

PARABOLIC SEMI-ORTHOGONAL DECOMPOSITIONS
AND KUMMER FLAT INVARIANTS OF LOG SCHEMES

SARAH SCHEROTZKE, NICOLÒ SIBILLA, AND MATTIA TALPO

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ABSTRACT. We construct semi-orthogonal decompositions on triangulated categories of parabolic sheaves on certain kinds of logarithmic schemes. This provides a categorification of the decomposition theorems in Kummer flat K-theory due to Hagihara and Niziol. Our techniques allow us to generalize Hagihara and Niziol's results to a much larger class of invariants in addition to K-theory, and also to extend them to more general logarithmic stacks.

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1 INTRODUCTION

In this paper we carry forward the study of the derived category of parabolic sheaves we initiated in [46], where we established the Morita invariance of parabolic sheaves under logarithmic blow-ups. Our main result is the construction of a special kind of semi-orthogonal decompositions on derived categories of parabolic sheaves. This provides in particular a categorification of structure theorems for the Kummer flat K-theory of log schemes due to Hagihara [20] and Niziol [36]. Additionally we generalize Hagihara and Niziol's result in two ways:

- we obtain uniform structure theorems which hold across all (Kummer flat) invariants of logarithmic schemes, including Hochschild and cyclic homology;
- our techniques allow us to extend these results to a much larger class of log schemes (and log stacks) than those considered by Hagihara and Niziol.

Our main results, Theorem A, Theorem B and Corollary C, hold over an arbitrary ground ring.

1.1 PARABOLIC SHEAVES, LOG SCHEMES, AND INFINITE ROOT STACKS

Parabolic sheaves were defined by Mehta and Seshadri in the 80's, for Riemann surfaces with marked points, as coherent sheaves equipped with flags at the marked points. They are the key ingredient to extend the Narasimhan–Seshadri correspondence to the non-compact setting. The theory over the last fifty years has undergone massive generalizations. It was extended first to pairs (X, D) where D is a normal crossing divisor in any dimension, and more recently to an even broader class of *logarithmic schemes*. The work of Borne, Vistoli and the third author shows that parabolic structures are best viewed within the framework of log geometry [6], [51], [49] and this is the perspective that we will adopt throughout the paper.

Logarithmic (log) geometry emerged in the 80's through the collective efforts of several authors including Deligne, Faltings, Fontaine, Illusie and Kato [28]. The theory was initially designed for applications to arithmetic geometry, but over the last twenty years it has become a key organizing principle in areas as diverse as algebraic geometry, symplectic geometry, and homotopy theory. Log geometric techniques lie at the core of the Gross–Siebert program in mirror symmetry [19], and feature prominently in recent approaches to Gromov–Witten theory, see [18] and references therein.

One of the main hurdles in working with log schemes is that they encode both classical geometric and combinatorial data. For this reason transporting familiar geometric constructions to the log setting is often delicate: see for instance [40], [21] for the definition of the cotangent complex and the Chow groups of log schemes. A definition of K-theory for log schemes was first proposed by Hagihara [20] and Nizioł [36]. We will refer to it as *Kummer flat (resp. étale) K-theory*, and it is the algebraic K-theory of the Kummer flat (resp. étale) topos, a logarithmic analogue of the classical flat (resp. étale) topos.

Our main result is a construction of infinite semi-orthogonal decompositions on categories of parabolic sheaves. This can be viewed as a categorification of an important structure theorem due to Hagihara and Nizioł for Kummer flat K-theory: if X is a regular scheme equipped with a simple normal crossing divisor $D \subset X$, the Kummer flat K-theory of (X, D) splits as an explicit direct sum indexed by the strata of D [36, Theorem 1.1]. We will show that our methods yield, in particular, substantial generalizations of Hagihara and Nizioł's results. Before stating our main result we review its two key ingredients: the *infinite root stack*, and *semi-orthogonal decompositions*.

INFINITE ROOT STACKS

The infinite root stack of a log scheme was introduced in [51]: it is a limit of tame Artin stacks (Deligne–Mumford in characteristic 0) which encodes log

information as stacky data. The infinite root stack captures the geometry of the underlying log scheme, and this point of view informs several recent works by the authors and their collaborators [9, 46, 52, 50], see also [44] for recent applications to Hall algebras and quantum groups. If X is a log scheme, we denote its infinite root stack by $\infty\sqrt{X}$. One of the key properties of the infinite root stack is that the Kummer flat topos of X is equivalent, as a ringed topos, to the fppf topos of $\infty\sqrt{X}$ [51, Theorem 6.16]. In particular Hagihara and Nizioł’s logarithmic algebraic K-theory of X coincides with the ordinary algebraic K-theory of the infinite root stack $\infty\sqrt{X}$. Thus we can study logarithmic algebraic K-theory by probing the geometry and the sheaf theory of infinite root stacks.

SEMI-ORTHOGONAL DECOMPOSITIONS

We view the algebraic K-theory of a stack as an invariant of its ∞ -category of perfect complexes. We study the category of perfect complexes of the infinite root stack, and show that Nizioł’s direct sum decomposition of Kummer flat K-theory is the shadow of a factorization that holds directly at the categorical level. The appropriate concept of factorization for categories is given by semi-orthogonal decompositions (*sod*-s, for short). These were introduced by Bondal and Orlov in [5]. In the setting of ∞ -categories, semi-orthogonal decompositions (of length two) were considered in [4] under the name of *split-exact sequences* of ∞ -categories.

THE MAIN THEOREM

Recent work of Ishii and Ueda [25] and Bergh, Lunts, and Schnürer [3] shows that the categories of perfect complexes of finite root stacks of pairs (X, D) , where X is a scheme (or stack) equipped with a simple normal crossing divisor $D \subset X$, admit a canonical sod where the summands are labelled by the strata of D . The infinite root stack is the limit of all finite root stacks, however the pull-back functors along root maps do not preserve the canonical sod-s. This issue can be obviated via a recursion that gives rise a sequence of nested sod-s as the root index grows, and that ultimately yields an infinite sod on $\text{Perf}(\infty\sqrt{X})$. Below we formulate our first main result for log scheme of the form (X, D) , where X is a scheme and D is a simple normal crossing divisor. We refer the reader to Theorem 3.16 in the main text for a sharper and more general statement, that applies for instance to all finite type tame algebraic stacks. Let $\{D_i\}_{i \in I}$ be the set of irreducible components of D . The divisor D determines a stratification of X where strata are intersections of the irreducible components of D . Strata are in bijection with the subsets of I : if S is a stratum $S = \cap_{j \in J} D_j$ for some $J \subset I$, and \overline{S} is the closure, we set $|\overline{S}| := |J|$. If N is a natural number we set $(\mathbb{Q}/\mathbb{Z})^{N,*} := (\mathbb{Q}/\mathbb{Z} \setminus \{0\})^N$.

THEOREM A (Theorem 3.16). *The ∞ -category of perfect complexes of the infinite root stack $\infty\sqrt{(X, D)}$ admits a semi-orthogonal decomposition*

$$\text{Perf}(\infty\sqrt{(X, D)}) = \langle \mathcal{A}_{\overline{S}}, \overline{S} \in S_D \rangle$$

such that all objects in $\mathcal{A}_{\overline{S}}$ are supported on \overline{S} . Additionally, for all \overline{S} the category $\mathcal{A}_{\overline{S}}$ carries a semi-orthogonal decomposition indexed by $(\mathbb{Q}/\mathbb{Z})^{|\overline{S}|,*}$ whose factors are equivalent to $\text{Perf}(\overline{S})$.

1.2 ADDITIVE INVARIANTS

Theorem A recovers in particular Hagihara and Nizioł’s results, but is a much stronger statement. In order to clarify this point let us refer to the notion of *additive invariants* of ∞ -categories. Let us denote by $\text{Cat}_{\infty}^{\text{perf}}$ the ∞ -category of stable ∞ -categories. A functor $H: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \mathcal{P}$, where \mathcal{P} is a stable presentable ∞ -category, is an *additive invariant* if it preserves zero objects and filtered colimits, and it maps *split exact sequences* to cofiber sequences (split exact sequences are the analogue in the ∞ -setting of a sod with two factors). Most homological invariants of algebras and categories are additive: algebraic K-theory and non-connective K-theory, (topological) Hochschild homology and negative cyclic homology are all additive invariants.

The theory of *non-commutative motives* was developed by Tabuada and others [48, 11, 4, 41] in analogy with the classical theory of motives. Non-commutative (additive) motives encode the universal additive invariant, exactly as classical motives are universal among Weil cohomologies. Noncommutative motives form a presentable and stable ∞ -category Mot^{add} which is the recipient of the universal additive invariant

$$U: \text{Cat}_{\infty}^{\text{perf}} \longrightarrow \text{Mot}^{\text{add}}.$$

Every additive invariant $H: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \mathcal{P}$ factors uniquely as a composition

$$\begin{array}{ccc} \text{Cat}_{\infty}^{\text{perf}} & \xrightarrow{H} & \mathcal{P} \\ \downarrow U & \nearrow \overline{H} & \\ \text{Mot}^{\text{add}} & & \end{array}$$

If H is an additive invariant and X is a stack we set $U(X) := U(\text{Perf}(X))$, $H(X) := H(\text{Perf}(X))$. Let (X, D) be a scheme equipped with a normal crossing divisor. Then Theorem A yields a direct product decomposition of the non-commutative motive of $\sqrt[\infty]{X}$.

THEOREM B (Corollary 5.6). *There is a canonical direct sum decomposition*

$$U(\sqrt[\infty]{(X, D)}) \simeq \bigoplus_{\overline{S} \in S_D} \left(\bigoplus_{(\mathbb{Q}/\mathbb{Z})^{|\overline{S}|,*}} U(\overline{S}) \right).$$

The K-theory of root stacks was also studied in [14], although from a different perspective.

KUMMER ÉTALE K-THEORY

Theorem B implies uniform direct product decompositions across all additive invariants of infinite root stacks, and recovers in particular Hagihara and Niziol’s structure theorems for Kummer flat K-theory. Let X be a log scheme, and denote X_{Kfl} its Kummer flat topos. Let $\text{Perf}(X_{\text{Kfl}})$ be its ∞ -category of perfect complexes. If \mathbb{H} is an additive invariant, we set $\mathbb{H}_{\text{Kfl}}(X) := \mathbb{H}(\text{Perf}(X_{\text{Kfl}}))$. When $\mathbb{H}(-) = K(-)$ is algebraic K-theory, this definition recovers Hagihara and Niziol’s Kummer flat K-theory.

Work of Vistoli and the third author [51] identifies the Kummer flat topos with the “small fppf topos” of the infinite root stack. As a consequence, under suitable assumptions, there is an equivalence of stable ∞ -categories $\text{Perf}(X_{\text{Kfl}}) \simeq \text{Perf}(\sqrt[\infty]{X})$. This, together with Theorem B, yields the following immediate corollary.

COROLLARY C. *If $\mathbb{H}: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \mathcal{P}$ is an additive invariant then there is a direct sum decomposition*

$$\mathbb{H}_{\text{Kfl}}(X, D) \simeq \bigoplus_{\bar{S} \in S_D} \left(\bigoplus_{(\mathbb{Q}/\mathbb{Z})^{|\bar{S}|}, * } \mathbb{H}(\bar{S}) \right).$$

In particular, the Kummer flat K-theory of (X, D) decomposes as a direct sum of spectra

$$K_{\text{Kfl}}(X, D) \simeq \bigoplus_{\bar{S} \in S_D} \left(\bigoplus_{(\mathbb{Q}/\mathbb{Z})^{|\bar{S}|}, * } K(\bar{S}) \right).$$

The second half of the statement recovers the first part of Niziol’s [36, Theorem 1.1] (see Remark 5.9 for some comments about the second part). Niziol’s result holds under the restrictive assumption that (X, D) is a log smooth pair given by a regular scheme X and a simple normal crossing divisor D . Corollary C holds with milder smoothness assumption on X : indeed in the main body of the paper we work with a finite type algebraic stacks X equipped with a simple normal crossing divisor D . In particular X needs not be regular outside of D .

In fact we can extend the the decomposition given by Corollary C to even more general log stacks. We clarify this by explaining three applications of our techniques. They require working over a field κ of characteristic zero. Additionally for the second one we need to assume $\kappa = \mathbb{C}$.

- *General normal crossing divisors.* We extend the decomposition of Theorem A and Corollary C to *general normal crossing divisors*, removing the simplicity assumption required by Hagihara and Niziol. An interesting new feature emerges in this setting. The semi-orthogonal summands appearing in the analogue of Theorem A for general normal crossing log stacks (X, D) are no longer equivalent to the category of perfect complexes on the strata: instead, they are equivalent to perfect complexes on the *normalization of the*

strata (see Theorem 4.6 in the main text). This is reflected by the Kummer flat K-theory of a general normal crossing log stack: it breaks up as a direct sum indexed by the strata, where the summands compute the K-theory of the normalization of the strata (see Theorem 5.8).

- *Simplicial log structures.* We generalize Corollary C to log smooth schemes with *simplicial log structure*, i.e. pairs (X, D) where D is a divisor with *simplicial singularities*. The decomposition formula holds only for the complexification of $K_{\text{K\AA}}(X)$, and under suitable additional assumptions on (X, D) . The main differences with the normal crossings case is that the formula has additional summands keeping track of the singularities of D , and that it depends on the *G-theory*, rather than the K-theory, of the strata. As the statement is somewhat technical we do not include it in this introduction, but refer the reader directly to Proposition 5.10 in the main text.
- *Logarithmic Chern character.* Having at our disposal a general definition of additive invariants of log schemes, we introduce a construction of the *logarithmic Chern character*. The availability of structure theorems valid across all additive invariants of log schemes allows us to study some of its fundamental properties. This includes a Grothendieck–Riemann–Roch statement, which we establish in the restrictive setting of strict maps of log schemes, leaving generalizations to future work. These results are contained in Section 5.3 of the text.

1.3 TOWARDS LOGARITHMIC DT INVARIANTS

Donaldson–Thomas invariants are part of the rich array of enumerative invariants inspired by string theory. One of the outstanding open questions in the area is to construct a theory of log DT invariants, analogous to the theory of log GW invariants developed in [18, 10, 1]. This would have applications to degeneration formulas for DT invariants. DT theory counts *Bridgeland stable* objects in triangulated categories. Thus building a theory of log DT invariants requires, first, to introduce a viable concept of derived category for log schemes; and, second, to define and study stability conditions over it. For the first requirement, one of the viable options is to try to use parabolic sheaves on (X, D) , which in turn are equivalent to sheaves on ${}^{\infty}\sqrt{X}$. Thus a first step towards defining log DT invariants consists in constructing *Bridgeland stability condition* on $\text{Perf}({}^{\infty}\sqrt{X})$. Results obtained in [12] give a means to glue stability conditions across semi-orthogonal decompositions. Adapting these techniques to the *sod-s* on $\text{Perf}({}^{\infty}\sqrt{X})$ obtained in Theorem A, we can already obtain stability conditions in some cases, such as many toric log pairs (X, D) . It is too early to tell whether they will be relevant from the viewpoint of log DT theory, but this seems an interesting avenue for future investigation.

1.4 RELATION TO WORK OF OTHER AUTHORS

Several different approaches to the definition of log motives and log invariants have been considered in the literature.

- A definition of log motives has been proposed in [26], and in [23].
- Constructions of Hochschild homology and topological Hochschild homology in the log setting have been proposed by Hesselholt and Madsen [22] Rognes, Sagave, and Schlichtkrull [42], Leip [32] and Olsson [38].

It would be very interesting to compare these approaches to the one we pursue in this paper, however there is a key difference in perspective between some of these works and our own. The constructions of log Hochschild homology considered in [22], [42], and [32] are closely related to the log de Rham complex, and therefore to the cohomology of the *complement* of the locus where the log structure is concentrated. In this paper, on the other hand, we investigate log schemes through the lenses of their *Kummer flat* topos and their infinite root stack.

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CONVENTIONS

We work over an arbitrary noetherian commutative ring κ (that could be the ring of integers \mathbb{Z}). In later parts of the paper (Sections 5.2 and 5.3) we will impose more restrictive assumptions on κ . All algebraic stacks (in the sense of [47, Tag 026O]) will be of finite type over κ . All monoids will be commutative and “toric”, i.e. finitely generated, sharp, integral and saturated.

2 PRELIMINARIES

2.1 LOG STRUCTURES FROM BOUNDARY DIVISORS AND ROOT STACKS

In this section we briefly recall how certain boundary divisors D on a scheme or algebraic stack X give rise to log structures and root stacks.

DEFINITION 2.1. Let X be a scheme over κ , and $D \subset X$ an effective Cartier divisor. Recall that the divisor D is:

- *simple normal crossings* if for every $x \in D$ the local ring $\mathcal{O}_{X,x}$ is regular, and there is a system of parameters $a_1, \dots, a_n \in \mathcal{O}_{X,x}$ such that the ideal of D in $\mathcal{O}_{X,x}$ is generated by a_1, \dots, a_k for some $1 \leq k \leq n$, and

- *normal crossings* if every $x \in D$ has an étale neighbourhood where D becomes simple normal crossings.

Note that if D is a normal crossing divisor on X , then every point of D is a regular point of X , but away from D the scheme X can well be singular.

If X is an algebraic stack and $D \subset X$ is an effective Cartier divisor, we say that D is (simple) normal crossings if the pull-back of D to some smooth presentation $U \rightarrow X$, where U is a scheme, is a (simple) normal crossing divisor on U .

REMARK 2.2. If κ is a field, then the divisor $D \subset X$ is normal crossings if and only if étale locally around every point $x \in D$, the pair (X, D) is isomorphic to the pair $(\mathbb{A}^n, \{x_1 \cdots x_k = 0\})$ for some $n \in \mathbb{N}$ and $0 \leq k \leq n$.

This is not necessarily true if κ is not a field. For example, if $\kappa = \mathbb{Z}$ the divisor $V(p) \cup V(x)$ for a prime number p is simple normal crossings on $X = \mathbb{A}^1 = \text{Spec } \mathbb{Z}[x]$, but X is not étale locally isomorphic to \mathbb{A}^2 around the point (p, x) .

Later on (Section 4.2) we will consider a generalization of this notion, where the divisor D is allowed to have *simplicial singularities*. Over a field κ one can define this by asking that étale locally around every point $x \in D$, the pair (X, D) is isomorphic to the pair $(\text{Spec } \kappa[P] \times \mathbb{A}^n, \Delta_P \times \mathbb{A}^n)$ for some simplicial monoid P and $n \in \mathbb{N}$. Recall that a sharp fine saturated monoid P is *simplicial* if the extremal rays of the rational cone $P_{\mathbb{Q}} \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ are linearly independent, and we denote by Δ_P the toric boundary in the affine toric variety $\text{Spec } \kappa[P]$, i.e. the complement of the torus $\text{Spec } \kappa[P^{\text{gp}}]$, equipped with the reduced subscheme structure. This weaker notion allows for some kinds of singularities along the divisor D itself.

It is not clear to us how to formulate this notion in the “absolute” case (i.e. for schemes over $\text{Spec } \mathbb{Z}$), so we will circumvent this problem by using a canonically defined root stack of a log scheme with simplicial log structure, and reducing to Definition 2.1 on this root stack (see Definition 4.11).

Next we recall how ((simple) normal crossing) divisors induce associated log structures and root stacks. We focus on the construction of root stacks, since the full formalism of logarithmic geometry will not play an important role in the paper. We refer the reader to [37] for an extensive introduction on log geometry, and to the appendix of [9] for a quick overview of the basic concepts. For more on root stacks, the reader can consult [7, 6, 51, 49].

Any effective Cartier divisor D in a scheme X induces a log structure, usually called the *compactifying log structure* of the open embedding $X \setminus D \subset X$, as follows. The sheaf $M_D = \{f \in \mathcal{O}_X \mid f|_{X \setminus D} \in \mathcal{O}_{X \setminus D}^{\times}\}$ is a sheaf of submonoids of \mathcal{O}_X (seen as a sheaf on the small étale site of X), where the monoid operation is multiplication of regular functions. The inclusion $\alpha: M_D \rightarrow \mathcal{O}_X$ gives rise to a log structure on X . If D is normal crossings, this log structure admits local charts, and in fact will also be fine and saturated. Morally, the sheaf M_D keeps track of how many branches of D intersect at a point of X , and how their local equations fit together in the local ring $\mathcal{O}_{X,x}$ (more precisely, in its strict henselization, since we are using the étale topology).

If X is a stack rather than a scheme, the above procedure gives a log structure on any given smooth presentation of X , and descent for fine log structures [39, Appendix] gives a fine saturated log structure on X itself. We will denote the resulting log scheme or stack by (X, D) .

2.1.1 ROOT STACKS ALONG A SINGLE REGULAR DIVISOR

Assume that $D \subset X$ is a simple normal crossing divisor with only one component, i.e. a regular connected divisor, and X is an algebraic stack. Recall that the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ functorially parametrizes pairs (L, t) where L is a line bundle and t is a global section of L .

As specified above, the effective Cartier divisor D on X induces a canonical log structure, that in this simple case can be described as follows: the divisor D determines a line bundle $\mathcal{O}_X(D)$ on X together with a section $\sigma \in \Gamma(\mathcal{O}_X(D))$ having D as its zero locus, and this yields a tautological morphism $s: X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$. This equips X with a log structure by pulling back the canonical log structure of $[\mathbb{A}^1/\mathbb{G}_m]$ (corresponding to the regular divisor $[\{0\}/\mathbb{G}_m] \subset [\mathbb{A}^1/\mathbb{G}_m]$) via s .

In this case, for $r \in \mathbb{N}$ the r -th root stack of the pair (X, D) can be described as the functor that associates to a scheme $T \rightarrow X$ over X the groupoid $\sqrt[r]{(X, D)}(T)$ of pairs (L, t) consisting of a line bundle on T with a global section, and with an isomorphism $(\mathcal{O}_X(D), \sigma)|_T \cong (L^{\otimes r}, t^{\otimes r})$.

The stack $\sqrt[r]{(X, D)}$ fits in a fiber square

$$\begin{array}{ccc}
 \sqrt[r]{(X, D)} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\
 g_{r,1} \downarrow & & \downarrow (-)^r \\
 X & \xrightarrow{s} & [\mathbb{A}^1/\mathbb{G}_m]
 \end{array} \tag{1}$$

where $(-)^r$ is induced by the r -th power maps on \mathbb{A}^1 and on \mathbb{G}_m , or equivalently is the functor sending a pair (L, t) of a line bundle with a global section to the pair $(L^{\otimes r}, t^{\otimes r})$. The reason for the notation $g_{r,1}$ will be apparent later (see Section 2.1.4). The construction of the stack $\sqrt[r]{(X, D)}$ is compatible with respect to pull-back along smooth morphisms towards X (in particular with respect to Zariski and étale localization on X), i.e. if $Y \rightarrow X$ is smooth, we have a canonical isomorphism $\sqrt[r]{(Y, D|_Y)} \simeq \sqrt[r]{(X, D)} \times_X Y$.

We denote by D_r the effective Cartier divisor on $\sqrt[r]{(X, D)}$ obtained by taking the reduction of the closed substack $g_{r,1}^{-1}(D) \subset \sqrt[r]{(X, D)}$, and we denote $i_r: D_r \rightarrow \sqrt[r]{(X, D)}$ the inclusion. We will refer to D_r as the *universal effective Cartier divisor* on $\sqrt[r]{(X, D)}$, since it is the universal r -th root of the divisor D on X .

Consider the commutative (non-cartesian) diagram

$$\begin{array}{ccc}
 \mathcal{B}\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\
 (-)^r \downarrow & & \downarrow (-)^r \\
 \mathcal{B}\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]
 \end{array} \tag{2}$$

where the left vertical arrow is induced by the r -th power map $(-)^r: \mathbb{G}_m \rightarrow \mathbb{G}_m$. As explained in [3], D_r can be defined equivalently as the top left vertex of the base change of diagram (2) along the tautological map $s: X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$: in particular, there is a fiber product

$$\begin{array}{ccc}
 D_r & \longrightarrow & \mathcal{B}\mathbb{G}_m \\
 f_{r,1} \downarrow & & \downarrow (-)^r \\
 D & \longrightarrow & \mathcal{B}\mathbb{G}_m.
 \end{array}$$

This implies that $D_r \rightarrow D$ is a μ_r -gerbe. Zariski locally on X , where the line bundle $\mathcal{O}_X(D)$ is trivial, the stack $\sqrt[r]{(X, D)}$ admits the following explicit description: assume also that $X = \text{Spec } A$ is affine, and let $f \in A$ correspond to the section σ of $\mathcal{O}_X(D)$ (so that D has equation $f = 0$). Then we have an isomorphism

$$\sqrt[r]{(X, D)} \simeq [\text{Spec } (A[x]/(x^r - f)) / \mu_r]$$

where μ_r acts by multiplication on x . The divisor $D_r \subset \sqrt[r]{(X, D)}$ is given by the global equation $x = 0$, and is therefore isomorphic to $[\text{Spec } (A/f) / \mu_r] \simeq D \times \mathcal{B}\mu_r$.

2.1.2 ROOT STACKS ALONG A SIMPLE NORMAL CROSSINGS DIVISOR

Assume now that D is a simple normal crossing divisor on X , and denote by D_1, \dots, D_N the irreducible components of D . In this case root stacks of (X, D) are indexed by elements of \mathbb{N}^N . For $\vec{r} = (r_1, \dots, r_N) \in \mathbb{N}^N$, the root stack $\sqrt[\vec{r}]{(X, D)}$ parametrizes tuples $((L_1, t_1), \dots, (L_N, t_N))$, where (L_i, t_i) is a r_i -th root of $(\mathcal{O}_X(D_i), \sigma_i)$. Each pair $(\mathcal{O}_X(D_i), \sigma_i)$ determines a morphism $s_i: X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$, and the stack $\sqrt[\vec{r}]{(X, D)}$ is the fiber product of the diagram

$$\begin{array}{ccc}
 \sqrt[\vec{r}]{(X, D)} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]^N \\
 \downarrow & & \downarrow (-)^{\vec{r}} \\
 X & \xrightarrow{s} & [\mathbb{A}^1/\mathbb{G}_m]^N
 \end{array}$$

where $s: X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]^N$ is determined by the s_i and $(-)^{\vec{r}}$ is the map induced by $(-)^{r_i}: [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ on the i -th component.

Equivalently, $\sqrt[r]{(X, D)}$ can be constructed by iteration from the previous case: from X we first construct the stack $\sqrt[r_1]{(X, D_1)}$ as in the previous section. The preimages $\widetilde{D}_2, \dots, \widetilde{D}_N$ of D_2, \dots, D_N to this stack give a simple normal crossing divisor \widetilde{D} , and we can replace (X, D) by the log stack $(\sqrt[r_1]{(X, D_1)}, \widetilde{D})$, and continue the process.

Finally, the stack $\sqrt[r]{(X, D)}$ can also be identified with the fibered product of the diagram

$$\begin{array}{ccccccc} \sqrt[r_1]{(X, D_1)} & & \sqrt[r_2]{(X, D_2)} & & \dots & & \sqrt[r_N]{(X, D_N)} \\ & \searrow & & \searrow & & & \swarrow \\ & & & & & & X \end{array}$$

As in the case of one divisor, the construction of $\sqrt[r]{(X, D)}$ is compatible with smooth base change.

On the stack $\sqrt[r]{(X, D)}$ there are N universal effective Cartier divisors D_{i,r_i} , obtained as the reduction of the preimage of D_i to $\sqrt[r]{(X, D)}$ via the projection to X , or equivalently as the preimage of the corresponding divisor D_{i,r_i} on $\sqrt[r_i]{(X, D_i)}$ constructed in the previous section (we abuse notation slightly and denote the two by the same symbol).

Zariski locally where $X = \text{Spec } A$ is affine and each D_i has a global equation $f_i = 0$ we have an isomorphism

$$\sqrt[r]{(X, D)} \simeq [\text{Spec } A[x_1, \dots, x_N]/(f_1 - x_1^{r_1}, \dots, f_N - x_N^{r_N}) / \prod_i \mu_{r_i}]$$

where each factor μ_{r_i} acts on x_i by multiplication and trivially on the other variables.

2.1.3 ROOT STACKS FOR NON-SIMPLE NORMAL CROSSINGS

If D is only normal crossings, then the “correct” notion of root stack of X along D is not the naive generalization of the simple normal crossings case, but rather the subtler one put forward in [6]. We refer to that paper and to [49, Section 2.2] for details about what follows.

We start by explaining with an example why the naive notion of root stacks along the lines of the previous two cases is not the one that we want to work with.

EXAMPLE 2.3. Consider an irreducible nodal cubic $D \subset \mathbb{P}^2$, and the pair (\mathbb{P}^2, D) . We could, as in Section 2.1.1, start with the morphism $\mathbb{P}^2 \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ that classifies the divisor D (i.e. determined by the pair $(\mathcal{O}_{\mathbb{P}^2}(D), \sigma_D)$, where σ_D is the tautological section of $\mathcal{O}_{\mathbb{P}^2}(D)$ cutting out D), and set $\sqrt[r]{(\mathbb{P}^2, D)}$ to be the fibered product of diagram (1). The unpleasant feature of this construction is that it adds a stabilizer μ_r along all of D , including at the node, where one would expect to have a $\mu_r \times \mu_r$ accounting for the two branches of D .

The point here is that the morphism $\mathbb{P}^2 \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ does not induce to the “correct” logarithmic structure on \mathbb{P}^2 corresponding to the pair (\mathbb{P}^2, D) , which is the compactifying log structure briefly mentioned in Section 2.1, and *does* distinguish the branches of D at the node.

The trouble with the previous example is remedied by working with sheaves of weights (as explained in [6]), that allow to locally distinguish the branches of D at the node, and to locally take roots separately along the branches.

We briefly recall what this entails: in general, the log structure of a fine saturated log scheme X can be seen as a “Deligne–Faltings” structure, a symmetric monoidal functor $L: A \rightarrow \text{Div}_X$ from a sheaf of saturated sharp monoids A on the small étale site of X to the symmetric monoidal stack Div_X of pairs (L, t) of line bundles with global section. The monoidal operation of Div_X is given by tensor product.

In the case of *simple* normal crossings, the canonical (compactifying) log structure $M_D \rightarrow \mathcal{O}_X$ induced by D can be completely described in the language of Deligne–Faltings structures by the symmetric monoidal functor $\mathbb{N}^N \rightarrow \text{Div}_X(X)$ sending the generator e_i to the pair $(\mathcal{O}_X(D_i), \sigma_i)$ (this is a “chart” for the log structure, in the sense of [6, Section 3.3]). If D is only normal crossings though, the *global* components of the divisor D will not capture the geometry of the situation faithfully (as for the nodal cubic of Example 2.3), and the compactifying log structure is *not* described by the symmetric monoidal functor $\mathbb{N}^N \rightarrow \text{Div}_X(X)$ defined above.

In both cases (simple and non-simple normal crossings), the sheaf A is the quotient M_D/\mathcal{O}_X^\times of the sheaf $M_D \subset \mathcal{O}_X$ consisting of regular functions that only possibly vanish along D (mentioned in Section 2.1) by the subgroup \mathcal{O}_X^\times of invertible functions, and the symmetric monoidal functor $L: A \rightarrow \text{Div}_X$ is induced by the inclusion $M_D \subseteq \mathcal{O}_X$ by passing to (stacky) quotients by \mathcal{O}_X^\times , obtaining

$$A = M_D/\mathcal{O}_X \rightarrow [\mathcal{O}_X/\mathcal{O}_X^\times] = [\mathbb{A}^1/\mathbb{G}_m]_X = \text{Div}_X.$$

If D has simple normal crossings, then the sheaf A is isomorphic to the direct sum $\bigoplus_{i=1}^N j_{i*}\mathbb{N}_{D_i}$, where D_i are the irreducible components of D , $j_i: D_i \rightarrow X$ is the inclusion and \mathbb{N}_{D_i} is the constant (étale) sheaf of monoids with stalk \mathbb{N} on D_i . The map $A \rightarrow \text{Div}_X$ in this case sends the generator 1_{D_i} of the summand $j_{i*}\mathbb{N}_{D_i}$ to the pair $(\mathcal{O}_X(D_i), \sigma_i)$ (where as usual σ_i is the section that cuts out D_i). If D is only normal crossings but non-simple, this description does not hold anymore (it only holds étale locally where D becomes simple normal crossings). For the nodal cubic of Example 2.3, the stalk of the sheaf A at the node is \mathbb{N}^2 (accounting for the two branches) rather than just \mathbb{N} .

For a general log scheme X and $r \in \mathbb{N}$, the root stack $\sqrt[r]{X}$ (in the sense of Borne and Vistoli) parametrizes lifts of the symmetric monoidal functor $L: A \rightarrow \text{Div}_X$ to a symmetric monoidal functor $\frac{1}{r}A \rightarrow \text{Div}_X$, where we are embedding A into $\frac{1}{r}A \cong A$ via the map $\cdot r: A \rightarrow A$. The image of the section $\frac{1}{r}a$ via this lift is an r -th root of the pair $L(a) = (L_a, s_a)$. In the case of the nodal cubic D of Example 2.3, i.e. of \mathbb{P}^2 equipped with the compactifying log structure induced

by D , this results in trivial stabilizers outside of D , a stabilizer μ_r over points of D that are not the node, and a stabilizer $\mu_r \times \mu_r$ at the node.

If $D \subset X$ is a normal crossing divisor and $U \rightarrow X$ is a surjective étale morphism such that the pull-back $D|_U$ is simple normal crossings, then the stack $\sqrt[r]{(X, D)}$ can be obtained from the root stack $\sqrt[r]{(U, D|_U)}$ constructed in the previous section (where \vec{r} is the vector (r, \dots, r)) by descent.

Moreover, In place of $\frac{1}{r}A$ one can use an arbitrary *Kummer extension* $A \rightarrow B$ of sheaves of monoids, an injective morphism such that every section of B locally has a multiple coming from A . This more general construction of the root stack $\sqrt[B]{X}$ for Kummer extensions $A \rightarrow B$, where B is not of the form $\frac{1}{r}A$, will come up only in Section 4.2. In general, the stack $\sqrt[B]{X}$ is a tame Artin stack (Deligne–Mumford in characteristic 0), and the projection $\sqrt[B]{X} \rightarrow X$ is a coarse moduli space morphism.

2.1.4 THE INFINITE ROOT STACK AND THE KUMMER FLAT TOPOS

In all the cases considered above, the various root stacks of (X, D) form an inverse system. Let us temporarily use the letter r to denote either a natural number, or a vector of natural numbers $\vec{r} \in \mathbb{N}^N$, depending on the context that we are considering. We write $r \mid r'$ to denote divisibility in the first case, and that $r_1 \mid r'_1, \dots, r_N \mid r'_N$ in the second case. We also write $\frac{r'}{r}$ for the vector $(\frac{r'_1}{r_1}, \dots, \frac{r'_N}{r_N})$.

With these conventions, if (X, D) is a pair where D is normal crossings and $r \mid r'$, there is a natural projection $g_{r',r}: \sqrt[r']{(X, D)} \rightarrow \sqrt[r]{(X, D)}$, roughly defined by raising the roots parametrized by the source stack to the r'/r -th power. Since $\sqrt[1]{(X, D)} \simeq X$, the map $g_{r,1}: \sqrt[r]{(X, D)} \rightarrow X$ is the natural projection. If D is a regular divisor, then the maps $g_{r',r}$ restrict to maps $f_{r',r}: D_{r'} \rightarrow D_r$ between the universal Cartier divisors on the two stacks (and analogously in the case where D is simple normal crossings).

Moreover, $g_{r',r}$ is a “relative root stack” morphism, in the sense that for a morphism $T \rightarrow \sqrt[r]{(X, D)}$ where T is a scheme, the pull-back of $g_{r',r}$ can be seen as the projection from a root stack of the scheme T . Consequently, the maps $g_{r',r}$ have all the properties of a projection to a coarse moduli space of a tame algebraic stack. More precisely, $\sqrt[r']{(X, D)}$ is canonically isomorphic to the $\frac{r'}{r}$ -th root stack of $\sqrt[r]{(X, D)}$. Under this isomorphism $g_{r',r}$ is identified with the projection

$$\sqrt[r']{(X, D)} \simeq \sqrt[\frac{r'}{r}]{(\sqrt[r]{(X, D)}, D_r)} \rightarrow \sqrt[r]{(X, D)}.$$

The maps $g_{r',r}$ equip the stacks $\{\sqrt[r]{(X, D)}\}_r$ with the structure of an inverse system. The inverse limit $\sqrt[\infty]{(X, D)} := \varprojlim_r \sqrt[r]{(X, D)}$ is the *infinite root stack* of (X, D) [51]. This, contrarily to the finite root stacks, is not algebraic, but has a local description as a quotient stack, that allows some control over quasi-coherent sheaves (and in particular perfect complexes) on it.

It will be convenient for us to work with a directed subsystem of root stacks which is cofinal in $\{\sqrt[r]{(X, D)}\}_r$. Namely we will consider the subsystem $\{\sqrt[n!]{(X, D)}\}_{n \in \mathbb{N}}$, where $\bar{n}! := (n!, \dots, n!)$ if D has more than one component. Note that the restriction of the ordering given by divisibility on \mathbb{N} to the subset $\{n!\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ coincides with the usual ordering of the naturals (i.e. $r! \mid s!$ if and only if $r \leq s$), and that this subsystem is cofinal, since given an index $\vec{r} = (r_1, \dots, r_N)$, we have $\vec{r} \mid \vec{M}!$ with $M = \text{lcm}(r_i)$. Therefore we have a canonical isomorphism $\sqrt[\infty]{(X, D)} \simeq \varprojlim_n \sqrt[n!]{(X, D)}$.

Let us also recall that every log scheme X has an associated Kummer flat topos X_{Kfl} . We omit a discussion of this construction, since for the present paper it can be safely taken as a black box. We refer the reader to [51, Section 6.2] or [36, Section 2] for the definition and basic properties. One can also define a “small fppf site” of the infinite root stack $\sqrt[\infty]{X}$, and as proven in [51, Theorem 6.16], the resulting topos $\sqrt[\infty]{X}_{\text{fppf}}$ is isomorphic to the Kummer flat topos X_{Kfl} of the fine saturated log scheme X .

We will also consider the Kummer flat topos X_{Kfl} for X a log algebraic stack. To the best of our knowledge this has not been discussed in the literature before. The conscientious reader can take the equivalence with the small fppf topos of the infinite root stack $\sqrt[\infty]{X}$ as a *definition* for X_{Kfl} in this setting. One can also write down a definition for the Kummer flat topos in analogy with the one for schemes, and use [51, Theorem 6.16] to prove that this is indeed equivalent to the small fppf topos of the infinite root stack.

2.2 ∞ -CATEGORIES AND CATEGORIES OF SHEAVES

Throughout the paper we will use the formalism of $(\infty, 1)$ -categories, the standard reference is Lurie’s work [34, 35]. The main reason for working with ∞ -categories is that additive invariants, and in particular algebraic K-theory, cannot be computed from the underlying triangulated categories alone: they are not invariants of triangulated categories but rather of their enhancements. We will work with *stable* $(\infty, 1)$ -categories as an enhancement of triangulated categories. We will need very little from the theory of ∞ -categories, and the reader could replace stable $(\infty, 1)$ -categories with κ -linear dg categories throughout. From now on we will refer to $(\infty, 1)$ -categories just as ∞ -categories. If \mathcal{C} is an ∞ -category, and A and B are objects in \mathcal{C} , we denote by $\text{Hom}_{\mathcal{C}}(A, B)$ the *mapping space* between A and B . All limits and colimits of ∞ -categories appearing in the paper are to be understood in the ∞ -categorical sense. We say that a diagram of ∞ -categories

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ G \downarrow & & \downarrow H \\ \mathcal{C}_3 & \xrightarrow{K} & \mathcal{C}_4 \end{array}$$

is *commutative* if there is a natural transformation $\alpha: HF \Rightarrow KG$ which is an

equivalence when passing to homotopy categories.

We will be mostly interested in *stable idempotent-complete* ∞ -categories. Stable ∞ -categories are an enhancement of triangulated categories: if \mathcal{C} is a stable ∞ -category its homotopy category $\mathrm{Ho}(\mathcal{C})$ is triangulated. An important source of stable ∞ -categories are the ∞ -categorical derived categories of Grothendieck abelian categories, see [35] Definition 1.3.5.8. We refer the reader to Section 2.2 of [4] for a summary of the theory of stable ∞ -categories which contains most of the facts that we will need in this paper. Small stable idempotent-complete ∞ -categories form a presentable ∞ -category which is denoted $\mathrm{Cat}_\infty^{\mathrm{perf}}$. In particular $\mathrm{Cat}_\infty^{\mathrm{perf}}$ has all small limits and colimits. As explained in [4] there is a well-defined notion of tensor product of stable idempotent-complete ∞ -categories. This endows $\mathrm{Cat}_\infty^{\mathrm{perf}}$ with a symmetric monoidal structure. If κ is a commutative ring we denote by $\mathrm{Perf}(\kappa)$ the symmetric monoidal stable ∞ -category of perfect κ -modules: if $\kappa\text{-mod}$ is the abelian category of κ -modules, $\mathrm{Perf}(\kappa)$ is the subcategory of compact objects in the ∞ -categorical derived category $\mathrm{D}(\kappa\text{-mod})$ (see Definition 1.3.5.8 of [35]). We denote $\mathrm{Cat}_{\infty,\kappa}^{\mathrm{perf}}$ the symmetric monoidal ∞ -category of idempotent-complete stable ∞ -categories tensored over $\mathrm{Perf}(\kappa)$, see Section 4.1 of [24] for more details on this construction. We will refer to objects in $\mathrm{Cat}_{\infty,\kappa}^{\mathrm{perf}}$ as κ -linear ∞ -categories. For later reference we recall from [43] a general result on colimits of ∞ -categories.

PROPOSITION 2.4. *Let I be a filtered category, $\{\mathcal{C}_i\}_{i \in I}$ be a filtered system of ∞ -categories and assume that all the structure maps $\alpha_{i \rightarrow j} : \mathcal{C}_i \rightarrow \mathcal{C}_j$ are fully faithful. Then the colimit $\mathcal{C} := \varinjlim \mathcal{C}_i$ is the ∞ -category with*

- objects given by the union $\bigcup_{i \in I} \mathrm{Ob}(\mathcal{C}_i)$, and
- the Hom complex between $A_i \in \mathcal{C}_i$ and $A_j \in \mathcal{C}_j$ is given by

$$\mathrm{Hom}_{\mathcal{C}}(A_i, A_j) = \mathrm{Hom}_{\mathcal{C}_l}(\alpha_{l \rightarrow i}(A_i), \alpha_{l \rightarrow j}(A_j))$$

where l is any object of I that is the source of morphisms $j \leftarrow l \rightarrow i$.

We will be working with stable categories of quasi-coherent sheaves on schemes and stacks. A survey of all basic facts and definitions on ∞ -categories of quasi-coherent sheaves can be found in Section 2 and 3 of [2]. Let X be a stack. We denote:

- by $\mathrm{qcoh}(X)$ the abelian category of quasi-coherent sheaves on X , and by $\mathrm{Qcoh}(X)$, the stable ∞ -category of quasi-coherent sheaves on X ;
- by $\mathrm{coh}(X)$ the abelian category of coherent sheaves on X , and by $\mathrm{Coh}(X)$ the stable ∞ -category of coherent sheaves on X .

REMARK 2.5. Recall from [2] that $\mathrm{Qcoh}(X)$ is defined as the limit

$$\mathrm{Qcoh}(X) := \varprojlim_{\mathrm{Spec}(A) \rightarrow X} \mathrm{Qcoh}(\mathrm{Spec}(A))$$

where $\text{Spec}(A) \rightarrow X$ runs over all maps from derived affine schemes with target X . In general $\text{Qcoh}(X)$ is different from the ∞ -categorical derived category $\text{D}(\text{qcoh}(X))$. By [17, Proposition 2.4.3], however, if X is a quasi-compact and quasi-separated classical Artin stack the respective subcategories of objects that are *bounded below* are equivalent

$$\text{Qcoh}(X)^+ \simeq \text{D}(\text{qcoh}(X))^+$$

The tensor product of quasi-coherent sheaves equips $\text{Qcoh}(X)$ with a symmetric monoidal structure. We define $\text{Perf}(X)$, the ∞ -category of perfect complexes, as the full subcategory of dualizable objects in $\text{Qcoh}(X)$. As proved in Proposition 3.6 of [2] this is equivalent to the ordinary definition of perfect complexes as objects that are locally equivalent to complexes of vector bundles. Let I be a small cofiltered category, and let $\{X_i\}_{i \in I}$ be a pro-object in stacks.

DEFINITION 2.6. We set $\text{Perf}(\{X_i\}_{i \in I}) := \varinjlim_i \text{Perf}(X_i)$ as an ∞ -category.

Let X be a log scheme. We will apply Definition 2.6 to the pro-object in stacks given by the root stacks of X together with the root maps between them, $\{\sqrt[r]{X}\}_{r \in \mathbb{N}}$. As we prove in Proposition 2.25 of [46], which we recall below, under appropriate assumptions there is an equivalence $\text{Perf}(\sqrt[\infty]{X}) \simeq \varinjlim_r \text{Perf}(\sqrt[r]{X})$. Although in [46] Proposition 2.25 was stated for the dg categories of perfect complexes, the proof given there works without variations for ∞ -categories of perfect complexes over an arbitrary ground ring.

PROPOSITION 2.7 ([46, Proposition 2.25]). *Let X be a noetherian fine saturated log algebraic stack with locally free log structure over κ . Then there is an equivalence of ∞ -categories*

$$\text{Perf}(\sqrt[\infty]{X}) \simeq \varinjlim_r \text{Perf}(\sqrt[r]{X}).$$

PROPOSITION 2.8. *Let X be a noetherian fine saturated log algebraic stack with locally free log structure over κ (this holds in particular if the log structure comes from a normal crossing divisor). Then there is an equivalence of ∞ -categories $\text{Perf}(X_{\text{Kfl}}) \simeq \text{Perf}(\sqrt[\infty]{X})$.*

Proof. The proof is very similar to the proof of Proposition 2.7 given in [46]. By Corollary 6.17 of [51] there is an equivalence of abelian categories $\text{coh}(X_{\text{Kfl}}) \simeq \text{coh}(\sqrt[\infty]{X})$, which yields an equivalence between the stable ∞ -categories $\text{Coh}(X_{\text{Kfl}}) \simeq \text{Coh}(\sqrt[\infty]{X})$. Also, the structure sheaves $\mathcal{O}_{X_{\text{Kfl}}}$ and $\mathcal{O}_{\sqrt[\infty]{X}}$ are coherent, see Proposition 4.9 of [51]. Thus, in the terminology of [16, Section 1.5], the ringed topoi X_{Kfl} and $\sqrt[\infty]{X}$ are eventually coconnective. As in the proof of Proposition 1.5.3 in [16], this implies that there are fully-faithful inclusions

$$\text{Perf}(X_{\text{Kfl}}) \subseteq \text{Coh}(X_{\text{Kfl}}), \quad \text{Perf}(\sqrt[\infty]{X}) \subseteq \text{Coh}(\sqrt[\infty]{X}).$$

Further, the categories of perfect complexes are the full subcategories of dualizable objects. In formulas, we can write

$$\text{Perf}(X_{\text{Kff}}) \simeq \text{Coh}(X_{\text{Kff}})^{\text{dual}}, \quad \text{Perf}(\sqrt[\infty]{X}) \simeq \text{Coh}(\sqrt[\infty]{X})^{\text{dual}}.$$

We obtain the following chain of equivalences, which implies the statement

$$\text{Perf}(X_{\text{Kff}}) \simeq \text{Coh}(X_{\text{Kff}})^{\text{dual}} \simeq \text{Coh}(\sqrt[\infty]{X})^{\text{dual}} \simeq \text{Perf}(\sqrt[\infty]{X}).$$

□

REMARK 2.9. Let X be a noetherian fine saturated log algebraic stack with locally free log structure over κ . In this paper we take $\text{Perf}(\sqrt[\infty]{X})$ as our chosen model for the category of perfect complexes over the log scheme X . By Proposition 2.8 this coincides with the category of perfect complexes on the Kummer flat topos of X and is therefore well suited to our applications to Kummer flat invariants of X . However we could have obtained an a priori different model of perfect complexes on X by working with the abelian category of quasi-coherent parabolic sheaves with rational weights on X , $\text{Par}(X)$, and then taking compact objects inside its ∞ -categorical derived category $\text{D}(\text{Par}(X))$. It turns out that this actually gives an equivalent category. Let us briefly explain why.

- By Theorem 7.3 of [51] the abelian categories $\text{Par}(X)$ and $\text{qcoh}(\sqrt[\infty]{X})$ are equivalent.
- From Remark 2.5 and Proposition 2.7 it easily follows that the subcategories of bounded-below objects in $\text{D}(\text{Par}(X)) \simeq \text{D}(\text{qcoh}(\sqrt[\infty]{X}))$ and in $\text{QCoh}(\sqrt[\infty]{X})$ are equivalent. Since compact objects are bounded, this implies that the subcategory of compact objects in $\text{D}(\text{Par}(X))$ coincides with $\text{Perf}(\sqrt[\infty]{X})$, as desired.

2.3 EXACT SEQUENCES OF ∞ -CATEGORIES

Let \mathcal{C} be a stable ∞ -category. We say that two objects A and A' are *equivalent* if there is a map $A \rightarrow A'$ that becomes an isomorphism in the homotopy category of \mathcal{C} . If $\iota: \mathcal{C}' \rightarrow \mathcal{C}$ is a fully faithful functor, we often view \mathcal{C}' as a subcategory of \mathcal{C} : accordingly, we will usually denote the image under ι of an object A of \mathcal{C}' simply by A rather than $\iota(A)$. We will always assume that subcategories are closed under equivalence. That is, if \mathcal{C}' is a full subcategory of \mathcal{C} , A is an object of \mathcal{C}' , and A' is an object of \mathcal{C} which is equivalent to A , we will always assume that A' lies in \mathcal{C}' as well.

Recall that if \mathcal{D} is a full subcategory of \mathcal{C} , $(\mathcal{D})^\perp$ denotes the *right orthogonal* of \mathcal{D} , i.e. the full subcategory of \mathcal{C} consisting of the objects B such that the Hom-space $\text{Hom}_{\mathcal{C}}(A, B)$ is contractible for every object $A \in \mathcal{D}$. Let $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ be a finite collection of stable subcategories of \mathcal{C} such that, for all $1 \leq i \leq n - 1$, $\mathcal{C}_i \subseteq (\mathcal{C}_{i+1})^\perp$. Then we denote by $\langle \mathcal{C}_1, \dots, \mathcal{C}_n \rangle$ the smallest stable subcategory of \mathcal{C} containing all the subcategories \mathcal{C}_i .

An *exact sequence* of stable ∞ -categories is a sequence

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \quad (3)$$

which is both a fiber and a cofiber sequence in $\text{Cat}_\infty^{\text{perf}}$. This concept captures the classical notion of Verdier localization of triangulated categories in the setting of ∞ -categories: as shown in Section 5.1 of [4], (3) is an exact sequence if and only if the sequence of homotopy categories

$$\text{Ho}(\mathcal{A}) \xrightarrow{\text{Ho}(F)} \text{Ho}(\mathcal{B}) \xrightarrow{\text{Ho}(G)} \text{Ho}(\mathcal{C})$$

is a classical Verdier localization of triangulated categories (up to idempotent-completion of $\text{Ho}(\mathcal{C})$). This implies in particular that the fully-faithfulness of a functor between stable ∞ -categories can be checked at the level of homotopy categories.

The functor F admits a right adjoint F^R exactly if G admits a right adjoint G^R , and similarly for left adjoints. This is proved in [29, Proposition 4.9.1] for triangulated categories but the extension to ∞ -categories is straightforward. If F (or equivalently G) admits a right adjoint we say that (3) is a *split exact sequence*. In this case the functor G^R is fully faithful and we have that $\mathcal{B} = \langle G^R(\mathcal{C}), \mathcal{A} \rangle$. As we indicated earlier, since G^R is fully faithful we will drop it from our notations whenever this is not likely to create confusion: thus we will denote $G^R(\mathcal{C})$ simply by \mathcal{C} , and write $\mathcal{B} = \langle \mathcal{C}, \mathcal{A} \rangle$.

2.4 PREORDERED SEMI-ORTHOGONAL DECOMPOSITIONS

In this section we introduce *preordered semi-orthogonal decompositions* of ∞ -categories. This concept was also discussed in [3]. See [31] for a survey of semi-orthogonal decompositions (sod-s). Let \mathcal{C} be a stable ∞ -category, and let P be a preordered set. Consider a collection of full stable subcategories $\iota_x: \mathcal{C}_x \rightarrow \mathcal{C}$ indexed by $x \in P$.

DEFINITION 2.10. We say that the subcategories \mathcal{C}_x form a *preordered semi-orthogonal decomposition (psod) of type P* if they satisfy the following three properties.

- For all $x \in P$, \mathcal{C}_x is a non-zero *admissible* subcategory: that is, the embedding ι_x admits a right adjoint and a left adjoint, which we denote by

$$r_x: \mathcal{C} \rightarrow \mathcal{C}_x \quad \text{and} \quad l_x: \mathcal{C} \rightarrow \mathcal{C}_x.$$

- If $y <_P x$, i.e. $y \leq_P x$, and $x \neq y$, then $\mathcal{C}_y \subseteq \mathcal{C}_x^\perp$.
- \mathcal{C} is the smallest stable subcategory of \mathcal{C} containing all the subcategories \mathcal{C}_x , $x \in P$.

DEFINITION 2.11. If \mathcal{C} is equipped with a psod of type (P, \leq) , we write $\mathcal{C} = \langle \mathcal{C}_x, x \in (P, \leq) \rangle$.

DEFINITION 2.12. Let $\mathcal{C} = \langle \mathcal{C}_x, x \in (P, \leq) \rangle$ and $\mathcal{D} = \langle \mathcal{D}_y, y \in (Q, \leq) \rangle$ be categories equipped with psod-s. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a fully-faithful functor. We say that F is *compatible with the psod-s* if for all $x \in P$ there exists $y \in Q$ such that $F(\mathcal{C}_x) = \mathcal{D}_y$.

REMARK 2.13. The definition of psod makes sense also for classical triangulated categories, and not just for ∞ -categories. We mentioned in Section 2.3 that a sequence of stable ∞ -categories is exact if and only if the corresponding sequence of homotopy categories is a Verdier localization. Also, $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ is an adjoint pair of functors between stable ∞ -categories if and only if

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{A}) \rightleftarrows \mathrm{Ho}(\mathcal{B}) : \mathrm{Ho}(G)$$

is an adjoint pair between the homotopy categories. As a consequence psod-s can be checked at the level of homotopy categories: that is, a collection of admissible subcategories \mathcal{C}_x gives a psod of type P of the ∞ -category \mathcal{C} if and only if $\mathrm{Ho}(\mathcal{C}_x)$ gives a psod of type P of the triangulated category $\mathrm{Ho}(\mathcal{C})$.

DEFINITION 2.14. Let P and Q be preordered sets. The *join of P and Q* is the preordered set

$$P * Q := (P \coprod Q, \leq_{P*Q})$$

defined as follows. Let x and y be in $P \coprod Q$:

- if $x, y \in P$, then $x \leq_{P*Q} y$ if and only if $x \leq_P y$;
- if $x, y \in Q$, then $x \leq_{P*Q} y$ if and only if $x \leq_Q y$;
- if $x \in P$ and $y \in Q$, then $x \leq_{P*Q} y$.

LEMMA 2.15. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be an exact sequence of stable idempotent-complete ∞ -categories. Assume that \mathcal{A} is admissible, and that \mathcal{A} and \mathcal{C} carry psod-s of type $P_{\mathcal{A}}$ and $P_{\mathcal{C}}$, respectively. Then \mathcal{B} carries a canonical psod of type $P_{\mathcal{B}} := P_{\mathcal{A}} * P_{\mathcal{C}}$ such that for all $x \in P_{\mathcal{B}}$ the component \mathcal{B}_x is equivalent to \mathcal{A}_x , if x is in $P_{\mathcal{A}}$, and to \mathcal{C}_x , if x is in $P_{\mathcal{C}}$.

Proof. It follows from the assumptions that the functor G has a right adjoint $G^R : \mathcal{C} \rightarrow \mathcal{B}$. The admissible subcategories \mathcal{C}_x , $x \in P_{\mathcal{C}}$, are admissible subcategories of \mathcal{B} under the image of G^R , and they are all right orthogonal to the image of \mathcal{A} under F . In particular, they are right orthogonal to $F(\mathcal{A}_x)$, $x \in P_{\mathcal{A}}$, and this concludes the proof. □

2.5 THE CHERN CHARACTER

In section 5 we will study the Chern character in the logarithmic setting. For later reference, we give below an account of the Chern character that follows closely [4].

Let \mathcal{S}_∞ be the ∞ -category of spectra. It follows from [4] that the Chern character can be defined in the abstract setting of ∞ -categories as a natural transformation between additive invariants

$$\text{ch}: K(-) \Rightarrow \text{HH}(-): \text{Cat}_{\infty, \kappa}^{\text{perf}} \longrightarrow \mathcal{S}_\infty \quad (4)$$

where:

- $K(-)$ is the algebraic K-theory,
- $\text{HH}(-)$ is the Hochschild complex viewed as an object in spectra.

As explained in Section 10 of [4] the Chern character (4) is uniquely determined by the choice of the element $1 \in \text{HH}(\text{Perf}(\kappa)) \simeq \kappa$. Assume now that κ is a field of characteristic 0. Then the Chern character (4) captures the ordinary de Rham Chern character, we refer to [8] for a thorough discussion of these aspects. More precisely, let X be a smooth and proper scheme over κ . Denote by $H_{\text{dR}}(X)$ be the de Rham cohomology of X , which is the hypercohomology of the de Rham complex. The HKR theorem gives an isomorphism $\text{HH}_0(\text{Perf}(X)) \cong \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(X)$. Then the composition

$$\text{ch}_{\text{dR}}: K_0(X) \xrightarrow{\text{ch}} \text{HH}_0(\text{Perf}(X)) \xrightarrow{\cong} \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(X),$$

recovers the ordinary Chern character taking values in the even de Rham cohomology of X .

3 PERFECT COMPLEXES OVER INFINITE ROOT STACKS

In this section we construct sod-s for infinite root stacks. In 3.1 and 3.2 we treat separately the case of root stacks of a single Cartier divisor, and of a simple normal crossing divisor with an arbitrary number of components. We start by reviewing the results obtained in [25] and [3] for finite root stacks of simple normal crossing divisors. We construct recursively compatible sod-s for the ∞ -categories of perfect complexes of these root stacks, for a cofinal subset of indices. This will be key to constructing sod-s on the infinite root stack.

In Section 4 we explain how to extend these results beyond the normal crossing case: in Section 4.1 we extend our investigation to root stacks of general (not necessarily simple) normal crossing divisors, and in Section 4.2 we discuss the case of log stacks with simplicial log structure.

3.1 ROOT STACKS OF A REGULAR DIVISOR

Let X be an algebraic stack, and $D \subset X$ a regular Cartier divisor. We use the notations of Section 2.1.1 in the preliminaries. Recall in particular that we denote by $g_{r', r}$ the projection $\sqrt[r']{X, D} \rightarrow \sqrt[r]{X, D}$ for $r \mid r'$, by D_r the universal Cartier divisor on $\sqrt[r]{X, D}$, i.e. the reduction of the preimage

$g_{r,1}^{-1}(D) \subset \sqrt[r]{(X, D)}$. Further, we denote by $i_r: D_r \rightarrow \sqrt[r]{(X, D)}$ the closed embedding.

LEMMA 3.1. *The category $\text{Perf}(D_r)$ splits as the direct sum of r copies of $\text{Perf}(D)$. More precisely, if \mathbb{Z}_r is the Cartier dual of μ_r , there are natural equivalences*

$$\text{Perf}(D_r) \stackrel{(1)}{\simeq} \bigoplus_{\chi \in \mathbb{Z}_r} \text{Perf}(D_r)_\chi \stackrel{(2)}{\simeq} \text{Perf}(D) \otimes \text{Perf}(\mathcal{B}\mu_r) \simeq \text{Perf}(D) \otimes \bigoplus_{\chi \in \mathbb{Z}_r} \text{Perf}(\kappa).$$

Proof. This is a well-known fact, that applies more generally to gerbes banded by a diagonalizable group scheme, so we limit ourselves to a brief sketch. Equivalence (1) comes from the character decomposition of $\text{Perf}(D_r)$. If χ is in \mathbb{Z}_r , let $\kappa(\chi)$ be the corresponding μ_r -representation. Then equivalence (2) maps the twisted structure sheaf \mathcal{O}_D^χ to $\mathcal{O}_D \otimes \kappa(\chi)$. □

LEMMA 3.2. *Let $r, r' \in \mathbb{N}$, and $r \mid r'$. Then the pull-back functors*

$$g_{r',r}^*: \text{Perf}(\sqrt[r]{(X, D)}) \longrightarrow \text{Perf}(\sqrt[r']{(X, D)})$$

are fully faithful.

Proof. Since $g_{r',r}$ is a relative coarse moduli space map, the natural map $\mathcal{O}_{\sqrt[r]{(X, D)}} \rightarrow g_{r',r,*} \mathcal{O}_{\sqrt[r']{(X, D)}}$ is an equivalence in $\text{Perf}(\sqrt[r]{(X, D)})$. This easily implies that $g_{r',r}^*$ is fully faithful by the adjunction with $g_{r',r,*}$ and the projection formula (see for instance [3, Lemma 4.4]). □

Let $\mathbb{Z}_r^* = \mathbb{Z}_r \setminus \{0\}$ be the set of non-trivial characters of μ_r . In order to keep track of indices it will be convenient to identify \mathbb{Z}_r and \mathbb{Z}_r^* with subsets of $\mathbb{Q}/\mathbb{Z} = \mathbb{Q} \cap (-1, 0]$ as follows:

$$\begin{aligned} \mathbb{Z}_r &\cong \left\{ -\frac{r-1}{r}, \dots, -\frac{1}{r}, 0 \right\} \subset \mathbb{Q} \cap (-1, 0], \\ \mathbb{Z}_r^* &\cong \left\{ -\frac{r-1}{r}, \dots, -\frac{1}{r} \right\} \subset \mathbb{Q} \cap (-1, 0]. \end{aligned}$$

We equip \mathbb{Z}_r and \mathbb{Z}_r^* with the total order \leq given by

$$-\frac{r-1}{r} < -\frac{r-2}{r} < \dots < -\frac{1}{r} < 0.$$

Here $-\frac{k}{r}$ should be thought of as the element $r-k$ in $\{1, \dots, r\}$, equipped with the standard ordering. This (perhaps unusual) identification will be convenient when we pass to the limit for $r \rightarrow \infty$.

Following Theorem 4.7 of [3], for every $\chi \in \mathbb{Z}_r^*$ we consider the fully faithful embedding

$$\Phi_\chi: \text{Perf}(D) \rightarrow \text{Perf}(\sqrt[r]{(X, D)})$$

given by the composite

$$\text{Perf}(D) \xrightarrow{\simeq} \text{Perf}(D_r)_\chi \xrightarrow{\subset} \text{Perf}(D_r) \xrightarrow{i_{r,*}} \text{Perf}(\sqrt[r]{(X, D)}).$$

Note that if we see χ as an element of $\{1, \dots, r\}$, in the notation of [3] the objects of the image of Φ_χ are actually equipped with the action corresponding to the character $-\chi \equiv r - \chi \in \mathbb{Z}_r$.

REMARK 3.3. Each individual summand $\text{Perf}(D_r)_\chi$ of $\text{Perf}(D_r)$ embeds fully faithfully in the category $\text{Perf}(\sqrt[r]{(X, D)})$ via Φ_χ , which is a restriction of $i_{r,*}$ to $\text{Perf}(D_r)_\chi$. However the functor $i_{r,*}$ itself is not fully faithful.

We introduce the following notations:

- Let χ be in \mathbb{Z}_r . If $\chi \neq 0$ we denote by $\mathcal{A}_\chi \subset \text{Perf}(\sqrt[r]{(X, D)})$ the image of $\text{Perf}(D)$ under Φ_χ , and we denote by $\mathcal{A}_0 \subset \text{Perf}(\sqrt[r]{(X, D)})$ the image of $\text{Perf}(X)$ under $g_{r,1}^*$.
- We denote by Q_r the subcategory of $\text{Perf}(\sqrt[r]{(X, D)})$ generated by the subcategories \mathcal{A}_χ for $\chi \in \mathbb{Z}_r^*$.

The next theorem is proved in [3] using the language of classical triangulated categories, but the proof applies without variations to the ∞ -setting.

PROPOSITION 3.4 ([3, Theorem 4.7]).

1. The category \mathcal{A}_0 is an admissible subcategory of $\text{Perf}(\sqrt[r]{(X, D)})$.
2. The right orthogonal of \mathcal{A}_0 inside $\text{Perf}(\sqrt[r]{(X, D)})$ is the subcategory Q_r introduced above. There is a psod of type (\mathbb{Z}_r^*, \leq)

$$Q_r = \langle \mathcal{A}_\chi, \chi \in (\mathbb{Z}_r^*, \leq) \rangle.$$

3. By items (1) and (2), the category $\text{Perf}(\sqrt[r]{(X, D)})$ has a psod of type (\mathbb{Z}_r, \leq)

$$\text{Perf}(\sqrt[r]{(X, D)}) = \langle Q_r, \mathcal{A}_0 \rangle = \langle \mathcal{A}_\chi, \chi \in (\mathbb{Z}_r, \leq) \rangle.$$

Consider the directed system of root stacks $\{\sqrt[n!]{(X, D)}\}_{n \in \mathbb{N}}$ whose indices are the factorials, with the natural projections

$$\dots \longrightarrow \sqrt[3!]{(X, D)} \xrightarrow{g_{3!,2!}} \sqrt[2!]{(X, D)} \xrightarrow{g_{2!,1!}} \sqrt[1!]{(X, D)} = X.$$

We point out once again that this subsystem of indices is identified with \mathbb{N} with its standard ordering, i.e. if $n \leq m$ we have a map $\sqrt[m!]{(X, D)} \xrightarrow{g_{m!,n!}} \sqrt[n!]{(X, D)}$, and these are compatible in the obvious sense. We will inductively construct a tower of compatible sod-s on the categories of perfect complexes on these root stacks. This is done in Proposition 3.5 below by applying iteratively Proposition 3.4. For $n \geq 2$, the sod-s we will construct on $\text{Perf}(\sqrt[n!]{(X, D)})$ will

be different from the sod-s on the category of perfect complexes on the root stacks of (X, D) given directly by Proposition 3.4 (see Example 3.6).

We will equip the set $\mathbb{Q}/\mathbb{Z} = \mathbb{Q} \cap (-1, 0]$ with a total order $\leq^!$ which is not the restriction of the usual ordering of the real numbers. First of all we define the order $\leq^!$ on $\mathbb{Z}_{n!}$ recursively, as follows.

- On $\mathbb{Z}_{2!} = \{-\frac{1}{2}, 0\}$ we set $-\frac{1}{2} <^! 0$.
- Having defined $\leq^!$ on $\mathbb{Z}_{(n-1)!}$, let us consider the natural short exact sequence

$$0 \rightarrow \mathbb{Z}_{(n-1)!} \rightarrow \mathbb{Z}_{n!} \xrightarrow{\pi_n} \mathbb{Z}_n \rightarrow 0,$$

where $\mathbb{Z}_n = \{-\frac{n-1}{n}, \dots, -\frac{1}{n}, 0\}$ is equipped with the standard order \leq described above. Given two elements $a, b \in \mathbb{Z}_{n!}$, we set $a \leq^! b$ if either $\pi_n(a) < \pi_n(b)$, or $\pi_n(a) = \pi_n(b)$ and $a \leq^! b$ in $\mathbb{Z}_{(n-1)!}$, where we are identifying the fiber $\pi_n^{-1}(\pi_n(a)) \subseteq \mathbb{Z}_{n!}$ with $\mathbb{Z}_{(n-1)!}$ in the canonical manner.

For example, on $\mathbb{Z}_{3!} = \{-\frac{5}{6}, -\frac{4}{6}, -\frac{3}{6}, -\frac{2}{6}, -\frac{1}{6}, 0\}$, the resulting ordering is described by

$$-\frac{5}{6} <^! -\frac{2}{6} <^! -\frac{4}{6} <^! -\frac{1}{6} <^! -\frac{3}{6} <^! 0.$$

Now every element in $\mathbb{Q} \cap (-1, 0]$ can be written as $-\frac{p}{n!}$ for some $p \in \mathbb{N}$ and $n \in \mathbb{N} \setminus \{0\}$. This expression is unique if we require n to be as small as possible, and we call this the *normal factorial form*. Let $\chi = -\frac{p}{n!}, \chi' = -\frac{q}{m!} \in \mathbb{Q} \cap (-1, 0]$ be in normal factorial form. We write $\chi <^! \chi'$ if:

- $n > m$, or
- $n = m$ and $-\frac{p}{n!} <^! -\frac{q}{n!}$ in $\mathbb{Z}_{n!}$.

For example, with this ordering we have $-\frac{1}{24} <^! -\frac{2}{6} <^! -\frac{4}{6} <^! -\frac{1}{2}$.

PROPOSITION 3.5. *For every $n \in \mathbb{N}$ the category $\text{Perf}(\sqrt[n]{(X, D)})$ has a psod of type $(\mathbb{Z}_{n!}, \leq^!)$*

$$\text{Perf}(\sqrt[n]{(X, D)}) = \langle \mathcal{A}_\chi^!, \chi \in (\mathbb{Z}_{n!}, \leq^!) \rangle,$$

where

- $\mathcal{A}_0^! \simeq \text{Perf}(X)$, and
- $\mathcal{A}_\chi^! \simeq \text{Perf}(D)$ for all $\chi \in \mathbb{Z}_{n!}^*$.

Further, for all $n \in \mathbb{N}$ the functor

$$g_{(n+1)!, n!}^* : \text{Perf}(\sqrt[n]{(X, D)}) \longrightarrow \text{Perf}(\sqrt[(n+1)]{(X, D)})$$

is compatible with the psod-s.

We remark again that for a given $\chi \in \mathbb{Z}_{n!}^*$, it is not necessarily the case that $\mathcal{A}_\chi = \mathcal{A}_\chi^!$ as subcategories of $\text{Perf}(\sqrt[n]{(X, D)})$ (see Example 3.6 below).

Proof. We construct the sod with the desired properties on $\text{Perf}(\sqrt[n]{(X, D)})$ inductively.

BASIS: $n = 2$. We take the sod on $\text{Perf}(\sqrt[2]{(X, D)})$ given by Proposition 3.4. We take \mathcal{A}_0 and $\mathcal{A}_{\frac{1}{2}}$ to be the same as in Proposition 3.4. This clearly gives a psod of $\text{Perf}(\sqrt[2]{(X, D)})$ of type (\mathbb{Z}_2, \leq^1) satisfying the properties of the claim.

INDUCTIVE STEP: $n - 1 \rightarrow n$. Recall that there is a natural identification

$$\sqrt[n]{(X, D)} \simeq \sqrt[n]{(\sqrt[(n-1)!]{(X, D)}, D_{(n-1)!})}. \tag{5}$$

Further, the projection

$$\sqrt[n]{(X, D)} \simeq \sqrt[n]{(\sqrt[(n-1)!]{(X, D)}, D_{(n-1)!})} \longrightarrow \sqrt[(n-1)!]{(X, D)}$$

coincides under this identification with the map $g_{n!, (n-1)!}$. We then apply Proposition 3.4 to (5). This yields a psod of type (\mathbb{Z}_n, \leq)

$$\text{Perf}(\sqrt[n]{(X, D)}) = \langle Q_n, \mathcal{B}_0 \rangle = \langle \mathcal{B}_\zeta, \zeta \in \mathbb{Z}_n \rangle,$$

where:

- the subcategory $\mathcal{B}_0 \simeq \text{Perf}(\sqrt[(n-1)!]{(X, D)})$ is given by the image of $g_{n!, (n-1)!}^*$, and
- the subcategory $\mathcal{B}_\zeta \simeq \text{Perf}(D_{n!})_\zeta \simeq \text{Perf}(D_{(n-1)!})$ is given by the image of Φ_ζ .

Now note that by Lemma 3.1 for all $\zeta \in \mathbb{Z}_n^*$ the category \mathcal{B}_ζ splits as a direct sum of categories labelled by characters in $\mathbb{Z}_{(n-1)!}$, that is:

$$\mathcal{B}_\zeta \simeq \text{Perf}(D_{(n-1)!}) \simeq \bigoplus_{\xi \in \mathbb{Z}_{(n-1)!}} \text{Perf}(D)_\xi =: \bigoplus_{\xi \in \mathbb{Z}_{(n-1)!}} \mathcal{B}_{\zeta, \xi}.$$

We identify the subcategories $\mathcal{B}_{\zeta, \xi}$ with the factors $\mathcal{A}_\chi^!$ for $\chi \in \mathbb{Z}_{n!}^*$ appearing in the statement of the proposition by setting

$$\mathcal{A}_{\frac{\zeta}{(n-1)!} + \xi}^! := \mathcal{B}_{\zeta, \xi}.$$

Note that we can make sense of the expression $\frac{\zeta}{(n-1)!} + \xi$ as an element of $\mathbb{Z}_{n!}^*$, because we have identified the sets of characters $\mathbb{Z}_{n!}$, $\mathbb{Z}_{(n-1)!}$ and \mathbb{Z}_n with subsets of $\mathbb{Q} \cap (-1, 0]$. It is easy to verify that we can write as sums of the form $\frac{\zeta}{(n-1)!} + \xi$ exactly the elements of $\mathbb{Z}_{n!}$ which are in the complement of $\mathbb{Z}_{(n-1)!}$, that is

$$\left\{ \frac{\zeta}{(n-1)!} + \xi \mid \zeta \in \mathbb{Z}_n^*, \xi \in \mathbb{Z}_{(n-1)!} \right\} = \mathbb{Z}_{n!} \setminus \mathbb{Z}_{(n-1)!} \subset \mathbb{Q} \cap (-1, 0].$$

Thus we conclude that the subcategory Q_n has a psod of type $(\mathbb{Z}_n! \setminus \mathbb{Z}_{(n-1)!}, \leq^!)$

$$Q_n = \langle \mathcal{A}_\chi^!, \chi \in (\mathbb{Z}_n! \setminus \mathbb{Z}_{(n-1)!}, \leq^!) \rangle,$$

such that $\mathcal{A}_\chi^! \simeq \text{Perf}(D)$ for all $\chi \in \mathbb{Z}_n! \setminus \mathbb{Z}_{(n-1)!}$.

By the inductive hypothesis $\mathcal{B}_0 \simeq \text{Perf}(\sqrt{(n-1)!}(X, D))$ carries a psod of type $(\mathbb{Z}_{(n-1)!}, \leq^!)$

$$\mathcal{B}_0 \simeq \text{Perf}(\sqrt{(n-1)!}(X, D)) = \langle \mathcal{A}_\chi^!, \chi \in (\mathbb{Z}_{(n-1)!}, \leq^!) \rangle$$

such that:

- $\mathcal{A}_0^! \simeq \text{Perf}(X)$, and
- $\mathcal{A}_\chi^! \simeq \text{Perf}(D)$ for all $\chi \in \mathbb{Z}_{(n-1)!}^*$.

We can thus write $\text{Perf}(\sqrt{n!}(X, D)) = \langle Q_n, \mathcal{B}_0 \rangle$ as

$$\langle \langle \mathcal{A}_\chi^!, \chi \in (\mathbb{Z}_n! \setminus \mathbb{Z}_{(n-1)!}, \leq^!) \rangle, \langle \mathcal{A}_\chi^!, \chi \in (\mathbb{Z}_{(n-1)!}, \leq^!) \rangle \rangle.$$

Note that the we have a canonical isomorphism of ordered sets

$$(\mathbb{Z}_n! \setminus \mathbb{Z}_{(n-1)!}, \leq^!) * (\mathbb{Z}_{(n-1)!}, \leq^!) \cong (\mathbb{Z}_n!, \leq^!).$$

Thus, by Lemma 2.15 the category $\text{Perf}(\sqrt{n!}(X, D))$ carries a psod of type $(\mathbb{Z}_n!, \leq^!)$,

$$\text{Perf}(\sqrt{n!}(X, D)) = \langle \mathcal{A}_\chi^!, \chi \in (\mathbb{Z}_n!, \leq^!) \rangle.$$

The compatibility with the psod-s of the pull-back along root maps follows by construction. This concludes the proof. □

EXAMPLE 3.6. For $n = 2$, by construction our psod coincides with the one given by Proposition 3.4. For $n = 3$ though, the psod given by that proposition for $\sqrt[3]{(X, D)}$ looks like

$$\text{Perf}(\sqrt[6]{(X, D)}) = \langle \mathcal{A}_{-\frac{5}{6}}, \mathcal{A}_{-\frac{4}{6}}, \mathcal{A}_{-\frac{3}{6}}, \mathcal{A}_{-\frac{2}{6}}, \mathcal{A}_{-\frac{1}{6}}, \text{Perf}(X) \rangle \tag{6}$$

(where $\mathcal{A}_{-\frac{k}{n}}$ is the factor Φ_k in Theorem 4.7 of [3]). The psod that was constructed in the previous proposition, on the other hand, has the form

$$\langle \mathcal{B}_{-\frac{2}{3}}, \mathcal{B}_{-\frac{1}{3}}, \text{Perf}(\sqrt[2]{(X, D)}) \rangle,$$

where $\mathcal{B}_{-\frac{k}{3}} \simeq \text{Perf}(D_2) \simeq \text{Perf}(D)_{-\frac{1}{2}} \oplus \text{Perf}(D)_0$, and $\text{Perf}(\sqrt[2]{(X, D)}) = \langle \mathcal{A}_{-\frac{1}{2}}^!, \text{Perf}(X) \rangle$, embedded in $\text{Perf}(\sqrt[6]{(X, D)})$ via pullback along the projection $\sqrt[6]{(X, D)} \rightarrow \sqrt[2]{(X, D)}$.

Following the proof of the previous proposition, the first term $\mathcal{B}_{-\frac{2}{3}}$ gives us $\mathcal{A}^1_{-\frac{5}{6}} \oplus \mathcal{A}^1_{-\frac{2}{6}}$, the second term $\mathcal{B}_{-\frac{1}{3}}$ gives $\mathcal{A}^1_{-\frac{4}{6}} \oplus \mathcal{A}^1_{-\frac{1}{6}}$, and $\mathcal{A}^1_{-\frac{3}{6}}$ is defined as the image of $\mathcal{A}^1_{-\frac{1}{2}} \subseteq \text{Perf}(\sqrt[2]{(X, D)})$. Overall, the psod looks like

$$\langle \mathcal{A}^1_{-\frac{5}{6}}, \mathcal{A}^1_{-\frac{2}{6}}, \mathcal{A}^1_{-\frac{4}{6}}, \mathcal{A}^1_{-\frac{1}{6}}, \mathcal{A}^1_{-\frac{3}{6}}, \text{Perf}(X) \rangle$$

(note that this reflects exactly the ordering $\leq^!$ on $\mathbb{Z}_{3!}$). In fact we could swap $\mathcal{A}^1_{-\frac{2}{6}}$ and $\mathcal{A}^1_{-\frac{4}{6}}$, and we have $\mathcal{A}^1_{-\frac{k}{6}} = \mathcal{A}^1_{-\frac{k}{6}}$ for all values of k except 3: the subcategory $\mathcal{A}^1_{-\frac{3}{6}}$ does not contain the structure sheaf of D equipped with character $-\frac{3}{6}$, but only thickened versions of it.

Set $\mathbb{Q}/\mathbb{Z}^* := \mathbb{Q}/\mathbb{Z} \setminus \{0\}$.

PROPOSITION 3.7. *The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a psod of type $(\mathbb{Q}/\mathbb{Z}, \leq^!)$*

$$\text{Perf}(\sqrt[\infty]{(X, D)}) = \langle \mathcal{A}^1_\chi, \chi \in (\mathbb{Q}/\mathbb{Z}, \leq^!) \rangle,$$

where:

- $\mathcal{A}^1_0 \simeq \text{Perf}(X)$, and
- $\mathcal{A}^1_\chi \simeq \text{Perf}(D)$ for all $\chi \in \mathbb{Q}/\mathbb{Z}^*$.

Proof. Factorials are cofinal in the filtered set of natural numbers ordered by divisibility. This together with Proposition 2.7 implies that $\text{Perf}(\sqrt[\infty]{(X, D)})$ is the colimit of the directed system of fully-faithful embeddings

$$\text{Perf}(X) \xrightarrow{g_{2!,1!}^*} \text{Perf}(\sqrt[2!]{(X, D)}) \xrightarrow{g_{3!,2!}^*} \text{Perf}(\sqrt[3!]{(X, D)}) \xrightarrow{g_{4!,3!}^*} \dots \quad (7)$$

By Proposition 3.5 the n -th category in the directed system (7) carries a psod of type $(\mathbb{Z}_{n!}, \leq^!)$. Further the structure functors are compatible with the sod-s: the compatibility is witnessed by the inclusion of preordered sets

$$(\mathbb{Z}_{(n-1)!}, \leq^!) \subset (\mathbb{Z}_{n!}, \leq^!),$$

which embeds the indexing set of the sod of $\text{Perf}(\sqrt[(n-1)!]{(X, D)})$ into the indexing set of the sod of $\text{Perf}(\sqrt[n!]{(X, D)})$.

By Proposition 2.4, $\text{Perf}(\sqrt[\infty]{(X, D)})$ is given by the union of the categories making up the directed system (7). This implies that $\text{Perf}(\sqrt[\infty]{(X, D)})$ carries a psod of type

$$\bigcup_{n \in \mathbb{N}} (\mathbb{Z}_{n!}, \leq^!) = (\mathbb{Q}/\mathbb{Z}, \leq^!).$$

It is immediate to see that this sod satisfies the properties (1) and (2) from the statement. This concludes the proof. \square

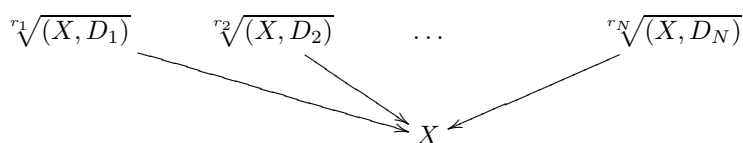
3.2 ROOT STACKS OF SIMPLE NORMAL CROSSING DIVISORS

As shown in [3], the categories of perfect complexes over the root stacks of simple normal crossing divisors carry a canonical sod. We review that result in a slightly different formulation, which is better adapted to our purposes. Further, we extend it to the infinite root stack. We start by fixing notations. Let X be an algebraic stack and D be a simple normal crossing divisor on X . We denote by D_1, \dots, D_N its irreducible components. As recalled in the preliminaries, this gives rise to a log stack (X, D) . Denote by I the set $\{1, \dots, N\}$. The stack X carries a canonical stratification, given by the closed substacks of D obtained as intersections of the D_i . If J is a subset of I , we denote $D_J := \cap_{j \in J} D_j$ if $J \neq \emptyset$, and we set $D_\emptyset := X$ otherwise. Let $S_I := \{J \subseteq I\}$ be the power set of I , and S_I^* denote the subset $S_I \setminus \{\emptyset\}$. We often equivalently regard I as the set of irreducible components of D , S_I as the set of strata of (X, D) , and S_I^* as the set of the strata of positive codimension. We equip S_I with a preorder \leq that keeps track of the inclusions of strata, but it is finer than that. Namely we let \leq be the coarsest preorder on S_I with the following two properties:

- if $J \subseteq J'$, then $J' \leq J$, and
- if d is the dimension of X , then the assignment $(S_I, \leq) \rightarrow (\mathbb{N}, \leq)$ mapping a subset J to $d - |J|$ is an order-reflecting map.

By “order-reflecting map” we mean a map between preordered sets $f : P \rightarrow Q$ such that $f(p) \leq f(p')$ implies $p \leq p'$. Note in particular that these conditions impose both $J \leq J'$ and $J' \leq J$ if $|J| = |J'|$.

Recall that the root stacks of (X, D) are indexed by multi-indices $\vec{r} = (r_1, \dots, r_N) \in \mathbb{N}^N$. For elements \vec{r}, \vec{r}' of \mathbb{N}^N we write $\vec{r} = (r_1, \dots, r_N) \mid (r'_1, \dots, r'_N) = \vec{r}'$ if $r_1 \mid r'_1, \dots, r_N \mid r'_N$. Recall also that the root stack $\sqrt[\vec{r}]{(X, D)}$ in this case can be realized as the limit of the diagram of stacks



If $\vec{r}, \vec{r}' \in \mathbb{N}^N$, $\vec{r} \mid \vec{r}'$, we denote the natural maps between root stacks by

$$g_{\vec{r}', \vec{r}} : \sqrt[\vec{r}]{(X, D)} \longrightarrow \sqrt[\vec{r}']{(X, D)}.$$

3.2.1 ROOT STACKS AND THE STRATA OF (X, D)

Let $\vec{r} \in \mathbb{N}^N$. We will use the following notation.

- Let $J = \{j_1, \dots, j_k\} \subseteq I$ be non-empty. We denote by $\vec{r}_J \in \mathbb{N}^N$ the index vector obtained from \vec{r} by setting to 1 all the entries whose index is not

in J . In formulas, letting \vec{e}_j be the size N vector with j -th entry 1 and all other entries equal to 0, we can write

$$\vec{r}_J = \sum_{j \in J} r_j \vec{e}_j + \sum_{j \notin J} \vec{e}_j.$$

Note that $\vec{r}_J \mid \vec{r}$.

- With slight abuse of notation, for all $i \in I$, we denote by D_{i,r_i} both the universal effective Cartier divisor on the stack $\sqrt[r_i]{(X, D_i)}$ (obtained as reduction of the preimage of D_i), and its pull-back to $\sqrt{\vec{r}}{(X, D)}$. Whenever we use this notation we will specify which of the two meanings is the intended one.

We denote by $D_{J,\vec{r}}$ the limit of the diagram of stacks

$$\begin{array}{ccccc} D_{j_1,r_{j_1}} & & D_{j_2,r_{j_2}} & & \dots & & D_{j_{k-1},r_{j_{k-1}}} & & D_{j_k,r_{j_k}} \\ & \searrow & \searrow & & & & \swarrow & & \swarrow \\ & & \sqrt{\vec{r}}{(X, D)} & & & & & & \end{array} \tag{8}$$

where the arrows are the embeddings $D_{j_i,r_{j_i}} \subset \sqrt{\vec{r}}{(X, D)}$. In general $D_{J,\vec{r}}$ is not a gerbe over D_J , as $D_{J,\vec{r}}$ will have larger isotropy groups along the higher codimensional strata $S \in S_I$ contained in D_J . However $D_{J,\vec{r}}$ is always a gerbe over D_J . We denote by $i_{J,\vec{r}} : D_{J,\vec{r}} \rightarrow \sqrt{\vec{r}}{(X, D)}$ the embedding.

We set $|J, \vec{r}| := \prod_{j \in J} r_j$, and

$$\begin{aligned} \mu_{J,\vec{r}} &:= \bigoplus_{j \in J} \mu_{r_j}, & \mathbb{Z}_{J,\vec{r}} &:= \bigoplus_{j \in J} \mathbb{Z}_{r_j}, & \mathbb{Z}_{J,\vec{r}}^* &:= \bigoplus_{j \in J} (\mathbb{Z}_{r_j} \setminus \{0\}), \\ (\mathbb{Q}/\mathbb{Z})_J &:= \bigoplus_{j \in J} \mathbb{Q}/\mathbb{Z}, & (\mathbb{Q}/\mathbb{Z})_J^* &:= \bigoplus_{j \in J} (\mathbb{Q}/\mathbb{Z} \setminus \{0\}). \end{aligned}$$

Note that $\mathbb{Z}_{J,\vec{r}}$ is the Cartier dual of $\mu_{J,\vec{r}}$.

REMARK 3.8. By definition, the set of strata of a pair (X, D) , where D is simple normal crossing, is the disjoint union of the intersections of the irreducible components of D . Thus strata are in bijection with the subsets of I . However it is sometime convenient to label the index sets we have introduced above via the strata themselves, without making an explicit reference to the corresponding subsets $J \subset I$. Thus if Z is a stratum of (X, D) , then $Z = \cap_{j \in J} D_j$ for some $J \subset I$, and we will sometime denote by

$$\mu_{Z,\vec{r}}, \quad \mathbb{Z}_{Z,\vec{r}}, \quad \mathbb{Z}_{Z,\vec{r}}^*, \quad (\mathbb{Q}/\mathbb{Z})_Z, \quad (\mathbb{Q}/\mathbb{Z})_Z^*$$

the index sets $\mu_{J,\vec{r}}, \mathbb{Z}_{J,\vec{r}}, \mathbb{Z}_{J,\vec{r}}^*, (\mathbb{Q}/\mathbb{Z})_J$, and $(\mathbb{Q}/\mathbb{Z})_J^*$.

LEMMA 3.9. *Let $J \subseteq I$ be a non-empty subset. Then the category $\text{Perf}(D_{J,\vec{r}_J})$ splits as the direct sum of $|J, \vec{r}|$ copies of $\text{Perf}(D_J)$. More precisely there are natural equivalences*

$$\text{Perf}(D_{J,\vec{r}_J}) \simeq \bigoplus_{\chi \in \mathbb{Z}_{J,\vec{r}}} \text{Perf}(D_{J,\vec{r}})_\chi \simeq \text{Perf}(D_J) \otimes \text{Perf}(\mathcal{B}\mu_{J,\vec{r}})$$

and

$$\text{Perf}(D_J) \otimes \text{Perf}(\mathcal{B}\mu_{J,\vec{r}}) \simeq \text{Perf}(D_J) \otimes \bigoplus_{\chi \in \mathbb{Z}_{J,\vec{r}}} \text{Perf}(\kappa).$$

Proof. The proof is similar to the proof of Lemma 3.1. □

3.2.2 THE MAIN THEOREM

Similarly to what we did in Section 3.1 for every $J \subset I$ we identify $\mathbb{Z}_{J,\vec{r}}, \mathbb{Z}_{J,\vec{r}}^*$ with subsets of $(\mathbb{Q}/\mathbb{Z})_I = \mathbb{Q}^I \cap (-1, 0]^I$ via

$$\mathbb{Z}_{J,\vec{r}}^* \subset \mathbb{Z}_{J,\vec{r}} = \bigoplus_{j \in J} \mathbb{Z}_{r_j} \subset \bigoplus_{i \in I} \mathbb{Z}_{r_i} = \mathbb{Z}_{I,\vec{r}} \subset \bigoplus_{i \in I} \mathbb{Q}/\mathbb{Z} = \mathbb{Q}^I \cap (-1, 0]^I,$$

where, on each factor, the inclusion of sets $\mathbb{Z}_{r_i} \subset \mathbb{Q}/\mathbb{Z} = \mathbb{Q} \cap (-1, 0]$ is the one we considered in Section 3.1. We also consider the inclusions

$$(\mathbb{Q}/\mathbb{Z})_J^* \subset (\mathbb{Q}/\mathbb{Z})_J \subset \bigoplus_{j \in J} \mathbb{Q}/\mathbb{Z} \subset \bigoplus_{i \in I} \mathbb{Q}/\mathbb{Z} = \mathbb{Q}^I \cap (-1, 0]^I.$$

The following lemma is immediate.

LEMMA 3.10. *We have decompositions as disjoint union of sets*

$$\mathbb{Z}_{I,\vec{r}} = \coprod_{J \subset I} \mathbb{Z}_{J,\vec{r}}^* \quad (\mathbb{Q}/\mathbb{Z})_I = \coprod_{J \subset I} (\mathbb{Q}/\mathbb{Z})_J^*. \quad \square$$

By Lemma 3.10 if χ is in $\mathbb{Z}_{I,\vec{r}}$ there is a unique $J \subset I$ such that χ is in $\mathbb{Z}_{J,\vec{r}}^*$, and similarly for $(\mathbb{Q}/\mathbb{Z})_I$ and $(\mathbb{Q}/\mathbb{Z})_J^*$.

DEFINITION 3.11. We equip $(\mathbb{Q}/\mathbb{Z})_I = \mathbb{Q}^I \cap (-1, 0]^I$ with the product partial order \leq : let

$$\chi = (\chi_1, \dots, \chi_N) \quad \text{and} \quad \chi' = (\chi'_1, \dots, \chi'_N) \quad \text{be in} \quad \mathbb{Q}^I \cap (-1, 0]^I$$

then we set $\chi \leq \chi'$ if $\chi_l \leq \chi'_l$ for all $l = 1, \dots, N$ for the restriction of the standard ordering of the real numbers. This restricts to a preorder on $\mathbb{Z}_{J,\vec{r}}, \mathbb{Z}_{J,\vec{r}}^*, (\mathbb{Q}/\mathbb{Z})_J$ and $(\mathbb{Q}/\mathbb{Z})_J^*$.

We introduce the following notations:

- Let $\vec{0} = (0, \dots, 0) \in \mathbb{Z}_{I,\vec{r}}$. We denote by $\mathcal{A}_{\vec{0}} \subset \text{Perf}(\sqrt[\vec{r}]{(X, D)})$ the image of $\text{Perf}(X)$ under $g_{\vec{r},1}^*$.

- Let $J \in S_I^*$ and let χ be in $\mathbb{Z}_{J,\bar{r}}^*$. We denote by $\mathcal{A}_\chi^J \subset \text{Perf}(\sqrt[r]{(X, D)})$ the image of $\text{Perf}(D_J)$ under the functor $\Phi_{J,\chi}$, which is defined as the composite

$$\begin{array}{c}
 \text{Perf}(D_J) \xrightarrow{(a)} \text{Perf}(D_{J,\bar{r}})_\chi \\
 \downarrow (b) \\
 \text{Perf}(D_{J,\bar{r}}) \\
 \downarrow (i_{J,\bar{r}})_* \\
 \text{Perf}(\sqrt[r]{(X, D)}) \\
 \downarrow (g_{\bar{r},\bar{r}})^* \\
 \text{Perf}(\sqrt[r]{(X, D)})
 \end{array}$$

where equivalence (a) and inclusion (b) are explained in Lemma 3.9.

- We denote by \mathcal{A}_J the subcategory of $\text{Perf}(\sqrt[r]{(X, D)})$ generated by the subcategories \mathcal{A}_χ^J for $\chi \in \mathbb{Z}_{J,\bar{r}}^*$.

The following statement is a rephrasing of [3, Theorem 4.9].

PROPOSITION 3.12.

1. The category $\text{Perf}(\sqrt[r]{(X, D)})$ has a psod of type S_I , $\text{Perf}(\sqrt[r]{(X, D)}) = \langle \mathcal{A}_J, J \in S_I \rangle$, such that:

- For all J , the subcategory \mathcal{A}_J is admissible.
- For all $J \in S_I^*$, the subcategory \mathcal{A}_J has a psod of type $(\mathbb{Z}_{J,\bar{r}}^*, \leq)$

$$\mathcal{A}_J = \langle \mathcal{A}_\chi^J, \chi \in (\mathbb{Z}_{J,\bar{r}}^*, \leq) \rangle$$

and, for all $\chi \in \mathbb{Z}_{J,\bar{r}}^*$, there is an equivalence $\mathcal{A}_\chi^J \simeq \text{Perf}(D_J)$.

2. The category $\text{Perf}(\sqrt[r]{(X, D)})$ has a psod of type $(\mathbb{Z}_{I,\bar{r}}, \leq)$

$$\text{Perf}(\sqrt[r]{(X, D)}) = \langle \mathcal{A}_\chi^J, \chi \in (\mathbb{Z}_{I,\bar{r}}, \leq) \rangle,$$

where $\mathcal{A}_\emptyset \simeq \text{Perf}(X)$ and for all $\chi \in \mathbb{Z}_{J,\bar{r}}^*$ there is an equivalence $\mathcal{A}_\chi^J \simeq \text{Perf}(D_J)$.

We will equip the set $(\mathbb{Q}/\mathbb{Z})_I = \mathbb{Q}^I \cap (-1, 0]^I$ with a total order $\leq^!$ (different from the one of Definition 3.11) which generalizes the preorder $(\mathbb{Q}/\mathbb{Z}, \leq^!)$ that we introduced in Section 3.1. We can write every element χ in $\mathbb{Q}^I \cap (-1, 0]^I$ as

$$\chi = \left(-\frac{p_1}{n!}, \dots, -\frac{p_N}{n!} \right)$$

for some p_1, \dots, p_N in \mathbb{N} and $n \in \mathbb{N}$. This expression is unique if we require n to be as small as possible, and we call this the *normal factorial form*.

DEFINITION 3.13. Let

$$\chi = \left(-\frac{p_1}{n!}, \dots, -\frac{p_N}{n!}\right), \quad \chi' = \left(-\frac{q_1}{m!}, \dots, -\frac{q_N}{m!}\right) \in \mathbb{Q}^I \cap (-1, 0]^I$$

be in normal factorial form. We write $\chi \leq^! \chi'$ if:

- $n > m$, or
- $n = m$ and $-\frac{p_i}{n!} \leq^! -\frac{q_i}{n!}$ in $\mathbb{Z}_{n!}$ for all $i = 1, \dots, N$, where $\leq^!$ is the ordering defined in Section 3.1.

For all $\vec{r} \in \mathbb{N}^N$ we obtain an induced ordering $\leq^!$ on $\mathbb{Z}_{I, \vec{r}}$ and $\mathbb{Z}_{I, \vec{r}}^*$.

If $n \in \mathbb{N}$ we set $\vec{n} := (n, \dots, n)$ and $\vec{n}! := (n!, \dots, n!) \in \mathbb{N}^N$.

PROPOSITION 3.14.

1. The category $\text{Perf}(\sqrt[n!]{(X, D)})$ has a collection of subcategories $\mathcal{A}_J^!$, $J \in S_I$, such that:

- For all J , the subcategory $\mathcal{A}_J^!$ is admissible.
- For all $J \in S_I^*$, the subcategory $\mathcal{A}_J^!$ has a psod of type $(\mathbb{Z}_{J, \vec{n}!}^*, \leq^!)$

$$\mathcal{A}_J^! = \langle \mathcal{A}_\chi^{J,!}, \chi \in (\mathbb{Z}_{J, \vec{n}!}^*, \leq^!) \rangle$$

and, for all $\chi \in \mathbb{Z}_{J, \vec{n}!}^*$, there is an equivalence $\mathcal{A}_\chi^{J,!} \simeq \text{Perf}(D_J)$.

2. The category $\text{Perf}(\sqrt[n!]{(X, D)})$ has a psod of type $(\mathbb{Z}_{I, \vec{n}!}, \leq^!)$

$$\text{Perf}(\sqrt[n!]{(X, D)}) = \langle \mathcal{A}_\chi^{J,!}, \chi \in (\mathbb{Z}_{I, \vec{n}!}, \leq^!) \rangle,$$

where $\mathcal{A}_\emptyset^! \simeq \text{Perf}(X)$ and for all $\chi \in \mathbb{Z}_{J, \vec{n}!}^*$ there is an equivalence $\mathcal{A}_\chi^{J,!} \simeq \text{Perf}(D_J)$.

3. For all $n \in \mathbb{N}$, $g_{(n+1)!, \vec{n}!}^* : \text{Perf}(\sqrt[n!]{(X, D)}) \longrightarrow \text{Perf}(\sqrt[(n+1)!]{(X, D)})$ is compatible with the psod-s.

The proof of Proposition 3.14 that we give below depends on a somewhat involved inductive argument. A much simpler proof is possible, at the price of a mild reduction of generality which still covers most examples of interest. We sketch it in Remark 3.15 below.

Proof. By Lemma 3.10 $\mathbb{Z}_{I, \vec{r}} = \coprod_{J \subset I} \mathbb{Z}_{J, \vec{r}}^*$ and thus the Proposition gives a description of all the factors appearing in the psod of $\text{Perf}(\sqrt[n!]{(X, D)})$.

It is actually more convenient to prove first part (2) of the Proposition, and deduce from there part (1). The compatibility with the psod-s, part (3), follows automatically. The proof involves a nested induction, first on the number N of irreducible components of D , and then on the index n appearing in the statement of Proposition 3.14. Let us clarify the structure of the induction.

- (a) The basis step of the induction on N consists in the proof of the statement of Proposition 3.14 for $N = 1$ and arbitrary n . This is given by Proposition 3.5.
- (b) The inductive step involves proving Proposition 3.14 in the case of a divisor D with N irreducible components: as inductive hypothesis we assume that Proposition 3.14 holds, for all $n \in \mathbb{N}$, in the case of a divisor D with M irreducible components, where M is any integer smaller than N .
- (c) We establish inductive step (b) via a second induction, this time on the index n . We will spend the rest of the proof explaining the basis step $n = 2$ and the inductive step $n - 1 \rightarrow n$. This will imply inductive step (b) and conclude the proof.

BASIS: $n = 2$. We have the sod on $\text{Perf}(\sqrt[2]{(X, D)})$ given by Proposition 3.12, with the required subcategories $\mathcal{A}_J^!$ for $J \in S_I^*$ and $\mathcal{A}_\chi^{J,!}$ with $\chi \in \mathbb{Z}_{I, \vec{2}}$.

INDUCTIVE STEP: $n - 1 \rightarrow n$. The proof is similar to the one of Proposition 3.5, thus we limit ourselves to an abbreviated treatment of the argument. We use the natural identification

$$\sqrt[n]{(X, D)} \simeq \sqrt[n]{\left(\sqrt[(n-1)]{(X, D)}, D_{\overrightarrow{(n-1)!}}\right)}. \tag{9}$$

Applying Proposition 3.12 to (9) yields a psod of type (S_I, \leq)

$$\text{Perf}(\sqrt[n]{(X, D)}) = \langle \mathcal{B}_J, J \in (S_I, \leq) \rangle.$$

Additionally each summand \mathcal{B}_J for $J \in S_I^*$ carries a psod

$$\mathcal{B}_J = \langle \mathcal{B}_\zeta, \zeta \in (\mathbb{Z}_{J, \vec{n}}^*, \leq) \rangle,$$

where for all $\zeta \in \mathbb{Z}_{J, \vec{n}}^*$ there is an equivalence $\mathcal{B}_\zeta \simeq \text{Perf}\left(D_{J, \overrightarrow{(n-1)!}}\right)$. We will realize the summands $\mathcal{A}_\chi^{J,!}$ appearing in the statement of Proposition 3.14 as semi-orthogonal factors of the categories \mathcal{B}_ζ .

Fix J in S_I^* . If $D_J = \emptyset$ then $\mathcal{B}_J = 0$, and we set $\mathcal{A}_\chi^{J,!} := 0$ for all $\chi \in \mathbb{Z}_{J, \vec{n}}^*$. Assume next that D_J is non-empty. Set $L := I \setminus J$ and $M := |L|$. The stratum D_J carries a simple normal crossing divisor ¹

$$D^L := \bigcup_{i \in L} (D_i \cap D_J) \subset D_J$$

Denote by $\overrightarrow{(n-1)!}^L \in \mathbb{N}^L$ the diagonal vector with entries all equal to $(n-1)!$

$$\overrightarrow{(n-1)!}^L := ((n-1)!, \dots, (n-1)!) \in \mathbb{N}^L.$$

¹Some, or even all, the intersections $D_i \cap D_J$, $i \in L$, might be empty. We can nonetheless argue as if D^L had M distinct irreducible components: the empty strata will give rise to zero categories, and therefore will give no contribution to the psod-s.

The substack $D_{J, \overrightarrow{(n-1)!}}$ is a $\mu_{J, \overrightarrow{(n-1)!}}$ -gerbe over the $\overrightarrow{(n-1)!}^L$ -th root stack of D_J with respect to D^L . Thus by Lemma 3.9 we have a decomposition

$$\mathrm{Perf}(D_{J, \overrightarrow{(n-1)!}}) \simeq \bigoplus_{\mathbb{Z}_{J, \overrightarrow{(n-1)!}}} \mathrm{Perf}\left(\overrightarrow{(n-1)!}^L \sqrt{(D_J, D^L)}\right) =: \bigoplus_{\xi \in \mathbb{Z}_{J, \overrightarrow{(n-1)!}}} \mathcal{B}_{\zeta, \xi}.$$

Additionally, since $M < N$, we can assume by the inductive hypothesis that Proposition 3.14 applies to the root stack $\overrightarrow{(n-1)!}^L \sqrt{(D_J, D^L)}$. This gives us a psod

$$\mathcal{B}_{\zeta, \xi} \simeq \mathrm{Perf}\left(\overrightarrow{(n-1)!}^L \sqrt{(D_J, D^L)}\right) = \left\langle \mathcal{B}_{\zeta, \xi, \rho}, \rho \in \left(\mathbb{Z}_{L, \overrightarrow{(n-1)!}}, \leq^!\right) \right\rangle.$$

Similarly to the proof of Proposition 3.5, we identify the subcategories $\mathcal{B}_{\zeta, \xi, \rho}$ with factors $\mathcal{A}_\chi^{J,!}$ for $\chi \in \mathbb{Z}_{I, \vec{n}!}$ appearing in the statement of the proposition by setting

$$\mathcal{A}_{\frac{\zeta}{\overrightarrow{(n-1)!} + \xi + \rho}}^{J,!} := \mathcal{B}_{\zeta, \xi, \rho}$$

where we are identifying ζ , ξ and ρ with elements of $\mathbb{Z}_{I, \vec{n}!}$ in the natural manner. Note the value of $\frac{\zeta}{\overrightarrow{(n-1)!} + \xi + \rho}$ ranges exactly over the set $\mathbb{Z}_{I, \vec{n}!} \setminus \mathbb{Z}_{I, \overrightarrow{(n-1)!}}$ for $\zeta \in \mathbb{Z}_{J, \vec{n}!}^*$, $\xi \in \mathbb{Z}_{J, \overrightarrow{(n-1)!}}$ and $\rho \in \mathbb{Z}_{L, \overrightarrow{(n-1)!}}$.

Now, by the inductive hypothesis $\mathcal{B}_{\vec{0}} \simeq \mathrm{Perf}\left(\overrightarrow{(n-1)!} \sqrt{(X, D)}\right)$ carries a psod of type $(\mathbb{Z}_{I, \overrightarrow{(n-1)!}}, \leq^!)$

$$\mathcal{B}_{\vec{0}} \simeq \mathrm{Perf}\left(\overrightarrow{(n-1)!} \sqrt{(X, D)}\right) = \left\langle \mathcal{A}_\chi^{J,!}, \chi \in (\mathbb{Z}_{I, \overrightarrow{(n-1)!}}, \leq^!) \right\rangle,$$

such that:

- $\mathcal{A}_{\vec{0}}^! \simeq \mathrm{Perf}(X)$, and
- $\mathcal{A}_\chi^{J,!} \simeq \mathrm{Perf}(D_J)$ for all $\chi \in \mathbb{Z}_{J, \overrightarrow{(n-1)!}}^*$.

As in the proof of Proposition 3.5 we conclude the categories $\mathcal{A}_\chi^{J,!}$, $\chi \in \mathbb{Z}_{I, \vec{n}!}$ that we have just constructed make up a psod of $\mathrm{Perf}\left(\overrightarrow{(n-1)!} \sqrt{(X, D)}\right)$ of type $(\mathbb{Z}_{I, \vec{n}!}, \leq^!)$ and this concludes the proof of part (2) of the Proposition.

Now it is easy to proceed backwards and prove part (1). For every $J \in S_I$ we define $\mathcal{A}_J^!$ as the subcategory of $\mathrm{Perf}\left(\overrightarrow{(n-1)!} \sqrt{(X, D)}\right)$ generated by the subcategories $\mathcal{A}_\chi^{J,!}$ as χ varies in $\mathbb{Z}_{J, \vec{n}!}^* \subset \mathbb{Z}_{I, \vec{n}!}$. By construction the subcategories $\mathcal{A}_J^!$ have the properties required by part (1) of the Proposition, and this concludes the proof. □

REMARK 3.15. It is possible to give a much simpler proof of Proposition 3.14 leveraging formal properties of the category of perfect complexes established in [2]. This however requires to reduce generality, and assume that X is a *perfect*

stack [2, Definition 3.2]: this is a large class of stacks containing for instance all quasi-compact schemes with affine diagonal.

Let us sketch the argument assuming for simplicity that $D = D_1 \cup D_2$ has two components. Recall that there is an equivalence

$$\sqrt[n!]{(X, D)} \simeq \sqrt[n!]{(X, D_1)} \times_X \sqrt[n!]{(X, D_2)}.$$

Since X is perfect, so are its root stacks. Then by [2, Theorem 1.2] there is an equivalence of categories

$$\text{Perf} \left(\sqrt[n!]{(X, D)} \right) \simeq \text{Perf} \left(\sqrt[n!]{(X, D_1)} \right) \otimes_{\text{Perf}(X)} \text{Perf} \left(\sqrt[n!]{(X, D_2)} \right). \tag{10}$$

Now by Proposition 3.12 we have psod-s

$$\text{Perf} \left(\sqrt[n!]{(X, D_1)} \right) = \langle \mathcal{B}_\xi^!, \xi \in (\mathbb{Z}_{n!}, \leq^!) \rangle, \tag{11}$$

$$\text{Perf} \left(\sqrt[n!]{(X, D_2)} \right) = \langle \mathcal{C}_{\xi'}^!, \xi' \in (\mathbb{Z}_{n!}, \leq^!) \rangle.$$

Equivalence (10) then implies that $\text{Perf} \left(\sqrt[n!]{(X, D)} \right)$ carries a psod whose semi-orthogonal factors are the tensor products of the factors appearing in (11): more precisely, for all $\chi = (\xi, \xi') \in \mathbb{Z}_{n!} \oplus \mathbb{Z}_{n!}$ set $\mathcal{A}_\chi := \mathcal{B}_\xi \otimes_{\text{Perf}(X)} \mathcal{C}_{\xi'}$. Then $\text{Perf} \left(\sqrt[n!]{(X, D)} \right)$ carries a psod with the categories \mathcal{A}_χ as factors

$$\text{Perf} \left(\sqrt[n!]{(X, D)} \right) = \langle \mathcal{A}_\chi = \mathcal{B}_\xi \otimes_{\text{Perf}(X)} \mathcal{C}_{\xi'}, \chi = (\xi, \xi') \in (\mathbb{Z}_{n!} \oplus \mathbb{Z}_{n!}, \leq^!) \rangle \tag{12}$$

For all $\xi, \xi' \in \mathbb{Z}_{n!}$ Theorem 1.2 of [2] yields equivalences

- $\mathcal{A}_{(0,0)} = \mathcal{B}_0 \otimes_{\text{Perf}(X)} \mathcal{C}_0 = \text{Perf}(X) \otimes_{\text{Perf}(X)} \text{Perf}(X) \simeq \text{Perf}(X)$,
- $\mathcal{A}_{(\xi,0)} = \mathcal{B}_\xi \otimes_{\text{Perf}(X)} \mathcal{C}_0 \simeq \text{Perf}(D_1) \otimes_{\text{Perf}(X)} \text{Perf}(X) \simeq \text{Perf}(D_1)$,
- $\mathcal{A}_{(0,\xi')} = \mathcal{B}_0 \otimes_{\text{Perf}(X)} \mathcal{C}_{\xi'} \simeq \text{Perf}(X) \otimes_{\text{Perf}(X)} \text{Perf}(D_2) \simeq \text{Perf}(D_2)$,
- $\mathcal{A}_{(\xi,\xi')} = \mathcal{B}_\xi \otimes_{\text{Perf}(X)} \mathcal{C}_{\xi'} \simeq \text{Perf}(D_1) \otimes_{\text{Perf}(X)} \text{Perf}(D_2) \simeq \text{Perf}(D_{\{12\}})$.

Thus psod (12) has the same properties required by Proposition 3.14 and in fact it is easy to see that it coincides with it.

THEOREM 3.16.

1. The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a collection of subcategories $\mathcal{A}_J^!$, $J \in S_I$, such that:

- For all J , the subcategory $\mathcal{A}_J^!$ is admissible.
- For all $J \in S_I^*$ the category $\mathcal{A}_J^!$ has a psod of type $((\mathbb{Q}/\mathbb{Z})_J^*, \leq^!)$

$$\mathcal{A}_J^! = \langle \mathcal{A}_\chi^{J,!}, \chi \in ((\mathbb{Q}/\mathbb{Z})_J^*, \leq^!) \rangle,$$

and, for all $\chi \in (\mathbb{Q}/\mathbb{Z})_J^*$, there is an equivalence $\mathcal{A}_\chi^{J,!} \simeq \text{Perf}(D_J)$.

2. The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a psod of type $((\mathbb{Q}/\mathbb{Z})_I, \leq^!)$

$$\text{Perf}(\sqrt[\infty]{(X, D)}) = \langle \mathcal{A}_X^{J,!}, \chi \in ((\mathbb{Q}/\mathbb{Z})_I, \leq^!) \rangle,$$

and for all $\chi \in (\mathbb{Q}/\mathbb{Z})_J^*$ there is an equivalence $\mathcal{A}_X^{J,!} \simeq \text{Perf}(D_J)$.

Proof. The proof is the same as the one of Proposition 3.7. □

REMARK 3.17. The psod-s constructed in Proposition 3.7 and Theorem 3.16 depend on the choice of a directed system which is cofinal in \mathbb{N}^N preordered by divisibility. We chose the set of diagonal vectors with factorial entries, but different choices were possible and would have given rise to different psod-s. At the level of additive invariants, and in particular K-theory, all choices yield identical splitting formulas. All these different psod-s should be connected via mutation patterns which give rise to canonical identifications of semi-orthogonal factors: we leave this to future investigation.

4 BEYOND SIMPLE NORMAL CROSSING DIVISORS

In this section we study sod-s of infinite root stacks of general normal crossing divisors, and of divisors with simplicial singularities. In both cases we will be able to reduce to the simple normal crossing setting. At the same time genuinely new phenomena will arise.

In the general normal crossing case, the factors making up the psod on the infinite root stacks are not equivalent to the category of perfect complexes on the strata, but on the *normalization of the strata*. In [46] it is proven that the categories of perfect complexes on infinite root stacks are invariant under some class of log blow-ups. This is the key ingredient in the construction of these psod-s. However since the results in [46] require working over a field of characteristic 0, we are bound to make the same assumption here.

Constructing sod-s for infinite root stacks in the setting of divisors with what we call “simple simplicial singularities” is more straightforward. We do this in Section 4.2. The proof depends on a cofinality argument which allows us to reduce directly to the (simple) normal crossing case.

4.1 ROOT STACKS OF NORMAL CROSSING DIVISORS

Throughout this section we work over a field κ of characteristic zero. We will establish sod-s for root stacks of normal crossing divisors which are not necessarily simple. As explained in the preliminaries (Section 2.1.3) root stacks of non-simple normal crossing divisors cannot be defined via an iterated root construction, as in [3]. This has to do with the fact that the self-intersections of the divisors create higher codimensional strata which are not correctly accounted for if we just take iterated roots of the divisors themselves. We rely instead on the general definition of root stacks of logarithmic schemes introduced by Borne and Vistoli [6].

Let X be an algebraic stack, and D a normal crossing divisor in X . In this section we consider the r -th root stacks $\sqrt[r]{(X, D)}$ for $r \in \mathbb{N}$ described in Section 2.1.3, and the infinite root stack $\sqrt[\infty]{(X, D)} = \varprojlim_r \sqrt[r]{(X, D)}$ of the log stack (X, D) . Note that if D happens to be simple normal crossings, then $\sqrt[r]{(X, D)}$ coincides with the root stack $\sqrt{\vec{r}}{(X, D)}$ of the previous section, where \vec{r} is the vector $(r, \dots, r) \in \mathbb{N}^N$ and N is the number of irreducible components of D .

4.1.1 STRICTIFICATION OF NORMAL CROSSING DIVISORS

Note that D equips X with a canonical stratification in locally closed substacks. This stratification is most easily expressed by saying that the locally closed strata are the connected substacks of X where the rank of the log structure (i.e. of the sheaf $\overline{M} = M/\mathcal{O}_X^\times$, using standard notation for log structures) remains constant. These are also the connected substacks where the number of points in the fiber of the normalization map $\tilde{D} \rightarrow D$ remains constant. We denote S_D the set of the closures of these strata, and we set $S_D^* := S_D - \emptyset$.

LEMMA 4.1. *Let X be an algebraic stack, and D a normal crossing divisor in X . Then there exists a finite sequence of log blow-ups*

$$(\tilde{X}, \tilde{D}) := (X_n, D_{X_n}) \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} (X_1, D_{X_1}) \xrightarrow{\pi_1} (X_0, D_{X_0}) := (X, D)$$

with the following properties:

1. For all $0 \leq i \leq n$, X_i is an algebraic stack with a normal crossing divisor D_i .
2. Denote by $S_{D_{X_i}}$ the set of closures of the strata. Then the map $\pi_i : (X_i, D_{X_i}) \rightarrow (X_{i-1}, D_{X_{i-1}})$ is the blow-up of a regular stratum $S \in S_{D_{X_{i-1}}}$.
3. The divisor \tilde{D} of \tilde{X} is simple normal crossing.

We say that (\tilde{X}, \tilde{D}) is a *strictification* of (X, D) . A proof of Lemma 4.1 is given in [13]. Let us explain briefly how to construct a strictification (\tilde{X}, \tilde{D}) , referring to [13] for further details.

Let $S \in S_D$ be a stratum of codimension d . We say that S is *non simple* if S is not a connected component of the intersection of d distinct irreducible components of D . Denote by Z_d be the substack of X given by the union of the non simple strata of X of codimension at most d . Let m be the maximal index such that $Z_m \neq \emptyset$. The substack Z_m is regular. There is a sequence of inclusions

$$Z_m \subset Z_{m-1} \subset \dots \subset Z_1 = D$$

Let $\pi_1 : (X_1, D_1) \rightarrow (X, D)$ be the blow-up at Z_m . The strict transform of Z_{m-1} under π_1 is regular, we denote it by \tilde{Z}_{m-1} . We denote by $\pi_2 : (X_2, D_2) \rightarrow (X_1, D_1)$ the blow-up at \tilde{Z}_{m-1} . The strictification (\tilde{X}, \tilde{D}) is obtained by iterating this procedure for $m - 1$ steps.

Let \tilde{I} be the set of irreducible components of (\tilde{X}, \tilde{D}) , and let $S_{\tilde{I}}$ be the set of strata. The iterated blow-up

$$\tilde{\pi} := \pi_n \circ \dots \circ \pi_1 : (\tilde{X}, \tilde{D}) \longrightarrow (X, D)$$

maps strata of (\tilde{X}, \tilde{D}) to strata of (X, D) . Let S in S_D be the image of the stratum \tilde{S} in $S_{\tilde{I}}$. The restriction of $\tilde{\pi}$ to \tilde{S}

$$\tilde{\pi}|_{\tilde{S}} : \tilde{S} \longrightarrow S$$

can be described explicitly in terms of the geometry of iterated blow-ups. This requires some combinatorial book-keeping which, although elementary, quickly becomes quite intricate.

For simplicity we will limit ourselves instead to give a qualitative description of the geometry of the strata of (\tilde{X}, \tilde{S}) . We introduce an auxiliary class of stacks whose geometry is related in a simple way to the geometry of the strata of (X, D) . This is done in Definition 4.2. It will be clear that all strata of (\tilde{X}, \tilde{S}) are of this form, and this will be sufficient for our applications.

DEFINITION 4.2. Let Y be an algebraic stack. We define recursively what it means for Y to be of type Z_i , where $m \geq i \geq 1$, starting from $i = m$:

1. We say that Y is of type Z_m if there exists a stratum $S \in S_D$, $S \subset Z_m \subset X$, and maps

$$Y = Y_\gamma \xrightarrow{\gamma} Y_\beta \xrightarrow{\beta} Y_\alpha \xrightarrow{\alpha} S$$

where α, β , and γ are morphisms of the following type.

- (a) Y_α is the disjoint union of finitely many copies of the normalization of S . The map $\alpha : Y_\alpha \rightarrow S$ restricts to the normalization map on each copy.
- (b) $\beta : Y_\beta \rightarrow Y_\alpha$ is a projective bundle.
- (c) $\gamma : Y_\gamma \rightarrow Y_\beta$ is an iterated blow-up having the following two properties:
 - it factors as a composite of blow-ups along regular centers, and
 - each of these centers is isomorphic to a projective bundle over a stratum $S' \in S_D$ such that $S' \subseteq S$.

2. We say that X is of type Z_i if there exists a stratum $S \in S_D$, $S \subset Z_i$, and maps

$$Y = Y_\gamma \xrightarrow{\gamma} Y_\beta \xrightarrow{\beta} Y_\alpha \xrightarrow{\alpha} S$$

where α, β are as above, and $\gamma : Y_\gamma \rightarrow Y_\beta$ factors as a composite of blow-ups along regular centers, and each of these centers is a stack of type Z_j for some $m \geq j > i$.

We say that Y is adapted to S_D if it is of type Z_i for some $m \geq i \geq 1$.

It follows from the definition that if X is adapted to S_D then X is regular.

LEMMA 4.3. *If X is adapted to S_D , $\text{Perf}(X)$ admits a sod such that all its semi-orthogonal factors are of the form $\text{Perf}(S^\vee)$, where S^\vee is the normalization of a stratum S in S_D .*

Proof. The category of perfect complexes over a disjoint union decomposes as a direct sum of the categories corresponding to each connected component. If $E \rightarrow X$ is a projective bundle then $\text{Perf}(E)$ admits a sod where all semi-orthogonal factors are equivalent to $\text{Perf}(X)$ [31, Example 3.2]. By Orlov blow-up formula, if X is regular and $Y \rightarrow X$ is a blow-up along a regular center $Z \subset X$, $\text{Perf}(Y)$ admits a sod whose factors are equivalent to either $\text{Perf}(X)$ or $\text{Perf}(Z)$ [31, Theorem 3.4]. Then the statement follows immediately from Definition 4.2. \square

LEMMA 4.4. *All strata \tilde{S} in $S_{\tilde{I}}$ are adapted to S_D .*

Proof. This follows because the strictification (\tilde{X}, \tilde{D}) is an iterated blow-up of (X, D) . \square

4.1.2 REDUCTION TO THE SIMPLE NORMAL CROSSING CASE

The next result, which was proved in [46], allows us to reduce to the simple normal crossing case, which was studied in Section 3.2. We stress that, as in [46], we need to assume that the ground ring κ is a field of characteristic zero.

PROPOSITION 4.5 (Proposition 3.9, [46]). *Let $(X', D') \rightarrow (X, D)$ be a log blow-up such that (X', D') is again an algebraic stack with a normal crossing divisor. Then there is an equivalence of ∞ -categories*

$$\text{Perf}(\sqrt[\infty]{(X', D')}) \simeq \text{Perf}(\sqrt[\infty]{(X, D)}). \tag{13}$$

Proof. Proposition 3.9 of [46] is formulated in terms of bounded derived categories and assumes that X is regular: however the same proof works in this more general setting. \square

Let X be an algebraic stack and let D be a normal crossing divisor. Let (\tilde{X}, \tilde{D}) be the strictification of (X, D) constructed in Section 4.1.1. As before let \tilde{I} be the set of divisors of (\tilde{X}, \tilde{D}) , and let $S_{\tilde{I}}$ be the preordered set of strata, and $S_{\tilde{I}}^* = S_{\tilde{I}} - \{\emptyset\}$.

THEOREM 4.6.

1. *The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a collection of subcategories $\mathcal{A}_J^!$, $J \in S_{\tilde{I}}$, such that:*
 - *For all J , $\mathcal{A}_J^!$ is admissible.*
 - *The category $\mathcal{A}_{\emptyset}^! = \text{Perf}(\tilde{X})$ has a psod whose semi-orthogonal factors are*

- (a) $\text{Perf}(X)$, and
- (b) factors of the form $\text{Perf}(S^\vee)$, where S^\vee is the normalization of a stratum S in S_D^* .
- For all $J \in S_{\tilde{I}}^*$ the category $\mathcal{A}_J^!$ has a psod of type $((\mathbb{Q}/\mathbb{Z})_J^*, \leq^!)$

$$\mathcal{A}_J^! = \langle \mathcal{A}_\chi^{J,!}, \chi \in ((\mathbb{Q}/\mathbb{Z})_J^*, \leq^!) \rangle.$$

Additionally, for all $\chi \in (\mathbb{Q}/\mathbb{Z})_J^*$, $\mathcal{A}_\chi^{J,!}$ has a psod whose semi-orthogonal factors are of the form $\text{Perf}(S^\vee)$, where S^\vee is the normalization of a stratum S in S_D^* .

2. The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a psod whose semi-orthogonal factors are given by the factors of the psod-s of $\mathcal{A}_{\emptyset}^! = \text{Perf}(\tilde{X})$ and $\mathcal{A}_\chi^{J,!}$ described above.

Proof. By Proposition 4.5 there is an equivalence $\text{Perf}(\sqrt[\infty]{(\tilde{X}, \tilde{D})}) \simeq \text{Perf}(\sqrt[\infty]{(X, D)})$. Since (\tilde{X}, \tilde{D}) is simple normal crossing, we can apply Theorem 3.16. Let us use the notations of Theorem 3.16: recall that the semi-orthogonal factors $\mathcal{A}_\chi^{J,!}$, $J \subset \tilde{I}$, considered there are equivalent to $\text{Perf}(\tilde{D}_J)$. The stack \tilde{D}_J is a stratum in $S_{\tilde{I}}$ and thus, by Lemma 4.4, is adapted to S_D . By Lemma 4.3, $\text{Perf}(\tilde{D}_J)$ carries a psod whose factors are of the form $\text{Perf}(S^\vee)$, where S^\vee is the normalization of a stratum S in S_D . This concludes the proof. □

4.2 ROOT STACKS OF DIVISORS WITH SIMPLICIAL SINGULARITIES

Let X be a log scheme with a *simplicial* log structure. This means that it is fine and saturated, and the stalks of the sheaf $\overline{M} = M/\mathcal{O}_X^\times$ are simplicial monoids. Recall that a sharp fine saturated monoid P is simplicial if the extremal rays of the rational cone $P_{\mathbb{Q}} \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ are linearly independent.

PROPOSITION 4.7. *Let X be a log scheme with simplicial log structure. Then there is a canonical minimal Kummer extension $\overline{M} \subseteq \mathcal{F}$ where \mathcal{F} is a coherent sheaf of monoids on X with free stalks.*

The minimality in the statement means that every Kummer extension $\overline{M} \rightarrow N$ where N is coherent with free stalks factors uniquely as $\overline{M} \rightarrow \mathcal{F} \rightarrow N$.

Proof. Because of the uniqueness part of the statement, it suffices to do the construction locally. Assume therefore that we have a global chart $X \rightarrow [\text{Spec } \kappa[P]/D(P^{\text{gp}})]$ for X (in the sense of [6, Section 3.3]), where P is a simplicial monoid, and $D(P^{\text{gp}})$ denotes the Cartier dual $\text{Hom}(P^{\text{gp}}, \mathbb{G}_m)$ of P^{gp} . Let p_1, \dots, p_n be the primitive generators in the lattice P^{gp} of the extremal rays of the rational cone $P_{\mathbb{Q}} \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by P . These are indecomposable elements of P . Let q_1, \dots, q_m be the remaining indecomposable elements of P . By simpliciality of P , for every index $i = 1, \dots, m$ we can express q_i uniquely

as a rational linear combination of the p_j . Let us write $q_i = \sum_{j=1}^n \frac{a_{ij}}{b_{ij}} \cdot p_j$ where for every pair of indices $\{i, j\}$, a_{ij} and b_{ij} are coprime non-negative integers. For every $j = 1, \dots, n$ let c_j be the lcm of the set $\{b_{1j}, \dots, b_{mj}\}$. Consider then the submonoid of $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the vectors $p_1/c_1, \dots, p_n/c_n \in P_{\mathbb{Q}}$. This is a free monoid \mathbb{N}^n containing P , and the inclusion $P \rightarrow \mathbb{N}^n$ is a Kummer morphism. It is easy to check that it is minimal among Kummer morphisms from P to a free monoid.

Now consider the map $P \rightarrow \overline{M}(X)$ corresponding to the chart for the log structure of X that we fixed above. Recall from [6, Section 3.3] that this map being a chart exactly means that the induced morphism $\phi: \underline{P} \rightarrow \overline{M}$ from the constant sheaf \underline{P} is a cokernel, i.e. it induces an isomorphism $\underline{P}/\ker \phi \cong \overline{M}$, where $\ker \phi$ denotes the preimage of the zero section (this is not always true in the category of monoids). In the same way, the map $P_{\mathbb{Q}} \rightarrow \overline{M}(X)_{\mathbb{Q}}$ gives a chart for the sheaf $\overline{M}_{\mathbb{Q}}$ over X . For the Kummer extension $P \subseteq \mathbb{N}^n \subset P_{\mathbb{Q}}$ constructed above, let us consider the image \mathcal{F} of the subsheaf $\underline{\mathbb{N}^n} \subset \underline{P_{\mathbb{Q}}}$ in $\overline{M}_{\mathbb{Q}}$. It is not hard to check that the natural map $\mathbb{N}^n \rightarrow \mathcal{F}(X)$ is a chart for \mathcal{F} (i.e. the corresponding $\underline{\mathbb{N}^n} \rightarrow \mathcal{F}$ is a cokernel), and that the induced morphism $\overline{M} \rightarrow \mathcal{F}$ is a Kummer extension with the universal property of the statement. \square

REMARK 4.8. The previous proposition is also true for log algebraic stacks with simplicial log structure, by passing to a smooth presentation and using uniqueness of the Kummer extension to produce descent data.

DEFINITION 4.9. Let X be a log algebraic stack with simplicial log structure, and let $\overline{M} \rightarrow \mathcal{F}$ be the canonical Kummer extension constructed above. We call the root stack $\sqrt[n]{X}$ the *canonical root stack* of X .

REMARK 4.10. Assume that κ is a field of characteristic 0. If $X = \text{Spec } \kappa[P]$ where P is a simplicial monoid, then we can consider the minimal Kummer extension $P \subseteq \mathbb{N}^n$ constructed in the proof of the previous proposition. The canonical root stack in this case is the quotient $[\text{Spec } \kappa[\mathbb{N}^n]/D(\mathbb{Z}^n/P^{\text{gp}})]$. Note that the quotient $\mathbb{Z}^n/P^{\text{gp}}$ is a finite group, so this quotient is a smooth Deligne–Mumford stack. In fact, in this case it coincides with the *canonical stack* of Fantechi–Mann–Nironi [15], which justifies its name.

Note that the natural log structure of the canonical root stack $\sqrt[n]{X}$ is locally free by construction.

DEFINITION 4.11. Let X be a scheme over κ , and $D \subset X$ be an effective Cartier divisor. We say that D has *simple simplicial singularities* if:

- the compactifying log structure M_D (whose definition is recalled in Section 2.1) is simplicial,
- the tautological log structure of the canonical root stack $X' = \sqrt[n]{(X, D)}$ is given by a simple normal crossing divisor $D' \subset X'$ (in the sense of Definition 2.1).

REMARK 4.12. In Definition 4.11 we have made the assumption that $D' \subset X'$ is a *simple* normal crossing divisor in order to simplify the exposition. However, in characteristic 0, the results from this section could be formulated more generally for the case when D' is a general normal crossing divisor. We leave it to the interested reader to recast Theorem 4.14 below in this greater generality using as input the psod constructed in the general normal crossing setting in Theorem 4.6.

In the rest of the paper, for convenience we will abbreviate “simple simplicial singularities” by just “simplicial singularities”.

If X is an algebraic stack and $D \subseteq X$ is an effective Cartier divisor, we say that D has simplicial singularities if the pull-back of D to some smooth presentation $U \rightarrow X$, where U is a scheme, has simplicial singularities in the sense of the previous definition.

REMARK 4.13. Assume that D is an effective Cartier divisor on X and that for every $x \in D$ the pair (X, D) is étale locally around x isomorphic to the pair $(\text{Spec } \kappa[P] \times \mathbb{A}^n, \Delta_P \times \mathbb{A}^n)$ for some simplicial monoid P and $n \in \mathbb{N}$. Then D has simplicial singularities in the sense Definition 4.11.

Now assume that $D \subset X$ has simplicial singularities, and consider the canonical root stack $(X', D') = \sqrt[r]{(X, D)}$. Since $(X', D') \rightarrow (X, D)$ is a root stack morphism we have a canonical isomorphism $\sqrt[\infty]{(X', D')} \simeq \sqrt[\infty]{(X, D)}$ and therefore in order to study the category of perfect complexes on $\sqrt[\infty]{(X, D)}$, we can pass to (X', D') . For future reference we state this as the following theorem. We will use it in Section 5.2 to obtain a formula for the Kummer flat K-theory of X .

Let I be the set of irreducible components of D' and let $S_{I'}$ be the preorder of strata of (X', D') .

THEOREM 4.14.

1. The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a collection of subcategories $\mathcal{A}_{J'}^1$, $J' \in S_{I'}$, such that:

- For all J' , the subcategory $\mathcal{A}_{J'}^1$ is admissible.
- For all $J' \in S_{I'}^*$, the category $\mathcal{A}_{J'}^1$ has a psod of type $((\mathbb{Q}/\mathbb{Z})_{J'}^*, \leq^!)$

$$\mathcal{A}_{J'}^1 = \langle \mathcal{A}_\chi^{J',!}, \chi \in ((\mathbb{Q}/\mathbb{Z})_{J'}^*, \leq^!) \rangle,$$

and, for all $\chi \in (\mathbb{Q}/\mathbb{Z})_{J'}^*$, there is an equivalence $\mathcal{A}_\chi^{J',!} \simeq \text{Perf}(D_{J'})$.

2. The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ has a psod of type $((\mathbb{Q}/\mathbb{Z})_{I'}, \leq^!)$

$$\text{Perf}(\sqrt[\infty]{(X, D)}) = \langle \mathcal{A}_\chi^{J',!}, \chi \in ((\mathbb{Q}/\mathbb{Z})_{I'}, \leq^!) \rangle,$$

and for all $\chi \in (\mathbb{Q}/\mathbb{Z})_{I'}^*$, there is an equivalence $\mathcal{A}_\chi^{J',!} \simeq \text{Perf}(D_{J'})$.

Proof. This follows immediately from the equivalence $\text{Perf}(\sqrt[\infty]{(X, D)}) \simeq \text{Perf}(\sqrt[\infty]{(X', D')})$ and the fact that by Theorem 3.16 we can equip $\text{Perf}(\sqrt[\infty]{(X', D')})$ with a psod of type $S_{I'}$. \square

REMARK 4.15. It would be very interesting to extend Theorem 4.14 to the general log smooth setting, however new ideas would be required. A promising avenue for further investigation might come from work of Satriano [45], which essentially extends the theory of canonical root stacks given in Definition 4.9 to general (non-necessarily simplicial) toric singularities.

5 NON-COMMUTATIVE MOTIVES OF LOG SCHEMES

In this section we associate to log stacks objects in the category of non-commutative motives. We start by giving a brief summary of the theory which follows closely the treatment given in [24, Section 5]. The reader can find accounts of the theory of non-commutative motives in [4] and [24].

Let \mathcal{T}_∞ be the ∞ -category of spaces, which is the homotopy coherent nerve of the simplicial category of Kan complexes. Let \mathcal{S}_∞ be the ∞ -category of spectra. The category \mathcal{S}_∞ is the stabilization of \mathcal{T}_∞ , and we denote by $\Sigma_+^\infty : \mathcal{T}_\infty \rightarrow \mathcal{S}_\infty$ the stabilization functor.

DEFINITION 5.1. Let \mathcal{C} be a small ∞ -category. We denote:

- by $\text{PSh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{T}_\infty)$ the ∞ -category of presheaves of ∞ -groupoids over \mathcal{C} ,
- by $\text{PSh}_{\mathcal{S}_\infty}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}_\infty)$ the ∞ -category of presheaves of spectra over \mathcal{C} ,
- by $\Sigma_+^\infty : \text{PSh}(\mathcal{C}) \rightarrow \text{PSh}_{\mathcal{S}_\infty}(\mathcal{C})$ the functor given, on objects, by stabilization.

Let $(\text{Cat}_{\infty, \kappa}^{\text{perf}})^\omega$ be the subcategory of compact objects in $\text{Cat}_{\infty, \kappa}^{\text{perf}}$. Let ϕ be the composite

$$\phi : \text{Cat}_{\infty, \kappa}^{\text{perf}} \longrightarrow \text{PSh}((\text{Cat}_{\infty, \kappa}^{\text{perf}})^\omega) \xrightarrow{\Sigma_+^\infty} \text{PSh}_{\mathcal{S}_\infty}((\text{Cat}_{\infty, \kappa}^{\text{perf}})^\omega),$$

where the first arrow is the restriction of the Yoneda to the subcategory $(\text{Cat}_{\infty, \kappa}^{\text{perf}})^\omega$.

DEFINITION 5.2. The *category of additive motives* Mot^{add} is the localization of $\text{PSh}_{\mathcal{S}_\infty}((\text{Cat}_{\infty, \kappa}^{\text{perf}})^\omega)$ at the class of morphisms $\phi(\mathcal{B})/\phi(\mathcal{A}) \rightarrow \phi(\mathcal{C})$ which are induced by split exact sequences $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ in $\text{Cat}_{\infty, \kappa}^{\text{perf}}$.

Let \mathcal{U} be the composite $\text{Cat}_{\infty, \kappa}^{\text{perf}} \xrightarrow{\phi} \text{PSh}_{\mathcal{S}_\infty}((\text{Cat}_{\infty, \kappa}^{\text{perf}})^\omega) \rightarrow \text{Mot}^{\text{add}}$, where the second arrow is given by the localization functor. An *additive invariant* is a functor $H : \text{Cat}_{\infty, \kappa}^{\text{perf}} \rightarrow \mathcal{P}$, where \mathcal{P} is a stable presentable ∞ -category, that preserves zero objects and filtered colimits, and that maps split exact sequences to cofiber sequences. The functor \mathcal{U} is the universal additive invariant. We formulate the precise statement below.

PROPOSITION 5.3 (Theorem 5.12 [24]). *Let \mathcal{P} be a presentable and stable ∞ -category, and let $H: \text{Cat}_{\infty, \kappa}^{\text{perf}} \rightarrow \mathcal{P}$ be an additive invariant. Then H factors uniquely as a composition*

$$\begin{array}{ccc} \text{Cat}_{\infty, \kappa}^{\text{perf}} & \xrightarrow{H} & \mathcal{P} \\ \mathcal{U} \downarrow & \nearrow \bar{H} & \\ \text{Mot}^{\text{add}} & & \end{array}$$

where \bar{H} is a colimit-preserving functor of presentable categories.

REMARK 5.4. Let $H: \text{Cat}_{\infty, \kappa}^{\text{perf}} \rightarrow \mathcal{P}$ be an additive invariant. If $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ is a split exact sequence in $\text{Cat}_{\infty, \kappa}^{\text{perf}}$, then there is a canonical splitting $H(\mathcal{B}) \simeq H(\mathcal{A}) \oplus H(\mathcal{C})$. Indeed, let $(F)^R$ be the right adjoint of F . Since H is additive

$$H(\mathcal{A}) \xrightarrow{H(F)} H(\mathcal{B}) \xrightarrow{H(G)} H(\mathcal{C}) \tag{14}$$

is a fiber sequence in \mathcal{P} . Further $H((F)^R)$ is a section of $H(F)$. Thus (14) splits, and $H(\mathcal{B})$ decomposes as the direct sum of $H(\mathcal{A})$ and $H(\mathcal{C})$.

LEMMA 5.5. *Let $\mathcal{C} = \langle \mathcal{C}_x, x \in P \rangle$ be a stable ∞ -category equipped with a psod of type P , and assume that P is finite and directed (i.e. it admits an order-reflecting map to the natural numbers \mathbb{N} , ordered in the standard manner). Then there is an equivalence $\mathcal{U}(\mathcal{C}) \simeq \bigoplus_{x \in P} \mathcal{U}(\mathcal{C}_x)$.*

Proof. Note that if P is directed we can choose a numbering $\{p_0, \dots, p_m\}$ of its elements with the property that, if $i < j$, then $\mathcal{C}_{p_i} \subseteq \mathcal{C}_{p_j}^\perp$. Thus we can write down a sod $\langle \mathcal{C}_{p_0}, \dots, \mathcal{C}_{p_m} \rangle$ for \mathcal{C} . Then the second statement is a simple consequence of Remark 5.4. \square

5.1 THE NON-COMMUTATIVE MOTIVE OF A LOG STACK

We introduce the following notations.

- If X is a stack we set $\mathcal{U}(X) := \mathcal{U}(\text{Perf}(X))$.
- If X is a log algebraic stack, we denote by X_{Kfl} the ringed Kummer flat topos over X . We set $\mathcal{U}(X_{\text{Kfl}}) := \mathcal{U}(\text{Perf}(X_{\text{Kfl}}))$.

We will apply Lemma 5.5 to the psod-s we constructed in sections 3.2 and 4.1. Let (X, D) be a log stack given by an algebraic stack X equipped with a normal crossing divisor D . Let S_I be the preorder of strata of (X, D) . In the statement below we use the same notations as in Section 3.2.

COROLLARY 5.6. *Let (X, D) be a log stack given by an algebraic stack X equipped with a simple normal crossing divisor D . Then there is an equivalence*

$$\mathcal{U}((X, D)_{\text{Kfl}}) \simeq \mathcal{U}(X) \bigoplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_I^*} \mathcal{U}(S) \right) \right).$$

Proof. Using Lemma 2.8 we obtain an equivalence $\text{Perf}((X, D)_{\text{Kfl}}) \simeq \text{Perf}(\sqrt[\infty]{(X, D)})$. The non-commutative motive of $\text{Perf}(\sqrt[n^!]{(X, D)})$ has a decomposition as in the statement (except the indexing set $(\mathbb{Q}/\mathbb{Z})_I^*$ has to be replaced by $\mathbb{Z}_{I, n^!}^*$): this follows from Proposition 3.14 and Lemma 5.5. The category $\text{Perf}(\sqrt[\infty]{(X, D)})$ is a filtered colimit of the categories $\text{Perf}(\sqrt[n^!]{(X, D)})$. Also, by Theorem 3.16, it carries a psod that is the colimit of the psod-s of the categories $\text{Perf}(\sqrt[n^!]{(X, D)})$. The statement follows because, by construction, $\mathcal{U}(-)$ commutes with filtered colimits. \square

Formulas exactly paralleling Corollary 5.6 can be obtained in the general normal crossing setting. This is straightforward, as explained in Section 4.1, but involves messy combinatorics. For this reason we give instead a simplified statement, which is contained in Corollary 5.7 below.

In the following statement, if S is a stratum of (X, D) , we denote by S^\vee its normalization.

COROLLARY 5.7. *Assume that the ground ring κ is a field of characteristic 0. Let (X, D) be a log stack given by an algebraic stack X equipped with a normal crossing divisor D . Then for each $S \in S_D^*$ there exists an infinite countable set \mathcal{I}_S , such that there is an equivalence*

$$\mathcal{U}((X, D)_{\text{Kfl}}) \simeq \mathcal{U}(X) \bigoplus \left(\bigoplus_{S \in S_D^*} \left(\bigoplus_{j \in \mathcal{I}_S} \mathcal{U}(S^\vee) \right) \right).$$

By the universal property of \mathcal{U} , Corollary 5.6 and 5.7 imply uniform direct sum decompositions across all additive invariants. As the case of algebraic K-theory is especially important we formulate it explicitly in the following corollary: this generalizes Hagihara and Nizioł’s as we drop the simplicity assumption on D , X can be a stack, and X need not be regular away from D .

COROLLARY 5.8.

- *Let (X, D) be a log stack given by an algebraic stack X equipped with a simple normal crossing divisor D . Then there is a direct sum decomposition of spectra*

$$K((X, D)_{\text{Kfl}}) \simeq K(X) \bigoplus \left(\bigoplus_{S \in S_D^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} K(S) \right) \right). \tag{15}$$

- *Assume that the ground ring κ is a field of characteristic 0. Let (X, D) be a log stack given by an algebraic stack X equipped with a normal crossing divisor D . Then there is a direct sum decomposition of spectra*

$$K((X, D)_{\text{Kfl}}) \simeq K(X) \bigoplus \left(\bigoplus_{S \in S_D^*} \left(\bigoplus_{j \in \mathcal{I}_S} K(S^\vee) \right) \right).$$

REMARK 5.9. The previous result has an analogue for the Kummer étale topos of (X, D) , parallel to the second part of the statement of Theorem 1.1 of [36] and the Main Theorem of [20]. In characteristic zero there is no difference, so this comment is relevant only if κ has positive or mixed characteristic, and, assuming that D is equicharacteristic as in [36], \mathbb{Q}/\mathbb{Z} has to be replaced by $(\mathbb{Q}/\mathbb{Z})' = \mathbb{Z}_{(p)}/\mathbb{Z}$ (where p is the characteristic over which D lives) in the formulas above.

This analogous formula for the Kummer étale K-theory follows from our methods, starting from the analogue of Proposition 2.8 for the Kummer étale site and a restricted version ${}^\infty\sqrt{(X, D)}$ of the infinite root stack, where we take the inverse limit only of root stacks ${}^r\sqrt{(X, D)}$ where r is not divisible by p . This statement in turn follows from the same argument used in the proof of 2.8, after proving Theorem 6.16 and Corollary 6.17 of [51] for the Kummer étale site and this restricted root stack. We leave the details to the interested reader.

5.2 LOG SCHEMES WITH SIMPLICIAL LOG STRUCTURE

Let D be a divisor with simplicial singularities in an algebraic stack X . Consider the associated log stack (X, D) with simplicial log structure and let ${}^{\mathcal{F}}\sqrt{(X, D)}$ be its canonical root stack, which is of the form (X', D') , where D' is a normal crossing divisor on X' . Our techniques allow us to derive a decomposition formula for the Kummer flat K-theory of X in terms of the geometry of (X', D') . In fact, we can formulate two such results.

Let $S_{I'}$ be the preorder of strata of (X', D') . By Theorem 4.14 $\text{Perf}((X, D)_{\text{Kfl}})$ carries a canonical psod of type $S_{I'}$, and this yields a decomposition of the noncommutative motive $\mathcal{U}((X, D)_{\text{Kfl}})$. In particular, we obtain an equivalence of spectra

$$K((X, D)_{\text{Kfl}}) \simeq K((X', D')_{\text{Kfl}}) \simeq K(X') \bigoplus \left(\bigoplus_{J \in S_{I'}^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})^*} K(S'_J) \right) \right). \tag{16}$$

Under some additional assumptions on (X, D) however, we can do better. We can refine (16) to a second decomposition formula for the (*complexified*) Kummer flat K-theory of (X, D) which is formulated in terms of the *G-theory* of the strata of X determined by the divisor D via the associated log structure. We do this in Proposition 5.10 below.

We will make use of results proved in [30]. Let (X, D) be a log scheme where D has simplicial singularities. We assume that

- (\star) $\kappa = \mathbb{C}$, X is quasi-projective, and (X, D) has a global chart $X \rightarrow [\text{Spec } \mathbb{C}[P]/D(P^{\text{gp}})]$ for a simplicial monoid P , which is a smooth morphism.

This implies that the canonical root stack ${}^{\mathcal{F}}\sqrt{(X, D)}$ is a quotient stack $[(Y, E)/G]$ where Y is a smooth quasi-projective scheme, $E \subset Y$ is a simple normal crossing divisor and G is a finite group acting on the pair (Y, E) .

Also, (X, D) is obtained by taking the coarse quotient for the action of G . We denote $\sqrt[\mathbb{F}]{(X, D)}$ by (X', D') , where $X' = [Y/G]$ and D' is the induced simple normal crossing divisor $[E/G]$. In particular, X' is smooth and has a quasi-projective coarse moduli space.

Let L, I' and I be, respectively, the set of irreducible components of the divisors $E \subset Y, D' \subset X'$, and $D \subset X$. As usual we denote the corresponding sets of strata by $S_L, S_{I'}$ and S_I . There is a canonical bijection between the sets $S_{I'}$ and S_I . The group G acts on S_L , and there is a map $p: S_L \rightarrow S_L/G \cong S_{I'} \cong S_I$ induced by the quotient $Y \rightarrow [Y/G] \cong X'$. Let F be the *disjoint* union of the sets of irreducible components of the fixed loci $Y^g \subset Y$, as g ranges over $G \setminus \{1_G\}$. The fixed loci are strata of Y , and this gives a map $F \rightarrow S_L$. In general this is not an injection, as the same stratum of Y might appear more than once in F if it is fixed by several distinct group elements.

If U is a stratum of Y we introduce the following notations,

- $F_U := \{T \in F \mid U \subseteq T\} \subseteq F$,
- $(\mathbb{Q}/\mathbb{Z})_{F_U}^* := (\mathbb{Q}/\mathbb{Z})_U^* \amalg \left(\amalg_{T \in F_U} (\mathbb{Q}/\mathbb{Z})_T^* \right)$, where the index sets on the right hand side are written according to the convention explained in Remark 3.8 : that is, they are labeled by strata, rather than by subsets of L . We will follow this convention throughout Section 5.2.

We extend this to X using the map $p: S_E \rightarrow S_D$. Namely, if S is in S_D we set:

- $F_S := \amalg_{U \in p^{-1}(S)} p(F_U)$, F_S is the *disjoint* union of the sets $p(F_U)$,
- $(\mathbb{Q}/\mathbb{Z})_{F_S}^* := (\mathbb{Q}/\mathbb{Z})_S^* \amalg \left(\amalg_{T \in F_S} (\mathbb{Q}/\mathbb{Z})_T^* \right)$.

If X is an algebraic stack, in the statement of Proposition 5.10, and throughout its proof, we denote $G_i(X)$ the i -th G-theory group of X with *complex coefficients*, i.e. $G_i(X) := K_i(\text{Coh}(X)) \otimes \mathbb{C}$.

PROPOSITION 5.10. *Let (X, D) be a log scheme given by a divisor D with simplicial singularities, satisfying assumption (\star) . Then for all $i \in \mathbb{N}$ there is a direct sum decomposition*

$$K_i((X, D)_{\text{Kfl}}) \otimes \mathbb{C} \cong G_i(X) \bigoplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_{F_S}^*} G_i(S) \right) \right). \tag{17}$$

Proposition 5.10 follows from (16) and an Atiyah–Segal-type formula expressing the G-theory of a stack in terms of the G-theory of the coarse moduli of its inertia. The most general version of such a formula in the literature was obtained in [30] (and holds over \mathbb{C}). The assumptions we impose on (X, D) mirror the assumptions made in [30]: they can be relaxed if more general versions of the Atiyah–Segal decomposition will become available in the future.

Proof. For simplicity we assume that X is an affine toric variety with simplicial singularities. Then $Y = \mathbb{A}^n$, and G is a finite group acting torically. The proof in the general case is the same, except the book-keeping of the summands on the right-hand side of (17) requires some extra care.

Throughout the proof, if X is a stack we denote by IX the inertia of X , and by \widetilde{IX} its coarse moduli space. For all $i \in \mathbb{N}$, formula (16) yields an isomorphism of abelian groups

$$K_i((X, D)_{\text{Kfl}}) \cong K_i(X') \bigoplus \left(\bigoplus_{S' \in S_{I'}} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_{S'}^*} K_i(S') \right) \right) \tag{18}$$

All the strata $S' \in S_{I'}$ are smooth, and thus their G-theory and K-theory are the same. By Theorem 1.1 of [30], for all $i \in \mathbb{N}$ there is an isomorphism

$$K_i(X') \otimes \mathbb{C} = G_i(X') \cong G_i(\widetilde{IX}') \cong \bigoplus_{g \in G} G_i(Y^g/G)$$

and

$$\bigoplus_{g \in G} G_i(Y^g/G) = G_i(X) \bigoplus \left(\bigoplus_{T \in F} G_i(T/G) \right).$$

Similarly, if $S' = [U/G] \in S_{I'}$ is a stratum, we have

$$K_i(S') \otimes \mathbb{C} = G_i(S') \cong G_i(\widetilde{IS}') \cong \bigoplus_{g \in G} G_i(Y^g \cap U/G)$$

and

$$\bigoplus_{g \in G} G_i(Y^g \cap U/G) = G_i(U/G) \bigoplus \left(\bigoplus_{T \in F} G_i(T \cap U/G) \right)$$

Thus, if we complexify formula (18), we find that $K_i((X, D)_{\text{Kfl}}) \otimes \mathbb{C}$ is isomorphic to

$$G_i(X) \bigoplus \left(\bigoplus_{T \in F} G_i(T/G) \right) \bigoplus \left(\bigoplus_{U \in S_L, U \neq Y} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_U^*} G_i(U/G) \bigoplus \left(\bigoplus_{T \in F} G_i(T \cap U/G) \right) \right) \right). \tag{19}$$

The main difference between the statement we need to prove and the decomposition (15) which holds in the simple normal crossings case is that, in general, the indexing set $(\mathbb{Q}/\mathbb{Z})_{F_S}^*$ corresponding to a stratum $S \in S_D$ is larger than the indexing set $(\mathbb{Q}/\mathbb{Z})_S^*$ which appears in (15). The reason is that bigger strata containing S might split off extra factors of the form $G_i(S)$ owing to the Atiyah–Segal decomposition encoded in (19). Formula (17) is then obtained by rearranging the factors on the right-hand side of (19) so as to group together all factors of the form $G_i(S)$.

More precisely, let $S = U/G$ be a stratum of X . Assume that there exists a pair $T \in F, V \in S_E$ such that $U = T \cap V$. Then the summand of (18) corresponding to the stratum $[V/G] \in S_{D'}$ is $\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_{[V/G]}^*} K_i([V/G]) \otimes \mathbb{C}$ and can be rewritten as

$$G_i(U/G) \bigoplus \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_V^*} G_i(T \cap V/G) \right) \bigoplus \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_V^*} \left(\bigoplus_{T' \in F, T' \neq T} G_i(T' \cap V/G) \right) \right) \cong$$

$$\cong G_i(U/G) \bigoplus \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_V^*} G_i(S) \right) \bigoplus \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_V^*} \left(\bigoplus_{T' \in F, T' \neq T} G_i(T' \cap V/G) \right) \right).$$

Thus it splits off a summand $\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_V^*} G_i(S)$. Taking into account the contributions coming from all pairs $T \in F, V \in S_E$ such that $U = T \cap V$, yields the summand $\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_{F_S}^*} G_i(S)$ which appears in (17). This concludes the proof. \square

5.3 LOGARITHMIC CHERN CHARACTER

In this last section we sketch one additional application of our techniques. Namely, we define a *logarithmic Chern character* and explain some of its basic properties. We conclude by formulating a Grothendieck-Riemann-Roch statement for the logarithmic Chern character. For simplicity in this section κ will be a field of characteristic 0.

Recall from Section 2.5 the definition of the Chern character morphism ch in the setting of ∞ -categories.

DEFINITION 5.11. Let X be a log algebraic stack. We define the *logarithmic Chern character* to be the morphism $\text{ch}: K(X_{\text{Kfl}}) \rightarrow \text{HH}(X_{\text{Kfl}})$.

To emphasize the fact that we are in the logarithmic setting, we will denote the logarithmic Chern character by ch^{log} . The next statement follows immediately from Corollary 5.6.

PROPOSITION 5.12. Let (X, D) be a log stack given by an algebraic stack X equipped with a simple normal crossing divisor D . Let I be the set of irreducible components of D and denote by S_I the set of strata. Then there is a commutative diagram

$$\begin{array}{ccc} K((X, D)_{\text{Kfl}}) & \xrightarrow{\text{ch}^{\text{log}}} & \text{HH}((X, D)_{\text{Kfl}}) \\ \cong \downarrow & & \downarrow \cong \\ K(X) \oplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} K(S) \right) \right) & \xrightarrow{\oplus \text{ch}} & \text{HH}(X) \oplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} \text{HH}(S) \right) \right) \end{array}$$

where $\oplus \text{ch}$ denotes the direct sum of the Chern character maps

$$\text{ch}: K(S) \rightarrow \text{HH}(S)$$

for $S \in S_I$.

DEFINITION 5.13. Let (X, D) be a log scheme given by a smooth and proper scheme X together with a simple normal crossing divisor D . Then we define the *de Rham logarithmic Chern character* $\text{ch}_{\text{dR}}^{\text{log}}$ to be the composite

$$\begin{array}{ccc} K_0((X, D)_{\text{Kfl}}) & \xrightarrow{\text{ch}^{\text{log}}} & \text{HH}_0((X, D)_{\text{Kfl}}) \\ & \searrow \text{ch}_{\text{dR}}^{\text{log}} & \downarrow \cong \\ & & \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(X) \oplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S) \right) \right) \end{array}$$

REMARK 5.14. The morphism $\text{ch}_{\text{dR}}^{\log}$ is closely related to the parabolic Chern character considered in [27]. One difference is that the authors in [27] work with finite rather than infinite root stacks.

We conclude by stating a Grothendieck–Riemann–Roch theorem for the logarithmic Chern character. We will place ourselves under quite restrictive assumptions. We will return to the problem of extending this logarithmic GRR formalism to a larger class of log stacks in future work. Let $f: (Y, E) \rightarrow (X, D)$ be a strict map of log schemes having the following properties:

- the underlying schemes Y and X are smooth and proper, and E and D are simple normal crossing divisors;
- the morphism between the underlying schemes $f: Y \rightarrow X$ is flat and proper.

Let L and I be the irreducible components of E and D and denote by S_L and S_I the sets of strata. Note that each stratum $S_Y \in S_L$ is mapped by f to a stratum $S_X \in S_I$. Further, for each stratum $S_Y \in S_L$, the classical Grothendieck–Riemann–Roch theorem gives a commutative diagram

$$\begin{CD}
 K_0(S_Y) @>\text{ch}^{\text{dR}}>> \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S_Y) \\
 @VVV @VVf_*(- \wedge \text{Td}_{S_Y/S_X})V \\
 K_0(S_X) @>\text{ch}^{\text{dR}}>> \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S_X)
 \end{CD} \tag{20}$$

where Td_{S_Y/S_X} is the Todd class of the relative tangent bundle. Taking the direct sum of the vertical morphism on the right of (20) over all strata we obtain a morphism

$$\begin{CD}
 \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(Y) \oplus \left(\bigoplus_{S \in S_L^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S) \right) \right) \\
 @VV \bigoplus f_*(- \wedge \text{Td}) V \\
 \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(X) \oplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S) \right) \right)
 \end{CD}$$

which we denote for simplicity $\bigoplus f_*(- \wedge \text{Td})$, dropping the indices from the Todd classes.

PROPOSITION 5.15. *Let $f: (Y, E) \rightarrow (X, D)$ be a map of log schemes satisfying the properties above. Then:*

1. *There is a commutative diagram in \mathcal{S}_∞*

$$\begin{CD}
 K((Y, E)_{\text{Kfl}}) @>\text{ch}^{\log}>> \text{HH}((Y, E)_{\text{Kfl}}) \\
 @Vf_*VV @VVf_*V \\
 K((X, D)_{\text{Kfl}}) @>\text{ch}^{\log}>> \text{HH}((X, D)_{\text{Kfl}}).
 \end{CD}$$

2. There is a commutative diagram of abelian groups

$$\begin{array}{ccc}
 K_0((Y, E)_{\text{Kfl}}) & \xrightarrow{\text{ch}_{\text{dR}}^{\log}} & \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(Y) \oplus \left(\bigoplus_{S \in S_L^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S) \right) \right) \\
 \downarrow f_* & & \downarrow \bigoplus f_*(- \wedge^{\text{Td}}) \\
 K_0((X, D)_{\text{Kfl}}) & \xrightarrow{\text{ch}_{\text{dR}}^{\log}} & \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(X) \oplus \left(\bigoplus_{S \in S_I^*} \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})_S^*} \bigoplus_{k \geq 0} H_{\text{dR}}^{2k}(S) \right) \right).
 \end{array}$$

Proof. Let us start with the first statement. We will use the equivalence $\text{Perf}((X, D)_{\text{Kfl}}) \simeq \text{Perf}(\infty\sqrt{(X, D)})$ from Proposition 2.8, and the identifications

$$K((Y, E)_{\text{Kfl}}) \simeq K(\infty\sqrt{(Y, E)}), \quad K((X, D)_{\text{Kfl}}) \simeq K(\infty\sqrt{(X, D)}),$$

$$\text{HH}((Y, E)_{\text{Kfl}}) \simeq \text{HH}(\infty\sqrt{(Y, E)}), \quad \text{HH}((X, D)_{\text{Kfl}}) \simeq \text{HH}(\infty\sqrt{(X, D)}).$$

Let $f_r: \sqrt[r]{(Y, E)} \rightarrow \sqrt[r]{(X, D)}$ and $f_\infty: \infty\sqrt{(Y, E)} \rightarrow \infty\sqrt{(X, D)}$, be the maps between the r -th and the infinite root stacks induced by f . For every $r \in \mathbb{N}$, f_r is flat and proper (therefore perfect) and thus by [33, Examples 2.2 (a)] it induces a push-forward $f_{r,*}: \text{Perf}(\sqrt[r]{(Y, E)}) \rightarrow \text{Perf}(\sqrt[r]{(X, D)})$. Taking the colimit over r we obtain the push-forward $f_{\infty,*}: \text{Perf}(\infty\sqrt{(Y, E)}) \rightarrow \text{Perf}(\infty\sqrt{(X, D)})$. Applying ch to $f_{\infty,*}$ yields the commutative diagram below, which gives statement (1)

$$\begin{array}{ccc}
 K(\infty\sqrt{(Y, E)}) & \xrightarrow{\text{ch}} & \text{HH}(\infty\sqrt{(Y, E)}) \\
 \downarrow f_{\infty,*} & & \downarrow f_{\infty,*} \\
 K(\infty\sqrt{(X, D)}) & \xrightarrow{\text{ch}} & \text{HH}(\infty\sqrt{(X, D)}),
 \end{array}$$

Let us consider the second statement next. For simplicity, we restrict to the case where D and E are irreducible (and this $f^{-1}(D) = E$, by strictness). The general case is similar. We need to prove that the push-forward $f_{\infty,*}$ functor preserves the summands of the psod-s of $\text{Perf}(\infty\sqrt{(Y, E)})$ and $\text{Perf}(\infty\sqrt{(X, D)})$. As f is strict the diagram

$$\begin{array}{ccc}
 \infty\sqrt{(Y, E)} & \xrightarrow{g_{\infty,1}} & Y \\
 \downarrow f_\infty & & \downarrow f \\
 \infty\sqrt{(X, D)} & \xrightarrow{g_{\infty,1}} & X
 \end{array} \tag{21}$$

is cartesian. Further, f is flat and thus base-change yields a commutative diagram

$$\begin{array}{ccc}
 \text{Perf}(Y) & \xrightarrow{g_{\infty,1}^*} & \text{Perf}(\infty\sqrt{(Y, E)}) \\
 \downarrow f_* & & \downarrow f_{\infty,*} \\
 \text{Perf}(X) & \xrightarrow{g_{\infty,1}^*} & \text{Perf}(\infty\sqrt{(X, D)}).
 \end{array}$$

This shows that $f_{\infty,*}$ maps the semi-orthogonal summand $\text{Perf}(Y) \subset \text{Perf}(\sqrt[r]{Y, E})$ to $\text{Perf}(X)$. Next, let us turn to the other summands of the sod of $\text{Perf}(\sqrt[r]{Y, E})$. Again, since f is strict, the square below is cartesian

$$\begin{array}{ccc} E_r & \longrightarrow & \sqrt[r]{Y, E} \\ f_r \downarrow & & \downarrow f_r \\ D_r & \longrightarrow & \sqrt[r]{X, D}, \end{array}$$

where as usual E_r and D_r denote the universal Cartier divisors of the two root stacks. Note that flatness of f implies flatness of f_r , and therefore E_r is also the derived fiber product of the diagram.

Base change yields a commutative diagram

$$\begin{array}{ccc} \bigoplus_{\chi \in \mathbb{Z}_r} \text{Perf}(E_r)_\chi \simeq \text{Perf}(E_r) & \longrightarrow & \text{Perf}(\sqrt[r]{Y, E}) \\ f_{r,*} \downarrow & & \downarrow f_{r,*} \\ \bigoplus_{\chi \in \mathbb{Z}_r} \text{Perf}(D_r)_\chi \simeq \text{Perf}(D_r) & \longrightarrow & \text{Perf}(\sqrt[r]{X, D}). \end{array}$$

Additionally, for every $\chi \in \mathbb{Z}_r$, the restriction of $f_{r,*}$ to $(\text{Perf}(E_r))_\chi$ coincides with f_* : more precisely, there is a commutative diagram

$$\begin{array}{ccc} (\text{Perf}(E_r))_\chi & \xrightarrow{\simeq} & \text{Perf}(E) \\ f_{r,*} \downarrow & & \downarrow f_* \\ (\text{Perf}(D_r))_\chi & \xrightarrow{\simeq} & \text{Perf}(D). \end{array}$$

This shows that f_* respects the summands of the sod-s of the r -th root stacks given by Proposition 3.4. We are actually interested in the compatibility with the sod-s of $n!$ -th root stacks constructed recursively in Proposition 3.5. Note however that the latter are obtained iterating the construction from Proposition 3.4: thus iterating the argument above also implies that f_* respects the sod given in Proposition 3.5.

This implies that the push-forward map $f_*: K((Y, E)_{\text{Kfl}}) \rightarrow K((X, D)_{\text{Kfl}})$ decomposes as a direct sum of push-forwards along f , which we denote by $\bigoplus f_*$,

$$\bigoplus f_*: K(Y) \bigoplus \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})^*} K(E) \right) \longrightarrow K(X) \bigoplus \left(\bigoplus_{\chi \in (\mathbb{Q}/\mathbb{Z})^*} K(D) \right). \tag{22}$$

Then the second statement follows by applying the ordinary Grothendieck–Riemann–Roch to each summand in (22). \square

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Sarah Scherotzke
Université du Luxembourg
Maison du Nombre
6, Avenue de la Fonte
L-4364 Esch-sur-Alzette
Luxembourg
sarah.scherotzke@uni.lu

Nicolò Sibilla
SMSAS, University of Kent
Canterbury, Kent CT2 7NF
UK
and
SISSA
Via Bonomea 265
34136 Trieste (TS)
Italy
nsibila@sissa.it

Mattia Talpo
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo 5
56127 Pisa (PI)
Italy
mattia.talpo@unipi.it

