Involutions of Azumaya Algebras

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Abstract. We consider the general circumstance of an Azumaya algebra A of degree n over a locally ringed topos $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ where the latter carries a (possibly trivial) involution, denoted λ. This generalizes the usual notion of involutions of Azumaya algebras over schemes with involution, which in turn generalizes the notion of involutions of central simple algebras. We provide a criterion to determine whether two Azumaya algebras with involutions extending λ are locally isomorphic, describe the equivalence classes obtained by this relation, and settle the question of when an Azumaya algebra A is Brauer equivalent to an algebra carrying an involution extending λ, by giving a cohomological condition. We remark that these results are novel even in the case of schemes, since we allow ramified, non-trivial involutions of the base object. We observe that, if the cohomological condition is satisfied, then A is Brauer equivalent to an Azumaya algebra of degree $2n$ carrying an involution. By comparison with the case of topological spaces, we show that the integer $2n$ is minimal, even in the case of a nonsingular affine variety X with a fixed-point free involution. As an incidental step, we show that if R is a commutative ring with involution for which the fixed ring S is local, then either R is local or R/S is a quadratic étale extension of rings.

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Contents

1 Introduction

1.1 Motivation

Let A be a central simple algebra over a field K and let $\tau : A \to A$ be an involution, i.e., an anti-automorphism satisfying $a^{\tau\tau} = a$ for all $a \in A$. Recall that τ can be of the *first kind* or of the *second kind*, depending on whether τ restricts to the identity on the centre K or not. We further say that τ is a λ-involution where $\lambda = \tau|_K$.

Central simple algebras and their involutions play a major role in the theory of classical algebraic groups, and also in Galois cohomology. For example, letting F denote the fixed field of $\lambda : K \to K$, it is well known that the absolutely simple adjoint classical algebraic groups over F are all given as the neutral connected component of projective unitary groups of algebras with involution (A, τ) as above, where K varies (here we also allow $K = F \times F$ with the switch involution), see [KMRT98, §26]. In fact, all simple algebraic groups of types A, B, C, D, excluding D_4, can be described by means of central simple algebras with involution. Involutions of central simple algebras also arise naturally in representation theory, either since group algebras admit a canonical involution, or in the context of orthogonal, unitary, or symplectic representations, see, for instance, [Rie01].

Azumaya algebras are generalizations of central simple algebras in which the base field is replaced with a ring, or more generally, a scheme. As with central simple algebras, Azumaya algebras and their involutions are important in the study of classical reductive group schemes, as well as in étale cohomology and in the representation theory of finite groups over rings; see [Knu91].

Suppose that K/F is a quadratic Galois extension of fields and let λ denote the non-trivial F-automorphism of K. A theorem of Albert, Riehm and Scharlau, [KMRT98, Thm. 3.1(2)], asserts that a central simple K-algebra A admits a λ-involution if and only if $[A]$, the Brauer class of A, lies in the kernel of the corestriction map $\mathrm{cores}_{K/F} : \mathrm{Br}(K) \to \mathrm{Br}(F)$. Saltman [Sal78, Thm. 3.1b] later showed that if K/F is replaced with a quadratic Galois extension of rings R/S, then the class $[A]$ lies in the kernel of $\mathrm{cores}_{R/S} : \mathrm{Br}(R) \to \mathrm{Br}(S)$ if and only if some representative $A' \in [A]$ admits a λ-involution. Here, in distinction to the case of fields, an arbitrary representative may not posses an involution. However, a later proof by Knus, Parimala and Srinivas [KPS90, Thm. 4.2], which applies to Azumaya algebras over schemes, implies that one can take $A' \in [A]$ such that $\deg A' = 2 \deg A$.

The aforementioned results all have counterparts for involutions of the first kind in which the condition $\mathrm{cores}_{R/S}[A] = 0$ is replaced by $2[A] = 0$.

In this article, we generalize this theory to more general sites and more general involutions. We have two purposes in doing so. The first, our initial motivation, is to demonstrate that the upper bound in Saltman's theorem, $\deg A' \leq 2 \deg A$ guaranteed by [KPS90, Thm. 4.2], cannot be improved in general for involutions of the second kind. The statement in the case of involutions of the first kind was established in [AFW19]. Our general approach here is similar to [AFW19], [AW14c] and related works. That is, the desired example is constructed by approximating a suitable classifying space, and topological obstruction theory is used to show that it has the required properties. In contrast with [AFW19] and [AW14c], the obstruction is obtained by means of equivariant homotopy theory.

We therefore introduce and study involutions of the second kind of Azumaya algebras on topological spaces. In fact, we develop the necessary foundations in the generality of connected locally ringed topoi with involution, and show that Saltman's theorem holds in this setting. In doing so, we stumbled into our second purpose, which we now explain.

Any involution of a field is either trivial or comes from a quadratic Galois extension, which is why the classical theory sees a dichotomy into involutions of the first or second kind. For a ring, the analogous involutions are the trivial involutions or those arising as the non-trivial automorphism of a quadratic étale extension R/S. Geometrically, these correspond to extreme cases where one has either a trivial action of the cyclic group $C_2 = \{1, \lambda\}$ on a scheme, or where the action is scheme-theoretically free. One may also view the free case as corresponding to an *unramified* map $\pi : X \to X/C_2$. This dichotomy has been preserved in the literature on involutions of Azumaya algebras over schemes, say for instance [KPS90] and [Knu91], by considering only trivial or unramified involutions of the base ring.

There are, of course, involutions $\lambda : R \to R$ which are neither trivial nor wholly unramified. For instance, one may encounter involutions of varieties that are generically free but fix a nonempty closed subscheme. Alternatively, there are involutions of nonreduced rings that restrict to trivial involutions of the reduction—these are geometrically ramified everywhere, but nonetheless non-trivial.

Our second purpose therefore became developing the theory of λ-involutions on Azumaya algebras with minimal assumptions on λ. We establish a generalization of Saltman's theorem, and present a classification of λ-involutions into *types*, generalizing the classification of involutions of central simple algebras as *orthogonal*, *symplectic*—both of the first kind—or *unitary*—of the second.

In more detail, given a field K of characteristic not 2 and an involution $\lambda : K \to K$, recall that two degree-n central simple K-algebras with involutions extending λ are of the same type if they become isomorphic after base change to a separable closure of the fixed field of λ. This definition extends naturally to the case of a general connected ring R in which 2 is a unit by replacing "a separable closure" by an étale extension of S, the fixed ring of $\lambda : R \to R$. It is natural to ask how many types are obtained in this manner, and how to distinguish them effectively. In the classically-considered cases of trivial or unramified involutions on R, the situation is known to be similar to case of fields: When $R = S$, there are at most two types — the orthogonal, which occurs for all n, and the symplectic, which occurs only for even n. When R/S is quadratic étale, only one type, called the unitary type, occurs for all n.

We describe the types for arbitrary $\lambda : R \to R$ and give a cohomological criterion to determine when two involutions are of the same type. This criterion implies in particular that the type of an Azumaya algebra with involution (A, τ) is determined entirely by the restriction of (A, τ) to the ramification locus of $\operatorname{Spec} R \to \operatorname{Spec} S$. More than two types may occur. With this new subtlety, one can further ask, in the context of Saltman's theorem, what are the types of λ-involutions which can be exhibited on representatives of a given Brauer class in $\operatorname{Br} R$. Our generalization of Saltman's theorem answers this question.

To demonstrate some of the ideas above, let us consider a field k of characteristic different from 2 and the ring $R := k[x, x^{-1}]$ of Laurent polynomials with the involution $\lambda : x \mapsto x^{-1}$. The fixed ring is $S := k[x + x^{-1}]$. The map $\operatorname{Spec} R \to \operatorname{Spec} S$ is ramified at two points, $x = 1$ and $x = -1$, and unramified elsewhere. Our results show that there are 4 types of λ-involution for even-degree algebras and 1 type in odd degrees. Furthermore, the type of a λ-involution is determined by the types — orthogonal or symplectic — obtained by specializing to $x = 1$ and $x = -1$. For example, consider the λ-involution of $\operatorname{Mat}_{2\times 2}(R)$ given by

$$\tau : \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} \mapsto \begin{bmatrix} d(x^{-1}) & x^{-1}b(x^{-1}) \\ xc(x^{-1}) & a(x^{-1}) \end{bmatrix}. \tag{1.1}$$

Evaluating at $x = 1$, the involution of (1.1) becomes orthogonal, whereas evaluating at $x = -1$ makes it symplectic. Our generalization of Saltman's theorem implies that if $\alpha \in \operatorname{Br} R$ is represented by an Azumaya R-algebra admitting a λ-involution, then each of the 4 types of λ-involutions is the type of a λ-involution of some representative of α.

It seems likely that our results on types could be used to extend the theory of involutive Brauer groups, intiated in [PS92] (see also [VV98]), to schemes carrying ramified involutions. We hope to address this in subsequent work.

We finally note that from the point of view of group schemes, the study of λ-involutions of Azumaya algebras in the case where is λ neither trivial nor unramified amounts to studying certain group schemes over $\operatorname{Spec} R$ which are generically reductive but degenerate on a closed subscheme. Specifically, the projective unitary group of an Azumaya algebra with a λ-involution is generically of type A and degenerates to types B, C or D on the connected components of the branch locus of $\operatorname{Spec} R \to \operatorname{Spec} S$. The study of degenerations of reductive groups have proved useful in many instances. Recent examples include [APS15] and [BFF17], but this manifests even more in the works of Bruhat and Tits on reductive groups over henselian discretely valued fields [BT72], [BT84], [BT87]. Broadly speaking, degenerations of reductive groups are encountered naturally when one attempts to extend a group scheme defined on a generic point of an integral scheme to the entire scheme, a process which is often considered in number theory.

1.2 Outline

Following is a detailed account of the contents of this paper, mostly in the order of presentation. While the majority of this work applies to schemes without assuming 2 is invertible, we make this assumption here in order to avoid certain technicalities.

Section 2 is devoted to technical preliminaries, largely to do with non-abelian cohomology in the context of Gorthendieck topoi.

Let X be a scheme and let $\lambda : X \to X$ be an involution. Our first concern is to specify an appropriate quotient of X by the group $C_2 = \{1, \lambda\}$. There is an evident choice when $X = \operatorname{Spec} R$ with R a ring, since one can take the quotient to be $\operatorname{Spec} S$, where S is the fixed ring of $\lambda : R \to R$. However, at the level of generality that we consider, there is often more than one plausible option. For instance, if the action of λ is not free, then $[X/C_2]$, a Deligne–Mumford stack, might serve just as well as the scheme or algebraic space X/C_2. The difference between these alternatives becomes particularly striking when C_2 acts trivially on X — the quotient $X \to X/C_2 = X$ can be regarded as a degenerate case of a double covering, ramified everywhere, whereas $X \to [X/C_2]$ is a C_2-Galois covering, ramified nowhere. From the point of view of the first quotient, all involutions will appear to be of the first kind, whereas with respect to the second quotient, all involutions will appear to be of the second kind.
We are therefore led to conclude that a chosen quotient $\pi : X \to Y$, in addition to X and λ, is necessary in order to discuss involutions in a way consistent with what is already done in the cases where λ is an involution of a ring.
We require a quotient to satisfy certain axioms, presented in Subsection 4.3, and prove that they are satisfied in a number of important examples, notably when the categorical quotient X/C_2 exists in the category of schemes and is a *good quotient*. Such quotients exist for instance if X is affine or projective, see Theorem 4.35. Thereafter in the development of the theory, we are usu-

ally agnostic about the quotient chosen. In examples, we often return to the motivating case of a good quotient.

Consider, therefore, a good quotient $\pi : X \to Y = X/C_2$. It is technically easier to work on Y than on X. Specifically, by virtue of our Theorem 4.28, there is an equivalence between Azumaya algebras with λ-involution on X on the one hand and Azumaya algebras with $\pi_*\lambda$-involution over the sheaf of rings $R := \pi_*(\mathcal{O}_X)$ on the other. We therefore study Azumaya algebras over R. While Y does not carry an involution, the ring sheaf R has an involution, namely, $\pi_*\lambda$, which we abbreviate to λ. A difficulty that we encounter here is that the sheaf of rings R is not a local ring object on Y, but rather a sheaf of rings with involution, the fixed subsheaf of which is the local ring object $(\pi_*\mathcal{O}_X)^{C_2} = \mathcal{O}_Y$. We devote considerable work to the study of commutative rings with involutions whose fixed subrings are local in Section 3, and conclude in Theorem 3.16 that any such ring is a semilocal ring, so that the sheaf R may be viewed as making Y a "semilocally ringed" space.

In Section 5, we introduce and study *types* of λ-involutions. Specifically, we define two Azumaya R-algebras with a λ-involution, (A, τ) and (B, σ), to be of the same *type* if some matrix algebra over (A, τ) is $Y_{\text{ét}}$-locally isomorphic to some matrix algebra over (B, σ). We show in Theorem 5.17 and Corollary 5.18 that the collection of types forms a 2-torsion group whose product rule is compatible with tensor products, and when $\deg A = \deg B$, the involutions τ and σ have the same type if and only if (A, τ) and (B, σ) are $Y_{\text{ét}}$-locally isomorphic, without the need to pass to matrix algebras. Thus, the definition given here agrees with the definition in Subsection 1.1. We then turn to the problem of calculating the group of types in specific cases.

Let $W \subset Y$ denote the *branch locus* of $\pi : X \to Y$. Then, away from W, the C_2-action on $V = X - \pi^{-1}(W)$ is unramified, hence there is only one possible type of λ-involution on $A|_V$, viz. unitary, and all involutions on $A|_V$ are locally isomorphic to the involution $\mathrm{Mat}_{n\times n}(R) \to \mathrm{Mat}_{n\times n}(R)$ given by applying the involution λ to each entry in the matrix and then taking the transpose, i.e., $M \mapsto (M^\lambda)^{\mathrm{tr}}$. In contrast, over a connected component Z_1 of $Z := \pi^{-1}(W)$, regarded as a reduced closed subscheme of X, the involution λ restricts to the identity (Proposition 4.47), and so λ-involutions of $A|_{Z_1}$ fall into one of two types — orthogonal or symplectic. This suggests that the types of λ-involutions over X should be in bijection with $\mathrm{H}^0(Z, \mu_2)$, where $\mu_2 := \{1, -1\}$ and 1 and -1 represent orthogonal and symplectic involutions respectively, and that two λ-involutions are of the same type if and only if they are of the same type when restricted to each connected component of Z. We prove the second statement in Theorem 5.37 and establish the first under the assumption that that Y is noetherian and regular in Corollary 6.23. We do not know whether the first statement holds in general. Determining the type of a given involution of a given algebra, $\tau : A \to A$, can now be carried out by considering the rank of the sheaf of τ-symmetric elements on the various components of W; see [KMRT98, Prp. 2.6].

In Section 6, we turn to the question of when a Brauer class $\alpha = [A] \in \mathrm{Br}(X)$ contains an algebra A' possessing a λ-involution. Saltman [Sal78, Thm. 3.1] gave necessary and sufficient conditions for this when λ is trivial or unramified. Specifically, A is Brauer equivalent to such an algebra if $2[A] = 0 \in \mathrm{Br}(X)$, in the case of a trivial action, or if $\mathrm{cores}_{X/Y}[A] = 0 \in \mathrm{Br}(Y)$, in the case of an unramified action. We unify these two results, and generalize to the cases that are neither trivial nor unramified, by defining a transfer map transf : $\mathrm{Br}(X) \to \mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m)$, and deducing in Theorem 6.10 that A is equivalent to an algebra admitting a λ-involution of type t if and only if $\mathrm{transf}([A]) = \Phi(t)$, where $\Phi(t) \in \mathrm{H}^2(Y, \mathbb{G}_m)$ is a cohomology class depending on the type. In both extreme cases of trivial and unramified actions, and in fact whenever Y is a nonsingular variety, $\Phi(t)$ is necessarily 0. Moreover, in the case of a trivial action, $\mathrm{transf}([A]) = 2[A]$, and in the unramified case, $\mathrm{transf}([A]) = \mathrm{cores}_{X/Y}[A]$, so we recover Saltman's theorem as a special case. We also show that if A is equivalent to an algebra with involution, then such an algebra can be constructed to have degree twice that of A, thus extending the analogous result of [KPS90, Thms. 4.1, 4.2]. We do not, however, follow [Sal78] and [KPS90] in considering the *corestriction algebra* of A, taking instead a purely cohomological approach. In fact, it is not clear whether a corestriction algebra of A can be defined in a meaningful way when $\lambda : X \to X$ is ramified. This problem was considered in [APS15, §5], where some positive results are given, and we leave its pursuit in the current level of generality to a future work.

Section 7 gives a number of examples of the workings out of the previous theory. In particular, we give examples of schemes X with involutions $\lambda : X \to X$ that are neither unramified nor trivial, along with a classification of the various types of λ-involutions of Azumaya algebras, e.g., Examples 7.6 and 7.7.

While this overview has so far been written in the language of schemes, the majority of the results are established in the setting of locally ringed Grothendieck topoi, of which the étale ringed topos of a scheme is a special case. The advantage of this generality is that all the results above also apply, essentially verbatim, to Azumaya algebras with involution over a topological C_2-space, or to Azumaya algebras with involutions on algebraic stacks. The applicability of our results in the context of other sites associated with schemes, e.g., the Zariski site, the fppf site, the Nisnevich site and some large sites, is discussed in Subsection 4.4.

Comparison of Azumaya algebras over schemes with topological Azumaya algebras has proved useful in the past, for instance in [AW14c], [AW15]. Having the previous theory available also in the topological context, we consider a finite type, regular $\mathbb{C}$-algebra R with an unramified involution λ and compare the theory of Azumaya R-algebras with involutions restricting to λ on the centre with the theory of topological Azumaya algebras with involution on the complex manifold $(\operatorname{Spec} R)_{\mathrm{an}}$. This is carried out in Subsection 4.2, specifically in Example 4.17.

By such comparison, we produce an example of an Azumaya algebra A of degree n, over a ring R with an unramified involution λ, having the property that A is Brauer equivalent to an algebra A' with λ-involution, but the least degree of such an A' is $2n$, Theorem 9.8; the bound $2n$ is the lowest possible by [KPS90, §4], which guarantees the existence of A' of degree $2n$ in general. An analogous example in the case where λ is assumed to be trivial was given in [AFW19]. The method of proof, which is carried out in Sections 8 and 9, is by using existing study of bundles with involution as a branch of equivariant homotopy theory, [May96]. In particular, we can find universal examples of topological Azumaya algebras with involution, which are valuable sources of counterexamples.

In an appendix, we give a proof that the stalks of the sheaf of continuous, complex-valued functions on a topological space X satisfy Hensel's lemma. This is used here and there in the body of the paper to treat this case at the same time as étale sites of schemes.

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2 Preliminaries

This section recalls necessary facts and sets notation for the sequel. Throughout, $\mathbf{X}$ denotes a Grothendieck topos. We reserve the term "ring" for commutative unital rings, whereas algebras are assumed unital but not necessarily commutative.

2.1 Generalities on Topoi

Recall that a Grothendieck topos is a category that is equivalent to the category of set-valued sheaves over a small site, or equivalently, a category satisfying Giraud's axioms; see [Gir71, Chap. 0]. In this paper we shall be particularly interested in the following examples:

(i) $\mathbf{X} = \mathbf{Sh}(X_{\text{ét}})$, the category of sheaves over the small étale site of a scheme X.

(ii) $\mathbf{X} = \mathbf{Sh}(X)$, the category of sheaves on a topological space X.

We will occasionally consider other sites associated with a scheme X. In particular, X_{Zar} and X_{fppf} will denote the small Zariski and small fppf sites of X, respectively.
The topos of sheaves over a singleton topological space, which is nothing but the category of sets, will be denoted **pt**.
We note that every topos $\mathbf{X}$ can be regarded as a site relative to its canonical topology. In this case, a collection of morphisms $\{U_i \to V\}_{i\in I}$ is a covering of V if and only if it is jointly surjective, and every sheaf over $\mathbf{X}$ is representable, so that $\mathbf{X} \cong \mathbf{Sh}(\mathbf{X})$. This allows us to define objects of $\mathbf{X}$ by specifying the sheaf that they represent, and to define morphisms between objects by defining them on sections.
The symbols $\emptyset_{\mathbf{X}}$ and $*_{\mathbf{X}}$ will be used for the initial and final objects of $\mathbf{X}$, respectively. When $\mathbf{X} = \mathbf{Sh}(X)$ for a site X, the sheaf $\emptyset_{\mathbf{X}}$ assigns an empty set to every non-initial object of X, and $*_{\mathbf{X}}$ is the sheaf assigning a singleton to every object in X. The subscript $\mathbf{X}$ will dropped when it may be understood from the context.
For every pair of objects A, U of $\mathbf{X}$ the U-sections of A are

$$A(U) := \mathrm{Hom}_{\mathbf{X}}(U, A)$$

and the global sections of A are $\mathrm{H}^0(\mathbf{X}, A) = \Gamma A = \Gamma_{\mathbf{X}} A := A(*)$. We will write $A_U = A \times U$, and will regard A_U as an object of the slice category $\mathbf{X}/U$.
By a *group* G in $\mathbf{X}$ we will mean a group object in $\mathbf{X}$. In this case, the U-sections $G(U)$ form a group for all objects U of $\mathbf{X}$. Similar conventions will apply to abelian groups, rings, G-objects, and so on.
If R is a ring object in some topos, then $\mu_{2,R}$ will denote the object of square-roots of 1 in R, that is, the object given sectionwise by $\mu_{2,R}(U) = \{x \in R(U) : x^2 = 1\}$. The bald notation μ_2 will denote the constant sheaf $\{+1, -1\}$.

2.2 TORSORS

DEFINITION 2.1. Let X be a site and let G be a sheaf of groups on X. A (right) G-*torsor* is a sheaf P on X equipped with a right action $P \times G \to P$ such that P is locally isomorphic to G as a right G-object.

Equivalently, and intrinsically to the topos $\mathbf{X} := \mathbf{Sh}(X)$, a (right) G-torsor is an object P of $\mathbf{X}$ equipped with a (right) G-action $m : P \times G \to P$ such that the unique morphism $P \to *$ is an epimorphism and such that the morphism $P \times G \xrightarrow{\pi_1 \times m} P \times P$ is an isomorphism. See [Gir71, Déf. III.1.4.1] where more general torsors over objects S of $\mathbf{X}$ are defined; our definition is that of torsors over the terminal object.
The equivalence of the two definitions of "torsor" is given by [Gir71, Prop. III.1.7.3].

The category of G-torsors, with G-equivariant isomorphisms as morphisms, will be denoted

$$\mathbf{Tors}(\mathbf{X}, G).$$

A G-torsor P is *trivial* if $P \cong G$ as right G-objects, and an object U is said to *trivialize* P if $P_U \cong G_U$ as G_U-objects. The latter holds precisely when $P(U) \neq \emptyset$.

Recall that if P is a G-torsor and X is a left G-object in $\mathbf{X}$, then $P \times^G X$ denotes the quotient of $P \times X$ by the equivalence relation $P \times G \times X \to (P \times X) \times (P \times X)$ given by $(p, g, x) \mapsto ((pg, x), (p, gx))$ on sections. We shall sometimes denote $P \times^G X$ by ${}^P X$ and call it the P-*twist* of X. We remark that X and ${}^P X$ are locally isomorphic in the sense that there exists a covering $U \to *$ in $\mathbf{X}$ such that $X_U \cong {}^P X_U$ — take any U such that $G_U \cong P_U$. If X posses some additional structure, for instance if X is an abelian group, and G respects this structure, then ${}^P X$ also posses the same structure and the isomorphism $X_U \cong {}^P X_U$ respects the additional structure. The general theory outlined here is established precisely in [Gir71, Chap. III].

Remark 2.2. There is another plausible definition of "torsor" on a site X, particularly when the topology is subcanonical and when the category X has finite products—i.e., X is a *standard site*. That is, one modifies the definition in 2.1 by requiring the objects G and P to be objects of the site X. These are the *representable torsors* as distinct from the *sheaf torsors* defined above. We will not consider the question of representability in this paper beyond the following remark: Suppose X is a scheme and G is a group scheme over X. Then G represents a group sheaf on the big flat site of X, also denoted G. If $G \to X$ is affine, then all sheaf G-torsors are representable by an X-scheme [Mil80, Thm. III.4.3].

2.3 Cohomology of Abelian Groups

The functor H^0 sending an abelian group A in $\mathbf{X}$ to its global sections is left exact. The i-th right derived functor of H^0 is denoted $\mathrm{H}^i(\mathbf{X}, A)$, as usual. If $\mathbf{X}$ is clear from the context, we shall simply write $\mathrm{H}^i(A)$. When $\mathbf{X} = \mathbf{Sh}(X_{\text{ét}})$ for a scheme X, we write $\mathrm{H}^i(\mathbf{X}, A)$ as $\mathrm{H}^i_{\text{ét}}(X, A)$, and likewise for other sites associated with X.

In the sequel, we shall make repeated use of Verdier's Theorem, quoted below, which provides a description of cohomology classes in terms of hypercoverings. We recall some details, and in doing so, we set notation. One may additionally consult [deJ17, Tag 01FX], [DHI04] or [AGV72b, Exp. V.7].

Let $\mathbf{\Delta}$ denote the category having $\{\{0, \dots, n\} \mid n = 0, 1, 2, \dots\}$ as its objects and the non-decreasing functions as its morphisms. Recall that a *simplicial object* in $\mathbf{X}$ is a contravariant functor $U_\bullet : \mathbf{\Delta} \to \mathbf{X}$. For every $0 \leq i \leq n$, we write $U_n = U_\bullet(\{0, \dots, n\})$ and set $d_i^n = U_\bullet(\delta_i^n)$ and $s_i^n = U_\bullet(\sigma_i^n)$, where $\delta_i^n : \{0, \dots, n-1\} \to \{0, \dots, n\}$ is the non-decreasing monomorphism whose image does not include i and $\sigma_i^n : \{0, \dots, n+1\} \to \{0, \dots, n\}$ is the non-decreasing epimorphism for which i has two preimages. We shall write d_i, s_i instead of d_i^n, s_i^n when n is clear from the context. Since the morphisms $\{\sigma_i^n, \delta_i^n\}_{i,n}$ generate $\mathbf{\Delta}$, in order to specify a simplicial object $U_\bullet$ in $\mathbf{X}$, it is enough to specify objects $\{U_n\}_{n\geq 0}$ and morphisms $s_i^n : U_n \to U_{n+1}$, $d_i^n : U_n \to U_{n-1}$ for all $0 \leq i \leq n$. Of course, the morphisms $\{s_i^n, d_i^n\}_{i,n}$ have to satisfy certain relations, which can be found in [May92], for instance.

For $n \geq 0$, let $\mathbf{\Delta}_{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ whose objects are $\{\{0\}, \dots, \{0, \dots, n\}\}$. The restriction functor $U_\bullet \mapsto U_{\leq n} : \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{X}) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leq n}, \mathbf{X})$ admits a right adjoint called the n-th coskeleton and denoted cosk_n. We also write $\mathrm{cosk}_n(U_\bullet)$ for $\mathrm{cosk}_n(U_{\leq n})$. The simplicial object $U_\bullet$ is called a *hypercovering* (of the terminal object) if $U_0 \to *$ is a covering and for all $n \geq 0$, the map $U_{n+1} \to \mathrm{cosk}_n(U_\bullet)_{n+1}$ induced by the adjunction is a covering. For example, when $n = 0$, the latter condition means that $(d_0^1, d_1^1) : U_1 \to U_0 \times U_0$ is a covering.

Hypercoverings form a category in the obvious manner, morphisms being natural transformations.

Example 2.3. Let $U \to *$ be a morphism in $\mathbf{X}$. Define $U_n = U \times \cdots \times U$ ($n+1$ times), let $d_i^n : U_n \to U_{n-1}$ be the projection omitting the i-th copy of U and let $s_i^n : U_n \to U_{n+1}$ be given by $(u_0, \dots, u_n) \mapsto (u_0, \dots, u_i, u_i, \dots, u_n)$ on sections. These data determine a simplicial object $U_\bullet$ which is a hypercovering if $U \to *$ is a covering. In this case, the map $U_{n+1} \to \mathrm{cosk}_n(U_\bullet)_{n+1}$ is an isomorphism for all n. The hypercovering $U_\bullet$ is called the *Čech hypercovering* associated to U. If $U_\bullet$ is an arbitrary hypercovering, then $\mathrm{cosk}_0(U_\bullet)$ is the Čech hypercovering associated to U_0.

The following lemma is fundamental.

Lemma 2.4 ([deJ17, Tag 01GJ] or [AGV72b, Thm. V.7.3.2]). *Let $U_\bullet$ be a hypercovering and let $V \to U_n$ be a covering. Then there exists a hypercovering morphism $U'_\bullet \to U_\bullet$ such that $U'_n \to U_n$ factors through $V \to U_n$.*

Let A be an abelian group object of $\mathbf{X}$. With any hypercovering $U_\bullet$ in $\mathbf{X}$ we associate a cochain complex $C^\bullet(U_\bullet, A)$ defined by $C^n(U_\bullet, A) = A(U_n)$ for $n \geq 0$ and $C^n(U_\bullet, A) = 0$ otherwise. The coboundary map $d^n : C^n(U_\bullet, A) \to C^{n+1}(U_\bullet, A)$ is given by $d^n(a) = \sum_{i=0}^{n+1}(-1)^i d_i^*(a)$, as usual; here $d_i^* = (d_i^{n+1})^* : A(U_n) \to A(U_{n+1})$ is the map induced by $d_i^{n+1} : U_{n+1} \to U_n$. The cocycles, coboundaries, and cohomology groups of the complex are denoted $Z^n(U_\bullet, A)$, $B^n(U_\bullet, A)$ and $\mathrm{H}^n(U_\bullet, A)$. Any morphism of hypercoverings $U'_\bullet \to U_\bullet$ induces a morphism $\mathrm{H}^n(U_\bullet, A) \to \mathrm{H}^n(U'_\bullet, A)$ in the obvious manner.

Theorem 2.5 (Verdier [AGV72b, Thm. V.7.4.1]). *Let $\mathbf{X}$ be a topos and A an abelian group object in $\mathbf{X}$. The functors*

$$A \mapsto \mathrm{H}^n(\mathbf{X}, A) \qquad \textit{and} \qquad A \mapsto \operatorname*{colim}_{U_\bullet} \mathrm{H}^n(U_\bullet, A)$$

from the category of abelian groups in $\mathbf{X}$ to the category of abelian groups are naturally isomorphic. Here, the colimit is taken over the category of hypercoverings.

Remark 2.6. If we were to take the colimit in the theorem over the category of the Čech hypercoverings, then the result would be the Čech cohomology of A. Consequently, the Čech cohomology and the derived-functor cohomology agree when every hypercovering admits a map from a Čech hypercovering. This is known to be the case when $\mathbf{X} = \mathbf{Sh}(X)$ for a paracompact Hausdorff topological space X [God73, Thm. 5.10.1], or $\mathbf{X} = \mathbf{Sh}(X_{\text{ét}})$ for a noetherian scheme X such that any finite subset of X is contained in an open affine subscheme [Art71, §4].

A short exact sequence of abelian groups $1 \to A' \to A \to A'' \to 1$ in $\mathbf{X}$ gives rise to a long exact sequence of cohomology groups. By the second proof of [deJ17, Tag 01H0], quoted as Theorem 2.5 here, the connecting homomorphism $\delta^n : \mathrm{H}^n(A'') \to \mathrm{H}^{n+1}(A')$ can be described as follows: Let $\alpha'' \in \mathrm{H}^n(A'')$ be a cohomology class represented by a cocycle $a'' \in Z^n(U_\bullet, A'')$ for some hypercovering $U_\bullet$. Since $A \to A''$ is an epimorphism, we can find a covering $V \to U_n$ such that $a'' \in A''(U_n)$ is the image of some $a \in A(V)$. By Lemma 2.4 there exists a morphism of hypercoverings $V_\bullet \to U_\bullet$ such that $V_n \to U_n$ factors through $V \to U_n$. We replace a with its image in $A(V_n)$. One easily checks that the image of $d^n(a) \in C^{n+1}(V_\bullet, A)$ in both $C^{n+1}(V_\bullet, A'')$ and $C^{n+2}(V_\bullet, A)$ is 0, and hence $d^n(a) \in Z^{n+1}(V_\bullet, A')$. Now, $\delta^n(\alpha'')$ is the cohomology class determined by $d^n(a) \in Z^{n+1}(V_\bullet, A')$.

2.4 Cohomology of Non-Abelian Groups

For a group object G of $\mathbf{X}$, not necessary abelian, we define the pointed set $\mathrm{H}^1(\mathbf{X}, G)$ by hypercoverings. Given a hypercovering $U_\bullet$ in $\mathbf{X}$, let $Z^1(U_\bullet, G)$ be the set of elements $g \in G(U_1)$ satisfying

$$d_2^* g \cdot d_0^* g \cdot d_1^* g^{-1} = 1 \tag{2.1}$$

in $G(U_2)$; here $d_i^* = (d_i^2)^* : G(U_1) \to G(U_2)$ is induced by $d_i^2 : U_2 \to U_1$. Two elements $g, g' \in Z^1(U_\bullet, G)$ are said to be cohomologous, denoted $g \sim g'$, if there exists $x \in G(U_0)$ such that $g = d_1^* x \cdot g' \cdot d_0^* x^{-1}$. We define the pointed set $\mathrm{H}^1(U_\bullet, G)$ to be $Z^1(U_\bullet, G)/{\sim}$ with the equivalence class of $1_{G(U_1)}$ as a distinguished element. A morphism of hypercoverings $U'_\bullet \to U_\bullet$ induces a morphism of pointed sets $\mathrm{H}^1(U_\bullet, G) \to \mathrm{H}^1(U'_\bullet, G)$. Now, following the literature, we define

$$\mathrm{H}^1(\mathbf{X}, G) := \operatorname*{colim}_{U_\bullet} \mathrm{H}^1(U_\bullet, G),$$

where the colimit is taken over the category of all hypercoverings in $\mathbf{X}$. We note that some texts take the colimit over the category of Čech hypercoverings, see Example 2.3, but this makes no difference thanks to the following lemma.

Lemma 2.7. *Let $U_\bullet$ be a hypercovering. Then the maps $Z^1(\operatorname{cosk}_0(U_\bullet), G) \to Z^1(U_\bullet, G)$ and $\mathrm{H}^1(\operatorname{cosk}_0(U_\bullet), G) \to \mathrm{H}^1(U_\bullet, G)$, induced by the canonical morphism $U_\bullet \to \operatorname{cosk}_0(U_\bullet)$, are isomorphisms.*

Proof. The proof shall require various facts about coskeleta. We refer the reader to [deJ17, Tag 0AMA] or any equivalent source for proofs.

Recall from Example 2.3 that $\mathrm{cosk}_0(U_\bullet)$ is nothing but the Čech hypercovering associated to U_0. Since $U_\bullet$ is a hypercovering, $(d_0, d_1) : U_1 \to \mathrm{cosk}_0(U_\bullet)_1 = U_0 \times U_0$ is a covering, and hence the induced map $G(U_0 \times U_0) \to G(U_1)$ is injective. Since the map $Z^1(\mathrm{cosk}_0(U_\bullet), G) \to Z^1(U_\bullet, G)$ is a restriction of the latter, it is also injective. This implies that if two cocycles in $Z^1(\mathrm{cosk}_0(U_\bullet), G)$ become cohomologous in $Z^1(U_\bullet, G)$, then they are also cohomologous in $Z^1(\mathrm{cosk}_0(U_\bullet), G)$, so $\mathrm{H}^1(\mathrm{cosk}_0(U_\bullet), G) \to \mathrm{H}^1(U_\bullet, G)$ is injective. It is therefore enough to show that $Z^1(\mathrm{cosk}_0(U_\bullet), G) \to Z^1(U_\bullet, G)$ is surjective.
We first observe that the canonical map $Z^1(\mathrm{cosk}_1(U_\bullet), G) \to Z^1(U_\bullet, G)$ is an isomorphism. This follows from the fact that $\mathrm{cosk}_1(U_\bullet)_1 = U_1$ and $U_2 \to \mathrm{cosk}_1(U_\bullet)_2$ is a covering, hence (2.1) is satisfied in $G(U_2)$ if and only if it is satisfied in $G(\mathrm{cosk}_1(U_\bullet)_2)$. Since $\mathrm{cosk}_0(\mathrm{cosk}_1(U_\bullet)) = \mathrm{cosk}_0(U_\bullet)$, we may replace $U_\bullet$ with $\mathrm{cosk}_1(U_\bullet)$. In this case, the construction of cosk_1 implies that U_2 is characterized by

$$U_2(X) = \{(e_{01}, e_{02}, e_{12}, v_0, v_1, v_2) \in U_1(X)^3 \times U_0(X)^3 : \\ d_0 e_{ij} = v_j \text{ and } d_1 e_{ij} = v_i \text{ for all legal } i, j\} .$$

for all objects X of $\mathbf{X}$. The maps $d_0, d_1, d_2 : U_2 \to U_1$ are then given by taking the e_{12}-part, e_{02}-part, and e_{01}-part, respectively. Geometrically, U_2 is the object of simplicial morphisms from the boundary of the 2-simplex to $U_\bullet$.
Let $g \in Z^1(U_\bullet, G) \subseteq G(U_1)$. We claim that g descends along (d_0, d_1) to $g' \in G(U_0 \times U_0)$. Write $V = U_1 \times_{U_0 \times U_0} U_1$ and let π_1, π_2 denote the first and second projections from V onto U_1. We need to show that $\pi_1^* g = \pi_2^* g$ in $G(V)$. For an object X, the X-sections of V can be described by

$$V(X) = \{(e_{01}, e'_{01}, v_0, v_1) \in X(U_1)^2 \times X(U_0)^2 : d_0(e_{01}) = d_0(e'_{01}) = v_1, \\ d_1(e_{01}) = d_1(e'_{01}) = v_0\} .$$

Define $\Psi : V \to U_2$ by

$$\Psi(e_{01}, e'_{01}, v_0, v_1) = (e_{01}, e'_{01}, s_0 v_1, v_0, v_1, v_1)$$

on sections. One readily checks that $d_0\Psi = s_0 d_0 \pi_1 = d_0 s_1 \pi_1$, $d_1\Psi = \pi_2$ and $d_2\Psi = \pi_1$. Now, applying $\Psi^* : G(U_2) \to G(V)$ to (2.1), we arrive at the equation $\pi_1^* g \cdot \pi_1^* d_0^* s_1^* g \cdot \pi_2^* g^{-1} = 1$ in $G(V)$, and applying $s_1^* : G(U_2) \to G(U_1)$ to (2.1), we find that $g \cdot s_1^* d_0^* g \cdot g^{-1} = 1$ in $G(U_1)$. Both equations taken together imply that $\pi_1^* g = \pi_2^* g$ in $G(V)$, hence our claim follows.
To finish the proof, it is enough to show that g' is a 1-cocycle. This will follow from the fact that g is a 1-cocycle if we show that the canonical map $U_2 \to \mathrm{cosk}_0(U_\bullet)_2 = U_0^3$ is a covering. The latter map is given by $(e_{01}, e_{02}, e_{12}, v_0, v_1, v_2) \mapsto (v_0, v_1, v_2)$ on sections. Since $(d_0, d_1) : U_1 \to U_0$ is a covering, for every pair of vertices $v_0, v_1 \in U_0(X)$, there exists a covering $X' \to X$ and an edge $e_{01} \in U_1(X')$ satisfying $d_0 e_{01} = v_1$, $d_1 e_{01} = v_0$. This easily implies that $U_2 \to U_0^3$ is locally surjective, finishing the proof. □

The following proposition summarizes the main properties of $\mathrm{H}^1(\mathbf{X}, G)$. As before, we shall suppress $\mathbf{X}$, writing $\mathrm{H}^1(G)$, when it is clear form the context.

Proposition 2.8. *Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups in $\mathbf{X}$.*

(i) $\mathrm{H}^1(G) = \mathrm{H}^1(\mathbf{X}, G)$ *is naturally isomorphic to the set of isomorphism classes of G-torsors; the distinguished element of* $\mathrm{H}^1(G)$ *corresponds to the isomorphism class of the trivial torsor.*

(ii) There is a long exact sequence of pointed sets

$$1 \to \mathrm{H}^0(G') \to \mathrm{H}^0(G) \to \mathrm{H}^0(G'') \xrightarrow{\delta^1} \mathrm{H}^1(G') \to \mathrm{H}^1(G) \to \mathrm{H}^1(G'') \ .$$

This exact sequence is functorial in $1 \to G' \to G \to G'' \to 1$, i.e., a morphism from it to another short exact sequence of groups gives rise a morphism between the corresponding long exact sequences.

(iii) When G' is central in G, one can extend the exact sequence of (ii) with an additional morphism $\delta^2 : \mathrm{H}^1(G'') \to \mathrm{H}^2(G')$, which is again functorial in $1 \to G' \to G \to G'' \to 1$.

(iv) When G', G, G'' are abelian, the exact sequence of (iii) is canonically isomorphic to the truncation of the usual long exact sequence of cohomology groups associated to $1 \to G' \to G \to G'' \to 1$. In particular, $\mathrm{H}^1(G)$ *defined here is naturally isomorphic to* $\mathrm{H}^1(G)$ *defined in 2.3 and regarded as a pointed set with distinguished element* 1.

The proposition is well known, but Giraud [Gir71, IV, 4.2.7.4, 4.2.10] is the only source we are aware of that treats all parts in the generality that we require. Since the treatment in [Gir71] is somewhat obscure, and since we shall need the definition of the maps δ^1 and δ^2 in the sequel, we include an outline of the proof here. Note that it is easier to prove (i) using the definition of $\mathrm{H}^1(G)$ via Čech hypercoverings, while it is easier to prove (iii) using the definition of $\mathrm{H}^1(G)$ via arbitrary hypercoverings, and these definitions are equivalent thanks to Lemma 2.7.

Proof (sketch). (i) Let P be a G-torsor. Choose a covering $U_0 \to *_{\mathbf{X}}$ such that $P(U_0) \neq \emptyset$ and fix some $x \in P(U_0)$. Form the Čech hypercovering $U_\bullet$ associated to U_0. Then there exists a unique $g \in G(U_1)$ such that $d_1^* x \cdot g = d_0^* x$ in $P(U_1)$. We leave it to the reader to check that $g \in Z^1(U_\bullet, G)$ and the construction $P \mapsto g$ induces a well-defined map from the isomorphism classes of $\mathbf{Tors}(G)$ to $\mathrm{H}^1(G)$ taking the trivial G-torsor to the special element of $\mathrm{H}^1(G)$.

In the other direction, let $\alpha \in \mathrm{H}^1(G)$. By Lemma 2.7, α is represented by some $g \in Z^1(U_\bullet, G)$ where $U_\bullet$ is a Čech hypercovering. Define P to be the object of $\mathbf{X}$ characterized by $P(V) = \{x \in G(U_0 \times V) \,:\, g_V \cdot d_0^* x = d_1^* x\}$; here, $d_i^* : G(U_0 \times V) \to G(U_1 \times V)$ is induced by $d_i \times \mathrm{id} : U_1 \times V \to U_0 \times V$. There is a right G-action on P given by $(x,h) \mapsto x \cdot h_{U_0}$ on sections. We leave it to the reader to check that P is indeed a G-torsor, and the assignment $\alpha \mapsto P$ defines an inverse to the map of the previous paragraph. In particular, note that $P \to *_{\mathbf{X}}$ is a covering because $g \in G(U_0)$; use the fact that $U_\bullet$ is a Čech hypercovering.

(ii) Define $\delta^1 : \mathrm{H}^0(G'') \to \mathrm{H}^1(G')$ as follows: Let $g'' \in \mathrm{H}^0(G'')$. There is a covering $U_0 \to *$ such that g'' lifts to some $g \in G(U_0)$. One easily checks that $g' := d_0^* g \cdot d_1^* g^{-1} \in G(U_1)$ lies in $Z^1(G', U_\bullet)$ where $U_\bullet$ is the Čech hypercovering associated to U_0. We define $\delta^1 g''$ to be the cohomology class represented by g', and leave it to the reader to check that this is well-defined. All other maps in the sequence are defined in the obvious manner and the exactness is easy to check.

(iii) Define $\delta^2 : \mathrm{H}^1(G'') \to \mathrm{H}^2(G')$ as follows: Let $\alpha \in \mathrm{H}^1(G'')$ be represented by $g'' \in Z^1(U_\bullet, G'')$ where $U_\bullet$ is a hypercovering. There is a covering $V \to U_1$ such that g'' lifts to some $g \in G(V)$. By Lemma 2.4, there is a morphism of hypercoverings $U'_\bullet \to U_\bullet$ such that $U'_1 \to U_1$ factors through $V \to U_1$. We replace g with its image in $G(U'_1)$. Let $g' := d_2^* g \cdot d_0^* g \cdot d_1^* g^{-1} \in G(U'_1)$. It is easy to check that g' lies in $G'(U'_1)$ and defines a 2-cocycle of G' relative to $U'_\bullet$. We define $\delta^2\alpha$ to be the cohomology class represented by $g' \in Z^2(U'_\bullet, G')$, and leave it to the reader to check that this is well-defined. The exactness of the sequence at $\mathrm{H}^1(G'')$ is straightforward to check.

(iv) Verdier's Theorem gives rise to an obvious isomorphism between $\mathrm{H}^1(G)$ as defined here and $\mathrm{H}^1(G)$ as defined in 2.3. It is immediate from the definitions that this isomorphism also induces an isomorphism between the long exact sequences; see the proofs of (ii), (iii) and the comment at the end of 2.3. □

2.5 Azumaya Algebras

Let R be a ring object of $\mathbf{X}$ and let n be a positive integer. Recall that an Azumaya R-algebra of degree n is an R-algebra A in $\mathbf{X}$ that is locally isomorphic to $\mathrm{M}_{n\times n}(R)$, i.e., there exists a covering $U \to *$ such that $A_U \cong \mathrm{M}_{n\times n}(R_U)$ as R_U-algebras. The Azumaya R-algebras of degree n together with R-algebra isomorphisms form a category which we denote by

$$\mathbf{Az}_n(\mathbf{X}, R) \ .$$

If A' is another Azumaya R-algebra, we let $\mathcal{H}om_{R\text{-alg}}(A, A')$ denote the subobject of the internal mapping object $(A')^A$ of $\mathbf{X}$ consisting of R-algebra homomorphisms. We define the group object $\mathcal{A}ut_{R\text{-alg}}(A)$ similarly.

Remark 2.9. We have defined here Azumaya algebras of constant degree only. When $\mathbf{X}$ is connected, these are all the Azumaya algebras, but in general, one has to allow the degree n to take values in the global sections of the sheaf $\mathbb{N}$ of positive integers on $\mathbf{X}$. For any such n one can define $\mathrm{M}_{n\times n}(R)$ and the definition of Azumaya algebras of degree n extends verbatim. We ignore this technicality, both for the sake of simplicity, and also since it is unnecessary for connected topoi, which are the topoi of interest to us.

Let $\mathcal{O}_{\mathbf{X}}$ be a ring object in $\mathbf{X}$. Recall that $\mathbf{X}$ is *locally ringed* by $\mathcal{O}_{\mathbf{X}}$, or $\mathcal{O}_{\mathbf{X}}$ is a *local ring object* in $\mathbf{X}$, if for any object U in $\mathbf{X}$ and $\{r_i\}_{i\in I} \subseteq \mathcal{O}_{\mathbf{X}}(U)$ with $\mathcal{O}_{\mathbf{X}}(U) = \sum_i r_i \mathcal{O}_{\mathbf{X}}(U)$, there exists a covering $\{U_i \to U\}_{i\in I}$ such that $r_i \in \mathcal{O}_{\mathbf{X}}(U_i)^\times$ for all $i \in I$. In fact, one can take $U_i = \emptyset_{\mathbf{X}}$ for almost all i. We remark that the condition should also hold when $I = \emptyset$, which implies that $\mathcal{O}_{\mathbf{X}}(U)$ cannot be the zero ring when $U \ncong \emptyset_{\mathbf{X}}$. When $\mathbf{X}$ has enough points, the condition is equivalent to saying that for every point $i : \mathbf{pt} \to \mathbf{X}$, the ring $i^*\mathcal{O}_{\mathbf{X}}$ is local (the zero ring is not considered local).
Suppose that $\mathcal{O}_{\mathbf{X}}$ is a local ring object. Then the group homomorphism $\mathrm{GL}_n(\mathcal{O}_{\mathbf{X}}) \to \mathcal{A}ut_{\mathcal{O}_{\mathbf{X}}\text{-alg}}(\mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}))$ given by $a \mapsto [x \mapsto axa^{-1}]$ on sections is surjective, [Gir71, V.§4]. This induces an isomorphism $\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}) := \mathrm{GL}_n(\mathcal{O}_{\mathbf{X}})/\mathcal{O}_{\mathbf{X}}^\times \to \mathcal{A}ut_{\mathcal{O}_{\mathbf{X}}\text{-alg}}(\mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}))$, which will be used to freely identify the source and target in the sequel. The following proposition is well established, again see [Gir71, V.§4].

Proposition 2.10. *If $\mathcal{O}_{\mathbf{X}}$ is a local ring object, then there is an equivalence of categories*

$$\mathbf{Tors}(\mathbf{X}, \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})) \xrightarrow{\sim} \mathbf{Az}_n(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$$

given by the functors

$$P \mapsto P \times^{\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})} \mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}) \qquad \textit{and} \qquad A \mapsto \mathcal{H}om_{\mathcal{O}_{\mathbf{X}}\text{-}alg}(\mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}), A).$$

The proposition holds for any ring object $\mathcal{O}_{\mathbf{X}}$ of $\mathbf{X}$ if one replaces the group object $\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})$ with $\mathcal{A}ut_{\mathcal{O}_{\mathbf{X}}\text{-alg}}(\mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}))$.

We continue to assume that $\mathcal{O}_{\mathbf{X}}$ is a local ring object. By Proposition 2.8(iii), the short exact sequence $1 \to \mathcal{O}_{\mathbf{X}}^\times \to \mathrm{GL}_n(\mathcal{O}_{\mathbf{X}}) \to \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}) \to 1$ gives rise to a pointed set map $\mathrm{H}^1(\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})) \to \mathrm{H}^2(\mathcal{O}_{\mathbf{X}}^\times)$; here, $\mathrm{H}^*(-) = \mathrm{H}^*(\mathbf{X}, -)$. As usual, the *Brauer group* of $\mathcal{O}_{\mathbf{X}}$ is

$$\mathrm{Br}(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = \mathrm{Br}(\mathcal{O}_{\mathbf{X}}) := \bigcup_{n\in\mathbb{N}} \mathrm{im}\left(\mathrm{H}^1(\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})) \to \mathrm{H}^2(\mathcal{O}_{\mathbf{X}}^\times)\right)$$

the addition being that inherited from the group $\mathrm{H}^2(\mathcal{O}_{\mathbf{X}}^\times)$. Since Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras correspond to $\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})$-torsors, which are in turn classified by $\mathrm{H}^1(\mathbf{X}, \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}))$, any Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra A gives rise to an element in $\mathrm{Br}(\mathcal{O}_{\mathbf{X}})$, denoted $[A]$ and called the *Brauer class* of A. By writing $A' \in [A]$ or saying that A' is *Brauer equivalent* to A, we mean that A' is an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra with $[A'] = [A]$. For more details, see [Gro68b] or [Gir71, Chap. V, §4].

Example 2.11. Let X be a topological space, let $\mathbf{X} = \mathbf{Sh}(X)$ and let $\mathcal{O}_{\mathbf{X}}$ be the sheaf of continuous functions from X to $\mathbb{C}$, denoted $\mathcal{C}(X, \mathbb{C})$. Then Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra are topological Azumaya algebras over X as studied in [AW14a].

Example 2.12. An Azumaya algebra A of degree n over a scheme X is a sheaf of $\mathcal{O}_X$-algebras that is locally, in the étale topology, isomorphic as an $\mathcal{O}_X$-algebra to $\mathrm{Mat}_{n\times n}(\mathcal{O}_X)$, [Gro68b, Para. 1.2]

Example 2.13. Let R be a ring. An Azumaya R-algebra of degree n is an R-algebra A for which there exists a faithfully flat étale R-algebra R' such that $A \otimes_R R' \cong \mathrm{M}_{n\times n}(R')$ as R'-algebras. This is equivalent to the definition of Example 2.12 in the case where $X = \operatorname{Spec} R$ by [Gro68b, Thm. 5.1, Cor. 5.2]. Consult [Knu91, III.§5] for other equivalent definitions and cf. Remark 2.9.

3 Rings with Involution

In this section we collect a number of results regarding involutions of rings that will be needed later in the paper. The main result is Theorem 3.16, which gives the structure of those rings with involution (R, λ) for which the fixed ring of λ is local. It is shown that in this case, R is a local ring in its own right, or R is a quadratic étale algebra over the fixed ring of λ. In particular, the ring R is semilocal.

Throughout, involutions will be written exponentially and the Jacobson radical of a ring R will be denoted $\mathrm{Jac}(R)$.

3.1 Quadratic Étale Algebras

Definition 3.1. Let S be a ring. A commutative S-algebra R is said to be *finite étale of rank* n if R is a locally free S-module of rank n, and the multiplication map $\mu : R \otimes_S R \to R$ may be split as a morphism of $R \otimes_S R$-modules, where R is regarded as an $R \otimes_S R$-algebra via μ. Finite étale S-algebras of rank 2 will be called *quadratic* étale algebras.

Remark 3.2. One common definition of étale for commutative S-algebras is that R should be flat over S, of finite presentation as an S-algebra, and unramified in the sense that $\Omega_{R/S}$, the module of Kähler differentials, vanishes. This is the definition in [Gro67, Sec. 17.6] in the affine case. Our finite étale algebras of rank n are precisely the étale algebras which are locally free of rank n.

Indeed, if R is locally free of rank n over S, then R is also finitely presented and flat as an S-module [deJ17, Tag 00NX], and hence of finite presentation as an R-algebra [Gro64, Prop. 1.4.7]. Furthermore, $\mu : R \otimes_S R \to R$ admits a splitting ψ if and only if the $R \otimes_S R$-ideal $\ker \mu$ is generated by an idempotent, namely $1 - \psi(1_R)$. For finitely generated S-algebras R, the existence of such an idempotent is equivalent to saying R is unramified over S by [deJ17, Tag 02FL].

Example 3.3. Let $f \in S[x]$ be a monic polynomial of degree n. It is well known that $S[x]/(f)$ is a finite étale S-algebra of rank n if and only if the discriminant of f is invertible in S. In particular, $S[x]/(x^2 + \alpha x + \beta)$ is a quadratic étale S-algebra if and only if $\alpha^2 - 4\beta \in S^\times$.

Every quadratic étale S-algebra R admits a canonical S-linear involution λ given by $r^\lambda = \mathrm{Tr}_{R/S}(r) - r$, see [Knu91, I.§1.3.6]. The fixed ring of λ is S and λ is the only S-automorphism of R with this property. Moreover, when S is connected, it is the only non-trivial S-automorphism of R.

Proposition 3.4. *Let S be a local ring with maximal ideal $\mathfrak{m}$ and residue field k, let R be a quadratic étale S-algebra, and let λ be the unique non-trivial S-automorphism of R. Then:*

(i) $\mathrm{Jac}(R) = R\mathfrak{m}$

(ii) $\overline{R} := R/R\mathfrak{m}$ *is either a separable quadratic field extension of k, or $\overline{R} \cong k \times k$. The automorphism that λ induces on $\overline{R}$ is the unique non-trivial k-automorphism of $\overline{R}$.*

(iii) ("Hilbert 90") For every $r \in R^\times$ with $r^\lambda r = 1$, there exists $a \in R^\times$ such that $r = a^{-1}a^\lambda$.

Proof. It is clear that $\overline{R}$ is a quadratic étale k-algebra, and hence a product of separable field extensions of k [DI71, Thm. II.2.5]. This implies the first assertion of (ii) as well as $\mathrm{Jac}(R) \subseteq R\mathfrak{m}$. The inclusion $R\mathfrak{m} \subseteq \mathrm{Jac}(R)$ holds because R is a finite S-module [Rei75, Thm. 6.15], so we have proved (i). The last assertion of (ii) follows from the fact that λ is given by $x \mapsto \mathrm{Tr}_{R/S}(x) - x$.
Let $\overline{r}$ denote the image of $r \in R$ in $\overline{R}$. To prove (iii), we first claim that there is $\overline{x} \in \overline{R}$ with $\overline{x} + \overline{x}^\lambda \overline{r} \in \overline{R}^\times$. This is easy to see if $\overline{R} = k \times k$. Otherwise, $\overline{R}$ is a field and such $\overline{x}$ exists unless $\overline{x} = -\overline{x}^\lambda \overline{r}$ for all $\overline{x} \in \overline{R}$. The latter forces $\overline{r} = -1$ (take $\overline{x} = 1$) and $\lambda_{\overline{R}} = \mathrm{id}_{\overline{R}}$, which is impossible by (ii), so $\overline{x}$ exists.
Let $x \in R$ be a lift of $\overline{x}$. Then $t := x + x^\lambda r$ is a lift of $\overline{x} + \overline{x}^\lambda \overline{r}$, which implies $t \in R^\times$ by (i). Since $r^\lambda = r^{-1}$, it is the case that $t^\lambda r = t$, and so $r = a^\lambda a^{-1}$ with $a = t^{-1}$. □

3.2 Quadratic Étale Algebras in Topoi

Our definition of a "quadratic étale algebra" extends directly to the case where S is a local ring object in a topos $\mathbf{X}$.

Definition 3.5. Given a ring object S in a topos $\mathbf{X}$, we say an S-algebra R is a *finite étale S-algebra of rank n* if R is a locally free S-module of rank n such that the multiplication map $\mu : R \otimes_S R \to R$ may be split as a morphism of $R \otimes_S R$-algebras. Finite étale S-algebras of rank 2 will be called *quadratic* étale algebras.

We alert the reader that if R is a quadratic étale S-algebra, then it is not true in general that $R(U)$ is a quadratic étale $S(U)$-algebra for all objects U of $\mathbf{X}$. In fact $R(U)$ may not be locally free of rank 2 over $S(U)$. Rather, one can always find a covering $V \to U$ such that $R(V)$ is a quadratic étale $S(V)$-algebra; for instance, one may take any $V \to U$ such that R_V is a free S_V-module of rank 2. We further note that in general there is no covering $U \to *$ such that

$R_U \cong S_U \times S_U$ as S_U-algebras, e.g. let $\mathbf{X} = \mathbf{pt}$ (the topos of sets) and take S and R to be $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$ respectively. While $\mu : R \otimes_S R \to R$ is easily seen to be split, there is no covering $U \to *$ such that $R_U \cong S_U \times S_U$.
As one might expect, being a finite étale algebra of rank n is a local property in that it may be tested on a covering.

Lemma 3.6. *Let S be a ring in $\mathbf{X}$, let R be an S-algebra and let $U \to *$ be a covering. Then R is a finite étale S-algebra of rank n if and only if R_U is a finite étale S_U-algebra of rank n in $\mathbf{X}/U$.*

Proof. Write $M = R \otimes_S R$. The only non-trivial thing to check is that if the multiplication map $\mu_U : M_U \to R_U$ admits a splitting $\psi : R_U \to M_U$ in $\mathbf{X}/U$, then so does $\mu : M \to R$ in $\mathbf{X}$. Let $\pi_1, \pi_2 : U \times U \to U$ denote the first and second projections, and let $\pi_i^*\psi : R_{U\times U} \to M_{U\times U}$ denote the pullback of ψ along π_i. We claim that $\mu_{U\times U} : M_{U\times U} \to R_{U\times U}$ admits at most one splitting. Provided this holds, we must have $\pi_1^*\psi = \pi_2^*\psi$ and so ψ descends to a map $\psi_0 : R \to M$ splitting μ as required.
The claim can be verified on the level of sections, namely, it is enough to check that any ring surjection $\mu : A \to B$ admits at most one A-linear splitting. If ψ is such a splitting and $e = \psi(1_B)$, then $\psi(1_B \cdot \alpha) = \alpha e$ for all $\alpha \in A$, so ψ is determined by the idempotent e. It is easy to check that $(1 - e)A = \ker \mu$ and that e is the only idempotent with this property, hence e is determined by μ. ∎

Example 3.7. Let $\pi : X \to Y$ be a quadratic étale morphism of schemes. That is, π is affine and Y can be covered by open affine subschemes $\{U_i\}_i$ such that the ring map corresponding to $\pi : \pi^{-1}(U_i) \to U_i$ is quadratic étale for all i. Then $\pi_*\mathcal{O}_X$ is a quadratic étale $\mathcal{O}_Y$-algebra in both $\mathbf{Sh}(Y_{\text{ét}})$ and $\mathbf{Sh}(Y_{\text{Zar}})$; this can be checked using Lemma 3.6.

Example 3.8. Let $\pi : X \to Y$ be a double covering of topological spaces and let $\mathcal{C}(X, \mathbb{C})$ and $\mathcal{C}(Y, \mathbb{C})$ denote the sheaves of continuous $\mathbb{C}$-values functions on X and Y, respectively. Then $\pi_*\mathcal{C}(X, \mathbb{C})$ is a quadratic étale $\mathcal{C}(Y, \mathbb{C})$-algebra in $\mathbf{Sh}(Y)$; again this can be checked with Lemma 3.6.

3.3 Rings with Involution

Throughout, R is an ordinary commutative ring, $\lambda : R \to R$ is an involution, and S is the fixed ring of λ. The purpose of this section is twofold. First, we show that the locus of primes $\mathfrak{p} \in \operatorname{Spec} S$ such that $R_\mathfrak{p}$ is a quadratic étale over $S_\mathfrak{p}$ is open in $\operatorname{Spec} S$. Second, we study the structure of R when S is local, showing, in particular, that R is quadratic étale over S, or R is local.
There are two pitfalls in the study of R over S. First of all, R may not be finite over S.

Example 3.9. Let I be any set, let R be the commutative $\mathbb{C}$-algebra freely generated by $\{x_i\}_{i\in I}$, and let $\lambda : R \to R$ be the $\mathbb{C}$-linear involution sending each x_i to $-x_i$. Then the fixed ring of λ is $S = \mathbb{C}[x_i x_j \,|\, i, j \in I]$. Let $\mathfrak{m} =$

$\langle x_i x_j \,|\, i,j \in I\rangle$. Since $R/\mathfrak{m}R \cong \mathbb{C}[x_i \,|\, i \in I]/\langle x_i x_j \,|\, i,j \in I\rangle$ and $S/\mathfrak{m} \cong \mathbb{C}$, it follows that R cannot be generated by fewer than $|I|$ elements as an S-algebra. Thus, when I is infinite, R is not finite over S. The same applies to the $S/\mathfrak{m}$-algebra $R/\mathfrak{m}R$, even though $S/\mathfrak{m}$ is noetherian. We further note that when $1 < |I| < \aleph_0$, the ring R is a smooth affine $\mathbb{C}$-algebra, but S is singular.

Second, the formation of fixed rings may not commute with extension of scalars. That is, if S' is a commutative S-algebra, then S' need not be the subring of λ-fixed elements in $R' := R \otimes_S S'$. In fact, $S' \to R'$ is *a priori* not one-to-one. Nevertheless, $S' \to R'$ restricts to an isomorphism $S' \to \{r \in R' : r = r^\lambda\}$ if S' is flat over S, or $2 \in S^\times$. To see this, consider the exact sequence of S-modules $0 \to S \to R \xrightarrow{\mathrm{id}_S - \lambda} R$. The statement amounts to showing that it remains exact after tensoring with S'. This is clear if S' is flat, and if $2 \in S^\times$, then it follows because $S \to R$ is split by $r \mapsto \frac{1}{2}(r + r^\lambda)$.

Remark 3.10. Voight [Voi11b, Corollary 3.2] showed that if R is locally free of rank at least 3 over S and 2 is not a zerodivisor in S, then R decomposes as $S \oplus M$ where M is an ideal of R such that $M^2 = 0$ and $\lambda|_M = -\mathrm{id}_M$. Voight calls such (commutative) rings with involution *exceptional*. This shows that if $\lambda : R \to R$ is not exceptional then either R is not locally free over S, or $\mathrm{rank}_S R \leq 2$. The case $R = \mathbb{C}[x_i \,|\, i \in I]/\langle x_i x_j \,|\, i,j \in I\rangle$ and $S = \mathbb{C}$ featuring in Example 3.9 is an example of an exceptional ring with involution, and essentially the only one if $S = \mathbb{C}$.

Having warned the reader of these pitfalls, we return to the main topic of the section, which is the study of R over S.

Lemma 3.11. *Assume that there exists $r \in R$ with $r - r^\lambda \in R^\times$. Then R is a quadratic étale S-algebra.*

Proof. For $a \in R$, write $t_a = a + a^\lambda$ and $n_a = a^\lambda a$, and observe that $t_a, n_a \in S$ and $a^2 - t_a a + n_a = 0$.
Suppose that $R = S[r]$. Since $r^2 - t_r r + n_r = 0$, it follows that $R = S + Sr$. Furthermore, if $\alpha r \in S$ for $\alpha \in S$, then $0 = (\alpha r) - (\alpha r)^\lambda = \alpha(r - r^\lambda)$, so $\alpha = 0$ because $r - r^\lambda \in R^\times$. It follows that the S-algebra map $S[x]/(x^2 - t_r x + n_r) \to R$ sending x to r is an isomorphism. Since $t_r^2 - 4n_r = (r - r^\lambda)^2 \in S^\times$, we conclude that R is a quadratic étale S-algebra.
We now show that $R = S[r]$. Write $u = r - r^\lambda = 2r - t_r \in S[r]$ and $a = u^{-1}r$. One verifies that $a = n_u^{-1}(n_r - r^2)$, so that $a \in S[r]$. Since $u^\lambda = -u$, we have $a + a^\lambda = u^{-1}r - u^{-1}r^\lambda = u^{-1}u = 1$. Let $b \in R$ and $b' = b - a(b + b^\lambda)$. Straightforward computation shows that $b'^\lambda = -b'$, hence $(u^{-1}b')^\lambda = u^{-1}b'$ and $u^{-1}b' \in S$. It follows that $b' \in uS \subseteq S[r]$ and thus, $b = b' + a(b + b^\lambda) \in S[r] + aS = S[r]$. □

Lemma 3.12. *Suppose that S is local and R is a quadratic étale S-algebra. Then there exists $r \in R$ such that $r - r^\lambda \in R^\times$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of S. By Proposition 3.4(i), we may replace R with $R/\mathfrak{m}R$. The claim then follows easily from Proposition 3.4(ii). □

Corollary 3.13. *The set of prime ideals $\mathfrak{p} \in \operatorname{Spec} S$ such that $R_{\mathfrak{p}}$ is a quadratic étale $S_{\mathfrak{p}}$-algebra is open in $\operatorname{Spec} S$. Equivalently for every $\mathfrak{p} \in \operatorname{Spec} S$ such that $R_{\mathfrak{p}}$ is a quadratic étale $S_{\mathfrak{p}}$-algebra, there exists $s \in S - \mathfrak{p}$ such that R_s is a quadratic étale S_s-algebra.*

Proof. By Lemma 3.12, there exists $r \in R_{\mathfrak{p}}$ with $r - r^{\lambda} \in R_{\mathfrak{p}}^{\times}$. We can find $s_1 \in S - \mathfrak{p}$ and $a, b \in R$ such that $r = as_1^{-1}$ and $r - r^{\lambda} = bs_1^{-1}$ in $R_{\mathfrak{p}}$. Since $bs_1^{-1} \in R_{\mathfrak{p}}^{\times}$, we can find $s_2 \in S - \mathfrak{p}$ such that bs_1^{-1} is invertible in $R_{s_1 s_2}$. Now take $s = s_1 s_2$ and apply Lemma 3.11 to R_s with $r = as_1^{-1}$. □

Remark 3.14. Corollary 3.13 is well known when R is locally free of finite rank over S; see, for instance, [deJ17, Tag 0C3J].

We momentarily consider an arbitrary finite group acting on R.

Proposition 3.15. *Let G be a finite group acting on a ring R and let S be the subring of elements fixed under G. If S is local then the maximal ideals of R form a single G-orbit. In particular, R is semilocal.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of S and, for the sake of contradiction, suppose $\mathfrak{p}$ and $\mathfrak{q}$ are maximal ideals of R lying in distinct G-orbits. Let $\mathfrak{p}' = \bigcap_{g \in G} g(\mathfrak{p})$ and $\mathfrak{q}' = \bigcap_{g \in G} g(\mathfrak{q})$. Since $g(\mathfrak{p}) + h(\mathfrak{q}) = R$ for all $g, h \in G$, we have $\mathfrak{p}' + \mathfrak{q}' = R$. Using the Chinese Remainder Theorem, choose $r \in \mathfrak{q}'$ such that $r \equiv 1 \bmod \mathfrak{p}'$. Replacing r with $\prod_{g \in G} g(r)$, we may assume that $r \in S$. Since $\mathfrak{p}' \cap S$ and $\mathfrak{q}' \cap S$ are both contained in $\mathfrak{m}$, this means that r lies both in $(1 + \mathfrak{p}') \cap S \subseteq 1 + \mathfrak{m}$ and in $\mathfrak{q}' \cap S \subseteq \mathfrak{m}$, which is absurd. □

We derive the main result of this section by specializing Proposition 3.15 to the case of a group with 2 elements.

Theorem 3.16. *Suppose R is a ring and $\lambda : R \to R$ is an involution with fixed ring S such that S is local. Let $\overline{R}$ denote $R/\operatorname{Jac}(R)$ and $\overline{\lambda}$ the restriction of λ to $\overline{R}$.*

(i) If $\overline{\lambda} \neq \mathrm{id}$, then R is a quadratic étale algebra over S.

(ii) If $\overline{\lambda} = \mathrm{id}$, then R is a local ring that is not quadratic étale over S.

In either case, R is semilocal.

Proof. Let $\mathfrak{M}$ be a maximal ideal of R. Taking $G = \{1, \lambda\}$ in Proposition 3.15, we see that the maximal ideals of R are $\{\mathfrak{M}, \mathfrak{M}^{\lambda}\}$. We consider the cases $\mathfrak{M} \neq \mathfrak{M}^{\lambda}$ and $\mathfrak{M} = \mathfrak{M}^{\lambda}$ separately.
Suppose that $\mathfrak{M} \neq \mathfrak{M}^{\lambda}$. By the Chinese Remainder Theorem, $\overline{R} = R/(\mathfrak{M} \cap \mathfrak{M}^{\lambda}) \cong R/\mathfrak{M} \times R/\mathfrak{M}^{\lambda}$, and under this isomorphism, $\overline{\lambda}$ acts by sending $(a + \mathfrak{M}, b + \mathfrak{M}^{\lambda})$ to $(b^{\lambda} + \mathfrak{M}, a^{\lambda} + \mathfrak{M}^{\lambda})$. This implies that $\overline{\lambda} \neq \mathrm{id}$, so we are in the situation of (i). Furthermore, by taking $a = 1$ and $b = 0$, we see that there exists $\overline{r} \in R$ such that $\overline{r - r^{\lambda}} \in \overline{R}^{\times}$, or equivalently, $r - r^{\lambda} \in R^{\times}$. Thus, by Lemma 3.11, R is quadratic étale over S.

Suppose now that $\mathfrak{M} = \mathfrak{M}^\lambda$. Then R is local and $\overline{R} = R/\mathfrak{M}$ is a field. If $\overline{\lambda} \neq \mathrm{id}$, then there exists $\overline{r} \in \overline{R}$ with $\overline{r - r^\lambda} \in \overline{R}^\times$ and again we find that R is quadratic étale over S. On the other hand, if $\overline{\lambda} = \mathrm{id}$, then R cannot be quadratic étale over S by Proposition 3.4(ii).

We have verified (i) and (ii) in both cases, so the proof is complete. ☐

Notation 3.17. A *henselian* ring is a local ring in which Hensel's lemma, [Eis95, Thm. 7.3], holds. A *strictly henselian* ring is a henselian ring for which the residue field is separably closed.

Lemma 3.18. *Let G be a finite group acting on a ring R and let S be the subring of R fixed under G. If S is local with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}R \subseteq \mathrm{Jac}(R)$. In particular, $\mathfrak{m}R \cap S = \mathfrak{m}$.*

Proof. Let $a \in \mathfrak{m}$. To prove $\mathfrak{m}R \subseteq \mathrm{Jac}(R)$, we need to show that $aR \subseteq \mathrm{Jac}(R)$, or equivalently, that $1 + aR$ consists of invertible elements. Let $b \in R$ and let $\{g_1, \dots, g_n\}$ denote the distinct elements of G. Then

$$\prod_{g \in G} g(1 + ab) = 1 + \sigma_1(g_1 b, \dots, g_n b)a + \sigma_2(g_1 b, \dots, g_n b)a^2 + \dots$$

where σ_i denotes the i-th elementary symmetric polynomial on n letters. Since $a \in \mathfrak{m}$, and since $\sigma_i(g_1 b, \dots, g_n b)$ is invariant under G, the right hand side lies in $1 + \mathfrak{m} \subseteq S^\times$. Thus, $1 + ab \in R^\times$. ☐

Corollary 3.19. *Let R be a ring, let $\lambda : R \to R$ be an involution, and let S denote the fixed ring of λ. Suppose that S is a strictly henselian ring with maximal ideal $\mathfrak{m}$. Then R is a finite product of strictly henselian rings.*

Proof. By Theorem 3.16, either R is a quadratic étale S-algebra, or R is local. In the former case, R is finite over S, and the lemma follows from [deJ17, Tag 04GH] so we assume R is local. Write $\mathfrak{M}$ for the maximal ideal of R, k for its residue field, and denote by $r \mapsto \overline{r}$ the surjection $R \to k$.

Observe first that R is integral over S since for all $r \in R$, we have $r^2 - (r^\lambda + r)r + r^\lambda r = 0$ and $r^\lambda + r, r^\lambda r \in S$. This implies that k is algebraic over the residue field of S, which is separably closed, hence k is separably closed.

Now, let $f \in R[x]$ be a monic polynomial such that $\overline{f} \in k[x]$ has a simple root $\eta \in k$. We will show that f has root $y \in R$ with $\overline{y} = \eta$. Let $a_0, \dots, a_{n-1} \in R$ be the coefficients of f and let $b \in R$ be any element with $\overline{b} = \eta$. Since R is integral over S, there is a finite S-subalgebra $R_0 \subseteq R$ containing $a_0, \dots, a_{n-1}, b$. By [deJ17, Tag 04GH], R_0 is a product of henselian rings. Since R has no non-trivial idempotents, this means R_0 is a henselian ring. Write $\mathfrak{M}_0$ for the maximal ideal of R_0 and $k_0 = R_0/\mathfrak{M}_0$. Since R_0 is finite over S, there is a natural number n such that $\mathfrak{M}_0^n \subseteq \mathfrak{m}R_0 \subseteq \mathfrak{M}_0$ by [Rei75, Thm. 6.15]. This implies that $(R\mathfrak{M}_0)^n = R\mathfrak{M}_0^n \subseteq R\mathfrak{m}$ and the latter is a proper ideal of R, by Lemma 3.18. Therefore, $R\mathfrak{M}_0 \subseteq \mathfrak{M}$ and we have a well-defined homomorphism of fields $k_0 \to k$ given by $x + \mathfrak{M}_0 \mapsto x + \mathfrak{M}$. Since $a_0, \dots, a_{n-1}, b \in R_0$, we have $\overline{f} \in k_0[x]$ and $\eta \in k_0$. As R_0 is a henselian ring, there is $y \in R_0$ with $f(y) = 0$ and $y + \mathfrak{M}_0 = \eta$. This completes the proof. ☐

4 Ringed Topoi with Involution

Unless indicated otherwise, throughout this section, $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ denotes a locally ringed topos. Our interest is in the following examples:

1. $\mathbf{X} = \mathbf{Sh}(X_{\text{ét}})$ for a scheme X, and $\mathcal{O}_{\mathbf{X}}$ is the structure sheaf of X which sends $(U \to X)$ to $\Gamma(U, \mathcal{O}_U)$.
2. $\mathbf{X} = \mathbf{Sh}(X)$ for a topological space X, and $\mathcal{O}_{\mathbf{X}}$ is the sheaf of continuous $\mathbb{C}$-valued functions, denoted $\mathcal{C}(X, \mathbb{C})$.

We write the cyclic group with two elements as C_2 and, when applicable, the non-trivial element of C_2 will always be denoted λ.
When there is no risk of confusion, we shall refer to $\mathbf{X}$ as a ringed topos, in which case the ring object is understood to be $\mathcal{O}_{\mathbf{X}}$.

4.1 Involutions of Ringed Topoi

Definition 4.1. Suppose $\mathbf{X}$ is a topos. An *involution* of $\mathbf{X}$ consists of an equivalence of categories $\Lambda : \mathbf{X} \to \mathbf{X}$ and a natural isomorphism $\nu : \Lambda^2 \Rightarrow \mathrm{id}$ satisfying the coherence condition that $\nu_{\Lambda X} = \Lambda \nu_X$ for all objects X of $\mathbf{X}$.

The natural isomorphism ν will generally be suppressed from the notation.

Definition 4.2. Suppose $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is a ringed topos. An *involution* of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ consists of an involution (Λ, ν) of $\mathbf{X}$ and an isomorphism λ of ring objects $\lambda : \mathcal{O}_{\mathbf{X}} \to \Lambda \mathcal{O}_{\mathbf{X}}$ such that $\Lambda\lambda \circ \lambda = \nu_{\mathcal{O}_{\mathbf{X}}}^{-1}$.

Suppressing ν from the last equation, we say $\Lambda\lambda \circ \lambda = \mathrm{id}$.

Remark 4.3. (i) The functor Λ is left adjoint to itself with the unit and counit of the adjunction being ν^{-1} and ν, respectively. Thus, if we write $\Lambda^* = \Lambda_* = \Lambda$, then the adjoint pair (Λ^*, Λ_*) defines a geometric automorphism of $\mathbf{X}$. Moreover, $(\Lambda^*, \Lambda_*, \lambda^{-1} : \Lambda_* \mathcal{O}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}})$ defines an automorphism of the ringed topos $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$.
(ii) Topoi and ringed topoi form 2-categories in which there is a notion of a weak C_2-action. An involution of a topos, resp. ringed topos, as defined here induces such a weak C_2-action in which the non-trivial element λ acts as the morphism $\Lambda = (\Lambda^*, \Lambda_*)$, resp. $(\Lambda^*, \Lambda_*, \lambda^{-1})$, and the trivial element acts as the identity. It can be checked that all C_2-actions with the latter property arise in this manner. Since an arbitrary weak C_2-action is equivalent to one in which 1 acts as the identity, specifying an involution is essentially the same as specifying a weak C_2-action.

Notation 4.4. Henceforth, when there is no risk of confusion, involutions of ringed topoi will always be denoted (Λ, ν, λ). In fact, we shall often abbreviate the triple (Λ, ν, λ) to λ.

It is convenient to think of Λ, ν as the "geometric data" of the involution and of λ as the "arithmetic data" of the involution. The following are the motivating examples.

Example 4.5. Let X be a scheme and let $\sigma : X \to X$ be an involution. The direct image functor $\Lambda := \sigma_*$ defines an involution of $\mathbf{X} := \mathbf{Sh}(X_{\text{ét}})$; the suppressed natural isomorphism $\nu : \Lambda^2 \Rightarrow \text{id}$ is the identity. Let $\mathcal{O}_{\mathbf{X}}$ be the structure sheaf of X on $X_{\text{ét}}$. The involution of X gives rise to an isomorphism $\lambda : \mathcal{O}_{\mathbf{X}} \to \Lambda\mathcal{O}_{\mathbf{X}}$ as follows: For an étale morphism $U \to X$, define $U^\sigma \to X$ via the pullback diagram

$$\begin{array}{ccc} U^\sigma & \longrightarrow & X \\ \downarrow{\scriptstyle \sigma_U} & & \downarrow{\scriptstyle \sigma} \\ U & \longrightarrow & X \end{array}$$

By definition, $\Lambda\mathcal{O}_{\mathbf{X}}(U \to X) = \sigma_*\mathcal{O}_{\mathbf{X}}(U \to X) = \mathcal{O}_{\mathbf{X}}(U^\sigma \to X) = \Gamma(U^\sigma, \mathcal{O}_{U^\sigma})$, and we define $\lambda_{U\to X} : \Gamma(U, \mathcal{O}_U) = \mathcal{O}_{\mathbf{X}}(U \to X) \to \Lambda\mathcal{O}_{\mathbf{X}}(U \to X) = \Gamma(U^\sigma, \mathcal{O}_{U^\sigma})$ to be the isomorphism induced by σ_U. It is easy to check that $\Lambda\lambda \circ \lambda = \text{id}$ and so (Λ, ν, λ) is an involution of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$. When $X = \operatorname{Spec} R$ for a ring R and σ is induced by an involution of R, we can recover that involution from $\lambda : \mathcal{O}_{\mathbf{X}} \to \Lambda\mathcal{O}_{\mathbf{X}}$ by taking global sections.
The small étale site $X_{\text{ét}}$ can be replaced by other sites, for instance the site X_{fppf}.

Example 4.6. Let X be topological space with a continuous involution $\sigma : X \to X$. Write $\mathbf{X} = \mathbf{Sh}(X)$ and let $\mathcal{O}_{\mathbf{X}} = \mathcal{C}(X, \mathbb{C})$, the ring sheaf of continuous functions into $\mathbb{C}$. Then the direct image functor $\Lambda := \sigma_*$ defines an involution of $\mathbf{X}$; the isomorphism $\nu : \Lambda^2 \Rightarrow \text{id}$ is the identity.
In particular, $(\Lambda\mathcal{O}_{\mathbf{X}})(U) = \mathcal{O}_{\mathbf{X}}(\sigma U) = \mathcal{C}(\sigma U, \mathbb{C})$. Let $\lambda : \mathcal{O}_{\mathbf{X}} \to \Lambda\mathcal{O}_{\mathbf{X}}$ be given sectionwise by precomposition with σ, namely

$$\lambda_U : C(U, \mathbb{C}) \to C(\sigma U, \mathbb{C}), \quad \phi \mapsto \phi \circ \sigma.$$

Then (Λ, ν, λ) is an involution of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$.

Example 4.7. Let $\mathbf{X}$ be any topos, let R be a ring object in $\mathbf{X}$ and let $\lambda : R \to R$ be an involution. Then $(\text{id}_{\mathbf{X}}, \nu := \text{id}, \lambda)$ is an involution of $(\mathbf{X}, R)$.

Definition 4.8. The *trivial involution* of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is $(\Lambda, \nu, \lambda) = (\text{id}_{\mathbf{X}}, \text{id}, \text{id}_{\mathcal{O}_{\mathbf{X}}})$. Any other involution (Λ, ν, λ) of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ will be said to be *weakly trivial* if it is equivalent to the trivial involution of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ in the following sense: There exists a natural isomorphism $\theta : \Lambda \Rightarrow \text{id}$ such that $\theta_{\mathcal{O}_X} = \lambda^{-1}$ and $\theta_X \circ \theta_{\Lambda X} = \nu_X$ for all objects X in $\mathbf{X}$.

Example 4.9. The involution of Example 4.5 (resp. Example 4.6) is trivial if and only if $\sigma : X \to X$ is the identity.

Let (Λ, ν, λ) be an involution of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ and let M be an $\mathcal{O}_{\mathbf{X}}$-module. Then ΛM carries a $\Lambda\mathcal{O}_{\mathbf{X}}$-module structure. We shall always regard ΛM as an $\mathcal{O}_{\mathbf{X}}$-module by using the morphism $\lambda : \mathcal{O}_{\mathbf{X}} \to \Lambda\mathcal{O}_{\mathbf{X}}$, that is, by twisting the structure.

Using the equality $\Lambda\lambda \circ \lambda = \nu_{\mathcal{O}_{\mathbf{X}}}^{-1}$, one easily checks that $\nu_M : \Lambda\Lambda M \to M$ is an isomorphism of $\mathcal{O}_{\mathbf{X}}$-modules, which we suppress from the notation henceforth. Notice that contrary to the case of λ-twisting of modules over ordinary rings, it is not in general true that M can be identified with its λ-twist as an abelian group object in $\mathbf{X}$. For instance, in the context of Example 4.5, suppose that $X = \operatorname{Spec} R$ and σ exchanges two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2 \lhd R$, and let M_1, M_2 denote the quasicoherent $\mathcal{O}_{\mathbf{X}}$-modules corresponding to the R-modules $R/\mathfrak{m}_1$ and $R/\mathfrak{m}_2$, respectively. Then $M_2 = \Lambda M_1$, but M_1 is not isomorphic to M_2 as abelian group objects in $\mathbf{X}$ because M_1 is supported at $\{\mathfrak{m}_1\}$ while M_2 is supported at $\{\mathfrak{m}_2\}$.

If A is an $\mathcal{O}_{\mathbf{X}}$-algebra, then ΛA, besides being an $\mathcal{O}_{\mathbf{X}}$-module, carries an $\mathcal{O}_{\mathbf{X}}$-algebra structure. Letting A^{op} denote the opposite algebra of A, we have $\Lambda(\Lambda A^{\mathrm{op}})^{\mathrm{op}} = A$ up to the suppressed natural isomorphism ν_A.

Definition 4.10. A *λ-involution* on an $\mathcal{O}_{\mathbf{X}}$-algebra A is a morphism of $\mathcal{O}_{\mathbf{X}}$-algebras $\tau : A \to \Lambda A^{\mathrm{op}}$ such that $\Lambda\tau \circ \tau = \mathrm{id}_A$. In this case, (A, τ) is called an *$\mathcal{O}_{\mathbf{X}}$-algebra with a λ-involution*. If A is an Azumaya algebra, it will be called an *Azumaya algebra with λ-involution*.

If (A, τ) and (A', τ') are $\mathcal{O}_{\mathbf{X}}$-algebras with λ-involutions, then a morphism from (A, τ) to (A', τ') is a morphism of $\mathcal{O}_{\mathbf{X}}$-algebras $\phi : A \to A'$ such that $\tau' \circ \phi = \Lambda\phi \circ \tau$.

Notice that applying Λ to both sides of $\Lambda\tau \circ \tau = \mathrm{id}_A$ gives $\tau \circ \Lambda\tau = \mathrm{id}_{\Lambda A}$.

Notation 4.11. The category where the objects are degree-n Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras with λ-involutions and where the morphisms are isomorphisms of $\mathcal{O}_{\mathbf{X}}$-algebras with λ-involutions shall be denoted $\mathbf{Az}_n(\mathbf{X}, \mathcal{O}_{\mathbf{X}}, \lambda)$, or just $\mathbf{Az}_n(\mathcal{O}_{\mathbf{X}}, \lambda)$.

Given a ring with involution (R, λ) and an R-algebra A, it is reasonable to define a λ-involution of A to be an involution $\tau : A \to A$ satisfying $(ra)^\tau = r^\lambda a^\tau$ for all $r \in R$, $a \in A$. Following Gille [Gil09, §1], this can be generalized to the context of schemes: Given a scheme X, an involution $\sigma : X \to X$, and an $\mathcal{O}_X$-algebra $A \in \mathbf{Sh}(X_{\mathrm{Zar}})$, then a *$\sigma$-involution* of A is an $\mathcal{O}_X$-algebra morphism $\tau : A \to \sigma_* A^{\mathrm{op}}$ such that $\sigma_*\tau \circ \tau = \mathrm{id}_A$. We reconcile these elementary definitions with Definition 4.10 in the following example.

Example 4.12. Let (R, λ) be a ring with involution, let $X = \operatorname{Spec} R$ and write $\sigma : X \to X$ for the involution induced by λ. Abusing the notation, let (Λ, ν, λ) denote the involution induced by σ on the étale ringed topos $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ of X, see Example 4.5.

Every R-module M gives rise to an $\mathcal{O}_{\mathbf{X}}$-module, also denoted M, the sections of which are given by $M(\operatorname{Spec} R' \to \operatorname{Spec} R) = M \otimes_R R'$ for any $(\operatorname{Spec} R' \to$

Spec R) in $X_{\text{ét}}$. In fact, this defines an equivalence of categories between R-modules and quasicoherent $\mathcal{O}_{\mathbf{X}}$-modules [deJ17, Tag 03DX]. Straightforward computation shows that the $\mathcal{O}_{\mathbf{X}}$-module ΛM corresponds to M^λ, the R-module obtained from M by twisting via λ.

Now let A be an R-algebra and let $\tau : A \to A$ be an involution satisfying $(ra)^\tau = r^\lambda a^\tau$ for all $r \in R$, $a \in A$. Realizing A as an $\mathcal{O}_{\mathbf{X}}$-algebra in $\mathbf{X}$, the algebra ΛA^{op} corresponds to the R-algebra $(A^\lambda)^{\mathrm{op}}$, and so the involution τ induces a λ-involution $A \to \Lambda A^{\mathrm{op}}$, also denoted τ. By taking global sections, we see that all λ-involutions of the $\mathcal{O}_{\mathbf{X}}$-algebra A are obtained in this manner. Likewise, if X is a scheme, $\sigma : X \to X$ is an involution, and A is a quasicoherent $\mathcal{O}_X$-algebra in $\mathbf{Sh}(X_{\mathrm{Zar}})$, then the λ-involutions of the $\mathcal{O}_{\mathbf{X}}$-algebra associated to A in $\mathbf{X} = \mathbf{Sh}(X_{\text{ét}})$ (see [deJ17, Tag 03DU]) are in one-to-one correspondence with the σ-involutions of A in the sense of Gille [Gil09, §1]. The quasicoherence assumption applies in particular when A is an Azumaya algebra over X.

Example 4.13. Let $\lambda = (\Lambda, \nu, \lambda)$ be an involution of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ and let n be a natural number. Then $\mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}})$ admits a λ-involution given by $(\alpha_{ij})_{i,j} \mapsto (\alpha_{ji}^\lambda)_{i,j}$ on sections and denoted $\lambda\mathrm{tr}$. If the sections α_{ij} lie in $\mathcal{O}_{\mathbf{X}}(U)$, then the sections α_{ji}^λ lie in $(\Lambda\mathcal{O}_{\mathbf{X}})(U)$.

4.2 Morphisms

We now give the general definition of a morphism of ringed topoi with involution. Only very few examples of these will be considered in the sequel.

Recall that a geometric morphism of topoi $f : \mathbf{X} \to \mathbf{X}'$ consists of two functors $f_* : \mathbf{X} \to \mathbf{X}'$, $f^* : \mathbf{X}' \to \mathbf{X}$ together with an adjunction between f^* and f_* and such that f^* commutes with finite limits. We shall usually denote the unit and counit natural transformations associated to the adjunction by $\eta^{(f)} : \mathrm{id}_{\mathbf{X}'} \Rightarrow f_* f^*$ and $\varepsilon^{(f)} : f^* f_* \Rightarrow \mathrm{id}_{\mathbf{X}}$, dropping the superscript f when there is no risk of confusion. If $(\mathbf{X}, \mathcal{O})$ and $(\mathbf{X}', \mathcal{O}')$ are ringed topoi, then a morphism $f : (\mathbf{X}, \mathcal{O}) \to (\mathbf{X}', \mathcal{O}')$ consists of a geometric morphism of topoi $f : \mathbf{X} \to \mathbf{X}'$ together with a ring homomorphism $f_\# : \mathcal{O}' \to f_*\mathcal{O}$, which then corresponds to a ring homomorphism $f^\# : f^*\mathcal{O}' \to \mathcal{O}$ via the adjunction.

Now let $(\mathbf{X}, \mathcal{O})$ and $(\mathbf{X}', \mathcal{O}')$ be ringed topoi with involutions (Λ, ν, λ) and $(\Lambda', \nu', \lambda')$, respectively. Regarding Λ and Λ' as geometric automorphisms of $\mathbf{X}$ and $\mathbf{X}'$, see Remark 4.3, a morphism of ringed topoi with involution $(\mathbf{X}, \mathcal{O}) \to (\mathbf{X}', \mathcal{O}')$ should intuitively consist of a morphism of ringed topoi f such that $f \circ \Lambda$ is "equivalent" to $\Lambda' \circ f$. The specifications of this equivalence, which we now give, are somewhat technical.

Definition 4.14. With the previous notation, a morphism of ringed topoi with involution $(\mathbf{X}, \mathcal{O}) \to (\mathbf{X}', \mathcal{O}')$ consists of a morphism of ringed topoi $f : (\mathbf{X}, \mathcal{O}) \to (\mathbf{X}', \mathcal{O}')$ together with natural isomorphisms $\alpha_* : f_*\Lambda \Rightarrow \Lambda' f_*$ and $\alpha^* : f^*\Lambda' \Rightarrow \Lambda f^*$ satisfying the following coherence conditions for all objects X in $\mathbf{X}$ and X' in $\mathbf{X}'$:

(1) The following diagram, the columns of which are induced by α^* and α_* and the rows of which are induced by the relevant adjunctions, commutes.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{X}}(\Lambda f^*X', X) & \longrightarrow & \mathrm{Hom}_{\mathbf{X}'}(X', f_*\Lambda X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{X}}(f^*\Lambda' X', X) & \longrightarrow & \mathrm{Hom}_{\mathbf{X}'}(X', \Lambda' f_* X) \end{array}$$

(2) $f_*\nu_X = \nu'_{f_*X} \circ \Lambda'\alpha_{*,X} \circ \alpha_{*,\Lambda X}$

(3) $\Lambda' f_\# \circ \lambda' = \alpha_{*,\mathcal{O}} \circ f_*\lambda \circ f_\#$

(4) $f^*\nu'_{X'} = \nu_{f^*X'} \circ \Lambda\alpha^*_{X'} \circ \alpha^*_{\Lambda' X'}$

(5) $\lambda \circ f^\# = \Lambda f^\# \circ \alpha^*_{\mathcal{O}'} \circ f^*\lambda'$

We say that f is *strict* when $\alpha_* = \mathrm{id}$ and $\alpha^* = \mathrm{id}$, so that $f_*\Lambda = \Lambda' f_*$ and $\Lambda f^* = f^*\Lambda'$.
We call f an equivalence when f_* is an equivalence of categories and $f_\#$ is an isomorphism, in which case the same holds for f^* and $f^\#$.

Remark 4.15. Yoneda's lemma and condition (1) imply that α_* and α^* determine each other. Explicitly, this is given as

$$\alpha^*_{X'} = \varepsilon^{(f)}_{\Lambda f^* X'} \circ f^*\nu'_{f_*\Lambda f^* X'} \circ f^*\Lambda'\alpha_{*,\Lambda f^* X'} \circ f^*\Lambda' f_* \nu^{-1}_{f^*X'} \circ f^*\Lambda'\eta_{X'} \ .$$

Furthermore, provided (1) holds, conditions (2) and (4) are equivalent, and so are (3) and (5). Thus, in practice, it is enough to either specify $\alpha_* : f_*\Lambda \Rightarrow \Lambda' f_*$ and verify (2) and (3), or specify $\alpha^* : f^*\Lambda' \Rightarrow \Lambda f^*$ and verify (4) and (5).

In accordance with Remark 4.3, we will sometimes call morphisms of ringed topoi with involution C_2-*equivariant* morphisms.
We will usually suppress α_* and α^* in computations, identifying $f_*\Lambda$ with $\Lambda' f_*$ and Λf^* with $f^*\Lambda'$. The coherence conditions guarantee that this will not cause inconsistency or ambiguity.

Example 4.16. Let (Λ, ν, λ) be a weakly trivial involution of $(\mathbf{X}, \mathcal{O})$ and let $(\Lambda', \nu', \lambda')$ be the trivial involution on $(\mathbf{X}, \mathcal{O})$, see Definition 4.8. Then there is $\theta : \Lambda \Rightarrow \mathrm{id}$ such that $\theta_{\mathcal{O}} = \lambda^{-1}$ and $\theta_X \circ \theta_{\Lambda X} = \nu_X$ for all X in $\mathbf{X}$, and one can readily verify that $(f, \alpha_*, \alpha^*) := (\mathrm{id}_{(\mathbf{X},\mathcal{O})}, \theta, \theta^{-1})$ defines an equivalence from $(\mathbf{X}, \mathcal{O}, \Lambda, \nu, \lambda)$ to $(\mathbf{X}, \mathcal{O}, \Lambda', \nu', \lambda')$, and that every such equivalence is of this form.
More generally, if (Λ, ν, λ) is arbitrary and there exists an equivalence (f, α_*, α^*) from $(\mathbf{X}, \mathcal{O}, \Lambda, \nu, \lambda)$ to a ringed topos with a trivial involution, then (Λ, ν, λ) is weakly trivial in the sense of Definition 4.8; take $\theta_X = \varepsilon_X \circ f^*\alpha_{*,X} \circ \varepsilon^{-1}_{\Lambda X}$.

If $f : (\mathbf{X}_1, \mathcal{O}_1, \Lambda_1, \nu_1, \lambda_1) \to (\mathbf{X}_2, \mathcal{O}_2, \Lambda_2, \nu_2, \lambda_2)$ is a morphism of ringed topoi with involution, and if A is an Azumaya $\mathcal{O}_2$-algebra with a λ_2-involution τ, then $f^*A \otimes_{f^*\mathcal{O}_2} \mathcal{O}_1$ is an Azumaya algebra on $(\mathbf{X}_1, \mathcal{O}_1)$ with a λ_1-involution given by $f^*\tau \otimes \lambda_1$. This induces transfer functors

$$\mathbf{Az}_n(\mathcal{O}_2) \to \mathbf{Az}_n(\mathcal{O}_1) \qquad \text{and} \qquad \mathbf{Az}_n(\mathcal{O}_2, \lambda_2) \to \mathbf{Az}_n(\mathcal{O}_1, \lambda_1) \ .$$

We shall need a particular instance of these transfer maps later.

Example 4.17. If X is a complex variety with involution $\lambda : X \to X$, then the étale ringed topos of X, denoted $\mathbf{X}_{\text{ét}}$, becomes a locally ringed topos with involution, as in Example 4.5. On the other hand, one may form the topological space $X(\mathbb{C})$, equipped with the analytic topology, which then has an associated site and consequently an associated topos $\mathbf{X}_{\text{top}}$. One may endow the topos $\mathbf{X}_{\text{top}}$ with several different local ring objects depending on the kind of geometry one wishes to carry out. There is the sheaf $\mathcal{H}$ of holomorphic functions, as set out in [Gro60], and there is also the sheaf $\mathcal{C}$ of continuous $\mathbb{C}$-valued functions.
We claim that $(\mathbf{X}_{\text{top}}, \mathcal{C})$ is a locally ringed topos and that there is a "realization" morphism $(\mathbf{X}_{\text{top}}, \mathcal{C}) \to (\mathbf{X}_{\text{ét}}, \mathcal{O}_X)$ of ringed topoi with involution. An outline of the argument follows.
For every complex variety U, there is a unique, functorially-defined *analytic topology* on $U(\mathbb{C})$; this is established in [GR71, Exp. XII]. The functor $U \mapsto U(\mathbb{C})$ preserves finite limits. Moreover, if $\{U_i \to X\}_{i \in I}$ is an étale covering of X, then the family of maps $\{U_i(\mathbb{C}) \to X(\mathbb{C})\}_{i \in I}$ is a jointly surjective family of local homeomorphisms, and may therefore be refined by a jointly surjective family $\{V_i \to X(\mathbb{C})\}_{i \in I'}$ of open inclusions. Since families of this form generate the usual Grothendieck topology on the topological space $X(\mathbb{C})$, it follows that there is a morphism of sites $f : (X(\mathbb{C}), \text{top}) \to X_{\text{ét}}$, and therefore a morphism of topoi, [AGV72a, III.1 and IV.4.9.4]. Complex realization may be applied to $\mathbb{A}^1_{\mathbb{C}}$, the representing object for $\mathcal{O}_X$, to obtain $\mathbb{C}$, the representing object for $\mathcal{C}$ which is the local ring object on $X(\mathbb{C})$. Therefore, f is a morphism of locally ringed topoi. It is routine to verify that since the involution on X becomes the obvious involution on $X(\mathbb{C})$ after realization, the morphism $f : \mathbf{X}_{\text{top}} \to \mathbf{X}_{\text{ét}}$ extends to a strict morphism of locally ringed topoi with involution. In this instance, all the "coherence" natural isomorphisms appearing are, in fact, identities.

4.3 Quotients by an Involution

Let $\lambda = (\Lambda, \nu, \lambda)$ be an involution of a locally ringed topos $\mathbf{X}$. We would like to consider a quotient of $\mathbf{X}$ by the action of λ, or equivalently, by the (weak) C_2-action it induces. In general, however, it is difficult to define a specific quotient topos in a geometrically reasonable way. For example, if $(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = (\mathbf{Sh}(X_{\text{ét}}), \mathcal{O}_X)$ for a scheme X admitting a C_2-action, then the étale ringed topoi of both the geometric quotient X/C_2, if exists, and the stack $[X/C_2]$ may *a priori* serve as reasonable quotients of $\mathbf{X}$.
We therefore ignore the problem of constructing or specifying a quotient of a locally ringed topos with involution and instead enumerate the properties that such a quotient should possess, declaring any locally ringed topos possessing these properties to be satisfactory.

To be precise, we ask for a locally ringed topos $\mathbf{Y}$, endowed with the trivial involution, together with a C_2-equivariant morphism $\pi : \mathbf{X} \to \mathbf{Y}$ which satisfy certain axioms. Recall from Subsection 4.2 that the data of π consists of a geometric morphism of topoi $\pi = (\pi^*, \pi_*) : \mathbf{X} \to \mathbf{Y}$, a ring homomorphism $\pi_\# : \mathcal{O}_{\mathbf{Y}} \to \pi_*\mathcal{O}_{\mathbf{X}}$ (or equivalently, $\pi^\# : \pi^*\mathcal{O}_{\mathbf{Y}} \to \mathcal{O}_{\mathbf{X}}$) and natural transformations $\alpha_* : \pi_*\Lambda \Rightarrow \pi_*$, $\alpha^* : \pi^* \Rightarrow \Lambda\pi^*$ satisfying the relations of Definition 4.14. We will often suppress α_* and α^*, identifying $\pi_*\Lambda$ with π_* and $\Lambda\pi^*$ with π^*. In fact, in many of our examples, α_* and α^* will both be the identity.

Definition 4.18. Let $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ be a locally ringed topos with involution $\lambda = (\Lambda, \nu, \lambda)$, let $(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$ be a locally ringed topos with a trivial involution, and let $\pi : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$ be a C_2-equivariant morphism of ringed topoi. We say that π is an *exact quotient* (of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ by the given C_2-action) if

(E1) $\pi_\# : \mathcal{O}_{\mathbf{Y}} \to \pi_*\mathcal{O}_{\mathbf{X}}$ is the equalizer of $\pi_*\lambda : \pi_*\mathcal{O}_{\mathbf{X}} \to \pi_*\Lambda\mathcal{O}_{\mathbf{X}} = \pi_*\mathcal{O}_{\mathbf{X}}$ and the identity map $\mathrm{id}_{\pi_*\mathcal{O}_{\mathbf{X}}}$,

(E2) π_* preserves epimorphisms.

Remark 4.19. An exact quotient is in particular a morphism of locally ringed topoi, i.e., a morphism of ringed topoi $\pi : \mathbf{X} \to \mathbf{Y}$ satisfying the additional condition that $\mathcal{O}_{\mathbf{Y}}^\times \to \mathcal{O}_{\mathbf{Y}}$ is the pullback of $\pi_*\mathcal{O}_{\mathbf{X}}^\times \to \pi_*\mathcal{O}_{\mathbf{X}}$ along $\pi_\#$. Indeed, given $V \in \mathbf{Y}$, the V-sections of the pullback consist of pairs $(x, y) \in \pi_*\mathcal{O}_{\mathbf{X}}^\times(V) \times \mathcal{O}_{\mathbf{Y}}(V)$ with $\pi_\# y = x$ in $\pi_*\mathcal{O}_{\mathbf{X}}(V)$. By (E1), we have $\pi_*\lambda(x) = x$ in $\pi_*\mathcal{O}_{\mathbf{X}}(V)$. Since $x \in \pi_*\mathcal{O}_{\mathbf{X}}^\times(V) = \pi_*\mathcal{O}_{\mathbf{X}}(V)^\times$, this means that $\pi_*\lambda(x^{-1}) = x^{-1}$. Thus, again by (E1), there exists unique $y' \in \mathcal{O}_{\mathbf{Y}}(V)$ with $\pi_\# y' = x^{-1}$. In particular, $\pi_\#(yy') = xx^{-1} = 1$ in $\pi_*\mathcal{O}_{\mathbf{X}}(V)$. Since $\mathcal{O}_{\mathbf{Y}}$ is a subobject of $\pi_*\mathcal{O}_{\mathbf{X}}$ via $\pi_\#$ (again, by (E1)), this means that $yy' = 1$ in $\mathcal{O}_{\mathbf{Y}}(V)$, so $y \in \mathcal{O}_{\mathbf{Y}}(V)^\times = \mathcal{O}_{\mathbf{Y}}^\times(V)$. As $x = \pi_\# y$, it follows that (x, y) is the image of a (necessarily unique) V-section of $\mathcal{O}_{\mathbf{Y}}^\times$ under the natural map $\mathcal{O}_{\mathbf{Y}}^\times \to \pi_*\mathcal{O}_{\mathbf{X}}^\times \times_{\pi_*\mathcal{O}_{\mathbf{X}}} \mathcal{O}_{\mathbf{Y}}$, as required.

The name "exact" comes from condition (E2), which implies in particular that π_* preserves exact sequences of groups. We shall see below that this condition is critical for transferring cohomological data from $\mathbf{X}$ to $\mathbf{Y}$. Condition (E1) informally means that $\mathcal{O}_{\mathbf{Y}}$ behaves as one would expect from the subring of $\mathcal{O}_{\mathbf{X}}$ fixed by λ — such an object cannot be defined in $\mathbf{X}$ because the source and target of λ are not, in general, canonically isomorphic.

To motivate Definition 4.18, we now give two fundamental examples of exact quotients. However, in order not to digress, we postpone the proof of their exactness to Subsection 4.4, where further examples and non-examples are exhibited.

Example 4.20. Let X be a scheme and let $\lambda : X \to X$ be an involution. A morphism of schemes $\pi : X \to Y$ is called a *good quotient* of X relative to the action of $C_2 = \{1, \lambda\}$ if π is affine, C_2-invariant, and $\pi_\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ defines an isomorphism of $\mathcal{O}_Y$ with $(\pi_*\mathcal{O}_X)^{C_2}$. By [GR71, Prp. V.1.3] and the going-up theorem, these conditions imply that π is universally surjective and so this agrees with the more general definition in [deJ17, Tag 04AB]. A good C_2-quotient of X exists if and only if every C_2-orbit in X is contained in an affine open subscheme, in which case it is also a categorical quotient in the category of schemes, hence unique up to isomorphism [GR71, Prps. V.1.3, V.1.8].

Let $(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = (\mathbf{Sh}(X_{\text{ét}}), \mathcal{O}_X)$, and let $\lambda = (\Lambda, \nu, \lambda)$ be the involution of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ induced by $\lambda : X \to X$, see Example 4.5. Given a good C_2-quotient, $\pi : X \to Y$, we define an exact quotient $\pi : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$ relative to λ by taking $(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}) = (\mathbf{Sh}(Y_{\text{ét}}), \mathcal{O}_Y)$, letting $\pi = (\pi^*, \pi_*) : \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}(Y_{\text{ét}})$ and defining $\pi_\# : \mathcal{O}_{\mathbf{Y}} \to \pi_* \mathcal{O}_{\mathbf{X}}$ to be the canonical extension of $\pi_\# : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ in $\mathbf{Sh}(X_{\text{Zar}})$ to the corresponding ring objects in $\mathbf{Sh}(X_{\text{ét}})$. The suppressed natural transformations α_* and α^* are both the identity.

Example 4.21. Let X be a Hausdorff topological space, let $\lambda : X \to X$ be a continuous involution, let $Y = X/\{1, \lambda\}$ and let $\pi : X \to Y$ be the quotient map. Let $(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = (\mathbf{Sh}(X), \mathcal{C}(X, \mathbb{C}))$ and let $\lambda = (\Lambda, \nu, \lambda)$ be the involution induced by $\lambda : X \to X$, see Example 4.6. We define an exact quotient $\pi : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$ relative to λ by taking $(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}) = (\mathbf{Sh}(Y), \mathcal{C}(Y, \mathbb{C}))$, letting $\pi = (\pi^*, \pi_*) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ be the geometric morphism induced by $\pi : X \to Y$, and defining $\pi_\# : \mathcal{C}(Y, \mathbb{C}) \to \pi_* \mathcal{C}(X, \mathbb{C})$ to be the morphism sending a section $f \in \mathcal{C}(U, \mathbb{C})$ to $f \circ \pi \in \mathcal{C}(\pi^{-1}(U), \mathbb{C}) = \pi_* \mathcal{C}(X, \mathbb{C})(U)$. Again, the suppressed natural transformations α_* and α^* are both the identity.

We also record the following trivial example.

Example 4.22. Suppose that the involution $\lambda = (\Lambda, \nu, \lambda)$ on $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is weakly trivial, namely, there is a natural isomorphism $\theta : \Lambda \Rightarrow \text{id}$ such that $\theta_X \circ \theta_{\Lambda X} = \nu_X$, and $\lambda = \theta^{-1}_{\mathcal{O}_{\mathbf{X}}}$. Then the identity morphism $\text{id} : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ defines an exact quotient upon taking $\alpha_* = \theta$ and $\alpha^* = \theta^{-1}$. We call it the *trivial quotient* of $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$.

More generally, an arbitrary exact C_2-quotient $\pi : \mathbf{X} \to \mathbf{Y}$ will be called trivial when π is an equivalence of ringed topoi with involution. As noted in Example 4.16, such a quotient can only exist when the involution of $\mathbf{X}$ is weakly trivial.

We shall see below (Remark 4.51) that a locally ringed topos with involution may admit several non-equivalent exact quotients.

We turn to establish some properties of exact quotients that will arise in the sequel. The most crucial of these will be the fact that when $\pi : \mathbf{X} \to \mathbf{Y}$ is an exact C_2-quotient, π_* induces an equivalence between the Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras and the Azumaya $\pi_* \mathcal{O}_{\mathbf{X}}$-algebras, and similarly for Azumaya algebras with a λ-involution.

The following theorem is a consequence of condition (E2).

Theorem 4.23. *Let $\pi : \mathbf{X} \to \mathbf{Y}$ be a geometric morphism of topoi such that π_* preserves epimorphisms, and let G be a group in $\mathbf{X}$. Then:*

(i) π_* induces an equivalence of categories $\mathbf{Tors}(\mathbf{X}, G) \to \mathbf{Tors}(\mathbf{Y}, \pi_* G)$.

(ii) There is a canonical isomorphism $\mathrm{H}^i(\mathbf{Y}, \pi_* G) \cong \mathrm{H}^i(\mathbf{X}, G)$ *when* $i = 0, 1$, *and for all* $i \geq 0$ *when* G *is abelian. For* $i = 0$, *this is the canonical isomorphism* $\mathrm{H}^0(\mathbf{Y}, \pi_* G) = \mathrm{H}^0(\mathbf{X}, G)$. *For* $i = 1$, *this isomorphism agrees with the one induced by (i) and Proposition 2.8(i).*

(iii) If $0 \to G' \to G \to G'' \to 0$ *is a short exact sequence of abelian groups then the isomorphism of (ii) gives rise to an isomorphism between the cohomology long exact sequence of* $G' \to G \to G''$ *and the cohomology long exact sequence of* $\pi_* G' \to \pi_* G \to \pi_* G''$. *The same holds for the truncated long exact sequence of parts (ii) and (iii) of Proposition 2.8 when* G', G, G'' *are not assumed to be abelian.*

Proof. (i) We treat the topos as a site in its own canonical topology. In this language, [Gir71, Chap. V, Sect. 3.1.1.1] says that π_* induces an equivalence $\mathbf{Tors}(\mathbf{X}, G)^{\mathbf{Y}} \to \mathbf{Tors}(\mathbf{Y}, \pi_* G)$, where the source is the category of G-torsors P for which there exists a covering $U \to *_{\mathbf{Y}}$ such that $P_{\pi^* U} \cong G_{\pi^* U}$. We claim that this applies to all G-torsors, and so π_* induces an equivalence $\mathbf{Tors}(\mathbf{X}, G) \to \mathbf{Tors}(\mathbf{Y}, \pi_* G)$.

Since a G-torsor P is trivialized by itself, it is also trivialized by any object mapping to P, for instance by $\pi^* \pi_* P$. It is therefore sufficient to show that the map $\pi_* P \to *_{\mathbf{Y}}$ is an epimorphism. Since $P \to *_{\mathbf{X}}$ is an epimorphism, our assumption implies that $\pi_* P \to \pi_*(*_{\mathbf{X}}) = *_{\mathbf{Y}}$ is also an epimorphism, so the claim is verified.

(ii) Suppose first that G is abelian. The fact that π_* is exact implies that the family of functors $\{G \mapsto \mathrm{H}^i(\mathbf{Y}, \pi_* G)\}_{i \geq 0}$ from abelian groups in $\mathbf{X}$ to abelian groups forms a δ-functor. Thus, the universality of derived functors implies that the canonical isomorphism $\mathrm{H}^0(\mathbf{X}, G) \xrightarrow{\sim} \mathrm{H}^0(\mathbf{Y}, \pi_* G)$ gives rise to a unique natural transformation $\mathrm{H}^i(\mathbf{X}, G) \to \mathrm{H}^i(\mathbf{Y}, \pi_* G)$ for any abelian G. Since π_* takes injective abelian groups to injective abelian groups, $\{G \mapsto \mathrm{H}^i(\mathbf{Y}, \pi_* G)\}_{i \geq 0}$ is an effaceable δ-functor, hence universal. Applying the universality of the latter to the natural isomorphism $\mathrm{H}^0(\mathbf{Y}, \pi_* G) \xrightarrow{\sim} \mathrm{H}^0(\mathbf{X}, G)$ implies that $\mathrm{H}^i(\mathbf{X}, G) \to \mathrm{H}^i(\mathbf{Y}, \pi_* G)$ is an isomorphism.

We note that if we use Verdier's Theorem, see Subsection 2.3, to describe $\mathrm{H}^i(\mathbf{X}, G)$ and $\mathrm{H}^i(\mathbf{Y}, \pi_* G)$, then the isomorphism is given by sending the cohomology class represented by $g \in Z^i(U_\bullet, G)$ to the cohomology class represented by $\pi_* g \in Z^i(\pi_* U_\bullet, \pi_* G)$. Notice that $\pi_* U_\bullet$, which is just $\pi_* \circ U_\bullet : \mathbf{\Delta} \to \mathbf{Y}$, is a hypercovering since π_* preserves epimorphisms and commutes with cosk_n for all n. This isomorphism coincides with the one in the previous paragraph because they coincide on the 0-th cohomology.

When $i = 1$, the map we have just described is defined for an arbitrary group G, and we take it to be the canonical morphism $\mathrm{H}^1(\mathbf{X}, G) \to \mathrm{H}^1(\mathbf{Y}, \pi_* G)$. The construction in the proof of Proposition 2.8(i) implies that this map agrees with the one induced by (i) and Proposition 2.8(i), and thus $\mathrm{H}^1(\mathbf{X}, G) \to \mathrm{H}^1(\mathbf{Y}, \pi_* G)$ is an isomorphism.

(iii) In the abelian case, this follows from the argument given in (ii), which shows that $\{G \mapsto \mathrm{H}^i(\mathbf{X}, G)\}_{i \geq 0}$ and $\{G \mapsto \mathrm{H}^i(\mathbf{Y}, \pi_* G)\}_{i \geq 0}$ are isomorphic δ-functors. In the nonabelian case, this follows from the proof of Proposition 2.8, parts (ii) and (iii). □

Remark 4.24. When $\pi : \mathbf{X} \to \mathbf{Y}$ is a geometric morphism of topoi, with no further assumptions, one still has natural transformations $\mathrm{H}^i(\mathbf{Y}, \pi_* G) \to \mathrm{H}^i(\mathbf{X}, G)$ for all G abelian and $i \geq 0$, or G non-abelian and $i = 0, 1$, and if π_* preserves epimorphisms, they coincide with the inverses of the isomorphisms of Theorem 4.23. In the abelian case, the construction is given as follows: Using the exactness of π^*, one finds that the canonical map $\mathrm{H}^0(\mathbf{Y}, A) \to \mathrm{H}^0(\mathbf{X}, \pi^* A)$ gives rise to natural transformations $\mathrm{H}^i(\mathbf{Y}, A) \to \mathrm{H}^i(\mathbf{X}, \pi^* A)$. Taking $A = \pi_* G$ and composing with the map $\mathrm{H}^i(\mathbf{X}, \pi^* \pi_* G) \to \mathrm{H}^i(\mathbf{X}, G)$, induced by the counit $\pi^* \pi_* G \to G$, one obtains a natural transformation $\mathrm{H}^i(\mathbf{Y}, \pi_* G) \to \mathrm{H}^i(\mathbf{X}, G)$. This map can be written explicitly on the level of cocycles, using Verdier's Theorem, and be adapted to the non-abelian case when $i = 0, 1$.

Henceforth, let $\pi : \mathbf{X} \to \mathbf{Y}$ be an exact quotient relative to an involution $\lambda = (\Lambda, \nu, \lambda)$ on $\mathbf{X}$. We write

$$R = \pi_* \mathcal{O}_{\mathbf{X}} \qquad \text{and} \qquad S = \mathcal{O}_{\mathbf{Y}}$$

for brevity, and, abusing the notation, we let $\lambda : R \to R$ denote the involution $\pi_* \lambda : \pi_* \mathcal{O}_{\mathbf{X}} \to \pi_* \Lambda \mathcal{O}_{\mathbf{X}}$.

We shall use the following lemma freely to identify $\pi_* \mathrm{GL}_n(\mathcal{O}_{\mathbf{X}})$ with $\mathrm{GL}_n(R)$ and $\pi_* \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})$ with $\mathrm{PGL}_n(R)$.

Lemma 4.25. *For all $n \in \mathbb{N}$, there are canonical isomorphisms $\pi_* \mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}) \cong \mathrm{M}_{n\times n}(R)$, $\pi_* \mathrm{GL}_n(\mathcal{O}_{\mathbf{X}}) \cong \mathrm{GL}_n(R)$ and $\pi_* \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}) \cong \mathrm{PGL}_n(R)$.*

Proof. Let $U \in \mathbf{Y}$. Thanks to the adjunction between π^* and π_*, we have a natural isomorphism $\pi_* \mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}})(U) \cong \mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}})(\pi^* U) = \mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}}(\pi^* U)) \cong \mathrm{M}_{n\times n}(\pi_* \mathcal{O}_{\mathbf{X}}(U))$. This establishes the first isomorphism.

The second isomorphism is obtained in the same manner.

The last isomorphism is deduced from the following ladder of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R^\times & \longrightarrow & \mathrm{GL}_n(R) & \longrightarrow & \mathrm{PGL}_n(R) & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow \cong & & \vdots \cong & & \\ 1 & \longrightarrow & \pi_*(\mathcal{O}_{\mathbf{X}}^\times) & \longrightarrow & \pi_*(\mathrm{GL}_n(\mathcal{O}_{\mathbf{X}})) & \longrightarrow & \pi_* \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}) & \longrightarrow & 1. \end{array}$$

Here, the left and middle isomorphisms follow from the previous paragraph, and the bottom row is exact since π_* preserves epimorphisms. □

We shall use the following lemma to identify $\mathrm{PGL}_n(R)$ with $\mathcal{A}ut_{R\text{-alg}}(\mathrm{M}_{n\times n}(R))$ henceforth.

Lemma 4.26. *The canonical group homomorphism* $\mathrm{PGL}_n(R) \to \mathcal{A}ut_{R\text{-}alg}(\mathrm{M}_{n\times n}(R))$ *is an isomorphism.*

The lemma would follow immediately from the discussion in Subsection 2.5 if R were a local ring object, but this is not the case in general. Using Theorem 3.16, we shall see that R functions as a "semilocal" ring object and therefore the isomorphism still holds.

Proof. We need to show that for all $U \in \mathbf{Y}$, any R_U-automorphism ψ of $\mathrm{M}_{n\times n}(R_U)$ becomes an inner automorphism after passing to a covering $V \to U$. Let $\mathfrak{p} \in \operatorname{Spec} S(U)$. Then $B := S(U)_{\mathfrak{p}}$ is the λ-fixed subring of $A := R(U)_{\mathfrak{p}}$. By Theorem 3.16, A is semilocal. It is then well known that ψ_A is an inner automorphism, [Knu91, III.§5.2]. Write $\psi_A(x) = a_{\mathfrak{p}} x a_{\mathfrak{p}}^{-1}$ for some $a_{\mathfrak{p}} \in \mathrm{GL}_n(A)$. There exists $f_{\mathfrak{p}} \in S(U) - \mathfrak{p}$ such that $a_{\mathfrak{p}}$ is the image of an element in $\mathrm{GL}_n(R(U)_{f_{\mathfrak{p}}})$, also denoted $a_{\mathfrak{p}}$, and such that $\psi_{R(U)_{f_{\mathfrak{p}}}}$ agrees with $x \mapsto a_{\mathfrak{p}} x a_{\mathfrak{p}}^{-1}$ on $\mathrm{M}_{n\times n}(R(U)_{f_{\mathfrak{p}}})$, e.g. if they agree on an $R(U)$-basis of $\mathrm{M}_{n\times n}(R(U))$. Since $S = \mathcal{O}_{\mathbf{Y}}$ is a local ring object, and since $S(U) = \sum_{\mathfrak{p}\in\operatorname{Spec} S(U)} S(U) f_{\mathfrak{p}}$, there exists a covering $\{V_{\mathfrak{p}} \to U\}_{\mathfrak{p}}$ such that $f_{\mathfrak{p}} \in S(V_{\mathfrak{p}})^{\times}$. By construction, $R(U) \to R(V_{\mathfrak{p}})$ factors through $R(U)_{f_{\mathfrak{p}}}$, and thus ψ is inner on $V_{\mathfrak{p}}$ for all $\mathfrak{p}$, as required. □

Lemma 4.27. *Let $\pi : \mathbf{X} \to \mathbf{Y}$ be a geometric morphism of topoi such that π_* preserves epimorphisms. Then π_* preserves quotients by equivalence relations. In particular, for any group object G, any G-torsor P, and any G-set X, there is a canonical isomorphism $\pi_*(P \times^G X) \cong \pi_* P \times^{\pi_* G} \pi_* X$.*

Proof. Recall that in a topos $\mathbf{X}$, any equivalence relation $Q \to A \times A$ is effective, meaning that $Q = A \times_B A$ for some epimorphism $A \to B$—in fact, $A \to B$ must be isomorphic to $A \to A/Q$.
Since π_* preserves epimorphisms and limits, this means that $\pi_*(A/Q)$ is canonically isomorphic to $\pi_* A/\pi_* Q$. □

We now come to the main result of this section, which allows passage from Azumaya algebras in the locally ringed topos $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ to Azumaya algebras in the ringed topos $(\mathbf{Y}, R)$.

Theorem 4.28. *Suppose $\mathbf{X}$ is a locally ringed topos with involution $\lambda = (\Lambda, \nu, \lambda)$, and that $\pi : \mathbf{X} \to \mathbf{Y}$ is an exact quotient relative to λ. Let $R = \pi_* \mathcal{O}_{\mathbf{X}}$ and $n \in \mathbb{N}$. Then the following categories are equivalent:*

(a) $\mathbf{Az}_n(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$*, the category of Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras of degree n.*

(b) $\mathbf{Tors}(\mathbf{X}, \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}))$*, the category of* $\mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}})$*-torsors on* $\mathbf{X}$.

(c) $\mathbf{Az}_n(\mathbf{Y}, R)$*, the category of Azumaya R-algebras of degree n.*

(d) $\mathbf{Tors}(\mathbf{Y}, \mathrm{PGL}_n(R))$*, the category of* $\mathrm{PGL}_n(R)$*-torsors on* $\mathbf{Y}$.

The equivalence between (a) and (b), resp. (c) and (d), is the one given in Proposition 2.10, and the equivalence between (a) and (c), resp. (b) and (d), is given by applying π_*.

In the context of Example 4.20, where our exact quotient is induced from a good C_2-quotient of schemes $\pi : X \to Y$, the theorem says that every Azumaya algebra A over X admits an étale covering U of Y (not of X) such that A becomes a matrix algebra after base change to $X \times_Y U$, and every automorphism ψ of A admits an étale covering V of Y (again, not of X) such that ψ becomes an inner automorphism after passing to $X \times_Y V$. Theorem 4.28 can be regarded as a generalization of this fact.

Proof. The equivalence between (a) and (b), resp. (c) and (d), is Proposition 2.10; here we identified $\mathrm{PGL}_n(R)$ with $\mathcal{A}ut_R(\mathrm{M}_{n\times n}(R))$ as in Lemma 4.26. The equivalence between (b) and (d) is Theorem 4.23(i), together with the isomorphism $\pi_* \mathrm{PGL}_n(\mathcal{O}_{\mathbf{X}}) \cong \mathrm{PGL}_n(R)$ of Lemma 4.25. It now follows from Lemma 4.27 that the induced equivalence between (a) and (c) is given by applying π_*. □

Remark 4.29. The exact quotient $\pi : \mathbf{X} \to \mathbf{Y}$ gives rise to a morphism of ringed topoi with involution $\hat{\pi} : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{Y}, R)$ by setting $\hat{\pi}_\# : R \to \pi_*\mathcal{O}_{\mathbf{X}}$ to be the identity. The induced transfer functor $\mathbf{Az}_n(R) \to \mathbf{Az}_n(\mathcal{O}_{\mathbf{X}})$ is then an inverse to the equivalence $\pi_* : \mathbf{Az}_n(\mathcal{O}_{\mathbf{X}}) \to \mathbf{Az}_n(R)$.

Remark 4.30. As phrased, Theorem 4.28 addresses Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras of constant degree n only. These constitute all Azumaya algebras when $\mathbf{X}$ is connected, but not in general. If we replace n with a global section of the constant sheaf $\mathbb{N}$ on $\mathbf{X}$, then Theorem 4.28 still holds, provided that n is fixed by $\Lambda : \mathrm{H}^0(\mathbf{X}, \mathbb{N}) \to \mathrm{H}^0(\mathbf{X}, \Lambda\mathbb{N}) = \mathrm{H}^0(\mathbf{X}, \mathbb{N})$, in which case n can be understood as an element of $\mathrm{H}^0(\mathbf{Y}, \mathbb{N})$. Since Theorem 4.28 is used throughout, we tacitly assume henceforth that all Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras have degrees that are fixed under Λ. This makes little difference in practice, because any Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra is Brauer equivalent to another Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra of degree which is fixed by Λ.

We define an involution on the ringed topos $(\mathbf{Y}, R)$ by setting $\Lambda = \mathrm{id}_{\mathbf{Y}}$, $\nu = \mathrm{id}$ and $\lambda_{(\mathbf{Y},R)} = \pi_*\lambda_{(\mathbf{X},\mathcal{O}_{\mathbf{X}})}$. Since π_* preserves products, for any $\mathcal{O}_{\mathbf{X}}$-algebra A, we have $\pi_*\Lambda A^{\mathrm{op}} = \Lambda\pi_* A^{\mathrm{op}}$ as R-algebras. However, we alert the reader that while $\Lambda\pi_* A^{\mathrm{op}} = \pi_* A^{\mathrm{op}}$ as noncommutative rings, the R-algebra structure of $\Lambda\pi_* A^{\mathrm{op}}$ is obtained from the R-algebra structure of $\pi_* A^{\mathrm{op}}$ by twisting via $\lambda : R \to \Lambda R = R$, as explained in Subsection 4.1. Theorem 4.28 now implies:

Corollary 4.31. *For all $n \in \mathbb{N}$, the functor π_* induces an equivalence between $\mathbf{Az}_n(\mathcal{O}_{\mathbf{X}}, \lambda)$, the category of degree-n Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras with λ-involution, and $\mathbf{Az}_n(R, \lambda)$, the category of degree-n Azumaya R-algebras with λ-involution.*

At this point we conclude that results proved so far allow one to shift freely between $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ and $(\mathbf{Y}, R)$, at least when Azumaya algebras, possibly with a λ-involution, are concerned. Of these two contexts, we shall work most often in the second, since this is technically easier. That said, the starting point is always a locally ringed topos $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ with involution $\lambda = (\Lambda, \nu, \lambda)$, and the choice of the corresponding $\mathbf{Y}$, R and $\lambda_{(\mathbf{Y},R)}$ is not in general uniquely determined by the initial data.

4.4 Examples of Exact Quotients

We now turn to providing various examples of exact C_2-quotients. In particular, we will prove that Examples 4.20 and 4.21 are exact C_2-quotients. It will also be shown that any locally ringed topos with involution admits an exact quotient.

Of the two conditions of Definition 4.18, condition (E2) is harder to establish. The following lemma is our main tool in proving it.

Lemma 4.32. *Let $\mathbf{X}$, $\mathbf{Y}$ be topoi and let $\pi : \mathbf{X} \to \mathbf{Y}$ be a geometric morphism. Suppose that $\mathbf{Y}$ has a conservative family of points $\{p_i : \mathbf{pt} \to \mathbf{Y}\}_{i\in I}$ with the property that for each $i \in I$, there exists a set of points $\{j_n : \mathbf{pt} \to \mathbf{X}\}_{n\in N_i}$ such that the functors $U \mapsto p_i^*\pi_*U$ and $U \mapsto \prod_{n\in N_i} j_n^*U$ from $\mathbf{X}$ to $\mathbf{pt}$ are isomorphic. Then π_* preserves epimorphisms.*

Proof. If $p_i : \mathbf{pt} \to \mathbf{Y}$ is a point as in the lemma, then $p_i^*\pi_*$ preserves epimorphisms because each j_n^* does. By assumption, a morphism ψ in $\mathbf{Y}$ is an epimorphism if and only if $p_i^*\psi$ is an epimorphism for any $p_i : \mathbf{pt} \to \mathbf{Y}$ as in the statement, so π_* preserves epimorphisms. □

Informally, a geometric morphism satisfying the conditions of Lemma 4.32 can be regarded as having "discrete fibres". It can also be thought of as a generalization of a finite morphism in algebraic geometry, thanks to the following corollary.

Corollary 4.33. *Let $\pi : X \to Y$ be a finite morphism of schemes. Then the direct image functor $\pi_* : \mathbf{Sh}(X_{ét}) \to \mathbf{Sh}(Y_{ét})$ preserves epimorphisms.*

This arises in the proof that the higher direct images vanish for cohomology with abelian coefficients, [deJ17, Tag 03QN]. We have included a proof here in order to present a modification later.

Proof. Recall ([AGV72b, Exp. VIII, §3–4]) that the points of $\mathbf{Sh}(Y_{ét})$ are constructed as follows: Given $y \in Y$, choose a cofiltered system of étale neighbourhoods $\{(U_\alpha, u_\alpha) \to (Y, y)\}_\alpha$ such that $\lim_\alpha U_\alpha = \operatorname{Spec} B$, where B is a strictly henselian ring, necessarily isomorphic to the strict henselization of $\mathcal{O}_{Y,y}$. Then the functor $i^* : F \mapsto \operatorname{colim}_\alpha F(U_\alpha)$ from $\mathbf{Sh}(Y_{ét})$ to $\mathbf{pt}$ and its right adjoint i_* define a point $i : \mathbf{pt} \to \mathbf{Sh}(Y_{ét})$, and these points form a conservative family, [AGV72b, Thm. VIII.3.5].
Let $i : \mathbf{pt} \to \mathbf{Sh}(Y_{ét})$ be such a point and write $V_\alpha = U_\alpha \times_Y X$. For all sheaves F on $X_{ét}$, we have $i^*\pi_*F = \operatorname{colim}_\alpha F(V_\alpha)$. Note that $\lim_\alpha V_\alpha = \lim U_\alpha \times_Y X =$

$\operatorname{Spec} B \times_Y X$. Since π is finite, $\operatorname{Spec} B \times_Y X = \operatorname{Spec} A$ where A is a finite B-algebra.
By [deJ17, Tag 04GH], $A = \prod_{n=1}^{t} A_n$ where each A_n is a henselian ring. Since the residue field of each A_n is finite over the separably closed residue field of B, each A_n is strictly henselian. Letting $e_1, \dots, e_t$ be the primitive idempotents of A, we may assume, by appropriately thinning the family $\{U_\alpha \to Y\}_\alpha$, that $e_1, \dots, e_t$ are defined as compatible global sections on each V_α. This allows us to write V_α as $\bigsqcup_n V_{n,\alpha}$ such that $\lim_\alpha V_{n,\alpha} = \operatorname{Spec} A_n$ for all n. Let x_n and $v_{n,\alpha}$ denote the images of the closed point of $\operatorname{Spec} A_n$ in X and $V_{n,\alpha}$, respectively, and let $j_n : \mathbf{pt} \to \mathbf{Sh}(X_{\text{ét}})$ denote the point corresponding to the filtered system $\{(V_{n,\alpha}, v_{n,\alpha}) \to (X, x_n)\}_\alpha$. Since we can commute a directed colimit past a finite limit, we have shown that

$$i^*\pi_* F = \operatorname*{colim}_{\alpha} F(V_\alpha) = \operatorname*{colim}_{\alpha} \prod_n F(V_{n,\alpha}) = \prod_n j_n^* F \ ,$$

and the result now follows from Lemma 4.32. □

Corollary 4.34. *Let $\pi : X \to Y$ be a continuous morphism of topological spaces such that:*

(1) For any $y \in Y$ and any open neighbourhood $U \supseteq \pi^{-1}(y)$, there exists an open neighbourhood V of y such that $\pi^{-1}(V) \subseteq U$.

(2) For any $y \in Y$, the fibre $\pi^{-1}(y)$ is finite and, letting $x_1, \dots, x_t \in X$ denote the points lying over y, there exist disjoint open sets $\{U_i\}_{i=1}^t$ such that $x_i \in U_i$.

Then $\pi_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ preserves epimorphisms.

The hypotheses are satisfied when $\pi : X \to Y$ is a finite covering space map of Hausdorff spaces, or a closed embedding of Hausdorff spaces, for instance.
It is also easy to see that condition (1) is equivalent to π being closed, and condition (2) is equivalent to π having finite fibres and being separated in the sense that the image of the diagonal map $X \to X \times_Y X$ is closed.

Proof. Again, we use Lemma 4.32. For $\mathbf{Sh}(Y)$, the points are induced by inclusion maps $i : \{y\} \to Y$ as y rages over Y, [MLM92, Chap. VII, §5]. Fix such an inclusion, let $x_1, \dots, x_t$ denote the points in $\pi^{-1}(y)$, and let $j_n : \{x_n\} \to X$ denote the inclusion maps. The corresponding morphisms on the topoi of sheaves will be denoted by the same letters.
By definition, for any sheaf F on X, we have

$$i^*\pi_*\mathcal{F} = \operatorname*{colim}_{U \ni y} \mathcal{F}(\pi^{-1}(U))$$

with the colimit taken over all open neighbourhoods of y. Using condition (2), choose disjoint open neighbourhoods $\{V_n\}_{n=1}^t$ with $x_n \in V_n$. Condition (1) implies that the family $\{V_n \cap \pi^{-1}(U) \,|\, U \text{ is an open neighbourhood of } y\}$ is a basis of open neighbourhoods of x_n, hence

$$\operatorname*{colim}_{U\ni y} \mathcal{F}(\pi^{-1}(U)) = \prod_{n=1}^{t} \operatorname*{colim}_{U\ni y} \mathcal{F}(\pi^{-1}(U)\cap V_n) = \prod_{n=1}^{t} j_n^*\mathcal{F} \ .$$

It follows that $i^*\pi_* = \prod_{n=1}^t j_n^*$, so the proof is complete. □

Theorem 4.35. *Suppose that*

(i) X is a scheme, $\lambda : X \to X$ is an involution, $\pi : X \to Y$ is a good quotient relative to $\{1,\lambda\}$, $(\mathbf{X},\mathcal{O}_{\mathbf{X}}) = (\mathbf{Sh}(X_{\acute{e}t}),\mathcal{O}_X)$ and $(\mathbf{Y},\mathcal{O}_{\mathbf{Y}}) = (\mathbf{Sh}(Y_{\acute{e}t}),\mathcal{O}_Y)$ (Example 4.20), or

(ii) X is a Hausdorff topological space, $\lambda : X \to X$ is an involution, $Y = X/\{1,\lambda\}$ and $\pi : X \to Y$ is the quotient map, $(\mathbf{X},\mathcal{O}_{\mathbf{X}}) = (\mathbf{Sh}(X),\mathcal{C}(X,\mathbb{C}))$ and $(\mathbf{Y},\mathcal{O}_{\mathbf{Y}}) = (\mathbf{Sh}(Y),\mathcal{C}(Y,\mathbb{C}))$ (Example 4.21).

Then the morphism $\pi : (\mathbf{X},\mathcal{O}_{\mathbf{X}}) \to (\mathbf{Y},\mathcal{O}_{\mathbf{Y}})$ induced by $\pi : X \to Y$ is an exact quotient relative to the involution induced by λ.

Proof. (i) The fact that $\pi_* : \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}(Y_{\text{ét}})$ preserves epimorphisms is shown as in the proof of Corollary 4.33, except one has to replace [deJ17, Tag 04GH] by Corollary 3.19. Checking that $\mathcal{O}_{\mathbf{Y}}$ is the coequalizer of $\lambda, \mathrm{id} : \pi_*\mathcal{O}_{\mathbf{X}} \to \pi_*\mathcal{O}_{\mathbf{X}}$, amounts to showing that for any étale morphism $U \to Y$, $\mathrm{H}^0(U,\mathcal{O}_U)$ is the fixed ring of λ in $\mathrm{H}^0(U\times_Y X,\mathcal{O}_{U\times_Y X})$. In fact, it is enough to check this after base changing to an open affine covering $\{Y_i \to Y\}$, so we may assume that $U \to Y$ factors as $U \to Y_0 \to Y$ with Y_0 open and affine. Write $Y_0 = \operatorname{Spec} B$ and $U = \operatorname{Spec} B'$. Since $X \to Y$ is affine, we may further write $Y_0 \times_Y X = \operatorname{Spec} A$. The assumption that $X \to Y$ is a good quotient relative to $\{1,\lambda\}$ implies that the sequence of B-modules $0 \to B \to A \xrightarrow{a\mapsto a-a^\lambda} A$ is exact. Since B' is flat over B, the sequence $0 \to B' \to A\otimes_B B' \xrightarrow{a\mapsto a-a^\lambda} A\otimes_B B'$ is exact, and hence $B' = \mathrm{H}^0(U,\mathcal{O}_U)$ is the fixed ring of λ in $A\otimes_B B' = \mathrm{H}^0(U\times_Y X,\mathcal{O}_{U\times_Y X})$.

(ii) Conditions (1) and (2) of Corollary 4.34 are easily seen to hold, hence π_* preserves epimorphisms. It remains to show that $\mathcal{O}_Y$ is the equalizer of $\pi_*\mathcal{O}_X$ under the action of $C_2 = \{1,\lambda\}$. To this end, let $U \subseteq Y$ be an open set. The C_2-action on X restricts to an action on $\pi^{-1}(U)$ and $\pi^{-1}(U)/C_2 = U$. In particular, $\mathcal{O}_Y(U) = C(U,\mathbb{C})$ is in natural bijection with the set of functions in $\pi_*\mathcal{O}_X(U) = \mathcal{O}_X(\pi^{-1}(U)) = C(\pi^{-1}(U),\mathbb{C})$ that are fixed under the C_2-action. This means that $\mathcal{O}_Y$ is the fixed subsheaf of $\pi_*\mathcal{O}_X$ under the action of C_2. □

Remark 4.36. The proofs of Theorem 4.35(i) and Corollary 4.33 can be modified to work for the Nisnevich site of a scheme — simply replace étale neighbourhoods by Nisnevich neighbourhoods and strictly henselian rings by henselian rings. Disregarding set-theoretic problems, the large étale and Nisnevich sites

can be handled similarly, using suitable conservative families of points, provided one assumes in Theorem 4.35(i) that $\mathcal{O}_Y \to \pi_*\mathcal{O}_X \xrightarrow{\lambda-\mathrm{id}} \pi_*\mathcal{O}_X$ splits in the middle, which is the case when $2 \in \mathcal{O}_X^\times$. This assumption guarantees that the sequence remains exact after base-change to any Y-scheme U, not necessarily flat.
Likewise, the proofs Theorem 4.35(ii) and Corollary 4.34 can be modified to work for the large site of a topological space.

The next examples bring several situations where condition (E1) of Definition 4.18 is satisfied while condition (E2) is not. They also show that some of the assumptions made in Theorem 4.35 cannot be removed in general.

Example 4.37. Let R be a strictly henselian discrete valuation ring with fraction field K. Let X denote the scheme obtained by gluing two copies of $Y := \operatorname{Spec} R$ along $\operatorname{Spec} K$, and let $\lambda : X \to X$ denote the involution exchanging these two copies. The morphism $\pi : X \to Y$ which restricts to the identity on each of the copies of $\operatorname{Spec} R$ is a geometric quotient relative to $C_2 = \{1, \lambda\}$ in the sense of [deJ17, Tag 04AD], namely, $\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^{C_2}$, $Y = X/C_2$ as topological spaces, and the latter property holds after base change. In particular, $\pi : X \to Y$ is the C_2-quotient of X in the category of schemes. However, the induced C_2-equivariant morphism $\pi : (\mathbf{Sh}(X_{\text{ét}}), \mathcal{O}_X) \to (\mathbf{Sh}(Y_{\text{ét}}), \mathcal{O}_Y)$ is not an exact quotient, the reason being that π_* does not preserve epimorphisms.
To see this, fix a non-trivial abelian group A, which will be regarded as a constant sheaf on the appropriate space, and let $i, j : \operatorname{Spec} R \to X$ denote the inclusions of the two copies of $\operatorname{Spec} R$ in X. Since i is an open immersion, we can form the extension-by-0 functor $i_! : \mathbf{Sh}((\operatorname{Spec} R)_{\text{ét}}) \to \mathbf{Sh}(X_{\text{ét}})$, which is left adjoint to i^*. Let $F = i_!i^*A \oplus j_!j^*A$. The counit maps $\varepsilon^{(i)} : i_!i^*A \to A$ and $\varepsilon^{(j)} : j_!j^*A \to A$ give rise to a morphism $\psi : F \to A$ given by $(x \oplus y) \mapsto (\varepsilon^{(i)}x + \varepsilon^{(j)}y)$ on sections. This morphism is surjective, as can be easily seen by checking the stalks. However $\pi_*\psi : \pi_*F \to \pi_*A$ is not surjective, as can be seen by noting that $\pi_*F(Y) = 0$, $\pi_*A(Y) = A \neq 0$, and any étale covering of Y has a section, because R is strictly henselian.
This example does not stand in contradiction to Theorem 4.35(i) because $\pi : X \to Y$ is not affine, and hence not a good quotient.

Example 4.38. Let X be an infinite set endowed with the cofinite topology, let $\lambda : X \to X$ be an involution acting freely on X, and let $\pi : X \to Y = X/C_2$ be the quotient map. Then the induced C_2-equivariant morphism $\pi : (\mathbf{Sh}(X), \mathcal{C}(X, \mathbb{C})) \to (\mathbf{Sh}(Y), \mathcal{C}(Y, \mathbb{C}))$ is not an exact C_2-quotient, because π_* fails to preserve epimorphisms. This is shown as in Example 4.37, except here one chooses $x \in X$ and uses the open embeddings $i : X - \{x\} \to X$ and $j : X - \{\lambda(x)\} \to X$. We conclude that in Theorem 4.35(ii), the assumption that X is Hausdorff in cannot be removed in general, even when λ acts freely on X.

Example 4.39. Let R be a principal ideal domain admitting exactly two maximal ideals, $\mathfrak{a}$ and $\mathfrak{b}$. Suppose that there exists an involution $\lambda : R \to R$ exchanging $\mathfrak{a}$ and $\mathfrak{b}$, and moreover, that the fixed ring of λ, denoted S, is a discrete valuation ring. Let $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ and let $\pi : X \to Y$ be the morphism adjoint to the inclusion $S \to R$. Then $\pi : X \to Y$ is a good quotient relative to $\lambda : X \to X$, but the induced C_2-equivariant morphism $\pi : (\mathbf{Sh}(X_{\mathrm{Zar}}), \mathcal{O}_X) \to (\mathbf{Sh}(Y_{\mathrm{Zar}}), \mathcal{O}_Y)$ is not an exact quotient, because, yet again, π_* does not preserve epimorphisms. Again, this is checked as in Example 4.37 by using the open embeddings $i : \operatorname{Spec} R_{\mathfrak{a}} \to \operatorname{Spec} R$ and $j : \operatorname{Spec} R_{\mathfrak{b}} \to \operatorname{Spec} R$. This shows that we cannot, in general, replace the étale site with the Zariski site in Theorem 4.35(i), even when $\pi : X \to Y$ is quadratic étale.

Remark 4.40. Let X be a scheme, let $\lambda : X \to X$ be an involution and let $\pi : X \to Y$ be a good quotient relative to $\{1, \lambda\}$. Then the associated morphism of fppf topoi $\pi : (\mathbf{Sh}(X_{\mathrm{fppf}}), \mathcal{O}_X) \to (\mathbf{Sh}(Y_{\mathrm{fppf}}), \mathcal{O}_Y)$ is not exact in general, even when π is an fppf morphism.

For example, let k be a field characteristic $\neq 2$, and consider $X = \operatorname{Spec} k[\varepsilon \,|\, \varepsilon^2 = 0]$, $Y = \operatorname{Spec} k$ and the k-involution λ sending ε to $-\varepsilon$. Then $x \mapsto x^2 : \mathcal{O}_X \to \mathcal{O}_X$ is surjective as morphism in $\mathbf{Sh}(X_{\mathrm{fppf}})$, but its pushforward to $\mathbf{Sh}(Y_{\mathrm{fppf}})$ is not, because ε is not in the image of $x \mapsto x^2 : A[\varepsilon \,|\, \varepsilon^2 = 0] \to A[\varepsilon \,|\, \varepsilon^2 = 0]$ for all commutative k-algebras A.

Nevertheless, when $\pi : X \to Y$ is finite and locally free, Theorem 4.28 and Corollary 4.31 still hold, the reason being that $\mathrm{PGL}_n(\mathcal{O}_X)$ and $\mathrm{PGL}_n(R) = \pi_* \mathrm{PGL}_n(\mathcal{O}_X)$ are both represented by smooth affine group schemes over X (use [BLR90, Prop. 7.6.5(h)]), and hence their étale and fppf cohomologies coincide [Gro68c, Thm. 11.7, Rmk. 11.8(3)]. As a result, some theorems in the next sections, e.g. Theorems 5.17 and 6.10, also hold in the context of fppf ringed topoi associated to a finite locally free good C_2-quotient of schemes $\pi : X \to Y$.

We finish with demonstrating that every locally ringed topos $\mathbf{X}$ with involution $\lambda = (\Lambda, \nu, \lambda)$ admits a canonical exact quotient, sometimes called the "homotopy fixed points", as in [Mer17, Section 2]. We denote this exact quotient by $\pi : \mathbf{X} \to [\mathbf{X}/C_2]$.

As the notation suggests, when $(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = (\mathbf{Sh}(X_{\text{ét}}), \mathcal{O}_X)$ for a scheme X, the ringed topos $[\mathbf{X}/C_2]$ will be equivalent to the étale ringed topos of the Deligne–Mumford stack $[X/C_2]$. Indeed, the objects of $[\mathbf{X}/C_2]$ will be C_2-*equivariant sheaves*, the data of which are equivalent to specifying a sheaf on the étale site of $[X/C_2]$; this is explained for coherent sheaves in [Vis89, Example 7.21], but the principle works for set-valued sheaves (in the sense of [deJ17, Tag 06TN]) as well.

Construction 4.41. Define the category $[\mathbf{X}/C_2]$ as follows: The objects of $[\mathbf{X}/C_2]$ consist of pairs (U, τ), where U is an object of $\mathbf{X}$ and $\tau : U \to \Lambda U$ is a morphism satisfying $\Lambda\tau \circ \tau = \mathrm{id}_U$. In other words, the objects of $[\mathbf{X}/C_2]$ are objects of $\mathbf{X}$ equipped with an involution, or a C_2-action. Morphisms in

$[\mathbf{X}/C_2]$ are defined as commuting squares

$$\begin{array}{ccc} U & \xrightarrow{\tau} & \Lambda U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Lambda f} \\ U' & \xrightarrow{\tau'} & \Lambda U'. \end{array}$$

Define $\pi^* : [\mathbf{X}/C_2] \to \mathbf{X}$ to be the forgetful functor $(U,\tau) \mapsto U$, and define $\pi_* : \mathbf{X} \to [\mathbf{X}/C_2]$ to be the functor sending U to $(U \times \Lambda U, \tau_U)$ where $\tau_U : U \times \Lambda U \to \Lambda(U \times \Lambda U) = \Lambda U \times U$ is the interchange morphism. For a morphism $\phi : U \to V$ in $\mathbf{X}$, let $\pi_*\phi = \phi \times \Lambda\phi$. The functor π^* is easily seen to be left adjoint to π_* with the unit and counit of the adjunction given by $u \mapsto (u, \tau u) : (U,\tau) \to \pi_*\pi^*(U,\tau) = (U \times \Lambda U, \tau_U)$ and $(v, v') \mapsto v : \pi^*\pi_* V = V \times \Lambda V \to V$ on the level of sections (in $\mathbf{X}$).
For objects V in $\mathbf{X}$ and (U,τ) in $[\mathbf{X}/C_2]$, let $\alpha_{*,V}$ denote the interchange morphism $(\Lambda V \times V, \tau_{\Lambda V}) \to (V \times \Lambda V, \tau_V)$ and let $\alpha^*_{(U,\tau)}$ denote $\tau : U \to \Lambda U$. Then α_* is a natural isomorphism $\pi_*\Lambda \Rightarrow \pi_*$ and α^* is a natural isomorphism $\pi^* \Rightarrow \Lambda\pi^*$. We alert the reader that these natural isomorphisms are in general not the identity transformations, even when the involution λ is trivial.
Define the ring object $\mathcal{O}_{[\mathbf{X}/C_2]}$ in $[\mathbf{X}/C_2]$ to be $(\mathcal{O}_\mathbf{X}, \lambda)$ with the obvious ring structure. Finally, define $\pi_\# : \mathcal{O}_{[\mathbf{X}/C_2]} \to \pi_*\mathcal{O}_\mathbf{X}$ to be $(\mathcal{O}_\mathbf{X}, \lambda) \to (\mathcal{O}_\mathbf{X} \times \Lambda\mathcal{O}_\mathbf{X}, \tau_{\mathcal{O}_\mathbf{X}})$, where the underlying morphism $\mathcal{O}_\mathbf{X} \to \mathcal{O}_\mathbf{X} \times \Lambda\mathcal{O}_\mathbf{X}$ is given by $x \mapsto (x, x^\lambda)$ on sections.

Proposition 4.42. *In Construction 4.41, the following hold:*

(i) $[\mathbf{X}/C_2]$ *is a Grothendieck topos.*

(ii) $\pi := (\pi^*, \pi_*) : \mathbf{X} \to [\mathbf{X}/C_2]$ *is an essential geometric morphism of topoi.*

(iii) A family of morphisms $\{(U_i,\tau_i) \to (U,\tau)\}_{i\in I}$ *in* $[\mathbf{X}/C_2]$ *is a covering if and only if* $\{U_i \to U\}_{i\in I}$ *is a covering in* $\mathbf{X}$.

(iv) $\mathcal{O}_{[\mathbf{X}/C_2]}$ *is a local ring object in* $[\mathbf{X}/C_2]$.

(v) $(\pi^*, \pi_*, \pi_\#, \alpha^*, \alpha_*)$ *defines an exact quotient* $\pi : (\mathbf{X}, \mathcal{O}_\mathbf{X}) \to ([\mathbf{X}/C_2], \mathcal{O}_{[\mathbf{X}/C_2]})$ *relative to* λ.

Proof. We first introduce the functor $\pi_! : \mathbf{X} \to [\mathbf{X}/C_2]$ given by sending an object U to $(U \sqcup \Lambda U, \sigma_U)$ where $\sigma_U : U \sqcup \Lambda U \to \Lambda(U \sqcup \Lambda U) = \Lambda U \sqcup U$ is the interchange morphism. It is routine to check that $\pi_!$ is left adjoint to π^*; the unit map is the canonical embedding $V \to \pi^*\pi_! V = V \sqcup \Lambda V$ and the counit map is the map $\pi_!\pi^*(U,\tau) = (U \sqcup \Lambda U, \sigma_U) \to (U,\tau)$ restricting to id_U on U and to τ^{-1} on ΛU. The existence of adjoints implies formally that π^* is continuous and cocontinuous, and that π^* and $\pi_!$ preserve epimorphisms. We now turn to the proof itself.

(i) We verify Giruad's axioms for $[\mathbf{X}/C_2]$. Briefly, if $\{G_i\}_{i\in I}$ is a set of generators for $\mathbf{X}$, then $\{\pi_! G_i\}_{i\in I}$ is a set of generators for $[\mathbf{X}/C_2]$. Indeed, let $f,g : U \to V$ be distinct morphisms in $[\mathbf{X}/C_2]$. Since π^* is faithful, $\pi^* f, \pi^* g : \pi^* U \to \pi^* V$ are distinct in $\mathbf{X}$, hence there exist $i \in I$ and $h : G_i \to \pi^* U \to \pi^* V$ such that $\pi^* f \circ h \neq \pi^* g \circ h$. By the adjunction between $\pi_!$ and π^*, the morphism h corresponds to a morphism $h' : \pi_! G_i \to U$ in $[\mathbf{X}/C_2]$ such that $f \circ h' \neq g \circ h'$, as required.

That sums are disjoint and equivalence relations are effective in $[\mathbf{X}/C_2]$ can be checked with the help of the forgetful functor π^* and the fact that these properties hold in $\mathbf{X}$. Finally, the existence of colimits and the fact that they commute with fiber products can be checked directly.

(ii) This is immediate from the adjunctions between $\pi_!$, π^* and π_* noted above.

(iii) We may replace $\{(U_i,\tau_i)\}_{i\in I}$ with their disjoint union, denoted (U',τ'), to assume that I consists of a single element. We need to show that $U' \to U$ is an epimorphism in $\mathbf{X}$ if and only if $(U',\tau') \to (U,\tau)$ is an epimorphism in $[\mathbf{X}/C_2]$. The "if" part follows from the fact that π^* preserves epimorphisms, being a left adjoint. To see the converse, it is enough to show that the composition $\pi_! U' = \pi_! \pi^*(U',\tau') \xrightarrow{\text{counit}} (U',\tau') \to (U,\tau)$ is an epimorphism. Let $(V,\sigma) \in [\mathbf{X}/C_2]$. Then $\mathrm{Hom}_{[\mathbf{X}/C_2]}(\pi_! U', (V,\sigma)) \cong \mathrm{Hom}_{\mathbf{X}}(U', \pi^*(V,\sigma)) = \mathrm{Hom}_{\mathbf{X}}(U', V)$, and under this isomorphism the map

$$\mathrm{Hom}_{[\mathbf{X}/C_2]}((U,\tau),(V,\sigma)) \to \mathrm{Hom}_{[\mathbf{X}/C_2]}(\pi_! U', (V,\sigma))$$

is the composition $\mathrm{Hom}_{[\mathbf{X}/C_2]}((U,\tau),(V,\sigma)) \hookrightarrow \mathrm{Hom}_{\mathbf{X}}(U,V) \to \mathrm{Hom}_{\mathbf{X}}(U',V)$, which is injective since $U' \to U$ an epimorphism. Thus, $(U',\tau') \to (U,\tau)$ is an epimorphism.

(iv) Let $\{r_i\}_{i\in I}$ be (U,τ)-sections of $(\mathcal{O}_{\mathbf{X}},\lambda)$ generating the unit ideal. Then $\{\pi^* r_i\}_{i\in I}$ generate the unit ideal in $\mathcal{O}_{\mathbf{X}}(U)$, and hence there exists a covering $\{\alpha_i : U_i \to U\}_{i\in I}$ such that $r_i \in \mathcal{O}_{\mathbf{X}}(U_i)^\times$ for all i. (We remark that $\mathcal{O}_{\mathbf{X}}(\emptyset)$ is the 0-ring, in which the unique element is invertible; it is possible that some of the U_i called for in this covering are initial objects.)

Fix $i \in I$. The adjunction between $\pi_!$ and π^* gives rise to a morphism $\beta_i : \pi_! U_i \to (U,\tau)$, adjoint to $\alpha_i : U_i \to U = \pi^*(U,\tau)$, and an isomorphism of rings $\mathcal{O}_{\mathbf{X}}(U_i) = \mathrm{Hom}_{\mathbf{X}}(U_i,\mathcal{O}_{\mathbf{X}}) \cong \mathrm{Hom}_{[\mathbf{X}/C_2]}(\pi_! U_i, (\mathcal{O}_{\mathbf{X}},\lambda)) = \mathcal{O}_{[\mathbf{X}/C_2]}(\pi_! U_i)$. Unfolding the definitions, one finds that under this isomorphism, $r_i|_{\pi_! U_i} = r_i \circ \beta_i \in \mathcal{O}_{[\mathbf{X}/C_2]}(\pi_! U_i)$ corresponds to $\pi^* r_i|_{U_i} = \pi^* r_i \circ \alpha_i$, which is invertible. Thus, r_i is invertible in $\mathcal{O}_{[\mathbf{X}/C_2]}(\pi_! U_i)$. By (iii), the collection $\{\beta_i : \pi_! U_i \to (U,\tau)\}_{i\in I}$ is a covering, so we have shown that $[\mathbf{X}/C_2]$ is locally ringed by $\mathcal{O}_{[\mathbf{X}/C_2]}$.

(v) One checks that $\pi_*\lambda : \pi_*\mathcal{O}_{\mathbf{X}} \to \pi_*\mathcal{O}_{\mathbf{X}}$ is the morphism

$$(\mathcal{O}_{\mathbf{X}} \times \Lambda\mathcal{O}_{\mathbf{X}}, \tau_{\mathcal{O}_{\mathbf{X}}}) \xrightarrow{(x,y)\mapsto(y^\lambda,x^\lambda)} (\mathcal{O}_{\mathbf{X}} \times \Lambda\mathcal{O}_{\mathbf{X}}, \tau_{\mathcal{O}_{\mathbf{X}}}),$$

and so $\mathcal{O}_{[\mathbf{X}/C_2]}$ is the equalizer of $\mathrm{id}, \pi_*\lambda : \pi_*\mathcal{O}_{\mathbf{X}} \to \pi_*\mathcal{O}_{\mathbf{X}}$. That π_* preserves epimorphisms can be checked directly using the definitions and (iii). The verification of the coherence conditions in Definition 4.14 is routine. □

4.5 Ramification

Let $\mathbf{X}$ be a locally ringed topos with an involution $\lambda = (\Lambda, \nu, \lambda)$, and let $\pi : \mathbf{X} \to \mathbf{Y}$ be an exact quotient, see Definition 4.18. For brevity, write

$$R = \pi_*\mathcal{O}_{\mathbf{X}} \qquad \text{and} \qquad S = \mathcal{O}_{\mathbf{Y}} \ .$$

As in Subsection 4.3, we write $\pi_*\lambda : R \to \pi_*\Lambda\mathcal{O}_{\mathbf{X}} = R$ as λ. The automorphism $\lambda : R \to R$ is an involution the fixed ring of which is S.

Definition 4.43. Let V be an object of $\mathbf{Y}$. We say that π is *unramified* (relative to λ) along V if R_V is a quadratic étale S_V-algebra in $\mathbf{Y}/V$, see Subsection 3.2. Otherwise, π is said to be *ramified* along V.
The morphism π is said to be *unramified* if it is unramified along $*_{\mathbf{Y}}$, and *ramified* otherwise. It is *everywhere ramified* if π is ramified along every non-initial object of $\mathbf{Y}$.

We alert the reader that, contrary to the common use of the term “ramification”, we consider trivial C_2-quotients as everywhere ramified.

Example 4.44. Suppose $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ is given a weakly trivial involution and π is the trivial C_2-quotient, namely, the identity map $\mathrm{id} : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ (Example 4.22). Then π is everywhere ramified. Indeed, in this case $R = S = \mathcal{O}_{\mathbf{X}}$ and and $\lambda = \mathrm{id}_R$. Since $\mathcal{O}_{\mathbf{X}}$ is a local ring object, for any $V \ncong \emptyset$ in $\mathbf{X}$, the ring $\mathcal{O}_{\mathbf{X}}(V)$ is nonzero, and so R_V cannot be locally free of rank 2 over S_V.

For any object V of $\mathbf{Y}$, define $U(V)$ to be a singleton if π is unramified along V, and an empty set otherwise. It follows from Lemma 3.6 that $V \mapsto U(V)$ defines a sheaf (the action of U on morphisms in $\mathbf{Y}$ is uniquely determined), which is then represented by an object of $\mathbf{Y}$, denoted $U = U_\pi$. We call U the *unbranched locus* of π. It is a subobject of $*_{\mathbf{Y}}$. Clearly, π is unramified if and only if $U = *_{\mathbf{Y}}$, and π is everywhere ramified if and only if $U = \emptyset_{\mathbf{Y}}$.
The following propositions give a more concrete description of the unbranched locus when $\pi : \mathbf{X} \to \mathbf{Y}$ is induced by a C_2-quotient of schemes or topological spaces, see Examples 4.20 and 4.21.

Proposition 4.45. *In the situation of Example 4.20, i.e., when $\pi : \mathbf{X} \to \mathbf{Y}$ is obtained from a good C_2-quotient of schemes $\pi : X \to Y$ by taking étale ringed topoi, the unbranched locus of π, defined above, is represented by an open subscheme $U \subseteq Y$. The latter can be defined in any of the following equivalent ways:*

(a) U is the largest open subset of Y such that $\pi_U : \pi^{-1}(U) \to U$ is quadratic étale.

(b) *The (Zariski) points of U are those points $y \in Y$ such that $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y} \to \operatorname{Spec} \mathcal{O}_{Y,y}$ is quadratic étale.*

(c) *The (Zariski) points of U are those points $y \in Y$ such that $X \times_Y \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{Y,y} \to \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{Y,y}$ is quadratic étale; here, $\mathcal{O}^{\mathrm{sh}}_{Y,y}$ is the strict henselization of $\mathcal{O}_{Y,y}$.*

(d) *The (Zariski) points of $Y - U$ are those points $y \in Y$ such that the set $\pi^{-1}(y)$ is a singleton $\{x\}$, and λ induces the identity on $k(x)$.*

Consequently, $\pi : \mathbf{X} \to \mathbf{Y}$ is unramified if and only if $\pi : X \to Y$ is quadratic étale.

Proof. By virtue of Lemma 3.6, if $V \to Y$ is an étale morphism having image U in Y, then $\pi_V : X \times_Y V \to V$ is quadratic étale if and only if $\pi_U : \pi^{-1}(U) \to U$ is quadratic étale. It follows that there exists a maximal open subset U of Y with the property that π_U is quadratic étale, and any $V \to X$ as above factors through the inclusion $U \subseteq X$. We also let U denote the sheaf it represents in $\mathbf{Y} = \mathbf{Sh}(Y_{\text{ét}})$.

Since U is a subobject of $*_{\mathbf{Y}}$, the set $U(V)$ is a singleton or empty for all $V \in \mathbf{Y}$. To show that U is the unbranched locus of $\pi : \mathbf{X} \to \mathbf{Y}$, it is enough to show that π is unramified along an object V of $\mathbf{Y}$ if and only if there exists a morphism $V \to U$. For any such V, we can find a covering $\{V_i \to V\}_i$ with each V_i represented by some $(\tilde{V}_i \to Y)$ in $Y_{\text{ét}}$. By Lemma 3.6, π unramified along V if and only if π is unramified along each V_i. Furthermore, if for each $i \in I$ there is a morphism $V_i \to U$ (in $\mathbf{Y}$), then these morphisms must patch to a morphism $V \to U$, because $U(V_i \times_V V_j)$ is a singleton. It is therefore enough to show that π is unramified along V_i if and only if there exists a morphism $\tilde{V}_i \to U$ in $Y_{\text{ét}}$. The latter holds precisely when $\operatorname{im}(\tilde{V}_i \to Y) \subseteq U$, and so the claim follows from the definition of U.

We finish by showing that the different characterizations of U are equivalent. The equivalence of (a) and (b) follows from Corollary 3.13, and the equivalence of (b) and (d) follows from Theorem 3.16. That the condition in (b) implies the condition in (c) is clear. It remains to prove the converse. Since $\mathcal{O}^{\mathrm{sh}}_{Y,y}$ is faithfully flat over $\mathcal{O}_{Y,y}$, this is a consequence of Lemma 3.6 applied to the fpqc site of $\operatorname{Spec} \mathcal{O}_{Y,y}$. □

Proposition 4.46. *In the situation of Example 4.21, i.e., when $\pi : \mathbf{X} \to \mathbf{Y}$ is induced by a C_2-quotient of Hausdorff topological spaces $\pi : X \to Y = X/C_2$, the unbranched locus is represented by an open subset $U \subseteq Y$. Specifically, $U = \{x \in X : x^\lambda \neq x\}/C_2$. Consequently, $\pi : \mathbf{X} \to \mathbf{Y}$ is unramified if and only if C_2 acts freely on X.*

Proof. This is similar to the previous proof and is left to the reader. □

We refer to the situations of Examples 4.20 and 4.21 as the scheme-theoretic case and topological case, respectively. In both cases, we define the *branch locus* of π to be the complement $W := Y - U$, where U is as in Proposition 4.45 or Proposition 4.46. The *ramification locus* of π is $Z := \pi^{-1}(W)$. In the scheme-theoretic case, we also endow W and Z with the reduced induced closed subscheme structure.

Notice that in the topological case, $\pi_U : \pi^{-1}(U) \to U$ is a double covering, while $\pi_W : \pi^{-1}(W) \to W$ is a homeomorphism. With slight modification, a similar statement holds for schemes.

Proposition 4.47. *In the notation of Proposition 4.45, let $W = U - Y$, and regard W and $\pi^{-1}(W)$ as reduced closed subschemes of Y and X, respectively. Then:*

(i) $\pi_U : \pi^{-1}(U) \to U$ *is quadratic étale.*

(ii) $\lambda : X \to X$ *restricts to the identity morphism on* $\pi^{-1}(W)$.

(iii) $\pi|_{\pi^{-1}(W)} : \pi^{-1}(W) \to W$ *is a homeomorphism, and when* 2 *is invertible on* Y*, it is an isomorphism of schemes.*

Proof. (i) This immediate from condition (a) in Proposition 4.45.

(ii) Condition (d) of Proposition 4.45 implies that λ fixes the (Zariski) points of $\pi^{-1}(W)$. Let f be a (Zariski) section of $\mathcal{O}_{\pi^{-1}(W)}$ and let $S \subseteq \pi^{-1}(W)$ be the largest open subset on which $f - f^\lambda$ is invertible. Let $s \in S$. Then $f - f^\lambda$ is invertible in $k(s)$, which is impossible by condition (d) of Proposition 4.45. Thus, $S = \emptyset$. Since $\pi^{-1}(W)$ is reduced, we conclude that $f - f^\lambda = 0$.

(iii) It is enough to prove the claim after restricting to an open affine covering of Y, so assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $W = \operatorname{Spec} B/I$ with I a radical ideal of B, and $\pi^{-1}(W) = \operatorname{Spec} A/\sqrt{IA}$, where $\sqrt{IA}$ denotes the radical of IA. The morphism $\pi|_{\pi^{-1}(W)} : \pi^{-1}(W) \to W$ is adjoint to the evident homomorphism $B/I \to A/\sqrt{IA}$, and $\lambda : X \to X$ induces an involution $\lambda : A \to A$ having fixed ring B.

We know that $\pi : \pi^{-1}(W) \to W$ is continuous and it is a set bijection since its set-theoretic fibers consist of singletons by condition (d) of Proposition 4.45. Thus, proving that $\pi|_{\pi^{-1}(W)}$ is a homeomorphism amounts to checking that it is closed. Since any $a \in A$ satisfies $a^2 - (a^\lambda + a)a + (a^\lambda a) = 0$, the morphism $\operatorname{Spec} A/\sqrt{IA} \to \operatorname{Spec} B/I$ is integral, and therefore closed by [deJ17, Tag 01WM].

Suppose now that $2 \in B^\times$. We need to show that $B/I \to A/\sqrt{IA}$ is bijective. Let $a \in A$. By virtue of (ii), λ induces the identity involution on $A/\sqrt{IA}$, and thus $a \equiv \frac{1}{2}(a + a^\lambda) \mod \sqrt{IA}$. Since $a + a^\lambda \in B$, we have established the surjectivity of $B/I \to A/\sqrt{IA}$. Next, write $J = \ker(B/I \to A/\sqrt{AI})$. Since $\operatorname{Spec} A/\sqrt{AI} \to \operatorname{Spec} B/I$ is bijective, J is contained in every prime ideal of B/I, and since B/I is reduced, $J = 0$. □

Remark 4.48. (i) In Proposition 4.47(iii), it is in general necessary to assume that 2 is invertible in order to conclude that $\pi : \pi^{-1}(W) \to W$ is an isomorphism. Consider, for example, a DVR S with $2 \neq 0$ having a non-perfect residue field K of characteristic 2, let $\alpha \in S^\times$ be an element such that its image in K is not a square, let $R = S[x \,|\, x^2 = \alpha]$, and let $\lambda : R \to R$ be the S-involution sending x to $-x$. Taking $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$, the set W consists of the closed point y of Y, but the induced map $k(y) \to k(\pi^{-1}(y))$ not an isomorphism.

(ii) Let $\pi : X \to Y$ be a good C_2-quotient of schemes which is everywhere ramified, and suppose 2 is invertible on Y. Then Proposition 4.47 implies that the induced morphism $\pi : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is an isomorphism. However, in general, and in contrast to Proposition 4.46, it can happen that $\pi : X \to Y$ is not an isomorphism. For example, take $X = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ and let λ be the $\mathbb{C}$-involution taking ε to $-\varepsilon$.

Remark 4.49. For a general exact C_2-quotient $\pi : \mathbf{X} \to \mathbf{Y}$ with unbranched locus U, it is possible to define the "branch topos" $\mathbf{W}$ of π as the full subcategory of $\mathbf{Y}$ consisting of objects W such that the projection $U \times W \to U$ is an isomorphism. In the situation of Examples 4.20 and 4.21, this turns out to give the topos of sheaves over the set-theoretic or scheme-theoretic branch locus of $\pi : X \to Y$ defined above. We omit the details as they will not be needed in this work.

We finish with showing that the exact quotient of Construction 4.41 is unramified. Thus, any locally ringed topos with involution admits an unramified exact quotient. When $\mathbf{X}$ is the étale ringed topos of a scheme X, this generalizes the well-known fact that the morphism from X to its quotient stack $[X/C_2]$ is quadratic étale.

Proposition 4.50. *The exact quotient $\pi : \mathbf{X} \to [\mathbf{X}/C_2]$ of Construction 4.41 is unramified.*

Proof. Recall from Construction 4.41 that $S = (\mathcal{O}_\mathbf{X}, \lambda)$ and $R = (\mathcal{O}_\mathbf{X} \times \Lambda \mathcal{O}_\mathbf{X}, \tau_{\mathcal{O}_\mathbf{X}})$, where $\tau_{\mathcal{O}_\mathbf{X}}$ is the interchange involution, and the morphism $\pi_\# : S \to R$ is given by $x \mapsto (x, x^\lambda)$ on sections (in $\mathbf{X}$). We shall make use of $\pi_! : \mathbf{X} \to [\mathbf{X}/C_2]$, the left adjoint of π^* constructed in the proof of Proposition 4.42.
Write $D := \pi_!(*_\mathbf{X}) = (* \sqcup \Lambda *, \sigma_*)$ and observe that the unique map $D \to (*_\mathbf{X}, \mathrm{id}) = *_{[\mathbf{X}/C_2]}$ is a covering by Proposition 4.42(iii). By Lemma 3.6, it is enough to show that R_D is a quadratic étale S_D-algebra. In fact, we will show that $R_D \cong S_D \times S_D$.
We first observe that the slice category $[\mathbf{X}/C_2]/D$ is equivalent to $\mathbf{X}$; the equivalence is given by mapping $(U, \tau) \to D$ in $[\mathbf{X}/C_2]/D$ to $* \times_{(*\sqcup\Lambda *)} U$, and by $\pi_!$ in the other direction. Now, consider R_D and S_D as sheaves on $\mathcal{O}_{[\mathbf{X}/C_2]}/D$. Then $R' = R_D \circ \pi_!$ and $S' = S_D \circ \pi_!$ are sheaves of rings on the equivalent topos $\mathbf{X}$. Since $\pi_!$ is left adjoint to π^*, for all objects V of $\mathbf{X}$, we

have natural isomorphisms of rings

$$\begin{aligned} R'(V) &= R(\pi_! V) = \mathrm{Hom}_{[\mathbf{X}/C_2]}(\pi_! V, (\mathcal{O}_{\mathbf{X}} \times \Lambda\mathcal{O}_{\mathbf{X}}, \tau_{\mathcal{O}_{\mathbf{X}}})) \cong \mathcal{O}_{\mathbf{X}}(V) \times \Lambda\mathcal{O}_{\mathbf{X}}(V) \\ S'(V) &= S(\pi_! V) = \mathrm{Hom}_{[\mathbf{X}/C_2]}(\pi_! V, (\mathcal{O}_{\mathbf{X}}, \lambda)) \cong \mathcal{O}_{\mathbf{X}}(V) \end{aligned}$$

and under these isomorphisms, the embedding $S' \to R'$ induced by $\pi_\# : S \to R$ is given by $x \mapsto (x, x^\lambda)$ on sections. From this it follows that $R' \cong S' \times S'$ as S'-algebras, and hence $R_D \cong S_D \times S_D$ as S_D-algebras. □

Remark 4.51. Propositions 4.45 and 4.46 provide plenty of examples where a locally ringed topos with involution admits a ramified exact quotient. However, Proposition 4.50 shows that these examples also admit unramified exact quotients. It follows that exact quotients are not unique in general.

5 Classifying Involutions into Types

The purpose of this section is to classify involutions of Azumaya algebras into types in such a way which generalizes the familiar classification of involutions of central simple algebras over fields as orthogonal, symplectic or unitary.

Throughout, $\mathbf{X}$ denotes a locally ringed topos with an involution $\lambda = (\Lambda, \nu, \lambda)$ and $\pi : \mathbf{X} \to \mathbf{Y}$ is an exact quotient relative to λ, see Definition 4.18. For brevity, we write

$$R = \pi_* \mathcal{O}_{\mathbf{X}} \qquad \text{and} \qquad S = \mathcal{O}_{\mathbf{Y}},$$

and, abusing the notation, denote the involution $\pi_*\lambda : R \to R$ by λ. Theorem 4.28 and Corollary 4.31 imply that Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras with a λ-involution are equivalent to Azumaya R-algebras with a λ-involution, and the latter are easier to work with.

In fact, most of the results of this section can be phrased with no direct reference to $\mathbf{X}$ or the quotient map π, assuming only a locally ringed topos $(\mathbf{Y}, S)$, an S-algebra R, and an involution $\lambda : R \to R$ having fixed ring S.

We remind the reader that Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras are always assumed to have a degree which is fixed by Λ, see Remark 4.30. This is automatic when $\mathbf{X}$ is connected.

5.1 Types of Involutions

Suppose K is a field and $\lambda : K \to K$ is an involution, the fixed subfield of which is F. Classically, when $\lambda = \mathrm{id}_K$, the λ-involutions of central simple K-algebras are divided into two types, *orthogonal* and *symplectic*, whereas in the case $\lambda \neq \mathrm{id}_K$, they are simply called *unitary*; see [KMRT98, §2]. This classification satisfies the following two properties:

(i) If (A, τ) and (A', τ') are central simple K-algebras with λ-involutions such that $\deg A = \deg A'$, then τ and τ' are of the same type if and only if (A, τ) and (A', τ') become isomorphic as algebras with involution over an algebraic closure of F.

(ii) If (A, τ) is a central simple K-algebra with λ-involution, then τ has the same type as $\tau\mathrm{tr} : \mathrm{M}_{n\times n}(A) \to \mathrm{M}_{n\times n}(A)$ given by $(a_{ij})_{i,j} \mapsto (a_{ji}^{\tau})_{i,j}$.

Of these two properties, it is mainly the first that motivates the classification into types. The second property should not be disregarded as it guarantees, at least when $2 \in K^{\times}$, that involutions adjoint to symmetric bilinear forms (resp. alternating bilinear forms, hermitian forms) of arbitrary rank all have the same type, see [KMRT98, §4].

Our aim in this section is to partition the λ-involutions of Azumaya R-algebras into equivalence classes, called types, so that properties analogous to (i) and (ii) hold. To this end, we simply take the minimal equivalence relation forced by the "if" part of condition (i) and condition (ii).

DEFINITION 5.1. Let (A, τ), (A', τ') be two Azumaya R-algebras with a λ-involution. Let $\tau\mathrm{tr}$ denote the involution $\tau \otimes \lambda\mathrm{tr}$ of $\mathrm{M}_{n\times n}(A) \cong A \otimes \mathrm{M}_{n\times n}(R)$. On sections, it is given by $(a_{ij})_{i,j} \mapsto (a_{ji}^{\tau})_{i,j}$.

We say that τ and τ' are of the same λ-*type* or *type* if there exist $n, n' \in \mathbb{N}$ and a covering $U \to *$ in $\mathbf{Y}$ such that

$$(\mathrm{M}_{n\times n}(A_U), \tau_U\mathrm{tr}) \cong (\mathrm{M}_{n'\times n'}(A'_U), \tau'_U\mathrm{tr})$$

as R_U-algebras with involution.

Being of the same λ-type is an equivalence relation. The equivalence classes will be called λ-*types* or just *types*, and the type of τ will be denoted

$$\mathrm{t}(\tau) \qquad \text{or} \qquad \mathrm{t}(A, \tau)\ .$$

The set of all λ-types will be denoted $\mathrm{Typ}(\lambda)$. Tensor product of Azumaya algebras with involution endows $\mathrm{Typ}(\lambda)$ with a monoid structure. We write the neutral element, represented by (R, λ), as 1.

We shall shortly see that our definition gives the familiar types in the case of fields, as well as in a number of other cases. It is no longer clear whether two involutions of the same degree and type are locally isomorphic, however, and the majority of this section will be dedicated to showing that this is indeed the case under mild assumptions. Another drawback of the definition is that it is not clear how to enumerate the types it yields, and nor does it provide a way to test whether two involutions are of the same type. These problems will also be addressed, especially in the situation of Theorem 4.35, namely, in cases induced by a good C_2-quotient of schemes on which 2 is invertible, or by a C_2-quotient of Hausdorff topological spaces.

Remark 5.2. Let (A, τ), (A', τ') be two Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras in $\mathbf{X}$ with a λ-involution. Using Corollary 4.31, we say that τ and τ' have the same λ-*type (relative to* π*)* when the same holds after applying π_*. The equivalence class of (A, τ) is denoted $\mathrm{t}_\pi(\tau)$ or $\mathrm{t}_\pi(A, \tau)$ and the monoid of types is denoted $\mathrm{Typ}_\pi(\lambda)$. We warn the reader that the λ-type of a λ-involution of an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra depends on the choice of the quotient $\pi : \mathbf{X} \to \mathbf{Y}$, which is why we include π in the notation.

For instance, we shall see below in Theorem 5.21 that in the case where $\mathbf{X}$ is given the trivial involution and $\mathbf{X}$ is connected, then under mild assumptions, taking π to be the trivial exact quotient $\mathrm{id} : \mathbf{X} \to \mathbf{X}$ results in two λ-types, whereas taking π to be the exact quotient $\mathbf{X} \to [\mathbf{X}/C_2]$ of Construction 4.41 results in only one λ-type.

Example 5.3. (i) Let X be a connected scheme on which 2 is invertible, let $\lambda : X \to X$ be the trivial involution and let $Y := X/C_2 = X$. Consider the exact quotient obtained from $\pi : X \to Y$ by taking étale ringed topoi, see Example 4.20. In this case, $\mathbf{X} = \mathbf{Y} = \mathbf{Sh}(X_{\text{ét}})$, $R = S = \mathcal{O}_X$ and $\lambda = \mathrm{id}_R$. Thus, an Azumaya R-algebra with a λ-involution is simply an Azumaya $\mathcal{O}_X$-algebra with an involution of the first kind on X. It is well known that there are two λ-types: *orthogonal* and *symplectic*. The orthogonal type is represented by (R, id_R) and the symplectic type is represented by $(\mathrm{M}_{2\times 2}(R), \mathrm{sp})$, where sp is given by $x \mapsto h_2 x^{\mathrm{tr}} h_2^{-1}$ on sections and $h_2 = [\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}]$. Moreover, every Azumaya R-algebra of degree n with an involution of the first kind is locally isomorphic either to $(\mathrm{M}_n(R), \mathrm{tr})$ or to $(\mathrm{M}_n(R), \mathrm{sp})$, where in the latter case $n = 2m$ and sp is given sectionwise by $x \mapsto h_n x^{\mathrm{tr}} h_n^{-1}$ with $h_n = [\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}]$; see [Knu91, III.§8.5] or [PS92, §1.1]. In this case, $\mathrm{Typ}(\lambda)$ is isomorphic to the group $\{\pm 1\}$.

(ii) Let $\pi : X \to Y$ be a quadratic étale morphism, and let $\lambda : X \to Y$ denote the canonical Y-involution of X. Again, let $\pi : \mathbf{X} \to \mathbf{Y}$ denote the exact quotient obtained from $\pi : X \to Y$ by taking étale ringed topoi. In this case, $R = \pi_* \mathcal{O}_X$ is quadratic étale over $S = \mathcal{O}_Y$, and λ-involutions are known as *unitary* involutions. There is only one type in this situation, and moreover, any Azumaya R-algebra of degree n with a λ-involution is locally isomorphic to $(\mathrm{M}_n(R), \lambda\mathrm{tr})$; these well-known facts can be found in [PS92, §1.2] without proof, but they follow from the results in the sequel. We were unable to find a source providing complete proofs.

Example 5.4. Let K be a perfect field of characteristic 2, and consider the case of the trivial involution on K. As in Example 5.3(i), this corresponds to taking $\mathbf{X} = \mathbf{Y} = \mathbf{Sh}((\operatorname{Spec} K)_{\text{ét}})$, $R = S = \mathcal{O}_{\operatorname{Spec} K}$ and $\lambda = \mathrm{id}$. Azumaya R-algebras with a λ-involution are therefore central simple K-algebras with an involution of the first kind. There are again two types in this case, again called orthogonal and symplectic, but $\mathrm{Typ}(\lambda)$ is isomorphic to the multiplicative monoid $\{0, 1\}$ with 0 corresponding to the symplectic type; see [KMRT98, §2]. This shows that the theory in characteristic 2 is substantially different from that in other characteristics.
The assumption that K is perfect can be dropped if one replaces the étale site with the fppf site (consult Remark 4.40).

5.2 Coarse types

In Subsection 5.1, we defined the type of a λ-involution of an Azumaya R-algebra in terms of the entire class of algebras and not as an intrinsic invariant of the involution. We now introduce another invariant of λ-involutions, called the *coarse type*, which, while in general coarser than the type, will enjoy an intrinsic definition. It will turn out that under mild assumptions the invariants are equivalent, and this will be used to address the questions raised in Subsection 5.1. Apart from that, coarse types will also be needed in proving the main results of Section 6.

We begin by defining the abelian group object N of $\mathbf{Y}$ via the exact sequence

$$1 \to N \to R^{\times} \xrightarrow{x \mapsto x^{\lambda} x} S^{\times} \tag{5.1}$$

and the abelian group object T of $\mathbf{Y}$ via the short exact sequence

$$1 \to R^{\times}/S^{\times} \xrightarrow{x \mapsto x^{\lambda} x^{-1}} N \to T \to 1 \ . \tag{5.2}$$

The group N should be regarded as the group of elements of λ-norm 1. We call the global sections of T *coarse λ-types* and write

$$\mathrm{cTyp}(\lambda) = \mathrm{H}^0(T) \ .$$

The following example and propositions give some hints about the structure of T.

Example 5.5. If the involution of $\mathbf{X}$ is trivial and the quotient map is the identity, then $N = \mu_{2,R}$ and the map $x \mapsto x^{-1}x^{\lambda} : R^{\times}/S^{\times} \to N$ is trivial, hence $T = \mu_{2,R}$ and $\mathrm{cTyp}(\lambda) = \mathrm{H}^0(\mu_{2,R}) = \{x \in \mathrm{H}^0(R) : x^2 = 1\}$.

Proposition 5.6. *If π is unramified along an object U of $\mathbf{Y}$, see Subsection 4.5, then $T_U = 1$ in $\mathbf{Y}/U$. In particular, when π is unramified, $T = 1$ and $\mathrm{cTyp}(\lambda) = \{1\}$.*

When $\mathbf{Y}$ has enough points, it is possible to argue at stalks, and therefore the proposition follows from our version of Hilbert's Theorem 90, Proposition 3.4(iii). The following argument applies even without the assumption of enough points.

Proof. We must show that for all objects U of $\mathbf{Y}$ and all $r \in R(U)$ satisfying $r^{\lambda} r = 1$, there is a covering $V \to U$ and $a \in R^{\times}(V)$ such that $a^{-1}a^{\lambda} = r$ in $R(V)$.
Refining U if necessary, we may assume that $R(U)$ is a quadratic étale $S(U)$-algebra, see Subsection 3.2. By Proposition 3.4(iii), for all $\mathfrak{p} \in \operatorname{Spec} S(U)$, there is $a_{\mathfrak{p}} \in R(U)_{\mathfrak{p}}^{\times}$ such that $a_{\mathfrak{p}}^{-1} a_{\mathfrak{p}}^{\lambda} = r$. For each $\mathfrak{p}$, choose some $f_{\mathfrak{p}} \in S(U) - \mathfrak{p}$ such that $a_{\mathfrak{p}}$ is the image of an element in $R(U)_{f_{\mathfrak{p}}}^{\times}$, also denoted $a_{\mathfrak{p}}$, which satisfies $a_{\mathfrak{p}}^{-1} a_{\mathfrak{p}}^{\lambda} = r$ in $S(U)_{f_{\mathfrak{p}}}$. The set $\{f_{\mathfrak{p}}\}_{\mathfrak{p}}$ is not contained in any proper ideal of $S(U)$ and therefore generates the unit ideal. Since S is a local ring object, there exists a covering $\{U_{\mathfrak{p}} \to U\}_{\mathfrak{p} \in \operatorname{Spec} S(U)}$ such that the image of $f_{\mathfrak{p}}$ is invertible in $S(U_{\mathfrak{p}})$ for all $\mathfrak{p}$. Now take $V = \bigsqcup_{\mathfrak{p}} U_{\mathfrak{p}}$ and a to be the image of $(a_{\mathfrak{p}})_{\mathfrak{p}}$ in $R(V) = \prod_{\mathfrak{p}} R(U_{\mathfrak{p}})$. □

Proposition 5.7. *T is a 2-torsion abelian sheaf.*

Proof. Let U be an object of $\mathbf{Y}$ and $t \in T(U)$. By passing to a covering of U, we may assume that t is the image of some $r \in N(U)$. Since $r^\lambda r = 1$, we have $r^2 = (r^{-1})^\lambda (r^{-1})^{-1}$, and thus $t^2 = 1$ in $T(U)$. □

Let $n = \deg A$. Using Lemma 4.26, we shall freely identify the group $\mathrm{PGL}_n(R)$ with $\mathcal{A}ut_{R\text{-alg}}(\mathrm{M}_{n\times n}(R))$. We denote by

$$-\lambda\mathrm{tr}$$

the automorphism of $\mathrm{GL}_n(R)$ given by $x \mapsto (x^{-1})^{\lambda\mathrm{tr}}$ on sections. This automorphism induces an automorphism on $\mathrm{PGL}_n(R)$, which is also denoted $-\lambda\mathrm{tr}$. We need the following lemma.

Lemma 5.8. *Let g be a section of $\mathrm{PGL}_n(R) = \mathrm{Aut}_{R\text{-alg}}(\mathrm{M}_{n\times n}(R)) = \mathrm{Aut}_{R\text{-alg}}(\Lambda\mathrm{M}_{n\times n}(R)^{op})$ and view $\lambda\mathrm{tr}$ as an R-algebra isomorphism $\mathrm{M}_{n\times n}(R) \to \Lambda\mathrm{M}_{n\times n}(R)^{op}$. Then $g \circ \lambda\mathrm{tr} = \lambda\mathrm{tr} \circ g^{-\lambda\mathrm{tr}}$.*

Proof. Suppose $g \in \mathrm{PGL}_n(R)(U)$ for some object U of $\mathbf{X}$. It is enough to prove the equality after passing to a covering $V \to U$. We may therefore assume that g is inner, and the lemma follows by computation. □

Construction 5.9. Let (A, τ) be a degree-n Azumaya R-algebra with a λ-involution. We now construct an element

$$\mathrm{ct}(\tau) \in \mathrm{cTyp}(\lambda) = \mathrm{H}^0(T)$$

and call it the *coarse λ-type* of τ. This construction, which is concluded in Definition 5.11, will play a major role in the sequel.

Choose a covering $U \to *_{\mathbf{Y}}$ such that there exists an isomorphism of R_U-algebras $\psi : A_U \xrightarrow{\sim} \mathrm{M}_{n\times n}(R_U)$. The isomorphism ψ gives rise to a λ_U-involution

$$\sigma = \psi \circ \tau_U \circ \psi^{-1} : \mathrm{M}_{n\times n}(R_U) \to \mathrm{M}_{n\times n}(R_U)\ .$$

From σ and the involution $\lambda\mathrm{tr}$, we construct

$$g := \lambda\mathrm{tr} \circ \sigma \in \mathrm{Aut}_{R_U\text{-alg}}(\mathrm{M}_{n\times n}(R_U)) = \mathrm{PGL}_n(R)(U)\ .$$

By Lemma 5.8, we have $\mathrm{id} = \sigma\circ\sigma = \lambda\mathrm{tr}\circ g\circ\lambda\mathrm{tr}\circ g = \lambda\mathrm{tr}\circ\lambda\mathrm{tr}\circ g^{-\lambda\mathrm{tr}}\circ g$, hence $g^{-\lambda\mathrm{tr}} g = 1$. Replacing U by a covering $U' \to U$ if necessary, we may assume that $g \in \mathrm{PGL}_n(R)(U)$ lifts to a section

$$h \in \mathrm{GL}_n(R)(U)\ .$$

From $g^{-\lambda\mathrm{tr}} g = 1$, we get

$$\varepsilon := h^{-\lambda\mathrm{tr}} h \in R^\times(U)\ . \tag{5.3}$$

Note that $\varepsilon^{\lambda\mathrm{tr}}\varepsilon = \varepsilon^{\lambda\mathrm{tr}} h^{-\lambda\mathrm{tr}} h = h^{-\lambda\mathrm{tr}} \varepsilon^{\lambda\mathrm{tr}} h = h^{-\lambda\mathrm{tr}}(h^{\lambda\mathrm{tr}} h^{-1}) h = 1$, hence $\varepsilon \in N(U)$. Let $\overline{\varepsilon}$ be the image of ε in $T(U)$.

Lemma 5.10. *The section $\overline{\varepsilon} \in T(U)$ determines a global section of T. It is independent of the choices made in Construction 5.9.*

Proof. Let $U_\bullet$ denote the Čech hypercovering associated to U—for the definition see Example 2.3. In particular, $U_0 = U$, $U_1 = U \times U$ and $d_0, d_1 : U_1 \to U_0$ are given by $d_i(u_0, u_1) = u_{1-i}$ on sections. Proving that $\overline{\varepsilon}$ determines a global section of T amounts to showing that there exists a covering $V \to U_1 = U \times U$ and $\beta \in R^\times(V)$ such that $d_1^*\varepsilon^{-1} \cdot d_0^*\varepsilon = \beta^{-1}\beta^\lambda$ holds in $R^\times(V)$.
For $i \in \{0,1\}$, let ψ_i denote the pullback of $\psi : A_U \to \mathrm{M}_{n\times n}(R_U)$ along $d_i : U_1 \to U_0 = U$. Define $\sigma_i : \mathrm{M}_{n\times n}(R_{U_1}) \to \mathrm{M}_{n\times n}(R_{U_1})$ similarly. Let

$$a := \psi_1 \circ \psi_0^{-1} : \mathrm{M}_{n\times n}(R_{U_1}) \to \mathrm{M}_{n\times n}(R_{U_1}).$$

and regard a as an element of $\mathrm{PGL}_n(R)(U_1)$. The fact that $\psi_i^{-1}\sigma_i\psi_i = \tau_{U_1}$ for $i = 0, 1$ implies that $\sigma_1 \circ a = a \circ \sigma_0$. Therefore, using Lemma 5.8, we get

$$d_1^*g \cdot a \quad = \lambda\mathrm{tr} \circ \sigma_1 \circ a = \lambda\mathrm{tr} \circ a \circ \sigma_0 = a^{-\lambda\mathrm{tr}} \circ \lambda\mathrm{tr} \circ \sigma_0 = a^{-\lambda\mathrm{tr}} \cdot d_0^*g\ ,$$

or equivalently, $a^{-\lambda\mathrm{tr}} \cdot d_0^*g \cdot a^{-1} \cdot d_1^*g^{-1} = 1$ in $\mathrm{PGL}_n(R)(U_1)$.
There exists a covering $V \to U_1$ such that the image of a in $\mathrm{PGL}_n(R)(V)$, lifts to

$$b \in \mathrm{GL}_n(R)(V)\ .$$

The relation $a^{-\lambda\mathrm{tr}} \cdot d_0^*g \cdot a^{-1} \cdot d_1^*g^{-1} = 1$ now implies that

$$\beta := b^{-\lambda\mathrm{tr}} \cdot d_0^*h \cdot b^{-1} \cdot d_1^*h^{-1} \in R^\times(V)\ . \tag{5.4}$$

Using (5.3) and (5.4) we get

$$\begin{aligned}
\beta^{-1}\beta^\lambda &= \beta^{-1}(b^{-\lambda\mathrm{tr}} \cdot d_0^*h \cdot b^{-1} \cdot d_1^*h^{-1})^{\lambda\mathrm{tr}} \\
&= \beta^{-1} \cdot d_1^*h^{-\lambda\mathrm{tr}} \cdot b^{-\lambda\mathrm{tr}} \cdot d_0^*h^{\lambda\mathrm{tr}} \cdot b^{-1} \\
&= d_1^*h^{-\lambda\mathrm{tr}} \cdot (b^{-\lambda\mathrm{tr}} \cdot d_0^*h \cdot b^{-1} \cdot d_1^*h^{-1})^{-1} \cdot b^{-\lambda\mathrm{tr}} \cdot d_0^*h^{\lambda\mathrm{tr}} \cdot b^{-1} \\
&= d_1^*h^{-\lambda\mathrm{tr}} \cdot d_1^*h \cdot b \cdot d_0^*h^{-1} \cdot b^{\lambda\mathrm{tr}} \cdot b^{-\lambda\mathrm{tr}} \cdot d_0^*h^{\lambda\mathrm{tr}} \cdot b^{-1} = d_1^*\varepsilon \cdot d_0^*\varepsilon^{-1}
\end{aligned}$$

in $\mathrm{GL}_n(R)(V)$. This establishes the first part of the lemma.
Let t denote the global section determined by $\overline{\varepsilon}$. The construction of t involves choosing U, ψ and $h \in \mathrm{GL}_n(U)$ above. Suppose that $t' \in \mathrm{H}^0(T)$ was obtained by replacing these choices with U', ψ' and $h' \in \mathrm{GL}_n(U')$. We need to show that $t = t'$.
Define $g', \sigma', \varepsilon'$ as above using U', ψ', h' in place of U, ψ, h. It is clear that refining the covering $U \to *$ does not affect t. Therefore, refining $U \to *$ and $U' \to *$ to $U \times U' \to *$, we may assume that $U = U'$. Write $\psi' = u \circ \psi$ with $u \in \mathrm{Aut}_{R_U}(\mathrm{M}_{n\times n}(R_U)) = \mathrm{PGL}_n(R)(U)$. Then $\sigma' = \psi' \circ \tau_U \circ \psi'^{-1} = u \circ \sigma \circ u^{-1}$, and using Lemma 5.8, we get $g' = \lambda\mathrm{tr} \circ \sigma' = u^{-\lambda\mathrm{tr}} g u^{-1}$. Refining $U \to *$ further, if necessary, we may assume that u lifts to $v \in \mathrm{GL}_n(R)(U)$. The relation

$g' = u^{-\lambda \mathrm{tr}} g u^{-1}$ implies that there is $\alpha \in R_U^\times$ such that $h' = \alpha^{-1} v^{-\lambda \mathrm{tr}} h v^{-1}$. Thus,

$$\begin{aligned} \varepsilon' = h'^{-\lambda \mathrm{tr}} h' &= (\alpha^{-1} v^{-\lambda \mathrm{tr}} h v^{-1})^{-\lambda \mathrm{tr}} \alpha^{-1} v^{-\lambda \mathrm{tr}} h v^{-1} \\ &= \alpha^\lambda v h^{-\lambda \mathrm{tr}} v^{\lambda \mathrm{tr}} \alpha^{-1} v^{-\lambda \mathrm{tr}} h v^{-1} = \alpha^\lambda \alpha^{-1} \varepsilon \end{aligned}$$

and $\overline{\varepsilon} = \overline{\varepsilon'}$ in $T(U)$. This completes the proof. □

Definition 5.11. Let (A, τ) be an Azumaya R-algebra with a λ-involution. The *coarse λ-type* or *coarse type* of τ is the global section of T determined by $\overline{\varepsilon} \in T(U)$ constructed above. It shall be denoted $\mathrm{ct}(\tau)$ or $\mathrm{ct}(A, \tau)$.

Remark 5.12. In accordance with Remark 5.2, we shall write the coarse type of a λ-involution τ of an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra, defined to be $\mathrm{ct}(\pi_* \tau)$, as $\mathrm{ct}_\pi(\tau)$.

Proposition 5.13. *Let (A, τ), (A', τ') be Azumaya R-algebras with λ-involutions. Then:*

(i) $\mathrm{ct}(A, \tau) = \mathrm{ct}(\mathrm{M}_{m \times m}(A), \tau \mathrm{tr})$ *for all m.*

(ii) $\mathrm{ct}(\tau \otimes_R \tau') = \mathrm{ct}(\tau) \cdot \mathrm{ct}(\tau')$ *in* $\mathrm{H}^0(T)$.

(iii) If there is a covering $V \to *$ such that $(A_V, \tau_V) \cong (A'_V, \tau'_V)$, then $\mathrm{ct}(\tau) = \mathrm{ct}(\tau')$.

Consequently, the map $\mathrm{t}(\tau) \mapsto \mathrm{ct}(\tau) : \mathrm{Typ}(\lambda) \to \mathrm{cTyp}(\lambda)$ *is a well-defined morphism of monoids.*

Proof. Write $n = \deg A$ and $n' = \deg A'$. Define $U, \psi, g, \sigma, h, \varepsilon$ as in Construction 5.9, and analogously, define $U', \psi', g', \sigma', h', \varepsilon'$ using (A', τ') in place of (A, τ).

(i) The isomorphism $\psi : A_U \to \mathrm{M}_{n \times n}(R_U)$ gives rise to an isomorphism

$$\psi_m : \mathrm{M}_{m \times m}(A)_U \to \mathrm{M}_{m \times m}(\mathrm{M}_{n \times n}(R_U)) = \mathrm{M}_{nm \times nm}(R_U).$$

Let $\sigma_m = \psi_m \circ \tau \mathrm{tr} \circ \psi_m^{-1}$, let $g_m := \lambda \mathrm{tr} \circ \sigma_m$ and let $h_m = (h \oplus \cdots \oplus h) \in \mathrm{GL}_{nm}(R)(U)$. Straightforward computation shows that the image of h_m in $\mathrm{PGL}_{nm}(R)(U)$ is g_m. This means that $\varepsilon_m := h_m^{-\lambda \mathrm{tr}} h_m$ coincides with $\varepsilon = h^{-\lambda \mathrm{tr}} h$ in $N(U)$, and thus $\mathrm{ct}(A, \tau) = \mathrm{ct}(\mathrm{M}_{m \times m}(A), \tau \mathrm{tr})$.

(ii) Consider $\tilde{\psi} = \psi \otimes \psi' : A_U \otimes A'_U \to \mathrm{M}_{n \times n}(R_U) \otimes \mathrm{M}_{n' \times n'}(R_U) = \mathrm{M}_{nn' \times nn'}(R_U)$, let $\tilde{\sigma} = \tilde{\psi} \circ (\tau \otimes \tau') \circ \tilde{\psi}^{-1} = \sigma \otimes \sigma'$, and let $\tilde{g} = \lambda \mathrm{tr} \circ \tilde{\sigma} = g \otimes g'$. Define $\tilde{h} := h \otimes h' \in \mathrm{GL}_{nn'}(R)(U)$. Then $\tilde{h}$ maps onto $\tilde{g}$, and we have $\tilde{h}^{-\lambda \mathrm{tr}} \tilde{h} = (h^{-\lambda \mathrm{tr}} h) \otimes (h'^{-\lambda \mathrm{tr}} h')$, which means $\mathrm{ct}(\tau \otimes_R \tau') = \mathrm{ct}(\tau) \cdot \mathrm{ct}(\tau')$.

(iii) Fix an isomorphism $\eta : (A'_V, \tau'_V) \to (A_V, \tau_V)$ and, in the construction of $\mathrm{ct}(\tau)$, choose a covering $U \to *$ factoring through $V \to *$. Taking $U' = U$ and $\psi' := \psi \circ \eta_U : A'_U \to \mathrm{M}_{n \times n}(R)$ in the construction of $\mathrm{ct}(A', \tau')$, we find that

$$\sigma' = \psi' \tau'_U \psi'^{-1} = \psi \eta_U \tau'_U \eta_U^{-1} \psi^{-1} = \psi \tau \psi^{-1} = \sigma ,$$

so $\mathrm{ct}(\tau) = \mathrm{ct}(\tau')$. □

Remark 5.14. There are examples where $\mathrm{Typ}(\lambda) \to \mathrm{cTyp}(\lambda)$ is not injective. For example, by Proposition 5.7, the image of $\mathrm{Typ}(\lambda) \to \mathrm{cTyp}(\lambda)$ is a subgroup, so this map is not injective when $\mathrm{Typ}(\lambda)$ is not a group, e.g. Example 5.4.

Definition 5.15. An abelian group object G of $\mathbf{Y}$ is said to have *square roots locally* if the the squaring map $x \mapsto x^2 : G \to G$ is an epimorphism. That is, for any object U of $\mathbf{Y}$ and $x \in G(U)$, there exists a covering $V \to U$ and $y \in G(V)$ such that $y^2 = x$.

Example 5.16. If $\mathbf{Y}$ is the topos of a topological space with the ring sheaf of continuous functions into $\mathbb{C}$ or the étale ringed topos of a scheme on which 2 is invertible, then the group $\mathcal{O}_{\mathbf{Y}}^{\times}$ has square roots locally. Indeed, this holds at the stalks, because the stalks of $\mathcal{O}_{\mathbf{Y}}$ are strictly henselian rings in which 2 is invertible—this is well known in the case of an étale ringed topos of a scheme, or proved in Appendix A in the case of a topological space. Furthermore, if $\mathbf{Y}$ is the fppf ringed topos of an arbitrary scheme Y, then $\mathcal{O}_{\mathbf{Y}}^{\times}$ has square roots locally, because any U-section has a square root over a degree-2 finite flat covering of U.

The main result of this section is the following theorem, which shows that under mild assumptions, λ-involutions of Azumaya algebras of the same degree having the same coarse type are locally isomorphic. As a consequence, an analogue of the desired property (i) from Section 5.1 holds under the same assumptions. The theorem holds in particular when $\pi : \mathbf{X} \to \mathbf{Y}$ is induced by a good C_2-quotient of schemes on which 2 is invertible (see Example 4.20), or by a C_2-quotient of Hausdorff topological spaces (see Example 4.21).

Theorem 5.17. *Let $\mathbf{X}$ be a locally ringed topos with involution λ, let $\pi : \mathbf{X} \to \mathbf{Y}$ be an exact quotient relative to λ, and write $R = \pi_* \mathcal{O}_{\mathbf{X}}$ and $S = \mathcal{O}_{\mathbf{Y}}$. Suppose that $S^{\times}$ has square roots locally and at least one of the following conditions holds:*

(1) $2 \in S^{\times}$.

(2) $\pi : \mathbf{X} \to \mathbf{Y}$ *is unramified.*

(3) n *is odd.*

Suppose (A, τ) and (A', τ') are two degree-n Azumaya R-algebras with λ-involutions. Then the following are equivalent:

(a) (A, τ) *and* (A', τ') *are locally isomorphic as R-algebras with involution.*

(b) (A, τ) *and* (A', τ') *have the same type.*

(c) (A, τ) *and* (A', τ') *have the same coarse type.*

Proof. Statement (b) is implied by (a) by virtue of the definition of "type", Definition 5.1. Then the implication of (c) by (b) is Proposition 5.13. The final implication, that of (a) by (c), is somewhat technical and it is given by Proposition 5.32 below. □

Corollary 5.18. *Suppose the assumptions of Theorem 5.17 hold and* $2 \in S^\times$*. Then the map* $\mathrm{t}(\tau) \mapsto \mathrm{ct}(\tau) : \mathrm{Typ}(\lambda) \to \mathrm{cTyp}(\lambda)$ *is injective.*

Proof. Suppose $\mathrm{ct}(A, \tau) = \mathrm{ct}(A', \tau')$ and write $n = \deg A$, $n' = \deg A'$. By Proposition 5.13, we may replace (A, τ) with $(\mathrm{M}_{n'\times n'}(A), \tau\mathrm{tr})$ and (A', τ') with $(\mathrm{M}_{n\times n}(A'), \tau'\mathrm{tr})$, and assume that $\deg A = \deg A'$. Now, by Theorem 5.17, (A, τ) is locally isomorphic to (A', τ') as an R-algebra with involution, and *a fortiori* it has the same type. □

Corollary 5.19. *With the assumptions of Corollary 5.18,* $\mathrm{Typ}(\lambda)$ *is a 2-torsion group.*

Proof. We know $\mathrm{cTyp}(\lambda)$ is a 2-torsion group by Proposition 5.7, and $\mathrm{Typ}(\lambda)$ is a submonoid by Corollary 5.18. □

Remark 5.20. We do not know whether the assumptions of Corollary 5.18 imply that the map $\mathrm{Typ}(\lambda) \to \mathrm{cTyp}(\lambda)$ is surjective. A more extensive discussion of this and some positive results will be given in Subsection 6.4.

The following theorem shows that the properties exhibited in Example 5.3 extend to our general setting under some assumptions.

Theorem 5.21. *With the assumptions of Theorem 5.17, the following hold:*

(i) If $2 \in S^\times$ *and* $\pi : \mathbf{X} \to \mathbf{Y}$ *is a trivial quotient (Example 4.22), then* $\mathrm{Typ}(\lambda)$ *is isomorphic to the group* $\mathrm{H}^0(\mu_{2,R})$*.*

(ii) If $\pi : \mathbf{X} \to \mathbf{Y}$ *is unramified, then* $\mathrm{Typ}(\lambda) = \{1\}$*.*

(iii) Let (A, τ) *be an Azumaya* R*-algebra with a* λ*-involution. If* $\deg A$ *is odd, then* $\mathrm{t}(\tau) = 1$*.*

We deduce Theorem 5.21 mostly as a corollary of Theorem 5.17.

Proof. (i) This follows from Theorem 5.17 and Example 5.5 if we show that for every $\varepsilon \in \mathrm{H}^0(\mu_{2,R}) = \mathrm{H}^0(T)$, there is an involution of coarse type ε. To see that, let $h = \left[\begin{smallmatrix} 0 & \varepsilon \\ 1 & 0 \end{smallmatrix}\right]$ and take $\tau : \mathrm{M}_{2\times 2}(R) \to \mathrm{M}_{2\times 2}(R)$ defined by $x \mapsto (hxh^{-1})^{\lambda\mathrm{tr}}$. That $\mathrm{ct}(\tau) = \varepsilon$ follows by applying Construction 5.9 with $U = *$ and h, ε just defined.
(ii) This follows from Theorem 5.17 and Proposition 5.6.
(iii) It is enough to show that $\mathrm{ct}(\tau) = 1$ whenever $\deg A = 2m + 1$. Define U, h and ε as in Construction 5.9. From (5.3), we have $h = \varepsilon h^{\lambda\mathrm{tr}}$. Taking the determinant of both sides yields $\det h = \varepsilon^{2m+1}(\det h)^\lambda$. Since $\varepsilon^\lambda = \varepsilon^{-1}$, this implies that $\varepsilon = \beta^{-1}\beta^\lambda$ for $\beta = \varepsilon^m \det h^{-1}$. This means that $\overline{\varepsilon}$, the image of ε in $T(U)$, is trivial, so $\mathrm{ct}(\tau) = 1$. □

We note that part (i) applies in the case where $\pi : \mathbf{X} \to \mathbf{Y}$ is induced by a scheme X on which 2 is invertible endowed with the trivial involution $\lambda = \mathrm{id} : X \to X$; see Example 5.3(i).

Part (ii) applies to the case where $\pi : \mathbf{X} \to \mathbf{Y}$ is induced by a quadratic étale morphism of schemes $\pi : X \to Y$ where X is given the canonical Y-involution; see Example 5.3(ii).

Our last application of Theorem 5.17 provides a concrete realization of the first cohomology set of the projective unitary group of an Azumaya R-algebra with a λ-involution (A, τ). As usual, the unitary group of (A, τ) is the group object $\mathrm{U}(A, \tau)$ in $\mathbf{Y}$ whose V-sections are $\{a \in A(V) : a^\tau a = 1\}$, and the projective unitary group of (A, τ) is the quotient

$$\mathrm{PU}(A, \tau) = \mathrm{U}(A, \tau)/N$$

where N, the group of λ-norm 1 elements in R defined above.
If (A', τ') is another Azumaya R-algebra with a λ-involution such that $\deg A = \deg A'$, we further define $\mathcal{H}om_R((A, \tau), (A', \tau'))$ to be the sheaf of R-linear isomorphisms from (A, τ) to (A', τ'), and $\mathcal{A}ut_R(A, \tau)$ to be the group sheaf of R-linear automorphisms of (A, τ).

Lemma 5.22. *Suppose $S^\times$ has square roots locally. Let (A, τ) be a degree-n Azumaya R-algebra with a λ-involution. Then the map $\mathrm{U}(A, \tau) \to \mathcal{A}ut_R(A, \tau)$ sending a section x to conjugation by x is an epimorphism with kernel N. Consequently, it induces an isomorphism $\mathrm{PU}(A, \tau) \cong \mathcal{A}ut_R(A, \tau)$.*

Proof. That the kernel is N follows easily from the fact that the centre of A is R. We turn to proving that the map is an epimorphism.
Let $V \in \mathbf{Y}$ and $\psi \in \mathcal{A}ut_R(A, \tau)(V) = \mathrm{Aut}_{R_V}(A_V, \tau_V)$. By replacing V with a suitable covering, we may assume that $A_V = \mathrm{M}_{n\times n}(R_V)$ and that $\psi_V \in \mathrm{PGL}_n(R)(V)$ is given sectionwise by $\psi_V(x) = hxh^{-1}$ for some $h \in \mathrm{GL}_n(R)(V)$. Since $\psi_V \circ \tau_V = \tau_V \circ \psi_V$, for any section $x \in A(V)$, we have $hx^\tau h^{-1} = (h^{-1})^\tau x^\tau h^\tau$, and thus $h^\tau h \in R^\times(V)$. In fact, since $(h^\tau h)^\lambda = h^\tau h$, we have $h^\tau h \in S^\times(V)$. By assumption, we can replace V with a suitable covering to assume that there is $\alpha \in S^\times(V)$ with $\alpha^2 = h^\tau h$. Replacing h with $h\alpha^{-1}$ yields $hh^\tau = 1$. We have therefore shown that over a covering of V, ψ lifts to a section of $\mathrm{U}(A, \tau)$. □

Corollary 5.23. *With the assumptions of Theorem 5.17, let (A, τ) be a degree-n Azumaya R-algebra with a λ-involution and identify $\mathcal{A}ut_R(A, \tau)$ with $\mathrm{PU}(A, \tau)$ as in Lemma 5.22. Then the functor $(A', \tau') \mapsto \mathcal{H}om_R((A, \tau), (A', \tau'))$ defines an equivalence between the full subcategory of $\mathbf{Az}_n(\mathbf{Y}, R, \lambda)$ consisting of R-algebras with a λ-involution of the same type as τ and $\mathbf{Tors}(\mathbf{Y}, \mathrm{PU}(A, \tau))$. Consequently, $\mathrm{H}^1(\mathbf{Y}, \mathrm{PU}(A, \tau))$ is in canonical bijection with isomorphism classes of the aforementioned algebras with involution.*

Proof. Theorem 5.17 implies that an R-algebra with a λ-involution (A', τ') is locally isomorphic to (A, τ) if and only if A is Azumaya of degree n and τ is of the same type as τ'. With this fact at hand, the equivalence is standard; see [Gir71, V.§4]. The last statement follows from Proposition 2.8(i). □

Many of the previous results require that $2 \in S^\times$. Indeed, our arguments build on using the coarse type, which is too coarse if 2 is not invertible—see Remark 5.14. Nevertheless, we ask:

Question 5.24. Does the equivalence between (a) and (b) in Theorem 5.17 hold without assuming any of the conditions (1), (2), (3)?

A particularly interesting case is the morphism of *fppf* ringed topoi associated to a finite locally free good C_2-quotient $\pi : X \to Y$, where X is a scheme on which 2 is not invertible (consult Remark 4.40).

5.3 Proof of Theorem 5.17

In this subsection we complete the proof of Theorem 5.17 by showing that condition (c) implies condition (a). This result is given as Proposition 5.32 below. The reader can skip this subsection without loss of continuity.

In the following lemmas, unless otherwise specified, A will be a ring, $\lambda : A \to A$ an involution, and B will be the fixed ring of λ. We will write $\overline{A}$ for $A/\mathrm{Jac}(A)$, and $\overline{\lambda} : \overline{A} \to \overline{A}$ for the involution induced on $\overline{A}$ by λ. Let $\varepsilon \in \{\pm 1\}$ and let $h \in \mathrm{GL}_n(A)$ be an $(\varepsilon, \lambda\mathrm{tr})$-hermitian matrix, which is to say

$$h = \varepsilon h^{\lambda\mathrm{tr}}.$$

Let $H : A^n \times A^n \to A$ be the (ε, λ)-hermitian form associated to h; it is given by $h(x, y) = x^{\lambda\mathrm{tr}} hy$ where $x, y \in A^n$ are written as column vectors. Let $\overline{H}$ denote the reduction of H to $\overline{A}$.

Lemma 5.25. *Assume B is local. If $\overline{\lambda} \neq \mathrm{id}_{\overline{A}}$, or $\varepsilon \neq -1$ in $\overline{A}$, or n is odd, then there exists $v \in \mathrm{GL}_n(A)$ such that $v^{\lambda\mathrm{tr}} hv$ is a diagonal matrix.*

Proof. Proving the lemma is equivalent to showing that H is diagonalizable, i.e., has an orthogonal basis.

We first claim that $\overline{H}$ has an orthogonal basis. This is well known when $\overline{A}$ is a field; see [Sch85, Thm. 7.6.3] for the case where $\overline{\lambda} \neq \mathrm{id}$ or $\varepsilon \neq -1$ in $\overline{A}$, and [Alb38, Thm. 6] for the case where $\overline{\lambda} = \mathrm{id}_A$, $\varepsilon = -1$ in $\overline{A}$ and n is odd. We note in passing that the second case can occur only when the characteristic of $\overline{A}$ is 2. If $\overline{A}$ is not a field, then Theorem 3.16 and Proposition 3.4(ii) imply that $\overline{A} \cong k \times k$, where k is the residue field of B, and $\overline{\lambda}$ acts by interchanging the two copies of k. In this case $\overline{H}$ is hyperbolic and the easy proof is left to the reader.

We now claim that any nondegenerate (ε, λ)-hermitian form H whose reduction $\overline{H}$ admits a diagonalization is diagonalizable, thus proving the lemma. Let $\{\overline{x}_1, \ldots, \overline{x}_n\} \subseteq \overline{A}^n$ be an orthogonal basis for $\overline{H}$ and let $x_1 \in A^n$ be an arbitrary lift of $\overline{x}_1$. Then $H(x_1, x_1) \in A^\times$ and hence $A^n = x_1 A \oplus x_1^\perp$. Write $P = x_1^\perp$ and $H_1 = H|_{P \times P}$. The A-module P is free because A is semilocal and P is projective of constant A-rank $n - 1$. Furthermore, since $\overline{P} = \sum_{i=2}^n \overline{x}_i \overline{A}$, the form $\overline{H_1}$ is diagonalizable by construction. We finish by applying induction to H_1. $\square$

Lemma 5.26. *Assume B is local, and suppose $\overline{\lambda} = \mathrm{id}_{\overline{A}}$, that $\varepsilon = -1$ and that $2 \in A^\times$. Then there exists $v \in \mathrm{GL}_n(A)$ such that $v^{\lambda\mathrm{tr}} hv$ is a direct sum of 2×2 matrices in $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] + \mathrm{M}_{2\times 2}(\mathrm{Jac}(A))$. In particular, n is even.*

Proof. We need to show that A^n admits a basis $\{x_1, y_1, x_2, y_2, \dots\}$ such that $x_i A + y_i A$ is orthogonal to $x_j A + y_j A$ whenever $i \neq j$ and such that $\left[\begin{smallmatrix} H(x_i,x_i) & H(x_i,y_i) \\ H(y_i,x_i) & H(y_i,y_i) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] + \mathrm{M}_{2\times 2}(\mathrm{Jac}(A)))$.
By Theorem 3.16, the assumption $\overline{\lambda} = \mathrm{id}_{\overline{A}}$ implies that A is local, hence $\overline{A}$ is a field of characteristic different from 2 and $\overline{H}$ is a nondegenerate alternating bilinear form. This means that n must be even. Choose a nonzero $\overline{x} \in \overline{A}^n$. Since $\overline{H}$ is nondegenerate, there is $\overline{y}$ such that $\overline{H}(\overline{x}, \overline{y}) = 1$. Since $\overline{H}$ is alternating, we also have $\overline{H}(\overline{y}, \overline{x}) = -1$ and $\overline{H}(\overline{x}, \overline{x}) = \overline{H}(\overline{y}, \overline{y}) = 0$. Let $x, y \in A^n$ be lifts of $\overline{x}$ and $\overline{y}$. The previous equations imply that $M := \left[\begin{smallmatrix} H(x,x) & H(x,y) \\ H(y,x) & H(y,y) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] + \mathrm{M}_{2\times 2}(\mathrm{Jac}(A))$. In particular, M is invertible, and so $A^n = (xA \oplus yA) \oplus \{x, y\}^\perp$. We proceed by induction on the restriction of H to $\{x, y\}^\perp$. □

Lemma 5.27. *Assume B is local and $2 \in B^\times$. Then $\mathrm{Jac}(A)^2 \subseteq \mathrm{Jac}(B)A$.*

Proof. Write $\mathfrak{m} = \mathrm{Jac}(B)$ and let $x, y \in \mathrm{Jac}(A)$. Then $x^\lambda + x, x^\lambda x \in \mathrm{Jac}(A) \cap B \subseteq \mathfrak{m}$. The equality $x^2 - (x^\lambda + x)x + (x^\lambda x) = 0$ implies $x^2 \in \mathfrak{m}A$. Likewise, $y^2, (x+y)^2 \in \mathfrak{m}A$. We finish by noting that $xy = \frac{1}{2}((x+y)^2 - x^2 - y^2)$. □

In the following lemmas, given a ring A and $a \in A$, we write $A[\sqrt{a}]$ to denote the ring $A[T]/(T^2 - a)$ and let $\sqrt{a}$ denote the image of T in $A[\sqrt{a}]$. By induction, we define $A[\sqrt{a_1}, \sqrt{a_2}, \dots] = A[\sqrt{a_1}][\sqrt{a_2}, \dots] \cong A[T_1, T_2, \dots]/(T_1^2 - a_1, T_2^2 - a_2, \dots)$. If $\lambda : A \to A$ is an involution with fixed ring B and $a \in B$, then λ extends to $A[\sqrt{a}]$ by setting $(\sqrt{a})^\lambda = \sqrt{a}$, and the fixed ring of $\lambda : A[\sqrt{a}] \to A[\sqrt{a}]$ is $B[\sqrt{a}]$.

Lemma 5.28. *Assume B is local and suppose $\overline{\lambda} = \mathrm{id}_{\overline{A}}$, that $\varepsilon = -1$ and $2 \in A^\times$. Suppose that h lies in $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] + \mathrm{M}_{2\times 2}(\mathrm{Jac}(A))$. Then there are $s, t \in B^\times$, $f_1, \dots, f_r \in B[\sqrt{s}, \sqrt{t}]$ and $v_i \in \mathrm{GL}_2(A[\sqrt{s}, \sqrt{t}]_{f_i})$ $(i = 1, \dots, r)$ such that $\sum_i f_i B[\sqrt{s}, \sqrt{t}] = B[\sqrt{s}, \sqrt{t}]$ and $v_i^{\lambda\mathrm{tr}} hv_i = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ in $A[\sqrt{s}, \sqrt{t}]_{f_i}$ for all i.*

Proof. Again, by Theorem 3.16, A is local. We write $\mathfrak{m} = \mathrm{Jac}(B)$, $\mathfrak{M} = \mathrm{Jac}(A)$ and let $\{x, y\}$ denote the standard A-basis of A^2. Once $s, t \in S$ and $f_1, \dots, f_r \in B[\sqrt{s}, \sqrt{t}]$ above have been chosen, we need to show that for all i, there are $\tilde{x}, \tilde{y} \in A[\sqrt{s}, \sqrt{t}]^2_{f_i}$ such that $\left[\begin{smallmatrix} H(\tilde{x},\tilde{x}) & H(\tilde{x},\tilde{y}) \\ H(\tilde{y},\tilde{x}) & H(\tilde{y},\tilde{y}) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.
Our assumption on h implies that $H(x, y) \in 1 + \mathrm{Jac}(A) \subseteq A^\times$. Replacing y with $H(x, y)^{-1} y$, we may assume that $H(x, y) = 1$ and so $H(y, x) = -1$. We further write $\alpha = H(x, x)$ and $\beta = H(y, y)$. Since $h = -h^{\lambda\mathrm{tr}}$, we have $\alpha^\lambda = -\alpha$ and $\beta^\lambda = -\beta$, and since $\overline{\lambda} = \mathrm{id}_{\overline{A}}$, it follows that $\alpha, \beta \in \mathfrak{M}$ and hence $\alpha^2, \beta^2 \in B \cap \mathfrak{M} \subseteq \mathfrak{m}$.
Observe that the polynomial $p(x) = -2x^2 + (2 - 2\alpha)x + \alpha \in A[x]$ has discriminant

$$s := (2 - 2\alpha)^2 - 4(-2\alpha) = 4 + 4\alpha^2$$

and that $s \in 4+\mathfrak{m} \subseteq B^\times$. The roots of $p(x)$ in $A[\sqrt{s}]$ are $c_1 = \frac{1}{4}(\sqrt{s}+2-2\alpha)$ and $c_2 = \frac{1}{4}(-\sqrt{s}+2-2\alpha)$ and we note that

$$16(c_1^\lambda c_1 + c_2^\lambda c_2) = (\sqrt{s}+2)^2 - 4\alpha^2 + (\sqrt{s}-2)^2 - 4\alpha^2 = 16 \in B^\times \; . \tag{5.5}$$

We repeat this construction with β in place of α, denoting the elements corresponding to s, c_1, c_2 by t, d_1, d_2.
Fix a maximal ideal $\mathfrak{p} \lhd B[\sqrt{s},\sqrt{t}]$ and write $B' := B[\sqrt{s},\sqrt{t}]_\mathfrak{p}$, $A' := A[\sqrt{s},\sqrt{t}]_\mathfrak{p}$, $\mathfrak{m}' = \mathrm{Jac}(B')$ and $\mathfrak{M}' = \mathrm{Jac}(A')$.
It is clear that B' is local. We claim that A' is also local and $\mathfrak{M}A' \subseteq \mathfrak{M}'$. Indeed, by [Rei75, Thm. 6.15], we have $\mathfrak{m}B[\sqrt{s},\sqrt{t}] \subseteq \mathrm{Jac}(B[\sqrt{s},\sqrt{t}]) \subseteq \mathfrak{p}$, and hence $\mathfrak{m}B' \subseteq \mathfrak{p}_\mathfrak{p} = \mathfrak{m}'$, which in turn implies $\mathfrak{m}A' \subseteq \mathfrak{m}'A'$. By Lemma 3.18, we have $\mathfrak{m}'A' \subseteq \mathfrak{M}'$, and by Lemma 5.27, $\mathfrak{M}^2 \subseteq \mathfrak{m}A$. Using the last three inclusions, we get $(\mathfrak{M}A')^2 = \mathfrak{M}^2A' \subseteq \mathfrak{m}A' \subseteq \mathfrak{m}'A' \subseteq \mathfrak{M}'$, and since $\mathfrak{M}'$ is semiprime, $\mathfrak{M}A' \subseteq \mathfrak{M}'$. The latter implies that $A' \to A'/\mathfrak{M}'$ factors through $A'/\mathfrak{M}A' \cong \overline{A} \otimes_B B'$, and hence the specialization of λ to $A'/\mathfrak{M}'$ is the identity. Since B' is flat over B, the local ring B' is the fixed ring of $\lambda : A' \to A'$ and Theorem 3.16 implies that A' is local.
Now, the inclusion $\mathfrak{M}A' \subseteq \mathfrak{M}'$ implies $\alpha, \beta \in \mathfrak{M}'$. By equation (5.5), there is $i \in \{1,2\}$ such that $c_i^\lambda c_i \in B'^\times$, and hence $c_i \in A'^\times$. In the same way, there is $j \in \{1,2\}$ such that $d_j \in A'^\times$.
Working in A', we have

$$\begin{aligned} c_i^{-1} + (c_i^{-1})^\lambda &= \frac{4}{\pm\sqrt{s}+2-2\alpha} + \frac{4}{\pm\sqrt{s}+2+2\alpha} \\ &= \frac{8(\pm\sqrt{s}+2)}{(\pm\sqrt{s}+2)^2 - 4\alpha^2} \\ &= \frac{8(\pm\sqrt{s}+2)}{s \pm 4\sqrt{s} + 4 - 4\alpha^2} \\ &= \frac{8(\pm\sqrt{s}+2)}{\pm 4\sqrt{s}+8} = 2 \end{aligned} \tag{5.6}$$

and likewise for d_j. Write $u = c_i^{-1} - 1$. Since $\alpha(c_i^{-1})^2 + (2-2\alpha)c_i^{-1} - 2 = c_i^{-2}p(c_i) = 0$, we have

$$\alpha u^2 + 2u - \alpha = 0 \; , \tag{5.7}$$

and since $\alpha \in \mathfrak{M}'$, this implies $u \in \mathfrak{M}'$. We further have $u^\lambda = -u$ by (5.6). In the same way, writing $w = d_j^{-1} - 1$, we have $\beta w^2 + 2w - \beta = 0$, $w \in \mathfrak{M}'$ and $w^\lambda = -w$. Let $\tilde{x} = (1+u)x + (1-w)y$. Then, using (5.7), we get

$$\begin{aligned} H(\tilde{x},\tilde{x}) &= (1+u)^\lambda(1+u)\alpha + (1+u)^\lambda(1-w) - (1-w)^\lambda(1+u) \\ &\quad + (1-w)^\lambda(1-w)\beta \\ &= (1-u^2)\alpha + (1-u)(1-w) - (1+w)(1+u) + (1-w^2)\beta \\ &= \alpha - \alpha u^2 - 2u + \beta - \beta w^2 - 2w = 0. \end{aligned}$$

Likewise, $\tilde{y} := (1-u)x - (1+w)y$ satisfies $H(\tilde{y},\tilde{y}) = 0$. Since $u, w \in \mathfrak{M}' = \mathrm{Jac}(A')$, the vectors $\tilde{x}, \tilde{y}$ span A'^2. Since H is nondegenerate, this forces $H(\tilde{x},\tilde{y}) \in A'^\times$. Replacing $\tilde{y}$ with $H(\tilde{x},\tilde{y})^{-1}\tilde{y}$, we find that $\left[\begin{smallmatrix} H(\tilde{x},\tilde{x}) & H(\tilde{x},\tilde{y}) \\ H(\tilde{y},\tilde{x}) & H(\tilde{y},\tilde{y}) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

Finally, for every maximal ideal $\mathfrak{p} \lhd B[\sqrt{s},\sqrt{t}]$, choose $f_\mathfrak{p} \in B[\sqrt{t},\sqrt{s}] - \mathfrak{p}$ such that the coefficients of $\tilde{x}$, $\tilde{y}$ constructed above are defined in $B[\sqrt{t},\sqrt{s}]_{f_\mathfrak{p}}$ and such that the identity $\left[\begin{smallmatrix} H(\tilde{x},\tilde{x}) & H(\tilde{x},\tilde{y}) \\ H(\tilde{y},\tilde{x}) & H(\tilde{y},\tilde{y}) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ holds in $B[\sqrt{s},\sqrt{t}]_{f_\mathfrak{p}}$. Then $\sum_\mathfrak{p} f_\mathfrak{p} B[\sqrt{s},\sqrt{t}] = B[\sqrt{s},\sqrt{t}]$. Since $B[\sqrt{s},\sqrt{t}]$ is a finite algebra over a local ring, it has only finitely many maximal ideals. The result follows. □

Lemma 5.29. *Maintaining the assumptions made at the beginning of this subsection, suppose that $h' \in \mathrm{GL}_n(A)$ is another $(\varepsilon, \lambda\mathrm{tr})$-hermitian matrix. Suppose further we are given a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, B$, units $s_1, \dots, s_\ell \in B_\mathfrak{p}^\times$, elements $f_1, \dots, f_r \in B_\mathfrak{p}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$ and $v_i \in \mathrm{GL}_n(A_\mathfrak{p}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]_{f_i})$ ($i = 1, \dots, r$) such that $f_1, \dots, f_r$ generate the unit ideal in $B_\mathfrak{p}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$ and such that $v_i^{\lambda\mathrm{tr}} h v_i = h'$ for all $1 \le i \le r$. Then there is $b \in B - \mathfrak{p}$ for which the previous condition holds upon replacing $B_\mathfrak{p}$, $A_\mathfrak{p}$ with B_b, A_b.*

Proof. There is $b \in B$ such that $s_1, \dots, s_\ell$ are in the image of $B_b^\times \to B_\mathfrak{p}^\times$. We may replace B, $\mathfrak{p}$ with B_b, $\mathfrak{p}_b$ and assume $s_1, \dots, s_\ell \in B^\times$ henceforth.

Write $B' = B[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$ and $A' = A[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$, and choose $g_1, \dots, g_r \in B'_\mathfrak{p}$ such that $\sum_i f_i g_i = 1$. Then there is $b \in B - \mathfrak{p}$ such that $f_1, \dots, f_r, g_1, \dots, g_r$ are images of elements in B'_b, also denoted $f_1, \dots, f_r, g_1, \dots, g_r$, and such that $\sum_i g_i f_i = 1$ in B'_b. Again, we replace B, $\mathfrak{p}$ with B_b, $\mathfrak{p}_b$ and assume $f_1, \dots, f_r \in B'$.

Fix $1 \le i \le r$. There are $v'_i \in \mathrm{M}_{n\times n}(A')$, $b \in B - \mathfrak{p}$ and $m \in \mathbb{N} \cup \{0\}$ such that $v_i = v'_i b^{-1} f_i^{-m}$ in $\mathrm{M}_{n\times n}((A'_\mathfrak{p})_{f_i})$. Since $v_i^{\lambda\mathrm{tr}} h v_i = h'$, we have $v_i'^{\lambda\mathrm{tr}} h v'_i = b^2 f_i^{2m} h'$ in $\mathrm{M}_{n\times n}((A'_\mathfrak{p})_{f_i})$, and hence there is $k \in \mathbb{N} \cup 0$ such that $f_i^k v_i'^{\lambda\mathrm{tr}} h v'_i = b^2 f_i^{2m+k} h'$ in $\mathrm{M}_{n\times n}(A'_\mathfrak{p})$. This in turn implies that there is $b' \in B - \mathfrak{p}$ such that $b' f_i^k v_i'^{\lambda\mathrm{tr}} h v'_i = b' b^2 f_i^{2m+k} h'$ in $\mathrm{M}_{n\times n}(A')$. Replacing B, $\mathfrak{p}$ with $B_{bb'}$, $\mathfrak{p}_{bb'}$, we may assume $b, b' \in B^\times$. Let $\tilde{v}_i = v'_i b^{-1} f_i^{-m} \in \mathrm{M}_{n\times n}(A'_{f_i})$. Then the image of $\tilde{v}_i$ in $\mathrm{M}_{n\times n}((A'_\mathfrak{p})_{f_i})$ is v_i and the equality $b' f_i^k v_i'^{\lambda\mathrm{tr}} h v'_i = b' b^2 f_i^{2m+k} h'$ implies that $\tilde{v}_i^{\lambda\mathrm{tr}} h \tilde{v}_i = h'$ in $\mathrm{M}_{n\times n}(A'_{f_i})$. Taking determinants, we see that $\tilde{v}_i \in \mathrm{GL}_n(A'_{f_i})$. The lemma follows by applying the previous paragraph to all $1 \le i \le r$. □

We now return to the context of ringed topoi.

Lemma 5.30. *Assume $2 \in S^\times$, and let $U \in \mathbf{Y}$, $\varepsilon \in N(U)$ and $t = \overline{\varepsilon} \in T(U)$. Then:*

(i) There is a covering $\{V_i \to U\}_{i=1,2}$ of U such that $1 + \varepsilon \in R^\times(V_1)$ and $1 - \varepsilon \in R^\times(V_2)$.

(ii) There exists a covering $\{V_i \to U\}_{i=1,2}$ of U such that $t|_{V_1} = 1$ and $t|_{V_2} = -1$. Equivalently, there exists a covering $\{V_i \to U\}_{i=1,2}$ of U and $\beta_i \in R(V_i)$ ($i = 1, 2$) such that $\beta_1^{-1}\beta_1^\lambda = \varepsilon|_{V_1}$ and $-\beta_2^{-1}\beta_2^\lambda = \varepsilon|_{V_2}$.

Proof. (i) We note that this statement requires proof because R is not a local ring object in general. Observe that $\varepsilon^{-1}(1\pm\varepsilon)^2 = \varepsilon^\lambda \pm 2 + \varepsilon$ and hence $\varepsilon^{-1}(1\pm\varepsilon)^2 \in S(U)$. Since $\varepsilon^{-1}(1+\varepsilon)^2 - \varepsilon^{-1}(1-\varepsilon)^2 = 4$ and S is a local ring object, the assumption $2 \in S^\times$ implies that there exists a covering $\{V_i \to U\}_{i=1,2}$ of U such that $\varepsilon^{-1}(1+\varepsilon)^2 \in S^\times(V_1)$ and $\varepsilon^{-1}(1-\varepsilon)^2 \in S^\times(V_2)$. It follows that $1+\varepsilon \in R^\times(V_1)$ and $1-\varepsilon \in R^\times(V_2)$.

(ii) Choose a covering $\{V_i \to U\}_{i=1,2}$ as in (i) and let $\beta_1 = \varepsilon^{-1}(1+\varepsilon)|_{V_1}$, $\beta_2 = \varepsilon^{-1}(1-\varepsilon)|_{V_2}$. Since $\varepsilon^\lambda = \varepsilon^{-1}$, we have $\beta_1^\lambda \varepsilon^{-1} = \beta_1$, and hence $\beta_1^{-1}\beta_1^\lambda = \varepsilon$. Likewise, $\beta_2^{-1}\beta_2^\lambda = -\varepsilon$. □

The following lemma is known when $\pi : \mathbf{X} \to \mathbf{Y}$ is unramified or trivial, i.e., when R is a quadratic étale S-algebra or $R = S$. The ramified situation that we consider appears not to have been considered before in the literature.

Lemma 5.31. *Let $U \in \mathbf{Y}$, let $\varepsilon \in N(U)$, and let $h, h' \in \mathrm{GL}_n(R)(U)$ be two $(\varepsilon, \lambda\mathrm{tr})$-hermitian matrices, i.e., $h = \varepsilon h^{\lambda\mathrm{tr}}$ and $h' = \varepsilon h'^{\lambda\mathrm{tr}}$. Assume $S^\times$ has square roots locally and that $2 \in S^\times$, or π is unramified, or n is odd. Then there exists a covering $V \to U$ and $v \in \mathrm{GL}_n(R)(V)$ such that $v^{\lambda\mathrm{tr}} h v = h'$ in $\mathrm{GL}_n(R)(V)$.*

Proof. Suppose first that $2 \in S^\times$. Let $\{V_i \to U\}_{i=1,2}$ and β_1, β_2 be as in Lemma 5.30(ii). We may replace h, h', U with $(\beta_i h, \beta_i h', V_i)_{i=1,2}$ and assume that $\varepsilon \in \{\pm 1\}$ henceforth.

We claim that it is enough to show that for all $\mathfrak{p} \in \operatorname{Spec} S(U)$, there are $s_1, \dots, s_\ell \in S(U)_\mathfrak{p}^\times$, $f_1, \dots, f_r \in S(U)_\mathfrak{p}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$ and $v_j \in \mathrm{GL}_n(R(U)_\mathfrak{p}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]_{f_j})$ $(j = 1, \dots, r)$ such that $f_1, \dots, f_r$ generate the unit ideal in $S(U)_\mathfrak{p}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$ and $v_j^{\lambda\mathrm{tr}} h v_j = h'$. If this holds, then Lemma 5.29 implies that for all $\mathfrak{p}$, we can find $b_\mathfrak{p} \in S(U) - \mathfrak{p}$ such that the previous condition holds upon replacing $S(U)_\mathfrak{p}, R(U)_\mathfrak{p}$ with $S(U)_{b_\mathfrak{p}}, R(U)_{b_\mathfrak{p}}$. Since $\sum_\mathfrak{p} b_\mathfrak{p} S(U) = S(U)$ and S is a local ring object, there is a covering $\{V_\mathfrak{p} \to U\}_\mathfrak{p}$ such that $b_\mathfrak{p} \in S(V_\mathfrak{p})^\times$. Fix some $\mathfrak{p}$ and let $b_\mathfrak{p}, s_1, \dots, s_\ell, f_1, \dots, f_r, v_1, \dots, v_r$ be as above. By construction, the images of $s_1, \dots, s_\ell$ in $S(V_\mathfrak{p})$ are invertible in $S(V_\mathfrak{p})$. Since $S^\times$ has square roots locally, we can replace $V_\mathfrak{p}$ with a suitable covering such that $s_1, \dots, s_\ell$ have square roots in $S(V_\mathfrak{p})$. In particular, $S(U) \to S(V_\mathfrak{p})$ factors through $S(U) \to S(U)_{b_\mathfrak{p}}[\sqrt{s_1}, \dots, \sqrt{s_\ell}]$. Applying the fact that S is a local ring object again, we see that there is a covering $\{V_{\mathfrak{p},i} \to V_\mathfrak{p}\}_{i=1}^r$ such that the image of f_i in $S(V_{\mathfrak{p},i})$ is invertible. It follows that $R(U) \to R(V_{\mathfrak{p},i})$ factors through $R(U)_{b_\mathfrak{p}}[\sqrt{s_1}, \dots, s_{s_\ell}]_{f_i}$ and hence there is $v_{\mathfrak{p},i} \in \mathrm{GL}_n(R(V_{\mathfrak{p},i}))$ — the image of v_i — such that $v_{\mathfrak{p},i}^{\lambda\mathrm{tr}} h v_{\mathfrak{p},i} = h'$. Finally, let $V = \bigsqcup_{\mathfrak{p},i} V_{\mathfrak{p},i}$ and take $v = (v_{\mathfrak{p},i})_{\mathfrak{p},i} \in \mathrm{GL}_n(R(V)) = \prod_{\mathfrak{p},i} \mathrm{GL}_n(R(V_{\mathfrak{p},i}))$.

Let $\mathfrak{p} \in \operatorname{Spec} S(U)$. We now prove the existence of $s_1, \dots, s_\ell, f_1, \dots, f_r$ and $v_1, \dots, v_r$ above. Write $B = S(U)_\mathfrak{p}$ and $A = R(U)_\mathfrak{p}$. Then B is local and it is the fixed ring of $\lambda : A \to A$. We write $\overline{A} = A/\mathrm{Jac}(A)$ and let $\overline{\lambda} : \overline{A} \to \overline{A}$ denote the involution induced by λ.

Suppose $\varepsilon = 1$ or $\overline{\lambda} \neq \mathrm{id}$. By Lemma 5.25, we may assume that h and h' are diagonal, say $h = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ and $h' \in \mathrm{diag}(\alpha'_1, \dots, \alpha'_n)$. Since h

and h' are $(\varepsilon,\lambda\mathrm{tr})$-hermitian, we have $(\alpha_i^{-1}\alpha_i')^\lambda = \alpha_i^{-1}\varepsilon\alpha_i'\varepsilon^{-1} = \alpha_i^{-1}\alpha_i'$, hence $\alpha_i^{-1}\alpha_i' \in B^\times$ for all i. Writing $s_i = \alpha_i^{-1}\alpha_i'$ and $v = \mathrm{diag}(\sqrt{s_1},\dots,\sqrt{s_n}) \in \mathrm{GL}_n(A[\sqrt{s_1},\dots,\sqrt{s_n}])$, we have $v^{\lambda\mathrm{tr}}hv = h'$, as required (take $f_1 = 1$).

Suppose now that $\varepsilon = -1$ and $\overline{\lambda} = \mathrm{id}$. Applying Lemma 5.26 to h and h', we may assume that h and h' are direct sums of 2×2 matrices in $\left[\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right] + \mathrm{M}_{2\times 2}(\mathrm{Jac}(A))$, say $h = h_1 \oplus\cdots\oplus h_m$, $h' = h_{m+1}\oplus\cdots\oplus h_n$ with $m = n/2$. Applying Lemma 5.28 to h_j, we obtain $s_j, t_j \in B^\times$, $f_{j1},\dots,f_{jr_j} \in B[\sqrt{s_j},\sqrt{t_j}]$, and $v_{ji} \in \mathrm{GL}_2(A[\sqrt{s_j},\sqrt{t_j}]_{f_{ji}})$ such that $v_{ji}^{\lambda\mathrm{tr}} h_j v_{ji} = \left[\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right]$.

Let $B' = B[\sqrt{s_1},\sqrt{t_1},\dots,\sqrt{s_n},\sqrt{t_n}]$, $A' = A[\sqrt{s_1},\sqrt{t_1},\dots,\sqrt{s_n},\sqrt{t_n}]$ and regard $\{f_{ji}\}_{j,i}$ as elements of B'. For every tuple $I = (i_1,\dots,i_n) \in \prod_j\{1,\dots,r_j\}$, let $f_I = \prod_j f_{ji_j}$, $v_I = v_{1i_1}\oplus\cdots\oplus v_{mi_m}$ and $v_I' = v_{(m+1)i_{m+1}}\oplus\cdots\oplus v_{ni_n}$, where v_I, v_I' are regarded as elements of $\mathrm{GL}_n(A'_{f_I})$. Then $\sum_I f_I B' = \prod_j(\sum_{i=1}^{r_j} f_{ji}B') = B'$ and $(v_I v_I'^{-1})^{\lambda\mathrm{tr}} h (v_I v_I'^{-1}) = h'$, which is what we want. This establishes the lemma when $2 \in S^\times$.

To prove the remaining cases, we observe that when π is unramified, or n is odd, the use of Lemmas 5.26, 5.28 and 5.30 can be avoided, and hence the assumption $2 \in S^\times$ is unnecessary.

When π is unramified, we apply Proposition 5.6 instead of Lemma 5.30(ii) and assume $\varepsilon = 1$ hereafter. Since in this case A is a quadratic étale B-algebra, Proposition 3.4(ii) implies that $\overline{\lambda} \neq \mathrm{id}$, and so the case $\overline{\lambda} = \mathrm{id}$ does not occur.

Suppose now that n is odd, say $n = 2m+1$. Taking the determinant of both sides of $h = \varepsilon h^{\lambda\mathrm{tr}}$ yields $\det h = \varepsilon^{2m+1}(\det h)^\lambda$. Since $\varepsilon^\lambda = \varepsilon^{-1}$, this implies that $\varepsilon = \beta^{-1}\beta^\lambda$ for $\beta = \varepsilon^m(\det h)^\lambda$. Replacing h, h' with $\beta h, \beta h'$, we may assume $\varepsilon = 1$. Since n is odd, we can now apply Lemma 5.25 even when $\overline{\lambda} = \mathrm{id}$ and finish the proof without using Lemmas 5.26, 5.28 or 5.30. □

We can finally complete the proof of Theorem 5.17.

PROPOSITION 5.32. *Let n be a positive integer. Suppose $S^\times$ has square roots locally and at least one of the following holds:*

(1) $2 \in S^\times$.

(2) $\pi : \mathbf{X} \to \mathbf{Y}$ *is unramified.*

(3) n *is odd.*

Let (A,τ) and (A',τ') be two degree-n Azumaya R-algebras with λ-involutions having the same coarse type. Then (A,τ) and (A',τ') are locally isomorphic as R-algebras with involution.

Proof. Following the construction of $\mathrm{ct}(A,\tau)$ in 5.2, define U, ψ, σ, g, h and $\varepsilon = h^{-\lambda\mathrm{tr}}h \in N(U)$ so that $\overline{\varepsilon} \in T(U)$ induces $\mathrm{ct}(A,\tau) \in \mathrm{H}^0(T)$. Repeating the construction with (A',τ') in place of (A,τ), we define $U',\psi',\sigma',g',h',\varepsilon'$ analogously. By refining both U and U', we may assume $U = U'$.

Since $\mathrm{ct}(A,\tau) = \mathrm{ct}(A',\tau')$, ε and ε' determine the same section in $T(U)$. Thus, there exists a covering $V \to U$ and $\beta \in R(V)$ such that $\varepsilon' = \beta^{-1}\beta^\lambda\varepsilon$. Replacing U with V, and h' with $\beta h'$, we may assume that $\varepsilon' = \varepsilon$.

Now, by Lemma 5.31, there exists a covering $V \to U$ and $v \in \mathrm{GL}_n(R(V))$ such that $v^{\lambda\mathrm{tr}} h v = h'$. Again, replace U with V. Letting u denote the image of v in $\mathrm{PGL}_n(R)(U)$, we deduce $gu = u^{-\lambda\mathrm{tr}} g'$. Unfolding the construction in 5.2, one finds that $\tau_U = \psi^{-1} \circ \sigma \circ \psi = \psi^{-1} \circ \lambda\mathrm{tr} \circ g \circ \psi$, and likewise $\tau'_U = \psi'^{-1} \circ \lambda\mathrm{tr} \circ g' \circ \psi'$. Let $\theta := \psi^{-1} \circ u \circ \psi' : A'_U \to A_U$. Then θ is an isomorphism of R-algebras, and since $gu = u^{-\lambda\mathrm{tr}} g'$, we have

$$\begin{aligned} \theta \circ \tau'_U &= \psi^{-1} \circ u \circ \psi' \circ \psi'^{-1} \circ \lambda\mathrm{tr} \circ g' \circ \psi' \\ &= \psi^{-1} \circ \lambda\mathrm{tr} \circ u^{-\lambda\mathrm{tr}} g' \circ \psi' \\ &= \psi^{-1} \circ \lambda\mathrm{tr} \circ gu \circ \psi' \\ &= \psi^{-1} \circ \lambda\mathrm{tr} \circ g \circ \psi \circ \psi^{-1} \circ u \circ \psi' = \tau_U \circ \theta \ . \end{aligned}$$

Thus, θ defines an isomorphism of algebras with involution $(A'_U, \tau'_U) \xrightarrow{\sim} (A_U, \tau_U)$. □

5.4 Determining types in specific cases

Under mild assumptions, Theorem 5.17 provides a cohomological criterion to determine whether two λ-involutions of Azumaya algebras have the same type, and Corollary 5.18 embeds the possible λ-types in $\mathrm{cTyp}(\lambda) = \mathrm{H}^0(T)$. We finish this section by making this criterion and the realization of the types even more explicit in case the exact quotient $\pi : \mathbf{X} \to \mathbf{Y}$ is induced by a C_2-quotient of schemes or topological spaces.

Notation 5.33. Throughout, we assume one of the following:

(1) X is a scheme on which 2 is invertible, $\lambda : X \to X$ is an involution and $\pi : X \to Y$ is a good quotient relative to $C_2 = \{1, \lambda\}$, see Example 4.20.

(2) X is a Hausdorff topological space, $\lambda : X \to X$ is a continuous involution, and $\pi : X \to Y = X/\{1, \lambda\}$ is the quotient map.

We will usually treat both cases simultaneously, but when there is need to distinguish them, we shall address them as the scheme-theoretic case and the topological case, respectively.

In the scheme-theoretic case, the terms sheaf, cohomology and covering should be understood as étale sheaf, étale cohomology and étale covering, whereas in the topological case, they retain their ordinary meaning relative to the relevant topological space. Furthermore, in the topological case, $\mathcal{O}_X$ stands for $\mathcal{C}(X, \mathbb{C})$, the sheaf of continuous functions into $\mathbb{C}$, and likewise for all topological spaces. As in Subsection 5.2, write $S = \mathcal{O}_Y$ and $R = \pi_* \mathcal{O}_X$, and define N to be the kernel of the λ-norm $x \mapsto x^\lambda x : R^\times \to S^\times$ and T to be the cokernel of $x \mapsto x^{-1} x^\lambda : R^\times \to N$. By means of Theorem 4.35, the results of the previous subsections can be applied, essentially verbatim, to Azumaya $\mathcal{O}_X$-algebras with λ-involution.

Recall from Propositions 4.45 and 4.46 that there is a maximal open subscheme, resp. subset, $U \subseteq Y$ such that $\pi_U : \pi^{-1}(U) \to U$ is unramified, i.e., a quadratic étale morphism or a double covering of topological spaces. We write

$$W = Y - U \qquad \text{and} \qquad Z = \pi^{-1}(W) \ .$$

Then W and Z are the branch locus and the ramification locus of $\pi : X \to Y$, respectively. We endow Z and W with the subspace topologies. In the scheme-theoretic case, we further endow them with the reduced induced closed subscheme structures in X and Y, respectively. Recall from Proposition 4.47 and the preceding comment that π induces an isomorphism of schemes, resp. topological spaces, $Z \to W$, and λ restricts to the identity map on Z. In particular, $W = Z/C_2$.

Recall that $\mu_{2,\mathcal{O}_W}$ denotes the sheaf of square roots of 1 in $\mathcal{O}_W$; we abbreviate this sheaf as $\mu_{2,W}$. Since 2 is invertible in $\mathcal{O}_W$, the sheaf $\mu_{2,W}$ is just the constant sheaf $\{\pm 1\}$. Similar notation applies to Z.

Let (A, τ) be an Azumaya $\mathcal{O}_X$-algebra with λ-involution and let $z \in Z$ be a point of the the ramification locus. Propositions 4.45 and 4.46 imply that $\lambda(z) = z$ and the specialization of λ to $k(z)$, denoted $\lambda_{k(z)}$, is the identity. Thus, the specialization of (A, τ) to $k(z)$, denoted $(A_{k(z)}, \tau_{k(z)})$, is a central simple $k(z)$-algebra with an involution of the first kind.

LEMMA 5.34. *With the above notation, the function $f_\tau : Z \to \{1, -1\}$ determined by*

$$f_\tau(z) = \begin{cases} 1 & \text{if } \tau_{k(z)} \text{ is orthogonal} \\ -1 & \text{if } \tau_{k(z)} \text{ is symplectic} \end{cases}$$

is locally constant, and therefore determines a global section $f_\tau \in \mathrm{H}^0(Z, \mu_{2,Z})$.

Proof. We may assume that $\deg A$ is constant; otherwise, decompose X into a disjoint union of components on which this holds and work componentwise.

Let (A_Z, τ_Z) denote the base change of (A, τ) from X to Z, namely $(i^*A, i^*\tau) \otimes_{i^*\mathcal{O}_X} (\mathcal{O}_Z, \lambda_Z^\# = \mathrm{id}_{\mathcal{O}_Z})$, where $i : Z \to X$ denotes the inclusion map. For any point $z \in Z$, the type of τ at $k(z)$ may be calculated relative to (A_Z, τ_Z). We may therefore replace X and (A, τ) with Z and (A_Z, τ_Z) to assume that $Z = X$ and $\lambda : X \to X$ is the trivial involution.

Let $A_+ = \ker(\mathrm{id}_A - \tau)$ and $A_- = \ker(\mathrm{id}_A + \tau)$. That is, A_+ and A_- are the sheaves of τ-symmetric and τ-antisymmetric elements in A. Since $\lambda = \mathrm{id}$, both A_+ and A_- are $\mathcal{O}_X$-modules, and since 2 is invertible on X, the sequence $0 \to A_- \hookrightarrow A \xrightarrow{\mathrm{id}+\tau} A_+ \to 0$ is split exact. Consequently, the sequence remains exact after base changing to $k(z)$ for all $z \in X$, and so we may identify $(A_+)_{k(z)}$ with the $\tau_{k(z)}$-symmetric elements of $A_{k(z)}$.

It is well known [KMRT98, §2A] that $\dim(A_+)_{k(z)}$ equals $\frac{1}{2}n(n+1)$ when $\tau_{k(z)}$ is orthogonal and $\frac{1}{2}n(n-1)$ when $\tau_{k(z)}$ is symplectic. Since A_+ is an $\mathcal{O}_X$-summand of A, it is locally free. Thus, the rank of A_+ is locally constant, and *a fortiori* so is f_τ. □

We will prove, after a number of lemmas, that the element f_τ determines the type of τ. In the course of the proof, we shall see that the sheaf T introduced in Subsection 5.2 is nothing but the pushforward to Y of the sheaf $\mu_{2,W}$ on W.

Lemma 5.35. *Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{v} & Y \end{array}$$

in which $\pi : X' \to Y'$ is a good C_2-quotient of schemes, resp. a C_2-quotient of Hausdorff topological spaces, and u is C_2-equivariant. Let λ' denote the involution of X' and let S', R', N', T' denote the sheaves corresponding to S, R, N, T and constructed with $\pi' : X' \to Y'$ in place of $\pi : X \to Y$. Then:

(i) There are commutative squares of ring sheaves on Y and Y', respectively:

$$\begin{array}{ccccc} v_*R' & = & \pi_*u_*\mathcal{O}_{X'} & \xleftarrow{\pi_*u_\#} & R \\ \uparrow{\scriptstyle v_*\pi'_\#} & & & & \uparrow{\scriptstyle \pi_\#} \\ v_*S' & & \xleftarrow{\quad v_\# \quad} & & S \end{array} \qquad \begin{array}{ccc} R' & \longleftarrow & v^*R \\ \uparrow{\scriptstyle \pi'_\#} & & \uparrow{\scriptstyle v^*\pi_\#} \\ S' & \xleftarrow{v^\#} & v^*S \end{array}$$

Here, the horizontal arrows of the right square are the adjoints of the horizontal arrows of the left square relative to the adjuntion between v^* and v_*. Furthermore, in both squares, the top horizontal arrows are morphisms of rings with involution.

(ii) The left square of (i) induces morphisms $N \to v_*N'$, $T \to v_*T'$ and $\mathrm{H}^0(T) \to \mathrm{H}^0(T')$. Furthermore, if (A, τ) is an Azumaya $\mathcal{O}_X$-algebra with a λ-involution and (A', τ') denotes the base change of (A, τ) to X', namely $(u^*A, u^*\tau) \otimes_{u^*\mathcal{O}_X} (\mathcal{O}_{X'}, \lambda')$, then the image of $\mathrm{ct}_\pi(\tau) \in \mathrm{H}^0(T)$ in $\mathrm{H}^0(T')$ is $\mathrm{ct}_{\pi'}(\tau')$.

Proof. Part (i) and the first sentence of (ii) are straightforward from the definitions. We turn to prove the last statement of (ii).
We first claim that

$$(\pi'_*A', \pi'_*\tau') \cong (v^*\pi_*A, v^*\pi_*\tau) \otimes_{v^*R} (R', \lambda') \ . \tag{5.8}$$

To see this, observe that the relevant counit maps induce a ring homomorphism

$$\begin{aligned} \pi'^*(v^*\pi_*A \otimes_{v^*R} R') &= \pi'^*v^*\pi_*A \otimes_{\pi'^*v^*\pi_*\mathcal{O}_X} \pi'^*\pi'_*\mathcal{O}_{X'} \\ &= u^*\pi^*\pi_*A \otimes_{u^*\pi^*\pi_*\mathcal{O}_X} \pi'^*\pi'_*\mathcal{O}_{X'} \to u^*A \otimes_{u^*\mathcal{O}_X} \mathcal{O}_{X'} = A' \end{aligned}$$

which respects the relevant involutions. This morphism is adjoint to a morphism

$$v^*\pi_*A \otimes_{v^*R} R' \to \pi'_*A' \tag{5.9}$$

which we claim to be the desired isomorphism. This is easy to see when $A = \mathrm{M}_{n\times n}(\mathcal{O}_X)$. In general, by Theorem 4.28, there exists a covering $U \to Y$ such that A becomes a matrix algebra after pulling back to X_U. Thus, $(v^*\pi_*A \otimes_{v^*R} R')_{Y'_U} \to (\pi'_*A')_{Y'_U}$ is an isomorphism, and we conclude that so does (5.9).
With (5.8) at hand, let $U, \psi, \sigma, g, h, \varepsilon$ be as in Construction 5.9, applied to (A, τ). We may assume that U is represented by a covering of Y, denoted $U \to Y$. Let $U' \to Y'$ be the pullback of $U \to Y$ along $v : Y' \to Y$, which corresponds to the sheaf v^*U in $\mathbf{Y}'$. Let $\psi' = v^*\psi \otimes_{v^*R_U} \mathrm{id}_{R'_{U'}}$ and let $\sigma' = \psi'\tau'\psi'^{-1} = v^*\sigma \otimes_{v^*R} \lambda'$. The right square of (i) induces canonical maps $v^*\,\mathrm{PGL}_n(R) = \mathrm{PGL}_n(v^*R) \to \mathrm{PGL}_n(R')$, $v^*\,\mathrm{GL}_n(R) = \mathrm{GL}_n(v^*R) \to \mathrm{GL}_n(R')$ and $v^*N \to N'$ (notice that v^* is exact). Let g' be the image of $v^*g \in v^*\,\mathrm{PGL}_n(R)(U')$ in $\mathrm{PGL}_n(R')(U')$, and define $h' \in \mathrm{GL}_n(R')(U')$ and $\varepsilon' \in N'(U')$ similarly. It is easy to check that we can apply Construction 5.9 to (A', τ') using $U', \psi', \sigma', g', h', \varepsilon'$. Consequently, the image of ε' in $T'(U')$ agrees with the image of $v^*\varepsilon$, which is exactly what we need to prove. ☐

Endowing Z with the trivial involution, we can apply Lemma 5.35 with the square

$$\begin{array}{ccc} Z & \xhookrightarrow{\;i\;} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ W & \xhookrightarrow{\;j\;} & Y \end{array} \tag{5.10}$$

where π' is the restriction of π to Z. By Example 5.5, the sheaf T' is just $\mu_{2,W}$ and hence Lemma 5.35(ii) gives rise to a morphism

$$\Psi : T \to j_*\mu_{2,W} \ .$$

LEMMA 5.36. *$\Psi : T \to j_*\mu_{2,W}$ is an isomorphism of abelian sheaves on Y.*

Proof. To show that Ψ is an isomorphism, it is enough to check the stalks. The topos-theoretic points of $\mathbf{Y}$ are recalled in the proofs of Corollaries 4.33 and 4.34; they are in correspondence with the set-theoretic points of Y.
Let $p : \mathbf{pt} \to \mathbf{Y}$ be a point, corresponding to $y \in Y$. Since p^* is exact, p^*N is the kernel of $x \mapsto x^\lambda x : p^*R^\times \to p^*R^\times$ and p^*T is the cokernel of $x \mapsto x^{-1}x^\lambda : p^*R^\times \to p^*N$.
Suppose that $y \notin W$. Then, since $j : W \to Y$ is a closed embedding, $p^*j_*\mu_{2,W} = 0$. On the other hand, since π is unramified at y, it is unramified at a neighborhood of y and hence $p^*T = 0$ by Proposition 5.6. Thus, $p^*\Psi : p^*T \to p^*j_*\mu_{2,W}$ is an isomorphism.
Suppose henceforth that $y \in W$. Then π is ramified at y. We claim that p^*R is local and λ induces the identity map on its residue field. This is evident from the definitions in the topological case, see Proposition 4.46. In the scheme-theoertic case, this follows from condition (c) in Proposition 4.45 and Theorem 3.16 after noting that $\operatorname{Spec} p^*R = X \times_Y \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{Y,y}$.

Now, we have $p^*j_*\mu_{2,W} = \{\pm 1\}$. With the notation of Lemma 5.35, applied to the square (5.10), the morphism $N \to j_*N'$ is just a restriction of the morphism $R \to j_*R' = j_*\mathcal{O}_W$. This implies that the images of $-1, 1 \in p^*N$ in p^*T are mapped under $p^*\Psi$ to $-1, 1 \in p^*j_*\mu_{2,W}$, respectively, so $p^*\Psi$ is surjective.
To finish, we show that p^*T consists of at most 2 elements. Every $t \in p^*T$ is represented by some $\varepsilon \in p^*N$. Since $2 \in p^*R^\times$ and p^*R is local, either $1+\varepsilon$ or $1-\varepsilon$ is invertible. Suppose $\beta := 1+\varepsilon \in p^*R^\times$. Since $\varepsilon^\lambda = \varepsilon^{-1}$, we have $\varepsilon\beta^\lambda = \beta$, or rather, $\varepsilon = (\beta^{-1})^\lambda\beta$, which implies $t = \overline{1}$. Similarly, when $1-\varepsilon \in p^*R^\times$, we find that $t = \overline{-1}$. It follows that $p^*T = \{\overline{1}, \overline{-1}\}$ and the proof is complete. □

We finally prove the main result of this subsection.

Theorem 5.37. *With Notation 5.33, Let (A,τ) and (A',τ') be Azumaya $\mathcal{O}_X$-algebras with λ-involutions, and let $f_\tau, f_{\tau'} \in \mathrm{H}^0(Z,\mu_{2,Z})$ be as in Lemma 5.34. Then:*

(i) τ and τ' have the same type if and only if $f_\tau = f'_\tau$.

(ii) There is a group isomorphism $\Phi : \mathrm{H}^0(T) \to \mathrm{H}^0(Z,\mu_{2,Z})$ such that $\Phi(\mathrm{ct}_\pi(\tau)) = f_\tau$ for all (A,τ).

Remark 5.38. We do not know whether every $f \in \mathrm{H}^0(Z,\mu_{2,Z})$ arises as f_τ for some (A,τ), see Remark 5.20.

Proof. By Theorem 5.17 and Example 5.16, in order prove (i), it is enough to prove that $\mathrm{ct}_\pi(\tau) = \mathrm{ct}_\pi(\tau')$, and this follows if we prove (ii).
Apply Lemma 5.35 and its notation to the square (5.10). The lemma gives rise to a morphism of sheaves $T \to j_*T' = j_*\mu_{2,W}$, which is an isomorphism by Lemma 5.36. This in turn induces an isomorphism

$$\mathrm{H}^0(Y,T) \to \mathrm{H}^0(Y,j_*T') = \mathrm{H}^0(W,T')\ ,$$

such that $\mathrm{ct}_\pi(A,\tau)$ is mapped to $\mathrm{ct}_{\pi'}(A_Z,\tau_Z)$, where (A_Z,τ_Z) denotes the base change of (A,τ) to Z. Since $\pi' : Z \to W$ is an isomorphism, this gives rise to an isomorphism

$$\mathrm{H}^0(Y,T) \to \mathrm{H}^0(W,T') = \mathrm{H}^0(W,\mu_{2,W}) \cong \mathrm{H}^0(Z,\mu_{2,Z})\ ,$$

which we take to be Φ. It remains to show that $\Phi(\mathrm{ct}_\pi(A,\tau)) = f_\tau$.
Since $f_{\tau_Z} = f_\tau$, and since the image of $\mathrm{ct}_\pi(A,\tau)$ in $\mathrm{H}^0(T')$ is $\mathrm{ct}_{\pi'}(A_Z,\tau_Z)$, it is enough to show that the image of $\mathrm{ct}_{\pi'}(A_Z,\tau_Z) \in \mathrm{H}^0(W,T')$ in $\mathrm{H}^0(Z,\mu_{2,Z})$ is f_{τ_Z}. To this end, we replace $\pi : X \to Y$ and (A,τ) with $\pi' : Z \to W$ and (A_Z,τ_Z). Now, λ is the trivial involution and we may assume that $Y = X$ and π is the identity map. The map $\mathrm{H}^0(W,T') \to \mathrm{H}^0(Z,\mu_{2,Z})$ is just the identity map $\mathrm{H}^0(X,\mu_{2,X}) \to \mathrm{H}^0(X,\mu_{2,X})$, see Example 5.5, and the proof reduces to showing that $\mathrm{ct}(\tau) = f_\tau$.

Let $x \in X$, and let $U, \sigma, h, \varepsilon$ be as in Construction 5.9, applied to (A, τ). We may assume that the sheaf U in $\mathbf{Y} = \mathbf{X}$ is represented by a covering $U \to X$. Since λ is the trivial involution, $\varepsilon \in N(U) = \mu_{2,X}(U)$, and since $\mu_{2,X}$ is the constant sheaf $\{\pm 1\}$ on X, there is a covering $U_1 \sqcup U_{-1} \to U$ such that $\varepsilon|_{U_{-1}} = -1$ and $\varepsilon|_{U_1} = 1$.
There is $c \in \{\pm 1\}$ and $u \in U_c$ such that x is the image of u under $U_c \to X$. It is immediate from the definition of $t := \mathrm{ct}(\tau)$ that $t(x) = c$. Let $k(u)$ denote the residue field of u. By construction, $(A_{U_c}, \tau_{U_c}) \cong (\mathrm{M}_{n\times n}(\mathcal{O}_X), \sigma)$, where σ is given sectionwise by $x \mapsto (hxh^{-1})^{\mathrm{tr}}$ and $ch^{\mathrm{tr}} = h$. Thus, τ_{U_c} is orthogonal when $c = 1$ and symplectic when $c = -1$. The same applies to $\tau_{k(u)} : A_{k(u)} \to A_{k(u)}$. Since $(A_{k(u)}, \tau_{k(u)}) = (A_{k(x)}, \tau_{k(x)}) \otimes_{k(x)} (k(u), \mathrm{id})$, it follows that $\tau_{k(x)} : A_{k(x)} \to A_{k(x)}$ is orthogonal when $t(x) = 1$ and symplectic when $t(x) = -1$. This means $t = f_\tau$, so we are done. □

6 Brauer Classes Supporting an Involution

6.1 Introduction

Let K be a field and let $\lambda : K \to K$ be an involution with fixed field F. The central simple K-algebras admitting a λ-involution were characterized by Albert, Riehm and Scharlau, see for instance [KMRT98, Thm. 3.1], who proved:

Theorem. *Let A be a central simple K-algebra. Then:*

(i) (Albert) When $\lambda = \mathrm{id}$, A admits a λ-involution if and only if $2[A] = 0$ in $\mathrm{Br}(K)$.

(ii) (Albert–Riehm–Scharlau) When $\lambda \neq \mathrm{id}$, A admits a λ-involution if and only if $[\mathrm{cores}_{K/F}(A)] = 0$ in $\mathrm{Br}(F)$.

Here, $\mathrm{cores}_{K/F}(A)$ is the *corestriction algebra* of A, whose definition we recall below.

The Albert–Riehm–Scharlau Theorem does not, in general, hold if we replace K with an arbitrary ring. However, in [Sal78], Saltman showed that the Brauer classes admitting a representative with a λ-involution can still be characterized similarly.

Theorem (Saltman [Sal78, Thm. 3.1]). *Let R be a ring, let $\lambda : R \to R$ be an involution and let S be the fixed ring of λ. Let A be an Azumaya R-algebra. Then:*

(i) When $\lambda = \mathrm{id}$, there exists $A' \in [A]$ such that A' admits a λ-involution if and only if $2[A] = 0$ in $\mathrm{Br}(R)$.

(ii) When R is quadratic étale over S, there exists $A' \in [A]$ such that A' admits a λ-involution if and only if $[\mathrm{cores}_{R/S}(A)] = 0$ in $\mathrm{Br}(S)$.

A later proof by Knus, Parimala and Srinivas [KPS90, Thms. 4.1, 4.2] applies in the generality of schemes and also implies that the representative A' can be chosen such that $\deg A' = 2 \deg A$.

In this section, we extend Saltman's theorem to locally ringed topoi with involution. We note that our generalization implies in particular that Salman's theorem applies to topological Azumaya algebras. Furthermore, while Saltman's theorem assumes that $\lambda = \mathrm{id}$, or R is quadratic étale over the fixed ring of λ, our result will apply without any restriction on the involution. Finally, we also characterize the possible types, or more precisely, coarse types, of the involutions of the various representatives $A' \in [A]$.

Notation 6.1. Throughout this section, let $\mathbf{X}$ be a locally ringed topos with ring object $\mathcal{O}_{\mathbf{X}}$ and involution $\lambda = (\Lambda, \nu, \lambda)$, and let $\pi : \mathbf{X} \to \mathbf{Y}$ be an exact quotient relative to λ, see 4.3. Recall that such quotients arise, for instance, from C_2-quotients of schemes or Hausdorff topological spaces as explained in Examples 4.20 and 4.21. In such cases, we shall work with the original schemes, resp. topological spaces, denoted X and Y, rather than the associated ringed topoi.
As in Section 5, we write $S = \mathcal{O}_{\mathbf{Y}}$ and $R = \pi_* \mathcal{O}_{\mathbf{X}}$.
We sometimes omit bases when evaluating cohomology; the base will always be clear from the context. If A is an abelian group in $\mathbf{X}$, we shall freely identify $\mathrm{H}^i(\mathbf{X}, A)$, written $\mathrm{H}^i(A)$, with $\mathrm{H}^i(\mathbf{Y}, \pi_* A)$, written $\mathrm{H}^i(\pi_* A)$, using Theorem 4.23.

6.2 The Cohomological Transfer Map

The corestriction map considered in the aforementioned theorems of Albert–Riehm–Scharlau and Saltman is a special case of the cohomological transfer map, which will feature in our generalization of Saltman's theorem.

Definition 6.2. The *cohomological λ-transfer* map $\mathrm{transf}_\lambda : \mathrm{H}^2(\mathbf{X}, \mathcal{O}_{\mathbf{X}}^\times) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$ is the composite of the isomorphism $\mathrm{H}^2(\mathbf{X}, \mathcal{O}_{\mathbf{X}}^\times) \xrightarrow{\sim} \mathrm{H}^2(\mathbf{Y}, \pi_* \mathcal{O}_{\mathbf{X}}^\times)$ induced by π_*, see Theorem 4.23, and the morphism $\mathrm{H}^2(\mathbf{Y}, \pi_* \mathcal{O}_{\mathbf{X}}^\times) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$ induced by the λ-norm map $x \mapsto x^\lambda x : \pi_* \mathcal{O}_{\mathbf{X}}^\times \to \mathcal{O}_{\mathbf{Y}}^\times$. When no confusion can arise, we shall omit λ, simply writing transf for transf_λ, and calling it the transfer map.

Example 6.3. If the involution λ of $\mathbf{X}$ is weakly trivial and $\pi : \mathbf{X} \to \mathbf{Y}$ is the trivial quotient, see Example 4.22, then the λ-norm is the squaring map $x \mapsto x^2 : \pi_* \mathcal{O}_{\mathbf{X}}^\times \to \pi_* \mathcal{O}_{\mathbf{X}}^\times = \mathcal{O}_{\mathbf{Y}}^\times$, and so $\mathrm{transf}_\lambda : \mathrm{H}^2(\mathcal{O}_{\mathbf{X}}^\times) \to \mathrm{H}^2(\mathcal{O}_{\mathbf{Y}}^\times) \cong \mathrm{H}^2(\mathcal{O}_{\mathbf{X}}^\times)$ is multiplication by 2.

Example 6.4. Let $\pi : X \to Y$ be a quadratic étale morphism of schemes, and let $\lambda : X \to X$ be the canonical Y-involution of X, given sectionwise by $x^\lambda = \mathrm{Tr}_{X/Y}(x) - x$. We consider the exact quotient obtained from π and λ by taking étale ringed topoi, see Example 4.20. In this case, the transfer map $\mathrm{transf}_\lambda : \mathrm{H}^2_{\text{ét}}(X, \mathcal{O}_X^\times) \to \mathrm{H}^2_{\text{ét}}(Y, \mathcal{O}_Y^\times)$ is, by definition, the corestriction map $\mathrm{cores}_{X/Y} : \mathrm{H}^2_{\text{ét}}(X, \mathcal{O}_X^\times) \to \mathrm{H}^2_{\text{ét}}(Y, \mathcal{O}_Y^\times)$. Moreover, $\mathrm{cores}_{X/Y}$ restricts to a map $\mathrm{cores}_{X/Y} : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ which can be described explicitly on the level of Azumaya algebras: Let A be an Azumaya $\mathcal{O}_X$-algebra. The corestriction algebra $\mathrm{cores}_{X/Y}(A)$ is an Azumaya $\mathcal{O}_Y$-algebra defined as the $\mathcal{O}_Y$-subalgebra of $\pi_*(A \otimes_{\mathcal{O}_X} \lambda^* A)$ fixed by the exchange automorphism, given by $x \otimes y \mapsto y \otimes x$ on sections. The map $\mathrm{cores}_{X/Y} : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is then given by $[A] \mapsto [\mathrm{cores}_{X/Y}(A)]$, see [KPS90, p. 68] (the diagram on that page contains a misprint, on the right column, both 'S's should be 'R's).

Remark 6.5. In contrast to the situation in Examples 6.3 and 6.4, we do not know whether

$$\mathrm{transf}_\lambda : \mathrm{H}^2(\mathbf{X}, \mathcal{O}_{\mathbf{X}}^\times) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$$

restricts to a map between the Brauer groups $\mathrm{Br}(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to \mathrm{Br}(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$, even in the cases induced by a good C_2-quotient of schemes $\pi : X \to Y$. Some positive results appear in [APS15, Lem. 5.1, Rmk. 5.2]. Also, when R is locally free of rank 2 over S, Ferrand [Fer98] constructs a universal norm functor taking R-algebras to S-algebras, which coincides with $\mathrm{cores}_{R/S}$ when R is quadratic étale over S, but it is *a priori* not clear whether it takes Azumaya R-algebras to Azumaya S-algebras in general. We hope to address this problem in a subsequent work.

We further note that without assuming that π is unramified, the construction of Example 6.4 may produce an algebra which is not Azumaya. For example, it can be checked directly that $\mathrm{cores}_{R/S}(\mathrm{M}_{2\times 2}(R))$ is not Azumaya over S when $S = \mathbb{C}$, $R = \mathbb{C}[x]/(x^2)$, and $\lambda : R \to R$ is the $\mathbb{C}$-involution taking x to $-x$.

Example 6.6. In the case where X is a Hausdorff topological space with a free C_2-action and $\pi : X \to Y := X/C_2$ is the corresponding 2-sheeted covering, the construction

$$\mathrm{transf} : \mathrm{H}^2(X, S^1) \cong \mathrm{H}^2(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(Y, \mathcal{O}_Y^\times) \cong \mathrm{H}^2(Y, S^1)$$

is a special case of the usual transfer map for a 2-sheeted cover. This can be proved by considering transf on the level of 2-cocycles. See also [Pia84, Sec. 3.3] and note that π^* takes $\mathcal{O}_Y^\times$, the sheaf of nonvanishing continuous complex-valued functions on Y, to $\mathcal{O}_X$ on $\mathbf{X}$.

Remark 6.7. There is a notion of transfer for ramified covers $X \to X/G$ where G is a finite group, in particular, when $G = C_2$. This may be found in [AP10]. It seems likely, that map transf_λ given here is a special case of that construction, but we do not pursue this further.

6.3 Brauer Classes Supporting a λ-Involution

In this subsection, we characterize those Brauer classes in $\mathrm{Br}(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ admitting a representative with a λ-involution, thus generalizing Saltman's theorem [Sal78, Thm. 3.1].

We remind the reader that the notational conventions of Notation 6.1 are still in effect. In particular, $S := \mathcal{O}_{\mathbf{Y}}$ is a local ring object in $\mathbf{Y}$ and $R := \pi_*\mathcal{O}_{\mathbf{X}}$ is a commutative S-algebra with involution λ such that the fixed ring of λ is S.

As in Subsection 5.2, we define N to be the kernel of the λ-norm $x \mapsto x^\lambda x : R^\times \to S^\times$ and let T be the quotient of N by the image of the map $x \mapsto x^\lambda x^{-1} : R^\times \to N$. Recall that $\mathrm{cTyp}(\lambda) := \mathrm{H}^0(T)$ is the group of coarse λ-types and there is a map $(A,\tau) \mapsto \mathrm{ct}_\pi(A,\tau) \in \mathrm{H}^0(T)$ associating an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra with a λ-involution to its coarse type, see Subsection 5.2.

The short exact sequence $1 \to R^\times/S^\times \xrightarrow{x\mapsto x^\lambda x^{-1}} N \to T \to 1$ induces the connecting homomorphism

$$\delta^0 : \mathrm{H}^0(T) \to \mathrm{H}^1(R^\times/S^\times)$$

and the short exact sequence $1 \to S^\times \to R^\times \to R^\times/S^\times \to 1$ induces a connecting homomorphism

$$\delta^1 : \mathrm{H}^1(R^\times/S^\times) \to \mathrm{H}^2(S^\times).$$

Notation 6.8. We denote the composite morphism $\delta^1 \circ \delta^0$ by Φ,

$$\Phi : \mathrm{cTyp}(\lambda) = \mathrm{H}^0(T) \to \mathrm{H}^2(S^\times).$$

Proposition 6.9. *The map Φ is the* 0*-map in the following cases:*

(i) When $\pi : \mathbf{X} \to \mathbf{Y}$ is a trivial quotient (Example 4.22), i.e., $R = S$.

(ii) When π is everywhere ramified (Definition 4.43), $2 \in S^\times$ and $S^\times$ has square roots locally.

(iii) When π is unramified (Definition 4.43), i.e., R is a quadratic étale S-algebra.

(iv) When $\pi : X \to Y$ is a good C_2-quotient of schemes, Y is noetherian and regular, and π is unramified at the generic points of Y; the corresponding exact quotient is obtained by taking étale ringed topoi as in Example 4.20.

Proof. (i) In this case, $R^\times/S^\times$ is trivial. As Φ factors through $\mathrm{H}^1(R^\times/S^\times) = 0$, the result follows.

(ii) We claim that squaring induces an automorphism of $R^\times/S^\times$, and hence of the group $\mathrm{H}^1(R^\times/S^\times)$. Since $\mathrm{H}^0(T)$ is a 2-torsion group (Proposition 5.7), this forces $\delta^0 : \mathrm{H}^0(T) \to \mathrm{H}^1(R^\times/S^\times)$ to vanish, implying Φ vanishes as well.

We show the surjectivity of $x \mapsto x^2 : R^\times/S^\times \to R^\times/S^\times$ by checking that $R^\times$ has square roots locally. Let U be an object of $\mathbf{Y}$ and $r \in R^\times(U)$. Since $S^\times$ has square roots locally, there a covering $V \to U$ and $s \in S^\times(V)$ such that $r^\lambda r = s^2$. Replacing r with rs^{-1} and U with V, we may assume $r^\lambda r = 1$. Now, by Lemma 5.30, there is a covering $\{V_i \to U\}_{i=1,2}$ and $\beta_i \in R^\times(V_i)$ such that

$r = \beta_1^{-1}\beta_1^{\lambda}$ in $R^{\times}(V_1)$ and $r = -\beta_2^{-1}\beta_2^{\lambda}$ in $R^{\times}(V_2)$. We may refine $V_2 \to V$ to assume that there is $a \in S^{\times}(V_2)$ such that $-\beta_2^{\lambda}\beta_2 = a^2$ and get $r = a^2\beta_2^{-2}$. Similarly, we refine V_1 to find a square root of r in $R^{\times}(V_1)$ and conclude that r has a square root on $V_1 \sqcup V_2$.

Next, let K denote the kernel of $x \mapsto x^2 : R^{\times}/S^{\times} \to R^{\times}/S^{\times}$. A section of K is represented by some $a \in R^{\times}(U)$ such that $a^2 \in S^{\times}(U)$, or rather, $a^2 = (a^{\lambda})^2$. Since $(a - a^{\lambda})^2 + (a + a^{\lambda})^2 = 4a^2 \in S^{\times}(U)$ and $(a - a^{\lambda})^2, (a + a^{\lambda})^2 \in S(U)$, and since S is a local ring object, there is a covering $\{U_i \to U\}_{i=1,2}$ such that $a - a^{\lambda} \in R^{\times}(U_1)$ and $a + a^{\lambda} \in R^{\times}(U_2)$. By virtue of Lemma 3.11, R_{U_1} is a quadratic étale over S_{U_1}, so our assumption that π is everywhere ramified forces $U_1 = \emptyset$. Thus, $U_2 \to U$ is a covering, implying that $a + a^{\lambda}$ is invertible in $R(U)$. Since $(a + a^{\lambda})(a - a^{\lambda}) = a^2 - (a^{\lambda})^2 = 0$, we must have $a - a^{\lambda} = 0$, so $a \in S^{\times}(U)$. It follows that a represents the 1-section in $R^{\times}/S^{\times}$, and thus $K = 0$.

(iii) In this case, a version of Hilbert's Theorem 90 applies in the form of Proposition 5.6, and $\mathrm{H}^0(\mathbf{Y}, T) = 0$. *A fortiori*, Φ is 0.

(iv) We may assume that Y is connected and therefore integral, otherwise we may work component by component.

Let $\xi : \operatorname{Spec} K \to Y$ denote the generic point of Y. Since ξ is flat, $\pi_{\xi} : X_{\xi} \to \operatorname{Spec} K$ is a good C_2-quotient relative to the action induced by λ; denote the sheaves corresponding to S, R, N, T by S', R', N', T'. We now apply Lemma 5.35 to the square

$$\begin{array}{ccc} X_{\xi} & \xrightarrow{\xi_{\pi}} & X \\ \downarrow{\scriptstyle \pi_{\xi}} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} K & \xrightarrow{\xi} & Y \end{array}$$

which gives rise to maps $\xi^* N \to N'$, $\xi^* T \to T'$, adjoint to the maps in the lemma. The exactness of ξ^* together with the natural homomorphism $\mathrm{H}^*(Y, -) \to \mathrm{H}^*(K, \xi^*(-))$ now give rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0_{\text{ét}}(Y, T) & \xrightarrow{\Phi} & \mathrm{H}^2_{\text{ét}}(Y, \mathcal{O}_Y^{\times}) \\ \downarrow & & \downarrow \\ \mathrm{H}^0_{\text{ét}}(K, T') & \xrightarrow{\Phi_{\xi}} & \mathrm{H}^2_{\text{ét}}(K, \mathcal{O}_{\operatorname{Spec} K}^{\times}) \end{array}$$

By [Gro68a, Cor. 1.8], the right vertical morphism is injective (here we need Y to be regular), and by (iii), $\Phi_{\xi} = 0$. Therefore, $\Phi = 0$. □

We are now ready to state our generalization of Saltman's theorem. Whereas Saltman's original proof [Sal78] and the later proof by Knus, Parimala and

Srinivas [KPS90] make use of the corestriction of an Azumaya algebra, we cannot employ this construction, as demonstrated in Remark 6.5. Rather, our proof is purely cohomological, phrased in the language set in Subsections 2.3 and 2.4. We remind the reader of our standing assumption from Remark 4.30 that the degrees of all Azumaya $\mathcal{O}_{\mathbf{X}}$-algebras considered are fixed under Λ, which is automatic when $\mathbf{X}$ is connected.

Theorem 6.10. *Let $\mathbf{X}$ be a locally ringed topos with involution λ, let $\pi : \mathbf{X} \to \mathbf{Y}$ be an exact quotient relative to λ, and consider the map $\mathrm{transf}_\lambda : \mathrm{Br}(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$ of Definition 6.2 and the map $\Phi : \mathrm{cTyp}(\lambda) = \mathrm{H}^0(T) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$ of Notation 6.8. Let A be an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra of degree n, and let $t \in \mathrm{H}^0(T)$. Then there exists $A' \in [A]$ admitting a λ-involution of coarse type t if and only if $\mathrm{transf}([A]) = \Phi(t)$ in $\mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$. The algebra A' can be chosen such that $\deg A' = 2n$.*

We recover Saltman's original theorem [Sal78, Thm. 3.1] and the improvement of Knus, Parimala and Sinivas [KPS90, Thms. 4.1, 4.2] from Theorem 6.10 by taking $\pi : \mathbf{X} \to \mathbf{Y}$ to be the exact quotient associated to a good C_2-quotient of schemes $\pi : X \to Y$ such that π is an isomorphism or quadratic étale, see Example 4.20. In this case, $\Phi = 0$ by Proposition 6.9, and the transfer map coincides with multiplication by 2 when $\pi = \mathrm{id}$, or with the corestriction map when π is quadratic étale, as demonstrated in Examples 6.3 and 6.4.
The relation between the type and the coarse type of an involution, as well as the question of when two involutions of the same type are locally isomorphic, had been studied extensively in Subsections 5.2 and 5.4.

Proof. Thanks to Theorems 4.23 and 4.28, we may replace A with $\pi_* A$ and work with R-algebras, rather than $\mathcal{O}_{\mathbf{X}}$-algebras. We abuse the notation and denote the map $\mathrm{H}^2(R^\times) \to \mathrm{H}^2(S^\times)$ induced by $x \mapsto x^\lambda x : R^\times \to S^\times$ as transf_λ.

Suppose first that there exists $[A'] \in A$ admitting a λ-involution τ of coarse type t. We may replace A with A'. We now invoke all the notation of Construction 5.9 and the proof of Lemma 5.10 through which t is constructed from (A, τ). Specifically:

- $U \to *_{\mathbf{Y}}$ is a covering such that there exists an isomorphism of R_U-algebras $\psi : A_U \to \mathrm{M}_{n\times n}(R_U)$,
- $\sigma := \psi \circ \tau_U \circ \psi^{-1}$ is an involution of $\mathrm{M}_{n\times n}(R_U)$,
- $g := \lambda\mathrm{tr} \circ \sigma$ is an element of $\mathrm{PGL}_n(R)(U)$,
- $h \in \mathrm{GL}_n(R)(U)$ is a lift of g (refine U if necessary),
- $\varepsilon := h^{-\lambda\mathrm{tr}} h$ is an element of $N(U)$, embedded diagonally in $\mathrm{GL}_n(R)(U)$,
- $U_\bullet$ is the Čech hypercovering corresponding to $U \to *$, see Example 2.3,
- $a = \psi_1 \circ \psi_0^{-1} \in \mathrm{PGL}_n(R)(U_1)$, where ψ_i is the pullback of ψ along $d_i : U_1 \to U_0$,

- b is a lift of a to $\mathrm{GL}_n(R)(V)$, where $V \to U_1$ is some covering,
- $\beta := b^{-\lambda\mathrm{tr}} \cdot d_0^* h \cdot b^{-1} \cdot d_1^* h^{-1}$ is an element $R^\times(V)$, embedded diagonally in $\mathrm{GL}_n(R)(V)$,
- $t = \mathrm{ct}(\tau)$ is the image of ε in $T(U)$; it descends to a global section of T since $d_1^*\varepsilon \cdot d_0^*\varepsilon^{-1} = \beta^{-1}\beta^\lambda$.

By Lemma 2.4, there is a hypercovering morphism $V_\bullet \to U_\bullet$ such that $V_1 \to U_1$ factors through $V \to U_1$. We replace V with V_1.
Recall from Theorem 4.28 that A corresponds to a $\mathrm{PGL}_n(R)$-torsor, which in turn corresponds to a cohomology class in $\mathrm{H}^1(\mathrm{PGL}_n(R))$. We claim that a is a 1-cocycle in $Z^1(U_\bullet, \mathrm{PGL}_n(R))$ which represents this cohomology class. Indeed, the $\mathrm{PGL}_n(R)$-torsor corresponding to a is $P := \mathcal{A}ut_R(\mathrm{M}_{n\times n}(R), A)$, and $\psi^{-1} \in P(U) = P(U_0)$ by construction. By the isomorphism given in the proof of Proposition 2.8(i), the cohomology class corresponding to P is represented by $d_1^*(\psi^{-1})^{-1} \cdot d_0^*(\psi^{-1}) = \psi_1 \circ \psi_0^{-1} = a$.
Consider the short exact sequence $1 \to R^\times \to \mathrm{GL}_n(R) \to \mathrm{PGL}_n(R) \to 1$ and its associated 7-term cohomology exact sequence, see Proposition 2.8(iii). It follows from the definition of $\delta^2 : \mathrm{H}^1(\mathrm{PGL}_n(R)) \to \mathrm{H}^2(R^\times)$, see the proof of Proposition 2.8(iii), that $[A] = \delta^2(a) \in \mathrm{H}^2(R^\times)$ is represented by

$$\alpha := d_2^* b \cdot d_0^* b \cdot d_1^* b^{-1} \in Z^2(V_\bullet, R^\times)\ . \tag{6.1}$$

and thus, $\mathrm{transf}([A])$ is represented by $\alpha^\lambda\alpha \in Z^2(V_\bullet, S^\times)$.
On the other hand, by the definition of $\delta^0 : \mathrm{H}^0(T) \to \mathrm{H}^1(R^\times/S^\times)$, see the beginning of this subsection and the end of 2.3, $\delta^0(t)$ is represented by the image of $\beta^{-1} \in R^\times(V_1)$ in $(R^\times/S^\times)(V_1)$, since $d_0^*\varepsilon \cdot d_1^*\varepsilon^{-1} = (\beta^{-1})^\lambda\beta$. Likewise, by the definition of $\delta^1 : \mathrm{H}^1(R^\times/S^\times) \to \mathrm{H}^2(S^\times)$, the class $\Phi(t) = \delta^1\delta^0(t)$ is represented by $d_0^*\beta^{-1} \cdot d_1^*\beta \cdot d_2^*\beta^{-1} \in Z^2(V_\bullet, S^\times)$.
In order to show that $\mathrm{transf}([A]) = \Phi(t)$, we check that $\alpha^\lambda\alpha = d_0^*\beta^{-1} \cdot d_1^*\beta \cdot d_2^*\beta^{-1}$ in $S^\times(V_2)$. For the computation, we shall make use of $d_0^*d_0^* = d_1^*d_0^*$, $d_0^*d_1^* = d_2^*d_0^*$, $d_1^*d_1^* = d_2^*d_1^*$ and the fact that if xyz is central in a group G, then $xyz = zxy = yzx$.

$$\begin{aligned}
&d_0^*\beta^{-1} \cdot d_1^*\beta \cdot d_2^*\beta^{-1}\\
&\quad= d_0^*\beta^{-1}(d_1^*b^{-\lambda\mathrm{tr}} \cdot d_1^*d_0^*h \cdot d_1^*b^{-1} \cdot d_1^*d_1^*h^{-1}) \cdot (d_2^*d_1^*h \cdot d_2^*b \cdot d_2^*d_0^*h^{-1} \cdot d_2^*b^{\lambda\mathrm{tr}})\\
&\quad= d_1^*b^{-\lambda\mathrm{tr}} \cdot d_1^*d_0^*h \cdot d_1^*b^{-1} \cdot d_2^*b \cdot d_2^*d_0^*h^{-1} \cdot (d_0^*\beta^{-1}) \cdot d_2^*b^{\lambda\mathrm{tr}}\\
&\quad= d_1^*b^{-\lambda\mathrm{tr}} \cdot d_1^*d_0^*h \cdot d_1^*b^{-1} \cdot d_2^*b \cdot d_2^*d_0^*h^{-1}\\
&\qquad\quad \cdot (d_0^*d_1^*h \cdot d_0^*b \cdot d_0^*d_0^*h^{-1} \cdot d_0^*b^{\lambda\mathrm{tr}}) \cdot d_2^*b^{\lambda\mathrm{tr}}\\
&\quad= d_1^*b^{-\lambda\mathrm{tr}} \cdot d_1^*d_0^*h \cdot (d_1^*b^{-1} \cdot d_2^*b \cdot d_0^*b) \cdot d_0^*d_0^*h^{-1} \cdot d_0^*b^{\lambda\mathrm{tr}} \cdot d_2^*b^{\lambda\mathrm{tr}}\\
&\quad= d_1^*b^{-\lambda\mathrm{tr}} \cdot d_1^*d_0^*h \cdot \alpha \cdot d_0^*d_0^*h^{-1} \cdot d_0^*b^{\lambda\mathrm{tr}} \cdot d_2^*b^{\lambda\mathrm{tr}}\\
&\quad= (d_2^*b \cdot d_0^*b \cdot d_1^*b^{-1})^{\lambda\mathrm{tr}} \cdot \alpha\\
&\quad= \alpha^\lambda\alpha
\end{aligned}$$

This completes the proof of the "only if" statement.

Suppose now that $\mathrm{transf}([A]) = \Phi(t)$. Define $U \to *$, $U_\bullet$, a, b, $V_\bullet$ and α as before. Using Lemma 2.4 twice, we can refine $V_\bullet$ to assume that t lifts to some $\varepsilon \in N(V_0)$ and there is $\beta \in R^\times(V_1)$ such that

$$d_0^*\varepsilon \cdot d_1^*\varepsilon^{-1} = (\beta^{-1})^\lambda \beta \tag{6.2}$$

in $N(V_1)$. As explained above, $\mathrm{transf}([A])$ is represented by $\alpha^\lambda\alpha \in Z^2(V_\bullet, S^\times)$ and $\Phi(t)$ is represented by $d_0^*\beta^{-1} \cdot d_1^*\beta \cdot d_2^*\beta^{-1}$. The assumption $\Phi(t) = \mathrm{transf}([A])$ therefore means that, after refining $V_\bullet$, there exists $\gamma \in S^\times(V_1)$ such that

$$d_0^*\gamma \cdot d_1^*\gamma^{-1} \cdot d_2^*\gamma \cdot d_0^*\beta^{-1} \cdot d_1^*\beta \cdot d_2^*\beta^{-1} = \alpha^\lambda\alpha\ .$$

We replace β with $\beta\gamma^{-1} \in R^\times(V_1)$, which does not affect (6.2) and allows us to assume

$$d_0^*\beta^{-1} \cdot d_1^*\beta \cdot d_2^*\beta^{-1} = \alpha^\lambda\alpha\ . \tag{6.3}$$

Writing in block form, define the $2n \times 2n$ matrices

$$h = \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix} \in \mathrm{GL}_{2n}(R)(V_0) \qquad \text{and} \qquad b' = \begin{bmatrix} b & 0 \\ 0 & \beta^{-1}b^{-\lambda\mathrm{tr}} \end{bmatrix} \in \mathrm{GL}_{2n}(R)(V_1)$$

and let $\sigma : \mathrm{M}_{2n\times 2n}(R_{V_0}) \to \mathrm{M}_{2n\times 2n}(R_{V_0})$ be the involution given by $x \mapsto (hxh^{-1})^{\lambda\mathrm{tr}} = h^{-\lambda\mathrm{tr}}xh^{\lambda\mathrm{tr}}$ on sections. Also, let a' be the image of b' in $\mathrm{PGL}_{2n}(R)(V_1)$, namely, $a' \in \mathrm{PGL}_{2n}(R)(V_1)$ is the automorphism of $\mathrm{M}_{2n\times 2n}(R_{V_1})$ given by $x \mapsto b'xb'^{-1}$ on sections.

We first observe that $a' \in Z^1(V_\bullet, \mathrm{PGL}_{2n}(R))$. Indeed, working in $\mathrm{GL}_{2n}(R)(V_2)$ and using (6.1) and (6.3), we find that

$$d_2^*b' \cdot d_0^*b' \cdot d_1^*b'^{-1} = \begin{bmatrix} \alpha & 0 \\ 0 & (d_0^*\beta \cdot d_2^*\beta \cdot d_1^*\beta^{-1})^{-1}\alpha^{-\lambda\mathrm{tr}} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \in R^\times(V_1)\ . \tag{6.4}$$

Let $\tilde{V}_\bullet$ denote the Čech hypercovering associated to $V_0 \to *$, see Example 2.3. By Lemma 2.7, $a' \in Z^1(V_\bullet, \mathrm{PGL}_{2n}(R))$ descends uniquely to a cocycle $\tilde{a}' \in Z^1(\tilde{V}_\bullet, \mathrm{PGL}_{2n}(R))$. The Čech 1-cocycle $\tilde{a}'$ defines descent data for $\mathrm{M}_{2n\times 2n}(R_{V_0})$ along $V_0 \to *$, giving rise to an Azumaya R-algebra A' of degree $2n$ and an isomorphism $\psi : A'_{V_0} \to \mathrm{M}_{2n\times 2n}(R_{V_0})$ such that $\tilde{a}' = \psi_1 \circ \psi_0^{-1}$, where ψ_i is the pullback of ψ along $d_i : \tilde{V}_1 \to \tilde{V}_0 = V_0$. Note that by construction, a' represents the class in $\mathrm{H}^1(\mathrm{PGL}_{2n}(R))$ corresponding to A', hence (6.4) implies that $[A'] = \alpha = [A]$.

We now claim that σ descends to an involution $\tau : A' \to A'$. Letting σ_i denote the pullback of σ along $d_i : V_1 \to V_0$, and noting that $(d_0, d_1) : V_1 \to V_0 \times V_0$ is a covering, see Subsection 2.3, this amounts to showing that $\sigma_1 a' = a'\sigma_0$. To see this, we first note that (6.2) and $\varepsilon^\lambda\varepsilon = 1$ imply that

$$b'^{-\lambda\mathrm{tr}} \cdot d_0^*h \cdot b'^{-1} \cdot d_1^*h^{-1} = \begin{bmatrix} \beta & 0 \\ 0 & \beta^\lambda \cdot d_0^*\varepsilon \cdot d_1^*\varepsilon^{-1} \end{bmatrix} = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix},$$

or equivalently,

$$b'^{-1} \cdot d_1^* h^{-1} = \beta \cdot d_0^* h^{-1} \cdot b'^{\lambda \mathrm{tr}}.$$

Using this, for any section x of $\mathrm{M}_{2n \times 2n}(R_{V_1})$, we have

$$\begin{aligned}\sigma_1(a'(x)) &= d_1^* h^{-\lambda \mathrm{tr}} (b' x b'^{-1})^{\lambda \mathrm{tr}} d_1^* h^{\lambda \mathrm{tr}} = (b'^{-1} d_1^* h^{-1})^{\lambda \mathrm{tr}} x (b'^{-1} d_1^* h^{-1})^{-\lambda \mathrm{tr}} \\ &= (\beta \cdot d_0^* h^{-1} \cdot b'^{\lambda \mathrm{tr}})^{\lambda \mathrm{tr}} x (\beta \cdot d_0^* h^{-1} \cdot b'^{\lambda \mathrm{tr}})^{-\lambda \mathrm{tr}} = a'(\sigma_0(x)) \ ,\end{aligned}$$

which is what we want.

We finish by checking that $\mathrm{ct}(\tau) = t$. To see this, apply Construction 5.9 to (A', τ) using $U := V_0$, ψ, σ and h defined above and note that $h^{-\lambda \mathrm{tr}} h = \left[\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{smallmatrix}\right]$. □

We now specialize Theorem 6.10 to Azumaya algebras over schemes and over topological spaces.

It is worth recalling at this point that in the situation of a good C_2-quotients of schemes $\pi : X \to Y$ such that 2 is invertible on Y (Example 4.20), or a C_2-quotient of Hausdorff topological spaces $\pi : X \to Y$ (Examples 4.21), the sheaf T is isomorphic to $i_* \mu_{2,W}$, where $i : W \to Y$ is the embedding of the branch locus of π in Y. Under this isomorphism, the coarse type of an involution $\tau : A \to A$ is the unique global section $f \in \mathrm{H}^0(W, \mu_{2,W}) = \mathcal{C}(W, \{\pm 1\})$ such that $f(w) = 1$ if $\tau_{k(\pi^{-1}(w))} : A_{k(\pi^{-1}(w))} \to A_{k(\pi^{-1}(w))}$ is orthogonal, and $f(w) = -1$ if $\tau_{k(\pi^{-1}(w))} : A_{k(\pi^{-1}(w))} \to A_{k(\pi^{-1}(w))}$ is symplectic, for all $w \in W$; see Subsection 5.4. Furthermore, in these situations, two λ-involutions of the same coarse type have the same type, and they are locally isomorphic if the degrees of their underlying Azumaya algebras agree; this follows from Theorem 5.17 and Corollary 5.18.

COROLLARY 6.11. *Let X be a scheme, let $\lambda : X \to X$ be an involution and let $\pi : X \to Y$ be a good quotient relative to $C_2 := \{1, \lambda\}$. Let A be an Azumaya $\mathcal{O}_X$-algebra and let $t \in \mathrm{cTyp}(\lambda)$ be a coarse type. Then there exists $A' \in [A]$ admitting a λ-involution of coarse type t if and only if $\Phi(t) = \mathrm{transf}_\lambda([A])$ in $\mathrm{H}^2_{\acute{e}t}(Y, \mathcal{O}_Y^\times)$. The algebra A' can be chosen such that $\deg A' = 2 \deg A$.*

Proof. This is a special case of Theorem 6.10. See Example 4.20 and Theorem 4.35. □

COROLLARY 6.12. *In the situation of Corollary 6.11, suppose that*

(1) $\lambda = \mathrm{id}$, or

(2) $\pi : X \to Y$ is quadratic étale, or

(3) Y is noetherian and regular, and π is unramified at the generic points of Y.

Then there exists $A' \in [A]$ admitting a λ-involution if and only if $\mathrm{transf}_\lambda([A]) = 0$. In this case, A' can be chosen to have a λ-involution of any prescribed coarse type (or any prescribed type, when 2 is invertible on Y) and to satisfy $\deg A' = 2 \deg A$.

Proof. This follows from Corollary 6.11 and Propositions 6.9 and 4.45. □

Corollary 6.13. *Let X be a Hausdorff topological space, let $\lambda : X \to X$ be a continuous involution, and let π denote the quotient map $X \to Y := X/\{1, \lambda\}$. Let A be an Azumaya $\mathcal{O}_X$-algebra and let $t \in \mathrm{cTyp}(\lambda)$ be a coarse type. Then there exists $A' \in [A]$ admitting a λ-involution of coarse type t if and only if $\Phi(t) = \mathrm{transf}_\lambda([A])$ in $\mathrm{H}^2_{\acute{e}t}(Y, \mathcal{O}_Y^\times)$. The algebra A' can be chosen such that* $\deg A' = 2 \deg A$.

Proof. This is a special case of Theorem 6.10, see Examples 4.21 and Theorem 4.35. □

Corollary 6.14. *In the situation of Corollary 6.13, if $\lambda = \mathrm{id}$, or λ acts freely on X, then there exists $A' \in [A]$ admitting a λ-involution if and only if $\mathrm{transf}_\lambda([A]) = 0$. In this case A' can be chosen to have a λ-involution of any prescribed type and to satisfy* $\deg A' = 2 \deg A$.

Proof. This follows from Corollary 6.13 and Propositions 6.9 and 4.46. □

Remark 6.15. Let R be a connected semilocal ring, and let $\lambda : R \to R$ be an involution with fixed ring S. When $R = S$ or R is quadratic étale over S, it was observed by Saltman [Sal78, Thm. 4.4] that an Azumaya R-algebra A that is Brauer equivalent to an algebra with a λ-involution already possesses a λ-involution. Otherwise said, in this special situation, we can choose $A' = A$ in Corollary 6.11.
We do not know whether this statement continues to hold if the assumption that $R = S$ or R is quadratic étale over S is dropped. In this case, the fact that R is semilocal implies that two Azumaya algebras of the same degree are isomorphic [OS71]. With this in hand, Corollary 6.11 implies that if A is equivalent to an Azumaya R-algebra admitting a λ-involution, then $\mathrm{M}_{2\times 2}(A)$ has a λ-involution. The problem therefore reduces to the question of whether the existence of a λ-involution on $\mathrm{M}_{2\times 2}(A)$ implies the existence of a λ-involution on A. The same question was asked for arbitrary non-commutative semilocal rings A in [Fir15, §12], where it was also shown that counterexamples, if any exist, are restricted. In particular, returning to the case of Azumaya algebras, it follows from [Fir15, Thm. 7.3] that if $\deg A$ is even, then A does possess a λ-involution when $\mathrm{M}_{2\times 2}(A)$ has one.

6.4 The Kernel of the Transfer Map

We continue to use R, S, N, T defined in Subsection 6.3.
Saltman's theorem can also be regarded as a result characterizing the kernel of the transfer map in terms of existence of certain involutions. We now use Theorem 6.10 to generalize this particular aspect, namely, describing the kernel of $\mathrm{transf}_\lambda : \mathrm{Br}(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times)$ in terms of the involutions that the Brauer classes support. For that purpose, we introduce the following families of λ-involutions.

DEFINITION 6.16. Let A be an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra. A λ-involution $\tau : A \to \Lambda A$ is called *semiordinary* if there exists a split Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra A' and a λ-involution $\tau' : A' \to \Lambda A'$ such that $(\pi_* A, \pi_* \tau)$ and $(\pi_* A', \pi_* \tau')$ are locally isomorphic. If A' can moreover be chosen to be $\mathrm{M}_{n\times n}(\mathcal{O}_{\mathbf{X}})$ and there is $a \in \mathrm{H}^0(A')$ such that τ' is given by $x \mapsto (axa^{-1})^{\lambda\mathrm{tr}}$ on sections, we say that τ is *ordinary.*
When τ is not semiordinary, we shall say it is *extraordinary.*

THEOREM 6.17. *With notation as in Theorem 6.10, let A be an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra of degree n. Then the following conditions are equivalent:*

(a) $\mathrm{transf}_\lambda([A]) = 0$,

(b) there exists $A' \in [A]$ admitting a semiordinary λ-involution,

(c) there exists $A' \in [A]$ admitting an ordinary λ-involution.

In (b), the algebra A' can be chosen to satisfy $\deg A' = 2 \deg A$ *and to have a semiordinary involution of any prescribed coarse type in* $\ker(\Phi : \mathrm{H}^0(T) \to \mathrm{H}^2(S^\times))$. *In (c), the algebra A' can be chosen to satisfy* $\deg A' = 2 \deg A$ *and to have an ordinary involution of any prescribed coarse type in* $\mathrm{im}(\mathrm{H}^0(N) \to \mathrm{H}^0(T))$.

We shall see below (Corollary 6.22) that in the situation of a scheme on which 2 is invertible and a trivial involution, or a quadratic étale covering of schemes with its canonical involution, all λ-involutions are ordinary. Thus, Theorem 6.17 recovers Saltman's Theorem when 2 is invertible.
More generally, it will turn out that under mild assumptions, all involutions are ordinary when $\pi : \mathbf{X} \to \mathbf{Y}$ is unramified or everywhere ramified.

Proof. As in the proof of Theorem 6.10, we switch to Azumaya R-algebras by applying π_*.
(c) $\Longrightarrow$ (b) is clear.
(b) $\Longrightarrow$ (a): Suppose $A' \in [A]$ admits a semiordinary involution τ and let (B, θ) be a split Azumaya R-algebra with a λ-involution that is locally isomorphic to (A', τ). Then by Theorem 6.10 and Proposition 5.13, $\mathrm{transf}_\lambda([A]) = \Phi(\mathrm{ct}(\tau)) = \Phi(\mathrm{ct}(\theta)) = \mathrm{transf}_\lambda([B]) = 0$.
(a) $\Longrightarrow$ (c): Let $t \in \mathrm{im}(\mathrm{H}^0(N) \to \mathrm{H}^0(T))$. Then t is the image of some $\varepsilon \in \mathrm{H}^0(N)$. We revisit the proof of the "if" part in Theorem 6.10 and apply it with our t and ε to obtain an Azumaya R-algebra with involution (A', τ) such that $A' \in [A]$, $\deg A' = 2n$, $\mathrm{ct}(\tau) = t$ and (A'_{V_0}, τ_{V_0}) is isomorphic to $(\mathrm{M}_{2n\times 2n}(R_{V_0}), \sigma)$ with σ being given by $x \mapsto ([\begin{smallmatrix}0&1\\ \varepsilon&0\end{smallmatrix}]x[\begin{smallmatrix}0&1\\ \varepsilon&0\end{smallmatrix}]^{-1})^{\lambda\mathrm{tr}}$ on sections. Since $\varepsilon \in \mathrm{H}^0(N)$, the involution σ descends to an involution on $\mathrm{M}_{2n\times 2n}(R)$, defined by the same formula as σ, hence τ is ordinary.

It remains to show that we can choose A' to have a semiordinary involution with a prescribed coarse type $t \in \ker \Phi$. Let $V \to *$ be a covering such that t lifts to some $\varepsilon \in N(V)$. Again, we apply the proof of the "if" part of Theorem 6.10

with t, ε and A to obtain an Azumaya R-algebra with involution (A', τ) satisfying $A' \in [A]$ and $\mathrm{ct}(\tau) = t$. We then reapply the proof with $\mathrm{M}_{n\times n}(R)$ in place of A to obtain another Azumaya R-algebra with involution (A'_1, τ_1) such that A'_1 is split and $\mathrm{ct}(\tau_1) = t$. By construction, after suitable refinement of $V \to *$, both (A'_V, τ_V) and $(A'_{1,V}, \tau_{1,V})$ are isomorphic to $(\mathrm{M}_{2n\times 2n}(R_V), \sigma)$, where is σ given by $x \mapsto ([\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix}] x [\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix}]^{-1})^{\lambda \mathrm{tr}}$ on sections. Consequently, (A', τ) and (A'_1, τ_1) are locally isomorphic and therefore τ is semiordinary. ☐

We now shift our attention from the involutions τ to the coarse types t.

Definition 6.18. Let $t \in \mathrm{H}^0(T)$ be a coarse λ-type. We say that t is *realizable* if there exists some Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra A and some λ-involution $\tau : A \to \Lambda A$ with coarse type t. We also say that t is *realizable in degree* n when A can be chosen so that $n = \deg A$. When τ can be chosen to be ordinary, resp. semiordinary, we call t *ordinary*, resp. *semiordinary*.

The following theorem characterizes the realizable, semiordinary, and ordinary coarse types in cohomological terms.

Theorem 6.19. *With notation as in Theorem 6.10, let $t \in \mathrm{H}^0(T)$ be coarse type and let $\delta^0 : \mathrm{H}^0(T) \to \mathrm{H}^1(R^\times/S^\times)$, $\delta^1 : \mathrm{H}^1(R^\times/S^\times) \to \mathrm{H}^2(S^\times)$ and $\Phi = \delta^1 \circ \delta^0$ be as in Subsection 6.3. Then:*

(i) t is realizable if and only if $\Phi(t) \in \mathrm{im}(\mathrm{transf}_\lambda : \mathrm{Br}(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to \mathrm{H}^2(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}^\times))$.

(ii) t is semiordinary if and only if $\Phi(t) = 0$.

(iii) t is ordinary if and only if $\delta^0(t) = 0$, or equivalently $t \in \mathrm{im}(\mathrm{H}^0(N) \to \mathrm{H}^0(T))$.

When (ii) or (iii) hold, t is realizable in degree 2, and hence in all even degrees.

Proof. (i) This follows form Theorem 6.10.

(ii) The "only if" part follows from Theorems 6.17 and 6.10. The "if" part and the assertion that t can be realized in degree 2 follow by applying Theorem 6.17 with $A = R$.

(iii) Suppose t is ordinary, say $t = \mathrm{ct}(\mathrm{M}_n(R), \tau)$ with τ given by $x \mapsto (hxh^{-1})^{\lambda \mathrm{tr}}$ on sections. Then we can apply Construction 5.9 by with $U = *$, $\psi = \mathrm{id}$, and h as above, resulting in $\varepsilon \in \mathrm{H}^0(N)$, which then maps onto $t \in \mathrm{H}^0(T)$.

The reverse implication follows by applying Theorem 6.17 with $A = R$. ☐

Corollary 6.20. *With the notation of Theorem 6.10, suppose $\mathcal{O}_{\mathbf{Y}}^\times$ has square roots locally, and assume further that $2 \in \mathcal{O}_{\mathbf{Y}}^\times$ or $\pi : \mathbf{X} \to \mathbf{Y}$ is unramified. Let (A, τ) be an Azumaya $\mathcal{O}_{\mathbf{X}}$-algebra with a λ-involution. Then τ is ordinary, resp. semiordinary, if and only if its coarse type is.*

Proof. The "only if" part is clear, so we turn to the "if" part. We replace (A,τ) with $(\pi_*A,\pi_*\tau)$, see Theorem 4.28 and Corollary 4.31, and write $t=\mathrm{ct}(\tau)$. In case $\mathbf{Y}$ is not connected, we express $*_{\mathbf{Y}}$ as $\bigsqcup_{n\in\mathbb{N}} Y_n$ such that A_{Y_n} has degree n, and work with each component separately. We may therefore assume that $n:=\deg A$ is constant.
By Theorem 5.17, it is enough to find an Azumaya R-algebra A' with an ordinary, resp. semiordinary, involution τ' such that $\deg A=\deg A'$ and $\mathrm{ct}(\tau)=\mathrm{ct}(\tau')$. If n is odd, then $t=1$ by Theorem 5.21(iii), and we can take $(A',\tau')=(\mathrm{M}_{n\times n}(R),\lambda\mathrm{tr})$. Otherwise, $n=2m$, and applying Theorem 6.17 to $\mathrm{M}_{m\times m}(R)$ yields an algebra with an ordinary, resp. semiordinary, involution (A',τ') such that $\mathrm{ct}(\tau')=\mathrm{ct}(\tau)$ and $\deg A'=\deg A$; here we used parts (ii) and (iii) of Theorem 6.19. □

Corollary 6.21. *With the notation of Theorem 6.10, suppose that*

(1) $\pi:\mathbf{X}\to\mathbf{Y}$ *is a trivial quotient (Example 4.22), or*

(2) $\pi:\mathbf{X}\to\mathbf{Y}$ *is everywhere ramified,* $2\in\mathcal{O}_{\mathbf{Y}}^\times$ *and* $\mathcal{O}_{\mathbf{Y}}^\times$ *has square roots locally, or*

(3) $\pi:\mathbf{X}\to\mathbf{Y}$ *is unramified.*

Then all coarse λ*-types are realizable and ordinary.*

Proof. It follows from the proof of Proposition 6.9 that in all three cases, $\delta^0:\mathrm{H}^0(T)\to\mathrm{H}^0(R^\times/S^\times)$ is the 0 map. Now apply Theorem 6.19(iii). □

Corollary 6.22. *With the notation of Theorem 6.10, suppose that* $\mathcal{O}_{\mathbf{Y}}^\times$ *has square roots locally and moreover*

(1) $\pi:\mathbf{X}\to\mathbf{Y}$ *is everywhere ramified and* $2\in\mathcal{O}_{\mathbf{Y}}^\times$*, or*

(2) $\pi:\mathbf{X}\to\mathbf{Y}$ *is unramified.*

Then all λ*-involutions are ordinary.*

Proof. This follows from Corollaries 6.20 and 6.21. □

Corollary 6.23. *Let* X *be a scheme, let* $\lambda:X\to X$ *be an involution, and let* $\pi:X\to Y$ *be a good quotient relative to* $\{1,\lambda\}$*. Assume* Y *is noetherian and regular, and* π *is quadratic étale on the generic points of* Y*. Then all coarse* λ*-types are realizable and semiordinary. If moreover* 2 *is invertible on* Y*, then all* λ*-involutions are semiordinary.*

Proof. The first assertion follows from Proposition 6.9(iv) and Theorem 6.19(ii). The second assertion then follows from Corollary 6.20. □

We conclude this section with two problems, both of which are open both in the context of varieties over fields of characteristic different from 2 with (ramified) involutions and in the context of topological spaces with (non-free) C_2-actions.

Problem 6.24. Is there an element $t \in \mathrm{cTyp}(\lambda) = \mathrm{H}^0(T)$ that is not the coarse type of any Azumaya algebra with λ-involution?

Problem 6.25. Is there an Azumaya algebra A with a λ-involution τ that is extraordinary (i.e. not semiordinary)?

By Theorem 5.37, the first problem can be phrased as follows: Suppse that X is a scheme with involution λ admitting a good quotient relative to $\{1, \lambda\}$, or X is a Hausdorff topological space with involution λ. Let Z be the locus of points where λ ramifies and let $Z = Z_{-1} \sqcup Z_1$ be a partition of Z into two closed subsets. Is it always possible to find an Azumaya algebra A over X admitting a λ-involution τ such that the specialization of τ to $k(z)$ is orthogonal if $z \in Z_1$ and symplectic if $z \in Z_{-1}$?

7 Examples and Applications

Example 7.1. Fix an exact quotient $\pi : (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to (\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$ and write $R = \pi_* \mathcal{O}_{\mathbf{X}}$, $S = \mathcal{O}_{\mathbf{Y}}$. We assume that $S^\times$ has square roots locally and $2 \in S^\times$. This assumption allows us to drop the distinction between types and coarse types for the most part (Corollary 5.18). As in the previous section, we use the notation N for the kernel the λ-norm map $x \mapsto x^\lambda x : R^\times \to S^\times$, and T for the quotient of N by the image of the map $R^\times \to N$ given by $r \mapsto r^{-1} r^\lambda$. The coarse types are then $\mathrm{H}^0(\mathbf{Y}, T)$.

Suppose t is an ordinary coarse type. By Theorem 6.19, this is equivalent to saying there exists some ε in $\mathrm{H}^0(\mathbf{Y}, N)$ mapping to t under the map $\mathrm{H}^0(\mathbf{Y}, N) \to \mathrm{H}^0(\mathbf{Y}, T)$. Such an ε can always be found if $\mathrm{H}^1(\mathbf{Y}, R^\times / S^\times)$ vanishes, for instance.

Let n be a natural number. Consider the matrix

$$h = h_{2n}(\varepsilon) = \begin{bmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{bmatrix}.$$

It is immediate that $\varepsilon h^{\lambda \mathrm{tr}} = h$. This equality implies that the map $\tau_\varepsilon : \mathrm{Mat}_{2n \times 2n}(R) \to \mathrm{Mat}_{2n \times 2n}(R)$ given on sections by

$$M \mapsto (h_{2n}(\varepsilon)\, M\, h_{2n}(\varepsilon)^{-1})^{\lambda \mathrm{tr}}.$$

is a λ-involution. The (coarse) type of τ is easily seen to be t, the image of ε in $\mathrm{H}^0(T)$. This follows from Construction 5.9.

In this case, any algebra of degree $2n$ with involution of coarse type t is locally isomorphic to $(\mathrm{Mat}_{2n \times 2n}(R), \tau)$, by Theorem 5.17. Thanks to Corollary 5.23, we may therefore place such algebras in bijective correspondence with G-torsors on $\mathbf{Y}$ where $G = \mathrm{PU}(\mathrm{Mat}_{2n \times 2n}(R), \tau_\varepsilon) \cong \mathcal{A}ut_R(\mathrm{Mat}_{2n \times 2n}(R), \tau_\varepsilon)$.

Example 7.2. As a special case of the previous example, we describe the Azumaya algebras with *symplectic involution* on a scheme or topological space with trivial involution. See Theorem 4.35 for the specific hypotheses on the underlying geometric object, and note that we assume 2 is invertible.

In this case, $\pi = \mathrm{id}$, $R = S$, and $N = \mu_{2,R}$. By Theorem 5.21(i), the group of coarse types is $\mathrm{H}^0(\mu_{2,R})$, which is just $\{1, -1\}$ when X is connected. We consider the (coarse) type -1, called the *symplectic* type.
Any Azumaya algebra with involution having this type is of even degree, $2n$, by Theorem 5.21(iii), and is locally isomorphic to the split degree-$2n$ algebra with symplectic involution

$$\mathrm{sp} : M \mapsto (h_{2n}(-1) M h_{2n}(-1)^{-1})^{\mathrm{tr}}.$$

The unitary group of $(\mathrm{M}_{2n\times 2n}(R), \mathrm{sp})$ is the familiar symplectic group $\mathrm{Sp}_{2n}(R)$, and it follows from Lemma 5.22 that the automorphism group of $(\mathrm{M}_{2n\times 2n}(R), \mathrm{sp})$ is

$$\mathrm{PSp}_{2n}(R) := \mathrm{Sp}_n(R)/\mu_{2,R}\ .$$

In particular, as noted in Corollary 5.23, the set of isomorphism classes of degree-$2n$ Azumaya algebras with symplectic involution is in canonical bijection with

$$\mathrm{H}^1(\mathbf{X}, \mathrm{PSp}_{2n}(R)).$$

Since the symplectic type is ordinary, by Theorem 6.17 and Example 6.3, we derive the well known fact that an Azumaya algebra A on X is Brauer equivalent to one having a symplectic involution if and only if the Brauer class of A is 2-torsion.

Example 7.3. Fix an exact quotient $\pi : \mathbf{X} \to \mathbf{Y}$ with ring objects $\mathcal{O}_{\mathbf{X}}$, and let $R = \pi_* \mathcal{O}_{\mathbf{X}}$ and $S = \mathcal{O}_{\mathbf{Y}}$. Let n be a natural number and assume that the hypotheses of Theorem 5.17 hold, namely $S^\times$ has square roots locally, and either $2 \in S^\times$, or π is unramified, or n is odd. We consider the trivial type, 1. This is the type of the involution

$$M \mapsto M^{\lambda \mathrm{tr}}$$

on the split algebra $\mathrm{Mat}_{n\times n}(R)$.
Any algebra with involution of the trivial type is locally isomorphic to this one, and therefore, as summarized in Corollary 5.23, these are classified by G torsors where $G = \mathcal{A}ut_R(\mathrm{M}_{n\times n}(R), \lambda\mathrm{tr}) \cong \mathrm{PU}(\mathrm{M}_{n\times n}(R), \lambda\mathrm{tr})$. We write the latter group as

$$\mathrm{PU}_n(R, \lambda)$$

and call it the *projective unitary group* of rank n for the involution λ. In accordance with this notation, the unitary group of $(\mathrm{M}_{n\times n}(R), \lambda\mathrm{tr})$ will be denoted $\mathrm{U}_n(R, \lambda)$.

Example 7.4. Consider the case of a scheme or a topological space X with trivial involution, as in the case of Example 7.2. The theory of Azumaya algebras with involution of type 1 can be established along the same lines as that of type -1. These algebras are called *orthogonal*. The automorphism group of $(\mathrm{M}_{2n\times 2n}(R), \mathrm{tr})$ is the quotient group $O_{2n}(R)/\mu_{2,R}$, which we denote

by $\mathrm{PO}_{2n}(R)$, the *projective orthogonal group*. This is special notation for the group $\mathrm{PU}_{2n}(R, \mathrm{id})$ of Example 7.3.
Again, by reference to Theorem 6.17 and Example 6.3, an algebra is Brauer equivalent to one having involution of this type if and only if the Brauer class is 2-torsion.

Example 7.5. In this example we discuss *unitary involutions*. As a special case of Example 7.3, we consider the case of an unramified double covering $\pi : X \to Y$ of schemes or topological spaces. Again, we refer to Theorem 4.35 for the specific hypotheses on the underlying geometric object.
In this case, the ring object R is a quadratic étale extension of S, see Propositions 4.45 and 4.46. Since π is unramified, Theorem 5.21(ii) implies that there is only one type of involution on Azumaya algebras, the trivial one, which is called *unitary* in this context. In particular, we are in a special case of Example 7.3.
The structure of the groups $\mathrm{U}_n(R, \lambda)$ and $\mathrm{PU}_n(R, \lambda) = \mathrm{U}_n(R, \lambda)/N$ depends on the nature of λ, so a complete description in the abstract is not possible. We can, however, find an étale, resp. open, covering $U \to Y$ such that $R_U \cong S_U \times S_U$. After specializing to U, the algebra $\mathrm{M}_{n\times n}(R)$ becomes $\mathrm{M}_{n\times n}(S) \times \mathrm{M}_{n\times n}(S)$ and the involution $\lambda\mathrm{tr}$ becomes the involution given sectionwise by $(x, y) \mapsto (y^{\mathrm{tr}}, x^{\mathrm{tr}})$. From this, one verifies that $\mathrm{U}_n(R, \lambda)_U \cong \mathrm{GL}_n(S)_U$ and $\mathrm{PU}_n(R, \lambda)_U \cong \mathrm{PGL}_n(S)_U$. Consequently, for a general degree-n Azumaya R-algebra with involution (A, τ), the groups $\mathrm{U}(A, \tau)$ and $\mathrm{PU}(A, \tau)$ are locally isomorphic to $\mathrm{GL}_n(S)$ and $\mathrm{PGL}_n(S)$, respectively.
This agrees with the well established fact that projective unitary group schemes of unitary involutions are of type A.

Example 7.6. In another instance of Example 7.1, we can produce an example of an Azumaya algebra with an involution that mixes the various classical types. This example also featured in the introduction.
We work with étale sheaves and étale cohomology throughout, see Example 4.20. Let k be an algebraically closed field and let $X = \operatorname{Spec} k[x, x^{-1}]$ with the k-linear involution λ sending x to x^{-1}. A good quotient of X by this involution exists, and is given by $Y = \operatorname{Spec} k[y]$ where $y = x + x^{-1}$.
Here, the ring object R is the ring $k[x, x^{-1}]$ viewed as a sheaf of rings on Y, and the ring object S is the structure sheaf of Y. The sheaf N is the sheaf of norm-1 elements in R, where the norm map sends a Laurent polynomial $p(x)$ to $p(x)p(x^{-1})$. Both the Picard and the Brauer groups of X and Y vanish, so that we can calculate $\mathrm{H}^1(Y, R^\times/S^\times) = 0$. The following sequence is therefore exact

$$1 \to \mathrm{H}^0(Y, R^\times/S^\times) \xrightarrow{\psi} \mathrm{H}^0(Y, N) \to \mathrm{H}^0(Y, T) \to 1.$$

Explicitly, we calculate that $\mathrm{H}^0(Y, R^\times/S^\times)$ consists of classes of monomials x^i for $i \in \mathbb{Z}$, and $\mathrm{H}^0(Y, N)$ consists of monomials of the form $\pm x^i$ for $i \in \mathbb{Z}$, but ψ maps the class of x^i to $x^i/x^{-i} = x^{2i}$. Therefore, the group of (coarse) types is isomorphic to the Klein 4-group: $\mathrm{H}^0(Y, T) = \{\overline{1}, \overline{-1}, \overline{x}, \overline{-x}\}$.

Since $\mathrm{H}^1(Y, R^\times/S^\times) = 0$, we are in the circumstance of Example 7.1 which provides models for each of the four types on even-degree split algebras. For instance, on $\mathrm{Mat}_{2\times 2}(R)$, we have the involution given by conjugating by $\left[\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right]$ and then applying $\lambda\mathrm{tr}$, explicitly

$$\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} \mapsto \begin{bmatrix} d(x^{-1}) & x^{-1}b(x^{-1}) \\ xc(x^{-1}) & a(x^{-1}) \end{bmatrix}.$$

Away from the fixed locus of $\lambda : X \to X$, namely, the points $x = 1$ and $x = -1$, this involution is unitary, whereas at $x = 1$ it specializes to be orthogonal and at $x = -1$ to be symplectic.

More generally, it follows from Theorem 5.37 that the type of any Azumaya X-algebra with involution (A, τ) is determined by the types seen upon specializing to $x = 1$ and $x = -1$.

Example 7.7. We now demonstrate that there exist involutions that are not locally isomorphic to involutions of the form exhibited in Example 7.1. Specifically, we will show that there are involutions which are not ordinary in the sense of Definition 6.16.

We consider a complex hyperelliptic curve X of genus g and a double covering $X \to Y =: \mathbb{P}^1_{\mathbb{C}}$. Explicitly: Let $a_1, \dots, a_{2g+1}$ be distinct complex numbers, and let

$$X_0 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - \prod_i (x - a_i)).$$

We complete X_0 by gluing it to $X_1 := \operatorname{Spec} \mathbb{C}[u, v]/(v^2 - u \prod_i (1 - a_i u))$ by mapping (x, y) to $(u, v) := (\frac{1}{x}, \frac{y}{x^{g+1}})$, and denote the resulting smooth complete curve by X. View $Y = \mathbb{P}^1_{\mathbb{C}}$ as the gluing of $Y_0 := \operatorname{Spec} \mathbb{C}[x]$ to $Y_1 := \operatorname{Spec} \mathbb{C}[u]$ via $(x : 1) \leftrightarrow (1 : u^{-1})$. Projection onto the x or u coordinate induces a double covering $\pi : X \to Y$ with ramification at the points $(a_1 : 0), \dots, (a_{2g+1} : 0)$ and $(0 : 1)$. The map $\lambda : X \to X$ given by $(x, y) \mapsto (x, -y)$, resp. $(u, v) \mapsto (u, -v)$, on the charts is an involution and π is a good quotient relative to $C_2 := \{1, \lambda\}$. Indeed, working with the affine covering $Y = Y_0 \cup Y_1$, we see that $\mathbb{C}[x]$ is the fixed ring of

$$\lambda^\# : \frac{\mathbb{C}[x, y]}{(y^2 - \prod_i (x - a_i))} \to \frac{\mathbb{C}[x, y]}{(y^2 - \prod_i (x - a_i))}, \qquad x \mapsto x, \quad y \mapsto -y,$$

and similarly on the other chart.

By Corollary 6.23, all coarse λ-types in $\mathrm{H}^0(T)$ are realizable and semiordinary. Since the branch locus of π consists of $2g+2$ points, it follows from Theorem 5.37 that there are 2^{2g+2} λ-types. Theorem 6.19 also says that the number of ordinary types is the cardinality of the image of the map $\mathrm{H}^0(N) \to \mathrm{H}^0(T)$, where N is the sheaf of sections of λ-norm 1 in $R := \pi_* \mathcal{O}_X$. Let $\varepsilon \in \mathrm{H}^0(N)$. Then $\varepsilon \in \mathrm{H}^0(Y, \pi_* \mathcal{O}_X) = \mathrm{H}^0(X, \mathcal{O}_X)$. Since X is a complete complex curve, the global sections of the structure sheaf are constant functions, meaning that $\varepsilon \in \mathbb{C}^\times$. Since $\varepsilon^\lambda \varepsilon = 1$, it follows that $\mathrm{H}^0(N) = \{\pm 1\}$. The images of $1, -1 \in \mathrm{H}^0(N)$ in $\mathrm{H}^0(T)$ are therefore the ordinary types. Thus, of the 2^{2g+2} possible coarse types, only 2 are ordinary, and the remaining $2^{2g+2}-2$ are merely semiordinary.

We remark that we have reached the latter conclusion without actually constructing Azumaya algebras with involution realizing any of the non-ordinary types. A construction is given in the proof of Theorem 6.19, and it can be made explicit in our setting with further work.

This example can also be carried with the affine models X_0 and Y_0. One can check directly that $\mathrm{H}^0(X_0, \mathcal{O}_{X_0}^\times) = \mathbb{C}^\times$ and thus it is still the case that $\mathrm{H}^0(Y, N) = \{\pm 1\}$. Since the ramification point $(0:1) \in \mathbb{P}^1_{\mathbb{C}} = Y$ was removed, in this case, there are 2^{2g+1} coarse types, all semiordinary, of which only 2 are ordinary.

Example 7.8. A surprising source of examples comes from Clifford algebras of quadratic forms with simple degeneration. We refer the reader to [ABB14, §1] or [APS15, §1] for all relevant definitions.

Let Y be a scheme on which 2 is invertible and let (E, q, L) be a line-bundle-valued quadratic space of even rank n over Y; when $L = \mathcal{O}_Y$ and $Y = \operatorname{Spec} S$, these data merely amount to specifying a quadratic space of rank n over the ring S. According to [APS15, Dfn. 1.9], q is said to have *simple degeneration* if for every $y \in Y$, the specialization of q to $k(y)$ is a quadratic form whose radical has dimension at most 1. In this case, it shown in [APS15, Prp. 1.11] that the even Clifford algebra $C_0(q)$, which is a sheaf of $\mathcal{O}_Y$-algebras, is Azumaya over its centre $Z(q)$. Furthermore, the sheaf $Z(q)$ corresponds to a flat double covering $\pi : X \to Y$, which ramifies at the points $y \in Y$ where $q_{k(y)}$ is degenerate. As such, π is a good quotient relative to the involution $\lambda : X \to X$ induced by the involution of $Z(q)$ given by $x \mapsto \operatorname{Tr}_{Z(q)/\mathcal{O}_Y}(x) - x$ on sections. Abusing the notation, we realize $C_0(q)$ as an Azumaya algebra over X.

Suppose q has simple degeneration. We moreover assume that Y is integral, regular and noetherian with generic point ξ and that $q_{k(\xi)}$ is nondegenerate, although it is likely that these assumptions are unnecessary. The algebra $C_0(q)$ has a canonical involution τ_0, see [Aue11, §1.8], and by applying [KMRT98, Prp. 8.4] to $C_0(q_{k(\xi)})$, we see that τ_0 is of the first kind when $n \equiv 0 \pmod 4$ and a λ-involution when $n \equiv 2 \pmod 4$. Nonetheless, in the case $n \equiv 0 \pmod 4$, we have $\operatorname{transf}_\lambda([C_0(q)]) = 0$ because $\operatorname{transf}_\lambda([C_0(q_{k(\xi)})]) = 0$ by [KMRT98, Thm. 9.12], and $\operatorname{Br}(Y) \to \operatorname{Br}(k(\xi))$ is injective by [Gro68a, Cor. 1.8] (or [AG60, Thm. 7.2] in the affine case). It therefore follows from Theorem 6.10 that there exists $A' \in [C_0(q)]$ with $\deg A' = 2 \deg C_0(q) = 2^n$ that admits a λ-involution. We expect that the choice of A' and its involution can be done canonically in q, and with no restrictions on Y.

With the observations just made, it is possible that our work could facilitate the study of Clifford invariants of non-regular quadratic forms, e.g. in [Voi11a], [ABB14], [APS15].

8 Topology and Classifying Spaces

The remainder of this paper is concerned with constructing a quadratic étale map of complex varieties $X \to Y$ and an Azumaya algebra A over over X such that A is Brauer equivalent to an algebra A' with a λ-involution, λ being the non-trivial Y-automorphism of X, but such that the smallest degree of any such A' is $2 \deg A$.

We recall that in this particular case, a Brauer equivalent algebra of degree $2 \deg A$ admitting a λ-involution is guaranteed to exist by a theorem of Knus, Parimala and Srinivas [KPS90, Thm. 4.2]; this has been generalized in Theorem 6.10. An analogous example in which $\lambda : X \to X$ is the trivial involution was exhibited in [AFW19].

The example, which is constructed in Section 9, will be obtained by means of topological obstruction theory, similarly to the methods of [AW14c], [AFW19] and related works. That is, the desired properties of A above will be verified by establishing them for the topological Azumaya algebra $A(\mathbb{C})$ over the complexification $X(\mathbb{C})$, whereas the latter will be done by means of certain homotopy invariants.

This section is foundational, describing in part an approach to topological Azumaya algebras with involution via equivariant homotopy theory. The main points are that Azumaya algebras with involution correspond to principal $\mathrm{PGL}_n(\mathbb{C})$-bundles with involution—a fact that is true even outside the topological context, but that we have not emphasized until now—, that there are equivariant classifying spaces for such bundles, and that their theory is tractable if one restricts to considering spaces X on which the C_2-action is trivial or free.

8.1 Preliminaries

In this section and the next, all topological spaces will be tacitly assumed to have a number of desirable properties. All spaces appearing will be assumed to be compactly generated, Hausdorff, paracompact and locally contractible.

Throughout, we work in the category of C_2-topological spaces and C_2-equivariant maps. There are two notions of homotopy one can consider for maps in this setting, the *fine*, in which homotopies are themselves required to be C_2-equivariant, and the *coarse*, where non-equivariant homotopies are allowed. These two notions each have model structures appropriate to them, the fine and the coarse. In the fine model structure, the weak equivalences are the equivariant maps $f : X \to Y$ inducing weak equivalences on fixed point sets $f : X^G \to Y^G$ where G is either the group C_2 or the subgroup $\{1\}$. In the coarse structure, it is required only that $f : X \to Y$ be a weak equivalence when the C_2-action is disregarded, that is, only the subgroup $\{1\}$ is considered. The identity functor is a left Quillen functor from the coarse to the fine. This is a synthesis of the theory of [DK84] with [Elm83].

Notation 8.1. The notation $[X, Y]$ is used to denote the set of maps between two (possibly unpointed) objects X and Y in a homotopy category. The notation $[X, Y]_{C_2}$ will be used to denote the set of maps between X and Y in the fine C_2-equivariant homotopy category, whereas $[X, Y]_{C_2\text{-coarse}}$ will be used for the coarse structure.

Remark 8.2. In the case of the coarse model structure, the cofibrant objects include the C_2-CW-complexes with free C_2-action, and if X is a C_2-CW-complex, then the construction $X \times EC_2 \to X$ furnishes a cofibrant replacement of X.

All spaces are fibrant in both the coarse and the fine model structures, which implies the following standard result.

Proposition 8.3. *If X is a free C_2-CW-complex and Y is a C_2-space, then there is a natural bijection*

$$[X, Y]_{C_2} \longleftrightarrow [X, Y]_{C_2\text{-}coarse}.$$

It is well known that C_2-equivariant homotopy theory in the coarse sense is equivalent to homotopy theory carried out over the base space BC_2. We refer to [Shu08, Sec. 8] for a sophisticated general account of this equivalence. Specifically, the Borel construction $X \mapsto X \times^{C_2} EC_2$ and the relative mapping space $Y \mapsto \mathrm{Map}_{BC_2}(EC_2, Y)$ form a Quillen equivalence between C_2-equivariant spaces with the coarse structure, and spaces over BC_2, endowed with what [Shu08] calls the "mixed" structure on spaces over BC_2.

Proposition 8.4. *Suppose X and Y are C_2-spaces with X being a C_2-CW-complex. Then the Borel construction $(\cdot) \times^{C_2} EC_2$ gives rise to a natural bijection*

$$\left\{ \begin{array}{c} \textit{coarse } C_2\textit{-homotopy classes} \\ \textit{of maps } X \to Y \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{homotopy classes of } BC_2\textit{-maps} \\ X \times^{C_2} EC_2 \to Y \times^{C_2} EC_2 \end{array} \right\}.$$

8.2 Equivariant Bundles and Classifying Spaces

There is a general theory of equivariant bundles and classifying spaces, more general indeed than what is required in this paper. All examples we consider are of the following form:

Definition 8.5. Suppose we are given a topological C_2-group G, or equivalently, a topological group G equipped with an involutary automorphism $\tau : G \to G$. A *principal G-bundle with a τ-involution*, or just *principal G-bundle with involution*, on a C_2-space X is a map $\pi : E \to X$ in C_2-spaces such that:

1. $\pi : E \to X$ is a principal G-bundle,
2. the actions of C_2 and of G on E are compatible, in the sense that if $c \in C_2$, $g \in G$ and $e \in E$, then

$$c \cdot (e \cdot g) = (c \cdot e) \cdot (c \cdot g).$$

Remark 8.6. This concept admits an equivalent definition. Any G-bundle E, equivariant or not, may be pulled back along the involution λ of X, in order to form $\lambda^* E \to X$. One may then twist the the G-action on $\lambda^* E$ by changing the structure group along $\tau : G \to G$, forming $E^* := G \times_\tau \lambda^* E$. This may be identified with $\lambda^* E$ as a topological space over X, but with a different G-action. The definition of principal G-bundle with involution given above is equivalent to asking that $\pi : E \to X$ be a principal G-bundle together with a G-bundle morphism f of order 2 from $\pi : E \to X$ to $\pi : E^* \to X$. On the underlying spaces, f must be an isomorphism of order 2 of E over X, which is equivalent to a C_2-action on E making $\pi : E \to X$ equivariant. The fact that $f : E \to E^*$ is an isomorphism of principal G-bundles is exactly the relation $c \cdot (e \cdot g) = (c \cdot e) \cdot (c \cdot g)$ above.

Because the automorphism $\tau : G \to G$ is not assumed to be trivial, this notion is more general than the most basic notion of 'equivariant principal G-bundle', but at the same time, because the sequence

$$1 \to G \to G \rtimes C_2 \to C_2 \to 1$$

is split, it is less general than the most general case considered in [May90]. One may construct a C_2-equivariant classifying space for C_2-equivariant principal G-bundles, as in [May90, Thm. 5]. We will take the time to explain the procedure, since some of the details will be important later [1]

NOTATION 8.7. The notation $EG \to BG$ will be used for a construction of the classifying space of a topological group G, functorial in G.

By functoriality, if G admits a C_2-action, then $EG \to BG$ admits a C_2-action. While the ordinary homotopy type of $EG \to BG$ is well defined, irrespective of the model we choose, the C_2-equivariant type is not. The construction $E_{C_2}G \to B_{C_2}G$ outlined below is a specific choice of such a type.
Start with $EG \to BG$. Now consider the space of continuous functions $\mathcal{C}(EC_2, EG)$. It is endowed with both a G-action, induced directly by the G-action on EG, and by a C_2-action given by conjugation of the map. The two actions together induce an action of $G \rtimes C_2$ on $\mathcal{C}(EC_2, EG)$, which is contractible, and consequently a C_2-action on $\mathcal{C}(EC_2, EG)/G$, which is a model for BG. The resulting map

$$\mathcal{C}(EC_2, EG) \to \mathcal{C}(EC_2, EG)/G$$

is a map of C_2-spaces, and will be denoted

$$E_{C_2}G \to B_{C_2}G.$$

[1]We remark that in our case, the group called Γ in [May90] is a semidirect product, so $EG \times EC_2$, with an appropriate Γ-action, is a model for $E\Gamma$. This allows us to replace the space of sections of $EC_2 \to E\Gamma$ by the space of maps $EC_2 \to EG$, an argument that appears in [GMM17, Sec. 5, p. 21].

We remark that in [May90] and other sources, May and coauthors denote these spaces $E(G; G \rtimes C_2)$ and $B(G; G \rtimes C_2)$.
Furthermore, the map $EC_2 \to *$ induces a map $EG = \mathcal{C}(*, EG) \to \mathcal{C}(EC_2, EG) = E_{C_2}G$. This map is $G \rtimes C_2$-equivariant, and induces a C_2-equivariant commutative square

$$\begin{array}{ccc} EG & \xrightarrow{\sim} & E_{C_2}G \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\sim} & B_{C_2}G, \end{array} \tag{8.1}$$

in which the horizontal maps are coarse, but not necessarily fine, C_2-weak equivalences. The map $E_{C_2}G \to B_{C_2}G$ is a classifying space for principal G-bundles with involution.

Proposition 8.8. *If X is a C_2-CW-complex, then there is a natural bijection between $[X, B_{C_2}G]_{C_2}$ and the set of isomorphism classes of principal G-bundles with involution on X.*

We refer to [May90, Thm. 5] for the proof.

Proposition 8.9. *If X is a free C_2-CW-complex, then the following are naturally isomorphic*

(a) $[X, B_{C_2}G]_{C_2}$,

(b) $[X, B_{C_2}G]_{C_2\text{-coarse}}$,

(c) $[X, BG]_{C_2}$,

(d) $[X, BG]_{C_2\text{-coarse}}$,

(e) The set of isomorphism classes of principal G bundles with involution on X.

Proof. The equivalences all follow from Propositions 8.3, 8.8 and Diagram (8.1). □

Remark 8.10. Proposition 8.9 means that if one is willing to restrict one's attention to spaces with free C_2-action, then the construction of $E_{C_2}G \to B_{C_2}G$ from $EG \to BG$ is not necessary. The C_2-action given by the functoriality of the construction of BG is sufficient.

Remark 8.11. Let G be a topological group. One may give $G \times G$ the C_2-action which interchanges the two factors. Then the resulting classifying space $BG \times BG$ also admits this action. In this instance, the space $B_{C_2}(G \times G)$ is C_2-equivalent to $BG \times BG$ with the interchange action, which may be verified by testing on C_2-fixed points, for instance.

The construction of taking a space Y and producing $Y \times Y$ with the C_2-action interchanging the factors is right adjoint to the forgetful functor. Suppose X is a C_2-space, then

$$[X, BG] \cong [X, BG \times BG]_{C_2} \tag{8.2}$$

where the set on the left is the set of maps in the nonequivariant homotopy category.

8.3 THE CASE OF PGL_n-BUNDLES

For the rest of this section, we write GL_n, PGL_n etc. for the Lie group of complex points, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{PGL}_n(\mathbb{C})$ and so on.

We now specify C_2-actions on groups that will appear in the sequel. There is a C_2-action on GL_n in which the non-trivial element acts via $A \mapsto A^{-\mathrm{tr}}$, the transpose-inverse. This passes to certain subquotients of GL_n, and we will use it as the C_2-action on the groups μ_n, $\mathbb{C}^\times$, SL_n and PGL_n, all viewed either as subgroups or as quotients of GL_n. Specifically, we write $-\mathrm{tr} : \mathrm{PGL}_n \to \mathrm{PGL}_n$ for the outer automorphism $A \mapsto A^{-\mathrm{tr}}$.

There is also a C_2-action on $\mathrm{GL}_n \times \mathrm{GL}_n$ given by interchanging the factors and then applying the transpose-inverse, so that the induced involution is

$$(A, B) \mapsto (B^{-\mathrm{tr}}, A^{-\mathrm{tr}}) \ .$$

This will be used for certain subquotients of this group, including $\mu_n \times \mu_n$, $\mathbb{C}^\times \times \mathbb{C}^\times$, $\mathrm{SL}_n \times \mathrm{SL}_n$ and $\mathrm{PGL}_n \times \mathrm{PGL}_n$.

There is a diagonal inclusion $\mathrm{GL}_n \to \mathrm{GL}_n \times \mathrm{GL}_n$, given by $A \mapsto (A, A)$. It is C_2-equivariant, and induces similar maps for the aforementioned subquotients of GL_n.

One may form C_2-equivariant classifying spaces for the groups named above, as outlined in Subsection 8.2. Among the possibilities, two are particularly useful to us: $B_{C_2}\,\mathrm{PGL}_n$ and $B_{C_2}(\mathrm{PGL}_n \times \mathrm{PGL}_n)$.

PROPOSITION 8.12. *Let X be a C_2-CW-complex with corresponding involution λ, and let n be a natural number. Then the following sets are in natural bijective correspondence:*

(a) Isomorphism classes of degree-n topological Azumaya algebras with λ-involution on X,

(b) Isomorphism classes of principal PGL_n-bundles with involution on X,

(c) $[X, B_{C_2}\,\mathrm{PGL}_n]_{C_2}$.

Proof. There is a well-known bijection between Azumaya algebras of degree n on X and principal PGL_n-bundles, since $\mathrm{PGL}_n(\mathbb{C})$ is the automorphism group of $M := \mathrm{Mat}_{n\times n}(\mathbb{C})$ as a $\mathbb{C}$-algebra, see Subsection 2.5. Let A be an Azumaya algebra of degree n on X and P the associated principal PGL_n-bundle.

The functor of taking opposite algebras on Azumaya algebras corresponds to the functor of change of group along $-\mathrm{tr} : \mathrm{PGL}_n \to \mathrm{PGL}_n$ of principal PGL_n-bundles; this can be seen at the level of clutching functions. Indeed, note that $m \mapsto m^{\mathrm{tr}} : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})^{\mathrm{op}}$ is a $\mathbb{C}$-algebra isomorphism. If one chooses coordinates for A on two open sets of X on which it trivializes, then the clutching function $f : \mathrm{Mat}_{n\times n}(\mathbb{C}) \times (U_1 \cap U_2) \to \mathrm{Mat}_{n\times n}(\mathbb{C}) \times (U_1 \cap U_2)$ given by $m \mapsto xmx^{-1}$, for some $x : (U_1 \cap U_2) \to \mathrm{PGL}_n(\mathbb{C})$. For the same choice of coordinates over both U_1 and U_2, the clutching function f^{op} of the opposite algebra is given by $m^{\mathrm{tr}} \mapsto (xmx^{-1})^{\mathrm{tr}} = x^{-\mathrm{tr}} m^{\mathrm{tr}} x^{\mathrm{tr}}$.

Therefore, the data of an isomorphism of $A \to A^{\mathrm{op}}$ of order 2 over the involution $\lambda : X \to X$ is equivalent to an order-2 self-map of the associated principal PGL_n-bundle, $P \to P^*$ over X, where P^* denotes the principal PGL_n-bundle

$$P^* := \mathrm{PGL}_n \times_{-\mathrm{tr}} \lambda^* P.$$

As explained in Remark 8.6, this is equivalent to the definition of principal PGL_n-bundle with involution in Definition 8.5; thus establishing the equivalence of (a) and (b).

The equivalence between (b) and (c) is an application of Proposition 8.8. □

The space $B_{C_2}(\mathrm{PGL}_n \times \mathrm{PGL}_n)$, by similar methods, is seen to classify ordered pairs of PGL_n-bundles on a C_2-space X, such that the one is obtained from the other by twisting relative to the involutions of X and PGL_n. But the category of such ordered pairs is identical to the category of ordinary PGL_n-bundles on the space X, forgetting the C_2-action.

This last fact also manifests itself algebraically via Remark 8.11 in the following way: Suppose G is a subgroup of GL_n closed under taking transposes, or a quotient of GL_n by such a subgroup, let $(G \times G, \alpha)$ denote the product group with the involution $(A, B) \mapsto (B^{-\mathrm{tr}}, A^{-\mathrm{tr}})$, and let $(G \times G, i)$ denote the product group with the involution exchanging A and B. Then $(A, B) \mapsto (A, B^{-\mathrm{tr}})$ is a C_2-equivariant isomorphism between these two groups with involution.

We will apply the classifying space theory developed above in the two extreme cases where the C_2-action on X is trivial and when it is free.

8.4 Trivial Action

Suppose X is equipped with a trivial C_2-action. Then principal PGL_n-bundles with involution on X are classified by $[X, B_{C_2}\,\mathrm{PGL}_n]_{C_2} = [X, (B_{C_2}\,\mathrm{PGL}_n)^{C_2}]$.

Proposition 8.13. *Let n be a positive integer. Then the fixed locus $(B_{C_2}\,\mathrm{PGL}_n)^{C_2}$ is homeomorphic to*

(i) $B\,\mathrm{PO}_n \sqcup B\,\mathrm{PSp}_n$ if n is even;

(ii) $B\,\mathrm{PO}_n$ if n is odd.

Proof. We may calculate the fixed-point-sets of $B_{C_2}(\mathrm{PGL}_n)$ by means of [May90, Thm. 7]. We explain the application of this theorem in the current case.

If $A \in \mathrm{PGL}_n$ is a matrix such that $AA^{-\mathrm{tr}} = I_n$, then $(A, -\mathrm{tr}) \in \Gamma := \mathrm{PGL}_n \rtimes C_2$ generates a subgroup that maps isomorphically onto C_2 and intersects PGL_n trivially. Denote by $\mathrm{PGL}_n^{(A,-\mathrm{tr})}$ the commutant of $(A, -\mathrm{tr})$ in PGL_n, i.e., the subgroup of PGL_n consisting of elements X such that $X^{-\mathrm{tr}} = A^{-1}XA$. We write $A \sim A'$ if $(A, -\mathrm{tr})$ and $(A', -\mathrm{tr})$ are conjugate under PGL_n, or equivalently, if there exists $X \in \mathrm{PGL}_n$ such that $XAX^{\mathrm{tr}} = A'$. Then the theorem asserts that

$$(B_{C_2}\,\mathrm{PGL}_n)^{C_2} = \bigsqcup_A B(\mathrm{PGL}^{(A,-\mathrm{tr})})$$

as A runs over equivalence classes of elements $A \in \mathrm{PGL}_n$ satisfying $A^{-\mathrm{tr}}A = I_n$. When n is even, say $n = 2m$, there are two such equivalence classes, namely the class of I_n and the class of $h_{2m}(-1)$, in the notation of Example 7.1, as can be calculated directly. The fixed points under the action are those matrices for which $B^{-\mathrm{tr}} = B$ in the first case and $B^{-\mathrm{tr}} = h_{2m}(-1)Bh_{2m}(-1)^{-1}$ in the second, which is to say, the subgroups of orthogonal and of symplectic matrices respectively. We therefore deduce

$$(B_{C_2}\,\mathrm{PGL}_n)^{C_2} = B\,\mathrm{PO}_n \sqcup B\,\mathrm{PSp}_n\,.$$

When n is odd, the argument is much the same, but only $B\,\mathrm{PO}_n$ occurs. □

Remark 8.14. By Theorem 5.37, we know that there are two types of involutions on Azumaya algebras over connected topological spaces with trivial action, the symplectic and orthogonal. By means of Examples 7.2 and 7.4, we know that the orthogonal and symplectic Azumaya algebras with involution are equivalent to principal bundles for the groups PO_n and PSp_n, the latter when n is even. Proposition 8.13 has recovered these observations via equivariant homotopy theory.

8.5 FREE ACTION

Now we address the case where the action of C_2 on X is free. In this case, the quotient map $X \to Y := X/C_2$ is a two-sheeted covering space map.

PROPOSITION 8.15. *Let X be a free C_2-CW-complex, with C_2 acting by the involution λ, and let $Y = X/C_2$. Consider Y as a space over BC_2, or alternatively, as a space equipped with a distinguished class $\alpha \in \mathrm{H}^1(Y, C_2)$. There are natural bijections between the following:*

(a) Isomorphism classes of degree-n topological Azumaya algebras over X equipped with a λ-involution.

(b) $[X, B\,\mathrm{PGL}_n]_{C_2}$.

(c) $[Y, B(\mathrm{PGL}_n \rtimes C_2)]_{BC_2}$.

(d) *Elements of the preimage of* α *under* $\mathrm{H}^1(Y, \mathrm{PGL}_n \rtimes C_2) \to \mathrm{H}^1(Y, C_2)$.

Proof. Propositions 8.12 and 8.9 give the bijection between (a) and (b), and Proposition 8.4 gives a bijection between (b) and (c). The one-to-one correspondence between (c) and (d) is standard. ☐

We continue to assume that X is a free C_2-CW-complex and let $Y = X/C_2$. We would like to have a classifying-space-level understanding of the cohomological transfer map $\mathrm{transf}_{X/Y} : \mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(Y, \mathbb{G}_m)$ considered in Subsection 6.2.

To that end, let μ denote the discrete group μ_n or the topological group $\mathbb{C}^\times$. We endow μ with the involution $a \mapsto a^{-1}$, give $\mu \times \mu$ the involution $(a, b) \mapsto (b^{-1}, a^{-1})$, and let μ^{triv} denote μ with the trivial action.

The map $\mu \times \mu \to \mu^{\mathrm{triv}}$ defined by $(a, b) \mapsto ab^{-1}$ is C_2-equivariant, and its kernel consists of pairs of the form (a, a), which is the image of the diagonal map $\mu \to \mu \times \mu$. That is, there is a C_2-equivariant short exact sequence of C_2-groups

$$1 \to \mu \to \mu \times \mu \to \mu^{\mathrm{triv}} \to 1$$

and therefore, a sequence of C_2-equivariant maps in which any three consecutive terms form a homotopy fibre sequence:

$$\mu \to \mu \times \mu \to \mu^{\mathrm{triv}} \to B\mu \to B(\mu \times \mu) \to B\mu^{\mathrm{triv}} \to B^2\mu \to \cdots .$$

Any such homotopy fibre sequence is a homotopy fibre sequence in the C_2-equivariant coarse structure. These constructions are plainly natural with respect to inclusion of subgroups of $\mathbb{C}^\times$.

Now, if X is a free C_2-CW-complex, then thanks to Proposition 8.9, one arrives at a long exact sequence of abelian groups

$$\cdots \to [X, B^i\mu]_{C_2} \to [X, B^i\mu \times B^i\mu]_{C_2} \to [X, B^i\mu^{\mathrm{triv}}]_{C_2} \to \cdots . \tag{8.3}$$

Since $\mu \times \mu$ with this action is isomorphic to $\mu \times \mu$ with the interchange action, it follows that $[X, B^i\mu \times B^i\mu]_{C_2} \cong [X, B^i\mu] = \mathrm{H}^i(X, \mu)$. Moreover, $[X, B^i\mu^{\mathrm{triv}}]_{C_2}$ is simply $[Y, B^i\mu] = \mathrm{H}^i(Y, \mu)$.

Therefore, the sequence of (8.3) reduces in this case to

$$\cdots \to [X, B^i\mu]_{C_2} \to \mathrm{H}^i(X, \mu) \xrightarrow{\mathrm{transf}} \mathrm{H}^i(Y, \mu) \to \cdots . \tag{8.4}$$

When $\mu = \mathbb{C}^\times$ and $i = 2$, the map denoted transf agrees with the transfer map defined in Subsection 6.2. Indeed, we know that the transfer map in Section 6 agrees with the ordinary transfer map for a 2-sheeted covering in the case at hand, Example 6.6. It suffices therefore to show that the map transf in (8.4) is the usual transfer map for a 2-sheeted cover. The trivial case $X = Y \times C_2$ is elementary. The general case where $\pi : X \to Y$ is merely locally trivial can be deduced from the trivial case by viewing $\mathrm{H}^i(Y, \mu)$ and $\mathrm{H}^i(X, \mu)$ as Čech cohomology groups and calculating each using covers $\mathcal{U}$ of Y and $\pi^{-1}\mathcal{U}$ of X where $\mathcal{U}$ trivializes the double cover π.

Proposition 8.16. *Let μ be μ_n or $\mathbb{C}^\times$, given the involution $z \mapsto z^{-1}$. Let X be a space with free C_2-action, let $Y = X/C_2$, and let $\xi : X \to B^i\mu$ be an equivariant map, representing a cohomology class $\xi \in \mathrm{H}^i(X, \mu)$. Then* $\operatorname{transf}_{X/Y} \xi = 0$.

Proof. Since $\xi : X \to B^n\mu$ is equivariant, and the action on X is free, ξ lies in the image of $[X, B^n\mu]_{C_2}$ in $[X, B^n\mu] = \mathrm{H}^n(X, \mu)$. Thus, the result follows from the exact sequence (8.4). ☐

9 Examples with no Involutions of the Second Kind

We finally construct the example promised at the beginning of Section 8.

Throughout, the notation $x\mathbb{Z}$ means a free cyclic group, written additively, with a named generator x. Recall that for a topological space X, the sheaf cohomology group $\mathrm{H}^2(X, \mathbb{C}^\times) := \mathrm{H}^2(X, \mathcal{C}(X, \mathbb{C}^\times))$ is isomorphic to the singular cohomology group $\mathrm{H}^3(X, \mathbb{Z})$, see [AFW19, §2.1], for instance. We shall use the latter group for the most part.

9.1 A Cohomological Obstruction

In all cases, the groups appearing in this subsection are the complex points of linear algebraic groups. In the interest of brevity, the relevant linear algebraic group, e.g. SL_n, will be written in place of the group itself, e.g. $\mathrm{SL}_n(\mathbb{C})$.

Unless otherwise specified, groups appearing will be endowed with a C_2-action. For the groups SL_n, the action is that sending A to $A^{-\mathrm{tr}}$, which restricts to the action $r \mapsto r^{-1}$ on the central subgroup μ_n. For the groups $\mathrm{SL}_n \times \mathrm{SL}_n$, the action is that given by $(A, B) \mapsto (B^{-\mathrm{tr}}, A^{-\mathrm{tr}})$, and similarly for $\mu_n \times \mu_n$. The maps $\mathrm{SL}_n \to \mathrm{SL}_n \times \mathrm{SL}_n$ and $\mu_n \to \mu_n \times \mu_n$ are given by diagonal inclusions.

Embed $\mu_n \hookrightarrow \mathrm{SL}_n \times \mathrm{SL}_n$ via $r \mapsto (rI_n, rI_n)$, and let Q_n denote the group obtained as the quotient of $\mathrm{SL}_n \times \mathrm{SL}_n$ by the image of μ_n.

In the following diagram, the horizontal arrows of the first two rows are C_2-equivariant. This induces C_2-actions on the groups in the third row so that all arrows become C_2-equivariant.

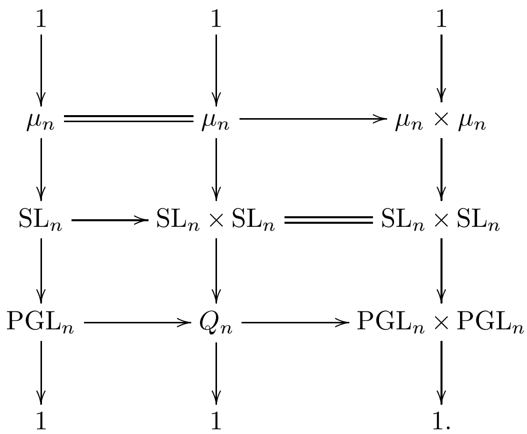


Figure 1: A Diagram of C_2 Groups.

Each of the groups appearing above is equipped with a C_2-action, and consequently each may be extended to a semidirect product with C_2, and equivariant classifying spaces of the form $B_{C_2}G$ may be constructed as in Subsection 8.2. Since we will consider equivariant maps with free C_2-action on the source, by Proposition 8.9, we may use any functorial model of BG with its functorially-induced C_2-action instead.

Proposition 9.1. *The C_2-action on PGL_n induces an action on $\mathrm{H}^*(B\,\mathrm{PGL}_n, \mathbb{Z})$. In low degrees, this action is summarized by Table 1.*

i	$\mathrm{H}^i(B\,\mathrm{PGL}_n, \mathbb{Z})$	Action
0	$\mathbb{Z}$	trivial
1	0	-
2	0	-
3	$\alpha\mathbb{Z}/n$	$\alpha \mapsto -\alpha$
4	$\tilde{c}_2\mathbb{Z}$	trivial

Table 1: The C_2-action on $\mathrm{H}^*(B\,\mathrm{PGL}_n, \mathbb{Z})$.

Proof. The compatible C_2-actions on the terms of the exact sequence $1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$ induce an action on the fibre sequence $B\,\mathrm{SL}_n \to B\,\mathrm{PGL}_n \to B^2\mu_n$, and therefore an action of C_2 on the associated Serre spectral sequence, which is illustrated in Figure 2.

The action on $\alpha\mathbb{Z}/n$ is the same as the action of C_2 on $\mu_n = \mathbb{Z}/n$ itself, which is the sign action. The action on $\mathrm{H}^4(\mathrm{SL}_n, \mathbb{Z}) = \mathbb{Z}c_2$ is calculated by identifying $\mathrm{H}^*(\mathrm{SL}_n, \mathbb{Z})$ as a subquotient of $\mathrm{H}^*(BT_n, \mathbb{Z})$, where T_n is the maximal

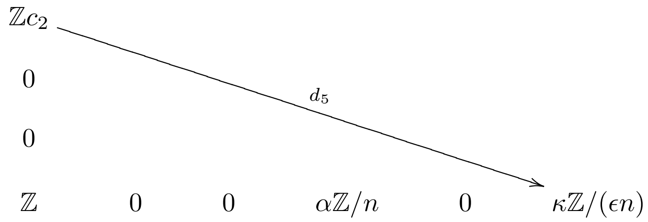


Figure 2: A portion of the Serre spectral sequence in cohomology associated to $B\,\mathrm{SL}_n \to B\,\mathrm{PGL}_n \to B^2\mu_n$. Here, ϵ is 2 if n is even and is 1 otherwise.

torus of diagonal matrices in GL_n. Specifically, $\mathrm{H}^*(BT_n, \mathbb{Z}) = \mathbb{Z}[\theta_1, \dots, \theta_n]$, where the C_2-action on θ_i is $\theta_i \mapsto -\theta_i$. Then the class c_2 in question may be identified with the image of the second elementary symmetric function in the θ_i in $\mathrm{H}^*(BT_n, \mathbb{Z})/(\sum_{i=1}^n \theta_i)$. It follows the action of C_2 on c_2 is trivial.
We know from [AW14b, Proposition 4.4] that the illustrated d_5 differential is surjective. Writing $\tilde{c}_2$ for $\epsilon n c_2$, it follows easily that the cohomology of $B\,\mathrm{PGL}_n$ takes the stated form, and carries the stated C_2-action. □

Proposition 9.2. *Fix a natural number n, and let $\epsilon = \gcd(n, 2)$. Let S be the subgroup of $c_2'\mathbb{Z} \oplus c_2''\mathbb{Z}$ consisting of terms $ac_2' + bc_2''$ where $a + b \equiv 0 \pmod{\epsilon n}$. The low-degree cohomology of BQ_n, along with its C_2-action, is summarized by Table 2. Moreover, the comparison map from $\mathrm{H}^i(BQ_n, \mathbb{Z})$ to $\mathrm{H}^i(B\,\mathrm{SL}_n, \mathbb{Z})$*

i	$\mathrm{H}^i(BQ_n, \mathbb{Z})$	Action
0	$\mathbb{Z}$	trivial
1	0	-
2	0	-
3	$\alpha\mathbb{Z}/n$	$\alpha \mapsto -\alpha$
4	S	$ac_2' + bc_2'' \mapsto bc_2' + ac_2''$

Table 2: The C_2-action on $\mathrm{H}^*(BQ_n, \mathbb{Z})$.

is the evident identification map when $i \le 3$. When $i = 4$, it is given by $ac_2' + bc_2'' \mapsto \frac{a+b}{\epsilon n}\tilde{c}_2$.

Proof. There is a fibre sequence $B(\mathrm{SL}_n \times \mathrm{SL}_n) \to BQ_n \to B^2\mu_n$. A portion of the associated Serre spectral sequence is shown in Figure 3. There is a comparison map of spectral sequences from this one to that of Figure 2. The map identifies the bottom row of the two E_2-pages, and sends c_2', c_2'' both to c_2. It is compatible with the C_2-actions. The claimed results except the C_2-action on $\mathrm{H}^4(BQ_n, \mathbb{Z})$ all follow from the comparison map and the values in Table 1. As for the action on $\mathrm{H}^4(BQ_n, \mathbb{Z})$, as in the proof of Proposition 9.1, this can

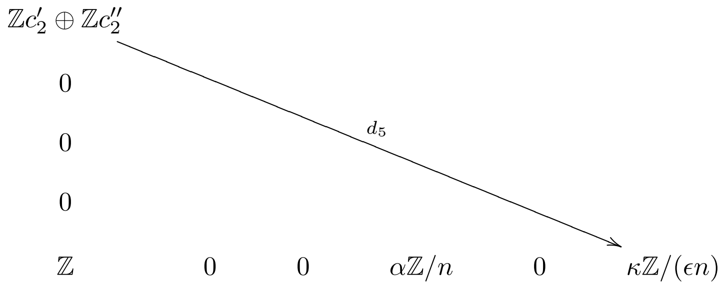


Figure 3: A portion of the Serre spectral sequence in cohomology associated to $B(\mathrm{SL}_n \times \mathrm{SL}_n) \to BQ_n \to B^2\mu_n$.

be deduced from the action on $\mathrm{H}^*(B(T_n \times T_n), \mathbb{Z}) = \mathbb{Z}[\theta_1', \dots, \theta_n', \theta_1'', \dots, \theta_n'']$, which is given by $\theta_i' \mapsto -\theta_i''$ and $\theta_i'' \mapsto -\theta_i'$. □

Remark 9.3. From Figures 2 and 3, we deduce that the maps $B\,\mathrm{PGL}_n \to B^2\mu_n$ and $BQ_n \to B^2\mu_n$ induced by Figure 1 both represent generators of the groups $\mathrm{H}^2(B\,\mathrm{PGL}_n, \mathbb{Z}/n) \cong \mathbb{Z}/n$ and $\mathrm{H}^2(BQ_n, \mathbb{Z}/n) \cong \mathbb{Z}/n$, respectively. Moreover, the image of the former class under the Bockstein map is a generator of $\mathrm{H}^3(B\,\mathrm{PGL}_n, \mathbb{Z})$, which is nothing but the tautological Brauer class α of $B\,\mathrm{PGL}_n$. That is, if $r : X \to B\,\mathrm{PGL}_n$ is the classifying map for a PGL_n-bundle, or equivalently, a degree-n topological Azumaya algebra, then the Brauer class of that algebra is $r^*(\alpha)$.

Our purpose in introducing the group Q_n is to construct a group which is as close to $\mathrm{PGL}_n \times \mathrm{PGL}_n$ (with the interchange action) as possible, but for which the transfer of all classes in $\mathrm{H}^2(BQ_n, \mathbb{Z}/n)$ vanish.

Proposition 9.4. *Let Q_n be as constructed above. Give the space $BQ_n \times EC_2$ the diagonal C_2-action. Then the transfer map, $\mathrm{H}^2(BQ_n \times EC_2, \mathbb{Z}/n) \to \mathrm{H}^2(BQ_n \times^{C_2} EC_2, \mathbb{Z}/n)$, considered at the end of Subsection 8.5, vanishes.*

The space $BQ_n \times EC_2$ is weakly equivalent to BQ_n, but carries a free C_2-action.

Proof. The action of C_2 on $BQ_n \times EC_2$ is free. As noted in Remark 9.3, one generator α of $\mathrm{H}^2(Q_n \times EC_2; \mathbb{Z}/n) \cong \mathrm{H}^2(Q_n; \mathbb{Z}/n)$ is given by the map $BQ_n \to B^2\mu_n$ arising from the short exact sequence defining Q_n. This map is C_2-equivariant when μ_n, and therefore $B^2\mu_n$, is given the standard involution, and therefore the result follows from Proposition 8.16. □

Proposition 9.5. *Let n be an even positive integer, and let a be an odd positive integer. Suppose $f : X \to BQ_n$ is a C_2-equivariant map and a 6-equivalence. Then there is no C_2-equivariant map $g : X \to B\,\mathrm{PGL}_{an}$ inducing a surjection on $\mathrm{H}^3(\,\cdot\,, \mathbb{Z})$.*

Proof. For the sake of contradiction, suppose that g exists.
By Remark 9.3, the composition

$$X \to BQ_n \to B^2\mu_n \to B^2\mathbb{C}^\times = B^3\mathbb{Z},$$

induced by f and the inclusion $\mu_n \to \mathbb{C}^\times$, represents a generator ξ of $\mathrm{H}^3(X,\mathbb{Z}) = \mathrm{H}^3(BQ_n,\mathbb{Z}) = \mathbb{Z}/n$. As a result, there is $t \in \mathbb{Z}$ such that the composition

$$X \xrightarrow{g} B\,\mathrm{PGL}_{an} \to B^2\mu_{an} \to B^2\mathbb{C}^\times = B^3\mathbb{Z}$$

represents $t\cdot\xi$. Consequently, the map $X \to BQ_n \to B^2\mu_n$ fits into a homotopy-commutative square

$$\begin{array}{ccc} X & \longrightarrow & B^2\mu_n \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\ B\,\mathrm{PGL}_{an} & \longrightarrow & B^2\mu_{an} \end{array}$$

in which s is the composition of $B^2\mu_n \to B^2\mu_{an}$ and the map $B^2\mu_{an} \to B^2\mu_{an}$ induced by $x \mapsto x^t : \mu_{an} \to \mu_{an}$. We extend this square into a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B^2\mu_n \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} \\ B\,\mathrm{SL}_{an} & \longrightarrow & B\,\mathrm{PGL}_{an} & \longrightarrow & B^2\mu_{an} \end{array}$$

where both rows are homotopy fibre sequences, so F is the homotopy fibre of $X \to B^2\mu_n$. Strictly speaking, we carry this out in the (fine) C_2-equivariant model structure on topological spaces, using the dual of [Hov99, Prop. 6.3.5] to deduce the existence of the dashed arrow in that category, so that it may be assumed to be C_2-equivariant. Moreover, the space F appearing in this argument has the appropriate non-equivariant homotopy type, since the functor forgetting the C_2-action is a right Quillen functor, and therefore preserves fibre sequences.

Each of the two fibre sequences is associated to a Serre spectral sequence in cohomology. In the case of the lower row, the E_2-page is represented in Figure 2, whereas in the case of the upper row, since X is 6-equivalent to BQ_n, it is isomorphic on the E_2-page to the spectral sequence represented in Figure 3. There is an induced map between these spectral sequences, and this map restricts to the following on the $\mathrm{E}_2^{*,0}$-line:

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & 0 & \mathbb{Z}/(an) & 0 & \kappa\mathbb{Z}/(2an) & \dots . \\ \| & & & \downarrow & & \downarrow & \\ \mathbb{Z} & 0 & 0 & \mathbb{Z}/(n) & 0 & \kappa\mathbb{Z}/(2n) & \dots \end{array}$$

We know the map on $\mathrm{H}^5(\,\cdot\,,\mathbb{Z})$ is surjective, because in each case, the group is generated by a class κ for which 2κ is $\beta(\iota^2)$, obtained by taking the canonical class ι in $\mathrm{H}^2(B^2\mu_n,\mathbb{Z}/n)$, resp. $\mathrm{H}^2(B^2\mu_{an},\mathbb{Z}/an)$, squaring it, and applying the Bockstein map with image $\mathrm{H}^5(\cdot,\mathbb{Z})$. This may be deduced from [Car54], or from the Serre spectral sequence associated to the path-loop fibration $B\mu_n \to * \to B^2\mu_n$.
Since the map g^* of spectral sequences is compatible with the C_2-action, it induces the following commutative square

$$\begin{array}{ccc} c_2\mathbb{Z} & \xrightarrow{d_5} & \kappa\mathbb{Z}/(2an) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\ c_2'\mathbb{Z}\oplus c_2''\mathbb{Z} & \xrightarrow{d_5} & \kappa\mathbb{Z}/(2n). \end{array}$$

in which all arrows are C_2-equivariant. Furthermore, the proofs of Propositions 9.1 and 9.2 imply that both horizontal maps are surjective, that C_2 acts trivially on $c_2\mathbb{Z}$, $\kappa\mathbb{Z}/(2an)$ and $\kappa\mathbb{Z}/(2n)$, and that the non-trivial element of C_2 interchanges c_2' and c_2''. Now, $g^*(c_2)$ lies in the C_2-fixed subgroup of $c_2'\mathbb{Z}\oplus c_2''\mathbb{Z}$, which is to say $g^*(c_2) = mc_2' + mc_2''$ for some integer m. Then $d_5(g^*(c_2))$ is $2m$ times a generator of $\kappa\mathbb{Z}/(2n)$, and hence not a generator of $\kappa\mathbb{Z}/(2n)$. On the other hand, $g^*(d_5(c_2))$ is a generator of $\mathbb{Z}/(2n)$ by the previous paragraph, a contradiction. □

9.2 An Algebraic Counterexample

In this section, we consider complex algebraic varieties. In particular, all algebraic groups are complex algebraic groups. Cohomology is understood to be étale cohomology in the context of varieties and singular cohomology in the context of topological spaces.
A C_2-action on a variety X is free if there exists a C_2-torsor $X \to Y$. In this case, Y coincides with categorical quotient X/C_2 in the category of varieties. Furthermore, if C_2 acts freely on X, then it also acts freely on $X(\mathbb{C})$. The converse holds when X is affine or projective, see Example 4.20 and Proposition 4.45, but not in general.

Fix an even positive integer n. We define the complex algebraic group Q_n by means of the short exact sequence

$$1 \to \mu_n \xrightarrow{x\mapsto(x,x)} \mathrm{SL}_n \times \mathrm{SL}_n \to Q_n \to 1$$

so that $Q_n(\mathbb{C})$ is the group Q_n considered in the previous subsection. There is a natural map $\mathrm{H}^1(\,\cdot\,,Q_n) \to \mathrm{H}^2(\,\cdot\,,\mu_n)$. Composing with the map $\mathrm{H}^2(\,\cdot\,,\mu_n) \to \mathrm{H}^2(\,\cdot\,,\mathbb{G}_m)$ induced by the inclusion $\mu_n \to \mathbb{G}_m$ allows us to associate with every Q_n-torsor $P \to X$ an n-torsion class in $\mathrm{H}^2(X,\mathbb{G}_m)$. This association is natural, and is, in particular, compatible with complex realization.

The first projection $\pi_1 : \mathrm{SL}_n \times \mathrm{SL}_n \to \mathrm{SL}_n$ induces a group homomorphism $\pi_1 : Q_n \to \mathrm{PGL}_n$ (it is not C_2-equivariant). Using this map, we associate to every Q_n-torsor P a PGL_n-torsor, namely, $P \times^{Q_n} \mathrm{PGL}_n$.

Lemma 9.6. *With the previous notation, let $P \to X$ be a Q_n-torsor, let α be its associated class in $\mathrm{H}^2(X, \mathbb{G}_m)$, and let $S \to X$ be its associated PGL_n-torsor. Then α is the image of S under the canonical map $\mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbb{G}_m)$. In particular, $\alpha \in \mathrm{Br}(X)$.*

Proof. This follows by considering the following morphism of short exact sequences and the induced morphism between the associated cohomology exact sequences.

$$\begin{array}{ccccc} \mu_n & \longrightarrow & \mathrm{SL}_n \times \mathrm{SL}_n & \longrightarrow & Q_n \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_1} \\ \mathbb{G}_m & \longrightarrow & \mathrm{GL}_n & \longrightarrow & \mathrm{PGL}_n \end{array}$$

Note that the vertical maps are not necessarily C_2-equivariant. □

Proposition 9.7. *Maintaining the previous notation, there exists a smooth affine complex variety X with free C_2-action, a Q_n-torsor $P \to X$ and a map $f : X(\mathbb{C}) \to BQ_n(\mathbb{C})$ such that the following hold:*

(i) The map $f : X(\mathbb{C}) \to BQ_n(\mathbb{C})$ is C_2-equivariant and a 6-equivalence.

(ii) The homotopy class of f corresponds to the principal $Q_n(\mathbb{C})$-bundle $P(\mathbb{C}) \to X(\mathbb{C})$.

(iii) The Brauer class $\alpha \in \mathrm{Br}(X) \subseteq \mathrm{H}^2(X, \mathbb{G}_m)$ associated with $P \to X$ has trivial image under $\mathrm{transf} : \mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(X/C_2, \mathbb{G}_m)$.

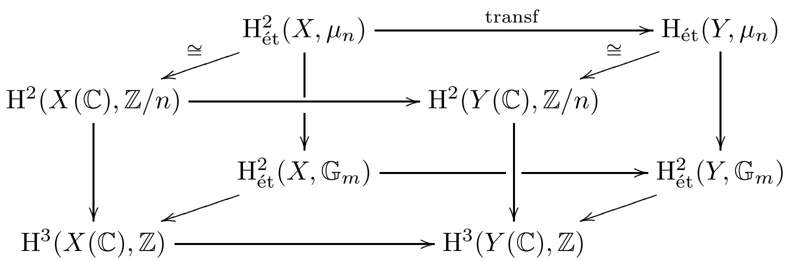
For later reference, and in keeping with the previous parts of this paper, we denote X/C_2 by Y.

Proof. As in Subsection 9.1, the group Q_n is an affine algebraic group equipped with an algebraic C_2-action. Consequently, the split exact extension Γ in

$$1 \to Q_n \to \Gamma \to C_2 \to 1$$

is also an affine algebraic group, [Mol77, Ex. 2.15 (c)].
Therefore it is possible to follow [Tot99] and construct affine spaces V on which Γ acts and such that V becomes a Γ-torsor after removing a locus, Z, of arbitrarily large codimension. In particular, $V - Z$ is a Q_n-torsor. Choose V so that Z has (complex) codimension at least 4.
Let $P = (V - Z)$ and $X = (V - Z)/Q_n$, and note that both P and X carry C_2-actions. The C_2-action on X is free since $V - Z$ is a Γ-torsor. Moreover, one checks directly that $P(\mathbb{C}) \to X(\mathbb{C})$ is a principal $Q_n(\mathbb{C})$-bundle with involution, see Definition 8.5. Since C_2 acts freely on $X(\mathbb{C})$, Proposition 8.9 implies that

this principal bundle is represented by a map $f : X(\mathbb{C}) \to BQ_n(\mathbb{C})$, which satisfies conditions (i) and (ii).
By means of the equivariant Jouanolou device, [Hoy17, Prop. 2.20], we may assume that X is a smooth affine variety with these properties.
Let $\alpha \in \mathrm{Br}(X)$ denote the Brauer class associated with $P \to X$. It remains to show that $\mathrm{transf}_{X/Y}(\alpha) = 0$ in $\mathrm{Br}(Y)$, where $Y = X/C_2$.
To that end, let ξ be the image of P under $\mathrm{H}^1(X, Q) \to \mathrm{H}^2(X, \mu_n)$, and similarly define $\xi(\mathbb{C})$ as the image of $P(\mathbb{C})$ under the analogous map in singular cohomology. It is enough to check that $\mathrm{transf}(\xi) \in \mathrm{H}^2(Y, \mu_n)$ vanishes. There is a commutative diagram



where each map from left to right is a transfer map, each map from back to front is a complex-realization map, and the maps from top to bottom are induced by the inclusion $\mu_n \to \mathbb{G}_m$. The two indicated maps are isomorphism by Artin's theorem. We also remark that $\mathrm{H}^2(\,\cdot\,, \mathbb{C}^\times) \cong \mathrm{H}^3(\,\cdot\,, \mathbb{Z})$, where the first group is understood as sheaf cohomology with coefficients in the sheaf of nonvanishing continuous complex-valued functions. Now, the transfer of $\xi(\mathbb{C}) \in \mathrm{H}^2(X(\mathbb{C}), \mathbb{Z}/n)$ is easily seen to be 0 by comparison with $\mathrm{H}^2(BQ_n \times EC_2, \mathbb{Z}/n)$, where it is known to vanish by Proposition 9.4. This completes the proof. □

Theorem 9.8. *For any even integer n, there exists a quadratic étale map $X \to Y$ of smooth affine complex varieties and an Azumaya algebra A of degree n over X such that:*

(i) The period and index of $\alpha = [A]$ are both n.

(ii) $\mathrm{transf}_{X/Y}(\alpha) = 0$ *in* $\mathrm{Br}(Y)$.

(iii) The degree of any Azumaya algebra Brauer equivalent to A and admitting a λ-involution is divisible by $2n$—here λ denotes the non-trivial involution of X over Y.

In particular, we see that the minimal degree of an Azumaya algebra Brauer equivalent to A and supporting a λ-involution is at least $2n$. This bound is sharp by Theorem 6.10.

Proof. Construct $P \to X$ as in Proposition 9.7, and let A be the Azumaya algebra corresponding to the PGL_n-torsor associated to P by means of $\pi_1 : Q_n \to \mathrm{PGL}_n$. The Azumaya algebra A has degree n, and the complex reaization of its Brauer class is a generator of $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}/n$, since the map $BQ_n(\mathbb{C}) \to B\,\mathrm{PGL}_n(\mathbb{C})$ induces an isomorphism on $\mathrm{H}^3(\cdot, \mathbb{Z})$, see Remark 9.3 and the proof of Proposition 9.2. In particular, the period and index of α must both be n.

Let a be an odd integer and suppose A were equivalent to an Azumaya algebra of degree an carrying a λ-involution. Then, by Proposition 8.12, the complex realization of this algebra would correspond to a topological $\mathrm{PGL}_{an}(\mathbb{C})$-bundle with involution. By Proposition 8.9, there would be a C_2-equivariant map $g : X(\mathbb{C}) \to B\,\mathrm{PGL}_{an}(\mathbb{C})$ such that $\alpha \in \mathrm{Br}(X(\mathbb{C}))$ was the image of the canonical Brauer class in $\mathrm{H}^3(B\,\mathrm{PGL}_{an}(\mathbb{C}), \mathbb{Z})$, and therefore g would induce a surjection in $\mathrm{H}^3(\cdot, \mathbb{Z})$, since the image of g^* would contain α. This is forbidden by Proposition 9.5. ☐

Question 9.9. Does Theorem 9.8 hold when n is odd?

A The Stalks of The Ring of Continuous Complex Functions

Let X be a topological space; we work throughout on the small site of X. Let $\mathcal{O}$ denote the sheaf of continuous $\mathbb{C}$-valued functions on X.

Let p be a point of X and consider $p^*\mathcal{O}$. It is a local ring with maximal ideal denoted $\mathfrak{m}$. An element $f \in p^*\mathcal{O}$ is the germ of a continuous $\mathbb{C}$-valued function at p, and the class $\bar{f} \in p^*\mathcal{O}/\mathfrak{m} \cong \mathbb{C}$ is the complex number $f(p)$.

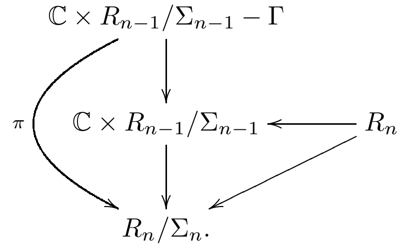
Proposition A.1. *The local ring $p^*\mathcal{O}$ is strictly henselian.*

Proof. It suffices to prove the ring is a henselian ring as the residue field is $\mathbb{C}$.

Consider $R_n := \mathbb{C}^n$ as the space of ordered sets of roots of a degree-n monic polynomial. There is a permutation action of the symmetric group Σ_n on R_n, and there is a homeomorphism $R_n/\Sigma_n \to \mathbb{C}^n$, where the map takes $(\alpha_1, \dots, \alpha_n)$ to the coefficients of the polynomial $\prod_{i=1}^n (t - \alpha_i)$, [BM83].

We embed $\Sigma_{n-1} \subset \Sigma_n$ as the permutations fixing the first element, and form R_n/Σ_{n-1}.

The space $R_n/\Sigma_{n-1} = \mathbb{C} \times R_{n-1}/\Sigma_{n-1}$ represents monic polynomials of degree n and a distinguished 'first' root, $(t - \alpha_1)q(t)$. There is a closed subset $\Gamma \subset R_n/\Sigma_{n-1}$, the locus where $q(\alpha_1) = 0$. Then $R_n/\Sigma_{n-1} - \Gamma \subset R_n/\Sigma_{n-1}$ is an open subset representing the set of monic, degree-n polynomials having a distinguished 'first' root which is not repeated. Since quotient maps of spaces given by finite group actions are open maps, in the following diagram, every map appearing is an open map:



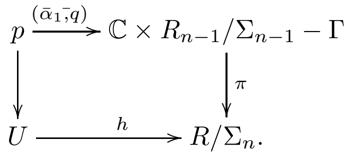
We denote the composite map $\mathbb{C} \times R_{n-1}/\Sigma_{n-1} - \Gamma \to R_n/\Sigma_n$ by π. It sends a pair $(\alpha_1, q(t))$, for which $q(\alpha_1) \neq 0$, to $(t - \alpha_1)q(t)$.

Let $(\alpha_1, q(t)) \in \mathbb{C} \times R_{n-1}/\Sigma_{n-1} - \Gamma$ be such a pair. We claim that there exists an open neighbourhood V of $(\alpha_1, q(t))$ such that $\pi|_V : V \to R_n/\Sigma_n$ is a homeomorphism onto its image. Choose an open neighbourhood of $(\alpha_1, q(t)) \in \mathbb{C} \times R_{n-1}/\Sigma_{n-1}$ of the form $V = B(\alpha_1; \epsilon) \times B(q(t); \epsilon)$, being the product of an ϵ-ball around α_1 and around $q(t)$, where ϵ is sufficiently small that none of the polynomials in $B(q(t); \epsilon)$ has any of the complex numbers in $B(\alpha_1; \epsilon)$ as roots. It is immediate that π is injective when restricted to this open set in $\mathbb{C} \times R_{n-1}/\Sigma_{n-1} - \Gamma$. Since π is an open map, $\pi|_V$ is a homeomorphism onto its image, establishing the claim.

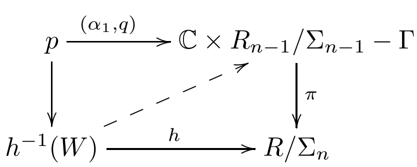
Suppose we are given a polynomial $h(t) \in p^*\mathcal{O}[t]$. Suppose further that a non-repeated root, $\bar{\alpha}_1$, of $\bar{h}(t)$ is given, where $\bar{h}(t)$ is the reduction of $h(t)$ to $(p^*\mathcal{O}/\mathfrak{m})[t] = \mathbb{C}[t]$. We can write $\bar{q}(t) = \bar{h}(t)/(t - \bar{\alpha}_1)$ in $\mathbb{C}[t]$. Note that we do not yet assert that $\bar{q}(t)$ and $\bar{\alpha}_1$ are the reductions of any specific elements in $p^*\mathcal{O}[t]$ or $p^*\mathcal{O}$. To prove that the ring is Henselian, we must find an element α_1 lifting $\bar{\alpha}_1$ and satisfying $h(\alpha_1) = 0$.

The germ $h(t)$ has an extension to an open neighbourhood $U \ni p$.

We have the data of a diagram



Around the image of $(\bar{\alpha}_1, \bar{q})$ in $\mathbb{C} \times R_{n-1}/\Sigma_{n-1} - \Gamma$ we can find an open set V such that $\pi|_V : V \to R/\Sigma_n$ is a homeomorphism onto the image, W. Then W is an open set in R/Σ_n containing $\bar{\alpha}_1\bar{q} = \bar{h}$. Since $h(p) = \bar{h}$, the preimage $h^{-1}(W)$ is an open subset of U containing p. Since $\pi|_V : V \to W$ is a homeomorphism, we may lift the map



as indicated.
That is to say, there is a neighbourhood, $h^{-1}(W)$, of p such that the factorization $\bar{h}(t) = (t - \bar{\alpha}_1)\bar{q}(t)$ can be extended on $h^{-1}(W)$ to a factorization $h(t) = (t - \alpha_1)q(t)$. In particular, the class of α_1 in $p^*\mathcal{O}$ is a root of the polynomial $h(t)$ extending $\bar{\alpha}_1$. This proves Hensel's lemma for $p^*\mathcal{O}$. □

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