Finiteness Properties of Affine Deligne-Lusztig Varieties

Paul Hamacher and Eva Viehmann[1]

Received: May 29, 2019
Revised: May 19, 2020

Communicated by Otmar Venjakob

Abstract. Affine Deligne-Lusztig varieties are closely related to the special fibre of Newton strata in the reduction of Shimura varieties or of moduli spaces of G-shtukas. In almost all cases, they are not quasi-compact. In this note we prove basic finiteness properties of affine Deligne-Lusztig varieties under minimal assumptions on the associated group. We show that affine Deligne-Lusztig varieties are locally of finite type, and prove a global finiteness result related to the natural group action. Similar results have previously been known for special situations.

2020 Mathematics Subject Classification: Primary 14G35, 20E42; Secondary 20G25
Keywords and Phrases: affine Deligne-Lusztig variety, Rapoport-Zink spaces, affine flag variety

1 Introduction

Let F be a local field, O_F its ring of integers, and $k_F = \mathbb{F}_q$ its residue field, a finite field of characteristic p. We denote by L the completion of the maximal unramified extension of F, and by O_L its ring of integers. Then the residue field k of L is an algebraic closure of $\mathbb{F}_q$. We denote by ϵ a uniformizer of F, which is then also a uniformizer of L. Let σ be the Frobenius of k over k_F and also of L over F. We denote by I the inertia group of F.
We consider a smooth affine group scheme $\mathscr{G}$ over O_F with reductive generic fibre. Let $P = \mathscr{G}(O_L)$ and let $G = \mathscr{G}_F$.

[1]The authors were partially supported by ERC Consolidator Grant 770936: NewtonStrat.

We denote by $\mathscr{F}\ell_{\mathscr{G}}$ the base change to k of the affine flag variety (over k_F) associated with $\mathscr{G}$ as in [PR08, § 1.c] and [BS17, Def. 9.4]. In particular, $\mathscr{F}\ell_{\mathscr{G}}$ is a sheaf on the fpqc-site of k-schemes ($\mathrm{char}\, F = p$) resp. of perfect k-schemes ($\mathrm{char}\, F = 0$) with

$$\mathscr{F}\ell_{\mathscr{G}}(k) = G(L)/P,$$

which is representable by an inductive limit of finite type schemes ($\mathrm{char}\, F = p$) resp. of perfectly of finite type schemes ($\mathrm{char}\, F = 0$); see [PR08, Thm. 1.4],[BS17, Cor. 9.6]. Hence we can define an underlying topological space of $\mathscr{F}\ell_{\mathscr{G}}$, which is Jacobson. This means that by mapping a subset of $\mathscr{F}\ell_{\mathscr{G}}$ to its intersection with the subset of closed points $\mathscr{F}\ell_{\mathscr{G}}(k) \subset \mathscr{F}\ell_{\mathscr{G}}$ we obtain a bijection between the open subsets of $\mathscr{F}\ell_{\mathscr{G}}$ and the open subsets of $\mathscr{F}\ell_{\mathscr{G}}(k)$ (same for closed and for locally closed subsets). Moreover, being a base change from k_F, we have an action of σ on $\mathscr{F}\ell_{\mathscr{G}}$.

To define affine Deligne-Lusztig varieties we fix an element $b \in G(L)$ and a locally closed subscheme Z of the loop group LG which is stable under P-σ-conjugation. Then we consider the functor on reduced k-schemes resp. reduced perfect k-schemes with

$$X_Z(b)(S) = \{g \in \mathscr{F}\ell_{\mathscr{G}}(S) \mid g_x^{-1} b\sigma(g_x) \in Z(\kappa_x) \text{ for every geom. point } x \text{ of } S\}.$$

Remark 1.1. The functor $X_Z(b)$ defines a locally closed reduced sub-indscheme of $\mathscr{F}\ell_{\mathscr{G}}$: Consider the functor $\tilde{X}_Z(b)$ on reduced k-schemes resp. perfect k-schemes with

$$\tilde{X}_Z(b)(S) = \{g \in LG(S) \mid g_x^{-1} b\sigma(g_x) \in Z(\kappa_x) \text{ for every geom. point } x \text{ of } S\}.$$

Then $\tilde{X}_Z(b)$ is the inverse image of Z under the morphism $LG \to LG$ with $g \mapsto g^{-1}b\sigma(g)$. Since Z is locally closed, also $\tilde{X}_Z(b)$ defines a locally closed reduced sub-ind-scheme of LG. Furthermore, $X_Z(b)$ is the image of $\tilde{X}_Z(b)$ under the quotient map $LG \to \mathscr{F}\ell_{\mathscr{G}}$, which is an $L^+\mathscr{G}$-torsor. Hence it is again a locally closed sub-ind-scheme.

Let J_b be the reductive group over F whose R-valued points for any F-algebra R are given by

$$J_b(R) = \{g \in G(R \otimes_F L) \mid gb = b\sigma(g)\}.$$

Then for every Z there is a natural action of $J_b(F)$ on $X_Z(b)$ given by left multiplication. Our main result is

Theorem 1.2. *Assume in addition that Z is bounded (see Section 3 for the definition of boundedness).*

(1) *$X_Z(b)$ is a scheme which is locally of finite type in the case that* $\mathrm{char}\, F = p$ *and locally perfectly of finite type in the case* $\mathrm{char}\, F = 0$.
(2) *The action of $J_b(F)$ on the set of irreducible components of $X_Z(b)$ has finitely many orbits.*

This theorem is related to the fact that they are the underlying reduced subscheme of moduli spaces of local G-shtukas and to the general expectation for the arithmetic case that (at least in the minuscule case) affine Deligne-Lusztig varieties are the reduction modulo p of integral models of local Shimura varieties. Their cohomology is conjectured to decompose according to the local Langlands and Jacquet-Langlands correspondences. In order to be able to apply the usual methods, one needs the cohomology groups to be finitely generated $J_b(F)$-representations, and thus the "infinite level" cohomology groups to be admissible. This follows from the above theorem by a formal argument once the integral model is constructed (see for example [Mie20, Thm. 4.4], [RV14, Prop. 6.1]).

Many particular cases of the theorem have been considered before. For the particular case of affine Deligne-Lusztig varieties arising as the underlying reduced subscheme of a Rapoport-Zink moduli space of p-divisible groups with additional structure of PEL type, questions as in Theorem 1.2 have been considered by several people. A recent general theorem along these lines is shown by Mieda [Mie20]. Also, the (rare) cases where an affine Deligne-Lusztig variety is even of finite type have been classified, compare [Gör10, Prop. 4.13].

In the case where $\mathscr{G}$ is reductive over O_F and Z is a single P-double coset, a complete description of the set of $J_b(F)$-orbits of irreducible components of $X_Z(b)$ is known. The present work was motivated by our own results in this direction in [HV18]. Recently, complete descriptions were given by Zhou and Zhu [ZZ20] and by Nie [Nie].

The main tool to prove Theorem 1.2 is to relate the claimed finiteness statements to finiteness properties of certain subsets of the extended Bruhat-Tits building of G, using previous work of Cornut and Nicole [CN16].

Acknowledgement. We are grateful to G. Prasad for pointing out some of his work on Bruhat-Tits theory to us. We thank the referee for his/her helpful comments.

2 Reduction to the parahoric case

As a first step, we reduce to the case that $\mathscr{G}$ is a parahoric group scheme. While most assertions in the following still hold true in the general setup, the assertion that $\mathscr{G}$ is parahoric will simplify the proofs and the notation.

By the fixed point theorem [Tit79, 2.3.1] the group $P \rtimes \langle\sigma\rangle$ has a fixed point x in the extended Bruhat-Tits building of G_L. We refer to the subsequent section for the relation between the extended Bruhat-Tits building and the "classical" Bruhat-Tits building. By definition the stabiliser P_x of x is σ-stable and contains P. We denote by $\mathscr{G}_x$ the corresponding group scheme over O_F in the sense of [Prab, 1.9] and [Praa, 2].

Lemma 2.1. *The fpqc quotient $L^+\mathscr{G}_x/L^+\mathscr{G}$ is representable by a finitely presented (resp. perfectly finitely presented) scheme.*

Proof. We denote $P_{x,n} := \ker(\mathscr{G}_x(O_L) \to \mathscr{G}_x(O_L/\epsilon^n))$. Since the $P_{x,n}$ form a neighbourhood basis of the unit element in $G(L)$ we have $P_{x,n} \subset P$ for some n. Thus the positive loop group L^+P contains the kernel of the reduction map into the truncated positive loop group $L^+P_x \to L_n^+P_x$. Indeed, we have just shown that this is true on geometric points and the kernel is an infinite dimensional affine space by Greenberg's structure theorem [Gre63, p. 263], thus in particular reduced. Hence we get $L^+\mathscr{G}_x/L^+\mathscr{G} \cong L_n^+\mathscr{G}_x/L_n^+\mathscr{G}$. Since the latter is a quotient of linear algebraic groups over k_F, the claim follows. □

Since $LG \to \mathscr{F}\ell_{\mathscr{G}}$ is an $L^+\mathscr{G}$-torsor, we get that $\mathscr{F}\ell_{\mathscr{G}}$ is étale locally isomorphic to $\mathscr{F}\ell_{\mathscr{G}_x} \times L^+\mathscr{G}_x/L^+\mathscr{G}$. In particular, the canonical projection $\mathscr{F}\ell_{\mathscr{G}} \to \mathscr{F}\ell_{\mathscr{G}_x}$ is relatively representable and of finite type. Thus Theorem 1.2 holds true for $\mathscr{G}$ if and only if it is true for $\mathscr{G}_x$, as it is enough to prove the theorem after enlarging Z so that becomes stable under P_x-σ-conjugation. Let $\mathscr{G}_x^\circ \subset \mathscr{G}_x$ be the parahoric group scheme associated to x. Repeating the argument above, we see that it suffices to prove Theorem 1.2 for $\mathscr{G}_x^\circ$ instead of $\mathscr{G}$.
Therefore we can (and will) assume from now on that $\mathscr{G}$ is a parahoric group scheme.

3 Some properties of Bruhat-Tits buildings

We consider the following group theoretical setup. Let $S_0 \subset G$ be a maximal L-split torus defined over F, let T_0 be its centraliser and let N_0 be the normaliser of T_0 in G. Then T_0 is a torus because G is quasi-split over L. Thus $W = N_0(L)/T_0(L)$ is the relative Weyl group of G over L. We denote by P_{T_0} the unique parahoric subgroup of T_0. The extended affine Weyl group is defined as

$$\tilde{W} := N_0(L)/P_{T_0} \cong X_*(T_0)_{\mathrm{Gal}(\overline{L}/L)} \rtimes W,$$

where $X_*(T_0)_{\mathrm{Gal}(\overline{L}/L)}$ denotes the group of Galois convariants of $X_*(T_0)$ over L. We may choose S_0 such that P stabilises a facet in the apartment of S_0 and denote $\tilde{W}^P = (N_0(L) \cap P)/P_{T_0} \subset \tilde{W}$. By [PR08, Appendix, Prop. 9] the embedding $N_0 \hookrightarrow G$ induces a bijection

$$\tilde{W}^P\backslash\tilde{W}/\tilde{W}^P \xrightarrow{1:1} P\backslash G(L)/P. \tag{3.1}$$

We call a subset $\tilde{X} \subset G(L)$ bounded if it is contained in a finite union of P-double cosets. The bounded subsets form a bornology on $G(L)$, which does not depend on the choice of P.
Let $\mathcal{B}^e(G,L)$ be the extended Bruhat-Tits building of G over L, that is

$$\mathcal{B}^e(G,L) = \mathcal{B}(G,L) \times V_0(G,L)$$

where $\mathcal{B}(G,L)$ is the "classical" Bruhat-Tits building of G and $V_0(G,L) := X_*(G^{\mathrm{ab}})_{\mathbb{R}}^{\mathrm{Gal}(\overline{L}/L)} \cong X_*(Z(G))_{\mathbb{R}}^{\mathrm{Gal}(\overline{L}/L)}$ with $Z(G)$ denoting the center of G. The extended apartment $\mathcal{A}^e(S,G) \subset \mathcal{B}^e(G,L)$ of a maximal L-split torus S

is defined as $\mathcal{A}(S,G;L) \times V_0(G,L)$ where $\mathcal{A}(S,G;L)$ denotes the apartment of S. We recall from Landvogt [Lan00, § 1.3] that $\mathcal{B}^e(G,L)$ is a polysimplicial complex with a metric d and a $G(L) \rtimes \langle\sigma\rangle$-action by isometries. Moreover, one can canonically identify $\mathcal{B}^e(G,F)$ with the set of σ-invariants $\mathcal{B}^e(G,L)^{\langle\sigma\rangle}$.
We consider the canonical map

$$i\colon G(L) \to \mathrm{Isom}(\mathcal{B}^e(G,L)),$$

where $\mathrm{Isom}(\mathcal{B}^e(G,L))$ denotes the space of self-isometries of $\mathcal{B}^e(G,L)$. A set $M \subset \mathrm{Isom}(\mathcal{B}^e(G,L))$ is called bounded if for some (or equivalently every) non-empty bounded set $A \subset \mathcal{B}^e(G,L)$ the set $\{f(x) \mid f \in M, x \in A\} \subset \mathcal{B}^e(G,L)$ is bounded. We have the following statement about the compatibility of bornological structure.

Proposition 3.2 ([BT84, Prop. 4.2.19]). *A subset $\tilde{X} \subset G(L)$ is bounded if and only if its image under i is.*

We consider the following maps between extended Bruhat-Tits buildings. Let $f\colon G \to G'$ be a morphism of reductive F-groups. A $G(L)$-equivariant map $g\colon \mathcal{B}^e(G,L) \to \mathcal{B}^e(G',L)$ is called *toral* if for every maximal L-split torus $S \subset G_L$ there exists a maximal L-split torus $S' \subset G'_L$ such that $f(S) \subset S'$ and g restricts to an $X_*(S)_{\mathbb{R}}$-translation equivariant map between the apartments of S and S'. In [Lan00], Landvogt proves that there always exists a $G(L) \rtimes \langle\sigma\rangle$-invariant toral map, which becomes an isometry after normalising the metric on $\mathcal{B}^e(G',L)$. However, this map depends on an auxiliary choice. We give a precise formulation of the result in the form and context that we need later on. For this consider the fixed element $b \in G(L)$ and denote by $\nu_b \in X_*(G)_{\mathbb{Q}}$ the Newton point of b (see [Kot85, § 4] for its precise definition). We fix an integer $s \gg 0$ such that $s \cdot \nu_b \in X_*(G)$. Denote by $M_b \subset G$ the Levi subgroup centralising ν_b (and thus $s \cdot \nu_b$). Then J_b is the inner form of M_b obtained by twisting the action of the Frobenius by b. We can thus use the following result to relate the buildings of G and J_b. A similar result is also shown in [CN16].

Proposition 3.3 ([Lan00, Prop. 2.1.5],[Rou77, Lemme 5.3.2]). *Let $f\colon M_b \hookrightarrow G$. Then there exists a toral $M_b(L) \rtimes \langle\sigma\rangle$-equivariant injective map*

$$f_*\colon \mathcal{B}^e(M_b,L) \to \mathcal{B}^e(G,L).$$

Moreover, f_* is injective and unique up to translation by an element of $V_0(G,L)^{\langle\sigma\rangle}$. In particular, its image is the same for every choice of f_* and equal to $\mathcal{B}^e(G,L)^{(s\cdot\nu_b)(O_L^\times)}$. After a suitable normalisation of the metric on $\mathcal{B}^e(G,L)$, this map becomes an isometry.

Remark 3.4. Since $J_{b,L} \cong M_{b,L}$, we obtain an identification of $\mathcal{B}^e(J_b,L)$ with $\mathcal{B}^e(G,L)^{(s\cdot\nu_b)(O_L^\times)}$. However, since J_b is an inner twist of a Levi subgroup of G, this identification will not respect the action of the Frobenius in general. In order to distinguish it from the action on $\mathcal{B}^e(G,L)$, we denote the

Frobenius action on $\mathcal{B}^e(J_b, L)$ (and $J_b(L)$) by σ_b. More explicitely, we have $\sigma_b = b\sigma|_{\mathcal{B}(J_b,L)} \times \sigma|_{V_0(J_b,L)}$. Indeed, by [Lan96, Lemma 3.3.1], the Frobenius action on the "classical" Bruhat-Tits building $\mathcal{B}(J_b, L)$ is uniquely determined by the equation $\sigma_b(j.x) = \sigma_b(j).\sigma_b(x) = (b\sigma(j)b^{-1}).\sigma_b(x)$ and thus has to be equal to $b\sigma$. It follows from the explicit description in [Lan96, (3.3.2)], that the Frobenius action on $V_0(J_b, L)$ remains the same.

Now assume that we have an embedding of reductive groups $f\colon G \hookrightarrow G'$. The following statement is the main result of [Lan00].

Proposition 3.5 ([Lan00, Thm. 2.2.1]). *There exists a $G(L) \rtimes \langle\sigma\rangle$-invariant toral map $f_*\colon \mathcal{B}^e(G, L) \to \mathcal{B}^e(G', L)$. Furthermore the metric on $\mathcal{B}^e(G, L)$ can be normalised in a way such that f_* becomes isometrical.*

To simplify the notation, we identify G with its image in G'. Now b, considered as element of G', induces a group J'_b which is an inner form of the centraliser of ν_b in G'. Since f_* preserves the fixed points of $\nu_b(O_L^\times)$, we obtain a commutative diagram by Proposition 3.3 and Remark 3.4,

$$\begin{array}{ccc} \mathcal{B}^e(J_b, L) & \xhookrightarrow{f_*} & \mathcal{B}^e(J'_b, L) \\ \big\downarrow & & \big\downarrow \\ \mathcal{B}^e(G, L) & \xhookrightarrow{f_*} & \mathcal{B}^e(G', L). \end{array} \tag{3.6}$$

Lemma 3.7. *The restriction $f_*|_{\mathcal{B}^e(J_b,L)}$ is σ_b-equivariant.*

A related statement is [CN16], 3.5. For the convenience of the reader, we provide the details of the proof here.

Proof. We denote by σ'_b the canonical Frobenius action on $\mathcal{B}^e(J'_b, L)$. Note that the action of $b\sigma$ and the actions of σ_b, σ'_b differ by the translations t_b, t'_b induced by the action of b on $V_0(J_b, L)$ and $V_0(J'_b, L)$ respectively. Since f_* is $b\sigma$-equivariant, it suffices to show that $f_* \circ t_b = t'_b \circ f_*$.
To prove this, consider the composition of f_* with the canonical projection $\mathcal{B}^e(J'_b, L) \twoheadrightarrow V_0(J'_b, L)$. We claim that this map factors through $V_0(J_b, L)$. This can be checked on extended apartments. Let $S \subset J_b, S' \subset J'_b$ be maximal split tori over L with $f(S) \subset S'$ and $f_*(\mathcal{A}^e(S, J_b; L)) \subset \mathcal{A}^e(S', J'_b; L)$. For the intersections with the derived groups of G, G' we have $S^{\mathrm{der}} \subset S'^{\mathrm{der}}$. Hence the composition $\mathcal{A}^e(S, J_b; L) \hookrightarrow \mathcal{A}^e(S', J'_b; L) \twoheadrightarrow V_0(J'_b; L)$ is $S^{\mathrm{der}}(L)$-invariant and thus factors through $\mathcal{A}^e(S, J_b; L)/\mathcal{A}^e(S^{\mathrm{der}}, J_b^{\mathrm{der}}; L) = V_0(J_b, L)$.
Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^e(J_b, L) & \xhookrightarrow{f_*} & \mathcal{B}^e(J'_b, L) \\ \big\downarrow p & & \big\downarrow p' \\ V_0(J_b, L) & \xhookrightarrow{f_*^{\mathrm{ab}}} & V_0(J'_b, L) \end{array}$$

Since p, p' and f_* commute with the action of b, so does f_*^{ab}. Thus $f_*^{\mathrm{ab}} \circ t_b = t'_b \circ f_*^{\mathrm{ab}}$, proving $f_* \circ t_b = t'_b \circ f_*$. □

4 BOUNDEDNESS PROPERTIES ON THE AFFINE FLAG VARIETY

We denote by $w_G\colon G(L) \to \pi_1(G)_I$ the Kottwitz homomorphism. For any subset $X \subset G(L)$ and $\omega \in \pi_1(G)_I$, we define $X^\omega := X \cap w_G^{-1}(\{\omega\})$. We remark that by [PR08, Thm. 5.1] and [Zhu17, Prop. 1.21] the connected components of $\mathscr{F}\ell_{\mathscr{G}}$ are precisely the subsets of the form $\mathscr{F}\ell_{\mathscr{G}}^\omega$.

For further considerations, it will be useful to fix a presentation of $\mathscr{F}\ell_{\mathscr{G}}^{\mathrm{red}}$ as a limit of schemes. For any $w \in \tilde{W}^P\backslash\tilde{W}/\tilde{W}^P$ we denote by

$$S_w^\circ := PwP/P$$
$$S_w := \bigcup_{w' \leq w} S_{w'}$$

the Schubert cell and the Schubert variety associated with w, respectively. Here, $\leq$ denotes the Bruhat order on $\widetilde{W}$ induced by any fixed choice of an Iwahori subgroup of P. By [PR08, § 8] and [BS17, Thm. 9.3] each Schubert variety (resp. cell) is a closed (resp. locally-closed) quasi-compact subscheme of $\mathscr{F}\ell_{\mathscr{G}}$, which is of finite type in the case $\operatorname{char} F = p$ and perfectly of finite type in the case $\operatorname{char} F = 0$. Note that by (3.1), we have that $\mathscr{F}\ell_{\mathscr{G}} = \bigcup S_w^\circ$ is a decomposition into locally closed subsets, hence we can write $\mathscr{F}\ell_{\mathscr{G}}^{\mathrm{red}} = \varinjlim S_w$.

We equip $\mathscr{F}\ell_{\mathscr{G}}(k)$ with the bornology induced by the canonical projection $G(L) \twoheadrightarrow G(L)/P = \mathscr{F}\ell_{\mathscr{G}}(k)$, that is a subset $X \subset \mathscr{F}\ell_{\mathscr{G}}(k)$ is bounded, if it is contained in a finite union of Schubert varieties. We obtain the following geometric characterisation of bounded subsets.

LEMMA 4.1. *A subset $X \subset \mathscr{F}\ell_{\mathscr{G}}(k)$ is bounded if and only if it is relatively quasi-compact (i.e. contained in a quasi-compact subset). In this case X is even quasi-compact itself.*

Proof. Since the S_w are quasi-compact, any bounded subset of $\mathscr{F}\ell_{\mathscr{G}}$ is relatively quasi-compact. The S_w are Noetherian, thus their subsets are quasi-compact themselves.

On the other hand, assume that X is not bounded. We prove that X is not quasi-compact by constructing an infinite discrete closed subset $Y \subset X$. By definition, the set $T := \{w \in \widetilde{W} \mid X \cap S_w^\circ \neq \emptyset\}$ is infinite. For each $w \in T$, choose an element $x_w \in X \cap S_w^\circ$. Then $Y := \{x_w \mid w \in T\}$ is infinite and discrete. Its intersection with every S_w for $w \in \widetilde{W}$ is closed, hence Y is closed. □

LEMMA 4.2. *Let $X \subset \mathscr{F}\ell_{\mathscr{G}}$ be a locally closed reduced sub-ind-scheme. Then X is a scheme if and only if every point of $X(k)$ has an open neighbourhood which is bounded as subset of $\mathscr{F}\ell_{\mathscr{G}}(k)$. In this case X is locally of finite type if* $\operatorname{char} F = p$, *respectively locally of perfectly finite type if* $\operatorname{char} F = 0$.

Proof. The "only if" direction follows from the previous lemma because every point of a scheme has a quasi-compact open neighbourhood.
To prove the "if" direction, we may assume that X is bounded, since its representability is a Zariski-local property. Then the embedding $X(k) \hookrightarrow \mathscr{F}\ell_{\mathscr{G}}$ factors through some finite union of Schubert varieties by the previous lemma, in particular $X(k)$ is a locally closed subvariety of this union. Since the Schubert varieties are (perfectly) of finite type, so is X. □

Remark 4.3. The analogous assertions of Lemmas 4.1 and 4.2 in $LG(k)$, the loop group of G, also hold true (with the exception of the last statement of Lemma 4.2). Indeed, since a set $X \subset G(L)$ is bounded if and only if $X \cdot P$ is bounded, it suffices to prove the assertion in the case that X is right P-invariant. Then the claim follows from the above lemmas since $LG \to \mathscr{F}\ell_{\mathscr{G}}$ is an $L^+\mathscr{G}$-torsor and thus relatively representable and quasi-compact.

5 Affine Deligne Lusztig varieties

We now prove that the first part of Theorem 1.2 implies the second. By Lemma 4.2 together with the first part of the theorem, its second assertion is equivalent to the following proposition, which we prove below.

Proposition 5.1. *Let Z a bounded subset of $G(L)$ and denote*

$$\tilde{X}_Z(b) \coloneqq \{g \in G(L) \mid g^{-1}b\sigma(g) \in Z\}.$$

Then there exists a bounded subset $\tilde{X}_0 \subset \tilde{X}_Z(b)$ such that $\tilde{X}_Z(b) = J_b(F) \cdot \tilde{X}_0$.

For the proof of the proposition we need some preparation.

Lemma 5.2. *The σ-conjugacy class of $b \in G(L)$ has a decent representative for which $\mathcal{B}^e(J_b, F) \cap \mathcal{B}^e(G, F) \neq \emptyset$ (viewed as subspaces of $\mathcal{B}^e(G, L)$).*

Here, an element $b \in G(L)$ is called decent if there is a natural number s with $(b\sigma)^s = s\nu(\epsilon)\sigma^s$.

Proof. In Remark 3.4 we identified the extended Bruhat-Tits building $\mathcal{B}^e(J_b, L)$ with $\mathcal{B}^e(M_b, L)$. We fix a maximal L-split torus $S \subset M_b$, denote by T its centraliser and by $\tilde{W}_{M_b}$ the associated extended affine Weyl group of M_b. Since any reductive group over F is residually quasi-split by [BT87, Thm. 4.1], there exists a σ-stable alcove $\mathfrak{a}$ in $\mathcal{A}(S, M_b, L)$. The Kottwitz homomorphism maps the stabiliser $\Omega \subset \tilde{W}_{M_b}$ of $\mathfrak{a}$ isomorphically onto $\pi_1(G)_I$. Since any basic σ-conjugacy class is uniquely determined by its Kottwitz point, we may assume that b (after replacing it by a $M_b(L)$-σ-conjugate if necessary) is a representative in $M_b(L)$ of an element of Ω. By [Kim19, Lemma 2.2.10] we may assume this representative to be decent. It now follows from the explicit description of σ_b in Remark 3.4 that we may take $p_0 \coloneqq (p_b, p_v)$ where $p_b \in \mathcal{B}(M_b, L)$ is the barycenter of $\mathfrak{a}$ and $p_v \in V_0(M_b, L)$ is any point fixed by σ. Then $p_0 \in \mathcal{B}^e(J_b, F) \cap \mathcal{B}^e(G, F)$. □

Thus after replacing b by a σ-conjugate if necessary, we fix $p_0 \in \mathcal{B}^e(J_b, F) \cap \mathcal{B}^e(G, F)$. In order to relate the bornologies on $G(L)$ and on $\mathcal{B}^e(G, L)$ directly, we consider the map

$$\iota\colon G(L) \to \mathcal{B}^e(G, L), g \mapsto g.p_0.$$

By the choice of p_0, the map ι is $G(L) \rtimes \langle\sigma\rangle$-equivariant and the restriction to $J_b(L)$ is moreover σ_b-equivariant, cf. Remark 3.4. By Proposition 3.2, for any $C' > 0$ the set $Z_{C'} := \{g \in G(L) \mid d(p_0, \iota(g)) < C'\}$ is a bounded set and for any bounded $Z \subset G(L)$ the constant $c_Z := \sup\{d(p_0, \iota(y)) \mid y \in Z\}$ is finite.
The following lemma translates the results of [CN16] into our terms.

Lemma 5.3. *Let G be a reductive group over F and $b \in G(L)$.*
(a) *For any $c > 0$ there exists a $C > 0$ such that if $x \in \mathcal{B}^e(G, L)$ satisfies $d(x, b\sigma(x)) < c$ then there exists $x_0 \in \mathcal{B}^e(J_b, F)$ with $d(x, x_0) < C$.*
(b) *For any $c > 0$ there exists a $C > 0$ such that if $x \in \iota(G(L))$ satisfies $d(x, b\sigma(x)) < c$ then there exists $x_0 \in \iota(J_b(F))$ with $d(x, x_0) < C$.*

Proof. Assertion (a) is proven in [CN16] by an elegant geometrical argument. By Theorem 3.3 of loc. cit. f_* identifies $\mathcal{B}^e(J_b, F)$ with the set

$$\mathrm{Min}(b\sigma) := \{x \in \mathcal{B}^e(G, L) \mid d(x, b\sigma(x)) \text{ attains its minimal possible value.}\},$$

Thus the statement (a) claims that if $d(x, b\sigma(x))$ is bounded, so is the distance to $\mathrm{Min}(b\sigma)$. This (together with an upper bound for C) is proven in [CN16, Prop. 8].
To show that (a) implies (b), we have to show that the distance of a point $x \in \mathcal{B}^e(G, L)$ to $\iota(G(L))$ is bounded above, or equivalently that there exists a bounded subset $M \subset \mathcal{B}^e(G, L)$ such that $G(L) \cdot M = \mathcal{B}^e(G, L)$ as well as the analogous assertion for $J_b(F)$. For this, we fix an isomorphism $X_*(Z)^I \cong \mathbb{Z}^r$, which yields an identification $V_0(G, L) = \mathbb{R}^r$. Then we may choose $M = \mathfrak{a} \times [0, 1]^r$, where $\mathfrak{a}$ is any alcove of the usual Bruhat-Tits building $\mathcal{B}(G, L)$. □

Proof of Proposition 5.1. Let $Z \subset G(L)$ be bounded. We fix $g \in \tilde{X}_Z(b)$ and denote $x := \iota(g)$. Then

$$d(x, b\sigma(x)) = d(g^{-1}.x, g^{-1}.b\sigma(x)) = d(p_0, g^{-1}b\sigma(g).p_0) < c_Z.$$

By Lemma 5.3(b), there exist a $C_Z > 0$ depending only on Z and a $j \in J_b(F)$ such that

$$d((j^{-1} \cdot g).p_0, p_0) = d(x, j.p_0) < C_Z,$$

i.e. $j^{-1} \cdot g \in Z_{C_Z}$. Hence $\tilde{X}_Z(b) = J_b(F) \cdot (\tilde{X}_Z(b) \cap Z_{C_Z})$. □

It remains to prove the first part of Theorem 1.2. By Lemma 4.2 it is equivalent to the following proposition.

Proposition 5.4. *Every $x_0 \in X_Z(b)(k)$ has a bounded open neighbourhood.*

Proof. The proof follows by an analogous argument as the last part of [HV11, Thm. 6.3]. Since the situation simplifies a lot by considering only the reduced structure, and since in loc. cit. only split groups, hyperspecial P, and certain Z are considered, we give the complete proof for the reader's convenience.
Let $\omega = w_G(x_0)$ and let again $X_Z(b)^\omega = X_Z(b) \cap w_G^{-1}(\omega)$. Since $X_Z(b)^\omega \subset X_Z(b)$ is open and closed, it suffices to prove the claim for $X_Z(b)^\omega$. We can define a $G(L)$- and σ-invariant semi-metric $d\colon G(L)^\omega \to \mathbb{N} \cup \{0\}$ by

$$d(g,h) \leq n \quad \Longleftrightarrow \quad h^{-1}g \in \overline{P\rho^\vee(\epsilon^{2n})P} = \bigcup_{w \leq 2n\rho^\vee} PwP$$

where $\rho^\vee$ denotes the half-sum of the positive coroots and $w \in \tilde{W}^P \backslash \tilde{W} / \tilde{W}^P$. Obviously this semi-metric descends to $\mathscr{F}\ell_G^\omega$. Then a subset is bounded with respect to the bornology defined before Lemma 4.1 if and only if it is bounded with respect to d.
We choose b as in Lemma 5.2. Let $s \in \mathbb{N}$ be as in the decency equation, i.e. $(b\sigma)^s = (s \cdot \nu)(\epsilon)$. Enlarging s and Z if necessary, we assume that ω and Z are both σ^s-invariant. Then $X_Z(b)^\omega$ is σ^s-stable and thus is defined over the extension k_s of degree s of k_F. The closed point x_0 defined over some finite extension of k_F. By enlarging k_s further if necessary, we assume that x_0 is a k_s-rational point. We denote by $\mathcal{M}$ the model of $X_Z(b)^\omega$ over k_s and for every $n \in \mathbb{N}$ we define the closed sub-ind-scheme

$$\mathcal{M}_n(k) := \{x \in \mathcal{M}(k) \mid d(x, x_0) \leq n\}.$$

Note that $\mathcal{M}_n$ is actually a (perfectly) finite type scheme by Lemma 4.2 and moreover defined over k_s since d is σ-invariant. Also note that $\mathcal{M} = \varinjlim \mathcal{M}_n$. The decency of b implies that $J_b(F) \subset G(F_s)$ where F_s is the unramified extension of F of degree s. Thus the $J_b(F)^0$-action stabilises $\mathcal{M}(k_s)$. Together with Proposition 5.1 (which we proved independently of the first assertion of Theorem 1.2) we obtain that there exists an $N_0 \in \mathbb{N}$ such that for every $x \in \mathcal{M}(k)$ there exists a $y_0 \in \mathcal{M}(k_s)$ with $d(x, y_0) \leq N_0$. For every $y_0 \in \mathcal{M}(k_s)$ define the closed subscheme $\mathcal{M}_n(y_0) \subset \mathcal{M}_n$ by

$$\mathcal{M}_n(y_0)(k) := \{y \in \mathcal{M}_n(k) \mid d(y_0, y) \leq N_0\}.$$

Now consider the open subset of $\mathcal{M}_n$

$$U_n := \mathcal{M}_n(x_0) \setminus \bigcup_{\substack{y \in \mathcal{M}(k_s) \\ d(x_0, y) > N_0}} \mathcal{M}_n(y).$$

The union on the right hand side is indeed finite (and hence closed): By the triangular inequality $\mathcal{M}_n(y)$ is empty unless $y \in \mathcal{M}_{N_0+n}(k_s)$; the latter set is finite since $\mathcal{M}_{N_0+n}$ is (perfectly) of finite type. We claim that the chain $U_1 \subseteq U_2 \subseteq \cdots$ stabilises at U_{2N_0} at the latest. To prove this, let $x \in U_n(k)$ for some n. We choose a rational point $y_0 \in \mathcal{M}(k_s)$ with $d(x, y_0) \leq N_0$. By definition of

U_n, we must have $d(x_0, y_0) \leq N_0$. Thus $d(x, x_0) \leq d(x, y_0) + d(y_0, x_0) \leq 2N_0$, i.e. $x \in U_{2N_0}$.
Since $\mathcal{M} = \varinjlim \mathcal{M}_n$, the subset $U_{2N_0} = \varinjlim U_n$ is open in $\mathcal{M}$. It is moreover bounded and contains x_0. It is thus a bounded open neighbourhood of x_0. ☐

REFERENCES

[BS17] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. 209 (2017), no. 2, 329–423. MR 3674218

[BT84] François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316 (86c:20042)

[BT87] François Bruhat and Jacques Tits, *Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), no. 3, 671–698. MR 927605

[CN16] Christophe Cornut and Marc-Hubert Nicole, *Cristaux et immeubles*, Bull. Soc. Math. France 144 (2016), no. 1, 125–143. MR 3481264

[Gör10] Ulrich Görtz, *Affine Springer fibers and affine Deligne-Lusztig varieties*, Affine flag manifolds and principal bundles, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010, pp. 1–50. MR 3013026

[Gre63] Marvin J. Greenberg, *Schemata over local rings. II*, Ann. of Math. (2) 78 (1963), 256–266. MR 0156855

[HV11] Urs Hartl and Eva Viehmann, *The Newton stratification on deformations of local G-shtukas*, J. Reine Angew. Math. 656 (2011), 87–129. MR 2818857

[HV18] Paul Hamacher and Eva Viehmann, *Irreducible components of minuscule affine Deligne-Lusztig varieties*, Algebra Number Theory 12 (2018), no. 7, 1611–1634. MR 3871504

[Kim19] Wansu Kim, *On central leaves of Hodge-type Shimura varieties with parahoric level structure*, Math. Z. 291 (2019), no. 1-2, 329–363. MR 3936073

[Kot85] Robert E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. 56 (1985), no. 2, 201–220. MR 809866 (87i:14040)

[Lan96] Erasmus Landvogt, *The functorial properties of the Bruhat-Tits building*, Ph.D. thesis, Westfälische Wilhelms-Universität Münster, 1996.

[Lan00] Erasmus Landvogt, *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. 518 (2000), 213–241. MR 1739403

[Mie20] Yoichi Mieda, *On irreducible components of Rapoport-Zink spaces*, Int. Math. Res. Not. IMRN (2020), no. 8, 2361–2407. MR 4090743

[Nie] Sian Nie, *Irreducible components of affine Deligne-Lusztig varieties*, arXiv:1809.03683, Preprint.

[PR08] George Pappas and Michael Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. 219 (2008), no. 1, 118–198, With an appendix by T. Haines and Rapoport. MR 2435422 (2009g:22039)

[Praa] Gopal Prasad, *Finite group actions on reductive groups and buildings and tamely-ramified descent in Bruhat-Tits theory*, Am. J. Math. 142 (2020), no. 4, 1239–1267. MR 4124119

[Prab] Gopal Prasad, *A new approach to unramified descent in Bruhat-Tits theory*, Am. J. Math. 142 (2020), no. 1, 215–253.

[Rou77] Guy Rousseau, *Immeubles des groupes réducitifs sur les corps locaux*, U.E.R. Mathématique, Université Paris XI, Orsay, 1977, Thèse de doctorat, Publications Mathématiques d'Orsay, No. 221-77.68. MR 0491992

[RV14] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. 7 (2014), no. 1, 273–326. MR 3271247

[Tit79] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69. MR 546588 (80h:20064)

[Zhu17] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) 185 (2017), no. 2, 403–492. MR 3612002

[ZZ20] Rong Zhou and Yihang Zhu, *Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties*, Camb. J. Math. 8, (2020), no. 1, 149–241.

Paul Hamacher
Technische Universität München
Fakultät für Mathematik - M11
Boltzmannstr. 3
85748 Garching bei München
Germany
hamacher@ma.tum.de

Eva Viehmann
Technische Universität München
Fakultät für Mathematik - M11
Boltzmannstr. 3
85748 Garching bei München
Germany
viehmann@ma.tum.de