# The Motive of a Smooth Theta Divisor

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Abstract. In this paper, we prove a motivic version of the Lefschetz hyperplane theorem for the motive of a smooth ample divisor on an Abelian variety.

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## 1 Introduction

Let $k$ be a field and let $\mathcal{M}_k$ be the category of Chow motives with *rational coefficients.* For a smooth projective variety $X$ over $k$, let

$$\mathfrak{h}(X) = (X, \Delta_X, 0) \in \mathcal{M}_k$$

denote its corresponding Chow motive, as usual. For a given choice of Weil cohomology $H^*$ with decomposition $H^*(X) = \bigoplus H^j(X)$, there is a fundamental question as to whether one can find Chow motives representing each degree of cohomology. More precisely, there is the following conjecture:

Conjecture 1.1 (Chow-Künneth). *There exists a direct sum decomposition:*

$$\mathfrak{h}(X) \cong \bigoplus \mathfrak{h}^i(X) \in \mathcal{M}_k$$

*such that $H^*(\mathfrak{h}^j(X)) = H^j(X)$ for a choice of Weil cohomology $H^*$.*

Stated differently, the conjecture asks for idempotents

$$\pi_{X,j} \in End_{\mathcal{M}_k}(\mathfrak{h}(X)) = CH^{dim(X)}(X \times X)$$

satisfying the following conditions:

(a) $\pi_{j,X} \circ \pi_{j',X} = 0$ for $j \neq j'$

(b) $\sum \pi_{j,X} = \Delta_X$

(c) $\pi_{j,X*}H^*(X) = H^j(X)$.

By setting $\mathfrak{h}^j(X) = (X, \pi_{j,X}, 0)$, it is straightforward to check that this is equivalent to the first conjecture. The Chow-Künneth conjecture is known to hold in several important cases: curves, surfaces ([15]), complete intersections in $\mathbb{P}^n$ ([16] Chapter 6), Abelian varieties ([3]), and modular varieties ([5]), among others. In the case of an Abelian varieties, one has the following:

Theorem 1.2 (Deninger-Murre, Jannsen). *Let $A$ be an Abelian variety of dimension $g$ over $k$.*

*(a) There exists a unique set of idempotents $\{\pi_{j,A}\} \in CH^g(A \times A)$ satisfying conditions (a), (b) and (c) above and the following relation for all $n \in \mathbb{Z}$:*

$$^t\Gamma_{n_A} \circ \pi_{j,A} = n^j \cdot \pi_{j,A} = \pi_{j,A} \circ {}^t\Gamma_{n_A}$$

*where $\Gamma_{n_A}$ denotes the graph of multiplication by $n$ on $A$.*

*(b) For any other choice of Chow-Künneth idempotents $\{\pi'_{j,A}\}$, there are isomorphisms:*

$$(A, \pi_{j,A}, 0) \cong (A, \pi'_{j,A}, 0).$$

*Proof.* See [3] Theorem 3.1 for (a) and for (b) note that, since $\pi_{j,A}$ and $\pi'_{j,A}$ have the same action on $H^*(A)$, it follows by [8] Lemma 3.1 (ii) that there exists a unit $u \in CH^g(A \times A)$ for which $\pi'_{j,A} = u \circ \pi_{j,A} \circ u^{-1}$. There is then a map of Chow motives:

$$\pi'_{j,A} \circ u \circ \pi_{j,A} : (A, \pi_{j,A}, 0) \to (A, \pi'_{j,A}, 0)$$

whose inverse is $\pi_{j,A} \circ u^{-1} \circ \pi'_{j,A}$ □

Remark 1.3. *We note more generally that whenever $X$ satisfies Conjecture 1.1 and Conjecture 2.6, the proof of Theorem 1.2 (b) shows that the $\mathfrak{h}^j(X)$ summands are unique up to some isomorphism.*

Our first goal will be to prove Conjecture 1.1 in a new case. Thus, for the remainder of the paper, let $A$ be an Abelian variety of dimension $g$ and let

$$i : \Theta \hookrightarrow A$$

be a smooth ample divisor on $A$ satisfying the following assumption:

Assumption 1.4. *There is some translate of $\Theta$ which is a symmetric divisor on $A$ (i.e., for which $\exists x \in A(k)$ such that $(-1)^*_A[t_x(\Theta)] = [t_x(\Theta)] \in CH^1(A)$).*

In this case, there is the Lefschetz hyperplane theorem, which gives isomorphisms:

$$i^* : H^j(A) \xrightarrow{\cong} H^j(\Theta) \text{ for } j < g-1$$
$$i_* : H^j(\Theta) \xrightarrow{\cong} H^{j+2}(A)(1) \text{ for } j > g-1.$$

There is also a surjective map $i_* : H^{g-1}(\Theta) \to H^{g+1}(A)(1)$ and its kernel:

Definition 1.5. *The* primal cohomology *of $\Theta$ is defined to be:*

$$H^{g-1}_{pr}(\Theta) := \ker[i_* : H^{g-1}(\Theta) \to H^{g+1}(A)(1)].$$

Since the cohomology of $\Theta$ is controlled by that of $A$ (except in the middle degree) and since $A$ possesses a Chow-Künneth decomposition, one would expect to be able to construct a Chow-Künneth decomposition for $\Theta$. Furthermore, one would hope to obtain a *motivic* version of the Lefschetz hyperplane theorem in the process. This is the content of our main result:

Theorem 1.6. *Let $k$ be a field and $A$ an Abelian variety of dimension $g$ with Chow-Künneth decomposition $\mathfrak{h}(A) \cong \bigoplus \mathfrak{h}^j(A)$. Further, let $\Theta \overset{i}{\hookrightarrow} A$ be a smooth ample divisor satisfying Assumption 1.4.*

(a) *There exists a Chow-Künneth decomposition of $\Theta$ :*

$$\mathfrak{h}(\Theta) \cong \bigoplus_{j=0}^{2(g-1)} \mathfrak{h}^j(\Theta) = \bigoplus_{j=0}^{2(g-1)} (\Theta, \pi_j, 0)$$

*such that there are isomorphisms:*

(i) $\mathfrak{h}^j(i) := \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{j,A} : \mathfrak{h}^j(A) \xrightarrow{\cong} \mathfrak{h}^j(\Theta)$ *for $j < g-1$*

(ii) ${}^t\mathfrak{h}^j(i) := \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} : \; h^j(\Theta) \xrightarrow{\cong} \mathfrak{h}^{j+2}(A)(1)$ *for $j > g-1$.*

(b) *For any ample divisor class $h \in Pic(\Theta)$, the corresponding Lefschetz correspondence $L_h = \Delta_*(h) \in CH^g(\Theta \times \Theta)$ induces an isomorphism of Chow motives for $j \leq g-1$:*

$$\pi_{2(g-1)-j,\Theta} \circ L_h^{g-1-j} \circ \pi_{j,\Theta} : \mathfrak{h}^j(\Theta) \xrightarrow{\cong} \mathfrak{h}^{2(g-1)-j}(\Theta)(g-1-j).$$

(c) *The map of Chow motives*

$${}^t\mathfrak{h}^{g-1}(i) := \pi_{g+1,A} \circ \Gamma_i \circ \pi_{g-1,\Theta} : \mathfrak{h}^{g-1}(\Theta) \to \mathfrak{h}^{g+1}(A)(1)$$

*is a split-surjective map. Moreover, there exists an idempotent $p \in CH^{g-1}(\Theta \times \Theta)$ with corresponding motive $P = (\Theta, p, 0)$ such that $H^*(P) = H^{g-1}_{pr}(\Theta)$ and such that there is an isomorphism of Chow motives:*

$$\mathfrak{h}^{g-1}(\Theta) \cong \mathfrak{h}^{g+1}(A)(1) \oplus P$$

Remark 1.7. *(a) The nontrivial part of the proof of Theorem 1.6 is constructing idempotents subject to the required conditions. Once this is done, it becomes a formal exercise to verify these conditions.*

*(b) Assumption 1.4 is satisfied as soon as $k$ is algebraically closed. In fact, for any field $k$, there is some finite extension $k \subset k'$ for which it is satisfied and, hence, for which the theorem applies. However, it is not known whether such a decomposition over $k'$ would descend to $k$ (see, for instance, [19] Question 1.6). The weaker property of having a Künneth decomposition, on the other hand, does descend under finite extensions (see [19] Proposition 1.7).*

Whenever a smooth projective variety satisfies the Chow-Künneth Conjecture, the summands are conjectured to be unique up to isomorphism. Indeed, as noted in Remark 1.3, this would follow from the *finite-dimensionality* of the motive of $\Theta$ (i.e., Conjecture 2.6 for $\Theta$). We will review this notion of finite-dimensionality and highlight its properties in the next section. As an indication of its desirability, we mention that Bloch's conjecture for zero cycles on a surface becomes trivial for surfaces known to be finite-dimensional (see [12]). Thus far, however, only motives that are summands of motives of Abelian varieties are known to be finite-dimensional. In particular, we cannot prove the finite-dimensionality of the motive of $\Theta$ nor the unicity of the summands, but we do note that the summands obtained here are minimal in some sense (see §3.3 below).

Finally, we will explore the complementary motive $P$ of Theorem 1.6 (b), which represents the primal cohomology of $\Theta$. For this, we will specialize to the case that $k = \mathbb{C}$ and $H^*$ is singular cohomology with $\mathbb{Q}$-coefficients, where Hodge theory can be of assistance. We will further simplify matters by specializing to the case where $\Theta$ is a principal polarization. We note, however, that when $g \leq 3$ the very general principally polarized Abelian variety is a Jacobian, and $\Theta$ is not smooth in this case. On the other hand, if we take $g = 4$, $\Theta$ is very generally smooth and the primal cohomology has Hodge level 1. Conjecturally, a motive over $\mathbb{C}$ whose singular cohomology has Hodge level 1 should correspond to an Abelian variety ([12] Remark 7.12). In this direction, we state the following result:

Theorem 1.8. *Suppose that $A$ is a very general principally polarized Abelian fourfold over $\mathbb{C}$ and let $\Theta$ be a divisor corresponding to the principal polarization. Let $\mathcal{M}_{hom,\mathbb{C}}$ denote the category of homological motives over $\mathbb{C}$ (using singular cohomology). Then, there exists an Abelian variety $J$ and an isomorphism in $\mathcal{M}_{hom,\mathbb{C}}$:*

$$P \cong \mathfrak{h}^1(J)(-1).$$

*Moreover, the following are equivalent:*

*(a) $\mathfrak{h}(\Theta)$ is finite-dimensional in the sense of Kimura,*

*(b) There is an isomorphism of Chow motives: $P \cong \mathfrak{h}^1(J)(-1) \in \mathcal{M}_{\mathbb{C}}$,*

*(c)* $CH_0(P) = 0$.

The author does not know of any instance for which any of the equivalent conditions (a)-(c) above holds. This more modest result should thus be viewed as an illustration of the intractability of the motive $P$ even in very simple cases. The plan of the paper will be as follows. In the second section, we will present some useful lemmas needed for the proof of Theorem 1.6, as well as summarize the notation to be used. In the third section, we will prove Theorem 1.6. In the final section, we will analyze the motive $P$ in the case of a very general principally polarized complex Abelian fourfold and prove Theorem 1.8.

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## 2 Preliminaries

### 2.1 Conventions on Chow motives

Throughout this paper, all Chow groups will be taken to have *rational coefficients.* We will let $\mathcal{M}_k$ denote the category of Chow motives over a field $k$ with *rational coefficients.* This is the category whose objects are triples

$$M = (X, \pi, m) \in \mathcal{M}_k,$$

where $X$ is a smooth projective variety, $\pi \in CH^{dim(X)}(X \times X)$ is an idempotent with respect to $\circ$ (as defined in [4] Chapter 16.1) and $m \in \mathbb{Z}$. Morphisms in $\mathcal{M}_k$ are given by *correspondences:*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}_k}((X,\pi,n),(X',\pi',n')) &:= \pi' \circ Cor^{n'-n}(X,X') \circ \pi \\ &= \pi' \circ CH^{dim(X)+n'-n}(X \times X') \circ \pi.\end{aligned}$$

We have a functor $\mathfrak{h} : \mathcal{V}_k^{opp} \to \mathcal{M}_k$ from smooth projective varieties to motives defined by $\mathfrak{h}(X) = (X, \Delta_X, 0)$. We also let $H^*$ denote a Weil cohomology theory (we refer the reader to [10] for a precise statement of the axioms) and require the following:

Assumption 2.1. *$H^*$ satisfies the Hard Lefschetz theorem and the Lefschetz hyperplane theorem.*

Remark 2.2. *We note that this assumption is satisfied if $H^*$ is singular cohomology (and $k = \mathbb{C}$) or if $H^*$ is $\ell$-adic cohomology (for $\ell \neq char\ k$).*

For a motive $M = (X, \pi, m)$, we will adopt the usual notation:

$$H^j(M) = \pi_* H^{j+2m}(X),\ CH^j(M) = \pi_* CH^{j+m}(X)$$

where the actions of a correspondence (such as $\pi$) on $H^*$ and $CH^*$ are defined as in [4] Chapter 16.1. Finally, for any $n \in \mathbb{Z}$, we will adopt the usual Tate twist notation:

$$M(n) := (X, \pi, m + n).$$

In the final section, we will also consider the category of homological motives $\mathcal{M}_{hom,\mathbb{C}}$, in which rational equivalence is replaced with homological equivalence (taking $H^*$ to be singular cohomology with $\mathbb{Q}$ coefficients).

### 2.2 Kimura Finite-dimensionality

Here we review the definition and properties of Kimura finite-dimensionality for motives. Recall that the category of Chow motives $\mathcal{M}_k$ is a tensor category with tensor product defined as:

$$(X, \pi, m) \otimes (Y, \tau, n) := (X \times Y, \pi \times \tau, m + n).$$

There is also an action of the symmetric group

$$\mathbb{Q}[\mathfrak{S}_n] \to End_{\mathcal{M}_k}(M^{\otimes n})$$

for $M \in \mathcal{M}_k$. Since $\mathcal{M}_k$ is a pseudo-Abelian category, all idempotents possess images in $\mathcal{M}_k$. So, for any idempotent in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$, there is a corresponding motive. In particular, we have

$$\begin{aligned} \mathrm{Sym}^n M &= \mathrm{Im}(\pi_{sym}) \\ \wedge^n M &= \mathrm{Im}(\pi_{alt}) \end{aligned}$$

for the symmetric and the alternating representation of $\mathfrak{S}_n$.

Definition 2.3 (Kimura). *A motive $M \in M_k$ is said to be oddly finite-dimensional if $Sym^n M = 0$ for $n >> 0$ and evenly finite-dimensional if $\wedge^n M = 0$ for $n >> 0$. $M$ is said to be finite-dimensional if $M = M_+ \oplus M_-$, where $M_+$ is evenly finite-dimensional and $M_-$ is oddly finite-dimensional.*

We have the following properties of finite-dimensional motives:

Theorem 2.4.

(a) *The motive of a smooth projective curve is finite-dimensional.*

(b) *If $M, N \in \mathcal{M}_k$ are finite-dimensional, then so are $M \oplus N$ and $M \otimes N$.*

(c) *If $f : M \to N$ is split-surjective and $M$ is finite-dimensional, then so is $N$. In particular, if $M \oplus N$ is finite-dimensional, then so are $M$ and $N$.*

(d) *Suppose $M$ and $N$ are finite-dimensional and $\Phi : M \to N$ is a map of Chow motives such that $H^*(\Phi) : H^*(M) \to H^*(N)$ is an isomorphism and such that $\exists \Psi \in \mathrm{Hom}_{\mathcal{M}_k}(N, M)$ with $H^*(\Psi) = H^*(\Phi)^{-1}$. Then, $\Phi$ is an isomorphism of Chow motives.*

*Proof.* Items (a)-(c) are all directly from [12]. Item (a) is Corollary 4.4. The $\oplus$ part of (b) follows from the isomorphism of Chow motives:

$$\wedge^n(M_1 \oplus M_2) \cong \bigoplus_{m_1+m_2=n} \wedge^{m_1} M_1 \otimes \wedge^{m_2} M_2$$

(and the corresponding isomorphism for Sym), while the $\otimes$ part is Corollary 5.11. Item (c) is Proposition 6.9. Finally, for (d), note by assumption that the endomorphisms $\Psi \circ \Phi \in \mathrm{End}_{\mathcal{M}_k}(M)$ and $\Phi \circ \Psi \in \mathrm{End}_{\mathcal{M}_k}(N)$ induce automorphisms of $H^*(M)$ and $H^*(N)$. It follows from [8] Lemma 3.1 (iii) that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are automorphisms of Chow motives. In particular, $\Phi : M \to N$ possesses both a left and right inverse in $\mathrm{Hom}_{\mathcal{M}_k}(N, M)$, which means $\Phi$ is an isomorphism of Chow motives. □

Remark 2.5. *As a consequence of (a) and (b), the motive of any product of smooth projective curves is finite-dimensional; from (c), so is any variety dominated by a product of curves (such as Abelian varieties). In [6] it is proved that varieties of dimension $\leq 3$ for which $CH_0(X)_{hom}$ is representable by an Abelian variety have finite-dimensional motive. (This is true, in particular, if $X$ is a rationally connected threefold.) Other than this, the conjecture below remains wide open.*

Conjecture 2.6 (Kimura, O'Sullivan). *Every motive $M \in \mathcal{M}_k$ is finite-dimensional.*

## 2.3 Hard Lefschetz and Lefschetz Standard

Suppose that $X$ is a smooth projective variety of dimension $d$ over a field $k$ and $D \in CH^1(X)$ is the class of an ample divisor. Let $\Delta : X \hookrightarrow X \times X$ denote the diagonal imbedding and let

$$L_D := \Delta_*(D) \in CH^{d+1}(X \times X) = \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}(X), \mathfrak{h}(X)(1))$$

denote the *Lefschetz correspondence* of $D$ (as in [13]). The following lemma then gives a familiar characterization of the action of this correspondence:

Lemma 2.7. *(a) For any $\alpha \in CH^j(X)$ (resp., $H^j(X)$), we have*

$$L_{D*}(\alpha) = \alpha \cdot D \in CH^{j+1}(X) \text{ (resp., } H^{j+2}(X)(1))$$

*(b) Suppose that $i : D \hookrightarrow X$ is a smooth divisor on $X$. Then,*

$$L_D = \Gamma_i \circ {}^t\Gamma_i \in CH^{d+1}(X \times X),$$

*where $\Gamma_i$ denotes the graph of $i$.*

*Proof.* For (a), see [13] Lemma 1.1. For (b), we begin by noting the obvious commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{\Delta} & D \times D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i\times i} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

From the functoriality of pushforward ([4] Chapter 1.6), we then have

$$L_D = (\Delta)_*(D) = (\Delta)_*(i_*1) = (i \times i)_*(\Delta_D) = \Gamma_i \circ \Delta_D \circ {}^t\Gamma_i = \Gamma_i \circ {}^t\Gamma_i,$$

where $\Delta_D \in CH^{g-1}(\Theta \times \Theta)$ denotes the class of the diagonal and where the penultimate step follows from Lemma 2.8 below. □

Lemma 2.8 (Liebermann). *Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms of smooth projective varieties and $\alpha \in CH^*(X \times Y)$. Then,*

$$(f \times g)_*(\alpha) = \Gamma_g \circ \alpha \circ {}^t\Gamma_f \in CH^*(X' \times Y').$$

*Proof.* See [4] Proposition 16.1.1. □

Since $H^*$ satisfies Assumption 2.1, for any ample divisor $D \in Pic(X)$, the Lefschetz correspondence induces an isomorphism:

$$\cup D^{d-j} : H^j(X) \xrightarrow{\cong} H^{2d-j}(X)(d-j) \tag{1}$$

for $j < d$. A natural question is then whether the inverse (cohomological) correspondence is algebraic. More precisely, we have

Conjecture 2.9 (Lefschetz standard conjecture). *There exists $\Lambda_{j,D} \in CH^{d-j}(X \times X)$ for which*

$$H^*(\Lambda_{j,D}) : H^{2d+j}(X)(j) \to H^{2d-j}(X)$$

*is the inverse of (1).*

Some cases for which the Lefschetz standard conjecture is known to hold include: curves, Abelian varieties ([10]), varieties for which the cycle class map is an isomorphism ([11]), as well as for uniruled threefolds, unirational fourfolds, the moduli space of stable vector bundles over a smooth projective curve, and for the Hilbert scheme $S^{[n]}$ of every smooth projective surface (see [1] Corollaries 4.3, 7.2 and 7.5 for these latter).

In the case of Abelian varieties, there is in fact a much stronger statement, given by the following result of Künnemann, which will be indispensable to our proof in the next section:

Theorem 2.10 (Motivic Hard Lefschetz). *Let $A$ be an Abelian variety of dimension $g$ and let $\Theta \in CH^1(A)$ be the class of a symmetric ample divisor and let $L_\Theta$ be the associated Lefschetz correspondence. Further, let $\{\pi_{j,A}\}$ be the Chow-Künneth idempotents of Theorem 1.2 (a). Then, there exists*

$$\Lambda_\Theta \in CH^{g-1}(A \times A)$$

*satisfying:*

*(i) Set $\pi_{j,A} = 0$ for all $j \notin \{0, 1, ...2g\}$. Then, we have*

$$L_\Theta \circ \pi_{j,A} = \pi_{j+2,A} \circ L_\Theta, \ \Lambda_\Theta \circ \pi_{j,A} = \pi_{j-2,A} \circ \Lambda_\Theta \tag{2}$$

*(ii) The correspondence $\pi_{2g-j,A} \circ L_\Theta^{g-j} \circ \pi_{j,A} : \mathfrak{h}^j(A) \to \mathfrak{h}^{2g-j}(A)(g-j)$ is an isomorphism of Chow motives for $j \leq g$; i.e., there are the following equalities of correspondences*

$$\begin{aligned} \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j} = \pi_{j,A} = \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j} \circ \pi_{j,A} \quad \text{for } j < g. \\ \pi_{j,A} \circ L_\Theta^{j-g} \circ \Lambda_\Theta^{j-g} = \pi_{j,A} = L_\Theta^{j-g} \circ \Lambda_\Theta^{j-g} \circ \pi_{j,A} \ \text{for } j > g. \end{aligned} \tag{3}$$

*Proof.* For (i), see [13] Theorem 4.1 and for (ii), see Theorem 5.2. □

Remark 2.11. *(a) It should be noted that the results in [13] actually hold more generally for the motives of Abelian schemes in the category of relative Chow motives over a smooth quasi-projective base scheme.*

*(b) The assumption that the divisor $\Theta$ be symmetric; i.e., that*

$$\Theta = (-1)^*\Theta \in CH^1(A)$$

*is essential for (i).*

### 2.4 A Shortcut to proving Conjecture 1.1

For convenience, we will state as a lemma a strategy often employed when finding Chow-Künneth idempotents. The strategy is essentially to construct idempotents for all but the middle degrees of cohomology and is used to prove Conjecture 1.1 for surfaces ([15]) and complete intersections in $\mathbb{P}^n$ ([16]) among others. This strategy is particularly well-suited for any variety whose "nontrivial" cohomology is concentrated in the middle degree, as is the case for theta divisors.

Lemma 2.12. *Let $X$ be a smooth projective variety of dimension $d$. Suppose there exist correspondences $\{\pi_0, \pi_1, \ldots \pi_{d-1}, \pi_{d+1}, \ldots \pi_{2d}\} \subset CH^d(X \times X)$ satisfying:*

*(i) $\pi_j^2 = \pi_j$*

*(ii)* $\pi_j \circ \pi_{j'} = 0$ *for* $j \neq j'$

*(iii)* $\pi_{j*}H^*(X) = H^j(X)$.

*Then, Conjecture 1.1 holds for X.*

*Proof.* The conditions of Conjecture 1.1 force the definition:

$$\pi_d = \Delta_X - \sum_{j \neq d} \pi_j.$$

It is then a straightforward computation to show that $\pi_d$ is mutually orthogonal to the remaining idempotents and that (iii) holds for $j = d$. □

## 3 Proof of Theorem 1.6

### 3.1 A Reduction and an Important Lemma

It suffices to prove the result in the case that $[\Theta] \in CH^1(A)$ is symmetric. Indeed, by assumption, there exists $x \in A(k)$, such that $[t_x(\Theta)]$ is symmetric. The translation morphism $t_x : A \to A$ then induces an isomorphism between $\Theta$ and $t_x(\Theta)$. The statement of Theorem 1.6 is then true for $\Theta$ if and only if it is true for $t_x(\Theta)$.

We can then define the following 2 sets of correspondences below which will be used in the proofs of both parts of the theorem.
For $j < g-1$:

$$\begin{array}{lll} \Phi_j := {}^t\Gamma_i \circ \pi_{j,A} & \in & \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^j(A), \mathfrak{h}(\Theta)) \\ \Psi_j := \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j-1} \circ \Gamma_i & \in & \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}(\Theta), \mathfrak{h}^j(A)) \end{array} \tag{4}$$

For $j > g-1$:

$$\begin{array}{lll} \Phi_j := \pi_{j+2,A} \circ \Gamma_i & \in & \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}(\Theta), \mathfrak{h}^{j+2}(A)(1)) \\ \Psi_j := {}^t\Gamma_i \circ L_\Theta^{j-g+1} \circ \Lambda_\Theta^{j-g+2} \circ \pi_{j+2,A} & \in & \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^{j+2}(A)(1), \mathfrak{h}(\Theta)) \end{array} \tag{5}$$

The lemma below summarizes their properties:

Lemma 3.1. *We have the following relations:*

*(a)* $\Psi_j \circ \Phi_{j'} = 0$ *for* $j \neq j' < g-1$.

*(b)* $\Phi_j \circ \Psi_{j'} = 0$ *for* $j \neq j' > g-1$.

*(c)* $\Psi_j \circ \Psi_{j'} = 0$ *for* $j < g-1 < j'$.

*(d)* $\Phi_{j'} \circ \Phi_j = 0$ *for* $j < g-1 < j'$.

*(e)* $\Phi_j \circ \pi_{j,A} = \Phi_j$ *for* $j < g-1$.

*(f)* $\pi_{j+2,A} \circ \Phi_j = \Phi_j$ *for* $j > g-1$.

*(g)* $\Psi_j \circ \Phi_j = \pi_{j,A}$ *for* $j < g-1$.

*(h)* $\Phi_j \circ \Psi_j = \pi_{j+2,A}$ *for* $j > g-1$.

*Proof.* For (a), we compute:

$$\begin{aligned}\Psi_j \circ \Phi_{j'} &= \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j',A} \\ &= \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j} \circ \pi_{j',A} \\ &= \pi_{j,A} \circ \pi_{j',A} = 0\end{aligned}$$

where the second equality follows by Lemma 2.7 (b) and the third from Theorem 2.10 (ii). The verification of (b) is essentially the same.
For (c), we compute:

$$\begin{aligned}\Psi_j \circ \Psi_{j'} &= \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ L_\Theta^{j'-g+1} \circ \Lambda_\Theta^{j'-g+2} \circ \pi_{j'+2,A} \\ &= \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j} \circ L_\Theta^{j'-g+1} \circ \Lambda_\Theta^{j'-g+2} \circ \pi_{j'+2,A} \\ &= \pi_{j,A} \circ L_\Theta^{j'-g+1} \circ \Lambda_\Theta^{j'-g+2} \circ \pi_{j'+2,A} \\ &= \pi_{j,A} \circ \pi_{j',A} \circ L_\Theta^{j'-g+1} \circ \Lambda_\Theta^{j'-g+2} = 0\end{aligned}$$

where, again, the second equality follows by Lemma 2.7 (b) and the third from Theorem 2.10 (ii). The fourth equality follows by applying 2.10 (i) repeatedly.
For (d), we have

$$\begin{aligned}\Phi_{j'} \circ \Phi_j &= \pi_{j'+2,A} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j,A} \\ &= \pi_{j'+2,A} \circ L_\Theta \circ \pi_{j,A} \\ &= \pi_{j'+2,A} \circ \pi_{j+2,A} \circ L_\Theta = 0\end{aligned}$$

where the second equality holds by Lemma 2.7 and the third holds by Theorem 2.10 (i).
For (e), we have:

$$\Phi_j \circ \pi_{j,A} = {}^t\Gamma_i \circ \pi_{j,A} \circ \pi_{j,A} = \Phi_j$$

using the fact that $\pi_{j,A}$ is an idempotent. The verification of (f) is similar.
For (g), we have

$$\begin{aligned}\Psi_j \circ \Phi_j &= \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j,A} \\ &= \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j} \circ \pi_{j,A} = \pi_{j,A}\end{aligned}$$

where the second equality is Lemma 2.7 and the third by Theorem 2.10 (ii). The verification of (h) is similar. □

## 3.2 Proof of Theorem 1.6

*Proof of Theorem 1.6 (a).* For the proof, we will first need to exhibit a set of correspondences $\{\pi_{0,\Theta}, \pi_{1,\Theta}, \ldots, \pi_{g-2,\Theta}, \pi_{g,\Theta}, \ldots \pi_{2g-2,\Theta}\}$ as in Lemma 2.12. These will give the required Chow-Künneth idempotents. We define these below:

$$\begin{aligned}\pi_{j,\Theta} &:= \Phi_j \circ \Psi_j \in \mathrm{End}_{\mathcal{M}_k}(\mathfrak{h}(\Theta)) \text{ for } j < g-1 \\ \pi_{j,\Theta} &:= \Psi_j \circ \Phi_j \in \mathrm{End}_{\mathcal{M}_k}(\mathfrak{h}(\Theta)) \text{ for } j > g-1\end{aligned} \tag{6}$$

We will be done with the proof, provided we can verify the following two bullet points:

- $\{\pi_{0,\Theta}, \pi_{1,\Theta}, \ldots, \pi_{g-2,\Theta}, \pi_{g,\Theta}, \ldots \pi_{2g-2,\Theta}\}$ satisfy the conditions Lemma 2.12.
- There are isomorphisms as in (i) and (ii) of Theorem 1.6 (a).

Towards the first bullet point, we first check condition (i) of Lemma 2.12. When $j < g-1$,

$$\pi_{j,\Theta}^2 = \Phi_j \circ (\Psi_j \circ \Phi_j) \circ \Psi_j = (\Phi_j \circ \pi_{j,A}) \circ \Psi_j = \Phi_j \circ \Psi_j = \pi_{j,\Theta}$$

where the second and third equalities hold by Lemma 3.1 (g) and (e), respectively. The verification in the case of $j > g-1$ is similar and uses instead Lemma 3.1 (h) and (f). We can also verify that these correspondences satisfy condition (ii). Indeed, there are the following 4 cases to consider:

$$\begin{aligned}
\pi_{j,\Theta} \circ \pi_{j',\Theta} &= \Phi_j \circ (\Psi_j \circ \Phi_{j'}) \circ \Psi_{j'} = 0 \text{ for } j \neq j' < g-1.\\
\pi_{j,\Theta} \circ \pi_{j',\Theta} &= \Psi_j \circ (\Phi_j \circ \Psi_{j'}) \circ \Phi_{j'} = 0 \text{ for } j \neq j' > g-1.\\
\pi_{j,\Theta} \circ \pi_{j',\Theta} &= \Phi_j \circ (\Psi_j \circ \Psi_{j'}) \circ \Phi_{j'} = 0 \text{ for } j < g-1 < j'.\\
\pi_{j',\Theta} \circ \pi_{j,\Theta} &= \Psi_{j'} \circ (\Phi_{j'} \circ \Phi_j) \circ \Psi_j = 0 \text{ for } j < g-1 < j'.
\end{aligned} \tag{7}$$

where the second equality in the above 4 cases holds by Lemma 3.1 (a)-(d), respectively.

Before checking that condition (iii) of Lemma 2.12 is satisfied, we verify the second bullet point and construct the isomorphisms in (i) and (ii) of Theorem 1.6 (a). Indeed, for $j \neq g-1$ set

$$\mathfrak{h}^j(\Theta) = (\Theta, \pi_{j,\Theta}, 0)$$

and we define:

$$\begin{aligned}
\Phi'_j &:= \pi_{j,\Theta} \circ \Phi_j \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^j(A), \mathfrak{h}^j(\Theta)) \text{ for } j < g-1\\
\Phi'_j &:= \Phi_j \circ \pi_{j,\Theta} \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^j(\Theta), \mathfrak{h}^{j+2}(A)(1)) \text{ for } j > g-1
\end{aligned} \tag{8}$$

Lemma 3.2. *The correspondences defined in (8) are isomorphisms of Chow motives. Hence, the motives $\mathfrak{h}^j(\Theta)$ satisfy conditions (i) and (ii) of Theorem 1.6 (a).*

*Proof.* We can construct the inverses explicitly:

$$\begin{aligned}
\Psi'_j &:= \Psi_j \circ \pi_{j,\Theta} \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^j(\Theta), \mathfrak{h}^j(A)) \text{ for } j < g-1\\
\Psi'_j &:= \pi_{j,\Theta} \circ \Psi_j \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^{j+2}(A)(1), \mathfrak{h}^j(\Theta)) \text{ for } j > g-1
\end{aligned} \tag{9}$$

To check that these are in fact inverses, we need to verify the following:

(A) $\Psi'_j \circ \Phi'_j = \pi_{j,A}$, $\Phi'_j \circ \Psi'_j = \pi_{j,\Theta}$ for $j < g-1$.

(B) $\Psi'_j \circ \Phi'_j = \pi_{j,\Theta}$, $\Phi'_j \circ \Psi'_j = \pi_{j+2,A}$ for $j > g-1$.

For (A), we verify the first condition:

$$\begin{aligned}\Psi'_j \circ \Phi'_j = \Psi_j \circ \pi_{j,\Theta} \circ \pi_{j,\Theta} \circ \Phi_j &= \Psi_j \circ \pi_{j,\Theta} \circ \Phi_j \\ &= (\Psi_j \circ \Phi_j) \circ (\Psi_j \circ \Phi_j) \\ &= \pi_{j,A}^2 = \pi_{j,A}\end{aligned}$$

using the definitions of $\pi_{j,\Theta}$, the fact that it is an idempotent and Lemma 3.1 (g) for the penultimate equality. For the second condition, we have:

$$\Phi'_j \circ \Psi'_j = \pi_{j,\Theta} \circ (\Phi_j \circ \Psi_j) \circ \pi_{j,\Theta} = \pi_{j,\Theta}^3 = \pi_{j,\Theta}$$

again using the definition of $\pi_{j,\Theta}$ and the fact that it is an idempotent. The verification of condition (B) is essentially the same. □

This completes the verification of the second bullet point. To finish the proof of the first bullet point, what remains then is to check that condition (iii) in Lemma 2.12 holds. To this end, we need to show that $\pi_{j,\Theta}$ acts as the identity on $H^j(\Theta)$ and trivially on $H^{j'}(\Theta)$ for $j \neq j'$ and any Weil cohomology $H^*$. Since $\pi_{j,\Theta} \circ \pi_{j',\Theta} = 0$ for $j \neq j'$, we are reduced to showing that $\pi_{j,\Theta}$ acts as the identity on $H^j(\Theta)$. We verify this for $j < g-1$, since the case of $j > g-1$ is similar. Indeed, we have by definition:

$$H^*(\pi_{j,\Theta}) = H^*(\Phi_j) \circ H^*(\Psi_j).$$

We observe that $H^*(\Phi_j) = i^* : H^j(A) \to H^j(\Theta)$ and, by the Lefschetz hyperplane theorem, this is an isomorphism. Further, since

$$\Psi_j \circ \Phi_j = \pi_{j,A}$$

this means $H^*(\Psi_j) : H^j(\Theta) \to H^j(A)$ is the inverse of $H^*(\Phi_j)$. This shows that $H^*(\pi_{j,\Theta}) = id|_{H^j(\Theta)}$, as required. □

*Proof of Theorem 1.6 (b).* It follows from Theorem 1.6 (b) and Remark 2.5 that the Chow motive $\mathfrak{h}^j(\Theta)$ is finite-dimensional for $j \neq g-1$. Thus, by Theorem 2.4 (d), it suffices to show that (for some fixed choice of Weil cohomology)

$$\cup h^{g-1-j} = H^*(L_h^{g-1-j}) : H^j(\Theta) \to H^{2(g-1)-j}(\Theta)(g-1-j)$$

is an isomorphism and its inverse is algebraic (i.e., Conjecture 2.9 is satisfied for $X = \Theta$ (and all $j \leq g$)). That the above map is an isomorphism is true by Assumption 2.1 and that Conjecture 2.9 is satisfied for $X = \Theta$ follows from [10] Prop. 2.12. □

*Proof of Theorem 1.6 (c).* We first define the following morphism:

$$\Phi_{g-1} := \pi_{g+1,A} \circ \Gamma_i \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}(\Theta), \mathfrak{h}^{g+1}(A)(1))$$

We would like to show it is split-surjective with right-inverse given by

$$\Psi_{g-1} := {}^t\Gamma_i \circ \Lambda_\Theta \circ \pi_{g+1,A} \in \mathrm{Hom}_{\mathcal{M}_k}(\mathfrak{h}^{g+1}(A)(1), \mathfrak{h}(\Theta))$$

i.e., that $\Phi_{g-1} \circ \Psi_{g-1} = \pi_{g+1,A}$. Indeed, we see that

$$\begin{aligned}\Phi_{g-1} \circ \Psi_{g-1} &= \pi_{g+1,A} \circ \Gamma_i \circ {}^t\Gamma_i \circ \Lambda_\Theta \circ \pi_{g+1,A}\\ &= \pi_{g+1,A} \circ L_\Theta \circ \Lambda_\Theta \circ \pi_{g+1,A} = \pi_{g+1,A}\end{aligned}$$

where the second equality holds by Lemma 2.7 and the third by Theorem 2.10 (ii). We can then define a correspondence:

$$\pi'_{g-1,\Theta} := \Phi_{g-1} \circ \Psi_{g-1} \in \mathrm{End}_{\mathcal{M}_k}(\mathfrak{h}(\Theta)) \tag{10}$$

A standard calculation then shows that $\pi'_{g-1,\Theta}$ is an idempotent, which then gives a Chow motive:

$$\mathfrak{h}_1^{g-1}(\Theta) := (\Theta, \pi'_{g-1,\Theta}, 0)$$

As in the proof of Theorem 1.6 (c), one check that there is an isomorphism:

$$\Phi'_{g-1} := \Phi_{g-1} \circ \pi'_{g-1,\Theta} : \mathfrak{h}_1^{g-1}(\Theta) \xrightarrow{\cong} \mathfrak{h}^{g+1}(A)(1)$$

whose inverse is

$$\Psi'_{g-1} := \pi'_{g-1,\Theta} \circ \Psi_{g-1} : \mathfrak{h}^{g+1}(A)(1) \to \mathfrak{h}_1^{g-1}(\Theta)$$

What remains is then to show that $\mathfrak{h}_1^{g-1}(\Theta)$ is a submotive of

$$\mathfrak{h}^{g-1}(\Theta) = (\Theta, \pi_{g-1,\Theta}, 0)$$

By the proof of Lemma 2.12, we have the forced definition:

$$\pi_{g-1,\Theta} := \Delta_\Theta - \sum_{j \neq g-1} \pi_{j,\Theta}$$

So, it suffices to check that $\pi'_{g-1,\Theta}$ is orthogonal to $\pi_{j,\Theta}$ for $j \neq g-1$. This verification is the same as that of (7) with $\pi_{j,\Theta}$ replaced by $\pi'_{g-1,\Theta}$. Indeed, we have

$$\pi'_{g-1,\Theta} = \Phi_{g-1} \circ \Psi_{g-1}$$

and the relations (a)-(d) of Lemma 3.1 (used to obtain (7)) still hold when $j = g-1$, given the definition of $\Phi_{g-1}$ and $\Psi_{g-1}$.
Thus, $\mathfrak{h}_1^{g-1}(\Theta) \cong \mathfrak{h}^{g+1}(A)(1)$ is a submotive of $\mathfrak{h}^{g-1}(\Theta)$ and we have a direct sum decomposition:

$$\mathfrak{h}^{g-1}(\Theta) = \mathfrak{h}_1^{g-1}(\Theta) \oplus P \cong \mathfrak{h}^{g+1}(A)(1) \oplus P$$

where $P := (\Theta, p, 0)$ and $p = \pi_{g-1,\Theta} - \pi'_{g-1,\Theta}$. □

Remark 3.3. *It should be noted that a result analogous to Theorem 1.6 (a) holds for a smooth (symmetric) ample divisor on an Abelian scheme $A \to S$ (where $S$ is a smooth quasi-projective variety) in the category of relative Chow motives over $S$ (since both Theorems 1.2 and 2.10 hold in the relative context).*

### 3.3 Conjectured unicity of the Chow-Künneth summands

As we note above, the unicity of the Chow-Künneth summands follows if the smooth projective variety has finite-dimensional Chow motive. While we cannot prove that the Chow motive of $\Theta$ is finite-dimensional, we do note that the summands $\mathfrak{h}^j(\Theta)$ are unique (up to isomorphism) for $j \neq g-1$ if one imposes the (motivic) Lefschetz hyperplane condition as in Theorem 1.6 (a). Additionally, there is the result below, which shows that the summands we obtain above are minimal in the following sense:

Lemma 3.4. *Retain the notation from Theorem 1.6. Also, let $\{\pi'_{j,\Theta}\}$ be another set of idempotents satisfying the conditions of the Chow-Künneth Conjecture. Denote the corresponding Chow motives by $\mathfrak{h}^j(\Theta)' := (\Theta, \pi'_{j,\Theta}, 0)$. Then, for each $j \neq g-1$, there exists a phantom motive $P_j \in \mathcal{M}_k$ (i.e., one having trivial cohomological realization) for which*

$$\mathfrak{h}^j(\Theta)' \cong \mathfrak{h}^j(\Theta) \oplus P_j \tag{11}$$

*Proof.* We show this for $j < g-1$ only (the proof in the case of $j > g-1$ being analogous). By Theorem 1.6 (a), it suffices to show that there is some $P_j$ as above for which:

$$\mathfrak{h}^j(\Theta)' \cong \mathfrak{h}^j(A) \oplus P_j$$

in $\mathcal{M}_k$. To this end, we observe that the pull-back along $\Theta \hookrightarrow A$ induces a morphism of Chow motives $\Phi'_j : \mathfrak{h}^j(A) \to \mathfrak{h}^j(\Theta)'$ that is an isomorphism on cohomology by the Lefschetz hyperplane theorem. Then, since the Lefschetz standard conjecture holds for $A$ and $\mathfrak{h}^j(A)$ is finite-dimensional, it is a standard argument to show that $\Phi'_j$ is split-injective (c.f., the proof of Theorem 1.6 (b)). This gives (11), as desired. □

## 4 The complementary motive P

In this section, we specialize to the case where $A$ is a principally polarized Abelian variety over $\mathbb{C}$, whose principal polarization is the class of $i : \Theta \to A$. We would like a case in which a very general $\Theta$ is smooth. As noted in the introduction, a general Abelian variety of dimension $\leq 3$ is a Jacobian (and, hence, $\Theta$ fails to be smooth). The next case is then when $g = 4$, where a well-known result of Mumford in [14] shows that $\Theta$ is very generally smooth. Moreover, we have the following:

Lemma 4.1. *The primal cohomology*

$$H^3_{pr}(\Theta, \mathbb{Q}) := ker\, [H^3(\Theta, \mathbb{Q}) \xrightarrow{i_*} H^5(A, \mathbb{Q})(1)]$$

*is a rational Hodge structure of level 1 and the intermediate Jacobian of this Hodge structure:*

$$J^3_{pr}(\Theta) := \frac{H^3_{pr}(\Theta, \mathbb{C})}{F^2 H^3_{pr}(\Theta, \mathbb{C}) + H^3_{pr}(\Theta, \mathbb{Z})}$$

*is an Abelian variety of dimension 5.*

*Proof.* See §1.3 of [7]. □

Definition 4.2. *For $X$ a smooth projective variety over $\mathbb{C}$, the the* coniveau filtration *is the descending filtration on the singular cohomology $H^j(X, R)$ (with coefficients in a ring $R \subset \mathbb{C}$) given by:*

$$N^c H^j(X, R) := \sum \ker[H^j(X, R) \to H^j(X \setminus Y, R)]$$

*where the sum ranges over all closed imbeddings of subvarieties $Y \hookrightarrow X$ of codimension $\geq c$.*

There is then the following well-known generalization of the Hodge conjecture due to Grothendieck:

Conjecture 4.3 (Generalized Hodge Conjecture, $GHC(c, j)$). *$N^c H^j(X, \mathbb{Q})$ is the largest $\mathbb{Q}$-Hodge substructure of level $j - 2c$ in $H^j(X, \mathbb{Q})$.*

Now, in the case at hand, we have the following result:

Theorem 4.4 (Izadi-Van Straten, [7]). *$GHC(1, 3)$ holds for $X = \Theta$, where $(A, \Theta)$ is very general in the moduli space of principally polarized Abelian varieties of dimension 4; i.e., $N^1 H^3(\Theta, \mathbb{Q})$ is the largest Hodge structure of level 1 in $H^3(\Theta, \mathbb{Q})$.*

The following result is perhaps well-known to the experts, but since this precise version was not found in the literature, we state it here for convenience:

Proposition 4.5. *Let $X$ be a smooth complex projective threefold that satisfies $GHC(1, 3)$. Then, the intermediate algebraic Jacobian*

$$J^3(X) := \frac{N^1 H^3(X, \mathbb{C})}{N^1 H^3(X, \mathbb{C}) \cap (F^2 H^3(X, \mathbb{C}) + H^3(X, \mathbb{Z}))}$$

*is an Abelian variety. Moreover, for every rational Hodge structure of level 1, $H \subset H^3(X, \mathbb{Q})$, there exists an (isogeny class of an) Abelian variety that appears in the Poincaré decomposition of $J^3(X)$ and an idempotent $\pi_H \in \mathrm{End}_{\mathcal{M}_{\mathbb{C}}}(\mathfrak{h}(X))$ for which:*

*(a) There is an isomorphism in $\mathcal{M}_{\mathbb{C}}$, $P_H = (X, \pi_H, 0) \cong \mathfrak{h}^1(A)(-1)$.*

*(b) $H^*(P_H) = \pi_{H*} H^*(X, \mathbb{Q}) = H$.*

*Proof.* The statement that the complex torus $J^3(X)$ is an Abelian variety follows from the fact $N^1H^3(X,\mathbb{C})$ is a level 1 Hodge structure and the cup product induces a polarization on it. Using [17] Theorem 2, there exists an idempotent $\pi \in End_{\mathcal{M}_{\mathbb{C}}}(\mathfrak{h}(X))$ for which:

- There is an isomorphism in $\mathcal{M}_{\mathbb{C}}$, $\phi : (X,\pi,0) \cong \mathfrak{h}^1(J^3(X))(-1)$.
- $\pi_* H^*(X,\mathbb{Q}) = N^1H^3(X,\mathbb{Q})$.

In particular, the correspondence $\phi$ induces an isomorphism

$$H^*(\phi) : N^1H^3(X,\mathbb{Q}) \to H^1(J^3(X),\mathbb{Q})(-1) \tag{12}$$

Since $X$ satisfies $GHC(1,3)$, any $H$ as in the statement of the proposition is a sub-Hodge structure of $N^1H^3(X,\mathbb{Q})$. Thus, the image of $H$ under (12) is a sub-Hodge structure of $H^1(J^3(X),\mathbb{Q})(-1)$ and so gives an Abelian variety, $A_H$, whose isogeny class appears in the decomposition of $J^3(X)$ and for which there is an isomorphism

$$H \cong H^1(A_H,\mathbb{Q})(-1) \tag{13}$$

induced by (12). Moreover, using [9] Proposition 2.1, there is a fully faithful functor

$$\mathfrak{h}^1 : AV_{\mathbb{C}}^{opp} \to \mathcal{M}_{\mathbb{C}}$$

which sends an Abelian variety $A$ to the Chow Künneth summand $\mathfrak{h}^1(A)$ from Theorem 1.2. Since $AV_{\mathbb{C}}^{opp}$ is a semi-simple category (using Poincaré irreducibility), there exists a summand $P_H$ of the Chow motive $(X,\pi,0)$ such that the isomorphism $\phi : (X,\pi,0) \cong \mathfrak{h}^1(J^3(X))(-1)$ induces an isomorphism $P_H \cong \mathfrak{h}^1(A_H)(-1)$. Because the isomorphism (13) is also induced by $\phi$, it follows that $H^*(P_H) = H$, and this proves the result. ☐

Remark 4.6. *It is straightforward that the Abelian variety $A_H$ in the previous theorem can be taken to be*

$$\frac{H_{\mathbb{C}}(1)}{F^1H_{\mathbb{C}}(1) + H_{\mathbb{Z}}(1)}$$

*where $H_{\mathbb{C}} = H \otimes \mathbb{C}$ and $H_{\mathbb{Z}} \subset H$ is a choice of lattice.*

We now have the following corollary, which gives the first statement in Theorem 1.8:

Corollary 4.7. *There is an isomorphism of homological motives, $P \cong \mathfrak{h}^1(J^3_{pr}(\Theta))(-1)$.*

*Proof.* We take $X = \Theta$ and $H = H^3_{pr}(\Theta,\mathbb{Q})$ as in Proposition 4.5. Using this latter, as well as Remark 4.6, there is a summand $P_H$ of $\mathfrak{h}(\Theta) \in \mathcal{M}_{\mathbb{C}}$ and an isomorphism in $\mathcal{M}_{\mathbb{C}}$:

$$P_H = (\Theta,\pi_H,0) \cong \mathfrak{h}^1(J^3_{pr}(\Theta))(-1).$$

It suffices then to show that $P = P_H \in \mathcal{M}_{\mathbb{C},hom}$; i.e., that $p = \pi_H \in \mathrm{End}_{\mathcal{M}_{\mathbb{C},hom}}(\mathfrak{h}(\Theta))$. Indeed, $p$ and $\pi_H$ are idempotents such that

$$p_* H^*(\Theta, \mathbb{Q}) = \pi_{H*} H^*(\Theta, \mathbb{Q}).$$

Thus, $p = \pi_H \in \mathrm{End}_{\mathbb{Q}}(H^*(\Theta, \mathbb{Q}))$. Since the Betti realization functor $\mathcal{M}_{\mathbb{C},hom} \to \mathrm{Vec}_{\mathbb{Q}}$ is faithful, it follows that $p = \pi_H \in \mathrm{End}_{\mathcal{M}_{\mathbb{C},hom}}(\mathfrak{h}(\Theta))$, as desired. □

*Proof of Theorem 1.8.* The first statement is precisely Corollary 4.7. We prove the equivalences below:

For (a) ⇒(b), note that if $\mathfrak{h}(\Theta)$ is finite-dimensional, then so is $P$ by Theorem 2.4 (c). By Lemma 4.7, there is an isomorphism of homological motives, $P \cong \mathfrak{h}^1(J^3_{pr}(\Theta))(-1)$. Since $P$ and $\mathfrak{h}^1(J^3_{pr}(\Theta))$ are finite-dimensional, it follows from Theorem 2.4(d) that we have an isomorphism:

$$P \cong \mathfrak{h}^1(J^3_{pr}(\Theta))(-1) \in \mathcal{M}_{\mathbb{C}} \tag{14}$$

For (b) ⇒ (c), apply $CH^3$ to (14) to obtain:

$$CH_0(P) = CH^3(P) \cong CH^3(\mathfrak{h}^1(J^3_{pr}(\Theta))(-1)) \cong CH^2(\mathfrak{h}^1(J^3_{pr}(\Theta))) = 0$$

For (c) ⇒ (a), we suppose that $CH_0(P) = 0$. To show that $\mathfrak{h}(\Theta) \in \mathcal{M}_{\mathbb{C}}$ is finite-dimensional, it suffices by Theorem 1.6 and Theorem 2.4 to show that $P$ is finite-dimensional. To this end, note by assumption that we have

$$p_* CH_0(\Theta) = 0 \tag{15}$$

Now, for $x \in \Theta$ let $j_x : x \times \Theta \hookrightarrow \Theta \times \Theta$ and then (15) yields

$$0 = p_*(x) = \pi_{2*}(\pi_1^*(x) \cdot p) = \pi_{2*}(j_{x*} j_x^* p) = j_x^* p.$$

Then, using Voisin's generalization of the Bloch-Srinivas decomposition-of-the-diagonal argument ([20] Theorem 3.1), we deduce that there exists some divisor $D \overset{j}{\hookrightarrow} \Theta$ such that $p \in CH^3(\Theta \times \Theta)$ is supported on $D \times \Theta$; i.e.,

$$p \in \mathrm{Im}\{CH^1(D \times \Theta) \xrightarrow{(j \times id)_*} CH^2(\Theta \times \Theta)\}.$$

Now, let $\epsilon : \tilde{D} \to D$ be a desingularization of $D$. Then, we have

$$p_* CH^2_{alg}(\Theta) \subset j_* \epsilon_* CH^1_{alg}(\tilde{D})$$

where $CH^*_{alg}$ is the group of algebraically trivial cycles ($\otimes\mathbb{Q}$). Then, using the representability of the Picard functor there is a divisor $\mathcal{P} \in CH^1(Pic^0(\tilde{D}) \times \tilde{D})$ for which

$$\mathcal{P}_* CH_0(Pic^0(\tilde{D}))_{alg} = CH^1_{alg}(\tilde{D}).$$

In fact, $\mathcal{P}_*$ factors through the Albanese map

$$CH_0(Pic^0(\tilde{D}))_{alg} \to Alb(Pic^0(\tilde{D})) \otimes \mathbb{Q}.$$

Then, using Bertini's theorem, one obtains a smooth ample curve $C \overset{\iota}{\hookrightarrow} Pic^0(\tilde{D})$ that induces a surjective map on the Albanese. Now, consider the correspondence

$$\Gamma := \Gamma_{j_D \circ \epsilon} \circ \mathcal{P} \circ \Gamma_\iota \in CH^2(C \times \Theta)$$

Using the above observations, we deduce that

$$p_* CH^2_{alg}(\Theta) \subset \Gamma_* CH^1_{alg}(C).$$

That is, $p_* CH^2_{alg}(\Theta)$ is *representable* (see [18] Definition 2.1). Since $p_* CH^3_{alg}(\Theta) = 0$ by assumption, [18] Theorem 3.4 applies and $P$ decomposes as

$$\bigoplus \mathbb{1}(i)^{\oplus n_i} \oplus \mathfrak{h}^1(J_i)(-i)$$

for integers $n_i$ and Abelian varieties $J_i$. In particular, $P$ is finite dimensional, and this proves (c) ⇒ (a). □

## References


[1] D. Arapura. *Motivation for Hodge cycles*, Adv. Math. 207 (2) (2006), 762-781.

[2] S. Bloch, V. Srinivas. *Remarks on correspondences and algebraic cycles*, Amer. J. Math. 105 (1983), 1235-1253.

[3] C. Deninger, J. Murre. *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. 88 (3) (1993), 333-353.

[4] W. Fulton. *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 2, Springer Verlag (1984).

[5] B. Brent Gordon, M. Hanamura, J.P. Murre. *Relative Chow-Künneth projectors for modular varieties*, J. Reine Angew. Math. 558 (2003), 1-14.

[6] S. Gorchinskiy, V. Guletskii. *Motives and representability of algebraic cycles on threefolds over a field*, J. Algebraic Geom. 21 (2) (2012), 347-373.

[7] E. Izadi, D. van Straten. *The intermediate Jacobians of the theta divisors of four-dimensional principally polarized abelian varieties*, J. Algebraic Geom. 4 (3) (1995), 557-590.

[8] U. Jannsen. *On finite-dimensional motives and Murre's conjecture*, Algebraic cycles and motives. Vol. 2, volume 344 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge (2007), 112-142.

[9] G. Kings. *Higher regulators, Hilbert modular surfaces, and special values of L-functions*, Duke Math. J. 92 (1998), 61-127.

[10] S. Kleiman. *Algebraic cycles and the Weil conjectures*, Dix exposés sur la cohomologie des schémas, Amsterdam: North-Holland, 1968, 359-386.

[11] S. Kleiman. *The Standard Conjectures*, Proceedings of Symposia in Pure Math., Vol. 55 (1994), Part 1, 3-20.

[12] S. Kimura. *Chow groups are finite dimensional in some sense*, Math. Ann. 331 (1) (2005), 173-201.

[13] K. Künnemann. *A Lefschetz decomposition for Chow motives of abelian schemes*, Inventiones Mathematicae 113 (1) (1993), 85-102.

[14] D. Mumford. *Prym varieties I*, Contributions to Analysis, Academic Press, New York (1974).

[15] J. Murre. *On the motive of an algebraic surface*, J. Reine Angew. Math. 409 (1990), 190-204.

[16] J. Murre, J. Nagel, C. Peters. *Lectures on the Theory of Pure Motives*, Amer. Mat. Soc., University Lecture Note Series 61 (2013).

[17] C. Vial. *Projectors on the intermediate algebraic Jacobian*, New York J. Math. 19 (2013), 793-822.

[18] C. Vial. *Pure motives with representable Chow groups*, Comptes Rendus 348 (2010), 1191-1195.

[19] C. Vial. *Remarks on motives of Abelian type*, Tohoku Math. J. 69 (2) (2017), 195-220.

[20] C. Voisin. *Chow rings, decomposition of the diagonal and the topology of families*, Annals of Math. Studies 187, Princeton University Press (2014).

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