Decay Rates of Bound States at the Spectral Threshold of Multi-Particle Schrödinger Operators

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Abstract. We consider N-body Schrödinger operators with $N \geq 3$ quantum particles interacting via short-range potentials in dimension $d \geq 3$, where the essential spectrum coincides with the half line $[0, \infty)$. We give the asymptotic behaviour of eigenfunctions corresponding to the eigenvalue at the threshold of the essential spectrum under the condition that the eigenfunctions are not orthogonal to the sum of the pair interactions. This condition is fulfilled when zero is the smallest eigenvalue and the pair interactions are negative. We also give examples of systems when this condition is not met.

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1 Introduction

Consider the N-particle Schrödinger operator

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{x_i} + \sum_{1\leq i<j\leq N} V_{ij}(x_{ij}), \qquad x_{ij} = x_i - x_j \tag{1.1}$$

with real valued pair interactions $V_{ij}(x_{ij})$ tending to zero as $|x_{ij}| \to \infty$ and denote by H_0 the Hamiltonian after separation of the center of mass motion. By the HVZ-Theorem the essential spectrum of H_0 is given by

$$\sigma_{\text{ess}}(H_0) = [\mu, \infty) \qquad \text{for some} \qquad \mu \leq 0. \tag{1.2}$$

It is well known that if H_0 has an eigenvalue $E < \mu$, then the corresponding eigenfunctions decay exponentially [1]. However, at the threshold μ the situation changes fundamentally. Here the behaviour of the potentials V_{ij} at infinity can cause quite different decay behaviour of solutions corresponding to the eigenvalue equation, which is important for many different physical phenomena, e.g. [10, 11]. It is known, that in case of Coulomb interactions and some other long-range potentials, such threshold eigenfunctions may have a sub-exponential decay rate, see [7, 8, 9, 6].

In this work we focus on the case of short-range potentials and the case $\mu = 0$. The threshold $\mu = 0$ in combination with short-range interactions is a particularly interesting case from many perspectives. For example, such zero-energy solutions are strong related to the so-called Efimov effect [18, 14] and their decay properties are crucial for its existence and non-existence [16]. In the context of the critical coupling constant threshold, see [11], D.K. Gridnev studied N-body systems in dimension three for a certain class of short-range potentials [4, 5], where the subsystems do not have resonances or bound states $E \leq 0$. Based on the analysis of integral equations for the solution of the Schrödinger equation, it was shown that for such systems zero is an eigenvalue. However, in order to study decay rates of the corresponding eigenfunctions, the method of such integral equations faces many difficulties. Note that the configuration space of the system of N three-dimensional particles has dimension $3(N-1)$ and the fundamental solution of the Laplace operator in this space decays as $c|x|^{-(3(N-1)-2)}$. Heuristically, by the Green function formalism this should lead to the same asymptotic behaviour of the corresponding zero energy bound state. However, even if every potential V_{ij} tends to zero at infinity, the sum of potentials in (1.1) does not necessarily have to. Even for compactly supported potentials this is not the case. This makes the implementation of the method of integral equations very difficult.

In the recent paper [3] S. Vugalter and the authors of this work considered systems of $N \geq 3$ particles in dimension $d \geq 3$. Using a purely variational approach it was shown that in case of short-range potentials such zero-energy bound states satisfy

$$(1+|x|)^{\alpha}\varphi_0 \in L^2 \qquad \text{for any} \qquad \alpha < \frac{d(N-1)}{2} - 2. \tag{1.3}$$

Choosing the constant α sufficiently close to it's right bound leads to an estimate from below of the same decay rate as the fundamental solution. However, it does not provide an upper estimate, which leaves open the question whether this estimate from below is close to optimal and whether it can vary for different classes of short-range potentials. In this work we answer this question by providing an explicit formula for the asympotitics of eigenfunctions corresponding to the eigenvalue zero, which gives an upper estimate of (1.3).

Note that both the method of integral equations and the variational approach in [3] are not sufficient in themselves to determine the decay rate of φ_0. But if we combine the two methods, it yields the asymptotic behaviour of φ_0. In [3] estimate (1.3) was deduced from

$$\nabla\left(|x|^\alpha \varphi_0\right) \in L^2 \qquad \text{for any} \quad 0 \le \alpha < \frac{d(N-1)}{2} - 1. \tag{1.4}$$

We will use it as an a priori estimate, which allows us to take advantage of the decay of $V_{ij}(x_{ij})$ in the direction x_{ij}. This way we can derive the asymptotic behaviour of φ_0 by studying its integral representation corresponding to the zero eigenvalue equation. Our proof is also an example how to obtain a two-sided estimate from a one-sided estimate.

2 Notation and Main Result

We consider a system of $N \ge 3$ particles in dimension $d \ge 3$ with masses $m_i > 0$ and position vectors $x_i \in \mathbb{R}^d,\ i = 1, \dots, N$. Such a system is described by the Hamiltonian H_N in (1.1), where we assume that the potentials V_{ij} satisfy

$$|V_{ij}(x_{ij})| \le c|x_{ij}|^{-2-\nu}, \quad x_{ij} \in \mathbb{R}^d,\ |x_{ij}| \ge A \tag{2.1}$$

for some constants $c, \nu, A > 0$ and we allow singularities of the type

$$\begin{cases} V_{ij} \in L^2_{\text{loc}}(\mathbb{R}^d), & \text{if } d = 3, \\ V_{ij} \in L^p_{\text{loc}}(\mathbb{R}^d) \text{ for some } p > 2, & \text{if } d = 4, \\ V_{ij} \in L^{\frac{d}{2}}_{\text{loc}}(\mathbb{R}^d), & \text{if } d \ge 5. \end{cases} \tag{2.2}$$

Under these assumptions on the potentials the operator H_N is essentially self-adjoint on $L^2(\mathbb{R}^{dN})$. Following [13], we define the space R_0 of relative motion of the system as

$$R_0 = \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^{dN} \ : \ \sum_{i=1}^N m_i x_i = 0 \right\} \tag{2.3}$$

and the scalar product

$$\langle x, \tilde{x} \rangle_1 = \sum_{i=1}^N m_i \langle x_i, \tilde{x}_i \rangle, \qquad |x|_1^2 = \langle x, x \rangle_1. \tag{2.4}$$

After the separation of the center of mass motion the Hamiltonian is given by

$$H_0 = -\Delta_0 + V(x), \tag{2.5}$$

where Δ_0 is the Laplace Beltrami operator on $L^2(R_0)$ and the potential V is given by $V(x) = \sum_{1 \le i < j \le N} V_{ij}(x_{ij})$. Our main assumption is that for all sufficiently small $\varepsilon > 0$ we have that

$$\sigma_{\text{ess}}\left(-(1-\varepsilon)\Delta_0 + V\right) = [0, \infty). \tag{2.6}$$

This is in particular the situation where for any subsystem the corresponding Hamiltonian does not have resonances or eigenvalues at the bottom of the essential spectrum. However, the Hamiltonian H_0 of the whole system might have eigenvalues $E \leq 0$.
For a fixed pair of particles $i \neq j$ and $k \neq i, j$ we set

$$R_{ij} = \{x = (x_1, \dots, x_N) \in R_0 \,:\, m_i x_i + m_j x_j = 0,\ x_k = 0\} \tag{2.7}$$

and $R_{ij}^{\perp} = R_0 \ominus R_{ij}$. Let P_{ij} and $P_{ij}^{\perp}$ be the projections in R_0 with respect to the scalar product $\langle \cdot, \cdot \rangle_1$ onto R_{ij} and $R_{ij}^{\perp}$, respectively. For $x \in R_0$ we denote

$$q_{ij} = P_{ij} x \qquad \text{and} \qquad \xi_{ij} = P_{ij}^{\perp} x. \tag{2.8}$$

Note that for any $1 \leq i < j \leq N$ it holds

$$|q_{ij}|_1 = \frac{\sqrt{m_i m_j}}{\sqrt{m_i + m_j}} |x_{ij}|, \tag{2.9}$$

which together with (2.1) implies

$$|V_{ij}(x_{ij})| \leq C |q_{ij}|_1^{-2-\nu} \qquad \text{for some } C > 0 \text{ and all } |x_{ij}| \geq A. \tag{2.10}$$

Our main result is the following

Theorem 2.1. *Assume that H_0 satisfies the conditions (2.1), (2.2) and (2.6). Suppose that φ_0 is an eigenfunction of H_0 corresponding to the eigenvalue zero. Then the following assertions hold.*

(i) *For all $1 \leq i < j \leq N$ we have*

$$V_{ij}(x_{ij}) \varphi_0(x) \in L^1(R_0). \tag{2.11}$$

(ii) *Let $\beta = d(N-1) - 2$ and denote by $|\mathbb{S}^{\beta-1}|$ the volume of the unit sphere in $\mathbb{R}^{\beta}$. Further, let*

$$C_0 = -\frac{1}{(\beta - 2)|\mathbb{S}^{\beta-1}|} \int_{R_0} \sum_{1 \leq i < j \leq N} V_{ij}(x_{ij}) \varphi_0(x) \, \mathrm{d}x. \tag{2.12}$$

Then the function φ_0 has the following asymptotics

$$\varphi_0(x) = \frac{C_0}{|x|_1^{\beta}} + g(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{2.13}$$

where the remainder g belongs to $L^p(R_0)$ for any p satisfying

$$\frac{\beta + 2}{\beta + \frac{\gamma^*}{1+\gamma^*}} < p < \frac{\beta + 2}{\beta} \qquad \text{with} \quad \gamma^* = \min\left\{\frac{d}{2} - 1, \nu\right\}. \tag{2.14}$$

Remark 2.2. *The most interesting question regarding the asymptotics* (2.13) *is whether the constant* C_0 *is zero or not. It turns out, that both cases are possible.*

(i) *The case of one-particle Schrödinger operators is not part of this work. However, the following example shows what can be expected in case of multi-particle Schrödinger operators. Let* $d = 5$ *and consider the Schrödinger operator* $-\Delta + V$ *in* $L^2(\mathbb{R}^5)$*, where the potential* V *is spherically symmetric, bounded and compactly supported. Let zero be an eigenvalue with* φ_0 *being a spherically symmetric eigenfunction. For arguments outside of the support of* V *one can easily show that* φ_0 *decays like* $c|x|^{-3}$ *with* $c \neq 0$*, which is the same decay rate as that of the fundamental solution of the Laplace operator. Hence, we have* $\langle V, \varphi_0 \rangle \neq 0$*. On the other hand, if* φ_0 *is a function of angular momentum* $l \geq 1$*, then we always have* $\langle V, \varphi_0 \rangle = 0$*. We emphasize that zero does not necessarily have to be the ground state in both cases.*

(ii) *Now we return to multi-particle systems and show that both cases can occur here as well. Consider the case of* $N \geq 3$ *particles in dimension* $d \geq 3$ *with the corresponding Hamiltonian* H_0*. Assume that zero is a ground state of* H_0*, then the corresponding eigenfunction* φ_0 *can be chosen to be strictly positive. Hence, if every potential* V_{ij} *satisfies* $V_{ij}(x) \leq 0$*, then we have* $C_0 \neq 0$*. In this case the leading term* $(1 + |x|_1)^{-\beta}$ *belongs to* $L^q(R_0)$*, only if* $q > \frac{\beta+2}{\beta}$*.*

(iii) *Consider a system of* $N \geq 3$ *identical bosons in dimension* $d \geq 3$ *and assume that the interactions are non-positive. If zero is a ground state, then according to remark* (ii) *the zero-energy ground state decays as* $C_0|x|_1^{-d(N-1)+2}$*.*

(iv) *Consider a system of* $N \geq 3$ *particles in dimension* $d \geq 3$*, which contains at least* $K \geq 3$ *identical fermions. If* φ_0 *is a zero energy bound state of the system, then* φ_0 *is orthogonal to all functions symmetric with respect to permutations of each pair of fermions. This implies* $C_0 = 0$*. Therefore, in this case by* (2.13) *the eigenfunction* φ_0 *always decays at least as fast as* $C|x|_1^{-\gamma}$*, where* $\gamma > d(N-1) - 2$*.*

(v) *Assume that the potentials* V_{ij} *are spherically symmetric. Then the operator* H_0 *is invariant under the action of the group* $\mathrm{SO}(d)$*, where* $d \geq 3$ *is the dimension of the corresponding particles. Consider the operator* H_0 *on a subspace of functions transformed according to a fixed irreducible representation of degree* $l = 0, 1, \dots$ *of the group* $\mathrm{SO}(d)$*. Assume that* φ_0 *is a zero energy bound state of* H_0 *with rotational symmetry of degree* $l \geq 1$*. Then, due to the orthogonality of functions corresponding to different irreducible representations, we have* $C_0 = 0$*.*

(vi) *The relation* (2.13) *shows that the decay rate of* φ_0 *does not depend on the potentials as long as the pair interactions are short-range and* $C_0 \neq 0$. *At the same time, since* $|x|_1^2 = \sum_{i=1}^N m_i |x_i|^2$, *the decay of* $\varphi_0(x)$ *depends on the respective direction of* x, *as long as the masses* m_i *are not all equal.*

Proof of Theorem 2.1. We will split the proof of the theorem into several propositions. The key argument of the proof is the following proposition, proved by S. Vugalter and the authors in [3].

Proposition 2.3 (A priori estimate). *The function* φ_0 *satisfies*

$$\nabla_0 \left(|x|_1^\alpha \varphi_0\right) \in L^2(R_0) \qquad \text{for any} \quad 0 \le \alpha < \frac{d(N-1)}{2} - 1. \tag{2.15}$$

For convenience of the reader we give the proof in the Appendix. To get the asymptotics (2.13) of φ_0 we will consider its integral representation as a convolution with the fundamental solution

$$\varphi_0(x) = \frac{-1}{(\beta-2)|\mathbb{S}^{\beta-1}|} \int_{R_0} |x-y|_1^{-\beta} V(y) \varphi_0(y) \, \mathrm{d}y. \tag{2.16}$$

The statement of assertion (i) of Theorem 2.1 is a special case of the following

Proposition 2.4 (The r.h.s. of (2.16) is well defined). *For all* $1 \le i < j \le N$ *and any* $0 < \gamma < \gamma^*$ *we have*

$$(1+|x|_1)^\gamma V_{ij}(x_{ij}) \varphi_0(x) \in L^1(R_0). \tag{2.17}$$

Proof. By Proposition 2.3, together with $|\nabla_{q_{ij}}| \le |\nabla_0|$ and Hardy's inequality in the space $H^1(R_{ij})$ we have

$$(1+|q_{ij}|_1)^{-1} (1+|x|_1)^\alpha \varphi_0 \in L^2(R_0). \tag{2.18}$$

Note that this non-symmetric estimate is crucial for the further proof. In fact, using the Hardy inequality in x instead of q_{ij} at this point would not be enough for us.

For any fixed $0 < \gamma < \gamma^*$ we write

$$(1+|x|_1)^\gamma V_{ij}(x_{ij}) \varphi_0(x) = (1+|q_{ij}|_1)^{-1} (1+|x|_1)^\alpha \varphi_0(x) \cdot f(x), \tag{2.19}$$

where

$$f(x) := (1+|x|_1)^{-\alpha+\gamma} (1+|q_{ij}|_1) V_{ij}(x_{ij}). \tag{2.20}$$

In view of (2.18) to prove Proposition 2.4 it suffices to show that f belongs to $L^2(R_0)$. Note that by definition of R_{ij} and $R_{ij}^\perp$ we have

$$L^2(R_0) = L^2(R_{ij}) \otimes L^2(R_{ij}^\perp). \tag{2.21}$$

Hence, we decompose the function f as

$$f(x) = f(x)\chi_{\{|x_{ij}|<A\}} + f(x)\chi_{\{|x_{ij}|\geq A\}} \tag{2.22}$$

and estimate the functions $f(x)\chi_{\{|x_{ij}|<A\}}$ and $f(x)\chi_{\{|x_{ij}|\geq A\}}$ separately, starting with the function $f(x)\chi_{\{|x_{ij}|<A\}}$.
Due to (2.9) and assumption (2.2) we obviously have

$$(1+|q_{ij}|_1)V_{ij}(x_{ij})\chi_{\{|x_{ij}|<A\}} \in L^2(R_{ij}). \tag{2.23}$$

Therefore, in order to show $f(x)\chi_{\{|x_{ij}|<A\}} \in L^2(R_0)$ we only need to prove that the function $(1+|x|_1)^{-\alpha+\gamma}$ belongs to $L^2(R_{ij}^{\perp})$. Since $\dim(R_{ij}^{\perp}) = d(N-2)$, we have

$$(1+|\xi_{ij}|_1)^{-\alpha+\gamma} \in L^2(R_{ij}^{\perp}) \quad \text{if and only if} \quad \alpha-\gamma > \frac{d(N-2)}{2}. \tag{2.24}$$

Recall that $\gamma < \gamma^*$, which in particular implies that $\gamma < \frac{d}{2}-1$. Therefore, the condition in (2.24) is satisfied if we choose α close enough to $\frac{d(N-1)-2}{2}$. Hence, by the use of $(1+|x|_1)^{-1} \leq (1+|\xi_{ij}|_1)^{-1}$ we have $(1+|x|_1)^{-\alpha+\gamma} \in L^2(R_{ij}^{\perp})$ and therefore

$$f(x)\chi_{\{|x_{ij}|<A\}} \in L^2(R_0). \tag{2.25}$$

In order to prove that the function $f(x)\chi_{\{|x_{ij}|\geq A\}}$ belongs to the space $L^2(R_0) = L^2(R_{ij}) \otimes L^2(R_{ij}^{\perp})$, we show that it can be estimated as

$$|f(x)\chi_{\{|x_{ij}|\geq A\}}| \leq |f_1(q_{ij})| \cdot |f_2(\xi_{ij})|, \tag{2.26}$$

where $f_1 \in L^2(R_{ij})$ and $f_2 \in L^2(R_{ij}^{\perp})$. Here, we will use the assumption that the potential $V_{ij}(x_{ij})$ decays faster than $|q_{ij}|_1^{-2}$ as $|x_{ij}| \to \infty$. Recall that $\dim(R_{ij}) = d$ and $\dim(R_{ij}^{\perp}) = d(N-2)$, which implies that for any $0 < \varepsilon < \nu-\gamma$ we have

$$f_1(q_{ij}) := (1+|q_{ij}|_1)^{-\frac{d}{2}-\varepsilon} \in L^2(R_{ij}) \tag{2.27}$$

and

$$f_2(\xi_{ij}) := (1+|\xi_{ij}|_1)^{-\alpha+\gamma-\nu+\varepsilon+\frac{d}{2}-1} \in L^2(R_{ij}^{\perp}). \tag{2.28}$$

Note that we can always assume $\nu < \frac{d}{2}-1$. By the use of $|q_{ij}|_1, |\xi_{ij}|_1 \leq |x|_1$ we get

$$(1+|x|_1)^{-\alpha+\gamma} \leq (1+|\xi_{ij}|_1)^{-\alpha+\gamma-\nu+\varepsilon+\frac{d}{2}-1}(1+|q_{ij}|_1)^{1-\frac{d}{2}+\nu-\varepsilon}. \tag{2.29}$$

This, together with (2.10) yields

$$|f(x)\chi_{\{|x_{ij}|\geq A\}}| \leq C|f_1(q_{ij})| \cdot |f_2(\xi_{ij})| \tag{2.30}$$

and therefore $f(x)\chi_{\{|x_{ij}|\geq A\}} \in L^2(R_0)$, which completes the proof of Proposition 2.4. □

Now we turn to the proof of statement (ii) of Theorem 2.1. Since

$$H_0\varphi_0 = (-\Delta_0 + V)\,\varphi_0 = 0 \tag{2.31}$$

and due to Proposition 2.4 $V\varphi_0 \in L^1(R_0)$, we can apply Theorem 6.21 in [12] to conclude

$$\varphi_0(x) = \frac{-1}{(\beta-2)|\mathbb{S}^{\beta-1}|}\int_{R_0} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y. \tag{2.32}$$

We derive the asymptotics (2.13) by studying the integral representation of φ_0 in (2.32). We will see that only certain regions contribute to the leading term of φ_0. We write

$$\varphi_0(x) = \frac{-1}{(\beta-2)|\mathbb{S}^{\beta-1}|}\left(I_1(x) + I_2(x)\right), \tag{2.33}$$

where

$$\begin{aligned} I_1(x) &= \int_{\{|x-y|_1\le 1\}} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y, \\ I_2(x) &= \int_{\{|x-y|_1> 1\}} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y. \end{aligned} \tag{2.34}$$

First we show that the function I_1 belongs to the remainder g in (2.13).

Proposition 2.5 (Remainder term of the asymptotics). *The function I_1 is an element of $L^p(R_0)$ for all $1 \le p < \frac{\beta+2}{\beta}$.*

Proof. Due to $\dim(R_0) = d(N-1)$ and $\beta = d(N-1) - 2$ we have

$$|x|_1^{-\beta}\chi_{\{|x|_1\le 1\}} \in L^p(R_0) \quad \text{for all} \quad 1 \le p < \frac{\beta+2}{\beta}. \tag{2.35}$$

By Proposition 2.4 we have $V\varphi_0 \in L^1(R_0)$, which together with Young's inequality yields the claim of Proposition 2.5. □

Now we show that only a part of I_2 gives the leading term in (2.13). Let $\eta = \frac{1}{1+\gamma^*}$. For $x \in R_0$ we define

$$\begin{aligned} \Omega_1(x) &= \{y \in R_0 \ : \ |x-y|_1 > 1,\ |y|_1 > |x|_1^\eta\}, \\ \Omega_2(x) &= \{y \in R_0 \ : \ |x-y|_1 > 1,\ |y|_1 \le |x|_1^\eta\} \end{aligned} \tag{2.36}$$

and

$$I_{2,k}(x) = \int_{\Omega_k(x)} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y, \qquad k = 1, 2. \tag{2.37}$$

At first we prove that the function $I_{2,1}$ belongs to the remainder g in (2.13).

Proposition 2.6 (Remainder term of the asymptotics). *Let $I_{2,1}$ be given by* (2.36) *and* (2.37)*, then we have*

$$I_{2,1} \in L^p(R_0) \quad \text{for all} \quad \frac{\beta+2}{\beta+\frac{\gamma^*}{1+\gamma^*}} < p < \frac{\beta+2}{\beta}. \tag{2.38}$$

Proof. Let $\gamma < \gamma^*$. By the use of $|y|_1 > |x|_1^\eta$ for $y \in \Omega_1(x)$ we get

$$\begin{aligned} |I_{2,1}(x)| &\leq \int_{\Omega_1(x)} |x-y|_1^{-\beta} |V(y)\varphi_0(y)| \,\mathrm{d}y \\ &\leq (1+|x|_1^\eta)^{-\gamma} \int_{\Omega_1(x)} |x-y|_1^{-\beta} (1+|y|_1)^\gamma \,|V(y)\varphi_0(y)| \,\mathrm{d}y \\ &= (1+|x|_1^\eta)^{-\gamma}\, \tilde{I}_{2,1}(x). \end{aligned} \tag{2.39}$$

We show that for any fixed p satisfying (2.38) we find a constant $\gamma < \gamma^*$, such that the function on the r.h.s. of (2.39) belongs to $L^p(R_0)$. Note that $\frac{\gamma^*}{1+\gamma^*} = \eta\gamma^*$, which for γ sufficiently close to γ^* implies

$$p > \frac{\beta+2}{\beta+\eta\gamma}. \tag{2.40}$$

By Proposition 2.4 and Young's inequality we have

$$\tilde{I}_{2,1}(x) = \int_{\Omega_1(x)} |x-y|_1^{-\beta} (1+|y|_1)^\gamma |V(y)\varphi_0(y)| \,\mathrm{d}y \in L^s(R_0) \tag{2.41}$$

with $s > \frac{d(N-1)}{d(N-1)-2}$. Now we apply Hölder's inequality to the r.h.s. of (2.39). For this purpose we fix s and define

$$t_1 = \frac{s}{s-p} \geq 1 \quad \text{and} \quad t_2 = \frac{s}{p} \geq 1 \quad \text{with} \quad \frac{1}{t_1} + \frac{1}{t_2} = 1. \tag{2.42}$$

Then we formally get

$$\int_{R_0} \frac{|\tilde{I}_{2,1}(x)|^p}{(1+|x|_1^\eta)^{\gamma p}} \,\mathrm{d}x \leq \left(\int_{R_0} (1+|x|_1^\eta)^{-\gamma p t_1} \mathrm{d}x\right)^{\frac{1}{t_1}} \left(\int_{R_0} |\tilde{I}_{2,1}(x)|^{p t_2} \,\mathrm{d}x\right)^{\frac{1}{t_2}}. \tag{2.43}$$

Since $pt_2 = s$ and $\tilde{I}_{2,1} \in L^s(R_0)$, the second integral on the r.h.s of (2.43) is finite. Due to $\dim(R_0) = d(N-1)$, to prove the finiteness of the first integral on the r.h.s of (2.43) it suffices to show that $\eta\gamma p t_1 > d(N-1)$. By definition of t_1 this is equivalent to

$$\begin{aligned} \eta\gamma s p > d(N-1)(s-p) \quad &\Leftrightarrow \quad p(\eta\gamma s + d(N-1)) > d(N-1)s \\ &\Leftrightarrow \quad \frac{1}{p} < \frac{\eta\gamma}{d(N-1)} + \frac{1}{s}. \end{aligned} \tag{2.44}$$

Since $p > \frac{d(N-1)}{d(N-1)-2+\gamma\eta}$, we see that the condition in (2.44) is fulfilled if s is chosen sufficiently close to $\frac{d(N-1)}{d(N-1)-2}$. It remains to use $\beta = d(N-1)-2$ to complete the proof. □

Now we finally show that $I_{2,2,}$ yields the leading term of φ_0 in (2.13).

Proposition 2.7 (Leading term of the asymptotics). *Let $I_{2,2}$ be given by (2.37), then we have*

$$I_{2,2}(x) = |x|_1^{-\beta} \int_{\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y + h(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{2.45}$$

where

$$h \in L^p(R_0) \quad \text{for all} \quad p > \frac{\beta+2}{\beta + \frac{\gamma^*}{1+\gamma^*}}. \tag{2.46}$$

Proof. For $y \in \Omega_2(x)$ we have (cf. [2])

$$|x|_1^{-1}\big(1 - |x|_1^{\eta-1}\big) \le |x-y|_1^{-1} \le |x|_1^{-1}\big(1 + c|x|_1^{\eta-1}\big) \tag{2.47}$$

for some $c > 0$. We apply this inequality to the positive and the negative part of the integrand in the definition of $I_{2,2}$ separately. Let

$$(V\varphi_0)_+(x) = \max\{V(x)\varphi_0(x), 0\}, \qquad (V\varphi_0)_- = -(V\varphi_0 - (V\varphi_0)_+), \tag{2.48}$$

then we have

$$|x|_1^{-\beta}\big(1 - |x|_1^{\eta-1}\big)^\beta \int_{\Omega_2(x)} \big(V\varphi_0\big)_\pm(y)\,\mathrm{d}y \le \int_{\Omega_2(x)} \frac{\big(V\varphi_0\big)_\pm(y)}{|x-y|_1^\beta}\,\mathrm{d}y \tag{2.49}$$

and

$$\int_{\Omega_2(x)} \frac{\big(V\varphi_0\big)_\pm(y)}{|x-y|_1^\beta}\,\mathrm{d}y \le |x|_1^{-\beta}\big(1 + c|x|_1^{\eta-1}\big)^\beta \int_{\Omega_2(x)} \big(V\varphi_0\big)_\pm(y)\,\mathrm{d}y. \tag{2.50}$$

Since $\dim(R_0) = d(N-1)$ we conclude from (2.49) and (2.50) that there exist functions

$$h_\pm \in L^p(R_0), \qquad p > \frac{d(N-1)}{d(N-1)-2+1-\eta}, \tag{2.51}$$

such that for sufficiently large $|x|_1$ it holds

$$\int\limits_{\Omega_2(x)} \frac{\big(V\varphi_0\big)_\pm(y)}{|x-y|_1^\beta}\,\mathrm{d}y = |x|_1^{-\beta} \int\limits_{\Omega_2(x)} (V\varphi_0(y))_\pm\,\mathrm{d}y + h_\pm(x). \tag{2.52}$$

Hence, we obtain

$$I_{2,2}(x) = |x|_1^{-\beta} \int\limits_{\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y + h(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{2.53}$$

where $h = h_+ - h_-$ belongs to $L^p(R_0)$ for p given in (2.51). Note that we have $1-\eta = \frac{\gamma^*}{1+\gamma^*}$ and $\beta = d(N-1)-2$. This concludes the proof. □

By combining Propositions 2.5, 2.6 and 2.7 we get

$$\varphi_0(x) = \frac{-|x|_1^{-\beta}}{(\beta-2)|\mathbb{S}^{\beta-1}|} \int\limits_{\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y + g(x) \qquad \text{as} \qquad |x|_1 \to \infty \tag{2.54}$$

with

$$g \in L^p(R_0) \qquad \text{for} \qquad \frac{\beta+2}{\beta+\frac{\gamma}{1+\gamma}} < p < \frac{\beta+2}{\beta}. \tag{2.55}$$

Note that the integral on the r.h.s of (2.54) is over the set $\Omega_2(x)$, in contrast to (2.13), where the integral is over the whole space R_0. Therefore, to complete the proof of Theorem 2.1 it remains to show that

$$|x|_1^{-\beta} \int\limits_{R_0\setminus\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y \tag{2.56}$$

does not contribute to the leading term in the asymptotic estimate of φ_0. Due to Proposition 2.4 it is easy to see that for any $\gamma < \gamma^*$ we have

$$\Big|\int\limits_{R_0\setminus\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y\Big| \leq C\,(1+|x|_1)^{-\eta\gamma} \tag{2.57}$$

for $|x|_1$ sufficiently large. This implies

$$|x|_1^{-\beta} \int\limits_{R_0\setminus\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y \in L^p(R_0) \quad \text{for} \quad p > \frac{\beta+2}{\beta+\frac{\gamma}{1+\gamma}}. \tag{2.58}$$

Choosing $\gamma < \gamma^*$ sufficiently close to γ^* and combining (2.54) and (2.58) completes the proof of the theorem. □

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A Proof of Proposition 2.3

The proof of Proposition 2.3 is based on Theorem A.1, which is a special case of a result of S. Vugalter and the authors of this work, see [3].
Consider the Schrödinger operator

$$h = -\Delta + V, \tag{A.1}$$

acting in $L^2(\mathbb{R}^d)$ with $d \geq 3$. Assume that the potential V is relatively form-bounded with relative bound less than one and denote by

$$q[\varphi] = \|\nabla\varphi\|^2 + \langle V\varphi, \varphi\rangle \tag{A.2}$$

the quadratic form of h with form domain $H^1(\mathbb{R}^d)$.

Theorem A.1. *Assume there exist constants $\gamma_0, b > 0$ and $\alpha_0 > 1$, such that for any function $\varphi \in H^1(\mathbb{R}^d)$ with $\operatorname{supp}(\varphi) \subset \{x \in \mathbb{R}^d : \ |x| \geq b\}$ it holds*

$$q[\varphi] - \gamma_0\|\nabla\varphi\|^2 - \alpha_0^2\langle |x|^{-2}\varphi, \varphi\rangle \geq 0. \tag{A.3}$$

If zero is an eigenvalue of h, then a corresponding eigenfunction $\varphi_0 \in H^1(\mathbb{R}^d)$ satisfies

$$\nabla(|x|^{\alpha_0}\varphi_0) \in L^2(\mathbb{R}^d) \qquad \textit{and} \qquad (1+|x|)^{\alpha_0-1} \in L^2(\mathbb{R}^d). \tag{A.4}$$

Proof. Since φ_0 is an eigenfunction corresponding to the eigenvalue zero, it holds

$$\langle \nabla\varphi_0, \nabla\psi\rangle + \langle V\varphi_0, \psi\rangle = 0 \tag{A.5}$$

for every function $\psi \in H^1(\mathbb{R}^d)$. For any $\varepsilon > 0$ and $R > 0$ we define the function

$$G_\varepsilon(x) = \frac{|x|^{\alpha_0}}{1+\varepsilon|x|^{\alpha_0}}\chi_R(x), \tag{A.6}$$

where χ_R is a C^∞ cutoff function satisfying

$$\chi_R(x) = \begin{cases} 0, & |x| \leq R \\ 1, & |x| \geq 2R. \end{cases} \tag{A.7}$$

By setting $\psi = G_\varepsilon^2\varphi_0$ we obtain

$$\langle \nabla\varphi_0, \nabla\left(G_\varepsilon^2\varphi_0\right)\rangle + \langle V\varphi_0, G_\varepsilon^2\varphi_0\rangle = 0. \tag{A.8}$$

Note that

$$\operatorname{Re}\langle\nabla\varphi_0, \nabla(G_\varepsilon^2\varphi_0)\rangle = \operatorname{Re}\langle\nabla(\varphi_0 G_\varepsilon), \nabla(\varphi_0 G_\varepsilon)\rangle - \operatorname{Re}\langle\varphi_0\nabla G_\varepsilon, \varphi_0\nabla G_\varepsilon\rangle. \tag{A.9}$$

Furthermore, for $|x| > 2R$ we can estimate

$$|\nabla G_\varepsilon| = \frac{\alpha_0|x|^{\alpha_0-1}}{(1+\varepsilon|x|^{\alpha_0})^2} \leq \alpha_0|x|^{-1}|G_\varepsilon| \tag{A.10}$$

and for $|x| \in [R, 2R]$ the function $|\nabla G_\varepsilon|$ is uniformly bounded in ε. Therefore, by (A.8) and (A.9) we have

$$\|\nabla(\varphi_0 G_\varepsilon)\|^2 + \langle VG_\varepsilon\varphi_0, G_\varepsilon\varphi_0\rangle - \alpha_0^2\int_{\{|x|>2R\}}\frac{|G_\varepsilon\varphi_0|^2}{|x|^2}\,\mathrm{d}x \leq C, \tag{A.11}$$

where $C > 0$ does not depend on $\varepsilon > 0$. Since $G_\varepsilon \varphi_0$ is supported outside the ball with radius $R > 0$, for any $R > b$ it satisfies (A.3). Hence, we conclude

$$\frac{\gamma_0}{2} \|\nabla(G_\varepsilon \varphi_0)\|^2 \leq C. \tag{A.12}$$

Taking $\varepsilon \to 0$ yields $\|\nabla \left(|x|^{\alpha_0} \varphi_0\right)\| < \infty$, which together with Hardy's inequality completes the proof. □

Proof of Proposition 2.3. The complete proof can be found in [3]. We will only give a sketch of the proof.
By the assumptions (2.2) and (2.1) on the pair interaction V_{ij}, together with Theorem A.1, to prove Proposition 2.3 we only need to show that there exist constants $\gamma_0 > 0$ and $b > 0$, such that

$$(1 - \gamma_0)\|\nabla_0 \varphi\|^2 + \langle V\varphi, \varphi\rangle - \alpha_0^2 \||x|_1^{-1}\varphi\| \geq 0 \tag{A.13}$$

holds for any $0 \leq \alpha_0 < \frac{d(N-1)-2}{2}$ and any function $\varphi \in H^1(R_0)$ with $\sup \varphi \subset \{x \in R_0 : |x|_1 \geq b\}$. Inequalities of this type are often used in geometric methods for many-particle Hamiltonians with short-range potentials, which usually play an essential role in proving the finiteness of the discrete spectrum, see for example [16, 19, 17]. In particular, in case $\gamma_0 = 0$ and $0 \leq \alpha_0 < \frac{1}{4}$ inequality (A.13) is a simplified form of the inequality proved in a different context in the work [17] of S. Vugalter and G. Zhislin, see the proof of Theorem I on page 56. Following the same approach as in [15] we make a partition of the unity of the configuartion space R_0, corresponding to different breakings of the system into different clusters. By doing so we can systematically separate cones, corresponding to where particles belonging to the same cluster in a breaking are close to each other and the other clusters are far away. In contrast to the situation in [17], the assumption (2.6) allows us to compensate the term $-\alpha_0^2 |x|_1^{-1}$ in such cones with a small part of the kinetic energy by applying the Poincaré-Friedrichs inequality instead of Hardy's inequality as in [17]. Outside of all cones, where all particles are far away from each other, using assumption (2.1) on the potentials and Hardy's inequality yields (A.13). By applying Theorem A.1 we conclude the proof. □

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