in 2-Cartesian diagrams

$$\begin{array}{ccccc}
 \Theta_U & \longrightarrow & \Theta_{\mathcal{X}} & \longrightarrow & \mathrm{Spec} k \\
 \downarrow & & \downarrow \pi & & \downarrow \\
 [\Theta_U/S_n] & \longrightarrow & \mathcal{Z} & \longrightarrow & \mathcal{B}_k S_n \\
 \downarrow & & \downarrow h & & \\
 U & \longrightarrow & \mathcal{X} & & 
 \end{array}$$

and a map  $\Theta_{\mathcal{X}} \rightarrow \mathcal{Y}$  fitting in the 2-diagram

$$\begin{array}{ccc} \Theta_{\mathcal{X}} & \longrightarrow & \mathcal{Y} \\ & \searrow h\pi & \swarrow f \\ & & \mathcal{X} \end{array}$$

for all smooth  $U \rightarrow \mathcal{X}$  with  $U$  being connected, where  $\mathcal{Z}$  is the following category fibred over  $\mathcal{X}$ : for each  $t : T \rightarrow \mathcal{X}$ ,  $\mathcal{Z}(T)$  is the category consisting of diagrams

$$\begin{array}{ccccc} P & \xrightarrow{x} & \Theta_T & \longrightarrow & \Theta_{\mathcal{X}} \\ & \searrow y & \downarrow & & \downarrow h\pi \\ & & T & \xrightarrow{t} & \mathcal{X} \end{array}$$

where  $y : P \rightarrow T$  is a torsor under  $S_n$ ,  $\Theta_T$  is the pullback, and  $x$  is a map of schemes which is  $S_n$ -equivariant. Just as in the case when  $\Theta_{\mathcal{X}}$  is a scheme and  $\mathcal{Z} = [\Theta_{\mathcal{X}}/S_n]$ , one can show that the diagram

$$\begin{array}{ccc} \Theta_{\mathcal{X}} & \longrightarrow & \text{Spec } k \\ \downarrow \pi & & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{B}_k S_n \end{array}$$

is Cartesian. In particular  $\mathcal{Z}$  is an algebraic stack of finite type over  $k$ . Since  $\text{Vect}(\mathcal{X})_f \subseteq \text{Vect}(\mathcal{X})_{h\pi}$  is fully faithful, by [TZ, Remark B7] we can assume  $\mathcal{Y} = \Theta_{\mathcal{X}}$  and  $f = h\pi$  for both Theorem I and Theorem II, (2). Consider the following 2-Cartesian diagram

$$\begin{array}{ccccc} \Theta_{\mathcal{X}} & \xrightarrow{f} & \mathcal{X} & \longrightarrow & \text{Spec } k \\ \downarrow \pi & & \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{u} & \mathcal{X} \times \mathcal{B}_k S_n & \longrightarrow & \mathcal{B}_k S_n \\ & \searrow h & \downarrow v & & \downarrow \\ & & \mathcal{X} & \longrightarrow & \text{Spec } k \end{array}$$

Since  $u$  is finite and  $v$  is proper we can conclude that the map  $h : \mathcal{Z} \rightarrow \mathcal{X}$  maps coherent sheaves to coherent sheaves. Thus  $\mathcal{Z}$  is a strongly pseudo-proper and normal algebraic stack of finite type over  $k$ .

### 3.6 PROOF OF $\mathcal{O}_{\mathcal{X}} = h_* \mathcal{O}_{\mathcal{Z}}$

We are going to show that  $\mathcal{O}_{\mathcal{X}} \rightarrow h_* \mathcal{O}_{\mathcal{Z}}$  is an isomorphism. In particular it will follow that  $H^0(\mathcal{O}_{\mathcal{Z}}) = k$  and therefore that  $\mathcal{Z}$  is inflexible over  $k$ . Given  $U = \text{Spec } A \rightarrow \mathcal{X}$  a smooth map with  $U$  being connected and written  $\Theta_U =$

Spec  $B$ , the push forward of the structure sheaf along  $[\Theta_U/S_n] \rightarrow U$  is  $B^{S_n}$ , and we must show that  $B^{S_n} \subseteq A$ . Set  $K_A$  for the function field of  $A$  and  $K_B$  for the generic fibre of  $B/A$ , that is  $K_B = S_{L_U/K_A}$ . We have  $B^{S_n} \subseteq B \cap K_B^{S_n} = B \cap K_A = A$  because  $A$  is normal and  $K_B$  is an  $S_n$ -torsor over  $K_A$ .

### 3.7 END OF THE PROOF OF THEOREM I

Let  $V$  be a vector bundle on  $\mathcal{X}$  which is trivialized by  $f$ . Since  $\mathcal{Z}$  is pseudo-proper and inflexible, by 2.5  $h^*V$  is essentially finite. Thus there is a finite gerbe  $\Gamma$  and a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \Gamma \\ \downarrow h & & \downarrow \lambda \\ \mathcal{X} & \xrightarrow{\phi} & \mathcal{BGL}_m \end{array}$$

where  $\phi$  corresponds to the vector bundle  $V$ . Replacing  $\Gamma$  by its image under  $\lambda$  we may assume that  $\lambda$  is faithful. Since  $\Gamma$  is finite it follows that  $\lambda$  is affine thanks to [TZ, Remark B7]. As  $h_*\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_{\mathcal{X}}$  the unique map  $\mathcal{Z} \rightarrow \mathcal{X} \times_{\mathcal{BGL}_m} \Gamma$  induces

$$\mathrm{Spec}(h_*\mathcal{O}_{\mathcal{Z}}) = \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{BGL}_m} \Gamma \rightarrow \Gamma$$

Then  $\phi$  factors through  $\lambda : \Gamma \rightarrow \mathcal{BGL}_m$ , so that  $V$  is essentially finite. This ends the proof of Theorem I.

### 3.8 END OF THE PROOF OF THEOREM II, (2)

Since  $h_*\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_{\mathcal{X}}$ , we have that the pullback  $\mathrm{Vect}(\mathcal{X}) \rightarrow \mathrm{Vect}(\mathcal{Z})$  is fully faithful (see [BV, Lemma 7.17]). In particular we get a fully faithful monoidal map  $\mathrm{Vect}(\mathcal{X})_f \rightarrow \mathrm{Vect}(\mathcal{Z})_{\pi}$ . By 2.5  $\mathrm{Vect}(\mathcal{Z})_{\pi}$  corresponds to a gerbe  $\Gamma_{\pi}$  affine over  $\mathcal{B}_k S_n$  and thus finite and étale. By [TZ, Remark B7] it follows that the gerbe associated with  $\mathrm{Vect}(\mathcal{X})_f$  is a quotient of  $\Gamma_{\pi}$  as required.  $\square$

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