Essential Dimension and Genericity for Quiver Representations

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Abstract. We study the essential dimension of representations of a fixed quiver with given dimension vector. We also consider the question of when the genericity property holds, i.e., when essential dimension and generic essential dimension agree. We classify the quivers satisfying the genericity property for every dimension vector and show that for every wild quiver the genericity property holds for infinitely many of its Schur roots. We also construct a large class of examples, where the genericity property fails. Our results are particularly detailed in the case of Kronecker quivers.

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1 Introduction

Let k be a field, and let A be a k-algebra. It is a natural goal to understand the category of representations of A, and if possible to give a classification. Initially one would like to describe representations when k is algebraically closed. However, it is also interesting to study representations of A when k is arbitrary. A template for this approach is provided by the classical theory of representations of finite groups (or equivalently, their group algebras), as summarized, e.g., in the books [10] or [30]. In particular, it is interesting to understand which representations are defined over which fields. This leads to the study of essential dimension in representation theory; see [18], [1] and [26].
In this paper we will focus on representations of quiver path algebras. This is a large and interesting family of algebras, which has found numerous applications

in algebraic geometry, Lie theory and physics. An important distinguishing feature of this family of algebras is that here representation-theoretic results can often be expressed in combinatorial (graph-theoretic) language. We initiated the study of essential dimension of quiver representations in the second half of [26]. This paper is a sequel to [26], with a focus on the genericity property.

Let k be a field. Following P. Brosnan, Z. Reichstein and A. Vistoli [6], we define the essential dimension $\mathrm{ed}_k \mathcal{X}$ of an algebraic stack $\mathcal{X}$ over k as the minimal number of parameters required to describe any object of $\mathcal{X}$. If $\mathcal{X}$ is integral, we define the generic essential dimension $\mathrm{ged}_k \mathcal{X}$ as the essential dimension of a generic object of $\mathcal{X}$. We say that the genericity property holds for $\mathcal{X}$ if $\mathrm{ged}_k \mathcal{X} = \mathrm{ed}_k \mathcal{X}$; see Section 3 for the precise definitions.

The genericity property fails in general (see [6, Example 6.5]) but holds for smooth algebraic stacks with reductive automorphism groups [25] (and in particular, tame Deligne-Mumford stacks [6]). In many interesting examples where these conditions are not satisfied, the genericity property continues to hold [3, 25]. This phenomenon is poorly understood; one of the goals of this paper is to investigate the genericity property of stacks of quiver representations. In particular, we produce large families of examples where genericity holds and where it fails.

Representations of dimension vector α of a fixed quiver Q are parametrized by an integral stack $\mathcal{R}_{Q,\alpha}$ of finite type over k (see Section 3), and it makes sense to consider the generic essential dimension of $\mathcal{R}_{Q,\alpha}$. In Remark 4.1 we give an equivalent definition of $\mathrm{ged}_k \mathcal{R}_{Q,\alpha}$, not involving stacks.

In this work, we study $\mathrm{ged}_k \mathcal{R}_{Q,\alpha}$ and the genericity property for $\mathcal{R}_{Q,\alpha}$. On the one hand, this improves our understanding of the essential dimension of representations of algebras. On the other hand, this is the first appearance of a large family of counterexamples to the genericity property. The algebraic stacks $\mathcal{R}_{Q,\alpha}$ are smooth, but their automorphism groups are often non-reductive, and so it is natural to investigate what happens in this case.

Our first result summarizes our understanding of the generic essential dimension of $\mathcal{R}_{Q,\alpha}$. We refer the reader to Section 2 for the definition of Schur roots.

Theorem 1.1. *Let Q be a quiver, and let α be a Schur root of Q. We have*

$$\mathrm{ged}_k \mathcal{R}_{Q,\alpha} \leq 1 - \langle \alpha, \alpha \rangle + \sum_p (p^{v_p(\gcd(\alpha_i))} - 1), \tag{1.1}$$

where the sum is over all prime numbers p. One has equality if Conjecture 5.1 holds for $d = \gcd(\alpha_i)$.

We generalize this result to the case when α is an arbitrary root of Q in Corollary 7.7.

The genericity property implies that the same formulas are true for essential dimension, making the question of when the property holds very natural. We have two results in this direction. A quiver Q is connected if the underlying graph of Q is connected.

Theorem 1.2. *Let Q be a connected quiver. Then $\mathcal{R}_{Q,\alpha}$ satisfies the genericity property for every dimension vector α if and only if Q is of finite representation type or admits at least one loop at every vertex.*

As an important special case, the combination of Theorem 1.1 and Theorem 1.2 gives us a formula for the essential dimension of the n-dimensional representations of the r-loop quiver; see Example 11.1.

Theorem 1.3. *Let Q be a wild quiver. There are infinitely many Schur roots α such that the genericity property holds for $\mathcal{R}_{Q,\alpha}$.*

For a constructive variant of this result, see Remark 12.3.
Our final result concerns generalized Kronecker quivers:

$$1 \overset{r}{\rightrightarrows} 2$$

The genericity property does not hold for them in general. Nevertheless, we find that it does in a certain range.

Theorem 1.4. *Assume that $r \geq 3$ and let K_r be the r-Kronecker quiver. Let $\alpha = (a,b)$ belong to the fundamental region of K_r, that is, $\frac{2b}{r} \leq a \leq \frac{rb}{2}$. Then the genericity property holds for $\mathcal{R}_{K_r,\alpha}$. In particular:*

$$\operatorname{ed}_k \operatorname{Rep}_{K_r,\alpha} \leq 1 - a^2 - b^2 + rab + \sum_p (p^{v_p(\gcd(a,b))} - 1),$$

with equality when Conjecture 5.1 holds for $d = \gcd(a,b)$.

Notational conventions

A base field k will be fixed throughout. We will denote by A an associative unital k-algebra. For a field extension K/k, we will write A_K for the tensor product $A \otimes_k K$. When considering an A_K-module M, we will always assume that M is a finite-dimensional K-vector space. For a field extension L/K, we will denote $M \otimes_K L$ by M_L.

2 Representations of quivers

The purpose of this section is to briefly recall the definitions and results from the theory of quiver representations that are relevant to our discussion.
Recall that a quiver Q is given by a set of vertices Q_0, a set of arrows Q_1, and two maps $s,t : Q_1 \to Q_0$, called source and target.
Let K/k be a field extension. A K-representation (M,φ) of Q is given by a finite-dimensional K-vector space M_i for each vertex i of Q, together with a linear map $\varphi_a : M_{s(a)} \to M_{t(a)}$ for every arrow $a \in Q_1$. If (M',φ') is another representation of Q, a homomorphism of representations $f : M' \to M$ is given by K-linear maps $f_i : M'_i \to M_i$ for every vertex i, such that for each arrow a

one has $\varphi_a \circ f_{s(a)} = f_{t(a)} \circ \varphi'_a$. It is a basic fact that there is an equivalence of categories between KQ-modules and K-linear representations of Q, functorial with respect to field extensions L/K, see [27, Theorem 5.4].

The *dimension vector* of the representation M is the vector $(\dim M_i)_{i\in Q_0}$. The support of α is the subset $\operatorname{supp}\alpha \subseteq Q_0$ of vertices i such that $\alpha_i \neq 0$.

A quiver Q is said to be of finite representation type, tame or wild if its path algebra kQ is so. The notion of representation type of an algebra is classical (see [11]), and can be restated in simple terms using essential dimension (see [26]). An algebra is of finite representation type, tame or wild if the essential dimension of its representations of dimension $\leq n$ is bounded, grows linearly, or grows quadratically as a function of n, respectively.

The connected quivers of finite representation type are classified: they are exactly those whose underlying graph is a Dynkin diagram of type A, D or E. The quiver Q is tame if and only if its underlying graph is an extended Dynkin diagram of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$.

The Tits form of Q is the bilinear form $\langle\cdot,\cdot\rangle : \mathbb{R}^{Q_0} \times \mathbb{R}^{Q_0} \to \mathbb{R}$ given by

$$\langle \alpha, \beta\rangle := \sum_{i\in Q_0} \alpha_i\beta_i - \sum_{a\in Q_1} \alpha_{s(a)}\beta_{t(a)}.$$

We also let $(\alpha, \beta) := \langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle$.

The *Weyl group* of Q is the subgroup $W \subseteq \operatorname{Aut}(\mathbb{Z}^{Q_0})$ generated by the *simple reflections*

$$s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$$
$$\alpha \mapsto \alpha - (\alpha, e_i)e_i$$

where i is a loop-free vertex of Q, and $e_i \in \mathbb{Z}^{Q_0}$ is the standard basis element corresponding to i. The *fundamental region* is the set F of non-zero $\alpha \in \mathbb{N}^{Q_0}$ with connected support and $(\alpha, e_i) \leq 0$ for all i. The *real roots* for Q are the dimension vectors that belong to an orbit of $\pm e_i$ (for $i \in Q_0$ loop-free) under the Weyl group. The *imaginary roots* for Q are the orbits of $\pm\alpha$ (for $\alpha \in F$) under W. An imaginary root α is called *isotropic* if $\langle\alpha, \alpha\rangle = 0$ and *anisotropic* if $\langle\alpha, \alpha\rangle < 0$. Collectively, real roots and imaginary roots are called roots. It can be shown that every root has either all non-negative components or all non-positive components. Hence we may speak of *positive* and *negative* roots.

A dimension vector α is called a *Schur root* if there exists a field extension K/k and a K-representation M of Q of dimension vector α such that $\operatorname{End}_K(M) = K$. Such M is called a *brick*. If $K \subseteq L$ is a field extension and M is a K-representation, $\operatorname{End}_K(M) \otimes_K L = \operatorname{End}_L(M_L)$, hence the property of being a brick is invariant under base change, and may be checked over an algebraically closed field.

Given a dimension vector α, there exists a partition $\alpha = \sum \beta_j$ such that a generic α-dimensional representation M of Q is a direct sum $M = \oplus M_j$ of indecomposable representations, where M_j has dimension vector β_j. This is called the *canonical decomposition* of α. For details, see [15] and [29].

3 Essential dimension of functors

We denote by Fields_k the category of field extensions of k. Consider a functor $F : \mathrm{Fields}_k \to \mathrm{Sets}$. We say that an element $\xi \in F(L)$ is *defined over a field* $K \subseteq L$, or that K is a *field of definition* for ξ, if ξ belongs to the image of $F(K) \to F(L)$. The *essential dimension* of ξ is

$$\mathrm{ed}_k\, \xi := \min_K \mathrm{trdeg}_k\, K$$

where the minimum is taken over all fields of definition K of ξ.
The *essential dimension* of F is defined to be

$$\mathrm{ed}_k\, F := \sup_{(K,\xi)} \mathrm{ed}_k\, \xi$$

where the supremum is taken over all pairs (K, ξ), where K is a field extension of k, and $\xi \in F(K)$.
Given a dimension vector α, we define the functor

$$\mathrm{Rep}_{Q,\alpha} : \mathrm{Fields}_k \to \mathrm{Sets}$$

by setting

$$\mathrm{Rep}_{Q,\alpha}(K) := \{\text{Isom. classes of } \alpha\text{-dimensional } K\text{-representations of } Q\}.$$

If $K \subseteq L$ is a field extension, the corresponding map $\mathrm{Rep}_{Q,\alpha}(K) \to \mathrm{Rep}_{Q,\alpha}(L)$ is given by tensor product.

Example 3.1. Let Q be the 1-loop quiver. Then isomorphism classes of n-dimensional representations of Q correspond to conjugacy classes of $n \times n$ matrices up to conjugation. The existence of the rational canonical form implies $\mathrm{ed}_k \mathrm{Rep}_{Q,n} \leq n$. On the other hand, a matrix in rational canonical form with characteristic polynomial $t^n + a_1 t^{n-1} + \cdots + a_n$, where the a_i are algebraically independent over k, is defined over $k(a_1, \ldots, a_n)$ but not over any proper subfield. This proves that in fact $\mathrm{ed}_k \mathrm{Rep}_{Q,n} = n$. See [24] for the details.

Example 3.2. Let α be a real root for the quiver Q. If K is a field, the unique indecomposable representation of dimension vector α is defined over the prime field of K. This was first proved by Kac [14, Theorem 1] when K is algebraically closed and $\mathrm{char}\, K > 0$. Later, Schofield noted that Kac's proof works over arbitrary fields of positive characteristic [28, p. 293], and extended Kac's result in characteristic zero; see [28, Theorem 8]. To our knowledge, this is the first result related to fields of definitions of quiver representations.

In [1] and [26], the following related functors are studied. Let A be an associative unital k-algebra. For any non-negative integer n, we define the functor

$$\mathrm{Rep}_A[n] : \mathrm{Fields}_k \to \mathrm{Sets}$$

by setting

$$\operatorname{Rep}_A[n](K) := \{\text{Isomorphism classes of } n\text{-dimensional representations of } A_K\}$$

for every field extension K/k. For an inclusion $K \subseteq L$, the corresponding map $\operatorname{Rep}_A[n](K) \to \operatorname{Rep}_A[n](L)$ is induced by tensor product. In [26], representations of dimension $\le n$ are considered in the definition of $\operatorname{Rep}_A[n]$. By [26, Proposition 6.5], the two definitions are equivalent when A admits a one-dimensional k-representation, e.g. when $A = kQ$ for some quiver Q.
For a quiver Q, we may consider the functors $\operatorname{Rep}_{Q,\alpha}$ for each dimension vector α, and the functors $\operatorname{Rep}_{kQ}[n]$ for each non-negative integer n. Since K-representations of a quiver Q correspond to K-representations of its path algebra, functorially in K, there is a clear relation between the two families of functors, namely

$$\operatorname{ed}_k \operatorname{Rep}_{kQ}[n] = \max_{\sum \alpha_i = n} \operatorname{ed}_k \operatorname{Rep}_{Q,\alpha}.$$

4 Essential dimension of stacks

If $\mathcal{X}$ is an algebraic stack over k, we obtain a functor

$$F_{\mathcal{X}} : \mathrm{Fields}_k \to \mathrm{Sets}$$

sending a field K containing k to the set of isomorphism classes of objects in $\mathcal{X}(\operatorname{Spec} K)$. If $\xi \in \mathcal{X}(K)$, we define its *essential dimension* $\operatorname{ed}_k \xi$ to be the essential dimension of its isomorphism class in $F_{\mathcal{X}}$. We define the *essential dimension* of $\mathcal{X}$ as

$$\operatorname{ed}_k(\mathcal{X}) := \operatorname{ed}_k(F_{\mathcal{X}}).$$

Let $\mathcal{X}$ be an integral algebraic stack of finite type over a field k. The *generic essential dimension* of $\mathcal{X}$ is defined as

$$\operatorname{ged}_k \mathcal{X} := \sup\{\operatorname{ed}_k \eta |\ \eta : \operatorname{Spec} K \to \mathcal{X} \text{ is dominant}\}.$$

We say that the stack $\mathcal{X}$ satisfies the *genericity property* if

$$\operatorname{ed}_k \mathcal{X} = \operatorname{ged}_k \mathcal{X}.$$

Let Q be a quiver. It is well known that one may view K-representations of Q as K-orbits of a suitable action. Let

$$X_{Q,\alpha} := \prod_{a\in Q_1} \mathrm{Mat}_{\alpha_{t(a)}\times\alpha_{s(a)},k}, \qquad G_{Q,\alpha} := \prod_{i\in Q_0} \mathrm{GL}_{\alpha_i,k}$$

be an affine space and an algebraic group over k, respectively. There is an action of $G_{Q,\alpha}$ over $X_{Q,\alpha}$, given by

$$(g_i)_{i\in Q_0} \cdot (P_a)_{a\in Q_1} := (g_{t(a)} P_a g_{s(a)}^{-1})_{a\in Q_1}.$$

By [5, Remark 2.2.4(1)], the scheme-theoretic stabilizers for this action are smooth in arbitrary characteristic. We denote by $\mathcal{R}_{Q,\alpha}$ the quotient stack $[X_{Q,\alpha}/G_{Q,\alpha}]$.
By [6, Example 2.6], for every field extension K/k, there is a natural correspondence between the orbits of this action defined over K, that is, K-points of $\mathcal{R}_{Q,\alpha}$, and the isomorphism classes of representations of Q of dimension vector α. Therefore

$$\operatorname{ed}_k \operatorname{Rep}_{Q,\alpha} = \operatorname{ed}_k \mathcal{R}_{Q,\alpha}.$$

Remark 4.1. The construction of $X_{Q,\alpha}$ comes with an α-dimensional representation M^{gen} of Q over the generic point $K := k(X_{Q,\alpha})$ of $X_{Q,\alpha}$, corresponding to the natural inclusion $\operatorname{Spec} K \hookrightarrow X_{Q,\alpha}$. One can show that

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = \operatorname{ed}_k M^{\mathrm{gen}};$$

see [2, Proposition 14.1].

For any k-scheme S, objects of $\mathcal{R}_{Q,\alpha}$ over S are pairs

$$E := (\{E_i\}_{i\in Q_0}, \{\varphi_a\}_{a\in Q_1}),$$

where E_i is a locally free $\mathcal{O}_S$-module of rank α_i for each vertex i and $\varphi_a : E_{s(a)} \to E_{t(a)}$ is a morphism of $\mathcal{O}_S$-modules for each $a \in Q_1$. A morphism $E' \to E$ is given by isomorphisms $E'_i \to E_i$ for each vertex i, satisfying the usual commutativity conditions.
We conclude this section with some considerations on the genericity property for non-wild quivers.

Proposition 4.2. *(a) Let Q be a quiver of finite representation type. Then*

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = \operatorname{ed}_k \operatorname{Rep}_{Q,\alpha} = 0$$

for every dimension vector α.
(b) Let Q be a tame quiver and δ be its null root. Then

$$\operatorname{ged}_k \mathcal{R}_{Q,n\delta} = \operatorname{ed}_k \operatorname{Rep}_{Q,n\delta} = n.$$

Proof. (a) It suffices to show that $\operatorname{ed}_k \operatorname{Rep}_{Q,\alpha} = 0$ for every α. Let M be representation of Q over some field K. By Gabriel's Theorem [27, Theorem 8.12(2)], the dimension vector of every indecomposable summand of $M_{\overline{K}}$ is a real root. By Example 3.2, it follows that every indecomposable summand of $M_{\overline{K}}$ is defined over the prime field of K. By Noether-Deuring's Theorem, it follows that M is defined over the prime field of K. In particular, $\operatorname{ed}_k M = 0$.
(b) We proved in [26, Proposition 9.3] that $\operatorname{ed}_k \operatorname{Rep}_{Q,n\delta} = n$ for each $n \geq 0$. Therefore, it suffices to prove that $\operatorname{ged}_k \mathcal{R}_{Q,n\delta} \geq n$. This follows from the proof of [26, Proposition 9.3], but we repeat the argument here.
We may assume that k is algebraically closed. There is a one-parameter family of δ-dimensional indecomposable representations of Q. By [29, Theorem

3.8], the canonical decomposition of $n\delta$ is $\sum_{h=1}^{n}\delta$. It follows that there exists a $G_{Q,n\delta}$-invariant dense open subset $Z_n \subseteq X_{Q,n\delta}$ such that for every representation M parametrized by Z_n we have $M = \oplus_{h=1}^{n} M_h$, where each M_h is indecomposable and has dimension vector δ. Consider n copies of an infinite family of indecomposable representations of dimension vector δ parametrized by an open subset of $\mathbb{A}^1_k$. This gives an generically finite dominant rational map

$$\rho : \mathbb{A}^n_k \dashrightarrow Z_n$$

such that a general $G_{Q,n\delta}$-orbit intersects the image of ρ, and does so in at most finitely many points (by the Krull-Schmidt Theorem). It follows that the Rosenlicht quotient $Z_n/G_{Q,n\delta}$ has dimension $\geq n$. Let now $\operatorname{Spec} K \to \mathcal{R}_{Q,n\delta}$ be a dominant morphism. Then $\operatorname{Spec} K$ maps to the generic point of $Z_n/G_{Q,n\delta}$, hence $\operatorname{ged}_k \mathcal{R}_{Q,n\delta} \geq n$. □

5 The Colliot-Thélène - Karpenko - Merkurjev Conjecture

As noted in the Introduction, part of the statement of Theorem 1.1 depends on a conjecture due to Colliot-Thélène, Karpenko and Merkurjev, formulated in [9, §1]. Following [3], we rephrase this conjecture in a way that is better suited to our needs.

Let A be a finite-dimensional k-algebra. We say that a projective right A-module M of finite dimension over k has *rank* $r \in \mathbb{Q}_{>0}$ if the direct sum $M^{\oplus n}$ is free of rank nr for some $n \in \mathbb{N}$ with $nr \in \mathbb{N}$. We let $\operatorname{Mod}_{A,r}$ be the functor of isomorphism classes of projective A-modules of rank r.

Recall from [23, §4a] that a functor $F : \mathrm{Fields}_k \to \mathrm{Sets}$ is called a *detection functor* if $|F(K)| \leq 1$ for every field extension K/k. By [3, Proposition 2.4], for every positive rational number r, $\operatorname{Mod}_{A,r}$ is a detection functor. If $A = D$ is a division algebra, and K/k is a field extension, by definition $\operatorname{Mod}_{D,r}(K) \neq \emptyset$ if and only if $X_D(K) \neq \emptyset$, where X_D is the Severi-Brauer variety of $(\deg D)$-dimensional right-ideals in D.

Let X be a smooth projective k-variety. We denote by $\operatorname{cd}(X)$ the *canonical dimension* of X, that is, the minimum dimension of a subvariety Y of X such that $Y(k(X)) \neq \emptyset$. Let G be a split reductive group over k, and let B be a k-split Borel subgroup of G. We define the *canonical dimension* $\operatorname{cd}(G)$ of G as the maximum of the canonical dimensions of the K-varieties T/B_K, where K/k is a field extension and T is a G_K-torsor. We refer the reader to [2], [17] for an extensive treatment of the canonical dimension of varieties and algebraic groups, and to [25, §2.2] for the definition of canonical dimension of a gerbe and for a useful summary. By [23, §4a], $\operatorname{ed}_k \operatorname{Mod}_{D,1/\deg D} = \operatorname{cd}(X_D)$.

The following conjecture and proposition were originally stated using canonical dimension and incompressibility of X_D in [9, §1]. For our purposes, it is better to rephrase them using the functor $\operatorname{Mod}_{D,1/\deg D}$, as is done in [3, Conjecture 3.10].

Conjecture 5.1. *Let $d \geq 1$. If D is a central division algebra of degree d over k, then*

$$\mathrm{ed}_k(\mathrm{Mod}_{D,1/d}) = \sum_{p|d} (p^{v_p(d)} - 1),$$

the sum being over all primes p.

Proposition 5.2. *Let $d \geq 1$. If D is a central division algebra of degree d over k, then*

$$\mathrm{ed}_k(\mathrm{Mod}_{D,1/d}) \leq \sum_{p|d} (p^{v_p(d)} - 1),$$

the sum being over all primes p. Equality holds if d is a prime power or 6.

Proof. See [3, Corollary 3.8]. The inequality is proved in [9, §1]. The equality is proved in [16, Corollary 4.4] when d is a prime power, and in [9, Theorem 1.3] when $d = 6$. □

6 Elementary examples

The following examples serve to illustrate the difference between essential dimension and generic essential dimension, in the context of quiver representations. They show that the failure of the genericity property is quite frequent.

Example 6.1. Let Q be the 2-Kronecker quiver:

$$1 \rightrightarrows 2$$

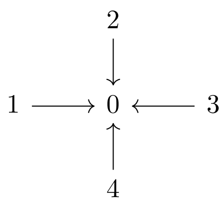
Representations of Q have been classified over an arbitrary field; see [7, Theorem 3.6]. The quiver Q is tame. The real roots of Q are the dimension vectors of the form $(n, n+1)$ and $(n+1, n)$, for each $n \geq 0$. The null root of Q is $\delta = (1,1)$, therefore the imaginary roots of Q are of the form $n\delta = (n,n)$. The generic representation of Q of dimension vector $(n, n+1)$ is indecomposable, and by Example 3.2 it is defined over the prime field. It follows that

$$\mathrm{ged}_k\, \mathcal{R}_{Q,(n,n+1)} = 0.$$

On the other hand, by [26, Proposition 9.3] we have

$$\mathrm{ed}_k\, \mathrm{Rep}_{Q,(n,n+1)} = \mathrm{ed}_k\, \mathrm{Rep}_{Q,n\delta} = n.$$

Example 6.2. Let m, n be non-negative integers, and consider the quiver Q_m with $m+1$ vertices labeled $0, 1, \ldots, m$, and one arrow a_i such that $s(a_i) = i$ and $t(a_i) = 0$ for every $i = 1, \ldots, m$. Here is a picture when $m = 4$.



The quiver Q_m is of finite representation type when $m \leq 3$, tame when $m = 4$, and wild for $m \geq 5$. As dimension vector, choose

$$\alpha_{m,n} := (n+1, 1, \dots, 1) \in \mathbb{R}^{m+1}.$$

An $\alpha_{m,n}$-dimensional representation of Q_m over K is given by at most m lines in K^{n+1}, up to linear automorphisms of K^{n+1}. It is basically the datum of at most m points in $\mathbb{P}^n_K$ up to projective equivalence. More precisely, consider the functor $\mathrm{Rep}_{Q_m,\alpha_{m,n}}$ and

$$\begin{aligned} L_{m,n} : \mathrm{Fields}_k &\to \mathrm{Sets} \\ K &\mapsto \{\mathrm{PGL}_{n+1}\text{-orbits in } (\mathbb{P}^n \cup \{0\})^m(K)\} \end{aligned}$$

where PGL_{n+1} acts diagonally on $(\mathbb{P}^n)^r$ for every $0 \leq r \leq m$, and fixes 0.
There is a morphism of functors $\Phi : \mathrm{Rep}_{Q_m,\alpha_{m,n}} \to L_{m,n}$ constructed as follows. If (M, φ) is a K-representation, fix an isomorphism $\mathbb{P}(M_0) \cong \mathbb{P}^n_K$. Then Φ sends (M, φ) to the orbit of the K-point of $(\mathbb{P}^n \cup \{0\})^m(K)$ whose r-th component is $\mathrm{Im}\,\varphi_{\alpha_i}$ when $\varphi_{\alpha_i} \neq 0$, and the point 0 otherwise. Of course, the orbit associated to (M, φ) in this way does not depend on the choice of an isomorphism $M_0 \cong K^{n+1}$.
We want to show that Φ is an isomorphism. It is immediate to check that if two K-representations map to the same orbit, then they are isomorphic, so Φ is injective. Given a K-orbit $\mathcal{O}$ of $(\mathbb{P}^n \cup \{0\})^m$, choose a K-point $(L_1, \dots, L_m) \in \mathcal{O}$. Set $M_0 := K^n, M_i := K$ for $i \geq 1$ and let φ_{α_i} be the zero map if $L_i = 0$, and send 1 to any non-zero vector lying on the line L_i otherwise. This defines a representation (M, φ) such that $\Phi(M, \varphi) = \mathcal{O}$, so Φ is surjective. Therefore Φ is an isomorphism. In particular, $\mathrm{ed}_k \mathrm{Rep}_{Q_m,\alpha_{m,n}} = \mathrm{ed}_k L_{m,n}$.
We start by computing $\mathrm{ged}_k \mathcal{R}_{Q_m,\alpha_{m,n}}$. If a morphism $\mathrm{Spec}\, K \to \mathcal{R}_{Q_m,\alpha_{m,n}}$ is dominant, the corresponding orbit in $(\mathbb{P}^n \cup \{0\})^m(K)$ consists of m-uples of points in $(\mathbb{P}^n)^m$ in general position. If $m \leq n+2$ then PGL_{n+1} acts transitively on m-uples of points in general position. If $m > n+2$ and the points are in general position, we may assume after acting with PGL_{n+1} that $n+2$ of them will be of the form

$$(1 : 0 : \cdots : 0), (0 : 1 : 0 : \cdots : 0), \dots, (0 : \cdots : 0 : 1), (1 : \cdots : 1). \tag{6.1}$$

The PGL_{n+1}-orbit of this m-tuple is then completely determined by the remaining $m-n-2$ points. Since any one of them is determined by $n+1$ coordinates up to simultaneous rescaling, each of the $m-n-2$ points contributes at most n to the essential dimension. Moreover, consider the configuration of m points, where the first $n+2$ are as in (6.1), and the remaining $m-n-2$ are of the form

$$(1 : a_{i1} : \cdots : a_{in}), \quad i = 1, \dots, m-n-2,$$

where the a_{ij} are independent variables over k. This configuration has a minimal field of definition $K := k(a_{ij})_{i,j}$, so that $\mathrm{trdeg}_k K = n(m-n-2)$. Moreover, the corresponding map $\operatorname{Spec} K \to \mathcal{R}_{Q_m,\alpha_{m,n}}$ is dominant. We obtain:

$$\operatorname{ged}_k \mathcal{R}_{Q_m,\alpha_{m,n}} = \begin{cases} 0 \text{ if } m \leq n+2, \\ n(m-n-2) \text{ if } m > n+2. \end{cases}$$

We now determine $\operatorname{ed}_k \operatorname{Rep}_{Q_m,\alpha_{m,n}}$. In order to compute it, we may clearly restrict ourselves to representations (M,φ) such that $\varphi_{\alpha_i} \neq 0$ for every i, that is, PGL_{n+1}-orbits in $(\mathbb{P}^n)^m$. Consider a configuration of points spanning a subspace H of $\mathbb{P}^n$ of dimension $r \leq \min(n, m-1)$. After a translation by an element of PGL_{n+1}, we may assume that H is given by the vanishing of the last $n-r$ coordinates. If $m = r+1$, PGL_{n+1} acts transitively on m-uples of points of H. If $m \geq r+2$, the action of PGL_{n+1} may be used to put $r+2$ points in the form

$$(1:0:\cdots:0), \ldots, (0:\cdots:0:1:0:\cdots:0), (1:\cdots:1:0:\cdots:0).$$

The remaining $m-r-2$ points are now fixed, and are determined by $r+1$ coordinates up to scaling. Using the inequality $ab \leq \frac{1}{4}(a+b)^2$, it is easy to see that $\operatorname{ed}_k \operatorname{Rep}_{Q_m,\alpha_{m,n}}$ is at most:

$$\max_{1 \leq r \leq \min(n,m-1)} r(m-r-2) = \begin{cases} \frac{1}{4}(m-2)^2 \text{ if } m \leq 2n \text{ is even}, \\ \frac{1}{4}(m-1)(m-3) \text{ if } m \leq 2n \text{ is odd}, \\ n(m-n-2) \text{ if } m > 2n. \end{cases} \tag{6.2}$$

Moreover, one can construct examples showing that equality actually holds, in a way which is totally analogous to what we did for $\operatorname{ged}_k \mathcal{R}_{Q_m,\alpha_{m,n}}$, so $\operatorname{ed}_k \operatorname{Rep}_{Q_m,\alpha_{m,n}}$ is given by (6.2).
This gives a very explicit class of examples for which the genericity property does not hold. The simplest among these examples is when $m = 4$ and $n = 2$. In this case $Q = \widetilde{D}_4$ is tame, and $\alpha_{4,2} = (3,1,1,1,1)$. Since PGL_3 acts transitively on 4-uples of points in $\mathbb{P}^2$ in general position, the generic essential dimension is zero. On the other hand, if the 4 points lie on a common line, and they are chosen generically on that line, the essential dimension of the configuration is 1.

7 Proof of Theorem 1.1

Let $\mathcal{X}$ be an irreducible algebraic stack. Then $\mathcal{X}$ admits a *generic gerbe*, defined as the residual gerbe at any dominant point $\operatorname{Spec} K \to \mathcal{X}$ (see [21, Chapitre 11]). If α is a Schur root for the quiver Q, the generic α-dimensional representation of Q is a brick. By [5, Remark 2.2.4(1)], the scheme-theoretic stabilizers of the $G_{Q,\alpha}$-action on $X_{Q,\alpha}$ are smooth. In particular, the stabilizer of a brick is isomorphic to $\mathbb{G}_{\mathrm{m}}$. It follow that the residue gerbe of a brick is a $\mathbb{G}_{\mathrm{m}}$-gerbe, and so gives rise to a Brauer class in $\operatorname{Br}(k(\mathcal{G}))$; see [13, Lemma 4.10] (we will recall the construction in Lemma 7.6 below).

Let A be a central simple algebra over K. Recall that the index of $[A] \in \mathrm{Br}(K)$ is the degree over K of the unique central division algebra D such that $A \cong M_n(D)$; see [12, Definition 2.8.1]. It is also the greatest common divisor of the degrees of the finite separable field extensions L/K that split A; see [12, Proposition 4.5.1].

If $\mathcal{G}$ is the residue gerbe of a brick, we define the index of $\mathcal{G}$ as the index of the corresponding Brauer class in $\mathrm{Br}(k(\mathcal{G}))$. Our strategy will be to first compute the index of the generic gerbe of $\mathcal{R}_{Q,\alpha}$, and then combine this information with Proposition 5.2 to deduce an upper bound for the essential dimension of the generic gerbe.

Lemma 7.1. *Let $\mathcal{G}$ be the residue gerbe of a brick of $\mathcal{R}_{Q,\alpha}$. Then* $\operatorname{ind}\mathcal{G}$ *divides* $\gcd_{i\in Q_0}(\alpha_i)$.

Proof. Since $\mathcal{G}$ parametrizes bricks, it is a $\mathbb{G}_{\mathrm{m}}$-gerbe, so its index is well-defined. By [13, Lemma 4.10] we know that $\operatorname{ind}\mathcal{G}$ is the greatest common divisor of the ranks of all the twisted sheaves (i.e., vector bundles of weight 1, as defined in [13, Definition 4.1]) on some open substack of $\mathcal{R}_{Q,\alpha}$.

To prove that $\operatorname{ind}\mathcal{G}$ divides $\gcd(\alpha_i)$, it is therefore sufficient to exhibit for every $i \in Q_0$ a twisted sheaf on $\mathcal{R}_{Q,\alpha}$ of rank α_i. Recall that a vector bundle of rank r on $\mathcal{R}_{Q,\alpha}$ is a 1-morphism $\mathcal{V} : \mathcal{R}_{Q,\alpha} \to \mathcal{V}ect_r$. If S is a scheme over k, an object of $\mathcal{R}_{Q,\alpha}(S)$ is a pair $E := (\{E_i\}_{i\in Q_0}, \{\varphi_a\}_{a\in Q_1})$, where E_i is a locally free sheaf over S of rank α_i for each vertex i and $\varphi_a : E_{s(a)} \to E_{t(a)}$ is a morphism $\mathcal{O}_S$-modules for each arrow a. Fix a vertex $i_0 \in Q_0$, and set $\mathcal{V}(E) := E_{i_0}$. Now let $E \in \mathcal{R}_{Q,\alpha}(S)$ and $E' := (\{E'_i\}_{i\in Q_0}, \{\varphi'_a\}_{a\in Q_1}) \in \mathcal{R}_{Q,\alpha}(S')$, where S' is also a scheme over k and let $f := (f_i : E'_i \to E_i)_{i\in Q_0}$ be a morphism from E' to E in $\mathcal{R}_{Q,\alpha}$, set $\mathcal{V}(f) := f_{i_0}$. By definition, $\mathcal{V}$ is a vector bundle of weight 1 and rank α_{i_0}. □

Lemma 7.2. *Suppose that there is a line bundle $\mathcal{L}$ of weight $w \in \mathbb{Z}$ on an open substack $\mathcal{U}$ of $\mathcal{R}_{Q,\alpha}$. Then we may extend $\mathcal{L}$ to a line bundle $\mathcal{L}'$ on $\mathcal{R}_{Q,\alpha}$ of the same weight.*

Proof. We make use of the following standard result, proved in [21, Corollaire 15.5].

Fact 7.3. *Let $\mathcal{X}$ be a noetherian algebraic stack over k and $\mathcal{U}$ an open substack of $\mathcal{X}$. Denote by $j : \mathcal{U} \to \mathcal{X}$ the inclusion 1-morphism. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module and $\mathcal{N}$ a coherent $\mathcal{O}_{\mathcal{U}}$-submodule of $j^*\mathcal{M}$. Then there exists a coherent $\mathcal{O}_{\mathcal{X}}$-submodule $\mathcal{N}'$ of $\mathcal{M}$ such that $j^*\mathcal{N}' = \mathcal{N}$.*

In our case we take $\mathcal{X} = \mathcal{R}_{Q,\alpha}$, $\mathcal{N} = \mathcal{L}$ and $\mathcal{M} = j_*\mathcal{L}$. Since $\mathcal{R}_{Q,\alpha}$ is noetherian, $\mathcal{M}$ is quasi-coherent. The lemma gives us a coherent subsheaf $\mathcal{F} \subseteq j_*\mathcal{L}$. Then the double dual $\mathcal{L}' := \mathcal{F}^{**}$ is a reflexive coherent sheaf of rank one on a smooth stack. By [4, VII 4.2], it follows that $\mathcal{L}'$ is a line bundle. The weight of $\mathcal{L}'$ is w because this may be checked on $\mathcal{U}$, where $\mathcal{L}'$ restricts to $\mathcal{L}$. □

Lemma 7.4. *Suppose that there is a line bundle $\mathcal{L}$ of weight $w \in \mathbb{Z}$ on an open substack $\mathcal{U}$ of $\mathcal{R}_{Q,\alpha}$. Then* $\gcd(\alpha_i)$ *divides* w.

Proof. By the previous lemma we may assume that $\mathcal{L}$ is defined on $\mathcal{R}_{Q,\alpha}$. Denote by $S_\alpha \in \mathcal{R}_{Q,\alpha}(k)$ the trivial representation of Q of dimension vector α over k, for which the linear maps are all zero. Then the central $\mathbb{G}_{\mathrm{m}} \subseteq \mathrm{GL}_\alpha := \prod_i \mathrm{GL}_{\alpha_i} = \mathrm{Aut}(S_\alpha)$ acts with weight w on the fiber of $\mathcal{L}$ over S_α. Since any one-dimensional representation of GL_α is of the form

$$(A_1, \dots, A_r) \mapsto \det(A_1)^{m_1} \cdot \ldots \cdot \det(A_r)^{m_r},$$

we get $w = m_1\alpha_1 + \cdots + m_r\alpha_r$, by restricting the above formula to r-uples of diagonal matrices. Hence w is a multiple of $\gcd(\alpha_i)$. □

Proposition 7.5. *Let α be a Schur root. The index of the generic gerbe of $\mathcal{R}_{Q,\alpha}$ is equal to* $\gcd(\alpha_i)$.

Proof. Let us call $\mathcal{G}$ the generic gerbe of $\mathcal{R}_{Q,\alpha}$. By Lemma 7.1, $\operatorname{ind}\mathcal{G}$ divides $\gcd(\alpha_i)$, so it suffices to show that $\gcd(\alpha_i)$ divides $\operatorname{ind}\mathcal{G}$. Let $\mathcal{V}$ be a vector bundle of rank n and weight w on some open substack $\mathcal{U}$. Define $\mathcal{L} := \det(\mathcal{M})$, then $\mathcal{L}$ is a line bundle of weight nw. In particular, if $\mathcal{V}$ has weight 1, $\mathcal{L}$ has weight n, so by Lemma 7.4 $\gcd(\alpha_i)$ divides n. We conclude that $\gcd(\alpha_i)$ divides $\operatorname{ind}\mathcal{G}$, as desired. □

Let $\mathcal{G}$ be the residue gerbe of a brick in $\mathcal{R}_{Q,\alpha}$, for some Schur root α. Since $\mathcal{G}$ parametrizes bricks, it is a $\mathbb{G}_{\mathrm{m}}$-gerbe, and so admits a Brauer class in $\mathrm{Br}(k(\mathcal{G}))$. On the other hand, $\mathcal{G}$ admits a smooth cover which is of finite type over $k(\mathcal{G})$, so by the Nullstellensatz there exists a field extension $l/k(\mathcal{G})$ of finite degree d such that $\mathcal{G}(l)$ is non-empty. If $V \in \mathcal{G}(l)$,

$$R := \mathrm{End}_{k(\mathcal{G})}(V)$$

is a central simple algebra over $k(\mathcal{G})$ split by l. It is not hard to check that this class is independent of the chosen field extension $l/k(\mathcal{G})$.

Lemma 7.6. *The Brauer classes of $\mathcal{G}$ and R in* $\mathrm{Br}(k(\mathcal{G}))$ *coincide.*

Proof. We briefly recall the construction of the Brauer class of $\mathcal{G}$, as given in [13, Lemma 4.10]. One starts by choosing a field extension $l/k(\mathcal{G})$ of finite degree d such that $\mathcal{G}(l)$ is non-empty. This means that $\mathcal{G}_l \cong B\,\mathbb{G}_{\mathrm{m}}$, so it admits a line bundle $\mathcal{L}_1$ of weight 1, corresponding to the tautological 1-dimensional representation of $\mathbb{G}_{\mathrm{m}}$. If $\pi : \mathcal{G}_l \to \mathcal{G}$ denotes the natural projection, $\mathcal{V} := \pi_*\mathcal{L}_1$ is a vector bundle of rank d and weight 1 on $\mathcal{G}$. The algebra bundle $\mathcal{E}nd(\mathcal{V})$ on $\mathcal{G}$ has weight 0, and so descends to a central simple algebra A over $k(\mathcal{G})$ split by l. By definition, the Brauer class of $\mathcal{G}$ is that of A. One then checks that this definition does not depend on the choice of the extension $l/k(\mathcal{G})$.

There is a chain of isomorphisms of $k(\mathcal{G})$-vector spaces:

$$\begin{aligned} R = \operatorname{Hom}_{k(\mathcal{G})}(V,V) &\cong \operatorname{Hom}_l(V \otimes_{k(\mathcal{G})} l, V) \\ &\cong \operatorname{Hom}_l(V^d, V) \\ &\cong \operatorname{Hom}_l(V,V)^d \\ &\cong \operatorname{Hom}_l(\mathcal{L}_1(V), \mathcal{L}_1(V))^d \\ &\cong \operatorname{Hom}_l(\mathcal{L}_1(V)^d, \mathcal{L}_1(V)) \\ &\cong \operatorname{Hom}_l(\pi^*\mathcal{V}(V), \mathcal{L}_1(V)) \\ &\cong \operatorname{Hom}_{k(\mathcal{G})}(\mathcal{V}(V), \mathcal{V}(V)) = A. \end{aligned}$$

The map $\operatorname{Hom}_l(V,V) \to \operatorname{Hom}_l(\mathcal{L}_1(V), \mathcal{L}_1(V))$ is the one induced by the functor $\mathcal{L}_1$, and the map $\operatorname{Hom}_{k(\mathcal{G})}(V,V) \to \operatorname{Hom}_{k(\mathcal{G})}(\mathcal{V}(V), \mathcal{V}(V))$ is exactly the map given by the functor $\mathcal{V}$, hence both respect compositions. Thus the map $R \to A$ is an isomorphism of $k(\mathcal{G})$-algebras. □

Proof of Theorem 1.1. Let $\operatorname{Spec} K \to \mathcal{R}_{Q,\alpha}$ be a dominant map, corresponding to an α-dimensional K-representation M. Since α is a Schur root, M is a brick. We have

$$\operatorname{trdeg}_k k(M) \leq \dim \operatorname{Aut}(M) + \dim \mathcal{R}_{Q,\alpha} = 1 - \langle \alpha, \alpha \rangle .$$

Let $\mathcal{G}$ be the residue gerbe of M. It is a $\mathbb{G}_{\mathrm{m}}$-gerbe with residue field $k(M)$. From [26, Theorem 4.4] we see that

$$\operatorname{ed}_{k(M)} \mathcal{G} = \operatorname{ed}_{k(M)}(\operatorname{Mod}_{R,1/\deg R}),$$

for some central simple algebra R over $k(M)$ split by l. By Lemma 7.6 and Lemma 7.1, the index of R divides $\gcd(\alpha_i)$.
Write $R = M_n(D)$, for some central division algebra D over $k(M)$ and some $n \geq 1$. We have $\deg R = n \deg D$ and $\operatorname{ind} R = \operatorname{ind} D$. By [3, Proposition 3.4], we obtain

$$\operatorname{ed}_{k(M)}(\operatorname{Mod}_{R,1/\deg R}) = \operatorname{ed}_{k(M)}(\operatorname{Mod}_{D,n/\deg R}) = \operatorname{ed}_{k(M)}(\operatorname{Mod}_{D,1/\deg D}).$$

Inequality (1.1) now follows from Proposition 5.2. Furthermore, by Proposition 7.5 the index of the generic gerbe is $\gcd(\alpha_i)$, so equality in (1.1) follows from Conjecture 5.1 and Proposition 5.2, for $d = \gcd(\alpha_i)$. □

Corollary 7.7. *Let α be a root of Q. If the canonical decomposition of α consists only of real roots, then*

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = 0.$$

Otherwise, let β be the unique imaginary Schur root appearing in the canonical decomposition of α; see [29, Theorem 4.4]. If β is isotropic of multiplicity $m \geq 1$, then

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = m.$$

If β is anisotropic, then

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} \leq 1 - \langle \beta, \beta \rangle + \sum_p (p^{v_p(\gcd(\beta_i))} - 1). \tag{7.1}$$

One has equality if Conjecture 5.1 holds for $d = \gcd(\alpha_i)$.

Proof. Our argument will make use of the reflection functors. We refer the reader to [20, Section 3.2] for background material on reflection functors. We note that reflection functors may be defined over any field, and their formation commutes with extension of scalars. It is an immediate consequence of [20, Theorem 3.11] that if σ_i is a reflection at an admissible vertex i (a source or a sink), and α is a Schur root, then

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = \operatorname{ged}_k \mathcal{R}_{Q',\sigma_i(\alpha)}$$

where Q' is obtained from Q by reversing all the arrows at i.
Let now α be a root. By [29, Theorem 4.4] the canonical decomposition of α contains at most one imaginary root. If all roots are real, by Example 3.2 the generic representation is a direct sum of indecomposable representations, all of which are defined over the prime field of k by Example 3.2. Hence $\operatorname{ged}_k \operatorname{Rep}_{Q,\alpha} = 0$ in this case. Assume now that there exists an imaginary root β in the canonical decomposition of α, and let M be a generic α-dimensional representation.
Using a suitable sequence of reflection functors we may assume that β is in the fundamental region of Q. We remark that although the reflection functors change orientation of the arrows, the fundamental region does not change. By [22, Proposition 4.14], β is either an anisotropic Schur root, or is a multiple of the null root of some tame subquiver of Q. In the first case, one may apply [26, Lemma 6.4] and the first part Theorem 1.1 to conclude. In the second case, the result follows from [26, Lemma 6.4] and Proposition 4.2(b). □

Remark 7.8. If we consider generic essential p-dimension (see [23, §1.1]), the inequalities (1.1) and (7.1) of Theorem 1.1 become unconditional equalities:

$$\operatorname{ged}_{k,p} \mathcal{R}_{Q,\alpha} = 1 - \langle \alpha, \alpha \rangle + \max_p (p^{v_p(\gcd(\alpha_i))} - 1)$$

and

$$\operatorname{ged}_{k,p} \mathcal{R}_{Q,\alpha} = 1 - \langle \beta, \beta \rangle + \max_p (p^{v_p(\gcd(\beta_i))} - 1).$$

In the notation of (1.1) and (7.1), this gives unconditional lower bounds

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} \geq 1 - \langle \alpha, \alpha \rangle + \max_p (p^{v_p(\gcd(\alpha_i))} - 1)$$

and

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} \geq 1 - \langle \beta, \beta \rangle + \max_p (p^{v_p(\gcd(\beta_i))} - 1).$$

We now give an unconditional formula for $\operatorname{ged}_k \mathcal{R}_{Q,\alpha}$, involving canonical dimension; see Section 5 for references on this notion.

Proposition 7.9. *Let α be a Schur root for the quiver Q, and set $d := \gcd(\alpha_i)$. Then*

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = 1 - \langle \alpha, \alpha \rangle + \operatorname{cd}(G_{Q,\alpha}/\mu_d). \tag{7.2}$$

Recall that $G_{Q,\alpha} := \prod_i \mathrm{GL}_{\alpha_i,k}$, the product being over all $i \in Q_0$. Here μ_d is embedded in $G_{Q,\alpha}$ as the subgroup $\{(\zeta \cdot \mathrm{Id}_{\alpha_i})_{i\in Q_0} : \zeta^d = 1\}$. We will use Proposition 7.9 in the proof of Theorem 1.2.

Proof. This argument was inspired, in part, by the proof of [25, Proposition 7.1]. Let $\overline{G}_{Q,\alpha} := G_{Q,\alpha}/H$, where $H \cong \mathbb{G}_{\mathrm{m}}$ is the diagonal copy of $\mathbb{G}_{\mathrm{m}}$ inside $G_{Q,\alpha}$, and set $\overline{\mathcal{R}}_{Q,\alpha} := [X_{Q,\alpha}/\overline{G}_{Q,\alpha}]$. Let $U_{Q,\alpha}$ be a $G_{Q,\alpha}$-invariant dense open subscheme of $X_{Q,\alpha}$ parametrizing bricks. Since α is a Schur root, we may take $U_{Q,\alpha}$ to be the stable locus for the $G_{Q,\alpha}$-action on $X_{Q,\alpha}$; see [29, Theorem 6.1] and [19]. We define the open substacks $\mathcal{U}_{Q,\alpha} := [U_{Q,\alpha}/G_{Q,\alpha}]$ and $\overline{\mathcal{U}}_{Q,\alpha} := [U_{Q,\alpha}/\overline{G}_{Q,\alpha}]$ of $\mathcal{R}_{Q,\alpha}$ and $\overline{\mathcal{R}}_{Q,\alpha}$, respectively. We have a cartesian diagram

$$\begin{array}{ccc} \mathcal{U}_{Q,\alpha} & \longrightarrow & \mathcal{R}_{Q,\alpha} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \overline{\mathcal{U}}_{Q,\alpha} & \longrightarrow & \overline{\mathcal{R}}_{Q,\alpha} \end{array}$$

where the horizontal maps are open embeddings, and the vertical maps are $\mathbb{G}_{\mathrm{m}}$-gerbes. Since α is a Schur root, $U_{Q,\alpha}$, $\mathcal{U}_{Q,\alpha}$ and $\overline{\mathcal{U}}_{Q,\alpha}$ are non-empty. Moreover, $\overline{G}_{Q,\alpha}$ acts freely on $U_{Q,\alpha}$, so $\overline{\mathcal{U}}_{Q,\alpha}$ is an integral algebraic space of finite type. It has dimension $1 - \langle \alpha, \alpha \rangle$; see [19, Proposition 4.4]. We set $d := \gcd(\alpha_i)$.
Let $\mathcal{G}$ be the generic gerbe of $\mathcal{R}_{Q,\alpha}$, i.e. the generic fiber of π. Its residue field is $k(\mathcal{G}) := k(\overline{\mathcal{U}}_{Q,\alpha})$. Then $\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = 1 - \langle \alpha, \alpha \rangle + \operatorname{ed}_{k(\mathcal{G})} \mathcal{G}$. If γ denotes the class of $\mathcal{G}$ in $H^2(k(\mathcal{G}), \mathbb{G}_{\mathrm{m}})$, then by [25, Proposition 2.3(a)] $\operatorname{ed}_{k(\mathcal{G})} \mathcal{G} = \operatorname{cd} \gamma$.
The action of $\overline{G}_{Q,\alpha}$ on $X_{Q,\alpha}$ is linear and generically free, hence it gives rise to a versal $\overline{G}_{Q,\alpha}$-torsor $t \in H^1(k(\mathcal{G}), \overline{G}_{Q,\alpha})$, and γ is the image of t under the boundary map $H^1(k(\mathcal{G}), \overline{G}_{Q,\alpha}) \to H^2(k(\mathcal{G}), \mathbb{G}_{\mathrm{m}})$ associated with the exact sequence

$$1 \to \mathbb{G}_{\mathrm{m}} \to G_{Q,\alpha} \to \overline{G}_{Q,\alpha} \to 1.$$

Since t is versal, $\operatorname{cd} t = \operatorname{cd} \overline{G}_{Q,\alpha}$; see [25, §2.2]. On the other hand, by [25, Lemma 2.2(b)] we have $\operatorname{cd} t = \operatorname{cd} \gamma$. By [8, Corollary A.2] we have an isomorphism of functors

$$H^1(-, \overline{G}_{Q,\alpha}) \cong H^1(-, G_{Q,\alpha}/\mu_d),$$

hence $\operatorname{cd}(\overline{G}_{Q,\alpha}) = \operatorname{cd}(G_{Q,\alpha}/\mu_d)$. Combining these equalities we obtain

$$\operatorname{ed}_{k(\mathcal{G})}(\mathcal{G}) = \operatorname{cd} \gamma = \operatorname{cd} t = \operatorname{cd} \overline{G}_{Q,\alpha} = \operatorname{cd}(G_{Q,\alpha}/\mu_d). \qquad \square$$

The following general lemma will be used in the proof of Theorem 1.2 and Theorem 1.4.

Lemma 7.10. *Let Q be any quiver, α a Schur root for Q, and M an α-dimensional brick. Then*

$$\operatorname{ed}_k M \leq \operatorname{ged}_k \mathcal{R}_{Q,\alpha}.$$

Proof. Let M be an α-dimensional brick defined over L/k. We must show that $\operatorname{ed}_k M \leq \operatorname{ged}_k \mathcal{R}_{Q,\alpha}$. By Proposition 7.9, this is equivalent to

$$\operatorname{ed}_k M \leq 1 - \langle \alpha, \alpha \rangle + \operatorname{cd}(G_{Q,\alpha}/\mu_d),$$

where $d := \gcd(\alpha_i)$. We write $\overline{M}$ for the image of M in $\overline{\mathcal{R}}_{Q,\alpha}$. Consider a subextension $k \subseteq K \subseteq L$ such that $\overline{M}$ descends to K and $\operatorname{trdeg}_k K = \operatorname{ed}_k \overline{M}$. We have a cartesian diagram

$$\begin{array}{ccc} \mathcal{G}_M & \longrightarrow & \mathcal{U}_{Q,\alpha} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \overline{\mathcal{U}}_{Q,\alpha} \end{array}$$

where $\mathcal{G}_M$ is the residue gerbe of M, and $\mathcal{U}_{Q,\alpha}$ and $\overline{\mathcal{U}}_{Q,\alpha}$ are as in the proof of Proposition 7.9. Since M is defined over L, $\mathcal{G}_M$ is split by L, and the map $M : \operatorname{Spec} L \to \mathcal{U}_{Q,\alpha}$ factors through a map $M_0 : \operatorname{Spec} L \to \mathcal{G}_M$. Now M_0 (and so M) descends to some intermediate subfield $K \subseteq K_0 \subseteq L$ such that $\operatorname{trdeg}_k K_0 \leq \operatorname{ed}_k(\mathcal{G}_M)$. By [25, Proposition 2.3(a)] $\operatorname{ed}_k \mathcal{G}_M = \operatorname{cd} \mathcal{G}_M$. By Proposition 7.5 the generic gerbe has index d, hence it follows from [25, Lemma 2.4(a)] that $\operatorname{ind} \mathcal{G}_M$ divides d. Therefore, by [25, Lemma 2.2(c)], $\operatorname{cd}(\mathcal{G}_M) \leq \operatorname{cd}(\operatorname{GL}_d/\mu_d)$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_d & \longrightarrow & \operatorname{GL}_d & \longrightarrow & \operatorname{GL}_d/\mu_d & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_d & \longrightarrow & G_{Q,\alpha} & \longrightarrow & G_{Q,\alpha}/\mu_d & \longrightarrow & 1 \end{array}$$

with exact rows. Here GL_d is embedded in $G_{Q,\alpha}$ block-diagonally. The associated diagram in cohomology shows that for every field extension k'/k, the coboundary map $H^1(k', \operatorname{GL}_d/\mu_d) \to H^2(k', \mu_d) = \operatorname{Br}(k')[d]$ factors through the coboundary $H^1(k', G_{Q,\alpha}/\mu_d) \to H^2(k', \mu_d)$. Now apply [25, Lemma 2.2(b)] with $G = \operatorname{GL}_d$ or $G = G_{Q,\alpha}$, and $C = \mu_d$, to obtain $\operatorname{cd}(\operatorname{GL}_d/\mu_d) \leq \operatorname{cd}(G_{Q,\alpha}/\mu_d)$. By [25, Lemma 2.2(c)], it follows that $\operatorname{cd}(\mathcal{G}_M) \leq \operatorname{cd}(G_{Q,\alpha}/\mu_d)$. On the other hand, by [26, Corollary 8.2] we have $\operatorname{trdeg}_k k(M) \leq 1 - \langle \alpha, \alpha \rangle$. Thus

$$\operatorname{ed}_k M = \operatorname{trdeg}_k k(M) + \operatorname{ed}_{k(M)} M \leq 1 - \langle \alpha, \alpha \rangle + \operatorname{cd}(G_{Q,\alpha}/\mu_d).$$

Combining this with Proposition 7.9 yields $\operatorname{ed}_k M \leq \operatorname{ged} \mathcal{R}_{Q,\alpha}$, as desired. □

8 Fields of definition

If M is a representation of Q, we denote by $k(M)$ its residue field, i.e., the residue field of its residue gerbe (see [21, Chapitre 11]). Since $k(M)$ is contained in any field of definition for M, we have

$$\mathrm{ed}_k M = \mathrm{ed}_{k(M)} M + \mathrm{trdeg}_k k(M).$$

In this section, we address the first term of this sum, by presenting a strengthening of [26, Lemma 4.8] for quiver algebras.

Lemma 8.1. *Let M be a representation of Q, and let $\mathcal{G}$ be its residue gerbe in $\mathcal{R}_{Q,\alpha}$, with residue field $K := k(\mathcal{G})$. There exists a separable finite field extension l of K such that $\mathcal{G}(l) \neq \emptyset$.*

Proof. Let α be the dimension vector of M. Since $\mathcal{R}_{Q,\alpha}$ is of finite type over k, by [21, Théorème 11.3] the gerbe $\mathcal{G}$ is of finite type over K. We may find a smooth cover $U \to \mathcal{G}$ that is of finite type over K. Let $\mathrm{Spec}\, l \to U$ be a closed point. Then, by the Nullstellensatz, l is a finite extension of K. The composition $\mathrm{Spec}\, l \to U \to \mathcal{G}$ gives an l-point for $\mathcal{G}$, corresponding to an object $\xi \in \mathcal{G}(l)$. This is equivalent to $\mathcal{G}_l \cong B\,\mathrm{Aut}(\xi)$. Since $\mathrm{Aut}(\xi)$ is an open subscheme of a vector space, it is smooth over K, hence $\mathcal{G}$ is smooth over K. It follows that U is also smooth over K, and so we may take l to be separable over K. □

Proposition 8.2. *Let Q be a quiver, and let M be an indecomposable α-dimensional K-representation of Q, for some field K containing k. Then*

$$\mathrm{ed}_{k(M)} M \leq \min_{i \in \mathrm{supp}\, \alpha} \alpha_i - 1.$$

Proof. Let $\mathcal{G}$ be the residue gerbe of the point $\mathrm{Spec}\, K \to \mathcal{R}_{Q,\alpha}$ given by M. By Lemma 8.1 there exists a separable finite field extension l of the residue field $k(\mathcal{G}) = k(M)$ and an l-representation N of Q such that $N_L \cong M_L$ for any field L containing both K and l. We may assume that $l/k(M)$ is Galois. We let G be the Galois group of $l/k(M)$, and we set $d := [l : k(M)]$. We denote by $\overline{N}$ the $k(M)$-representation of Q obtained from N by restriction of scalars. Let

$$\overline{N} = \oplus_{h=1}^{s} N_h^{\oplus r_h}$$

be the decomposition of $\overline{N}$ in indecomposable $k(M)$-representations of Q, where $N_h \cong N_{h'}$ if and only if $h = h'$. We may write

$$\overline{N} \otimes_{k(M)} l = \oplus_{\sigma \in G} N^{\sigma}.$$

Let $L = lK$ be a compositum of l and K. Since M is defined over K and is indecomposable, the Galois group of L/K acts transitively on the set of isomorphism classes of indecomposable summands of M_L. It follows that all indecomposable summands of $N_L \cong M_L$ have the same dimension vector β.

Since every N_h is a summand of $\overline{N}$, we deduce that each N_h has dimension vector multiple of β. In particular, $\operatorname{supp}\beta = \operatorname{supp}\alpha$. For every h, we write $\dim_{k(M)} N_h = n_h\beta$, where $n_h \geq 1$. By definition

$$d\alpha = \dim_{k(M)} \overline{N} = \sum_{h=1}^{s} r_h \dim_{k(M)} N_h = (\sum_{h_1}^{s} r_h n_h)\beta.$$

Consider

$$A := \operatorname{End}_{k(M)}(N)/j(\operatorname{End}_{k(M)}(N))$$

and

$$A_h := \operatorname{End}_{k(M)}(N_h)/j(\operatorname{End}_{k(M)}(N_h)),$$

for $h = 1, \dots, s$. We may write $A_h = M_{r_h}(D_h)$ for some division algebra D_h. Fitting's lemma and [3, Corollary 3.7] imply

$$A \cong \prod_{h=1}^{s} A_h.$$

Let $i \in \operatorname{supp}\alpha = \operatorname{supp}\beta$. By [26, Lemma 4.7], $\dim_{k(M)} D_h \leq \dim(N_i)_h$ for every h. By [3, Corollary 3.7], we have

$$\operatorname{ed}_{k(M)}(\operatorname{Mod}_{A_h,1/d}) < \frac{r_h}{d} \dim_{k(M)}(N_h)_i.$$

Using [3, Proposition 3.3 and Proposition 3.2], we get

$$\operatorname{ed}_{k(M)}(\operatorname{Mod}_{A,1/d}) \leq \sum_h \operatorname{ed}_{k(M)}(\operatorname{Mod}_{A_h,1/d}) < \frac{1}{d} \sum_h r_h \dim_{k(M)}(N_h)_i = \alpha_i$$

for each vertex $i \in \operatorname{supp}\alpha$. The claimed inequality now follows from an application of [26, Theorem 4.4]. □

9 Beginning of the proof of Theorem 1.2

Let Q be a connected quiver. In this section we show that if Q is of finite representation type or admits at least one loop at every vertex, then $\mathcal{R}_{Q,\alpha}$ satisfies the genericity property for every dimension vector α. This will establish one direction of Theorem 1.2; we will prove the other direction in Section 11. By Proposition 4.2(a), the genericity property holds when Q is of finite representation type. We may thus assume that Q has at least one loop at every vertex. We start by reducing the problem to the following assertion. A dimension vector $\alpha \in \mathbb{N}^{Q_0}$ is called *sincere* if $\alpha_i \neq 0$ for every $i \in Q_0$.

Claim 9.1. Let Q be a connected quiver having at least one loop at every vertex. Assume that Q is not the 1-loop quiver. Then for every sincere dimension vector α, and for every α-dimensional representation M of Q that is not a brick, we have

$$\operatorname{ed}_k M \leq - \langle \alpha, \alpha \rangle .$$

Lemma 9.2. *Assume that Claim 9.1 holds. Let Q be a connected quiver with at least one loop at every vertex. Then for every dimension vector α the stack $\mathcal{R}_{Q,\alpha}$ satisfies the genericity property.*

Proof. Since Q has at least one loop at every vertex, every dimension vector α belongs to the fundamental region, hence by [22, Proposition 4.14] either α has tame support or is an imaginary anisotropic Schur root. On the other hand, the only tame quiver with at least one loop at every vertex is the 1-loop quiver, so if α has tame support then $\alpha = me_i$ for some $m \geq 1$ and some vertex i. For such α the genericity property holds (see Example 3.1 or Proposition 4.2(b)). Assume now that α is an imaginary anisotropic Schur root. The subquiver Q' of Q defined by $Q'_0 = \operatorname{supp}\alpha$ and Q'_1 the set of all arrows in Q_1 between vertices in $\operatorname{supp}\alpha$ also has one loop at each vertex, thus we are reduced to the case when α is sincere.
When α is sincere, by Claim 9.1 we have $\operatorname{ed}_k M \leq -\langle\alpha,\alpha\rangle$ for every representation M that is not a brick. By Remark 7.8, $\operatorname{ged}_k \mathcal{R}_{Q,\alpha} \geq 1 - \langle\alpha,\alpha\rangle$, so the maximum must be attained among bricks. The conclusion now follows from Lemma 7.10. ∎

The combination of Lemma 9.2 and Claim 9.1 proves the first implication of Theorem 1.2. The purpose of the remaining part of this section is the proof of Claim 9.1.
Let Q be as in Claim 9.1. For each vertex i of Q, let l_i be the number of loops at i. Since Q has at least one loop at every vertex, we have $l_i \geq 1$ for every $i \in Q_0$, so in the Tits form

$$\langle\beta,\beta\rangle = \sum_{i\in Q_0}(1-l_i)\beta_i^2 - \sum_{a\in Q_1}\beta_{s(a)}\beta_{t(a)}$$

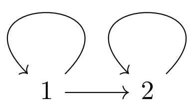
every monomial appears with a non-positive coefficient.
We split the proof into several lemmas.

Lemma 9.3. *Let Q be as in Claim 9.1, and let α be a dimension vector.*

(a) We have

$$-\langle\alpha,\alpha\rangle \geq \min_{i\in Q_0}\alpha_i,$$

with equality if and only if Q is the 2-loop quiver and $\alpha = (1)$ or the quiver



(9.1)

and $\alpha = (1,1)$.

(b) Let $i_0 \in Q_0$ *satisfy* $l_{i_0} \geq 2$*, and write* $\alpha = \sum_{h=1}^{r} \beta_h$*, for some* $\beta_h \in \mathbb{N}^{Q_0} \setminus \{0\}$ *and* $r \geq 2$*. Then*

$$-\sum_{h=1}^{r} \langle \beta_h, \beta_h \rangle \leq -\langle \alpha, \alpha \rangle - \alpha_{i_0}.$$

Proof. (a). The monomials in the Tits form of Q can only appear with negative coefficients (or zero). Since $\alpha_i \alpha_j \geq \alpha_i$ when $\alpha_j \neq 0$, the inequality immediately follows. In order to have equality, it is necessary that the Tits form consists of exactly one monomial. If $l_i \geq 2$ for some i, this implies that Q is a 2-loop quiver, and then it is clear that $\alpha = (1)$ as well. If $l_i = 1$ for every i, then there are two vertices (just one is excluded, because Q is not the 1-loop quiver) connected by exactly one arrow, so the quiver is (9.1) and $\alpha = (1, 1)$. This proves (a).

(b). If $\alpha_{i_0} \geq 2$, then

$$\sum_{h=1}^{r} \beta_{h,i_0}^2 \leq \alpha_{i_0}^2 - 2\alpha_{i_0} + 2 \leq \alpha_{i_0}^2 - \alpha_{i_0};$$

see [3, Lemma 6.5]. We also have the trivial inequalities

$$\sum_{h=1}^{r} \beta_{h,i}^2 \leq \alpha_i^2, \qquad \sum_{h=1}^{r} \beta_{h,s(a)} \beta_{h,t(a)} \leq \alpha_{s(a)} \alpha_{t(a)},$$

for each vertex $i \neq i_0$ and each arrow a. Adding all of these inequalities together gives the conclusion. If $\alpha_{i_0} = 1$, then we need only show that

$$-\sum_{h=1}^{r} \langle \beta_h, \beta_h \rangle < -\langle \alpha, \alpha \rangle,$$

but this is clear because all monomials appear with a positive (or zero) coefficient and $r \geq 2$. ☐

Lemma 9.4. *Let Q be as in Claim 9.1, and let α be a dimension vector. Let M be an indecomposable α-dimensional representation over an algebraically closed field, and assume that M is not a brick. Then*

$$\operatorname{trdeg}_k k(M) \leq 1 - \langle \alpha, \alpha \rangle - \min_{i \in Q_0} \alpha_i.$$

Proof. Using [26, Corollary 8.2], we may write

$$\operatorname{trdeg}_k k(M) \leq 1 - \sum_h \langle \beta_h, \beta_h \rangle,$$

where β_h is the dimension vector of $\operatorname{im} \varphi^{h-1} / \operatorname{im} \varphi^h$ for a generic $\varphi \in \operatorname{End}(M)$. All the entries of β_1 are non-zero, and since the generic φ is non-zero there exists a vertex i_0 such that $\beta_{2,i_0} \neq 0$. If there is a vertex i' with two or more loops, then by Lemma 9.3(b) we have

$$\operatorname{trdeg}_k k(M) \leq 1 - \sum_h \langle \beta_h, \beta_h \rangle \leq 1 - \langle \alpha, \alpha \rangle - \alpha_{i'}$$

and the conclusion follows. Hence we may assume that $l_i = 1$ for every $i \in Q_0$. In this case, since Q is not the 1-loop quiver, Q has at least two vertices. If $j \neq i_0$ is another vertex of Q, then

$$\begin{aligned} \alpha_{i_0}\alpha_j =& (\sum_h \beta_{h,i_0})(\sum_{h'} \beta_{h',j}) \\ =& \sum_h \beta_{h,i_0}\beta_{h,j} + \sum_{h\neq h'} \beta_{h,i_0}\beta_{h',j} \\ \geq& \sum_h \beta_{h,i_0}\beta_{h,j} + \beta_{2,i_0}\beta_{1,j} + \beta_{1,i_0} \sum_{h'\geq 2} \beta_{h',j} \\ \geq& \sum_h \beta_{h,i_0}\beta_{h,j} + \alpha_j. \end{aligned}$$

Fix an arrow a such that $s(a) = i_0$ and $t(a) = i_1$. We consider the estimate above for the term corresponding to a (that is, by letting $j = i_1$), and the inequality

$$\sum_h \beta_{h,s(a')}\beta_{h,t(a')} \leq \beta_{s(a')}\beta_{t(a')}$$

for every arrow $a' \neq a$. Summing up all these inequalities yields

$$-\sum_h \langle \beta_h, \beta_h \rangle \leq -\langle \alpha, \alpha \rangle - \alpha_j \leq -\langle \alpha, \alpha \rangle - \min_{i\in Q_0} \alpha_i. \qquad \square$$

Lemma 9.5. *Let Q and α be as in Claim 9.1, and let K be a field containing k. If M is an indecomposable K-representation of dimension vector α and is not a brick, then*

$$\operatorname{ed}_k M \leq -\langle \alpha, \alpha \rangle.$$

Proof. Consider the decomposition $M_{\overline{K}} = \oplus_{h=1}^s N_h$ in indecomposable representations. By [26, Lemma 12.1], this decomposition is defined over K^{sep}, hence over a finite Galois extension L/K. Since M is defined over K, the Galois group of L/K acts transitively on the set of isomorphism classes of indecomposable summands of M_L. We deduce that if some N_h is a brick all the other summands are bricks as well, and that for each h, h' the iterated images of the generic nilpotent endomorphisms of N_h and $N_{h'}$ have the same dimension vectors. We let $\alpha = \dim_K M$, $\beta = \dim_K N_h$, so that $\alpha = s\beta$.

Assume that N_h is a brick for every h. Then, since M is not a brick, necessarily $s \geq 2$. We have $\operatorname{trdeg}_k k(N_h) \leq 1 - \langle \beta, \beta \rangle$ by [26, Corollary 8.2]. By Lemma 9.3, we have $\min \beta_i \leq -\langle \beta, \beta \rangle - 1$, with the exception of the 2-loop quiver and

$\beta = (1)$, and of the quiver (9.1) and $\beta = (1,1)$. If $\min \beta_i \leq -\langle \beta, \beta \rangle - 1$, using Proposition 8.2 and [26, Corollary 8.2], we obtain:

$$\begin{aligned} \mathrm{ed}_k M &= \mathrm{ed}_{k(M)} M + \mathrm{trdeg}_k k(M) \\ &\leq \mathrm{ed}_{k(M)} M + \sum_h \mathrm{trdeg}_k k(N_h) \\ &\leq s \min_{i \in Q_0} \beta_i - 1 + s(1 - \langle \beta, \beta \rangle) \\ &\leq -s(1 + \langle \beta, \beta \rangle) - 1 + s(1 - \langle \beta, \beta \rangle) \\ &< -2s \langle \beta, \beta \rangle \leq -s^2 \langle \beta, \beta \rangle = -\langle \alpha, \alpha \rangle . \end{aligned}$$

If Q is the 2-loop quiver and $\beta = (1)$, we have $\langle \beta, \beta \rangle = -1$ and $\langle \alpha, \alpha \rangle = -s^2$. If $s \geq 3$, following the same steps as above we obtain

$$\mathrm{ed}_k M \leq 3s - 1 < s^2 = -\langle \alpha, \alpha \rangle .$$

If $s = 2$, we may choose a basis so that M is represented by 2 matrices A_1, A_2 commuting with the nilpotent Jordan block of size 2. This implies that

$$A_i = \begin{pmatrix} a_i & 0 \\ b_i & a_i \end{pmatrix}, \qquad i = 1, 2$$

so $\mathrm{ed}_k M \leq 4 = -\langle \alpha, \alpha \rangle$.
If Q is the quiver (9.1) and $\beta = (1,1)$, we have again $\langle \beta, \beta \rangle = -1$ and $\langle \alpha, \alpha \rangle = -s^2$. If $s \geq 3$, the same computation yields

$$\mathrm{ed}_k M \leq 3s - 1 < s^2 = -\langle \alpha, \alpha \rangle .$$

Assume that $s = 2$, and let a be the unique arrow with $s(a) = 1$ and $t(a) = 2$. Notice that $\varphi_a : M_1 \to M_2$ splits, upon base change to L, into the direct sum of two linear maps of the same rank (they are L-conjugate), so $\mathrm{rank}\, \varphi_a$ is either 0 or 2. In the first case $\varphi_a = 0$ and M is the direct sum of two representations of dimension $(2,0)$ and $(0,2)$, and it is easy to see that $\mathrm{ed}_k M \leq 4$. If φ_a is an isomorphism we may identify M_1 with M_2 via φ_a, so that M becomes a representation of the 2-loop quiver, so $\mathrm{ed}_k M \leq 4$ by the previous case.
Assume now that the N_h are not bricks. Note that this time s might be 1. Combining Proposition 8.2 with Lemma 9.4, we get:

$$\begin{aligned} \mathrm{ed}_k M &\leq \mathrm{ed}_{k(M)} M + \sum_h \mathrm{trdeg}_k k(N_h) \\ &\leq s \min_{i \in Q_0} \beta_i - 1 + s(1 - \langle \beta, \beta \rangle - \min_{i \in Q_0} \beta_i) \\ &< -s \langle \beta, \beta \rangle + s - 1 \leq -\langle \alpha, \alpha \rangle , \end{aligned}$$

the last inequality being equivalent to $-\langle \beta, \beta \rangle s(s-1) \geq s - 1$, which is true because α is sincere and so $\langle \beta, \beta \rangle = s^{-2} \langle \alpha, \alpha \rangle < 0$. This concludes the proof of Lemma 9.5. □

Proof of Claim 9.1. Let M be a K-representation that is not a brick, for some field extension K/k. If M is indecomposable, then $\operatorname{ed}_k M \leq -\langle \alpha, \alpha \rangle$ by Lemma 9.5. If M is decomposable, denote by $M_1, \ldots, M_s$ its indecomposable summands, for some $s \geq 2$. Applying Proposition 8.2 and [26, Corollary 8.2] to every M_l, we obtain

$$\operatorname{ed}_k M \leq \sum_{l=1}^{s} \operatorname{ed}_{k(M_l)} M_l + \sum_{l=1}^{s} \operatorname{trdeg}_k k(M_l) \leq \min_{i \in Q_0} \alpha_i - \sum_h \langle \beta_h, \beta_h \rangle ,$$

where $\sum \beta_h = \alpha$. To prove that $\operatorname{ed}_k M \leq -\langle \alpha, \alpha \rangle$, it suffices to show that

$$-\langle \alpha, \alpha \rangle + \sum_h \langle \beta_h, \beta_h \rangle \geq \min_{i \in Q_0} \alpha_i .$$

Assume first that there exists a vertex j such that the sum $\alpha_j = \sum \beta_{h,j}$ has at least two non-zero terms. Consider an arrow a having j as one of its endpoints, and let i be the other endpoint of a (possibly $i = j$). We have

$$\alpha_i \alpha_j - \sum_h \beta_{h,i} \beta_{h,j} = \sum_h \beta_{h,i}(\alpha_j - \beta_{h,j}) \geq \sum_h \beta_{h,i} = \alpha_i .$$

For every other arrow a', we have

$$\alpha_{s(a')} \alpha_{t(a')} - \sum_h \beta_{h,s(a')} \beta_{h,t(a')} \geq 0 \tag{9.2}$$

The claim follows from adding up all of these inequalities.
Assume now that $\beta_{h,i} \in \{0, \alpha_i\}$ for each vertex i and every h. Since M is decomposable, there exists an arrow a with endpoints i and j (possibly $i = j$) and two distinct positive integers $h_1 \neq h_2$ such that $\beta_{h_1,i} = \alpha_i$ and $\beta_{h_2,j} = \alpha_j$. Then $\beta_{h,i} = 0$ for all $h \neq h_1$, and $\beta_{h_2,j} \neq 0$ for all $h \neq h_2$. Thus

$$\alpha_i \alpha_j - \sum_h \beta_{h,i} \beta_{h,j} = \alpha_i \alpha_j \geq \alpha_i .$$

The claim follows by adding this to the inequalities (9.2), for all arrows $a' \neq a$. ☐

10 Subquivers

If Q is a quiver, recall that a subquiver of Q is a quiver Q' such that $Q'_0 \subseteq Q_0$ and whose arrows are all the arrows of Q between vertices in Q'_0.
To finish the proof of Theorem 1.2, we will need the following combinatorial lemma.

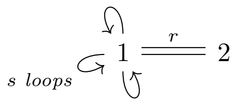
Lemma 10.1. *Let Q be a connected quiver that is not of finite representation type and does not admit at least one loop at every vertex. Then Q contains a subquiver of one of the following types:*

1. *a tame quiver,*

2. *a quiver with two vertices and $r \geq 3$ arrows, none of which is a loop and not necessarily pointing in the same direction,*

$$1 \overset{r}{\equiv} 2$$

3. *a quiver with two vertices, one of which has $s \geq 2$ loops, and with $r \geq 1$ arrows between the two vertices.*



Proof. Note that Q has at least two vertices, otherwise it would be the trivial quiver with one vertex (which is of finite representation type) or an r-loop quiver (which has at least one loop per vertex).
Assume first that Q admits at least one loop. Then, since Q is connected, we can find two adjacent vertices i and j such that there is at least one loop at i and there are no loops at j. If there is exactly one loop at i, then Q admits a 1-loop quiver as a subquiver, and this is tame. If there are at least two loops at i, then Q admits a subquiver of type (3).
Consider now the case when that Q does not have any loops. Assume first that there are two vertices i and j connected by $r \geq 2$ arrows. If $r = 2$, then Q admits a tame subquiver of type $\widetilde{A}_2$. If $r \geq 3$, then it contains a subquiver of type (2). Assume now that Q does not have multiple arrows. If Q admits a cycle, then it admits a tame subquiver of type $\widetilde{A}_n$. The last case to consider is that of a quiver Q without cycles and multiple arrows. By assumption, Q is not of finite representation type. Let Q' be a maximal subquiver of finite representation type of Q. Since Q is not of finite representation type, $Q \neq Q'$, and so Q contains a subquiver Q'' obtained from Q' by adding one new vertex j to Q', connected to a single $i \in Q'_0$ via a unique arrow. One patiently considers all cases for j, and concludes that either Q'' is of finite representation type, or it contains a tame subquiver. More precisely:

- if Q' is of type A, then either Q'' is of type A, D, E, or it contains a subquiver of type $\widetilde{E}$;
- if Q' is of type D, then either Q'' can be of type D, E or it contains a subquiver of type $\widetilde{D}, \widetilde{E}$;
- if Q' is of type E_6, then Q'' either is of type $\widetilde{E}_6, \widetilde{E}_7, E_8$ or contains a subquiver of type $\widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6$, and similarly in the case when Q' is of type E_7 and E_8.

Since Q' is maximal among subquivers of Q of finite representation type, Q'' may not be of finite representation type, and so it contains a tame subquiver. Therefore, Q contains a tame subquiver. ∎

11 End of the proof of Theorem 1.2

Let Q be a connected quiver. In Section 9 we showed that $\mathcal{R}_{Q,\alpha}$ has the genericity property for every dimension vector α if Q is of finite representation type, or if Q has at least one loop at every vertex. In this section we will establish the converse, thus completing the proof of Theorem 1.2.

Assume that for every dimension vector α, the stack $\mathcal{R}_{Q,\alpha}$ satisfies the genericity property. Then the same is true for every subquiver of Q. This is because a representation of a subquiver Q' may be extended to a representation of Q by associating zero vector spaces and zero linear transformations to the vertices and arrows in Q but not in Q'. This gives rise to an isomorphism between the functors $\mathrm{Rep}_{Q,\alpha}$ and $\mathrm{Rep}_{Q',\alpha'}$ where $\alpha \in \mathbb{N}^{Q_0}$ is obtained from $\alpha' \in \mathbb{N}^{Q'_0}$ by filling in zeros for the missing vertices.

Therefore, it suffices to find for every quiver of the list of Lemma 10.1 a dimension vector for which the genericity property does not hold. We will argue in the following way. Suppose that we may find a real root α and a dimension vector β such that $\beta_i \leq \alpha_i$ for each vertex i and such that $\mathrm{ed}_k \mathrm{Rep}_{Q,\beta} > 0$. By Example 3.2 we have $\mathrm{ged}_k \mathcal{R}_{Q,\alpha} = 0$, but on the other hand by [26, Lemma 7.2] one has $\mathrm{ed}_k \mathrm{Rep}_{Q,\alpha} \geq \mathrm{ed}_k \mathrm{Rep}_{Q,\beta} > 0$, so the genericity property does not hold for $\mathcal{R}_{Q,\alpha}$.

Consider first the case when Q is a tame quiver, and let $\beta = \delta$ be its null root. By [20, Theorem 7.8(1)], there exists a real root α such that $\alpha_i \geq \delta_i$ for each vertex i of Q.

Let now Q be of the second type. The dimension vector (n, n) is a Schur root of generic essential dimension at least $1 + (r-1)n^2$, since after fixing an isomorphism between the two vector spaces using one of the arrows, one is reduced to the $(r-1)$-loop quiver. We now construct a suitable real root α. One can easily compute the two simple reflections for Q:

$$(x_1, x_2) \mapsto (rx_2 - x_1, x_2), \qquad (x_1, x_2) \mapsto (x_1, rx_1 - x_2).$$

If we apply them to $(1, 0)$, we get

$$(1, 0) \mapsto (1, r-1) \mapsto (r^2 - r - 1, r - 1).$$

Since $r \geq 3$, we have $r^2 - r - 1 > r - 1$, hence it suffices to choose $\alpha = (r^2 - r - 1, r - 1)$ and $\beta = (r - 1, r - 1)$.

Let now Q be of the third type. Assume first that $r \geq 2$. One can see as in the previous case that the dimension vector (n, n) is a Schur root of generic essential dimension at least $1+(r-1)n^2$. The fundamental region of Q is given by those vectors (x_1, x_2) satisfying

$$rx_1 - 2x_2 \geq 0.$$

The vector $(2, 1)$ is in the fundamental region and is therefore a Schur root. There is only one simple reflection, given by

$$\sigma : (x_1, x_2) \mapsto (x_1, rx_1 - x_2)$$

By Theorem 1.1, the Schur root $\alpha = (2, 2r - 1)$ obtained by reflecting $(2, 1)$ satisfies:

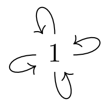
$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = 1 - \langle \alpha, \alpha \rangle = 2r + 4s - 4.$$

On the other hand, since $r \geq 2$, the vector $\beta = (2, 2)$ is component-wise smaller than α, and

$$\operatorname{ged}_k \mathcal{R}_{Q,\beta} = 4r + 4s - 7 > 2r + 4s - 4,$$

thus the genericity property does not hold for α.
If $r = 1$, one may choose $\alpha = (4, 3)$ and $\beta = (4, 2)$. The vector β belongs to the fundamental region $\{(x_1, x_2) : x_1 - 2x_2 \geq 0\}$, hence it is a Schur root and has generic essential dimension 5. The vector α is obtained by reflecting $(2, 1)$, which belongs to the fundamental region. Hence α is also a Schur root, and has generic essential dimension 4. This completes the proof of Theorem 1.2.

Example 11.1. Let $r \geq 1$, and consider the r-loop quiver L_r, here depicted for $r = 4$.



The case $r = 1$ has been considered in Example 3.1. Representations of L_r correspond to representations of the free algebra on r generators. It follows from Theorem 1.1 and Theorem 1.2 that

$$\operatorname{ed}_k \mathcal{R}_{L_r,n} \leq 1 + (r - 1)n^2 + \sum_p (p^{v_p(n)} - 1),$$

with equality when [3, Conjecture 3.10] holds for n.
This example was originally worked out by Z. Reichstein and A. Vistoli (unpublished). Their proof is in the spirit of [25].

12 Proof of Theorem 1.3

The starting point for the proofs of Theorem 1.3 and Theorem 1.4 is the following lemma.

Lemma 12.1. *Let Q be a quiver, and let α be a dimension vector in the fundamental region of Q such that $\alpha_i > 0$ for each vertex i. Write $\alpha = \sum_{h=1}^r \beta_h$ for some dimension vectors $\beta_h \in \mathbb{N}^{Q_0}$.*
(a) We have

$$-\sum_{h=1}^{r} \langle \beta_h, \beta_h \rangle \leq -\langle \alpha, \alpha \rangle .$$

(b) Assume further that for each vertex i there exist at least two vectors β_h satisfying $\beta_{h,i} \neq 0$. Then

$$-\sum_{h=1}^{r} \langle \beta_h, \beta_h \rangle \leq -\langle \alpha, \alpha \rangle - \sum_{i \in Q_0} 2(\alpha_i - 1)\Big(\sum_{a:\, t(a)=i} \frac{\alpha_{s(a)}}{2\alpha_i} + \sum_{a:\, s(a)=i} \frac{\alpha_{t(a)}}{2\alpha_i} - 1 \Big).$$

Proof. Since α belongs to the fundamental region, for each vertex i we have:

$$(\alpha, e_i) = 2\alpha_i - \sum_{a\colon t(a)=i} \alpha_{s(a)} - \sum_{a\colon s(a)=i} \alpha_{t(a)} \leq 0.$$

Since $\alpha_i > 0$, this may be rewritten as

$$\sum_{a\colon t(a)=i} \frac{\alpha_{s(a)}}{2\alpha_i} + \sum_{a\colon s(a)=i} \frac{\alpha_{t(a)}}{2\alpha_i} - 1 \geq 0 \tag{12.1}$$

By algebraic manipulations, starting from

$$(\alpha_i\beta_{h,j} - \alpha_j\beta_{h,i})^2 \geq 0,$$

we obtain

$$\beta_{h,j}\beta_{h,i} \leq \frac{\alpha_i}{2\alpha_j}\beta_{h,j}^2 + \frac{\alpha_j}{2\alpha_i}\beta_{h,i}^2$$

for every $i, j \in Q_0$ and every $h = 1, \dots, r$. Hence

$$\begin{aligned}
1 - \sum_{h=1}^{r} \langle \beta_h, \beta_h \rangle =& 1 - \sum_{i\in Q_0,h} \beta_{h,i}^2 + \sum_{a\in Q_1,h} \beta_{h,s(a)}\beta_{h,t(a)} \\
\leq& 1 - \sum_{i,h} \beta_{h,i}^2 + \sum_{a,h} \frac{\alpha_{s(a)}}{2\alpha_{t(a)}}\beta_{h,t(a)}^2 + \sum_{a,h} \frac{\alpha_{t(a)}}{2\alpha_{s(a)}}\beta_{h,s(a)}^2 \\
=& 1 - \sum_{i,h} \beta_{h,i}^2 + \sum_{i,h} \Big(\sum_{a\colon t(a)=i} \frac{\alpha_{s(a)}}{2\alpha_i}\beta_{h,i}^2 + \sum_{a\colon s(a)=i} \frac{\alpha_{t(a)}}{2\alpha_i}\beta_{h,i}^2 \Big) \\
=& 1 + \sum_{i,h} \beta_{h,i}^2 \Big(\sum_{a\colon t(a)=i} \frac{\alpha_{s(a)}}{2\alpha_i} + \sum_{a\colon s(a)=i} \frac{\alpha_{t(a)}}{2\alpha_i} - 1 \Big).
\end{aligned}$$

(a) By (12.1), the quantities in the parentheses are non-negative. Since $\sum_h \beta_{h,i} = \alpha_i$ and $\beta_{h,i} \geq 0$, clearly

$$\sum_h \beta_{h,i}^2 \leq \alpha_i^2. \tag{12.2}$$

Substituting (12.2) into the previous inequality, we get

$$\begin{aligned}
1 - \sum_{h=1}^{r} \langle \alpha_h, \alpha_h \rangle \leq& 1 + \sum_i \alpha_i^2 \Big(\sum_{a\colon t(a)=i} \frac{\alpha_{s(a)}}{2\alpha_i} + \sum_{a\colon s(a)=i} \frac{\alpha_{t(a)}}{2\alpha_i} - 1 \Big) \\
=& 1 - \langle \alpha, \alpha \rangle .
\end{aligned}$$

(b) The conclusion follows from using, for every vertex i, the inequality

$$\sum_h \beta_{h,i}^2 \leq \alpha_i^2 - 2\alpha_i + 2$$

(see [3, Lemma 6.5]) instead of (12.2). □

Lemma 12.2. *Let $r \geq 3$, and Q be a quiver whose underlying graph has the following form:*

$$1 \overset{r}{\equiv\equiv} 2$$

Then, for infinitely many n, the genericity property holds for the dimension vector (n, n).

Proof. Let M be a K-representation of Q of dimension vector $\alpha = (n, n)$, and let $1 \leq d \leq 2n$ be the number of indecomposable summands in a Krull-Schmidt decomposition of M. By Proposition 8.2 and [26, Corollary 8.2], we may write

$$\mathrm{ed}_k M \leq n - 1 + d - \sum \langle \beta_h, \beta_h \rangle$$

for some dimension vectors β_h satisfying $\sum \beta_h = \alpha$ (note that the β_h are not necessarily the dimension vectors of the summands of M). If $\beta_h = (n, 0)$ or $\beta_h = (0, n)$ for some h, then it is clear that $\mathrm{ed}_k M = 0$. In all other cases, by Lemma 12.1(b) we have

$$-\sum \langle \beta_h, \beta_h \rangle \leq (r-2)n^2 - 4(n-1)(\frac{r}{2} - 1) \leq (r-2)n^2 - 2n + 2.$$

We deduce that

$$\mathrm{ed}_k M \leq d - n + 1 + (r-2)n^2.$$

Assume that n is the power of a prime. Then by Theorem 1.1 we have

$$\mathrm{ged}_k \mathcal{R}_{Q,\alpha} = (r-2)n^2 + n.$$

If $d \leq 2n - 1$, the result follows. On the other hand, if $d = 2n$, then M is a direct sum of representations of dimension vectors $(1, 0)$ or $(0, 1)$, and so $\mathrm{ed}_k M = 0$. We conclude that the genericity property holds when n is the power of a prime. □

Proof of Theorem 1.3. By Theorem 1.2, we may assume that Q does not have at least one loop at every vertex. Moreover, we are allowed to pass to a subquiver of Q. By Lemma 10.1, we may assume that Q is of one of the following types:

1. a quiver obtained from a tame quiver Q' by connecting one extra vertex i_0 without loops to a single vertex i_1 of Q' with $r \geq 1$ arrows,
2. a quiver with two vertices and $r \geq 3$ arrows, none of which is a loop,
3. a quiver with two vertices, such that one vertex has $s \geq 2$ loops and the other vertex has none, and with $r \geq 1$ arrows between the two vertices.

Types (2) and (3) come directly from Lemma 10.1. Type (1) of the list needs further explanation. Assume that Q contains a tame quiver, and let $i_0 \in Q_0$ be connected to at least one vertex i_1 of Q'. If there are $s \geq 1$ loops based at

i_0, then the subquiver of Q whose vertices are i_0 and i_1 is of type (1) if $s = 1$, and of type (3) if $s \geq 2$. Assume now that there are no loops based at i_0. If i_0 is connected to more than one vertex of Q', then Q contains a cycle and admits at least one vertex with at least 3 arrows starting from or pointing to it. This means that Q has a wild proper subquiver Q'', and we may consider the smaller quiver Q'' instead of Q. Iterating this procedure, we eventually arrive to a quiver of type (1).

In case (3), Q has a subquiver with at least one loop at every vertex, so the claim holds. Case (2) has been treated in Lemma 12.2. If Q is of type (1), let δ be the null root of the tame subquiver Q'. Fix $m \geq 0$ and define a dimension vector α of Q by setting $\alpha_{i_0} = 1$ and $\alpha_i = m\delta_i$ for each $i \neq i_0$. In other words, $\alpha = m\delta + e_{i_0}$, where δ is viewed as a vector in $\mathbb{R}^{Q_0}$ by extension to zero. Notice that α belongs to the fundamental region of Q for $m \geq 2$, since

$$(\alpha, e_i) = \begin{cases} 2 - mr\delta_{i_1} & \text{if } i = i_0 \\ -r & \text{if } i = i_1 \\ 0 & \text{otherwise.} \end{cases}$$

By [22, Proposition 4.14] α is an anisotropic Schur root for every $m \geq 2$. We also have

$$\langle \alpha, \alpha \rangle = \langle m\delta, m\delta \rangle + (m\delta, e_{i_0}) + \langle e_{i_0}, e_{i_0} \rangle = 1 - r\alpha_{i_1}.$$

Since $\gcd(\alpha_i) = 1$, by Theorem 1.1 we get

$$\operatorname{ged}_k \mathcal{R}_{Q,\alpha} = 1 - \langle \alpha, \alpha \rangle = r\alpha_{i_1}.$$

Let now K be a field containing k, and let M be an α-dimensional K-representation of Q. By Proposition 8.2, $\operatorname{ed}_{k(M)} M = 0$. We may write $M_{\overline{K}} = M_1 \oplus M_2 \oplus M_3$, where M_1 is the unique indecomposable summand with $(M_1)_{i_0} \neq 0$, M_2 is the direct sum of all imaginary indecomposable summands of $M_{\overline{K}}$, and M_3 is the direct sum of the real ones. Write $\alpha = \beta + c\delta + \gamma$ for the corresponding decomposition of the dimension vector of M. By [26, Corollary 8.2], we may write

$$\operatorname{trdeg}_k k(M_1) \leq 1 - \sum_h \langle \beta_h, \beta_h \rangle$$

for some decomposition $\beta = \sum \beta_h$. Among the β_h, only one is not supported on the tame subquiver Q', and we denote it by β'. For every other β_h, we have $\langle \beta_h, \beta_h \rangle \geq 0$. Writing $\beta' = e_{i_0} + \beta''$, for some $\beta'' \in \mathbb{N}^{Q'_0}$, we obtain

$$\operatorname{trdeg}_k k(M_1) \leq 1 - \langle \beta', \beta' \rangle = 1 - \langle e_{i_0}, e_{i_0} \rangle - \langle \beta'', \beta'' \rangle - (\beta'', e_{i_0}) \leq 1 + r\beta'_{i_1}.$$

From Proposition 4.2(b), we have

$$\operatorname{trdeg}_k k(M_2) \leq c.$$

Example 3.2 gives

$$\operatorname{trdeg}_k k(M_3) = 0.$$

Thus

$$\operatorname{ed}_k M = \operatorname{trdeg}_k k(M) \leq r\beta'_{i_1} + c \leq r(\beta_{i_1} + c\delta_{i_1}) \leq r\alpha_{i_1}.$$

Therefore, the genericity property holds for the dimension vector α. □

Remark 12.3. If Q is of type (1) in the list above, δ is the null root of its tame subquiver Q', $\alpha = m\delta + e_{i_0}$, we have just shown that (when $m \geq 2$) α is a Schur root and the genericity property holds for α. If Q is of type (2), we have shown in the proof of Lemma 12.2 that the genericity property holds for (n, n) when n is the power of a prime. Finally, if Q is of type (3), it contains the s-loop quiver for $s \geq 2$ as a subquiver with unique vertex i_0, and so by Theorem 1.2 the genericity property holds for me_{i_0} for every $m \geq 0$.
By Lemma 10.1, every wild quiver contains at least one subquiver of type (1), (2) or (3). To produce Schur roots for which the genericity property holds, it thus suffices to identify one of these subquivers.

13 Proof of Theorem 1.4

For a positive integer r, let K_r be the r-Kronecker quiver.
Let $\alpha = (a, b)$ be a dimension vector for K_r. The quiver K_1 is of finite representation type, hence by Proposition 4.2(a) we have

$$\operatorname{ed}_k \operatorname{Rep}_{K_1,\alpha} = 0.$$

The indecomposable representations of K_2 were classified by Kronecker (see [7, Theorem 3.6] for a description over an arbitrary field). It follows from the classification that

$$\operatorname{ed}_k \operatorname{Rep}_{K_2,\alpha} = \min(a, b).$$

The purpose of this section is the proof Theorem 1.4. Recall that we have already shown in the course of proving Theorem 1.2 that the genericity property fails for the Schur root $(r^2 - r - 1, r - 1)$. Therefore one cannot expect Theorem 1.4 to hold for every Schur root.
The argument follows steps similar to those of the proof of Claim 9.1. We start with a simple estimate.

Lemma 13.1. *Assume that $\alpha = (a, b)$ is in the fundamental region of K_r, $r \geq 3$. Let*

$$f(a, b) := 2(a-1)(\frac{rb}{2a} - 1) + 2(b-1)(\frac{ra}{2b} - 1).$$

Then

$$f(a, b) \geq \min(a, b) - 1.$$

Proof. Since (a,b) belongs to the fundamental region of K_r, we have $2a \leq rb$ and $2b \leq ra$. Moreover, since f is symmetric, we may assume that $a \geq b$. Then $\frac{ra}{2b} \geq \frac{r}{2}$, so

$$f(a,b) \geq 2(b-1)(\frac{ra}{2b}-1) \geq (b-1)(r-2) \geq b-1. \qquad \square$$

Lemma 13.2. *Assume that M is an indecomposable α-dimensional representation of K_r over an algebraically closed field K, and that M is not a brick. Then*

$$\begin{aligned}\operatorname{trdeg}_k k(M) &\leq 2 - \langle \alpha, \alpha \rangle - \min_{i=1,2}(\alpha_i) \\ &= 2 - a^2 - b^2 + rab - \min(a,b).\end{aligned}$$

Proof. Let $\varphi \in \operatorname{End}_K(M)$ be a generic nilpotent endomorphism of M. Write

$$\alpha_h = (a_h, b_h) := \dim_K(\operatorname{Im}\varphi^{h-1}/\operatorname{Im}\varphi^h)$$

for every $h \geq 0$. If $a_1 = a$, this means that there exists a nilpotent endomorphism ψ of M such that $\psi_1 = 0$ and $\psi_2 \neq 0$. We may choose bases of M_1 and M_2 in such a way that ψ_2 is represented by a nilpotent matrix in Jordan form. With respect to these bases, the matrices $A_1, \dots, A_r$ corresponding to the r arrows of K_r all have at least one common row made of only zeros. This is impossible, since M was supposed to be indecomposable. An analogous reasoning proves that $b_1 \neq b$, so each of the decompositions $a = \sum a_h$ and $b = \sum b_h$ contains at least two summands. Using [26, Corollary 8.2] and Lemma 12.1(b), we obtain

$$\operatorname{trdeg}_k k(M) \leq 1 - \sum_{h=1}^{r} \langle \alpha_h, \alpha_h \rangle \leq 1 - \langle \alpha, \alpha \rangle - f(a,b),$$

where $f(a,b)$ is as in Lemma 13.1. By Lemma 13.1, we have $f(a,b) \geq \min(a,b) - 1$. Therefore

$$\operatorname{trdeg}_k k(M) \leq 1 - a^2 - b^2 + rab - \min(a,b) + 1. \qquad \square$$

Lemma 13.3. *Assume that M is an indecomposable α-dimensional representation of K_r over an arbitrary field K containing k. If M is not a brick, then*

$$\operatorname{ed}_k M \leq 1 - \langle \alpha, \alpha \rangle .$$

Proof. Consider the decomposition $M_{\overline{K}} = \oplus_{h=1}^{s} N_h$ in indecomposable representations. By [26, Lemma 12.1], this decomposition is defined over K^{sep}, hence over a finite Galois extension L/K. Since M is indecomposable, the Galois group of L/K acts transitively on isomorphism classes of indecomposable summands of M_L. We deduce that if one of the N_h is a brick all of them are,

and that for each h, h' the iterated images of the generic nilpotent endomorphisms of N_h and $N_{h'}$ have the same dimension vectors. We let $\alpha = \dim_K M$, $\beta = (\beta_1, \beta_2) = \dim_K N_h$, so that $\alpha = s\beta$.
Assume that N_h is a brick for every h. Since by assumption M is not a brick, necessarily $s \geq 2$. We have $\operatorname{trdeg}_k k(N_h) \leq 1 - \langle \beta, \beta \rangle$ by [26, Corollary 8.2]. Since β is in the fundamental region of K_r, it satisfies the inequalities

$$2\beta_1 - r\beta_2 \leq 0, \qquad 2\beta_2 - r\beta_1 \leq 0,$$

which imply

$$-\langle \beta, \beta \rangle = -\beta_1^2 - \beta_2^2 + r\beta_1\beta_2 \geq \max(\beta_1^2 - \beta_2^2, \beta_2^2 - \beta_1^2).$$

If $\beta_1 \neq \beta_2$, we obtain $-\langle \beta, \beta \rangle \geq \min \beta_i$, which is also true if $\beta_1 = \beta_2$. We use Proposition 8.2 and [26, Corollary 8.2] to obtain:

$$\begin{aligned}
\operatorname{ed}_k M &= \operatorname{ed}_{k(M)} M + \operatorname{trdeg}_k k(M) \\
&\leq \operatorname{ed}_{k(M)} M + \sum_h \operatorname{trdeg}_k k(N_h) \\
&\leq s \min_{i=1,2} \beta_i - 1 + s(1 - \langle \beta, \beta \rangle) \\
&\leq -2s \langle \beta, \beta \rangle + s - 1 \\
&\leq 1 - s^2 \langle \beta, \beta \rangle = 1 - \langle \alpha, \alpha \rangle .
\end{aligned}$$

The last inequality holds because it is equivalent to $-s(s-2)\langle \beta, \beta \rangle \geq s - 2$, which is clearly valid if $s = 2$, and reduces to the true statement $-s\langle \beta, \beta \rangle \geq 1$ when $s \geq 3$.
Assume now that the N_h are not bricks. We still have $-\langle \beta, \beta \rangle \geq \min \beta_i$. Note that this time s might be 1. Combining Lemma 13.2 with Proposition 8.2, we get:

$$\begin{aligned}
\operatorname{ed}_k M &\leq \operatorname{ed}_{k(M)} M + \sum_h \operatorname{trdeg}_k k(N_h) \\
&\leq s \min_{i=1,2} \beta_i - 1 + s(2 - \min_{i=1,2} \beta_i - \langle \beta, \beta \rangle) \\
&= 2s - 1 - s \langle \beta, \beta \rangle \\
&\leq 1 - s^2 \langle \beta, \beta \rangle = 1 - \langle \alpha, \alpha \rangle .
\end{aligned}$$

The last inequality is equivalent to $2(s-1) \leq -s(s-1)\langle \beta, \beta \rangle$, which is clearly satisfied for $s = 1$, and if $s \geq 2$ reduces to $2 \leq -s\langle \beta, \beta \rangle$, which is also true. □

Proof of Theorem 1.4. Let K be a field extension of k, and let M be an α-dimensional representation of K_r that is not a brick. It suffices to show that $\operatorname{ed}_k M \leq 1 - \langle \alpha, \alpha \rangle$.
If M is indecomposable, $\operatorname{ed}_k M \leq 1 - \langle \alpha, \alpha \rangle$ by Lemma 13.3. If M is decomposable, set $\alpha = (a, b)$, and write $M = \oplus_{h=1}^{s} M_h$ for the decomposition of M in

indecomposable representations, where $s \geq 2$. Let $\beta_h = (a_h, b_h) = \dim M_h$. If $\beta_h = (a_h, 0)$ for some h, then let $M' := \oplus_{j \neq h} M_j$. It is clear that M_h is defined over k, hence by [26, Lemma 6.4] we have $\mathrm{ed}_k M = \mathrm{ed}_k M'$. Moreover

$$\langle \alpha - \beta_h, \alpha - \beta_h \rangle \geq \langle \alpha, \alpha \rangle \,,$$

since this reduces to $a_h(2a - rb - a_h) \leq 0$, which is true because $2a - rb \leq 0$. Since M' has dimension smaller than M, we may assume that the claim holds for M'. Thus

$$\mathrm{ed}_k M = \mathrm{ed}_k M' \leq 1 - \langle \alpha - \beta_h, \alpha - \beta_h \rangle \leq 1 - \langle \alpha, \alpha \rangle \,.$$

The case when some β_h is of the form $(0, b_h)$ is similar. Therefore, we may assume that $a_h, b_h \neq 0$ for every h. Using in order Proposition 8.2, [26, Corollary 8.2], Lemma 12.1(b) and Lemma 13.1, we obtain:

$$\begin{aligned} \mathrm{ed}_k M &\leq \sum_h \mathrm{ed}_k M_h \\ &= \sum_h (\mathrm{ed}_{k(M_h)} M_h + \mathrm{trdeg}_k\, k(M_h)) \\ &\leq \sum_h (\min(a_h, b_h) - \langle \beta_h, \beta_h \rangle) \\ &\leq \min(a, b) - \langle \alpha, \alpha \rangle - f(\alpha) \\ &\leq 1 - \langle \alpha, \alpha \rangle \,. \end{aligned}$$

□

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