GSV-Index for Holomorphic Pfaff Systems

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Abstract. In this work we introduce a GSV type index for varieties invariant by holomorphic Pfaff systems (possibly non locally decomposables) on projective manifolds. We prove a non-negativity property for the index. As an application, we prove that the non-negativity of the GSV-index gives us an obstruction to the solution of the Poincaré problem for Pfaff systems on projectives spaces.

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1 Introduction

The GSV-index for vector fields tangent to hypersurfaces with isolated singularities was introduced by X. Gómez-Mont, J. Seade and A. Verjovsky [20] generalizing the Poincaré-Hopf index. The concept of GSV-index for vector fields tangent to complete intersections was extended by J. Seade and T. Suwa in [28, 29] and J.-P. Brasselet, J. Seade and T. Suwa in [2].
D. Lehmann, M. G. Soares and T. Suwa introduced in [22] the *virtual index* for vector fields on complex analytic varieties via Chern-Weil theory. They showed that this index coincides with the GSV-index if the variety is a local complete intersection variety with isolated singularities. X. Gomez-Mont in [19] defined the *homological index* for holomorphic vector fields on arbitrary varieties with an isolated normal singularity, and it coincides with the GSV-index when the variety in question is complete intersection. Recently, T. Suwa in [31] gave a new interpretation of the GSV-index as a residue arising from a certain localization of the Chern class of the ambient tangent bundle.
M. Brunella in [4] introduced the GSV-index for 1-dimensional singular foliations in complex surfaces in terms of the germs of 1-forms inducing the foliation

and established a relation between the GSV-index with the *Khanedani-Suwa variational index* [27] and *Camacho-Sad index* [7]. Moreover, he showed that the non-negativity of the GSV-index is the obstruction to the solution of the Poincaré problem in complex compact surfaces. We recall that the *Poincaré problem* is a question proposed by H. Poincaré in [24] of bounding the degree of algebraic solutions of an algebraic differential equation on the complex plane. Many authors have been working on the Poincaré problem and its generalizations for Pfaff systems, see for instance the papers by D. Cerveau and A. Lins Neto [9], M. G. Soares [32], M. Brunella and L. G. Mendes [5], E. Esteves and S. L. Kleiman [17], V. Cavalier and D. Lehmann [8], E. Esteves and J. D. A. Cruz [16], M. Corrêa and M. Jardim [10], and M. Corrêa and M. G. Soares [14, 15].

In this work, our main goal is to introduce a GSV type index for Pfaff systems on projective manifolds and demonstrate some of its important properties. More precisely, we prove the following Theorem.

Theorem 1.1. *Let X be a projective manifold and $V \subset X$ a reduced local complete intersection subvariety of codimension k invariant by a Pfaff system, of rank k, induced by a twisted form $\omega \in \mathrm{H}^0(X, \Omega_X^k \otimes \mathcal{N})$. Then the following hold:*

(a) there exists a complex number $\mathrm{GSV}(\omega, V, S_i)$ which depends only on the local representatives of ω, V and S_i;

(b) if $\mathrm{Sing}(\omega, V) := \mathrm{Sing}(\omega) \cap V$ has codimension one in V, then the following formula holds

$$\sum_i \mathrm{GSV}(\omega, V, S_i)[S_i] = c_1([\mathcal{N} \otimes \det(N_{V/X})^{-1}])|_V \frown [V],$$

where S_i denotes an irreducible component of $\mathrm{Sing}(\omega, V)$ and $N_{V/X}$ is the normal sheaf of the subvariety V.

The next result says us how to calculate the GSV-index which can be compared with Suwa's formula in [31, Proposition 5.1].

Theorem 1.2. *Setting as in Theorem 1.1. Let $x \in V$, let $\{f_1 = \cdots = f_k = 0\}$ be a local equation of V in a neighborhood U of x and consider*

$$\omega_{|U} = \sum_{|I|=k} a_I dZ_I, \quad a_I \in \mathcal{O}(U).$$

the holomorphic k-form inducing the Pfaff system $\omega \in \mathrm{H}^0(X, \Omega_X^k \otimes \mathcal{N})$ on U. Then the following formula holds

$$\mathrm{GSV}(\omega, V, S_i) = \mathrm{ord}_{S_i}(a_I\,|_V) - \mathrm{ord}_{S_i}(\Delta_I\,|_V). \tag{1}$$

where Δ_I is the $k \times k$ minor of the Jacobian matrix $\mathrm{Jac}(f_1, \dots, f_k)$, corresponding to the multi-index I.

The formula (1) allows us to prove the following non-negativity properties for the index:

(i) If $S_i \cap \mathrm{Sing}(V) = \emptyset$, then $\mathrm{GSV}(\omega, V, S_i) \geq 0$.

(ii) If V is smooth, then $\mathrm{GSV}(\omega, V, S_i) > 0$.

See corollaries 3.3 and 3.4 in the section 3.
As an application, we prove, in the section 4, that the non-negativity of the GSV-index gives an obstruction to the solution of the Poincaré Problem for Pfaff systems on projective complex space. More precisely, let $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+k+1))$ be a holomorphic Pfaff system of rank k and degree d. Let $V \subset \mathbb{P}^n$ be a reduced complete intersection variety, of codimension k and multidegree $(d_1, \ldots, d_k)$, invariant by ω. Suppose that $\mathrm{Sing}(\omega, V)$ has codimension one in V, then

$$\sum_i \mathrm{GSV}(\omega, V, S_i)\deg(S_i) \;=\; [d+k+1-(d_1+\cdots+d_k)]\cdot(d_1\cdots d_k),$$

where S_i denotes an irreducible component of $\mathrm{Sing}(\omega, V)$. Therefore, if $\mathrm{GSV}(\omega, V, S_i) \geq 0$, for all i, we have

$$d_1+\cdots+d_k \leq d+k+1.$$

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2 Preliminaries

2.1 Holomorphic Pfaff Systems

Given a complex manifold X of dimension n, we denote by Ω^p_X the sheaf of germs of holomorphic p-forms on X.

Definition 2.1. Let X be an n-dimensional complex manifold. A holomorphic Pfaff system of rank p $(1 \leq p \leq n)$ on X is a non-trivial section

$$\omega \in \mathrm{H}^0(X, \Omega^p_X \otimes \mathcal{N}),$$

where $\mathcal{N}$ is a holomorphic line bundle on X. The singular set of ω is defined by $\mathrm{Sing}(\omega) = \{z \in X;\ \omega(z) = 0\}$.

Given a Pfaff system ω of rank p on X, then ω is determined by the following:

(i) an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of X;

(ii) holomorphic p-forms $\omega_\alpha \in \Omega^p_{U_\alpha}$ satisfying

$$\omega_\alpha = (h_{\alpha\beta})\omega_\beta \qquad \text{on} \qquad U_\alpha \cap U_\beta \neq \emptyset,$$

where $h_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)^*$ determines the cocycle representing $\mathcal{N}$. For more details on Pfaff systems see [11, 12, 17].

Definition 2.2. We say that an analytic subvariety $V \subset X$ is *invariant* by a Pfaff system ω if $i^*\omega \equiv 0$, where $i : V \hookrightarrow X$ is the inclusion map.

Let ω be a Pfaff system of rank p on X and V an analytic subvariety of X of pure codimension k . Suppose that for each $\alpha \in \Lambda$ we have

$$V \cap U_\alpha = \{z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0\},$$

where $f_{\alpha,1}, \ldots, f_{\alpha,k} \in \mathcal{O}(U_\alpha)$. If V is invariant by ω, then for each $i \in \{1, \ldots, k\}$ there exist holomorphic $(p+1)$-forms $\theta^\alpha_{i1}, \ldots, \theta^\alpha_{ik} \in \Omega^{p+1}_{U_\alpha}$, such that

$$\omega_\alpha \wedge df_{\alpha,i} = f_{\alpha,1}\theta^\alpha_{i1} + \cdots + f_{\alpha,k}\theta^\alpha_{ik}. \tag{2}$$

2.2 Pfaff Systems on $\mathbb{P}^n$

Let $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(r))$ be a holomorphic Pfaff system of rank k on $\mathbb{P}^n$. Now take a generic non-invariant linearly embedded subspace $i : H \simeq \mathbb{P}^k \hookrightarrow \mathbb{P}^n$. We have an induced non-trivial section

$$i^*\omega \in \mathrm{H}^0(H, \Omega^k_H(r)) \simeq \mathrm{H}^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(-k-1+r)),$$

since $\Omega^k_{\mathbb{P}^k} = \mathcal{O}_{\mathbb{P}^k}(-k-1)$. The tangency set between ω and H, denoted by $Z(i^*\omega)$, is defined as the hypersurface of zeros of $i^*\omega$ on H. The *degree* of ω, denoted by $\deg(\omega)$, is defined as the degree of $Z(i^*\omega)$ in H and, therefore, is given by

$$\deg(\omega) = -k - 1 + r.$$

In particular, in this case, $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+k+1))$, where $\deg(\omega) = d$. A Pfaff system of degree d can be induced by a polynomial k-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$, see for instance [12, 13].

2.3 GSV-index on Surfaces

In this section we present Brunella's definition of the GSV-index for one-dimensional holomorphic foliations on surfaces, see [4].

Let X be a complex compact surface and $\mathscr{F}$ a one-dimensional holomorphic foliation on X. Let C be a reduced curve on X. Consider $\omega \in \mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{N})$

a rank one Pfaff system inducing $\mathscr{F}$. If C is invariant by $\mathscr{F}$ we say that $\mathscr{F}$ is *logarithmic along* C.
Given a point $x \in C$, let $f = 0$ be a local equation of C in a neighborhood U_α of x and let ω_α be the holomorphic 1-form inducing the foliation $\mathscr{F}$ on U_α. Since $\mathscr{F}$ is logarithmic along C, it follows from [25, 23, 30] that there are holomorphic functions g and ξ defined in a neighborhood of x, that do not vanish identically both of them simultaneously on C, such that

$$g\,\frac{\omega_\alpha}{f} = \xi\,\frac{df}{f} + \eta, \tag{3}$$

with η being a suitable holomorphic 1-form. M. Brunella in [4] showed that the GSV-index can be defined as follows:

Definition 2.3 (Brunella [4]). Let $\mathscr{F}$ be a one-dimensional holomorphic foliation on a complex compact surface X and logarithmic along a reduced curve $C \subset X$. Given $x \in C$, we define

$$\mathrm{GSV}(\mathscr{F}, C, x) = \sum_i \mathrm{ord}_x \left(\frac{\xi}{g}|_{C_i}\right),$$

where $C_i \subset C$ are irreducible components of C and $\mathrm{ord}_x\left(\frac{\xi}{g}|_{C_i}\right)$ denotes the order of vanishing of $\frac{\xi}{g}|_{C_i}$ at x.

Theorem 2.4 (Brunella [4]). *Let $\mathscr{F}$ be a one-dimensional holomorphic foliation on a complex compact surface X and logarithmic along a reduced curve $C \subset X$. Then*

$$\sum_{x \in Sing(\mathscr{F}) \cap C} \mathrm{GSV}(\mathscr{F}, C, x) = \mathcal{N} \cdot C - C \cdot C\,.$$

2.4 Decomposition of meromorphic forms

A. G. Aleksandrov in [1] introduced the concept of *multiple residues* of a logarithmic differential form with poles along a complete intersection which is a generalization of Saito's residues [25].
In this section, we make a brief presentation about a decomposition of meromorphic forms with poles along complete intersections which is used in the definition of the GSV-index for Pfaff systems.
Let U be a germ of n-dimensional complex manifold. Let D be an analytic reduced hypersurface on U and consider its decomposition into irreducible components

$$D = D_1 \cup \cdots \cup D_k,$$

and suppose that the analytic subvariety $V = D_1 \cap \cdots \cap D_k$ has pure codimension k. We assume that

$$V = \{z \in U : f_1(z) = \cdots = f_k(z) = 0\},$$

with $f_1, \ldots, f_k \in \mathcal{O}(U)$ and for each $i \in \{1, \ldots, k\}$,

$$D_i = \{z \in U : f_i(z) = 0\}.$$

Since V is a reduced variety, then the k-form $df_1 \wedge \ldots \wedge df_k$ is not identically zero on each irreducible component of V.
We denote by $\Omega^q_U(\hat{D}_i)$, $q \geq 1$, the $\mathcal{O}_U$-module of meromorphic differential q-forms with simple poles on the $\hat{D}_i = D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_k$, for each $i = 1, 2, \ldots, k$.

Theorem 2.5 (Aleksandrov [1]). *Let $\omega \in \Omega^q_U(D)$ be a meromorphic q-form with simple poles on D. If for each $j = 1, \ldots, k$,*

$$df_j \wedge \omega \in \sum_{i=1}^{k} \Omega^{q+1}_U(\hat{D}_i)$$

then, there exist a holomorphic function g, which is not identically zero on every irreducible component of V, a holomorphic $(q-k)$-form $\xi \in \Omega^{q-k}_U$ and a meromorphic q-form $\eta \in \sum_{i=1}^{k} \Omega^q_U(\hat{D}_i)$ such that the following decomposition holds

$$g\omega = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_k}{f_k} \wedge \xi + \eta. \tag{4}$$

Since g is not identically zero on every irreducible component of V, the restriction

$$\frac{\xi}{g}|_V$$

is well defined and it is called the *multiple residue* of the meromorphic q-form ω.

Proposition 2.6 (Aleksandrov [1]). *The multiple residues of the meromorphic q-form ω do not depend on the decomposition (4).*

Remark 2.7. *It follows from [1, remark 2] that in the decomposition (4), the function g belongs to the ideal of $\mathcal{O}(U)$ generated by all $k \times k$ minors of the Jacobian matrix* $\mathrm{Jac}(f_1, \ldots f_k)$ *of the map*

$$z \in U \longmapsto (f_1(z), \ldots, f_k(z)) \in \mathbb{C}^k.$$

In other words, if for each multi-index $I = (i_1, \ldots, i_k)$, $1 \leq i_1, \ldots, i_k \leq n$, we denote the corresponding minor of $\mathrm{Jac}(f_1, \ldots f_k)$ *by*

$$\Delta_I = \det \left[\frac{\partial f_i}{\partial z_{i_r}} \right], \quad 1 \leq i, r \leq k,$$

then g can be written as

$$g = \sum_{|I|=k} \lambda_I \, \Delta_I, \qquad \lambda_I \in \mathcal{O}(U). \tag{5}$$

Moreover, if the meromorphic q-form ω is represented in U by

$$\omega = \frac{1}{f_1 \cdot \ldots \cdot f_k} \sum_{|J|=q} a_J(z) \, dZ_J, \quad a_J \in \mathcal{O}(U) \ ,$$

then for each minor Δ_I the following identity holds:

$$\begin{aligned} \Delta_I \sum_{|J|=q} a_J \, dZ_J \quad &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6) \\ &= \quad df_1 \wedge \cdots \wedge df_k \wedge \left(\sum_{|I'|=q-k} a_{(I,I')} \, dZ_{I'} \right) + (f_1 \cdots f_k) \, \eta. \end{aligned}$$

Thus, for the special case where $q = k$, we have that $\xi \in \mathcal{O}(U)$ is given by

$$\xi = \sum_{|I|=k} \lambda_I \, a_I \, . \tag{7}$$

Proposition 2.8. *Let $\omega \in \mathrm{H}^0(X, \Omega_X^p \otimes \mathcal{N})$ be a Pfaff system of rank p on a complex manifold X, and $V \subset X$ a reduced local complete intersection subvariety of codimension k which is invariant by ω. Then for all local representations $\omega_\alpha = \omega|_{U_\alpha}$ of ω, and all local expressions of V in U_α*

$$V \cap U_\alpha = \{ z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0 \},$$

there exist a holomorphic function $g_\alpha \in \mathcal{O}(U_\alpha)$, a holomorphic $(p-k)$-form $\xi_\alpha \in \Omega_{U_\alpha}^{p-k}$ and a holomorphic p-form $\eta_\alpha \in \Omega_{U_\alpha}^p$, such that

$$g_\alpha \, \omega_\alpha = df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k} \wedge \xi_\alpha + \eta_\alpha. \tag{8}$$

Moreover, g_α is not identically zero on every irreducible component of V and η_α is given by

$$\eta_\alpha = f_{\alpha,1} \, \eta_{\alpha,1} + \cdots + f_{\alpha,1} \, \eta_{\alpha,k},$$

where each $\eta_{\alpha,i} \in \Omega_{U_\alpha}^p$ is a holomorphic p-form.

Proof. Consider for each $i \in \{1, \ldots, k\}$

$$D_i = \{ z \in U_\alpha : f_{\alpha,i}(z) = 0 \},$$

and

$$\hat{D}_i = D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_k.$$

Since V is invariant by ω it follows from the expression (2) that for each $i \in \{1, \ldots, k\}$, there exist differential $(p+1)$-forms $\theta^{\alpha}_{i1}, \ldots, \theta^{\alpha}_{ik} \in \Omega^{p+1}_{U_\alpha}$, such that

$$\omega_\alpha \wedge df_{\alpha,i} = f_{\alpha,1}\theta^{\alpha}_{i1} + \cdots + f_{\alpha,k}\theta^{\alpha}_{ik}.$$

With this, we deduce that

$$df_{\alpha,j} \wedge \frac{\omega_\alpha}{f_\alpha} \in \sum_{i=1}^{k} \Omega^p_{U_\alpha}(\hat{D}_i), \qquad j = 1, \ldots, k.$$

Hence, the meromorphic p-form $\dfrac{\omega_\alpha}{f_\alpha}$ satisfies the hypothesis of Theorem 2.5 and the decomposition (8) follows from decomposition (4). $\square$

The decomposition (8) will be called an *Aleksandrov-Saito's decomposition of* ω in U_α.

3 GSV-index for Pfaff system on projective manifolds

In this section we define the GSV-index for Pfaff systems on projective manifolds X.

Let X be a projective manifold. Consider a Pfaff system

$$\omega \in \mathrm{H}^0(X, \Omega^p_X \otimes \mathcal{N})$$

of rank p and V a reduced local complete intersection subvariety of pure codimension k invariant by ω. Let us denote $\mathrm{Sing}(\omega, V) := \mathrm{Sing}(\omega) \cap V$. We also assume that the rank of ω coincides with the codimension of V, i.e., $p = k$. Fixed an irreducible component S_i of $\mathrm{Sing}(\omega, V)$, let us take $\omega_\alpha = \omega|_{U_\alpha}$, a local representation of ω, such that $U_\alpha \cap S_i \neq \emptyset$. Assume that ω_α is defined by

$$\omega_\alpha = \sum_{|I|=k} a_I(z) dZ_I, \quad a_I \in \mathcal{O}(U_\alpha).$$

Also, we consider a local expression of V in U_α, given by

$$V \cap U_\alpha = \{z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0\}.$$

and let us take an Aleksandrov-Saito's decomposition of ω in U_α,

$$g_\alpha\, \omega_\alpha = (df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k})\, \xi_\alpha + \eta_\alpha, \tag{9}$$

with $\eta_\alpha = f_{\alpha,1}\, \eta_{\alpha,1} + \cdots + f_{\alpha,k}\, \eta_{\alpha,k}$, where $\eta_{\alpha,i} \in \Omega^k_{U_\alpha}$ and, furthermore, ξ_α being a holomorphic function.

In this context, we define the GSV-index:

Definition 3.1. Suppose that S is a codimension one subvariety of V. The GSV-index of ω relative to V in S is defined by

$$\mathrm{GSV}(\omega, V, S) := \sum_j \mathrm{ord}_S\left(\frac{\xi_\alpha}{g_\alpha}|_{V_j}\right),$$

where the sum is taken over all irreducible components V_j of V and $\mathrm{ord}_S\left(\frac{\xi_\alpha}{g_\alpha}|_{V_j}\right)$ denotes the order of vanishing of $\frac{\xi_\alpha}{g_\alpha}|_{V_j}$ along S.

We recall that for a rational function r the order of vanishing is defined by $\mathrm{ord}_S(r) = l_{\mathcal{O}_{S,V_j}}(\mathcal{O}_{S,V_j}/(r))$, where $l_{\mathcal{O}_{S,V_j}}(\mathcal{O}_{S,V_j}/(r))$ denotes the length of the $\mathcal{O}_{S,V_j}$-module $\mathcal{O}_{S,V_j}/(r)$, see [18].
Now we will prove our main result.

3.1 Proof of Theorem 1.1

Firstly, it follows from the definition and Proposition 2.6 that $\mathrm{GSV}(\omega, V, S_i)$ does not depend on chosen decomposition of ω. Now, we will prove that $\mathrm{GSV}(\omega, V, S_i)$ does not depend on the local representation ω_α of ω, and does not depend on local expression for V:

$$V \cap U_\alpha = \{z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0\}.$$

In fact, if we consider another local representation $\omega_\beta = \omega|_{U_\beta}$, such that $U_\beta \cap S_i \neq \emptyset$ and other local expression for V

$$V \cap U_\beta = \{z \in U_\beta : f_{\beta,1}(z) = \cdots = f_{\beta,k}(z) = 0\},$$

we obtain the Aleksandrov-Saito's decomposition of ω in U_β

$$g_\beta\, \omega_\beta = (df_{\beta,1} \wedge \cdots \wedge df_{\beta,k})\, \xi_\beta + \eta_\beta \tag{10}$$

and the decomposition of ω in U_α

$$g_\alpha\, \omega_\alpha = (df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k})\, \xi_\alpha + \eta_\alpha, \tag{11}$$

where $\eta_\beta|_V = \eta_\alpha|_V = 0$. Now, in the intersection $U_\alpha \cap U_\beta \neq \emptyset$ we have that

$$(df_{\alpha,1} \wedge \ldots \wedge df_{\alpha,k}) = m_{\alpha\beta}\,(df_{\beta,1} \wedge \ldots \wedge df_{\beta,k}) + \theta_{\alpha,\beta}, \tag{12}$$

where $m_{\alpha\beta}|_V \in \mathcal{O}_V(U_\alpha \cap U_\beta \cap V)^*$ is the cocycle of the determinant of the normal bundle $\det(N_{V/X})$ on V and $\theta_{\alpha,\beta}$ is a holomorphic k-form such that $\theta_{\alpha,\beta}|_V = 0$. Also, in $U_\alpha \cap U_\beta$ we have that

$$\omega_\alpha = h_{\alpha\beta}\omega_\beta, \tag{13}$$

where $h_{\alpha\beta}, \in \mathcal{O}(U_\alpha \cap U_\beta)^*$ is the cocycle of the line bundle $\mathcal{N}$. On the one hand, by using (12) and (13) in (11), we obtain

$$(g_\alpha h_{\alpha\beta})\omega_\beta = (m_{\alpha\beta}\xi_\alpha)(df_{\beta,1} \wedge \ldots \wedge df_{\beta,k}) + \eta_\alpha + \theta_{\alpha,\beta}. \tag{14}$$

On the other hand, by using (10) in (14), we obtain

$$(h_{\alpha\beta}\xi_\beta g_\alpha - m_{\alpha\beta}\xi_\alpha g_\beta)\,(df_{\beta,1}\wedge\ldots\wedge df_{\beta,k}) = \eta_\beta + g_\beta\eta_\alpha + g_\beta\theta_{\alpha,\beta}.$$

Since $(\eta_\beta + g_\beta\eta_\alpha + g_\beta\theta_{\alpha,\beta})|_V \equiv 0$, we conclude that

$$(h_{\alpha\beta}\xi_\beta g_\alpha - m_{\alpha\beta}\xi_\alpha g_\beta)\,(df_{\beta,1}\wedge\ldots\wedge df_{\beta,k}) \equiv 0(\mod (f_{\beta,1},\ldots,f_{\beta,k})).$$

It follows from [26] that there exists $r\in\mathbb{Z}$, with $r\geq 1$, such that

$$\mathscr{D}^r(h_{\alpha\beta}\xi_\beta g_\alpha - m_{\alpha\beta}\xi_\alpha g_\beta) \in df_{\beta,1}\wedge\Omega^{k-1}_{U_\beta} + \cdots + df_{\beta,k}\wedge\Omega^{k-1}_{U_\beta},$$

where $\mathscr{D}$ is the ideal of $\mathcal{O}_{U_\beta}$ generated by all minors of maximal order of the Jacobian matrix $\mathrm{Jac}(f_{\beta,1},\ldots,f_{\beta,k})$. Now, as observed in [1, Proposition 2] the image $\mathrm{Im}(\mathscr{D})$ of $\mathscr{D}$ in the ring $\mathcal{O}_{V\cap U_\beta}$ is not equal to $\mathrm{Ann}(\mathcal{O}_{V\cap U_\beta})$. In fact, since V is reduced, then $df_{\beta,1}\wedge\ldots\wedge df_{\beta,k}$ does not vanish identically on each irreducible component of V. Therefore, it follows from [6, Theorem 2.4. (1)] that the $\mathcal{O}_{V\cap U_\beta}$-depth of the ideal $\mathscr{D}^r$ is at least 1. Then, there is $D\in\mathscr{D}$ which is not a zero-divisor in $\mathcal{O}_{V\cap U_\beta}$ and

$$D^r(h_{\alpha\beta}\xi_\beta g_\alpha - m_{\alpha\beta}\xi_\alpha g_\beta) \in df_{\beta,1}\wedge\Omega^{k-1}_{U_\beta} + \cdots + df_{\beta,k}\wedge\Omega^{k-1}_{U_\beta}.$$

Thus, the class $D^r(h_{\alpha\beta}\xi_\beta g_\alpha - m_{\alpha\beta}\xi_\alpha g_\beta) = 0 \in \Omega^k_{V\cap U_\beta}$. Then

$$(h_{\alpha\beta}\xi_\beta g_\alpha - m_{\alpha\beta}\xi_\alpha g_\beta)|_V = 0$$

since $D\in\mathscr{D}$ is not a zero-divisor. We conclude that

$$\frac{\xi_\alpha}{g_\alpha}|_V = h_{\alpha\beta}\,(m_{\alpha\beta})^{-1}\,\frac{\xi_\beta}{g_\beta}|_V. \tag{15}$$

Thus, we obtain a family of meromorphic functions $\mathfrak{s} = \{(\xi_\alpha/g_\alpha)|_V\}_{\alpha\in\Lambda}$ which defines a rational section of the line bundle $[\mathcal{N}\otimes\det(N_{V/X})^{-1}]|_V$ on V whose support is $\mathrm{Sing}(\omega, V) = \cup_i S_i$. Since $h_{\alpha\beta}$ and $(m_{\alpha\beta})^{-1}$ are non-vanishing holomorphic functions we observe that

$$\mathrm{ord}_{S_i}\left(\frac{\xi_\alpha}{g_\alpha}|_V\right) = \mathrm{ord}_{S_i}\left(\frac{\xi_\beta}{g_\beta}|_V\right).$$

Therefore, we have the Cartier divisor associated to the section $\mathfrak{s}$ given by

$$(\mathfrak{s})_0 = \sum_i \mathrm{GSV}(\omega, V, S_i)[S_i].$$

That is

$$\mathcal{O}((\mathfrak{s})_0) \cong [\mathcal{N}\otimes\det(N_{V/X})^{-1}]|_V.$$

Thus, by normalization property of the Chern class [18, Thm. 3.2,(f)] we have

$$\begin{aligned} c_1([\mathcal{N} \otimes \det(N_{V/X})^{-1}]|_V) &= c_1(\mathcal{O}((\mathfrak{s})_0)) \\ &= \sum_i \mathrm{GSV}(\omega, V, S_i)[S_i]. \end{aligned}$$

Therefore

$$\sum_i \mathrm{GSV}(\omega, V, S_i)[S_i] = c_1([\mathcal{N} \otimes \det(N_{V/X})^{-1}]|_V) \frown [V].$$

The next result says us how to calculate the GSV-index which can be compared with Suwa's formula in [31, Proposition 5.1].

Theorem 3.2. *For each multi-index I, with $\mid I \mid = k$, the following formula holds*

$$\mathrm{GSV}(\omega, V, S_i) = \mathrm{ord}_{S_i}(a_I\,|_V) - \mathrm{ord}_{S_i}(\Delta_I\,|_V).$$

where Δ_I is the $k \times k$ minor of the Jacobian matrix $\mathrm{Jac}(f_{\alpha,1}, \ldots, f_{\alpha,k})$, corresponding to the multi-index I.

Proof. By the expression (6) we have

$$(\Delta_I . \sum_{|J|=k} a_J\, dZ_J)|_V = [(df_{\alpha,1} \wedge \ldots \wedge df_{\alpha,k})\, a_I]|_V.$$

Since

$$df_{\alpha,1} \wedge \ldots \wedge df_{\alpha,k} = \sum_{|J|=k} \Delta_J\, dZ_J,$$

for each J, with $\mid J \mid = k$, we get

$$(\Delta_I\, a_J)|_V = (\Delta_J\, a_I)|_V. \tag{16}$$

Thus,

$$\begin{aligned} \mathrm{GSV}(\omega, V, S_i) &= \mathrm{ord}_{S_i}\left(\frac{\xi_\alpha}{g_\alpha}|_V\right) \\ &= \mathrm{ord}_{S_i}\left(\frac{\xi_\alpha}{g_\alpha}|_V\right) + \mathrm{ord}_{S_i}(\Delta_I|_V) - \mathrm{ord}_{S_i}(\Delta_I|_V) \\ &= \mathrm{ord}_{S_i}\left(\frac{\xi_\alpha\, \Delta_I}{g_\alpha}|_V\right) - \mathrm{ord}_{S_i}(\Delta_I|_V). \end{aligned}$$

By (7) we have

$$\xi_\alpha = \sum_{|J|=k} \lambda_J \, a_J.$$

Hence, by using (16) we obtain

$$\begin{aligned} \xi_\alpha \, \Delta_I|_V &= \sum_{|J|=k} \lambda_J \, (a_J \, \Delta_I)|_V = \sum_{|J|=k} \lambda_J \, (\Delta_J \, a_I)|_V \\ &= \Big(\sum_{|J|=k} \lambda_J \, \Delta_J\Big) \, a_I|_V = g_\alpha \, a_I|_V, \end{aligned}$$

where in the last step we have used (5). Thus,

$$\mathrm{ord}_{S_i}\left(\frac{\xi_\alpha \, \Delta_I}{g_\alpha}|_V\right) = \mathrm{ord}_{S_i}\,(a_I\,|_V)\,.$$

Then

$$\mathrm{GSV}(\omega, V, S_i) = \mathrm{ord}_{S_i}(a_I\,|_V) - \mathrm{ord}_{S_i}(\Delta_I\,|_V).$$

□

Corollary 3.3. *If* $S_i \cap \mathrm{Sing}(V) = \emptyset$, *then* $\mathrm{GSV}(\omega, V, S_i) \geq 0$.

Proof. By Theorem 3.2, for each multi-index I, with $|\, I \,| = k$, we have

$$\mathrm{GSV}(\omega, V, S_i) = \mathrm{ord}_{S_i}(a_I\,|_V) - \mathrm{ord}_{S_i}(\Delta_I\,|_V). \tag{17}$$

Let $h_i \in \mathcal{O}_V(U_\alpha \cap V)$ be a function which defines locally S_i in $U_\alpha \cap V$, i.e.,

$$(U_\alpha \cap V) \cap S_i = \{z \in U_\alpha \cap V : h_i(z) = 0\}.$$

If $\mathrm{GSV}(\omega, V, S_i) < 0$ then, by (17), for each multi-index I, with $|\, I \,| = k$,

$$\mathrm{ord}_{S_i}(\Delta_I\,|_V) = \delta_I > 0.$$

Therefore, we obtain

$$\Delta_I|_V = h_i^{\delta_I} \, \mu_I|_V,$$

for some function $\mu_I \in \mathcal{O}_V(U_\alpha \cap V)$. This implies that for each multi-index I, with $|\, I \,| = k$, the following inclusion occurs

$$\{z \in U_\alpha \cap V : h_i(z) = 0\} \subset \{z \in U_\alpha \cap V : \Delta_I(z) = 0\}.$$

Thus, we conclude that

$$\begin{aligned}(U_\alpha \cap V) \cap S_i &= \{z \in U_\alpha \cap V : h_i(z) = 0\} \subset \\ &\subset \bigcap_{|I|=k} \{z \in U_\alpha \cap V : \Delta_I(z) = 0\} = U_\alpha \cap \mathrm{Sing}(V)\end{aligned}$$

and consequently, we get $S_i \cap \mathrm{Sing}(V) \neq \emptyset$. □

Corollary 3.4. *If V is smooth, then* $\mathrm{GSV}(\omega, V, S_i) > 0$*.*

Proof. If V is smooth, then there exists some $k \times k$ minor Δ_I (of Jacobian matrix $\mathrm{Jac}(f_{\alpha,1}, \dots, f_{\alpha,k})$) which is not zero along V and in this way, we obtain

$$\mathrm{ord}_{S_i}(\Delta_I|_V) = 0. \tag{18}$$

Therefore, it follows from Theorem 3.2 that

$$\mathrm{GSV}(\omega, V, S_i) = \mathrm{ord}_{S_i}(a_I\,|_V) - \mathrm{ord}_{S_i}(\Delta_I\,|_V) = \mathrm{ord}_{S_i}(a_I\,|_V) > 0.$$

□

4 An Application: Poincaré problem for Pfaff systems

In this section we show that the non-negativity of the GSV-index gives an obstruction to the solution of the Poincaré problem for Pfaff systems on projective spaces. This application was motivated by a result due to M. Brunella in [4].

Theorem 4.1. *Let $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+k+1))$ be a holomorphic Pfaff system of rank k and degree d. Let $V \subset \mathbb{P}^n$ be a reduced complete intersection variety, of codimension k and multidegree $(d_1, \dots, d_k)$, invariant by ω. Suppose that $\mathrm{Sing}(\omega, V)$ has codimension one in V, then*

$$\sum_i \mathrm{GSV}(\omega, V, S_i)\deg(S_i) \;=\; [d+k+1-(d_1+\dots+d_k)]\cdot(d_1\cdots d_k),$$

where S_i denotes an irreducible component of $\mathrm{Sing}(\omega, V)$. In particular, if $\mathrm{GSV}(\omega, V, S_i) \geq 0$, for all i, we have

$$d_1 + \dots + d_k \leq d + k + 1.$$

Proof. By applying Theorem 1.1 and taking degrees we obtain

$$\sum_i \mathrm{GSV}(\omega, V, S_i)\deg(S_i) \;=\; \deg\left(c_1([\mathcal{N} \otimes \det(N_{V/\mathbb{P}^n})^{-1}]|_V \frown [V]\right).$$

On the one hand, the normal bundle of V is given by

$$N_{V/\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k)|_V.$$

Thus

$$\det(N_{V/\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(d_1 + \cdots + d_k)|_V.$$

On the other hand, we have that

$$\mathcal{N} = \mathcal{O}_{\mathbb{P}^n}(d + k + 1).$$

Now, since

$$[V] = (d_1 \cdots d_k)\ c_1(\mathcal{O}(1))^k$$

we get that $\deg([\mathcal{N} \otimes \det(N_{V/\mathbb{P}^n})^{-1}]|_V \frown [V])$ is equal to

$$[(d + k + 1) - (d_1 + \cdots + d_k)]\ \cdot (d_1 \cdots d_k).$$

Therefore, we obtain

$$\sum_i \mathrm{GSV}(\omega, V, S_i) \deg(S_i) \quad = \quad [d + k + 1 - (d_1 + \cdots + d_k)]\ \cdot (d_1 \cdots d_k).$$

Now, if $\mathrm{GSV}(\omega, V, S_i) \geq 0$, for all i, we have

$$0 \leq \sum_i \mathrm{GSV}(\omega, V, S_i) \deg(S_i) \quad = \quad [d + k + 1 - (d_1 + \cdots + d_k)]\ \cdot (d_1 \cdots d_k).$$

This implies that

$$d_1 + \cdots + d_k \leq d + k + 1.$$

□

In the next result we obtain a bound similar to those by Esteves–Cruz [16] and Corrêa–Jardim [10].

Corollary 4.2. *Let $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+k+1))$ be a holomorphic Pfaff system of rank k and degree d. Let $V \subset \mathbb{P}^n$ be a reduced complete intersection variety of codimension k and multidegree $(d_1, \ldots, d_k)$ invariant by ω. Suppose that $\mathrm{Sing}(\omega, V)$ has codimension one in V and that $S_i \cap \mathrm{Sing}(V) = \emptyset$ for all i, then*

$$d_1 + \cdots + d_k \leq d + k + 1.$$

Moreover, if V is regular we have

$$d_1 + \cdots + d_k \leq d + k.$$

Proof. The result follows from Corollary 3.4 and Corollary 3.3. □

Now we give an optimal example.

Example 4.3. Consider the Pfaff system $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+k+1))$ given by

$$\omega = \sum_{0 \leq j \leq k} (-1)^j d_j f_j \; df_0 \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k,$$

where f_j is a homogeneous polynomial of degree d_j. We can see that

$$d_0 + d_1 + \cdots + d_k = d + k + 1.$$

Suppose that $\deg(f_0) = d_0 = 1$ and that $V = \{f_1 = \cdots = f_k = 0\}$ is smooth. We have that V is invariant by ω and

$$d_1 + \cdots + d_k = d + k.$$

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