On Semicontinuity of Multiplicities in Families

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Abstract. This paper investigates the behavior of Hilbert–Samuel multiplicity and Hilbert–Kunz multiplicity in families of ideals. We show that Hilbert–Samuel multiplicity is upper semicontinuous and that Hilbert–Kunz multiplicity is upper semicontinuous in families of finite type. As a consequence, F-rational signature, an invariant defined by Hochster and Yao as the infimum of relative Hilbert–Kunz multiplicities, is, in fact, a minimum. This gives a different proof for its main property: F-rational signature is positive if and only if the ring is F-rational. The tools developed in this paper can be also applied to families over $\mathbb{Z}$ and yield a solution to Claudia Miller's question on reduction mod p of Hilbert–Kunz function.

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1 Introduction

Hilbert–Kunz multiplicity is a multiplicity theory native to positive characteristic. Its definition mimics the definition of Hilbert–Samuel multiplicity but replaces regular powers I^n with Frobenius powers $I^{[p^e]} = \{x^{p^e} \mid x \in I\}$. The Hilbert–Kunz multiplicity of an $\mathfrak{m}$-primary ideal I of a local ring $(R, \mathfrak{m})$ is the limit

$$\mathrm{e}_{\mathrm{HK}}(I) = \lim_{e \to \infty} \frac{\lambda_R(R/I^{[p^e]})}{p^{e \dim R}}.$$

It is not easy to see that the above limit exists. Existence was shown by Monsky, who introduced the concept in [Mon83] as a continuation of earlier work of Kunz [Kun69, Kun76].

Hilbert–Kunz multiplicity is very hard to calculate, and Paul Monsky was a driving force behind most of the known examples. Several interesting families appear in literature: plane cubics ([Mon97, Mon11, BC97, Par94]), quadrics in characteristic two ([Mon98a, Mon98b]), and another family in [Mon05]. The most famous of these families is the one appearing in [Mon98b].

Example 1.1. Let K be an algebraically closed field of characteristic 2. For $\alpha \in K$ let $R_\alpha = K[x,y,z]/(z^4 + xyz^2 + (x^3+y^3)z + \alpha x^2y^2)$ localized at (x,y,z). Then

1. $\mathrm{e}_{\mathrm{HK}}(R_\alpha) = 3 + \frac{1}{2}$, if $\alpha = 0$,
2. $\mathrm{e}_{\mathrm{HK}}(R_\alpha) = 3 + 4^{-m}$, if $\alpha \neq 0$ is algebraic over $\mathbb{Z}/2\mathbb{Z}$, where $m = [\mathbb{Z}/2\mathbb{Z}(\lambda) : \mathbb{Z}/2\mathbb{Z}]$ for λ such that $\alpha = \lambda^2 + \lambda$
3. $\mathrm{e}_{\mathrm{HK}}(R_\alpha) = 3$ if α is transcendental over $\mathbb{Z}/2\mathbb{Z}$.

Monsky's computations were later used by him and Brenner to give in [BM10] a counter-example to an outstanding problem in the field: localization of tight closure, the problem originating already from the foundational treatise of Hochster and Huneke [HH90]. For this result, it is better to think about the example as a family of rings parametrized by $\operatorname{Spec} K[t]$ and the necessary phenomenon is the jump in the values between the generic fiber, corresponding to transcendental values, and special fibers, corresponding to algebraic values. Another consequence of Monsky's example was found by the author in [Smi19], where it was shown that Hilbert–Kunz multiplicity takes infinitely many values as a function on

$$\operatorname{Spec} K[x,y,z,t]/(z^4 + xyz^2 + (x^3+y^3)z + tx^2y^2)$$

by developing a technique of lifting this phenomenon from special fibers to the corresponding maximal ideals $\mathfrak{m}_\alpha = (x,y,z,t-\alpha)$.

Semicontinuity in Hilbert–Kunz theory was already studied by Kunz, who showed in [Kun76] upper semicontinuity of individual terms of the sequence (also, see [SB79]), but the real momentum was given by Enescu and Shimomoto in [ES05], where they investigated both semicontinuity of Hilbert–Kunz multiplicity as a function on the spectrum and in a one-parameter family. In both settings, they established weaker forms of semicontinuity [ES05, Theorem 2.5, Theorem 2.6]. The complete solution for the spectrum was obtained by the author in [Smi16, Smi19], and the goal of this article is to establish semicontinuity for a class of families similar to the situation in Example 1.1 (see Definition 3.8).

Our definition of a family is versatile enough to include another outstanding problem in the field: the behavior of Hilbert–Kunz multiplicity in reduction mod p. For an illustration, consider the family $\mathbb{Z} \to R := \mathbb{Z}[x,y,z]/(z^4+xyz^2+(x^3+y^3)z+x^2y^2)$. A natural way to define the Hilbert–Kunz multiplicity of the generic fiber $\mathbb{Q}[x,y,z]/(z^4 + xyz^2 + (x^3+y^3)z + x^2y^2)$, a ring of characteristic zero, would be by taking the limit of Hilbert–Kunz multiplicities of special fibers $\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(R(p))$, and the question is whether the limit exists.

Hilbert–Kunz multiplicity is independent of characteristic for several classes of "combinatorial" rings because it only depends on the combinatorial data, for example: Stanley–Reisner rings ([Con96]), toric rings ([Wat00]), monoid algebras [Eto02, Bru05], and binoid algebras, generalizing the previous cases ([BB]). Monsky's work provides examples where Hilbert–Kunz multiplicity depends on the characteristic ([GM]), but the only general case where this problem was solved is for graded rings of dimension two [Tri07, BLM12]. In an attempt to simplify the problem, in [BLM12] Claudia Miller asked whether it is possible to replace the double limit $\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(R(p))$ by a single limit of the individual terms $\lim_{p\to\infty} \lambda(R(p)/\mathfrak{m}^{[p^e]}R(p))/p^{ed}$ for a fixed $e \geq 1$. A positive answer to this question (and a more general statement) was recently announced by Pérez, Tucker, and Yao ([PTY]). The methods of this paper provide an easy proof of this result in a special case (Corollary 4.12) and generalize a recent result of Trivedi ([Tri19]) which was established in the graded case. However, neither this paper nor [PTY] provide new cases in which $\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(R(p))$ is known to exist, but rather make a step in Miller's approach.

Another application of this work is in the theory of F-rational signature, an invariant introduced by Hochster and Yao in [HY]. If $(R, \mathfrak{m})$ is a local ring, then its F-rational signature is defined by

$$\mathrm{s}_{\mathrm{rat}}(R) = \inf_u \{\mathrm{e}_{\mathrm{HK}}(\underline{x}) - \mathrm{e}_{\mathrm{HK}}(\underline{x}, u)\}$$

where the infimum is taken over socle elements u modulo a system of parameters $\underline{x}$. Proposition 4.14 proves that if the residue field is algebraically closed, then the infimum in the definition is attained. This gives a fundamentally different proof of the main property of F-rational signature ([HY, Theorem 4.1]): $\mathrm{s}_{\mathrm{rat}}(R)$ is positive if and only if R is F-rational.

Last, we want to mention that using results in [PTY] Carvajal-Rojas, Schwede, and Tucker [CRST] recently obtained results in the spirit of this work. However, their motivation is to study the behavior of Hilbert–Kunz multiplicity in a family of varieties, while this work focuses on a family of ideals which are not necessarily maximal.

The methods and the structure of the paper

This paper uses the uniform convergence method that was introduced by Tucker in [Tuc12] to show convergence of F-signature as a limit and was later extended by the author in [Smi16] to show semicontinuity of Hilbert–Kunz multiplicity. Polstra and Tucker in [PT18] gave a more "functorial" approach to the uniform convergence constants based on the discriminant technique in tight closure theory ([HH90, Section 6]). This approach was then applied by Polstra and the author [PS] to study Hilbert–Kunz multiplicity under small perturbations. The uniform convergence machinery of this paper is largely a mix of the techniques developed in [PS] and [Smi16]. Moreover, the appearing constants can be made independent of the characteristic, which gives a uniform convergence statement for fibers even if the base ring has characteristic zero (Corollary 4.9). It should be noted that [CRST, Proposition 4.5] can be used to get a version of Theorem 4.7 under stronger assumptions.

Section 2 slightly expands on [PT18] by further incorporating ideas from [HH90]. Section 3 presents old and new results on the behavior of Hilbert–Samuel function in families. Definition 3.8 introduces the assumptions of this work. The main results are presented in Section 4 and we finish with questions coming from this work.

2 Discriminants and separability

Definition 2.1. Let R be a ring and A a finite R-algebra which is a free R-module. If $e_1, \dots, e_n$ are a free basis of A, then the discriminant of A over R is defined as

$$\mathrm{D}_R(A) = \det \begin{pmatrix} \mathrm{Tr}(e_1^2) & \mathrm{Tr}(e_1e_2) & \cdots & \mathrm{Tr}(e_1e_n) \\ \mathrm{Tr}(e_2e_1) & \mathrm{Tr}(e_2^2) & \cdots & \mathrm{Tr}(e_2e_n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{Tr}(e_ne_1) & \mathrm{Tr}(e_ne_2) & \cdots & \mathrm{Tr}(e_n^2) \end{pmatrix},$$

where $\mathrm{Tr}(A)$ denotes the trace of the multiplication map $\times A$ on A. Up to multiplication by a unit of R, the discriminant is independent of the choice of basis. Discriminants are also functorial in R, for example, see [PS].

We start with a fundamental lemma provided by Hochster and Huneke in [HH90, Lemmas 6.4, 6.5].

Lemma 2.2. *Let R be a normal domain of characteristic $p > 0$ and A be a module-finite, torsion-free, and generically separable R-algebra. Let L be the fraction field of R, $L' = A \otimes_R L$, and $d = \mathrm{D}_L(L')$ computed using a basis of elements in A. Then $0 \neq d \in R$ and $dA^{1/p} \subseteq R^{1/p}[A] \cong R^{1/p} \otimes_R A$.*

The lemma also provides a way to define a discriminant of a non-free algebra. We will abuse the notation and still denote it by $\mathrm{D}_R(A)$. If A is not torsion-free, we will use the ideal $T_R(A) = \{a \in A \mid ar = 0 \text{ for some } 0 \neq r \in R\}$.

Corollary 2.3. *Let R be a normal domain and A be module-finite and generically separable R-algebra. Let L be the fraction field of R, $L' = A \otimes_R L$, and $d = \mathrm{D}_R(A)$ computed as in Lemma 2.2. If $c \in R$ such that $cT_R(A) = 0$, then we have maps $\alpha\colon R^{1/p} \otimes_R A \to F_*A$ and $\beta\colon F_*A \to R^{1/p} \otimes_R A$ such that $cd(\operatorname{coker} \alpha) = 0$ and $cd(\operatorname{coker} \beta) = 0$.*

Proof. Multiplication by c on A induces a map $A' := A/T_R(A) \xrightarrow{\times c} A$. Observe that A' is still generically separable over R, since $L' = A \otimes_R L = A' \otimes_R L$. Hence $dF_*A' \subseteq R^{1/p}[A'] \cong R^{1/p} \otimes_R A'$ by Lemma 2.2.

Now we construct the maps in the assertion as compositions:

$$\alpha\colon R^{1/p} \otimes_R A \to R^{1/p} \otimes_R A' \to F_* A' \xrightarrow{\times F_* c} F_* A,$$

where the first map is natural and the second map is the multiplication $F_* r \otimes a \mapsto F_* a^p r$, and

$$\beta\colon F_* A \to F_* A' \xrightarrow{\times d} R^{1/p} \otimes_R A' \xrightarrow{1 \otimes \times c} R^{1/p} \otimes_R A.$$

For the first map, we note that $cdF_* A \subseteq dF_* A' \subseteq R^{1/p}[A']$ by Lemma 2.2. Because $R^{1/p}[A']$ is the image of α, $\operatorname{coker} \alpha$ is annihilated by cd. In the second map, we note that $F_* A \to F_* A'$ is surjective, $R^{1/p}[A'] \subseteq F_* A'$, and $cA = cA'$, so it follows that the cokernel of β is annihilated by cd. ☐

The corollary becomes especially powerful after combining it with another result of Hochster and Huneke [HH90, Lemma 6.15].

Lemma 2.4. *Let A be a reduced ring, module-finite over a regular ring R of characteristic $p > 0$. Then for all sufficiently large e, $A \otimes_R R^{1/p^e}$ is module-finite and generically separable over R^{1/p^e}.*

Proof. Let L be the fraction field of R and $L' = A \otimes_R L$. Since A is reduced, L' is a product of fields. Tensoring with L we get that

$$A \otimes_R R^{1/p^e} \otimes_R L = (A \otimes_R L) \otimes_L (R^{1/p^e} \otimes_R L) = L' \otimes_L L^{1/p^e}.$$

Hence the statement is reduced to the field case. ☐

Corollary 2.5. *Let A be a reduced ring, module-finite over a regular ring R of characteristic $p > 0$. Let $c \in R$ such that there exists a free R-module $F \subseteq A$ such that $cA \subseteq F$. Then for large e we have exact sequences of A-modules*

$$R^{1/p^{e+1}} \otimes_R A \to F_*(A \otimes_R R^{1/p^e}) \to C_{1,e} \to 0$$

and

$$F_*(A \otimes_R R^{1/p^e}) \to R^{1/p^{e+1}} \otimes_R A \to C_{2,e} \to 0,$$

where the cokernels are annihilated by $c\,\mathrm{D}_{R^{1/p^e}}(A \otimes_R R^{1/p^e})$.

Proof. We take e large enough to satisfy Lemma 2.4. Let $A' = A \otimes_R R^{1/p^e}$, $R' = R^{1/p^e}$, and $F' = F \otimes_R R^{1/p^e}$. Because R^{1/p^e} is flat by [Kun69], $cA' \subseteq F'$, so $cT_{R'}(A') \subseteq cA' \subseteq F'$ and $cT_{R'}(A') = 0$ because F' is torsion-free. Now, we may use Corollary 2.3 for A' and R'. ☐

3 Families and semicontinuity

We adopt the following notion of a family from [Lip82]. Let R be a ring, A be an R-algebra, and $I \subset A$ be an ideal such that A/I is a finitely generated R-module. For any prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ define $A(\mathfrak{p}) := A \otimes_R k(\mathfrak{p})$ and $I(\mathfrak{p}) = IA(\mathfrak{p}) := I(A(\mathfrak{p}))$. By the assumption, $A(\mathfrak{p})/I(\mathfrak{p}) = A/I \otimes_R k(\mathfrak{p})$ has finite length. Thus, $I(\mathfrak{p})$ is a family of finite colength ideals in a family of rings $A(\mathfrak{p})$ parametrized by $\operatorname{Spec} R$. If M is a finite A-module, then $M(\mathfrak{p}) := M \otimes_R k(\mathfrak{p})$ is a finite $A(\mathfrak{p})$-module for all $\mathfrak{p} \in \operatorname{Spec} R$.

Hilbert–Kunz multiplicity (and Hilbert–Samuel multiplicity) is now a real-valued function on $\operatorname{Spec} R$ via $\mathfrak{p} \mapsto \mathrm{e}_{\mathrm{HK}}(I(\mathfrak{p}), A(\mathfrak{p}))$. An example of such function is given in Example 1.1 by a family $K[t] \to K[x,y,z,t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2)$ with $I = (x, y, z)$.

We also fix the following definition of semicontinuity.

Definition 3.1. Let X be a topological space and $(\Lambda, \prec)$ be a partially ordered set. We say that a function $f\colon X \to \Lambda$ is *upper semicontinuous* if for each $\lambda \in \Lambda$ the set

$$X_{\prec\lambda} = \{x \in X \mid f(x) \prec \lambda\}$$

is open.

In the literature, one can find an alternative definition of semicontinuity that instead requires the sets $X_{\preceq\lambda} = \{x \in X \mid f(x) \preceq \lambda\}$ to be open. This definition is stronger than Definition 3.1 but coincides if f is discretely valued. As it was observed by Enescu and Shimomoto ([ES05, Theorem 2.7]), Monsky's example shows that Hilbert–Kunz multiplicity is not an upper semicontinuous function in this, stronger sense (take $\lambda = 3$).

Remark 3.2. Nagata's criterion of openness ([Mat80, 22.B]) is often used to show that a function is semicontinuous. Namely, if R is Noetherian, then a function $f\colon \operatorname{Spec} R \to \Lambda$ is upper semicontinuous if and only if the following two conditions hold:

1. if $\mathfrak{p} \subset \mathfrak{q}$ then $f(\mathfrak{p}) \preceq f(\mathfrak{q})$,

2. if $f(\mathfrak{p}) \prec \lambda$ then there exists an elements $s \notin \mathfrak{p}$ such that for every $\mathfrak{q}$ with $s \notin \mathfrak{q}$ and $\mathfrak{p} \subseteq \mathfrak{q}$ we have $f(\mathfrak{q}) \prec \lambda$.

3.1 Hilbert–Samuel function in families

The theory of families of ideals originates from the work of Teissier ([Tei80]) on the principle of specialization of integral closure and was further developed by Lipman in [Lip82].

We start with a lemma found in the proof of [FM00, Proposition 4.2].

Lemma 3.3. *Let $R \to A$ be a map of Noetherian rings and I be an ideal of A such that $R \to A/I$ is finite. Suppose M is a finite A-module. If $\mathrm{Gr}_I(M)$ is flat over R, then for every finite R-module N the canonical map*

$$\mathrm{Gr}_I(M) \otimes_R N \to \mathrm{Gr}_I(M \otimes_R N)$$

is an A-isomorphism.

Proof. It is sufficient to show that for all n the natural map $I^n M \otimes_R N \to I^n(M \otimes_R N)$ is an A-isomorphism. Because R acts on $M \otimes_R N$ by multiplication on M, the map is surjective, so it remains to check injectivity.
Because $I^n M/I^{n+1}M$ is a flat R-module as a direct summand of $\mathrm{Gr}_I(M)$, there is an exact sequence

$$0 \to I^{n+1}M \otimes_R N \to I^n M \otimes_R N \to (I^n M/I^{n+1}M) \otimes_R N \to 0.$$

Using induction on n it is now easy to verify the natural maps $I^n M \otimes_R N \to I^n(M \otimes_R N)$ are injective. ☐

Using this lemma we are able to expand [Lip82, Proposition 3.1].

Theorem 3.4. *Let $R \to A$ be a map of Noetherian rings and I be an ideal in A such that A/I is a finite R-module. Let M be a finitely generated A-module. Then the following functions on $\operatorname{Spec} R$ are upper semicontinuous:*

1. $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} M(\mathfrak{p})/I^n M(\mathfrak{p})$ *for any n,*
2. $\mathfrak{p} \mapsto \big(\dim_{k(\mathfrak{p})} M(\mathfrak{p})/IM(\mathfrak{p}), \dim_{k(\mathfrak{p})} M(\mathfrak{p})/I^2M(\mathfrak{p}), \dots\big)$ *(with lex-order),*

Proof. It can be shown by induction that, for all n, the modules M/I^nM and $I^nM/I^{n+1}M$ are finitely generated R-modules. Observe that $M(\mathfrak{p})/I^nM(\mathfrak{p}) \cong R/I^n \otimes_R M(\mathfrak{p}) \cong M/I^nM \otimes_R k(\mathfrak{p})$. But for any finite R-module N, $\dim_{k(\mathfrak{p})} N \otimes_R k(\mathfrak{p})$ is the minimal number of generators of $N(\mathfrak{p})$, which is clearly an upper semicontinuous function, see for example [PT18, Lemma 2.2]. In particular, we obtain that the first condition of Nagata's criterion from Remark 3.2 is satisfied.
For the second condition, we provide a neighborhood of $\mathfrak{p}$ where the functions are constant. Observe that $\mathrm{Gr}_I(M)$ is a finitely generated module over a finitely generated R-algebra, because it is a finite $\mathrm{Gr}_I(A)$-module and $\mathrm{Gr}_I(A)$ is a finitely generated module over $A/I[x_1, \dots, x_N]$ where $x_1, \dots, x_N$ are homogeneous generators of $\mathrm{Gr}_I(A)$ of degree one. For a fixed prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, we may apply generic freeness ([Mat80, 22.A]) over $R/\mathfrak{p}$ for the module $\mathrm{Gr}_I(M/\mathfrak{p}M)$.
In the resulting neighborhood $D(s)$ where $\mathrm{Gr}_I(M/\mathfrak{p}M)$ is free, by Lemma 3.3 and flatness of localization, for all $\mathfrak{q} \in D(s) \cap V(\mathfrak{p})$ we have the isomorphism

$$\mathrm{Gr}_I(M/\mathfrak{p}M) \otimes_R k(\mathfrak{q}) \cong \mathrm{Gr}_I(M \otimes_R k(\mathfrak{q})).$$

Because each $(I^n + \mathfrak{p})M/(I^{n+1} + \mathfrak{p})M$ is projective, it follows that $\dim_{k(\mathfrak{q})} I^nM(\mathfrak{q})/I^{n+1}M(\mathfrak{q})$ is constant on $V(\mathfrak{p}) \cap D(s)$ for all n. ☐

COROLLARY 3.5 ([Lip82, Proposition 3.1]). *Let $R \to A$ be a map of Noetherian rings and $I \subset A$ be an ideal such that A/I is a finite R-module. If $\mathfrak{p} \subseteq \mathfrak{q} \subset R$ are prime ideals and M is a finitely generated A-module, then* $\dim M(\mathfrak{p}) \leq \dim M(\mathfrak{q})$ *and if* $\dim M(\mathfrak{p}) = \dim M(\mathfrak{q})$ *then* $\mathrm{e}(IM(\mathfrak{p})) \leq \mathrm{e}(IM(\mathfrak{q}))$.

COROLLARY 3.6. *Let $R \to A$ be a map of Noetherian rings and I be an ideal such that A/I is a finite R-module. Let $d = \max_{\mathfrak{m} \in \operatorname{Max} R} \dim A/\mathfrak{m}A$. Then there exists a constant C such that for all $\mathfrak{p} \in \operatorname{Spec} R$ and all n*

$$\dim_{k(\mathfrak{p})} A(\mathfrak{p})/I^n(\mathfrak{p}) < Cn^d.$$

Proof. First, note that if $\mathfrak{p} \subseteq \mathfrak{m}$ then $\dim A/\mathfrak{m}A = \dim A(\mathfrak{m}) \geq \dim A(\mathfrak{p})$. So, for every $\mathfrak{p}$, there is some constant $C(\mathfrak{p})$ that will work for all n. Given any C the set

$$\begin{aligned} U(C) &= \{\mathfrak{p} \mid \dim_{k(\mathfrak{p})} A(\mathfrak{p})/I^n(\mathfrak{p}) < Cn^d \text{ for all } n\} \\ &= \cap_{n=1}^{\infty} \{\mathfrak{p} \mid \dim_{k(\mathfrak{p})} A(\mathfrak{p})/I^n(\mathfrak{p}) < Cn^d\} \end{aligned}$$

is open by Theorem 3.4. Thus we can build C by Noetherian induction: we first choose C to be the maximum $C(\mathfrak{p})$ over the generic points and then keep increasing C by considering generic points of the complement of $U(C)$ until $U(C) = \operatorname{Spec} R$. □

The following result of Lipman ([Lip82, Proposition 3.3]) provides a natural sufficient condition for equidimensionality of a family.

LEMMA 3.7. *Let $R \to A$ be a map of Noetherian rings and I an ideal of A such that A/I is a finite R-module and $R \cap I = 0$. Furthermore, assume that*

1. $\operatorname{ht} \mathfrak{q} + \dim A/\mathfrak{q} = \dim A$ *for every prime ideal* $\mathfrak{q} \supseteq I$ *in* A,
2. $\dim A/\mathfrak{m}A + \dim R = \dim A$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

Then for every prime ideal $\mathfrak{p}$ of R we have $\dim A(\mathfrak{p}) = \dim A - \dim R = \operatorname{ht} I$.

Due to the fundamental nature of Lemma 3.7, we would like to call the map $R \to A$ satisfying its assumptions an I-family.

DEFINITION 3.8. We say that $R \to A$ is an *affine I-family* if A is a finitely generated R-algebra and $I \subset A$ is an ideal such that

1. A/I is a finite R-module,
2. $R \cap I = 0$,
3. $\operatorname{ht} \mathfrak{q} + \dim A/\mathfrak{q} = \dim A$ for every prime ideal $\mathfrak{q} \supseteq I$ in A,
4. $\dim A/\mathfrak{m}A + \dim R = \dim A$ for every maximal ideal $\mathfrak{m}$ of R.

The second condition guarantees that $I(\mathfrak{p}) \neq A(\mathfrak{p})$ for every $\mathfrak{p}$. We can always pass to such a family by factoring by $I \cap R$. If A is formally equidimensional then it satisfies (3), if A is a flat R-algebra, then it satisfies (4). In particular, Example 1.1 is coming from an affine (x, y, z)-family: localization does not change the Hilbert–Kunz multiplicity because the Frobenius powers are (x, y, z)-primary.

4 Semicontinuity

We want to show that $\mathrm{e}_{\mathrm{HK}}(I(\mathfrak{p}))$ is an upper semicontinuous function on $\operatorname{Spec} R$ in the sense of Definition 3.1. In order to build the uniform convergence machinery, we start with auxiliary lemmas.

Lemma 4.1. *Let $R \to A$ be a map of rings of characteristic $p > 0$ and I be an ideal in A such that A/I is a finite R-module. For each integer $e \geq 1$ the function $\mathfrak{p} \to \dim_{k(\mathfrak{p})}(A(\mathfrak{p})/I(\mathfrak{p})^{[p^e]})$ is upper semicontinuous on* $\operatorname{Spec} R$.

Proof. If I can be generated by h elements, then $I^{hp^e} \subseteq I^{[p^e]}$, so $A/I^{[p^e]}$ is a finite R-module as in Theorem 3.4. Thus $\mathfrak{p} \to \dim_{k(\mathfrak{p})}(A(\mathfrak{p})/I(\mathfrak{p})^{[p^e]})$ is the minimal number of generators of that module at $\mathfrak{p}$ and is an upper semicontinuous function. $\square$

Corollary 4.2. *Let $R \to A$ be an I-family as in Lemma 3.7. Then for every $\mathfrak{p} \subseteq \mathfrak{q}$ we have $\mathrm{e}_{\mathrm{HK}}(I(\mathfrak{p})) \leq \mathrm{e}_{\mathrm{HK}}(I(\mathfrak{q}))$.*

Proof. Observe that $\dim A(\mathfrak{p}) = \operatorname{ht} I$ by Lemma 3.7, so we may pass to the limit in Lemma 4.1. $\square$

Lemma 4.3. *Let R be a Noetherian ring and let A be an intersection flat R-algebra, i.e., $\cap_{\lambda \in \Lambda}(I_\lambda A) = (\cap_{\lambda \in \Lambda} I_\lambda)A$ for arbitrary Λ and ideals $I_\lambda \subset R$. Then for any element $f \in A$ the set*

$$V_R(f) := \{\mathfrak{p} \in \operatorname{Spec} R \mid f \in \mathfrak{p}A\}$$

is closed.

Proof. Let I be the intersection of all primes in $V_R(f)$. Then $f \in \cap_{\mathfrak{p} \in V_R(f)} \mathfrak{p}A = (\cap_{\mathfrak{p} \in V_R(f)} \mathfrak{p})A = IA$. Hence $V_R(f) = V(I)$. $\square$

Last, we record a crucial lemma that provides a uniform upper bound for the main result. Note that polynomial extensions are intersection flat.

Lemma 4.4. *Let R be a Noetherian domain, $A = R[T_1, \ldots, T_d]$, and I be an $(T_1, \ldots, T_d)$-primary ideal. Let M be a finite A-module annihilated by $0 \neq f \in A$. Then there exists a constant D with the following property: for any $e \geq 0$ and $\mathfrak{p}$ in the open subset $\operatorname{Spec} R \setminus V_R(f)$ with $p := \operatorname{char} k(\mathfrak{p})$ we have*

$$\dim_{k(\mathfrak{p})} M(\mathfrak{p})/I^{[p^e]}M(\mathfrak{p}) < Dp^{e(d-1)},$$

where the characteristic of $k(\mathfrak{p})$ may depend on $\mathfrak{p}$.

Proof. For every maximal ideal $\mathfrak{m} \notin V_R(f)$

$$\dim A/(f,\mathfrak{m})A = \dim R/\mathfrak{m}[T_1,\dots,T_d]/(f) \leq d-1.$$

Let N be such that $(T_1,\dots,T_d)^N \subseteq I$. Then we have inclusions

$$(T_1,\dots,T_d)^{Ndp^e} \subseteq ((T_1,\dots,T_d)^{[p^e]})^N \subseteq I^{[p^e]}.$$

Suppose that M can be (globally) generated by ν elements. We note that $\operatorname{Spec} R \setminus V_R(f)$ is a finite union of principal open set $D(c)$ and for each c we may apply Corollary 3.6 to the map $R_c \to A_c$ and estimate

$$\begin{aligned}\dim_{k(\mathfrak{p})} M(\mathfrak{p})/I^{[p^e]}M(\mathfrak{p}) &\leq \nu \dim_{k(\mathfrak{p})} A(\mathfrak{p})/I(\mathfrak{p})^{[p^e]} \\ &< \nu C(Ndp^e)^{d-1} = (\nu CN^{d-1}d^{d-1})p^{e(d-1)}. \qquad \square\end{aligned}$$

4.1 Main result

Before proceeding to the proof of the main theorem we recall two lemmas. The first is due to Kunz [Kun76].

Lemma 4.5. *Let R be a Noetherian ring of characteristic $p>0$. Then for every $\mathfrak{p}\subset\mathfrak{q}$*

$$[k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] = p^{e \dim R_{\mathfrak{q}}/\mathfrak{p}}[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})].$$

Second, we will need the following form of the Noether normalization theorem from [Nag62, 14.4].

Theorem 4.6. *Let R be a domain and A be a finitely generated R-algebra. Then there exists an element $0 \neq c \in R$ such that A_c is module-finite over a polynomial subring $R_c[z_1,\dots,z_d]$.*

Theorem 4.7. *Let R be a regular F-finite ring of characteristic $p>0$ and $R \to A$ be an affine I-family with reduced fibers of dimension $h = \operatorname{ht} I$. Then there exists an open set $U \subseteq \operatorname{Spec} R$ and a constant D such that for all $\mathfrak{q}\in U$ and all $e \geq 1$*

$$\left| \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^{e+1}]}A(\mathfrak{q})}{p^{(e+1)h}} - \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^e]}A(\mathfrak{q})}{p^{eh}} \right| < \frac{D}{p^e}.$$

Proof. Because $A(0)$ is reduced, after inverting an element of R we may assume that A is reduced. Next, by Theorem 4.6 we invert another element and assume that A is finite over $S = R[T_1,\dots,T_h]$.
Applying Lemma 2.4 to the pair $S \subseteq A$ we find e_0 such that $S^{1/p^{e_0}} \to A\otimes_S S^{1/p^{e_0}}$ is generically separable. Since S is a domain, there exists a free module $F \subseteq A$ and an element $0 \neq c \in S$ such that $cA \subseteq F$. Because $S^{1/p^{e_0}}$ is flat, $F \otimes_S S^{1/p^{e_0}} \subseteq A \otimes_S S^{1/p^{e_0}}$ is a free submodule and c still annihilates the cokernel. Let $d^{1/p^{e_0}}$ to be the discriminant of $A\otimes_S S^{1/p^{e_0}}$ over $S^{1/p^{e_0}}$.

Claim 1. *Let $\mathfrak{q}$ be a prime ideal in the open set* $\operatorname{Spec} R \setminus V_R(cd)$*. Then $F(\mathfrak{q})$ is a free submodule of $A(\mathfrak{q})$ such that $cA(\mathfrak{q}) \subseteq F(\mathfrak{q})$.*

Proof of the claim. We have the induced map $F \otimes_S S(\mathfrak{q}) \to A \otimes_S S(\mathfrak{q})$ whose cokernel is annihilated by the image of c in $S(\mathfrak{q})$. The image of c is nonzero by the assumption, $F_c \cong A_c$, and $c \notin \mathfrak{q}S$, so $F(\mathfrak{q})$ and $A(\mathfrak{q})$ are still generically isomorphic as $S(\mathfrak{q})$-modules. Thus, since $F(\mathfrak{q})$ is a free $S(\mathfrak{q})$-module and $S(\mathfrak{q}) \cong k(\mathfrak{q})[T_1, \dots, T_h]$ is a domain, the induced map $F \otimes_S S(\mathfrak{q}) \to A \otimes_S S(\mathfrak{q})$ is still an inclusion. □

By the functoriality of discriminants (as in [PS, Proposition 2.2]), the image of d is still a discriminant of $A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0}}$ over $S(\mathfrak{q})^{1/p^{e_0}}$. Since $d \notin \mathfrak{q}S$, the inclusion is still generically separable. Hence, by Lemma 2.5, we have sequences

$$A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0+1}} \to F_* \left(A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0}} \right) \to C_1 \to 0 \tag{4.1}$$

and

$$F_* \left(A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0}} \right) \to A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0+1}} \to C_2 \to 0, \tag{4.2}$$

where $cdC_1 = 0 = cdC_2$. Tensoring these exact sequences with $B := A/I^{[p^e]}$, we obtain that

$$\begin{aligned} &| \dim_{k(\mathfrak{q})} B(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0+1}} - \dim_{k(\mathfrak{q})} B \otimes_A F_* \left(A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0}} \right) | \\ &\qquad \leq \max \left(\dim_{k(\mathfrak{q})} C_1/I^{[p^e]}C_1, \dim_{k(\mathfrak{q})} C_2/I^{[p^e]}C_2 \right). \end{aligned} \tag{4.3}$$

Claim 2. *Denote $\alpha(\mathfrak{q}, e) := p^{eh}[k(\mathfrak{q})^{1/e} : k(\mathfrak{q})]$. There is a constant D independent of $\mathfrak{q}$ such that*

$$\dim_{k(\mathfrak{q})} C_1/I^{[p^e]}C_1, \dim_{k(\mathfrak{q})} C_2/I^{[p^e]}C_2 < Dp^{e(h-1)}\alpha(\mathfrak{q}, e_0 + 1).$$

Proof. Tensoring the exact sequence (4.1) by $\otimes_A A/(cd, I^{[p^e]})$ we obtain a surjection

$$F_* \left(A(\mathfrak{q})/(c^p d^p, I^{[p^{e+1}]})A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0}} \right) \to C_1/I^{[p^e]}C_1 \to 0.$$

Since $S(\mathfrak{q})$ is a polynomial ring of dimension h, by Lemma 4.5 $S(\mathfrak{q})^{1/p^{e_0}}$ is a free $S(\mathfrak{q})$-module of rank $\alpha(\mathfrak{q}, e_0)$. Then we may bound

$$\begin{aligned} \dim_{k(\mathfrak{q})} C_1/I^{[p^e]}C_1 &\leq \alpha(\mathfrak{q}, e_0) \dim_{k(\mathfrak{q})} F_*(A(\mathfrak{q})/(c^p d^p, I^{[p^{e+1}]})A(\mathfrak{q})) \\ &= \alpha(\mathfrak{q}, e_0)[k(\mathfrak{q})^{1/p} : k(\mathfrak{q})] \dim_{k(\mathfrak{q})} A(\mathfrak{q})/(c^p d^p, I^{[p^{e+1}]})A(\mathfrak{q}) \\ &= \alpha(\mathfrak{q}, e_0 + 1)p^{-h} \dim_{k(\mathfrak{q})} A(\mathfrak{q})/(c^p d^p, I^{[p^{e+1}]})A(\mathfrak{q}). \end{aligned}$$

Because A/I is a finite R-module, I is $(T_1, \dots, T_h)$-primary, so by Corollary 4.4 applied to $A/(cd)$ we may find a constant D independent of $\mathfrak{q}$ such that

$$\begin{aligned}\dim_{k(\mathfrak{q})}(A(\mathfrak{q})/(c^p d^p, I^{[p^{e+1}]})A(\mathfrak{q})) &\leq p \dim_{k(\mathfrak{q})}(A(\mathfrak{q})/(cd, I^{[p^{e+1}]})A(\mathfrak{q})) \\ &\leq pDp^{(e+1)(h-1)} = Dp^h p^{e(h-1)},\end{aligned}$$

thus $\dim_{k(\mathfrak{q})}(C_1/I^{[p^e]}C_1) < Dp^{e(h-1)}\alpha(\mathfrak{q}, e_0+1)$.
The second bound is similar: $C_2/I^{[p^e]}$ is an image of $A(\mathfrak{q})/(cd, I^{[p^e]})A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0+1}}$, thus

$$\begin{aligned}\dim_{k(\mathfrak{q})}(C_2/I^{[p^e]}C_2) &\leq \alpha(\mathfrak{q}, e_0+1) \dim_{k(\mathfrak{q})} A(\mathfrak{q})/(cd, I^{[p^e]})A(\mathfrak{q}) \\ &< Dp^{e(h-1)}\alpha(\mathfrak{q}, e_0+1). \qquad \square\end{aligned}$$

As in the proof Claim 2, we have $\dim_{k(\mathfrak{q})} A/I^{[p^e]} \otimes_A F_*(A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0}}) = \alpha(\mathfrak{q}, e_0+1)p^{-h} \dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^{e+1}]}A(\mathfrak{q})$, and

$$\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^e]}A(\mathfrak{q}) \otimes_{S(\mathfrak{q})} S(\mathfrak{q})^{1/p^{e_0+1}} = \alpha(\mathfrak{q}, e_0+1) \dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^e]}A(\mathfrak{q}).$$

Now, dividing (4.3) by $p^{eh}\alpha(\mathfrak{q}, e_0+1)$, from Claim 2 we obtain that

$$\left| \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^{e+1}]}A(\mathfrak{q})}{p^{(e+1)h}} - \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^e]}A(\mathfrak{q})}{p^{eh}} \right| < \frac{Dp^{e(h-1)}}{p^{eh}} \leq \frac{D}{p^e}. \qquad \square$$

4.2 Families over $\mathbb{Z}$

A careful analysis of the proof shows that it can be applied even when the characteristic varies in a family.

Theorem 4.8. *Let R be a regular ring of characteristic 0 and $R \to A$ be an affine I-family with reduced fibers of dimension h. Suppose that for every $\mathfrak{p} \in \operatorname{Spec} R$ the residue field $k(\mathfrak{p})$ is F-finite whenever it has positive characteristic. Then there exists an open set $U \subseteq \operatorname{Spec} R$ and a constant D with the following property: if $\mathfrak{q} \in U$ and $p := \operatorname{char} k(\mathfrak{q}) > 0$ then*

$$\left| \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^{e+1}]}A(\mathfrak{q})}{p^{(e+1)h}} - \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^e]}A(\mathfrak{q})}{p^{eh}} \right| < \frac{D}{p^e}.$$

Note that p, the characteristic of $k(\mathfrak{q})$, may vary in the family and D is independent of p.

Proof. After inverting an element if necessary, we choose a Noether normalization $S = R_f[x_1, \dots, x_d]$ of A_f. Note that $S \subseteq A$ is generically separable, because $A(0)$ has characteristic 0. So, we may proceed with the proof of Theorem 4.7 with $e_0 = 0$. The constant D in claim Claim 2 comes from Lemma 4.4 and does not depend on characteristic as it arises from the Hilbert–Samuel theory. $\square$

Corollary 4.9. *Let $R \to A$ be an affine I-family with reduced fibers of dimension h. Suppose that for every $\mathfrak{p} \in \operatorname{Spec} R$ the residue field $k(\mathfrak{p})$ is F-finite whenever it has positive characteristic (e.g., R is F-finite or $R = \mathbb{Z}$). Then there exists an open set $U \subseteq \operatorname{Spec} R$ and a constant D with the following property: if $\mathfrak{q} \in U$ and $p := \operatorname{char} k(\mathfrak{q}) > 0$ then*

$$\left| \mathrm{e}_{\mathrm{HK}}(I(\mathfrak{p})) - \frac{\dim_{k(\mathfrak{q})} A(\mathfrak{q})/I^{[p^e]}A(\mathfrak{q})}{p^{eh}} \right| < \frac{2D}{p^e}.$$

Proof. We may pass to $R/\mathfrak{p} \to A/\mathfrak{p}$ and assume that $\mathfrak{p} = 0$. An F-finite ring is excellent ([Kun76, Theorem 2.5]), so the regular locus of R is open and, by inverting an element, we assume that R is regular.
Let D be the constant provided by Theorem 4.7 or Theorem 4.8, then the claim follows from the proof of [PT18, Lemma 3.5]. ☐

Corollary 4.10. *Let R be an F-finite ring of characteristic $p > 0$ and $R \to A$ be an affine I-family with reduced fibers. Then the function $\mathfrak{p} \mapsto \mathrm{e}_{\mathrm{HK}}(I(\mathfrak{p}))$ is upper semicontinuous on* $\operatorname{Spec} R$.

Proof. We use uniform convergence to pass semicontinuity from the individual term to the limit as in [PT18, Smi16]. Each individual term, $\dim_{k(\mathfrak{p})} A(\mathfrak{p})/I^{[p^e]}A(\mathfrak{p})$ is the number of generators of $A/I^{[p^e]}$ at $\mathfrak{p}$ and, thus, is naturally upper semicontinuous. ☐

We have the following geometric consequence.

Corollary 4.11. *Let $X \to T$ together with a section $\sigma\colon T \to X$ be a flat family of finite type with reduced fibers over a variety T of characteristic $p > 0$. Then the function $t \mapsto \mathrm{e}_{\mathrm{HK}}(\sigma(t), X_t)$ is upper semicontinuous on T.*

The following corollary provides a positive answer to the question of Claudia Miller from [BLM12] and recovers the main result, [BLM12, Corollary 3.3]. Similar result was recently obtained by Trivedi in [Tri19, Corollary 1.2] for families of geometrically integral graded rings and $e \geq h - 1$. A much more general result about reductions mod p was announced in [PTY].

Corollary 4.12. *Let $\mathbb{Z} \to R$ be an affine I-family with reduced fibers of dimension h. Then for every $e \geq 1$*

$$\lim_{p \to \infty} \left(\mathrm{e}_{\mathrm{HK}}(IR(p)) - \frac{\lambda(R(p)/I(p)^{[p^e]})}{p^{eh}} \right) = 0.$$

Proof. By Corollary 4.9, we obtain that for all sufficiently large p

$$\left| \mathrm{e}_{\mathrm{HK}}(IR(p)) - \frac{\lambda(R(p)/I(p)^{[p^e]})}{p^{eh}} \right| < \frac{2D}{p^e}.$$

and the theorem follows. ☐

4.3 F-rational signature

In [HY] Hochster and Yao introduced the following definition.

Definition 4.13. Let $(R, \mathfrak{m})$ be a local ring. The *F-rational signature* of R is defined as

$$\mathrm{s}_{\mathrm{rat}}(R) = \inf_{u} \{\mathrm{e}_{\mathrm{HK}}(\underline{x}) - \mathrm{e}_{\mathrm{HK}}(\underline{x}, u)\}$$

where the infimum is taken over all systems of parameters $\underline{x}$ and socle elements u.

Proposition 4.14. *Let k be a field of characteristic $p > 0$, R be a finitely generated k-algebra, and $\mathfrak{m}$ be a maximal ideal of R. Then the infimum in the definition of $\mathrm{s}_{\mathrm{rat}}(R_{\mathfrak{m}})$ is achieved.*

Proof. In [HY, Theorem 2.5], it was shown that one can fix an arbitrary $\underline{x}$ in the definition. Thus, the assertion is equivalent to showing that for a system of parameters $\underline{x}$ the function $\mathrm{e}_{\mathrm{HK}}(\underline{x}, u)$ has a maximum as u varies through socle elements modulo $\underline{x}$.

Let $u_1, \ldots, u_N$ be a basis of $(\underline{x}) : \mathfrak{m}/(\underline{x})$ as a k-vector space. We may parametrize the socle ideals (I, u) via two affine families: $(\underline{x}, T_1 u_1 + \cdots + T_{N-1} u_{N-1} + u_N)$-family

$$S := k[T_1, \ldots, T_{N-1}] \to R[T_1, \ldots, T_{N-1}]$$

and, similarly, for $u_1 + T_2 u_2 + \cdots + T_N u_N$. By Corollary 4.10, the function $f \colon \mathfrak{p} \mapsto \mathrm{e}_{\mathrm{HK}}((\underline{x}, T_1 u_1 + \cdots + T_{N-1} u_{N-1} + u_N) R(\mathfrak{p}))$ is upper semicontinuous on $\operatorname{Spec} S$.

The claim now follows since an upper semicontinuous function satisfies the ascending chain condition. Namely, let u_n be a sequence of socle elements such that $\mathrm{e}_{\mathrm{HK}}(\underline{x}, u_n) < \mathrm{e}_{\mathrm{HK}}(\underline{x}, u_{n+1})$ for all n. Without loss of generality we may assume that u_n correspond to maximal ideals $\mathfrak{m}_n$ of S. Then $U_n = \{\mathfrak{p} \in \operatorname{Spec} S \mid f(\mathfrak{p}) < \mathrm{e}_{\mathrm{HK}}(\underline{x}, u_n))\}$ is an increasing family of open sets which cannot stabilize because $\mathfrak{m}_n \in U_{n+1} \setminus U_n$. Since $\operatorname{Spec} S$ is Noetherian, this is a contradiction. ☐

Remark 4.15. We want to note that Proposition 4.14 can be also applied when R is given as a quotient of a power series ring by an ideal generated by polynomials, since the lengths do not change under completion. By Artin's celebrated result ([Art69, Theorem (3.8)]) this gives us that the conclusion holds for complete isolated singularities with a perfect residue field.

As a consequence, we recover a special case of [HY, Theorem 4.1].

Corollary 4.16. *Let k be a field of characteristic $p > 0$, R be a finitely generated k-algebra, and $\mathfrak{m}$ be a maximal ideal of R. Then $\mathrm{s}_{\mathrm{rat}}(R_{\mathfrak{m}}) > 0$ if and only if $R_{\mathfrak{m}}$ is F-rational.*

Proof. Recall that R is F-rational if $\underline{x}$ is tightly closed or, equivalently, that $\mathrm{e}_{\mathrm{HK}}(\underline{x}) > \mathrm{e}_{\mathrm{HK}}(\underline{x}, u)$ for every socle element u. ☐

Remark 4.17. A variation of the Hochster–Yao definition, relative F-rational signature, was proposed in [ST]

$$\mathrm{s}_{\mathrm{rel}}(R) = \inf_{\underline{x} \subset I} \frac{\mathrm{e}_{\mathrm{HK}}(\underline{x}) - \mathrm{e}_{\mathrm{HK}}(I)}{\lambda(R/\underline{x}) - \lambda(R/I)},$$

where the infimum is taken over all $\mathfrak{m}$-primary ideals I containing a system of parameters $\underline{x}$. The paper shows that the definition also does not depend on the choice of $\underline{x}$ and that it might have better properties than $\mathrm{s}_{\mathrm{rat}}(R)$.
By considering higher degree Grassmannians of $(\underline{x}) : \mathfrak{m}/(\underline{x})$, from the proof of Proposition 4.14 we may also get that the relative F-rational signature is a minimum.

5 Questions

5.1 Nilpotents

Like the preceding work [PS], this paper has to assume that the family is reduced because of the lack of control in non-reduced rings. While Hilbert–Kunz multiplicity exists for non-reduced rings, the original proof in [Mon83] and its extensions pass to R_{red} by observing that $F_*^{e_0} R$ is an R_{red}-module for large e_0. This is not satisfactory for two reasons: the approach via discriminants does not adapt for modules and we do not see how to control the exponent e_0.

5.2 F-signature

F-signature is a measure of singularity in positive characteristic introduced by Huneke and Leuschke in [HL02]. Due to similarities between the two theories, it is natural to ask whether the results of this paper extend to F-signature.
A related statement was observed in [CRST, Theorem 4.9], however, it does not give lower semicontinuity since A is assumed to be of finite type over a field and cannot be localized to apply Nagata's criterion. In fact, F-signature is not lower semicontinuous in families, because an example of Singh shows that strong F-regularity is not open ([Sin99], see also [DSS]).

5.3 Localization of tight closure

As it was mentioned above, in [BM10] Brenner and Monsky showed that tight closure does not localize. However, we do not understand the underlying reasons. In particular, how does it relate to the results of [HH00] and how typical is this phenomenon? As [BM10] depends on an irregular behavior of Hilbert–Kunz multiplicity in a family, we hope that it should be possible to give a general procedure for producing counter-examples from such families, for example, the family in [Mon05]. The study of Hilbert–Samuel multiplicity in families was pioneered by Teissier ([Tei80]) to give a criterion of equimultiplicity: $\mathrm{e}(I(\mathfrak{p}))$ is independent of $\mathfrak{p}$ if and only if $\ell(I) = \mathrm{ht}(I)$. The author suspects that a study of equimultiplicity in families for Hilbert–Kunz multiplicity might explain the phenomenon presented in [BM10].

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