Cotilting Sheaves over Weighted Noncommutative Regular Projective Curves

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Abstract. We consider the category $\operatorname{Qcoh}\mathbb{X}$ of quasicoherent sheaves where $\mathbb{X}$ is a weighted noncommutative regular projective curve over a field k. This category is a hereditary, locally noetherian Grothendieck category. We classify all indecomposable pure-injective sheaves and all cotilting sheaves of slope ∞. In the cases of non-negative orbifold Euler characteristic this leads to a classification of pure-injective indecomposable sheaves and a description of all large cotilting sheaves in $\operatorname{Qcoh}\mathbb{X}$.

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Contents

1   Introduction

The study of large cotilting objects originates in the context of the representation theory of associative rings, where it amounts to the study of (tilting) derived equivalences between the module category and Grothendieck categories ([56]). A generalisation of cotilting to the setting of Grothendieck categories was provided in [8], and investigated in greater depth in [10]. As cotilting objects are automatically pure-injective (unlike the dual notion of a tilting object), the classifications of cotilting objects and of indecomposable pure-injective objects are strongly related to each other. In this paper we consider these classification problems for a certain class of Grothendieck categories that are not module categories: the categories $\mathrm{Qcoh}\,\mathbb{X}$ of quasicoherent sheaves over weighted noncommutative regular projective curves over a field k. We emphasize that each smooth projective curve is included in this setting as a special case.

Each such category $\mathrm{Qcoh}\,\mathbb{X}$ is determined by its full subcategory $\mathrm{coh}\,\mathbb{X}$ of finitely presented objects. The category $\mathrm{coh}\,\mathbb{X}$ is, by definition, a k-linear abelian category that shares important characteristics with classical categories of coherent sheaves over (commutative) projective curves. In fact, the categories $\mathrm{coh}\,\mathbb{X}$ have been axiomatised ([36]) and subsequently studied by several authors (for example, [31, 1]). In particular, the category $\mathrm{coh}\,\mathbb{X}$ is a small hereditary abelian category in which every object is noetherian.

The structure of the category $\mathrm{Qcoh}\,\mathbb{X}$ is less well-understood than $\mathrm{coh}\,\mathbb{X}$ and is likely to be beyond any hope of classification or description as a whole. In this article, we systematically study the full subcategory of pure-injective sheaves in $\mathrm{Qcoh}\,\mathbb{X}$, in the sense of [11, 19]. This subcategory properly contains the subcategory $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves and, moreover, it constitutes a tractible subcategory of $\mathrm{Qcoh}\,\mathbb{X}$, due to the fact that we may make use of the pure-exact structure.

For arbitrary $\mathbb{X}$ we are able to give the following description of the indecomposable pure-injective sheaves E of slope ∞, that is, those which satisfy additionally $\mathrm{Hom}(E, \mathrm{vect}\,\mathbb{X}) = 0$.

THEOREM (5.11). *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve over a field k. The following is a complete list of indecomposable pure-injective objects in $\mathrm{Qcoh}\,\mathbb{X}$ of slope ∞.*

(1) The indecomposable sheaves of finite length.

(2) The sheaf $\mathcal{K}$ of rational functions, the Prüfer and the adic sheaves.

Moreover, each pure-injective sheaf of slope ∞ is discrete and thus uniquely determined by its indecomposable summands.

If we assume that $\operatorname{coh}\mathbb{X}$ is of tame representation type (which means that the orbifold Euler characteristic of $\mathbb{X}$ is nonnegative), then we can extend this classification to the sheaves of rational and infinite slope. In the case of positive Euler characteristic, we describe all the indecomposable pure-injective sheaves in $\operatorname{Qcoh}\mathbb{X}$. In particular, we show that, when the orbifold Euler characteristic of $\mathbb{X}$ is nonnegative, the form of the indecomposable pure-injective sheaves is analogous to the case of modules over concealed canonical algebras ([2]). We recall that in case of orbifold Euler characteristic zero each indecomposable object has a slope, which is a real number or infinite, by [45, 1].

THEOREM (7.1 and 8.11). *Let $\chi'_{orb}(\mathbb{X})$ denote the orbifold Euler characteristic of $\mathbb{X}$. Then the following statements hold.*

(1) *If $\chi'_{orb}(\mathbb{X}) > 0$ (i.e. if $\mathbb{X}$ is a domestic curve), then each indecomposable pure-injective sheaf in $\operatorname{Qcoh}\mathbb{X}$ either has slope ∞, and thus is as in the preceding theorem, or is a vector bundle.*

(2) *If $\chi'_{orb}(\mathbb{X}) = 0$ (i.e. if $\mathbb{X}$ is a tubular or an elliptic curve), then the following is a complete list of indecomposable pure-injective sheaves of rational or infinite slope w in $\operatorname{Qcoh}\mathbb{X}$.*

   (a) *The indecomposable coherent sheaves.*
   (b) *The generic, the Prüfer and the adic sheaves of slope w.*

We also classify the cotilting sheaves in $\operatorname{Qcoh}\mathbb{X}$, which allows us to determine the existence of pure-injective sheaves of irrational slope. For arbitrary $\mathbb{X}$, we have the following parametrisation of the cotilting sheaves in $\operatorname{Qcoh}\mathbb{X}$ of slope ∞; for the complete statement we refer to Theorem 6.11. Note that branch sheaves are certain rigid coherent sheaves contained in non-homogeneous tubes and are defined in Section 6.

THEOREM (6.11). *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve over a field k. The cotilting sheaves C in $\operatorname{Qcoh}\mathbb{X}$ of slope ∞ are parametrized by pairs (B, V) where V is a subset of $\mathbb{X}$ and B a branch sheaf.*

In the theorem, the cotilting module C is uniquely determined by its torsion part, which is given as a direct sum of B and a coproduct of Prüfer sheaves concentrated in V; the set of the indecomposable summands of the torsionfree part is then given by certain "complementing" adic sheaves concentrated in $\mathbb{X} \setminus V$ (and $\mathcal{K}$, if $V = \emptyset$).

In the cases of nonnegative orbifold Euler characteristic we show that every large (=non-coherent) cotilting sheaf C in $\operatorname{Qcoh}\mathbb{X}$ has a well-defined slope w (see Theorem 7.1 and Theorem 8.13) and, moreover, the equivalence class of C is completely determined by a set of indecomposable pure-injective sheaves (see Proposition 3.19). We have the following parametrisation of the large cotilting sheaves in $\operatorname{Qcoh}\mathbb{X}$. Note that branch sheaves of rational slope are defined in Section 8.

Theorem (7.1 and 8.16). *Under the assumptions of first theorem above, the following statements hold.*

(1) *If $\chi'_{orb}(\mathbb{X}) > 0$, then all large cotilting sheaves in $\mathrm{Qcoh}\,\mathbb{X}$ are of slope ∞.*

(2) *If $\chi'_{orb}(\mathbb{X}) = 0$, then all large cotilting sheaves in $\mathrm{Qcoh}\,\mathbb{X}$ have a well-defined slope w and are parametrised as follows.*

   (a) *If w is rational or ∞, then the large cotilting sheaves of slope w are parametrised (up to equivalence) by pairs (B_w, V_w) where B_w is a branch sheaf of slope w and $V_w \subseteq \mathbb{X}_w$.*

   (b) *If w is irrational, then there is a unique large cotilting sheaf $\mathbf{W}_w$ of slope w (up to equivalence).*

In the case where w is rational or ∞, we provide an explicit description of the minimal set of indecomposable direct summands of C (see Theorem 8.16) in terms of the classification given in Theorem 5.11. Moreover, we describe the pure-injective sheaves of irrational slope in terms of the large cotilting sheaves given in Theorem 8.3.

Corollary (8.9). *If $\chi'_{orb}(\mathbb{X}) = 0$ and w is irrational, then $\mathrm{Prod}(\mathbf{W}_w)$ is the set of the pure-injective sheaves of slope w.*

The form of the indecomposable pure-injective sheaves of irrational slope is not known but we show that there is a direct connection between the indecomposable direct summands of $\mathbf{W}_w$ and the simple objects in the heart $\mathcal{G}_w$ of the HRS-tilted t-structure (see Proposition 3.19). This perspective therefore provides an interesting future strategy for investigating the indecomposable pure-injective sheaves of irrational slope. In particular, in relation to the recent description of some simple objects in $\mathcal{G}_w$ in the case where $\mathbb{X}$ is of tubular type (see [44] and a forthcoming preprint by A. Rapa and J. Šťovíček). Therefore we will exhibit and prove some basic properties of the categories $\mathcal{G}_w$ in the final section.

In some sense the results presented here are "dual" to the description of large tilting sheaves of finite type given by the first named author and L. Angeleri Hügel ([1]). In categories of modules over noetherian rings, there is a bijection between (1-)cotilting left modules and (1-)tilting right modules. This bijection is induced by a duality (see [14, Thm. 15.2, Thm. 15.31]), which, in the case of a finite dimensional k-algebra, can be taken to be the functor $D := \mathrm{Hom}_k(-, k)$. Moreover, each cotilting class is the dual definable subcategory of the corresponding tilting class (see [14, Rem. 15.35]). This close relationship is often used to deduce a classification of cotilting modules from a classification of tilting modules (for example, [3, 2]). Although we find a connection to the results of [1] as a consequence of our classification, we employ very different methods to reach our conclusion. In fact, in our setting there is no obvious notion analogous to "left" and "right" modules, let alone a concrete duality relating tilting and cotilting sheaves.

In the absence of an explicit duality, we observe some connection between large tilting sheaves of finite type and large cotilting sheaves in the following theorem, as well as in Proposition 6.10 and Lemma 8.12.

THEOREM (4.4). *If $T \in \mathrm{Qcoh}\,\mathbb{X}$ is a tilting sheaf of finite type there is a cotilting sheaf $C \in \mathrm{Qcoh}\,\mathbb{X}$ such that $^{\perp_1}C \cap \mathrm{coh}\,\mathbb{X} = {}^{\perp_1}(T^{\perp_1}) \cap \mathrm{coh}\,\mathbb{X}$, and conversely. The assignments $\Gamma\colon [T] \mapsto [C]$ and $\Theta\colon [C] \mapsto [T]$ induce mutually inverse bijections between the sets of*

- *equivalence classes $[T]$ of tilting sheaves T of finite type, and of*

- *equivalence classes $[C]$ of cotilting sheaves C.*

In Example 7.2, we observe that this bijection appears to be different in nature to the bijection obtained from dual definable subcategories.

We end this introduction with a summary of the structure of the paper. In Section 2 we introduce the main set of techniques we use to establish our results: the theory of purity in locally finitely presented Grothendieck categories and the theory of purity in compactly generated triangulated categories. We also prove some preliminary results in this setting. Next, in Section 3, we introduce the definition of a cotilting object in a Grothendieck category. We summarise the connections between properties of cotilting objects and the injective cogenerators in HRS-tilted categories. In Section 4 we introduce the categories of quasicoherent sheaves over weighted noncommutative regular projective curves. In Section 5 we classify the indecomposable pure-injective sheaves of slope ∞ and in Section 6 we classify the large cotilting sheaves of slope ∞; as mentioned, this is done for all orbifold Euler characteristics. In the final sections we extend these classifications of sheaves of slope infinity to include all slopes in the domestic and the tubular/elliptic cases, respectively. We then study the above-mentioned categories $\mathcal{G}_w$ in the case where w is irrational.

NOTATION

Let $\mathcal{X}$ be a class of objects in a Grothendieck category $\mathcal{A}$. We will use the following notation for orthogonal classes:

$$
\begin{aligned}
\mathcal{X}^{\perp_0} &= \{F \in \mathcal{A} \mid \mathrm{Hom}(\mathcal{X}, F) = 0\}, \quad \mathcal{X}^{\perp_1} = \{F \in \mathcal{A} \mid \mathrm{Ext}^1(\mathcal{X}, F) = 0\}, \\
{}^{\perp_0}\mathcal{X} &= \{F \in \mathcal{A} \mid \mathrm{Hom}(F, \mathcal{X}) = 0\}, \quad {}^{\perp_1}\mathcal{X} = \{F \in \mathcal{A} \mid \mathrm{Ext}^1(F, \mathcal{X}) = 0\}, \\
\mathcal{X}^{\perp} &= \mathcal{X}^{\perp_0} \cap \mathcal{X}^{\perp_1}, \quad {}^{\perp}\mathcal{X} = {}^{\perp_0}\mathcal{X} \cap {}^{\perp_1}\mathcal{X}.
\end{aligned}
$$

By $\mathrm{Add}(\mathcal{X})$ (resp. $\mathrm{add}(\mathcal{X})$) we denote the class of all direct summands of direct sums of the form $\bigoplus_{i \in I} X_i$, where I is any set (resp. finite set) and $X_i \in \mathcal{X}$ for all i. By $\mathrm{Gen}(\mathcal{X})$ we denote the class of all objects Y *generated by* $\mathcal{X}$, that is, such that there is an epimorphism $X \to Y$ with $X \in \mathrm{Add}(\mathcal{X})$ (and similarly $\mathrm{gen}(\mathcal{X})$ with $\mathrm{add}(\mathcal{X})$). As usual we write $X^{(I)}$ for $\bigoplus_{i \in I} X$.

By $\mathrm{Prod}(\mathcal{X})$ we denote the class of all direct summands of products of the form $\prod_{i \in I} X_i$, where I is any set and $X_i \in \mathcal{X}$ for all i. By $\mathrm{Cogen}(\mathcal{X})$ we denote the class of all objects Y *cogenerated by* $\mathcal{X}$, that is, such that there is a monomorphism $Y \to X$ with $X \in \mathrm{Prod}(\mathcal{X})$. We write X^I for $\prod_{i \in I} X$.

We denote by $\varinjlim \mathcal{X}$ the direct limit closure of $\mathcal{X}$ in $\mathcal{A}$. We will often use also the shorthand notation $\vec{\mathcal{X}} = \varinjlim \mathcal{X}$.

Let $(I, \leq)$ be an ordered set and $\mathcal{X}_i$ classes of objects for all $i \in I$, in any additive category. We write $\bigvee_{i \in I} \mathcal{X}_i$ for $\mathrm{add}(\bigcup_{i \in I} \mathcal{X}_i)$ if additionally $\mathrm{Hom}(\mathcal{X}_j, \mathcal{X}_i) = 0$ for all $i < j$ is satisfied. In particular, notation like $\mathcal{X}_1 \vee \mathcal{X}_2$ and $\mathcal{X}_1 \vee \mathcal{X}_2 \vee \mathcal{X}_3$ makes sense (where $1 < 2 < 3$).

2 PURE-INJECTIVITY

The notion of purity is of great importance in our setting. For details we refer to [43, 11]. Let $\mathcal{A}$ be an abelian category. We denote the full subcategory of finitely presented objects in $\mathcal{A}$ by $\mathrm{fp}(\mathcal{A})$.

- We say that $\mathcal{A}$ is *Grothendieck* if all set-indexed coproducts exist, direct limits are exact and $\mathcal{A}$ has a generator.

- We say that $\mathcal{A}$ is *locally finitely presented* if $\mathrm{fp}(\mathcal{A})$ is skeletally small and every object in $\mathcal{A}$ is a direct limit of objects in $\mathrm{fp}(\mathcal{A})$.

- We say that $\mathcal{A}$ is *locally coherent* if $\mathcal{A}$ is locally finitely presented and $\mathrm{fp}(\mathcal{A})$ is abelian.

- If $\mathcal{A}$ is k-linear over a field k, then $\mathcal{A}$ is called *Hom-finite* if $\mathrm{Hom}_{\mathcal{A}}(C, D)$ is a finite-dimensional k-vector space for every pair of objects C and D in $\mathcal{A}$.

- Let $\mathcal{A}$ be k-linear locally coherent and $\mathrm{D} := \mathrm{Hom}_k(-, k)$. Then $\mathrm{fp}(\mathcal{A})$ is said to satisfy *Serre duality* if $\mathrm{fp}(\mathcal{A})$ is Hom-finite and if there is an autoequivalence $\tau \colon \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{A})$ and an isomorphism $\mathrm{D}\,\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \tau X)$, natural in $X, Y \in \mathrm{fp}(\mathcal{A})$. Moreover, $\mathcal{A}$ is said to satisfy *(generalised) Serre duality* if additionally $\mathrm{D}\,\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \tau X)$ and $\mathrm{Ext}^1_{\mathcal{A}}(Y, \tau X) = \mathrm{D}\,\mathrm{Hom}_{\mathcal{A}}(X, Y)$ hold for all objects $Y \in \mathcal{A}$, $X \in \mathrm{fp}(\mathcal{A})$.

REMARK 2.1. Since we have assumed that $\mathcal{A}$ is abelian, we have that, if $\mathcal{A}$ is locally finitely presented, then $\mathcal{A}$ is Grothendieck. See, for example, [11, Sec. 2.4].

DEFINITION 2.2. Let $\mathcal{A}$ be a locally finitely presented abelian category.

(1) An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $\mathcal{A}$ is called *pure-exact*, if for every $F \in \mathrm{fp}(\mathcal{A})$ the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(F, A) \to \mathrm{Hom}_{\mathcal{A}}(F, B) \to \mathrm{Hom}_{\mathcal{A}}(F, C) \to 0$$

is exact. In this case α (resp. β) is called a *pure monomorphism* (resp. *pure epimorphism*), and A a *pure subobject* of B.

(2) A *pure-essential morphism* is a pure-monomorphism j in $\mathcal{A}$ such that, if fj is a pure monomorphism for some morphism f in $\mathcal{A}$, then f is a pure monomorphism.

(3) An object $E \in \mathcal{A}$ is called *pure-injective* if for every pure-exact sequence $0 \to A \to B \to C \to 0$ the induced sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(C, E) \to \operatorname{Hom}_{\mathcal{A}}(B, E) \to \operatorname{Hom}_{\mathcal{A}}(A, E) \to 0$$

is exact.

(4) For an object M in $\mathcal{A}$, a *pure-injective envelope* of M is a pure-essential morphism $M \to N$ where N is pure-injective.

(5) An object N is called *superdecomposable* if N has no nonzero indecomposable direct summands.

(6) An object $E \in \mathcal{A}$ is called Σ-*pure-injective* if the coproduct $E^{(I)}$ is pure-injective for every set I.

(7) An object $Y \in \mathcal{A}$ is called *fp-injective* if $\operatorname{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for every $X \in \operatorname{fp}(\mathcal{A})$.

For every locally finitely presented abelian category $\mathcal{A}$, there exists a locally coherent Grothendieck category $\mathcal{F}(\mathcal{A})$ and a fully faithful functor $d \colon \mathcal{A} \to \mathcal{F}(\mathcal{A})$ that identifies the pure-exact sequences in $\mathcal{A}$ with exact sequences in $\mathcal{F}(\mathcal{A})$ and the pure-injective objects in $\mathcal{A}$ with the injective objects in $\mathcal{F}(\mathcal{A})$; see [11, 19]. The pure-injective objects in $\mathcal{A}$ therefore inherit the following properties of injective objects in $\mathcal{F}(\mathcal{A})$.

PROPOSITION 2.3. *Let $\mathcal{A}$ be a locally finitely presented abelian category. The following statements hold.*

(1) *Every object M in $\mathcal{A}$ has a pure-injective envelope $M \to \operatorname{PE}(M)$ that is unique up to isomorphism.*

(2) *Every pure-injective object N has the following form*

$$N \cong \operatorname{PE}\left(\bigoplus_{i \in I} N_i \right) \oplus N_c$$

where $\{N_i\}_{i \in I}$ is the set of indecomposable pure-injective summands of N and N_c is superdecomposable.

(3) Let N be a pure-injective object and suppose

$$N \cong \mathrm{PE}\left(\bigoplus_{i \in I} N_i\right) \oplus N_c \cong \mathrm{PE}\left(\bigoplus_{j \in J} M_j\right) \oplus M_c$$

such that N_i, M_j are indecomposable for all $i \in I$, $j \in J$ and N_c, M_c are superdecomposable. Then there exists a bijection $\sigma \colon I \to J$ such that $N_i \cong M_{\sigma(i)}$ for all $i \in I$ and $N_c \cong M_c$.

Proof. Both (1) and (2) follow from the analogous result for injective objects in a Grothendieck category; see, for example, [54, Prop. X.2.5, Cor. X.4.3] for (1) and [42, Thm. E.1.9] for (2) and (3). □

If N is a pure-injective object as in Proposition 2.3 such that $N_c = 0$, we say that N is a *discrete* pure-injective object. The following statement provides an alternative characterisation of pure-injectivity; it is often called the Jensen-Lenzing criterion.

PROPOSITION 2.4 ([43, Thm. 5.4]). *An object E in a locally finitely presented abelian category $\mathcal{A}$ is pure-injective if and only if for any index set I the summation morphism $E^{(I)} \to E$ factors through the canonical embedding $E^{(I)} \to E^I$.* □

LEMMA 2.5. *Every indecomposable pure-injective object in $\mathcal{A}$ has a local endomorphism ring.*

Proof. Cf. [17, Cor. 7.5]. □

LEMMA 2.6. *Assume that $\mathcal{A}$ is a locally coherent abelian k-category over a field k and that $\mathrm{fp}(\mathcal{A})$ is Hom-finite. Then every object $F \in \mathrm{fp}(\mathcal{A})$ is Σ-pure-injective.*

Proof. By Hom-finiteness of $\mathrm{fp}(\mathcal{A})$ this follows directly from [11, (3.5) Thm. 2]. □

PURE-INJECTIVES AND EXT

If E is a pure-injective module over a ring R, then the functor $\mathrm{Ext}^1_R(-, E)$ sends direct limits to inverse limits. Here we show that pure-injective objects in our setting have a similar property.

THEOREM 2.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then, for any pure-injective object E in $\mathcal{A}$ and any directed system of objects M_i $(i \in I)$, we have that $\mathrm{Ext}^1_{\mathcal{A}}(\varinjlim M_i, E) = 0$ whenever $\mathrm{Ext}^1_{\mathcal{A}}(M_i, E) = 0$ for all $i \in I$.*

Proof. The exact sequence $0 \to K \xrightarrow{f} \coprod_{i \in I} M_i \xrightarrow{g} \varinjlim M_i \to 0$ where $K \cong \operatorname{Ker} g$ is pure-exact since finitely presented objects in $\mathcal{A}$ commute with direct limits and direct sums. Therefore, any morphism $K \to E$ factors through f. So if we apply $\operatorname{Hom}_{\mathcal{A}}(-, E)$, we obtain the exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(\coprod M_i, E) \to \operatorname{Hom}_{\mathcal{A}}(K, E) \xrightarrow{0} \operatorname{Ext}^1_{\mathcal{A}}(\varinjlim M_i, E) \to \prod \operatorname{Ext}^1_{\mathcal{A}}(M_i, E).$$

By our assumption, we may conclude that $\operatorname{Ext}^1_{\mathcal{A}}(\varinjlim M_i, E) = 0$. $\qquad\square$

2.8. If $\mathcal{A}$ is a locally noetherian Grothendieck category (that is, a Grothendieck category which has a family of noetherian generators) such that every object in $\mathcal{A}$ has finite injective dimension, it follows from [26, Prop. 2.3, Ex. 3.10] that the derived category $\mathcal{D}(\mathcal{A})$ is compactly generated and the full subcategory of compact objects coincides with $\mathcal{D}^b(\operatorname{fp}(\mathcal{A}))$. There is a well-developed notion of pure-injectivity in a compactly generated triangulated category; we refer to [22] for some background. We will make use of the interaction between the purity in $\mathcal{A}$ and the purity in $\mathcal{D}(\mathcal{A})$. In particular, we note that, by [22, Thm. 1.8], an object E in a compactly generated triangulated category is pure-injective if and only if, for every index set I, the summation morphism $E^{(I)} \to E$ factors as in Proposition 2.4. Moreover, we note that Lemma 2.5 is true in such a category.

In the following lemma we will need to distinguish between products taken in a Grothendieck category $\mathcal{A}$ and products taken in the derived category $\mathcal{D}(\mathcal{A})$. We will denote an S-indexed direct sum of copies of E in $\mathcal{A}$ by $\bigoplus_S^{\mathcal{A}} E$ and an S-indexed product of copies of E in $\mathcal{A}$ by $\prod_S^{\mathcal{A}} E$. Similar notation, $\bigoplus_S^{\mathcal{T}} E$ and $\prod_S^{\mathcal{T}} E$, will be used for S-indexed direct sums and products taken in $\mathcal{T} = \mathcal{D}(\mathcal{A})$.

LEMMA 2.9. *Let $\mathcal{A}$ be a locally noetherian hereditary Grothendieck category, i.e. $\operatorname{Ext}^2(-, -) = 0$. Then E is pure-injective in $\mathcal{A}$ if and only if E is pure-injective when considered as an object of $\mathcal{D}(\mathcal{A})$.*

Proof. If E is pure-injective in $\mathcal{D}(\mathcal{A})$, it follows from [32, Prop. 5.2] that E is pure-injective in $\mathcal{A}$.
For the converse, let E be pure-injective in $\mathcal{A}$, and let $f^{\mathcal{A}} \colon \bigoplus_S^{\mathcal{A}} E \to \prod_S^{\mathcal{A}} E$ be the canonical embedding and $\Sigma^{\mathcal{A}} \colon \bigoplus_S^{\mathcal{A}} E \to E$ be the summation morphism for some set S. By Proposition 2.4, there exists a morphism $h^{\mathcal{A}} \colon E^S \to E$ such that $h^{\mathcal{A}} f^{\mathcal{A}} = \Sigma^{\mathcal{A}}$.
Now consider the S-indexed product $\prod_S^{\mathcal{T}} E$ taken in $\mathcal{T} := \mathcal{D}(\mathcal{A})$. Observe that we have an isomorphism $\prod_S^{\mathcal{T}} E \cong H^0(\prod_S^{\mathcal{T}} E) \oplus H^1(\prod_S^{\mathcal{T}} E)[-1]$ (see, for example, [27, Sec. 1.6]) and we also have that $H^0(\prod_S^{\mathcal{T}} E) \cong \prod_S^{\mathcal{A}} E$. Clearly we have $\bigoplus_S^{\mathcal{A}} E \cong \bigoplus_S^{\mathcal{T}} E$ because coproducts are exact in $\mathcal{A}$.
Let $f_S^{\mathcal{T}} \colon \bigoplus_S^{\mathcal{T}} E \longrightarrow \prod_S^{\mathcal{T}} E$ be the canonical morphism from the coproduct to the product. Since $\operatorname{Hom}_{\mathcal{T}}(\bigoplus_S^{\mathcal{T}} E, H^1(\prod_S^{\mathcal{T}} E)[-1]) = 0$, we have that the projection

$\pi\colon \prod_S^{\mathcal{T}} E \to H^0(\prod_S^{\mathcal{T}} E) \cong \prod_S^{\mathcal{A}} E$ induces an isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(\bigoplus_S^{\mathcal{T}} E, \prod_S^{\mathcal{T}} E) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(\bigoplus_S^{\mathcal{T}} E, \prod_S^{\mathcal{A}} E).$$

Moreover, the universal properties of the canonical morphisms ensure that this isomorphism sends $f_S^{\mathcal{T}}$ to $f_S^{\mathcal{A}}$, that is $f_S^{\mathcal{A}} = \pi \circ f_S^{\mathcal{T}}$. Therefore, we have $(h^{\mathcal{A}} \circ \pi) \circ f_S^{\mathcal{T}} = \Sigma_S^{\mathcal{A}} = \Sigma_S^{\mathcal{T}}$, which shows that the object E is pure-injective in $\mathcal{D}(\mathcal{A})$. □

For a triangulated category $\mathcal{T}$ with arbitrary coproducts, we denote the full subcategory of compact objects in $\mathcal{T}$ by $\mathcal{T}^c$. The category $\mathcal{T}$ is *compactly generated* if $\mathcal{T}^c$ is skeletally small and, for every nonzero object X in $\mathcal{T}$, there exists a compact object C such that $\operatorname{Hom}_{\mathcal{T}}(C, X) \neq 0$. An important tool for studying $\mathcal{T}$ is the category Mod-$\mathcal{T}^c$ of additive functors from $(\mathcal{T}^c)^{\mathrm{op}}$ to the category Ab of abelian groups; see, for example, [26, Sec. 1.2]. We make use of the *restricted Yoneda functor* $\mathrm{y}\colon \mathcal{T} \to \operatorname{Mod}\text{-}\mathcal{T}^c$, which takes an object M in $\mathcal{T}$ to the functor $\mathrm{y}(M) := \operatorname{Hom}_{\mathcal{T}}(-, M)|_{\mathcal{T}^c}$. An object E in $\mathcal{T}$ is pure-injective if and only if $\mathrm{y}(E)$ is injective in $\operatorname{Mod} \mathcal{T}^c$; see [22, Thm. 1.8].

Pure subobjects of products of compact objects

Let k be a field and let $\mathcal{T}$ be a compactly generated triangulated k-linear category for a field k. We will denote the category of additive functors from $\mathcal{T}^c$ to Ab by $\mathcal{T}^c\text{-}\operatorname{Mod}$. Since $\mathcal{T}$ is k-linear, we have a functor $D\colon \operatorname{Mod}\text{-}\mathcal{T}^c \to \mathcal{T}^c\text{-}\operatorname{Mod}$ given by postcomposition with $\operatorname{Hom}_k(-, k)$. Similarly, we have $D\colon \mathcal{T}^c\text{-}\operatorname{Mod} \to \operatorname{Mod}\text{-}\mathcal{T}^c$.

We say that $\mathcal{T}^c$ has *Auslander-Reiten triangles* if for every indecomposable object C in $\mathcal{T}^c$, there exist objects A, B, D, E and Auslander-Reiten triangles

$$C \to D \to E \to C[1] \quad \text{and} \quad A \to B \to C \to A[1]$$

in $\mathcal{T}^c$. The definition of an Auslander-Reiten triangle can be found in [15, Sec. 4.1].

LEMMA 2.10 ([25, Lem. 4.1, Thm. 4.4]). *Suppose $\mathcal{T}$ is a compactly generated k-linear triangulated category such that $\mathcal{T}^c$ is Hom-finite. Then the following statements hold.*

(1) *There is a functor $T\colon \mathcal{T}^c \to \mathcal{T}$, together with a natural isomorphism*

$$\mathrm{D}\operatorname{Hom}_{\mathcal{T}}(C, X) \cong \operatorname{Hom}_{\mathcal{T}}(X, TC)$$

for every compact object C and every object X in $\mathcal{T}$.

(2) *The functor T restricts to an equivalence $\mathcal{T}^c \to \mathcal{T}^c$ if and only if $\mathcal{T}^c$ has Auslander-Reiten triangles.*

Remark 2.11. For a compactly generated triangulated k-linear category where $\mathcal{T}^c$ is Hom-finite, every compact object is endofinite and hence pure-injective by [21, Thm. 1.2].

Proposition 2.12. *Let k be a field and let $\mathcal{T}$ be a compactly generated triangulated k-linear category such that $\mathcal{T}^c$ is Hom-finite and has Auslander-Reiten triangles. Then every object in $\mathcal{T}$ is a pure subobject of a product of compact objects.*

Proof. Let $F := \operatorname{Hom}_{\mathcal{T}}(-, N)|_{\mathcal{T}^c}$ for some object N in $\mathcal{T}$. Then DF is cohomological so, by [22, Lem. 2.7] or [7, Rem. 8.12], it is a flat object of $\mathcal{T}^c$- Mod. By [40, Thm. 3.2], there exists a directed system of representable functors $\{\operatorname{Hom}_{\mathcal{T}^c}(C_i, -)\}_{i \in I}$ such that $DF \cong \varinjlim_{i \in I} \operatorname{Hom}_{\mathcal{T}}(C_i, -)$. It follows that there exists a canonical epimorphism $\bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{T}}(C_i, -) \to DF \to 0$. Applying the functor D again we obtain a monomorphism

$$0 \to D^2 F \to \prod_{i \in I} D \operatorname{Hom}_{\mathcal{T}}(C_i, -).$$

Now, applying Lemma 2.10, we have that

$$\prod_{i \in I} D \operatorname{Hom}_{\mathcal{T}}(C_i, -) \cong \prod_{i \in I} \operatorname{Hom}_{\mathcal{T}}(-, TC_i) \cong \operatorname{Hom}_{\mathcal{T}}(-, \prod_{i \in I} TC_i).$$

Moreover we have a natural family of monomorphisms from each vector space to its double dual and this induces a monomorphism $0 \to F \to D^2 F$. Composing these morphisms we obtain a monomorphism $0 \to F \to \operatorname{Hom}_{\mathcal{T}}(-, \prod_{i \in I} TC_i)$. Finally, since $\prod_{i \in I} TC_i$ is pure-injective, we have that this is induced by a pure monomorphism $N \to \prod_{i \in I} TC_i$ by [22, Thm. 1.8]. $\square$

Compact summands of products

In Remark 2.11 we have that, when $\mathcal{T}$ is compactly generated with $\mathcal{T}^c$ Hom-finite, every compact object C is pure-injective. In the next proof we show that if $\mathcal{T}^c$ also has Auslander-Reiten triangles, then $\mathrm{y}(C)$ is the injective envelope of a simple functor in Mod-$\mathcal{T}^c$. In particular, compact objects have the following property with respect to products of pure-injective objects in $\mathcal{T}$.

Proposition 2.13. *Let $\mathcal{T}$ be a compactly generated triangulated category such that $\mathcal{T}^c$ is Hom-finite and has Auslander-Reiten triangles. If a compact object C is a direct summand of a product $\prod_{i \in I} N_i$ of pure-injective objects $\{N_i\}_{i \in I}$ in $\mathcal{T}$, then C is a direct summand of N_i for some $i \in I$.*

Proof. Let $C \xrightarrow{f} D \to E \to C[1]$ be an Auslander-Reiten triangle and consider the functor $F := \operatorname{Ker}(\mathrm{y}(f))$.
First we show that F is a simple functor. Consider the category $\operatorname{Coh}(\mathcal{T})$ of coherent functors $\mathcal{T} \to \operatorname{Ab}$ i.e. covariant functors that are of the form

$\mathrm{Coker}(\mathrm{Hom}_{\mathcal{T}}(g,-))$ for some morphism g in $\mathcal{T}^c$. In [24], Krause shows that there exists a duality $(-)^{\vee}\colon \mathrm{mod}\text{-}\mathcal{T}^c \to \mathrm{Coh}(\mathcal{T})$ where $G^{\vee}(X) := \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{T}^c}(G, \mathrm{y}(X))$ for each functor G in $\mathrm{mod}\text{-}\mathcal{T}^c$ and object X in $\mathcal{T}$. By [4, Cor. 1.12], we have that the functor $F^{\vee}$ is isomorphic to $\mathrm{Coker}(\mathrm{Hom}_{\mathcal{T}}(f,-))$ which is a simple functor. It follows that F is a simple functor in $\mathrm{Mod}\text{-}\mathcal{T}^c$.

By assumption, there is a split monomorphism $C \to \prod_{i\in I} N_i$ and so its image $\mathrm{y}(C) \to \prod_{i\in I} \mathrm{y}(N_i)$ is a split monomorphism in $\mathrm{Mod}\text{-}\mathcal{T}^c$. Since $\mathrm{y}(C)$ is an indecomposable injective object, the monomorphism $F \to \mathrm{y}(C)$ must be an injective envelope and F must be essential in $\mathrm{y}(C)$. It follows that the composition $F \to \mathrm{y}(C) \to \prod_{i\in I} \mathrm{y}(N_i) \to \mathrm{y}(N_i)$ is a non-zero monomorphism $F \to \mathrm{y}(N_i)$ for some $i \in I$. But then the injective envelope $\mathrm{y}(C)$ of F must be a direct summand of the injective object $\mathrm{y}(N_i)$. By [22, Thm. 1.8] we have that C is a direct summand of N_i. □

3 Cotilting objects

Let $\mathcal{A}$ be a Grothendieck category.

Definition 3.1 ([10, Def. 2.4]). An object $C \in \mathcal{A}$ is called a *cotilting object* if $\mathrm{Cogen}(C) = {}^{\perp_1}C$, and if this class contains a generator for $\mathcal{A}$. Then $\mathrm{Cogen}(C)$ is called the associated *cotilting class*.

Lemma 3.2 ([10, Thm. 2.11]). *An object $C \in \mathcal{A}$ is cotilting if and only if the following conditions are satisfied:*

(CS0) C has injective dimension $\mathrm{id}(C) \leq 1$.

(CS1) $\mathrm{Ext}^1(C^I, C) = 0$ for every cardinal I.

(CS3) For every injective cogenerator W of $\mathcal{A}$ there is a short exact sequence

$$0 \to C_1 \to C_0 \to W \to 0$$

with $C_0, C_1 \in \mathrm{Prod}(C)$.

Each cotilting C moreover satisfies

(CS2) ${}^{\perp}C = 0$, that is: if $X \in \mathcal{A}$ satisfies $\mathrm{Hom}(X, C) = 0 = \mathrm{Ext}^1(X, C)$, then $X = 0$. □

We used this order of numbering since (CS0), (CS1) and (CS2) are the duals of the corresponding properties (TS0), (TS1), (TS2) for tilting sheaves in [1].

Theorem 3.3 ([10, Thm. 3.9]). *Let $C \in \mathcal{A}$ be cotilting. Let $\mathcal{F} = {}^{\perp_1}C = \mathrm{Cogen}(C)$ be the associated cotilting class. Then C is pure-injective and $\mathcal{F}$ is closed under direct limits in $\mathcal{A}$.* □

It follows that Definition 3.1 is equivalent to the definition of cotilting objects given in [8] for locally noetherian Grothendieck categories. The following is well-known and easy to show.

Lemma 3.4. *Let $C \in \mathcal{A}$ be cotilting with associated cotilting class $\mathcal{F}$.*

(1) $\mathcal{F} = \mathrm{Copres}(C)$, *the class of objects in $\mathcal{A}$ which are kernels of morphisms of the form $C^I \to C^J$.*

(2) $\mathcal{F} \cap \mathcal{F}^{\perp_1} = \mathrm{Prod}(C)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.5. *Let $\mathcal{A}$ be locally noetherian with the property that every object in $\mathcal{A}$ has finite injective dimension. Let $C \in \mathcal{A}$ be a cotilting object with cotilting class $\mathcal{F} = {}^{\perp_1}C$. If $B \in \mathrm{fp}(\mathcal{A})$ is indecomposable with $B \in \mathcal{F} \cap \mathcal{F}^{\perp_1}$, then B is a direct summand of C.* $\qquad\square$

Proof. By Lemma 3.4, we have that $\mathcal{F} \cap \mathcal{F}^{\perp_1} = \mathrm{Prod}(C)$ and so we may apply [10, Cor. 2.13] to obtain that products of copies of C in $\mathcal{A}$ coincide with products of copies of C in $\mathcal{D}(\mathcal{A})$. By Theorem 3.3, the cotilting object C is pure-injective in $\mathcal{A}$ so, by Lemma 2.9, it is also pure-injective in $\mathcal{D}(\mathcal{A})$. Finally, we may apply Proposition 2.13, to obtain that B is a direct summand of C in $\mathcal{D}(\mathcal{A})$ and hence in $\mathcal{A}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.6.   (1) Two cotilting objects $C, C' \in \mathcal{A}$ are *equivalent*, if they have the same cotilting class. This is equivalent to $\mathrm{Prod}(C) = \mathrm{Prod}(C')$.

(2) A cotilting object $C \in \mathcal{A}$ is called *minimal* if, for any other cotilting object C' with same cotilting class $\mathrm{Cogen}(C') = \mathrm{Cogen}(C)$, we have that C is a direct summand of C'.

Let $\mathcal{A}$ additionally be locally finitely presented.

Theorem 3.7 ([10, Thm. 3.13], [8, Thm. 1.13]). *Let $\mathcal{A}$ be locally noetherian. The torsionfree classes $\mathcal{F}$ in $\mathcal{A}$ associated to a cotilting object bijectively correspond to the torsion pairs $(\mathcal{T}_0, \mathcal{F}_0)$ in $\mathrm{fp}(\mathcal{A})$ where $\mathcal{F}_0$ is a generating class for $\mathrm{fp}(\mathcal{A})$. The correspondence is given by*

$$\mathcal{F} \mapsto \mathcal{F} \cap \mathrm{fp}(\mathcal{A}) \quad and \quad (\mathcal{T}_0, \mathcal{F}_0) \mapsto \varinjlim(\mathcal{F}_0).$$

Accordingly, two cotilting objects $C, C' \in \mathcal{A}$ are equivalent if and only if ${}^{\perp_1}C \cap \mathrm{fp}(\mathcal{A}) = {}^{\perp_1}C' \cap \mathrm{fp}(\mathcal{A})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.8. A cotilting object $C \in \mathcal{A}$ is called *large* if it is not equivalent to a coherent cotilting object.

Definition 3.9. Let $E \in \mathcal{A}$.

(1) E is called *rigid*, if $\mathrm{Ext}^1(E, E) = 0$.

(2) E is called *self-orthogonal*, if $\mathrm{Ext}^1(E^\alpha, E) = 0$ for every cardinal α.

(3) A self-orthogonal E is called *maximal self-orthogonal*, if $\mathrm{Prod}(E) \subseteq \mathrm{Prod}(F)$ implies $\mathrm{Prod}(E) = \mathrm{Prod}(F)$ for every self-orthogonal F.

PROPOSITION 3.10. *Let $\mathcal{A}$ be hereditary. Every cotilting object in $\mathcal{A}$ is maximal self-orthogonal.*

Proof. The proof in [8, Prop. 3.1] does also work in this more general situation. □

PROPOSITION 3.11. *Let $\mathcal{A}$ be locally noetherian Grothendieck and hereditary. An object C in $\mathcal{A}$ is cotilting if and only if (CS1) and (CS2) hold and $^{\perp_1}C \cap \mathrm{fp}(\mathcal{A})$ is generating.*

Proof. Let C satisfy (CS1) and (CS2). It is sufficient to show that $\mathrm{Cogen}(C) = {}^{\perp_1}C$. By (CS1) and since $\mathcal{A}$ is hereditary we easily get $\mathrm{Cogen}(C) \subseteq {}^{\perp_1}C$. For the reverse inclusion, we let $X \in {}^{\perp_1}C$ and consider the short exact sequences induced by the reject K of $\{C\}$ in X, that is, $0 \to K \to X \to U \to 0$ and $0 \to U \to C^I \to Y \to 0$ with $I = \mathrm{Hom}(X, C)$. By applying $\mathrm{Hom}(-, C)$ to these sequences and using again that $\mathcal{A}$ is hereditary, we obtain $\mathrm{Ext}^1(K, C) = 0 = \mathrm{Ext}^1(U, C) \cong \mathrm{Hom}(K, C)$, and then $K = 0$ by (CS2). We get $X \in \mathrm{Cogen}(C)$. □

COTILTING OBJECTS AND INJECTIVE COGENERATORS

Let $\mathcal{A}$ be a Grothendieck category and $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{A}$. We define a t-structure $(\mathcal{U}_\mathcal{T}, \mathcal{V}_\mathcal{F})$ on $\mathcal{D}(\mathcal{A})$ as follows:

$$\mathcal{U}_\mathcal{T} := \{X \in \mathcal{D}(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i > 0 \text{ and } H^0(X) \in \mathcal{T}\}$$

$$\mathcal{V}_\mathcal{F} := \{X \in \mathcal{D}(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i < -1 \text{ and } H^{-1}(X) \in \mathcal{F}\}.$$

We call $(\mathcal{U}_\mathcal{T}, \mathcal{V}_\mathcal{F})$ the *HRS-tilted t-structure* of $(\mathcal{T}, \mathcal{F})$, after [16].
The following full subcategory

$$\mathcal{G} = \{X \in \mathcal{D}(\mathcal{A}) \mid H^{-1}(X) \in \mathcal{F}, H^0(X) \in \mathcal{T} \text{ and } H^i(X) = 0 \text{ for } i \neq 0, -1\}$$

of $\mathcal{D}(\mathcal{A})$ is the *heart* of the t-structure $(\mathcal{U}_\mathcal{T}, \mathcal{V}_\mathcal{F})$. It is sometimes also called an *HRS-tilt* of $\mathcal{A}$. Then $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{G}$, and if $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair, then $\mathcal{D}(\mathcal{G}) = \mathcal{D}(\mathcal{A})$, cf. [57].
We will also consider (in Section 8) the category $\mathcal{G}[-1]$ instead and call it the *[-1]-shifted heart*. In this case we have that $(\mathcal{F}, \mathcal{T}[-1])$ is a torsion pair in $\mathcal{G}[-1]$. Since $[-1]$ is an automorphism of $\mathcal{D}(\mathcal{A})$ there is just a notational difference (which has some tradition in the theory of weighted projective lines). Besides the mentioned results from [10] we will also need the following:

THEOREM 3.12 ([52, Thm. 5.2]). *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then $\mathcal{G}$ is a locally coherent Grothendieck category (with $\mathcal{G} \cap \mathcal{D}^b(\mathrm{fp}\,\mathcal{A})$ the class of finitely presented objects) if and only if $\mathcal{F}$ is closed under direct limits.* □

Theorem 3.13 ([10, Prop. 4.4]). *Let $\mathcal{A}$ be a Grothendieck category and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair where $\mathcal{F}$ is a generating class. Then an object E in $\mathcal{G}$ is an injective cogenerator of $\mathcal{G}$ if and only if $E \cong C[1]$ where C is a cotilting object in $\mathcal{A}$ with $\mathcal{F} = \mathrm{Cogen}(C)$.* □

Together we obtain the following corollary:

Corollary 3.14. *Let $\mathcal{A}$ be a locally noetherian category. Then the following statements hold:*

(1) *If $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair in $\mathcal{A}$, then $\mathcal{G}$ is locally coherent.*

(2) *If $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair in $\mathcal{A}$ and $\mathcal{G}$ has a minimal injective cogenerator, then there exists a minimal cotilting object C in $\mathcal{A}$ with $\mathrm{Cogen}(C) = \mathcal{F}$.* □

Σ-pure-injective cotilting objects

Before we continue the discussion on minimal cotilting objects, we observe that we obtain the following criterion as a corollary of the above. An analogous result for modules over any ring can be found in [9, Thm. 5.3]. This is also shown in a more general setting in [32, Prop. 5.6].

Corollary 3.15. *Let $\mathcal{A}$ be a locally coherent Grothendieck category and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair with $\mathcal{F} = {}^{\perp_1} C$ associated with a cotilting object $C \in \mathcal{A}$. The following are equivalent:*

(1) *C is Σ-pure-injective in $\mathcal{A}$.*

(2) *$\mathcal{G}$ is locally noetherian.*

Proof. By the preceding discussion, $\mathcal{G}$ is locally coherent and C is pure-injective. $\mathrm{Prod}(C[1])$ (in $\mathcal{G}$) is the class of injective objects in $\mathcal{G}$.
By [54, Prop. V.4.3], $\mathcal{G}$ is locally noetherian if and only if each coproduct of injective objects is injective, that is, $\mathrm{Prod}(C[1])$ in $\mathcal{G}$ is closed under coproducts. By [10, Cor. 2.13] this is equivalent to $\mathrm{Prod}(C)$ in $\mathcal{A}$ being closed under coproducts. If this holds then in particular $C^{(I)}$ is pure-injective for each set I, that is, C is Σ-pure-injective. Conversely, if C is Σ-pure-injective, then by [11, (3.5) Thm. 2] so is each object in $\mathrm{Prod}(C)$ and is a coproduct of indecomposables. It follows that each injective object in $\mathcal{G}$ is a coproduct of indecomposable objects. Thus $\mathcal{G}$ is locally noetherian by [23, Thm. A.11]. □

Locally finitely generated Grothendieck categories and minimal injective cogenerators

Let $\mathcal{A}$ be a Grothendieck category. An object F in $\mathcal{A}$ is *finitely generated* if, whenever $F = \sum_{i \in I} F_i$ for a direct family of subobjects $\{F_i\}_{i \in I}$ of F, there exists an index $i_0 \in I$ such that $F = F_{i_0}$.

Remark 3.16. We define $\sum$ as follows ([54, pg. 88]). Let $\{C_i\}_{i \in I}$ be a family of subobjects of C, then the monomorphisms $C_i \to C$ induce a morphism $\alpha \colon \bigoplus_{i \in I} C_i \to C$. The image of α is denoted $\sum_{i \in I} C_i$ and is called the *sum* of the subobjects $\{C_i\}_{i \in I}$. By [54, Ch. IV, Ex. 8.3], if $\{C_i\}_{i \in I}$ is a direct family, then $\varinjlim_{i \in I} C_i$ is a subobject of C and coincides with $\sum_{i \in I} C_i$.

We say that $\mathcal{A}$ is *locally finitely generated* if there exists a family of finitely generated generators. By [54, Lem. 3.1(i)], if C is finitely generated, then the image of a morphism $C \to D$ is finitely generated. Thus $\mathcal{A}$ is locally finitely generated if and only if, for every object C in $\mathcal{A}$, there is a direct family $\{C_i\}_{i \in I}$ of finitely generated subobjects of C such that $C = \sum_{i \in I} C_i$.

Proposition 3.17 (Element-free version of [54, Prop. 6.6]). *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then an injective object E is a cogenerator if and only if it contains as a subobject an isomorphic copy of each simple object.*

Proof. If E is a cogenerator, then there exists a non-zero morphism $S \to E$ for each simple object S which is necessarily a monomorphism.
For the converse, it suffices to show that every finitely generated object M has a maximal proper subobject and hence a simple quotient $M \to S$. Since then, for any finitely generated object M in $\mathcal{A}$, there is a non-zero morphism $M \to S \hookrightarrow E$. For an arbitrary object N, there exists a non-zero finitely generated subobject $M \hookrightarrow N$ and so the non-zero morphism $M \to E$ extends to a non-zero morphism $N \to E$.
So, consider the collection $\mathcal{M}$ of proper subobjects of M, ordered by inclusion. Then let $\mathcal{L}$ be a totally ordered subset of $\mathcal{M}$ and consider the subobject $\bar{L} := \sum_{L \in \mathcal{L}} L$. If $\bar{L} = M$, then $M = L$ for some $L \in \mathcal{L}$ which contradicts the assumption that the objects of $\mathcal{M}$ are proper subobjects. Thus $\bar{L}$ is a proper subobject of M and so is an upper bound of the subset $\mathcal{L}$ in $\mathcal{M}$. Applying Zorn's lemma, we conclude that $\mathcal{M}$ has a maximal object as desired. $\square$

Using some standard arguments we obtain the following corollary.

Corollary 3.18. *Let $\mathcal{S}$ be a set of representatives of the isomorphism class of simple objects in $\mathcal{A}$. Then the object $\mathrm{E}(\bigoplus_{S \in \mathcal{S}} \mathrm{E}(S))$ is a minimal injective cogenerator of $\mathcal{A}$.* $\square$

Minimal cotilting objects in locally noetherian categories

Combining the previous two subsections, we obtain the following proposition.

Proposition 3.19. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then*

(1) *Every equivalence class of cotilting objects has a minimal representative C_0 that is a discrete pure-injective object.*

(2) *The indecomposable direct summands of C_0 are in bijection with the isomorphism classes of simple objects in $\mathcal{G}$.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a cotilting torsion pair. Then, by Corollary 3.14, the heart $\mathcal{G}$ in $\mathcal{D}(\mathcal{A})$ is locally coherent, so in particular, it is locally finitely generated. By the Corollary 3.18, the category $\mathcal{G}$ has a minimal injective cogenerator and so by Corollary 3.14, there exists a minimal cotilting object C such that $\mathrm{Cogen}(C) = \mathcal{F}$. Moreover, this minimal cotilting object is discrete since the minimal injective cogenerator has no superdecomposable part. □

LEMMA 3.20. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Let E be a discrete pure-injective object in $\mathcal{A}$ with* $\mathrm{id}(E) \leq 1$.

(1) *The class* $^{\perp_1} E$ *is closed under products.*

(2) *The following are equivalent:*

(a) $\mathrm{Ext}^1(E, E) = 0$.

(b) $\mathrm{Ext}^1(E', E'') = 0$ *for all indecomposable summands E', E'' of E.*

(c) *E is self-orthogonal, that is,* $\mathrm{Ext}^1(E^I, E) = 0$ *for each set I.*

Proof. (1) Since $\mathrm{id}(E) \leq 1$, the class $\mathcal{F} = {}^{\perp_1} E \cap \mathrm{fp}(\mathcal{A})$ is closed under subobjects and extensions, and $^{\perp_1} E$ is closed under direct limits by Theorem 2.7. As in [8, Prop. 1.8] then $^{\perp_1} E = \varinjlim \mathcal{F}$ is a torsionfree class, and thus closed under products.

(2) As in [8, Cor. 2.3]. □

4   Weighted noncommutative regular projective curves

We define the class of weighted noncommutative regular projective curves by the axioms (NC 1) to (NC 5) below. For details we refer to [36, 31, 1]. The content of this background section contains some overlap with [1, Sec. 2] since the settings are the same. We recall part of the material here for the convenience of the reader. At the end of this section we exhibit a very useful correspondence between cotilting sheaves and tilting sheaves of finite type, cf. Theorem 4.4.

The axioms

A noncommutative curve $\mathbb{X}$ is given by a category $\mathcal{H}$ which is regarded as the category $\mathrm{coh}\,\mathbb{X}$ of *coherent sheaves* over $\mathbb{X}$. Formally it behaves like a category of coherent sheaves over a (commutative) regular projective curve over a field k (we refer to [31]):

(NC 1) $\mathcal{H}$ is small, connected, abelian and noetherian.

(NC 2) $\mathcal{H}$ is a k-linear category with Hom- and Ext-spaces of finite k-dimension.

(NC 3) Serre duality holds in $\mathcal{H}$: For all objects $X, Y \in \mathcal{H}$ we have a natural isomorphism
$$\operatorname{Ext}^1_{\mathcal{H}}(X, Y) = \operatorname{D} \operatorname{Hom}_{\mathcal{H}}(Y, \tau X)$$
with $\operatorname{D} = \operatorname{Hom}_k(-, k)$ and with $\tau \colon \mathcal{H} \to \mathcal{H}$ an autoequivalence, called Auslander-Reiten translation. (It follows that $\mathcal{H}$ is a hereditary category without non-zero projective or injective object.)

(NC 4) There exist objects in $\mathcal{H}$ of infinite length.

Let $\mathcal{H}_0$ denote the class of finite length objects and $\mathcal{H}_+ = \operatorname{vect} \mathbb{X}$ the class of a torsionfree objects, also called vector bundles. Decomposing $\mathcal{H}_0$ in its connected components we have
$$\mathcal{H}_0 = \coprod_{x \in \mathbb{X}} \mathcal{U}_x,$$
where $\mathbb{X}$ is an index set (explaining the terminology $\mathcal{H} = \operatorname{coh} \mathbb{X}$) and every $\mathcal{U}_x$ is a connected uniserial length category, a so-called tube. We additionally assume that $\mathcal{H}$ has the following condition.

(NC 5) $\mathbb{X}$ consists of infinitely many points.

Then $\mathbb{X}$ (or $\mathcal{H}$) is called a *weighted noncommutative regular projective curve* over k. The following statement is shown in [31].

Proposition 4.1. *There are (up to isomorphism) only finitely many simple objects in $\mathcal{U}_x$, for all x, and for almost all x there is even only one.*   □

By $p(x)$ we denote the rank of the tube $\mathcal{U}_x$, which is the number of simple objects in $\mathcal{U}_x$ (up to isomorphism). The numbers $p(x)$ with $p(x) > 1$ are called the *weights*. The tubes $\mathcal{U}_x$ of rank 1 are called homogeneous, those finitely many of rank > 1 non-homogeneous or exceptional. If S_x is a simple object in $\mathcal{U}_x$, then all simple objects (up to isomorphism) in $\mathcal{U}_x$ are given by the Auslander-Reiten orbit $\tau S_x, \tau^2 S_x, \ldots, \tau^{p(x)} S_x = S_x$.

In the following, if not otherwise specified, let $\mathcal{H} = \operatorname{coh} \mathbb{X}$ be a weighted noncommutative regular projective curve.

The category of quasicoherent sheaves

In our focus will be a larger category, the Grothendieck category $\vec{\mathcal{H}}$. It is obtained from $\mathcal{H}$ as the category $\operatorname{Lex}(\mathcal{H}^{\mathrm{op}}, \operatorname{Ab})$, cf. [13, II. Thm. 1]. We write $\vec{\mathcal{H}} = \operatorname{Qcoh}(\mathbb{X})$ and call the objects quasicoherent sheaves. It is also of the form $\operatorname{Qcoh}(\mathcal{A})$, the category of quasicoherent modules over a certain hereditary order $\mathcal{A}$; we refer to [31, Thm. 7.11].

The category $\vec{\mathcal{H}}$ is hereditary abelian, and a locally noetherian Grothendieck category; every object in $\vec{\mathcal{H}}$ is a direct limit of objects in $\mathcal{H}$. The full abelian subcategory $\mathcal{H}$ consists of the coherent (= finitely presented = noetherian) objects in $\vec{\mathcal{H}}$, we also write $\mathcal{H} = \operatorname{fp}(\vec{\mathcal{H}})$. Every indecomposable coherent sheaf has a local endomorphism ring, and $\mathcal{H}$ is a Krull-Schmidt category.

Prüfer and adic sheaves

If S is a simple sheaf, then we denote by $S[n]$ the (unique) indecomposable sheaf of length n with socle S. The inclusions $S[n] \to S[n+1]$ $(n \geq 1)$ form a direct system, the *ray* starting in S, and their direct union is the *Prüfer sheaf* $S[\infty] = \varinjlim S[n]$ with respect to S. Cf. [46].

We denote by $S[-n]$ the (unique) indecomposable sheaf of length n with top S. The epimorphisms $S[-n-1] \to S[-n]$ $(n \geq 1)$ form an inverse system, the *coray* ending in S. We write $S[-\infty] = \varprojlim S[-n]$ for the inverse limit and call it the *adic sheaf* with respect to S. It will be shown in Lemma 5.9 that $S[-\infty]$ is indecomposable.

Rank. Line bundles

Let $\mathcal{H}/\mathcal{H}_0$ be the quotient category of $\mathcal{H}$ modulo the Serre category of sheaves of finite length, let $\pi\colon \mathcal{H} \to \mathcal{H}/\mathcal{H}_0$ the quotient functor, which is exact. The *function field* of $\mathcal{H}$ (or of $\mathbb{X}$) is the up to isomorphism unique skew field $k(\mathcal{H})$ such that $\mathcal{H}/\mathcal{H}_0 \cong \mathrm{mod}(k(\mathcal{H}))$. The $k(\mathcal{H})$-dimension on $\mathcal{H}/\mathcal{H}_0$ induces the *rank* of objects in $\mathcal{H}$, which induces a linear form $\mathrm{rk}\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$. The objects in $\mathcal{H}_0$ are just the objects of rank zero, every non-zero vector bundle has a positive rank. The vector bundles of rank one are called *line bundles*. For every line bundle L' the endomorphism ring $\mathrm{End}(L')$ is a skew field. Every vector bundle has a line bundle filtration, cf. [36, Prop. 1.6]. There exists a line bundle L, called structure sheaf, having certain additional properties (we refer to [31, 8.1+Sec. 13]).

The sheaf of rational functions

The *sheaf $\mathcal{K}$ of rational functions* is the injective envelope of any line bundle L in the category $\vec{\mathcal{H}}$; this does not depend on the chosen line bundle. It is torsionfree by [29, Lem. 14], and it is a generic sheaf in the sense of [33]; its endomorphism ring is the function field, $\mathrm{End}_{\vec{\mathcal{H}}}(\mathcal{K}) \cong \mathrm{End}_{\mathcal{H}/\mathcal{H}_0}(\pi L) \cong k(\mathcal{H})$, where $\pi\colon \mathcal{H} \to \mathcal{H}/\mathcal{H}_0$ is the quotient functor.

Orbifold Euler characteristic and representation type

Let $\mathcal{H}$ be a weighted noncommutative regular projective curve with structure sheaf L and $\bar{p}$ the least common multiple of the weights. Let $s = s(\mathcal{H})$ be the square root of the dimension of the function field $k(\mathcal{H})$ over its centre (called the (global) skewness). We have the *average Euler form* $\langle\!\langle E, F \rangle\!\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j E, F \rangle$, and then the normalized *orbifold Euler characteristic* of $\mathcal{H}$ is defined by $\chi'_{orb}(\mathbb{X}) = \frac{1}{s^2 \bar{p}^2} \langle\!\langle L, L \rangle\!\rangle$.

The orbifold Euler characteristic determines the representation type of the category $\mathcal{H} = \mathrm{coh}\, \mathbb{X}$.

- $\mathbb{X}$ is domestic: $\chi'_{orb}(\mathbb{X}) > 0$

- $\mathbb{X}$ is elliptic: $\chi'_{orb}(\mathbb{X}) = 0$, and $\mathbb{X}$ non-weighted ($\bar{p} = 1$)

- $\mathbb{X}$ is tubular: $\chi'_{orb}(\mathbb{X}) = 0$, and $\mathbb{X}$ properly weighted ($\bar{p} > 1$)

- $\mathbb{X}$ is wild: $\chi'_{orb}(\mathbb{X}) < 0$.

Degree and slope

With the structure sheaf L we define the *degree* function $\deg \colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ by

$$\deg(F) = \frac{1}{\kappa\varepsilon} \langle\!\langle L, F \rangle\!\rangle - \frac{1}{\kappa\varepsilon} \langle\!\langle L, L \rangle\!\rangle \operatorname{rk}(F). \tag{4.1}$$

Here, $\kappa = \dim_k \operatorname{End}(L)$, and ε is the positive integer such that the resulting linear form $\mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ becomes surjective. We have $\deg(L) = 0$, and $\deg$ is positive and τ-invariant on sheaves of finite length.

The *slope* of a non-zero coherent sheaf F is defined as $\mu(F) = \deg(F)/\operatorname{rk}(F) \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, and F is called *stable* (*semistable*, resp.) if for every non-zero proper subsheaf F' of F we have $\mu(F') < \mu(F)$ (resp. $\mu(F') \le \mu(F)$).

Torsion, torsionfree, divisible and reduced sheaves

4.2. The class of *torsionfree* (quasicoherent) sheaves is given by $\mathcal{F} = \mathcal{H}_0^{\perp_0}$. We have $\operatorname{vect}\mathbb{X} = \mathcal{F} \cap \mathcal{H}$. The class of *torsion* sheaves is given as the direct limit closure $\mathcal{T} = \vec{\mathcal{H}}_0 = \varinjlim \mathcal{H}_0$. The pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\vec{\mathcal{H}}$. Every $E \in \vec{\mathcal{H}}$ has a largest subsheaf from $\mathcal{T}$, the torsion subsheaf tE. The canonical sequence $0 \to tE \to E \to E/tE \to 0$ is pure-exact (cf. [1, Lem. 3.3]), and E/tE is torsionfree.

Let $V \subseteq \mathbb{X}$ be a subset. The class of *V-divisible* sheaves is $\mathcal{D}_V = \left(\coprod_{x \in V} \mathcal{U}_x\right)^{\perp_1}$. It is closed under direct summands, set-indexed direct sums, extensions and epimorphic images. If $V = \{x\}$, then we will refer to V-divisible sheaves as *x-divisible*. In case $V = \mathbb{X}$ we just say *divisible*.

The class $\mathcal{D} = \mathcal{D}_{\mathbb{X}}$ of divisible sheaves is a torsion class, and the corresponding torsion pair $(\mathcal{D}, \mathcal{R})$ in $\vec{\mathcal{H}}$ splits. The sheaves in $\mathcal{R}$ are called *reduced*. By [1, Lem. 3.3] we have $\mathcal{D} = \operatorname{Inj}(\vec{\mathcal{H}})$, the class of injective sheaves. Moreover, the indecomposable injective sheaves are (up to isomorphism) the sheaf $\mathcal{K}$ of rational functions and the Prüfer sheaves $S[\infty]$ ($S \in \mathcal{H}$ simple).

A tilting-cotilting correspondence

Proposition 4.3. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ be a weighted noncommutatuve regular projective curve and $\mathcal{H} = \operatorname{coh}\mathbb{X}$. A class $\mathcal{F} \subseteq \mathcal{H}$ is torsionfree and generating if and only if it is resolving (in the sense of [1, Def. 4.2]).*

Proof. This is [1, Cor. 4.17]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We recall that cotilting sheaves C, C' are equivalent if $\mathrm{Cogen}(C) = \mathrm{Cogen}(C')$. Similarly, tilting sheaves T, T' are equivalent if $\mathrm{Gen}(T) = \mathrm{Gen}(T')$. By slight abuse of notation we denote the equivalence classes in both cases in the same way as $[C]$ and $[T]$, respectively.

THEOREM 4.4. *If T is a tilting sheaf of finite type there is a cotilting sheaf C such that $^{\perp_1}C \cap \mathcal{H} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$, and conversely. The assignments $\Gamma \colon [T] \mapsto [C]$ and $\Theta \colon [C] \mapsto [T]$ induce mutually inverse bijections between the sets of*

- *equivalence classes $[T]$ of tilting sheaves T of finite type, and of*

- *equivalence classes $[C]$ of cotilting sheaves C.*

Proof. Follows from the preceding proposition by invoking Theorem 3.7 and [1, Thm. 4.14]. □

For simplicity, we will even just write $\Gamma(T) = C$, without brackets. In Example 7.2 we will see that this assignment appears to be unrelated to the usual bijection between tilting left modules and cotilting right modules over a noetherian ring.

5   Pure-injective sheaves of slope infinity

Let $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$ be a weighted noncommutative regular projective curve over a field k. Let

$$\mathcal{M}(\infty) = {}^{\perp_0}\mathrm{vect}\,\mathbb{X} = (\mathrm{vect}\,\mathbb{X})^{\perp_1}.$$

The sheaves in $\mathcal{M}(\infty)$ are said to have slope ∞. Examples are the torsion sheaves, but also the generic and the adic sheaves, which are torsionfree. Moreover:

LEMMA 5.1. *For every non-empty subset V of $\mathbb{X}$ the V-divisible sheaves have slope ∞: $\mathcal{D}_V \subseteq \mathcal{M}(\infty)$.*

Proof. Let E be x-divisible for some point x. If there is a non-zero morphism from E to a vector bundle, then there is also an epimorphism to a line bundle L'. Since there is an epimorphism from L' to a simple object S_x concentrated in x, we get with Serre duality a contradiction to x-divisibility. □

PROPOSITION 5.2. *Let $E \in \vec{\mathcal{H}}$ be pure-injective, torsionfree and of slope ∞. Then E is rigid.*

Proof. Since E is torsionfree, we have $E = \varinjlim E_i$ for a directed system of vector bundles $(E_i)_{i \in I}$. The claim then follows from Theorem 2.7. □

COROLLARY 5.3. *Let E, $F \in \{\mathcal{K}, S[-\infty] \mid S \text{ simple}\}$. Then $\mathrm{Ext}^1(E, F) = 0$.*

Proof. By the proposition, $E \oplus F$ is rigid. □

Definability

Let $\mathcal{A}$ be a locally coherent Grothendieck category. A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *definable* if it is closed under products, direct limits and pure subobjects.

Proposition 5.4. $\mathcal{M}(\infty)$ *is a definable subcategory of* $\mathrm{Qcoh}\,\mathbb{X}$.

Proof. Let X_i be a family of objects in $\mathcal{A}$ and $E \in \mathrm{vect}\,\mathbb{X}$. By [10, Cor. A.2] we have $\mathrm{Ext}^1(E, \prod_i X_i) = 0$ if and only if $\mathrm{Ext}^1(E, X_i) = 0$ for all i. Hence $\mathcal{M}(\infty)$ is closed under products.
Assume that the X_i form a directed set of objects. Then $\mathrm{Hom}(\varinjlim X_i, E) \cong \varprojlim \mathrm{Hom}(X_i, E)$, and thus $\mathcal{M}(\infty)$ is also closed under direct limits.
By applying $\mathrm{Hom}(E, -)$ to a pure-exact sequence $0 \to X \to Y \to Z \to 0$ with $Y \in (\mathrm{vect}\,\mathbb{X})^{\perp_1}$, the resulting long exact sequence shows $X \in (\mathrm{vect}\,\mathbb{X})^{\perp_1}$, and thus $\mathcal{M}(\infty)$ is closed under pure subobjects. $\square$

Pure-injectives of slope ∞

We wish to determine the indecomposable pure-injective objects in the class $\mathcal{M} := \mathcal{M}(\infty)$ of objects of slope ∞. In order to do this, we consider the equivalent category $\mathcal{M}$ that occurs as a subcategory of $\mathcal{D}(\vec{\mathcal{H}})$.
In the derived category $\mathcal{D}(\vec{\mathcal{H}})$ the compact objects are given by $\mathcal{D}^b(\mathcal{H}) = \bigvee_{n \in \mathbb{Z}} \mathcal{H}[n]$ (see the introduction for an explanation of this notation). Given this description of $\mathcal{D}^b(\mathcal{H})$, we may partition $\bigvee_{n \in \mathbb{Z}} \mathcal{H}[n]$ into three parts:

$$\mathbf{p} := \left(\bigvee_{n<0} \mathcal{H}[n]\right) \vee \mathrm{vect}\,\mathbb{X} \quad \mathbf{t} := \mathcal{H}_0 \quad \mathbf{q} := \left(\bigvee_{n>0} \mathcal{H}[n]\right).$$

We also consider the HRS-tilted t-structure $(\mathcal{U}_\mathcal{D}, \mathcal{V}_\mathcal{R})$ of the split torsion pair $(\mathcal{D}, \mathcal{R})$ in $\mathcal{D}(\vec{\mathcal{H}})$ where we take the class $\mathcal{D} := \{M \in \vec{\mathcal{H}} \mid \mathrm{Hom}(M, \mathcal{H}_0) = 0\}$ of divisible objects and the class $\mathcal{R} := \mathcal{D}^{\perp_0}$ of reduced objects (see Section 4). By Proposition 2.12 applied to $\mathcal{D}(\vec{\mathcal{H}})$ the pure-injective objects in $\mathcal{D}(\vec{\mathcal{H}})$ are exactly those in the class $\mathrm{Prod}(\mathcal{D}^b(\mathcal{H}))$ and so we will use the partition $(\mathbf{p}, \mathbf{t}, \mathbf{q})$, to find the indecomposable objects in $\mathrm{Prod}(\mathcal{D}^b(\mathcal{H})) \cap \mathcal{M}$.

Lemma 5.5. *Let* $\{X_i\}_{i \in I}$ *be a set of objects in* $\mathcal{H}_0$. *Then in* $\mathcal{T} := \mathcal{D}(\vec{\mathcal{H}})$ *we have* $\prod_{i \in I}^{\mathcal{T}} X_i \cong \prod_{i \in I}^{\vec{\mathcal{H}}} X_i$, *where the product on the left is taken in* $\mathcal{T}$ *and the product on the right is taken in* $\vec{\mathcal{H}}$.

Proof. Note that the class $\mathcal{F}$ of torsionfree objects in $\vec{\mathcal{H}}$ is a torsionfree class that contains a system of generators. Moreover, by definition, we have that $\mathcal{F} = \mathcal{H}_0^{\perp_0}$. By applying (generalised) Serre duality, we have that $\mathcal{H}_0 \subseteq \mathcal{F}^{\perp_1}$. Thus, by [10, Prop. 2.12], we have the desired result. $\square$

The following lemma is a derived version of [50, 2.2]. In the proof, we will make use of the following setup several times; we will refer to it as Setup (*). For

every pure-injective object X in $\mathcal{D}(\vec{\mathcal{H}})$, we have the following morphisms:

$$X \xrightarrow{\begin{pmatrix} f_{\mathbf{p}} \\ f_{\mathbf{t}} \\ f_{\mathbf{q}} \end{pmatrix}} X_{\mathbf{p}} \oplus X_{\mathbf{t}} \oplus X_{\mathbf{q}} \xrightarrow{(\,g_{\mathbf{p}}\ g_{\mathbf{t}}\ g_{\mathbf{q}}\,)} X$$

such that $g_{\mathbf{p}} f_{\mathbf{p}} + g_{\mathbf{t}} f_{\mathbf{t}} + g_{\mathbf{q}} f_{\mathbf{q}} = 1$ where $X_{\mathbf{p}}$ is a product of objects in $\mathbf{p}$, $X_{\mathbf{t}}$ is a product of objects in $\mathbf{t}$ and $X_{\mathbf{q}}$ is a product of objects in $\mathbf{q}$. All products and $\mathrm{Prod}(-)$ are taken in $\mathcal{D}(\vec{\mathcal{H}})$ unless otherwise stated.

LEMMA 5.6. *We have*

(1) *(i)* $\mathrm{Prod}(\mathbf{p}) \cap \mathcal{M} = 0$, *(ii)* $\mathrm{Prod}(\mathbf{t}) \subseteq \mathcal{M}$, *(iii)* $\mathrm{Prod}(\mathbf{q}) \cap \mathcal{M} = \mathcal{D}$.

(2) *The class* $\mathrm{Prod}(\mathbf{t})$ *is the class of pure-injective objects in* $\mathcal{R} \cap \mathcal{M}$.

(3) *The following are equivalent for* $X \in \vec{\mathcal{H}}$.

    (a) $X = X' \oplus X''$ *where* $X' \in \mathrm{Prod}(\mathbf{t})$ *and* $X'' \in \mathcal{D}$.

    (b) X *is pure-injective and belongs to* $\mathcal{M}$.

Proof. It follows from Proposition 2.13 that the classes $\mathrm{Prod}(\mathbf{p})$, $\mathrm{Prod}(\mathbf{t})$ and $\mathrm{Prod}(\mathbf{q})$ have pairwise zero intersections.
(1)(i) Let $M \in \mathcal{M}$, then $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(M, \bigvee_{n<0} \mathcal{H}[n]) = 0$ and $\mathrm{Hom}_{\vec{\mathcal{H}}}(M, \mathrm{vect}\,\mathbb{X}) = 0$. That is, we have $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(M, \mathbf{p}) = 0$ and so the first claim follows.

(ii) Next, note that $\mathcal{M}$ is closed under products in $\vec{\mathcal{H}}$ by Proposition 5.4. Then, as $\mathbf{t} \subseteq \mathcal{M}$, it follows from Lemma 5.5 that $\mathrm{Prod}(\mathbf{t}) \subseteq \mathcal{M}$.

(iii) For the third claim, let $X \in \mathcal{D}$. As $\mathcal{D}$ consists of pure-injective objects in $\vec{\mathcal{H}}$, it follows from Lemma 2.9 that X is a pure-injective object in $\mathcal{D}(\vec{\mathcal{H}})$ and so we are in Setup (*) above. We have that $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(\mathcal{D}, \mathrm{Prod}(\mathbf{t})) = 0$ because $\mathrm{Hom}_{\vec{\mathcal{H}}}(\mathcal{D}, \mathcal{H}_0) = 0$, so $f_{\mathbf{t}} = 0$. Similarly, we have that $f_{\mathbf{p}} = 0$ because $\mathrm{Hom}_{\vec{\mathcal{H}}}(\mathcal{D}, \mathrm{vect}\,\mathbb{X}) = 0$ and $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(\mathcal{D}, \bigvee_{n<0} \mathcal{H}[n]) = 0$. So $X \in \mathrm{Prod}(\mathbf{q})$. Moreover, we have that $\mathcal{D} \subseteq \mathcal{M}$ so we have shown that $\mathcal{D} \subseteq \mathrm{Prod}(\mathbf{q}) \cap \mathcal{M}$ in $\mathcal{D}(\vec{\mathcal{H}})$.

We wish to show that $\mathrm{Prod}(\mathbf{q}) \cap \mathcal{M} \subseteq \mathcal{D}$; in fact we will show that $\mathrm{Prod}(\mathbf{q}) \cap \vec{\mathcal{H}} \subseteq \mathcal{D}$. Let $Y \in \mathrm{Prod}(\mathbf{q}) \cap \vec{\mathcal{H}}$. We will show that $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(S_x[-1], Y) \cong \mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(S_x, Y[1]) \cong \mathrm{Ext}^1_{\vec{\mathcal{H}}}(S_x, Y) = 0$ for all simple $S_x \in \mathcal{H}_0$ i.e. $Y \in \mathcal{D}$. Suppose, for a contradiction, that there is a non-zero morphism $f\colon S_x[-1] \to Y$. Since $Y \in \mathrm{Prod}(\mathbf{q})$, it follows that there exists a non-zero morphism $g\colon Y \to Q$ with $Q \in \mathbf{q}$ indecomposable such that $gf \neq 0$. By definition, we have $Q \cong X[i]$ for some $X \in \mathcal{H}$ and $i > 0$. But then we have $0 \neq gf \in \mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(S_x[-1], X[i]) \cong \mathrm{Ext}^{i+1}_{\vec{\mathcal{H}}}(S_x, Y)$, which is a contradiction. Therefore we must have that $Y \in \mathcal{D}$.

(2) We have already seen that $\mathrm{Prod}(\mathbf{t})$ consists of pure-injective objects and also $\mathrm{Prod}(\mathbf{t}) \subseteq \mathcal{M}$. Since $(\mathcal{D}, \mathcal{R})$ is a split torsion pair in $\vec{\mathcal{H}}$, we can write any $X \in \mathrm{Prod}(\mathbf{t})$ as $X \cong X_{\mathcal{D}} \oplus X_{\mathcal{R}}$ where $X_{\mathcal{D}} \in \mathcal{D}$ and $X_{\mathcal{R}} \in \mathcal{R}$. But

then $X_{\mathcal{D}} \in \mathrm{Prod}(\mathbf{q}) \cap \mathrm{Prod}(\mathbf{t}) = 0$. So $X \cong X_{\mathcal{R}} \in \mathcal{R}$. We have shown that $\mathrm{Prod}(\mathbf{t}) \subseteq \mathcal{M} \cap \mathcal{R}$.

For the reverse inclusion, let $X \in \mathcal{M} \cap \mathcal{R}$ be pure-injective. Again, we are in Setup (*) above. In the proof of (1)(i) we saw that $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(X, \mathbf{p}) = 0$, therefore $f_{\mathbf{p}} = 0$. We will show that $g_{\mathbf{q}} = 0$. Note that $X_{\mathbf{q}} \cong \bigoplus_{i \geq 0} Q_i[i]$ where $Q_i \cong H^{-i}(X_{\mathbf{q}}) \in \vec{\mathcal{H}}$ for each $i \geq 0$ (see, for example, [27, Sec. 1.6]). Then $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(X_{\mathbf{q}}, X) \cong \prod_{i \geq 0} \mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(Q_i[i], X) \cong \mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(Q_0[0], X)$. In the proof of (1)(iii) we saw that $\mathrm{Prod}(\mathbf{q}) \cap \vec{\mathcal{H}}[0] \subseteq \mathcal{D}[0]$, and so we have $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(Q_0, X) = 0$ because $X \in \mathcal{R}$ and $Q_0 \in \mathcal{D}$. We therefore have that $X \in \mathrm{Prod}(\mathbf{t})$.

(3) (a) $\Rightarrow$ (b) This implication is clear since $\mathrm{Prod}(\mathbf{t})$ and $\mathcal{D}$ both consist of pure-injective objects.

(b) $\Rightarrow$ (a) Suppose $X \in \mathcal{M}$ is pure-injective. Using that split torsion pair $(\mathcal{D}, \mathcal{R})$, we may decompose X as $X_{\mathcal{D}} \oplus X_{\mathcal{R}}$. Then $X_{\mathcal{D}} \in \mathcal{D}$ and $X_{\mathcal{R}} \in \mathcal{R} \cap \mathcal{M}$. Since $X_{\mathcal{R}}$ is pure-injective, we have $X_{\mathcal{R}} \in \mathrm{Prod}(\mathbf{t})$ by part (2). $\qquad \square$

We obtain in particular:

LEMMA 5.7. *In $\vec{\mathcal{H}}$ we have that $\mathrm{Prod}(\mathcal{H}_0)$ is the class of pure-injective sheaves in $\mathcal{R} \cap \mathcal{M}(\infty)$.*

Proof. By Lemma 5.6, we have that $\mathrm{Prod}(\mathcal{H}_0[0])$ coincides with the class of pure-injective objects in $\mathcal{R}[0] \cap \mathcal{M}[0]$ in $\mathcal{D}(\vec{\mathcal{H}})$. The result then follows from Lemma 5.5. $\qquad \square$

LEMMA 5.8. *Let $M \in \vec{\mathcal{H}}$ be reduced and having no non-zero direct summand of finite length. Then M is torsionfree.*

Proof. The canonical sequence $0 \to tM \to M \to M/tM \to 0$, where tM is the largest torsion subsheaf of M, is pure-exact. Since finite length sheaves are pure-injective it follows that tM as well has no non-zero direct summand of finite length, and thus tM is a coproduct of Prüfer sheaves (cf. [1, Cor. 3.7(2)]). Since M is reduced, this coproduct must be empty, that is, $tM = 0$. $\qquad \square$

LEMMA 5.9. *Let $0 \neq M \in \vec{\mathcal{H}}$ be torsionfree and lying in $\mathrm{Prod}(\mathcal{U}_x)$ for some $x \in \mathbb{X}$. Then M has an indecomposable direct summand isomorphic to the adic $S_x[-\infty]$ for some simple $S_x \in \mathcal{U}_x$.*

Proof. (1) If $\vec{\mathcal{H}} = \mathrm{Qcoh}(\mathcal{A})$ with a hereditary order $\mathcal{A}$, let R_x be the endomorphism ring of $\mathcal{A}$ considered as an object in the quotient category $\vec{\mathcal{H}}_x = \vec{\mathcal{H}}/\varinjlim(\coprod_{y \neq x} \mathcal{U}_y)$. Since $\mathcal{U}_x \subseteq \{\mathcal{U}_y \mid y \neq x\}^{\perp} \simeq \mathrm{Mod}(R_x)$, we consider M as an R_x-module. Moreover, $\varinjlim \mathcal{U}_x$ is the class of torsion modules over R_x, and the above right-perpendicular category is closed in $\vec{\mathcal{H}}$ under limits (in particular: products) and direct limits (cf. [18]). Thus an R_x-module is (pure-) injective if and only if it is (pure-) injective in $\vec{\mathcal{H}}$. In particular, M is a reduced,

torsionfree and pure-injective R_x-module. Since M is a direct summand of a product of modules in $\mathcal{U}_x$, which are complete, it is also complete.

(2) We first treat the special case where $p(x) = 1$. By completeness, M is a V_x-module, where V_x is a complete discrete valuation domain with $\widehat{R}_x \cong \mathrm{M}_{e(x)}(V_x)$, where $\widehat{R}_x = \varprojlim R_x / \mathrm{rad}(R_x)^i$ is the $\mathrm{rad}(R_x)$-adic completion of R_x; we refer to [31, Prop. 3.16]. The class of torsion (resp., finite length) V_x-modules coincides with the class of torsion (resp., finite length) R_x-modules. In particular, it follows that M is also reduced, torsionfree and pure-injective as a V_x-module. Since M is reduced, it has a maximal submodule. Each $a \in M \setminus \mathrm{rad}\, M$ induces a monomorphism $f \colon V_x \to M$ which does not belong to $\mathrm{rad}(V_x, M)$. Since (cf. [28, after Ex. 11.9]) $\mathrm{End}_{R_x}(V_x) = \mathrm{End}_{V_x}(V_x) \cong V_x$ is local, we obtain that f splits, and thus V_x is a direct summand of M. (For a similar argument cf. [28, Cor. 11.6].) By construction, the R_x-module $V_x = \varprojlim V_x / \mathrm{rad}(V_x)^i$ corresponds to the adic $S_x[-\infty]$ in $\vec{\mathcal{H}}$, where S_x (corresponding to the R_x-module $V_x / \mathrm{rad}(V_x)$) is the only simple in $\mathcal{U}_x$.

(3) Now let $p = p(x)$ be arbitrary. Then we have to replace the complete ring V_x by the ring $H = H_p(V_x)$, see [31, Prop. 13.4]. Since $M \in \mathrm{Prod}(\mathcal{U}_x)$, and with the same arguments as in (2), M is a complete, torsionfree, reduced and pure-injective H-module. Since M is reduced, there is $a \in M \setminus \mathrm{rad}\, M$. This induces a monomorphism $f \colon H \to M$ with $f \notin \mathrm{rad}(H, M)$. Let $e_1, \ldots, e_p$ be the canonical complete set of primitive, othogonal idempotents of H. There is some i and a morphism $f_i \colon e_i H \to M$ with $f_i \notin \mathrm{rad}(e_i H, M)$. Moreover $\mathrm{End}(e_i H) = e_i H e_i \cong V_x$ is local. It follows that f_i is a split monomorphism. Thus $e_i H$ is an indecomposable direct summand of M, and it corresponds to the adic associated with some simple in $\mathcal{U}_x$ (cf. also [47, 4.4]). $\square$

REMARK 5.10. The lemma shows in particular that all the adics $S[-\infty]$ are indecomposable.

THEOREM 5.11. *The following is a complete list of the indecomposable pure-injective sheaves in $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$ of slope ∞:*

(1) *The indecomposable sheaves of finite length.*

(2) *The sheaf $\mathcal{K}$ of rational functions, the Prüfer and the adic sheaves.*

Moreover, each pure-injective sheaf E of slope ∞ is discrete, that is, has – unless zero – an indecomposable direct summand.

Proof. We assume that M is indecomposable pure-injective of slope ∞ and not coherent. Since M is indecomposable, M is either divisible or reduced. In the first case it is generic or Prüfer. Thus we can assume that M is reduced, and we have to show that M is an adic. By Lemma 5.7 we have $M \in \mathrm{Prod}(\mathcal{H}_0)$. Since M is indecomposable there is $x \in \mathbb{X}$ such that even $M \in \mathrm{Prod}(\mathcal{U}_x)$ (cf. [50, 2.3]; the arguments therein also hold in our setting). Since M is not of finite length, it is torsionfree by Lemma 5.8. By the Lemma 5.9 then M is an adic with respect to $\mathcal{U}_x$.

The additional statement follows also from that lemma. Indeed, it is sufficient to assume that E is reduced and moreover belonging to $\mathrm{Prod}(\mathcal{U}_x)$ for some x. □

PROPOSITION 5.12. *For every simple S there is a short exact sequence*

$$0 \to \tau S[-\infty] \to E \to S[\infty] \to 0$$

with E a direct sum of copies of $\mathcal{K}$.

Proof. Let $p \geq 1$ be the rank of the tube $\mathcal{U}_x$ containing S. As in [20] we get by an inverse limit construction (using $S[-p] \cong \tau^- S[p]$) a short exact sequence $0 \to \tau S[-\infty] \to \tau S[-\infty] \to S[p] \to 0$; for exactness of the inverse limit we note that we may form the inverse limit of a surjective inverse system in $\vec{\mathcal{H}}_x = \mathrm{Mod}(R_x)$ as in the proof of Lemma 5.9. Then by a direct limit construction we get a short exact sequence $0 \to \tau S[-\infty] \to E \to S[\infty] \to 0$; it follows as in [49, Prop. 4] that E is torsionfree and divisible, hence a direct sum of copies of $\mathcal{K}$. □

The proposition shows (cf. [8, Lem. 2.7]):

LEMMA 5.13. *Let x, $y \in \mathbb{X}$ and j be an integer. Then*

  (1) $\mathrm{Ext}^1(S_x[\infty], \tau^j S_y[-\infty]) \neq 0$ if and only if $x = y$.

  (2) $\mathrm{Ext}^1(S_x[-\infty], \tau^j S_y[\infty]) = 0$. □

6   COTILTING SHEAVES OF SLOPE INFINITY

We will classify all cotilting sheaves having slope ∞.

PROPOSITION 6.1. *We have the following.*

  (1) Let C be a cotilting sheaf and $\mathcal{F}_0 = {}^{\perp_1}C \cap \mathcal{H}$. Then C has slope ∞ if and only if $\mathrm{vect}\,\mathbb{X} \subseteq \mathcal{F}_0$.

  (2) Let C and C' be two equivalent cotilting sheaves. If one of them has slope ∞, then so has the other.

  (3) Each cotilting sheaf of slope ∞ is large.

  (4) Let C be a cotilting sheaf and T be a tilting sheaf such that $\Gamma(T) = C$, with Γ as in Theorem 4.4. Then C has slope ∞ if and only if T has slope ∞.

Proof. (1) is clear.
(2) Follows from $\mathrm{Prod}(C) = \mathrm{Prod}(C')$.
(3) Follows since there is no cotilting sheaf consisting only of indecomposable summands of finite length (cf. [1, Rem. 7.7]).
(4) Follows from Theorem 4.4 and (1) (and its analogue for tilting objects). □

Rigidity

The following basic splitting property will be crucial for our treatment of cotilting sheaves.

THEOREM 6.2 ([1, Thm. 3.8]). *Let $E \in \vec{\mathcal{H}}$ be a rigid sheaf, that is, $\mathrm{Ext}^1(E, E) = 0$ holds.*

(1) *The torsion subsheaf tE is a direct sum of Prüfer sheaves and exceptional sheaves of finite length. Accordingly it is pure-injective.*

(2) *The canonical exact sequence $0 \to tE \to E \to E/tE \to 0$ splits.* □

Given a rigid sheaf $E \in \vec{\mathcal{H}}$, we will use the notation

$$E = E_+ \oplus E_0$$

where $E_0 = tE$ denotes the *torsion part* and $E_+ \cong E/tE$ denotes the *torsionfree part* of E. We will say that E has a *large torsion part* if there is no coherent sheaf F such that $\mathrm{Add}(tE) = \mathrm{Add}(F)$.

COROLLARY 6.3. *Let $E \in \vec{\mathcal{H}}$ be rigid and indecomposable. Then E is either torsion or torsionfree.* □

PROPOSITION 6.4. *For any sheaf E, if the Prüfer sheaf $S[\infty]$ belongs to $\mathrm{Prod}(E)$, then it is a direct summand of E.*

Proof. Suppose $S[\infty]$ is a direct summand of E^I and let $s \colon S[\infty] \to E^I$ be the corresponding split monomorphism. Also let $m \colon S \to S[\infty]$ be an essential monomorphism, which exists since $S[\infty]$ is the injective envelope of the simple object S. Then $s \circ m$ is non-zero and hence $\pi \circ s \circ m$ is a non-zero monomorphism for some projection $\pi \colon N^I \to N$. As m is an essential monomorphism, it follows that $\pi \circ s$ is a monomorphism. As $S[\infty]$ is injective, the proposition follows. □

COROLLARY 6.5. *Let $C \in \vec{\mathcal{H}}$ be cotilting with torsionfree class $\mathcal{F} = {}^{\perp_1}C$. If $S[\infty] \in \mathcal{F}$, then it is a direct summand of C.*

Proof. Since $S[\infty]$ is injective, we have $S[\infty] \in \mathcal{F} \cap \mathcal{F}^{\perp_1} = \mathrm{Prod}(C)$. □

MAXIMAL SELF-ORTHOGONALITY W.R.T. TUBES

Let $\mathcal{U}$ be a tube. As in [8] we say that a pure-injective object M *belongs to $\mathcal{U}$* if every indecomposable direct summand of M is of the form $S[n]$ with $S \in \mathcal{U}$ simple and $n \in \mathbb{N} \cup \{\pm\infty\}$. The subcategory formed by all such objects is denoted by $\overline{\mathcal{U}}$. The *$\mathcal{U}$-component $M_{\mathcal{U}}$* of M is defined to be a maximal direct summand of M belonging to $\overline{\mathcal{U}}$. If $\mathcal{U} = \mathcal{U}_x$, we will also use the notation M_x. The $\mathcal{U}$-component is unique up to isomorphism. In this context it is useful to recall that each indecomposable pure-injective object has a local endomorphism ring and we have the exchange property for such objects U, cf. [42, Thm. E.1.53.];

for instance, if U is a direct summand of a direct sum $M \oplus N$, then it is a direct summand of M or of N. Moreover, the $\mathcal{U}$-component of a cotilting object M is said to be of *Prüfer type* (resp., *adic type*) if it has a Prüfer (resp., adic) summand; it will turn out that each component is either of Prüfer or of adic type, and not both.

The following lemma is shown as in [8, Prop. 3.3].

LEMMA 6.6. *Let C be cotilting of slope ∞ and $\mathcal{U}$ a tube. Then the $\mathcal{U}$-component $C_{\mathcal{U}}$ is maximal self-orthogonal with respect to all objects in $\overline{\mathcal{U}}$.* $\qquad\square$

BRANCHES

6.7. BRANCH SHEAVES. In this section we consider certain coherent sheaves, called branch sheaves, which turn out to be typical coherent summands of large cotilting sheaves. Let $\mathcal{U}_x$ be a tube of rank $p > 1$. The exceptional (i.e. indecomposable and rigid) sheaves E in $\mathcal{U}_x$ are exactly those of length $\leq p - 1$ and so there are only finitely many of them. The collection $\mathcal{W}$ of subquotients of an exceptional sheaf E is called the *wing rooted in E* and E is called the *root of $\mathcal{W}$*. The set of simple sheaves in $\mathcal{W}$ is called the *basis of $\mathcal{W}$*. The basis of any wing is of the form $S, \tau^- S, \ldots, \tau^{-(r-1)} S$ for a simple sheaf S where r is the length of the root E. We say that another wing $\mathcal{W}'$ is *not adjacent* to $\mathcal{W}$ if their bases are disjoint and neither τS nor $\tau^{-r} S$ is in $\mathcal{W}'$ (we say that the two wings $\mathcal{W}$ and $\mathcal{W}'$ are *non-adjacent*) [35, Ch. 3].

The full subcategory add $\mathcal{W}$ is equivalent to the category of modules over the path algebra of a linearly ordered Dynkin quiver of type $\mathbb{A}$, cf. [35, Ch. 3]. We define a *connected branch B in $\mathcal{W}$* in the following way: B has exactly r nonisomorphic indecomposable summands $B_1, \ldots, B_r$ such that $B_1 \cong E$ and for every j, the wing rooted in B_j contains exactly ℓ_j indecomposable summands of B where ℓ_j is the length of B_j. Each connected branch in $\mathcal{W}$ is a tilting object in the subcategory add $\mathcal{W}$ [48, p. 205].

A module B in $\mathcal{H}_0$ is called a *branch sheaf* if it is a multiplicity-free direct sum of connected branches in pairwise non-adjacent wings [35, Ch. 3]. Any branch sheaf B is rigid and decomposes as $B = \bigoplus_{x \in \mathbb{X}} B_x$. In fact, it is clear from the definition that $B_x = 0$ for all $x \in \mathbb{X}$ corresponding to homogeneous tubes and there are only finitely many isomorphism classes of branch sheaves.

Given a non-empty subset $V \subseteq \mathbb{X}$, we also write

$$B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$$

where $B_{\mathfrak{e}}$ is supported in $\mathbb{X} \setminus V$ and $B_{\mathfrak{i}}$ in V. In this case we will say that $B_{\mathfrak{e}}$ is *exterior* and $B_{\mathfrak{i}}$ is *interior* with respect to V. We will see in Theorem 6.11 that a pair (B, V) determines a cotilting module C, in which the exterior part of B with respect to V determines the adic summands of C and the interior part of B with respect to V determines the Prüfer summands of C.

LEMMA 6.8. *Let C be a cotilting sheaf of slope ∞ and x a point of weight $p = p(x) \geq 1$. There are two possible cases:*

(1) *Exterior "adic type" case:*

    (a) *The $\mathcal{U}_x$-component C_x of C contains no Prüfer sheaf. The torsion part of C_x consists of a direct sum of $0 \leq s \leq p-1$ indecomposable summands of finite length.*

    (b) *The finite length summands are arranged in connected branches in pairwise non-adjacent wings; let $\mathcal{W}$ denote the union of these wings in $\mathcal{U}_x$.*

    (c) *The $p - s$ adic sheaves $S_x[-\infty]$ such that $\mathrm{Hom}(S_x[-\infty], \tau\mathcal{W}) = 0$ are (torsionfree) direct summands of C.*

(2) *Interior "Prüfer type" case:*

    (a) *The $\mathcal{U}_x$-component C_x of C consists of a direct sum of $1 \leq s \leq p$ Prüfer sheaves, and precisely $p - s$ indecomposable summands of finite length.*

    (b) *The finite length summands belong to wings of the following form: if $S[\infty]$, $\tau^{-r}S[\infty]$ are summands of C with $2 \leq r \leq p$, but the Prüfer sheaves $\tau^- S[\infty], \ldots, \tau^{-(r-1)}S[\infty]$ in between are not, then there is a (unique) connected branch in the wing $\mathcal{W}$ rooted in $S[r-1]$ that occurs as a summand of C.*

    (c) *The torsionfree part C_+ of C is x-divisible; thus C is automatically of slope ∞ in this case.*

Proof. We assume without loss of generality that C is minimal cotilting and hence discrete; thus C is unqiuely determined by its indecomposable direct summands.

Suppose that C_x does not contain a Prüfer sheaf as a direct summand. Then, by Theorem 6.2, we must have that C_x is a direct sum of exceptional sheaves in $\mathcal{U}_x$. There are only finitely many exceptional sheaves in $\mathcal{U}_x$ and so let $B_1, \ldots, B_s$ denote the exceptional summands of C (up to isomorphism). Since $\bigoplus_{i=1}^{s} B_i$ is rigid, it has at most the number of summands of a tilting object in $\mathcal{U}_x$, such objects are well-known and it follows that $0 \leq s < p$, cf. [35, Ch. 3]. By Lemma 6.6, we must have that every adic $S_x[-\infty]$ such that $\mathrm{Hom}_{\bar{\mathcal{H}}}(S_x[-\infty], \tau B_i) = 0$ for all $1 \leq i \leq s$ is a direct summand of C; for future reference, let $\mathcal{A}$ denote the set of these adics. This is a consequence of generalised Serre duality and the fact that $\mathrm{Hom}_{\bar{\mathcal{H}}}(\tau^- B_i, S_x[-\infty]) = 0$ for all $1 \leq i \leq s$. Since $s < p$, there is at least one coray that does not contain any B_i, therefore $\mathcal{A}$ is non-empty. Now, by another application of Lemma 6.6, we have that the B_i are maximal self-orthogonal (and hence maximal rigid) among the sheaves in $\{U \in \mathcal{U}_x \mid \tau U \in \mathcal{A}^{\perp_0}\}$. It follows from a standard argument (e.g. [35, Ch. 3]) that $B_1, \ldots B_s$ must form a branch sheaf. It follows from the form of the branches that there are $p - s$ adics in $\mathcal{A}$.

Assume the interior case. By Lemma 5.13 no adic sheaf associated with the tube $\mathcal{U}_x$ can be a direct summand of C. Moreover, the same proof as in [1, Lem. 4.10] shows that C_+ is x-divisible. By Lemma 5.1 thus C has slope ∞. The proof concerning the claim for the wings in the interior case is completely dual to the arguments given in the proof of [1, Lem. 4.9], and we therefore omit it. □

Suppose C is a cotilting sheaf of slope ∞ such that C falls into case (2) of Lemma 6.8 with respect to $x \in \mathbb{X}$. It follows immediately from Corollary 3.5 that the branch summand B of C in $\mathcal{U}_x$ is given by

$$\mathrm{add}(B) = \mathrm{Prod}(C) \cap \mathcal{U}_x.$$

In particular, this shows that a cotilting sheaf C' with a different branch $B' \neq B$ in $\mathcal{U}_x$ will have ${}^{\perp_1}C' \neq {}^{\perp_1}C$, that is, C and C' cannot be equivalent.

The generating torsionfree classes

We now consider a pair (B, V) given by a branch sheaf $B \in \mathcal{H}$ and a subset $V \subseteq \mathbb{X}$, and we associate a generating torsionfree class in $\mathcal{H}$ to it. In order to do this we next define two pieces of notation.

Firstly, let $\mathcal{W}$ be the collection of pairwise non-adjacent wings in $\mathcal{U}_x$ determined by the branch sheaf B. Let $S, \tau^- S, \ldots, \tau^{-(r-1)}S$ be a basis for one of the wings in $\mathcal{W}$. Then $\mathcal{R}_x$ is the set of indices $j \in \{0, \ldots, p(x)-1\}$ such that $\tau^{j+1}S \notin \mathcal{W}$. Secondly, given a connected branch A with associated wing $\mathcal{W}_A$, we define the *undercut* of A as in [1, (4.9)]:

$$A^> := \begin{cases} A^{\perp_0} \cap \mathcal{W}_A & \text{if } A \text{ is interior,} \\ A^{\perp_0} \cap \tau\mathcal{W}_A & \text{if } A \text{ is exterior.} \end{cases}$$

The torsionfree class $\mathcal{F}_0$ associated to (B, V) will consist of all vector bundles, of the rays given by the sets $\mathcal{R}_x$, and of some objects determined by B. Up to τ-shift, these objects will belong to the wings defined by the undercut of B.

LEMMA 6.9. *Let $V \subseteq \mathbb{X}$ and $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ be a branch sheaf.*

(1) *The class*

$$\mathcal{F}_0 = \mathrm{add}\left(\mathrm{vect}\,\mathbb{X} \cup \tau^-(B^>) \cup \bigcup_{x \in V} \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\} \right) \quad (6.1)$$

is a torsionfree class in $\mathcal{H}$ which generates.

(2) *There is a cotilting sheaf C with cotilting class ${}^{\perp_1}C = \varinjlim \mathcal{F}_0$. For any such C its torsion part is (up to multiplicities) given by*

$$C_0 = B \oplus \bigoplus_{x \in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty].$$

(3) *If, moreover, C is assumed to be minimal cotilting, then the indecomposable summands of its torsionfree part C_+ are given by the adic sheaves $S_y[-\infty]$ with $y \in \mathbb{X} \setminus V$ and $S_y \in \mathcal{U}_y$ simple such that $\mathrm{Hom}(S_y[-\infty], \tau B_{\mathfrak{e}}) = 0$, and in case $V = \emptyset$, additionally by the sheaf $\mathcal{K}$ of rational functions.*

Proof. (1) It is shown in [1, Lem. 4.11] that $\mathcal{F}_0$ is a resolving class, which in our setting means, that $\mathcal{F}_0$ generates and is closed under subobjects and extensions.

(2) By Theorem 3.7 there is a cotilting object C with $^{\perp_1} C = \vec{\mathcal{F}}_0$. Given a simple object $S \in \mathcal{U}_x$, it follows from Corollary 6.5 and the fact that $\vec{\mathcal{F}}_0$ is is a torsion-free class that we have that $S[\infty]$ is a direct summand of C if and only if $\vec{\mathcal{F}}_0 = {}^{\perp_1} C$ contains the ray $\{S[n] \mid n \geq 1\}$. Therefore the objects $\tau^j S_x[\infty]$ with $x \in V$ and $j \in \mathcal{R}_x$ are precisely the Prüfer summands of C. Moreover, by Lemma 3.4, we have that

$$\mathcal{F}_0 \cap \mathcal{F}_0{}^{\perp_1} = \mathrm{Prod}(C) \cap \mathcal{H}.$$

It remains to show that this class coincides with $\mathrm{add}(B)$, which then shows that the torsion part C_0 is as indicated. Since, by Proposition 4.3, the resolving classes in $\mathcal{H}$ coincide with the generating torsion-free classes, we may proceed in exactly the same way as in the proof of [1, Lem. 4.11], replacing $\mathscr{S}$ with $\mathcal{F}_0$.

(3) After Lemma 6.8 (1) it only remains to show that C_+ does not have another indecomposable summand, except $\mathcal{K}$, if C is assumed to be minimal, since C is uniquely determined by its indecomposable direct summands (or also by part (2) of the preceding proposition). By Theorem 5.11 this could only be either the generic $\mathcal{K}$ or another adic. If $V \neq \emptyset$ then the generic is already in $\mathrm{Prod}(C)$ (since C contains a Prüfer summand), in case $V = \emptyset$ we have to add it. Additional adics are not possible, using Lemma 5.13. $\square$

As a consequence we obtain

PROPOSITION 6.10. *Let C be a cotilting sheaf of slope ∞.*

(1) *Let T be a corresponding tilting sheaf such that $\Gamma(T) = C$. Then the torsion parts C_0 and T_0 coincide up to "multiplicities": $\mathrm{Add}(C_0) = \mathrm{Add}(T_0)$.*

(2) *Up to equivalence, C is uniquely determined by its torsion part C_0.* $\square$

THE CLASSIFICATION

Next we present the main result of this section, which states that, for any pair (V, B), where $V \subseteq \mathbb{X}$ is non-empty and B is a branch sheaf, there is a uniquely determined cotilting sheaf C of slope ∞. Moreover, every cotilting sheaf of slope ∞ arises in this way. Each $x \in V$ dictates that $\mathcal{U}_x$ is of Prüfer type and the interior part of B with respect to x dictates which Prüfers belonging to $\mathcal{U}_x$ occur. Similarly, the set $\mathbb{X} \setminus V$ and the exterior part of B with respect to V control the tubes of adic type.

THEOREM 6.11. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, where $\mathbb{X}$ is a weighted noncommutative regular projective curve.*

(1) *Let $V \subseteq \mathbb{X}$ and $B \in \mathcal{H}_0$ be a branch sheaf. There is a unique large cotilting sheaf $C = C_+ \oplus C_0$ of slope ∞ up to equivalence such that*

$$C_0 = B \oplus \bigoplus_{x \in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty], \tag{6.2}$$

where the sets $\mathcal{R}_x \subseteq \{0, \ldots, p(x) - 1\}$ are non-empty and are uniquely determined by B.

(2) *Every cotilting sheaf of slope ∞ is, up to equivalence, as in (1) and C is V-divisible. (If $V \neq \emptyset$ hence C is automatically of slope ∞.)*

(3) *Assuming C to be minimal, the indecomposable summands of the torsion-free part C_+ are the following:*

- *the adic sheaves $\tau^\ell S_y[-\infty]$ with $y \in \mathbb{X}\backslash V$ and ℓ such that $\tau^\ell S_y \notin \tau\mathcal{W}$ for any wing $\mathcal{W}$ associated with an exterior branch part of B; if $V \neq \emptyset$ then C_+ is the pure-injective envelope of the coproduct of these adic sheaves;*

- *if $V = \emptyset$, additionally the sheaf of rational functions $\mathcal{K}$.*

Proof. The result is a consequence of Lemma 6.9 and the fact, that the classes (6.1) are just the (generating) torsionfree subclasses in $\mathcal{H}$ containing $\operatorname{vect}\mathbb{X}$, which follows from the corresponding results on resolving classes and tilting sheaves, cf. [1, Sec. 4]. $\square$

6.12. Let $V \subseteq \mathbb{X}$ and B be a branch sheaf. The minimal large cotilting sheaf $C = C_+ \oplus C_0$ from Theorem 6.11 will be denoted by $C_{(B,V)}$. With the large tilting sheaf $T_{(B,V)}$ of finite type from [1, (4.6)] if $V \neq \emptyset$ and $T_{(B,\emptyset)} = \mathbf{L}' \oplus B$ with $\mathbf{L}'$ a Lukas tilting sheaf (cf. [1, Prop. 4.5]) in $B^\perp$, we have $\Gamma(T_{(B,V)}) = C_{(B,V)}$ by construction. It follows that the following holds true:

$C_{(B,V)}$ *is tilting* $\Leftrightarrow$ $T_{(B,V)}$ *is cotilting* $\Leftrightarrow$ $T_{(B,V)}$ *is pure-injective* $\Leftrightarrow$ $V = \mathbb{X}$.

Indeed: because of Proposition 6.10 (1), saying that the pure-injective torsion parts C_0 and T_0 agree, we only need to consider the torsionfree parts C_+ and T_+, respectively. Here, C_+ is always pure-injective. By [1, Thm. 4.8] we have that T_+ is V-divisible, hence by Lemma 5.1 of slope ∞, unless $V = \emptyset$. It is sufficient to show that T_+ is pure-injective if and only if $V = \mathbb{X}$. If $V = \mathbb{X}$ then $T_+ \in \operatorname{Add}(\mathcal{K})$ is pure-injective (and up to multiplicities and summands isomorphic to $\mathcal{K}$, $T_{(B,\mathbb{X})} = C_{(B,\mathbb{X})}$). If $V = \emptyset$, then T_+ is not pure-injective: otherwise the Lukas sheaf $\mathbf{L}$ would be pure-injective and therefore $\mathbf{L}$ must have an adic summand $S[-\infty]$. Let S' be simple from a different, homogeneous tube. It is easy to see that $S'[-\infty]$ cogenerates every vector bundle, and hence also $\mathbf{L}$,

since $\mathbf{L}$ is $\mathrm{vect}(\mathbb{X})$-filtered by [1, Thm. 4.4] together with [53, Cor. 2.15(2)]. Then $S'[-\infty]$ also cogenerates $S[-\infty]$. But from Proposition 2.13 we deduce $\mathrm{Hom}(S[-\infty], S'[-\infty]) = 0$. Thus we can assume that $\emptyset \neq V \subsetneq \mathbb{X}$. Then again T_+ is not pure-injective: It follows from [1, Sec. 5] that we can assume that $\mathbb{X}$ is non-weighted and (compare also the proof of Lemma 5.9) that T_+ becomes a projective generator in $\vec{\mathcal{H}}/\mathcal{T}_V$, and $A_V := \mathrm{End}(T_+) = \varinjlim \mathrm{Hom}_{\mathcal{H}}(L', L)$ (where L is the structure sheaf and the direct limit runs over all sub-line bundles L' so that L/L' has support in V) is a noncommutative Dedekind domain ([31, Cor. 3.15] is the special case $A_{\mathbb{X} \setminus \{x\}} = R_x$), and it is PI since $A_V \subseteq k(\mathcal{H})$, with the latter finite over its centre. It follows from [38, Thm. 4.2], or [41, Thm. 1.6] (alternatively also from Lemma 5.9), that in case U is a pure-injective indecomposable summand of T_+, it must be the P-adic completion $\widehat{(A_V)}_P$ of A_V for some non-zero prime ideal P. On the other hand, U is a summand of A_V. But in A_V the partial sums of the first powers of a non-zero element in P yields a Cauchy sequence which has no limit in A_V. Thus, U is not pure-injective.

The discussion also shows:

C_+ is reduced if and only if $V \neq \emptyset$. In any case, the reduced part of C (resp. of C_+) is (up to Prod-equivalence) of the form $\widehat{T}$ (resp. $\widehat{T_+}$).

Here, the *completion* $\widehat{M}$ of a sheaf M is defined as $\prod_{x \in \mathbb{X}} \widehat{M}^x$, with the x-*completion* $\widehat{M}^x$ of M defined as $\varprojlim_{M/M' \in \mathcal{U}_x} M/M'$, like in [55], [50, 2.4]. Note that $\widehat{M}^x = 0$ if M is x-divisible, and $\widehat{M}^x = M$ if $M \in \mathcal{U}_x$. Since $\mathcal{U}_y^\perp$ is closed under limits, $\widehat{M}^x$ is y-divisible for any $y \neq x$, and thus has slope ∞ by Lemma 5.1. In $\mathrm{Mod}(R_x)$ we have that $\widehat{M}^x$ is pure-injective since it is complete (cf. [28, Thm. 11.4]), and hence it is pure-injective also in $\vec{\mathcal{H}}$. It follows then that $\widehat{M}^x$ lies in $\mathrm{Prod}(\mathcal{U}_x)$ and is discrete. We obtain that $\widehat{M}$ coincides with $\mathrm{PE}(\bigoplus_{x \in \mathbb{X}} \widehat{M}^x)$, since they have the same indecomposable (pure-injective) summands by using Theorem 5.11 and Proposition 2.13. $\qquad\square$

Maximal rigid objects in a large tube

Following [6], we call an object U in the direct limit closure $\vec{\mathcal{U}}$ of a tube $\mathcal{U}$ *maximal rigid* if it is rigid and every indecomposable $Y \in \vec{\mathcal{U}}$ satisfying $\mathrm{Ext}^1(U \oplus Y, U \oplus Y) = 0$ is a direct summand of U. With the preceding results we complement [1, Cor. 4.19] by the statements (2') and (3'):

COROLLARY 6.13. *The following statements are equivalent for an object $U \in \vec{\mathcal{U}}$.*

(1) U is maximal rigid in $\vec{\mathcal{U}}$.

(2) U is tilting in $\vec{\mathcal{U}}$.

(2') U is cotilting in $\vec{\mathcal{U}}$.

(3) *U is of Prüfer type and it coincides, up to multiplicities, with the sum-*
    mand $(tT)_x$ supported at x in the torsion part of some large tilting sheaf
    $T \in \vec{\mathcal{H}}$.

(3') *U is of Prüfer type and it coincides, up to multiplicities, with the sum-*
    mand $(tC)_x$ supported at x in the torsion part of some large cotilting sheaf
    $C \in \vec{\mathcal{H}}$.   □

7   The case of positive Euler characteristic

We assume that $\mathbb{X}$ is of domestic type, that is, the normalized *orbifold Euler characteristic* $\chi'_{orb}(\mathbb{X})$ is *positive*. Let $\delta(\omega)$ be the (negative) integer such that for the slopes $\mu(\tau E) = \mu(E) + \delta(\omega)$ holds for each indecomposable vector bundle E. The collection $\mathcal{E}$ of indecomposable vector bundles F such that $0 \leq \mu(F) < -\delta(\omega)$ forms a slice in the sense of [48, 4.2], and $T_{\mathrm{her}} := \bigoplus_{F \in \mathcal{F}} F$ is a tilting bundle having a finite-dimensional tame hereditary k-algebra H as endomorphism ring. We refer to [36, Prop. 6.5]. In particular $\mathcal{D}^b(\vec{\mathcal{H}}) = \mathcal{D}^b(\mathrm{Mod}(H))$, and this is also the repetitive category of $\mathrm{Mod}(H)$. Denote by $\mathbf{p}$ and $\mathbf{q}$ the preprojective and the preinjective component of H, respectively. Since H is hereditary, the tilting torsion pair $(\mathcal{T}, \mathcal{F})$ in $\vec{\mathcal{H}}$ induced by T_{her} splits. Moreover, in $\mathrm{Mod}(H)$ there is the (split) torsion pair $(\mathcal{Q}, \mathcal{C})$ with $\mathcal{Q} = \mathrm{Gen}\,\mathbf{q}$.

THEOREM 7.1. *Let $\mathbb{X}$ be a domestic curve.*

(1) *Each indecomposable pure-injective sheaf in $\mathrm{Qcoh}\,\mathbb{X}$ is either a vector bundle or has slope ∞.*

(2) *Each large cotilting sheaf in $\mathrm{Qcoh}\,\mathbb{X}$ has slope ∞.*

Hence the classifications of indecomposable pure-injective sheaves and of large cotilting sheaves in the domestic case are given by Theorem 5.11 (plus the indecomposable vector bundles) and Theorem 6.11, respectively.

Proof. (1) It is sufficient to proof the following: if E is indecomposable pure-injective and there is a non-zero morphism to a vector bundle then E is a vector bundle. The analogue in the module case is well-known (cf. [12, 3. Lem. 1]). Since E is indecomposable, either $E \in \mathcal{T}$ or $E \in \mathcal{F}$. We regard E as an object in $\mathcal{D}^b(\vec{\mathcal{H}}) = \mathcal{D}^b(\mathrm{Mod}(H))$. Thus either $E \in \mathrm{Mod}(H)$ or $E \in \mathrm{Mod}(H)[-1]$. We invoke Lemma 2.9 applied to $\mathcal{D}(\vec{\mathcal{H}})$. Let first $E \in \mathrm{Mod}(H)$. Since there is even an epimorphism from E to a vector bundle F, thus $F \in \mathcal{T}$ and hence $F \in \mathbf{p}$, the claim $E \in \mathbf{p} \subseteq \mathrm{vect}\,\mathbb{X}$ follows from the result in the module case. In case $E \in \mathrm{Mod}(H)[-1]$, we have $E \in \mathcal{Q}[-1]$. Then $E[1]$ is a direct summand of a product of finite dimensional H-modules. Since $(\mathcal{Q}, \mathcal{C})$ is a torsion pair and $\mathcal{C}$ closed under products, we obtain $E[1] \in \mathrm{Prod}(\mathbf{q})$. Thus, by [50, 2.2], either $E[1] \in \mathcal{C}$ or $\mathrm{Hom}(\mathbf{q}, E[1]) \neq 0$. The first case is not possible since $E \in \vec{\mathcal{H}}$. In the latter case we get $E \in \mathbf{q}[-1] \subseteq \mathrm{vect}\,\mathbb{X}$ from [12, 3. Lem. 1].

(2) We make use of Proposition 6.1 (4) and of the fact that the corresponding result for tilting sheaves is known, cf. [1, Sec. 6]. □

EXAMPLE 7.2. Let $\mathbb{X}$ be a domestic curve. Since we have that $\mathcal{D}^b(\vec{\mathcal{H}}) = \mathcal{D}^b(\mathrm{Mod}(H))$ and that the sheaves of slope ∞ are contained in the intersection of $\vec{\mathcal{H}}$ and $\mathrm{Mod}(H)$, we may observe that our classification coincides with the classification of infinite-dimensional cotilting left H-modules ([8]). We may therefore compare the correspondence in Theorem 4.4 with the bijection between tilting right H-modules and cotilting left H-modules ([3]).

In particular, we consider the case where H is the Kronecker algebra and $\mathbb{X} = \mathbb{P}_1(k)$ the projective line. In this case there are no branch sheaves so the following sets are parametrised by subsets V of $\mathbb{P}_1(k)$:

(1) Equivalence classes of large cotilting sheaves C_V in $\vec{\mathcal{H}}$ (Theorem 7.1).

(2) Equivalence classes of large tilting sheaves T_V of finite type in $\vec{\mathcal{H}}$ ([1, Cor. 6.5]).

(3) Equivalence classes of infinite-dimensional cotilting left H-modules M_V ([8, Cor. 3.10]).

(4) Equivalence classes of infinite-dimensional tilting right H-modules N_V ([3, Cor. 2.8]).

In each case, we have that $x \in V$ if and only if the Prüfer object $S_x[\infty]$ occurs as a direct summand X_V (where X is replaced with C, T, M, and N respectively). Under the bijective correspondence Γ given by Theorem 4.4 between the sets (1) and (2), we have that $\Gamma(T_V) = C_V$ (and by 6.12 the reduced part of C_V is obtained from T_V by completion). The sets (3) and (4) are also in bijective correspondence that is induced by the functor $D := \mathrm{Hom}_k(-, k)\colon \mathrm{Mod}(H^{op}) \to \mathrm{Mod}(H)$. Under this bijection we have that N_V is sent to M_U where $U = \mathbb{X} \setminus V$ (see [3, App.]).

8 The case of Euler characteristic zero

Throughout this section let $\mathbb{X}$ be a weighted noncommutative projective curve of orbifold Euler characteristic zero, and $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$.

For general information on the tubular case we refer to [34], [33], [45, Ch. 13], [30, Ch. 8] and [31, Sec. 13], on the elliptic case to [31, Sec. 9].

Let $\bar{p}$ denote the least common multiple of the weights $p_1, \ldots, p_t$, that is, $\bar{p} = 1$ in the case where $\mathbb{X}$ is elliptic and let $\bar{p} > 1$ if $\mathbb{X}$ is tubular. The formulae for the degree and for the slope of a non-zero object $E \in \mathcal{H}$ simplify to $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, with $\deg(E) = \frac{1}{\kappa\varepsilon}\langle\!\langle L, E \rangle\!\rangle$, cf. (4.1).

Since we are assuming $\chi'_{orb}(\mathbb{X}) = 0$, it follows that every indecomposable coherent sheaf is semistable. We therefore have the following result, which is analogous to Atiyah's classification [5].

Theorem 8.1 ([30, Prop. 8.1.6], [31, Thm. 9.7]). *For every $\alpha \in \widehat{\mathbb{Q}}$ the full subcategory $\mathbf{t}_\alpha$ of $\mathcal{H}$ formed by the semistable sheaves of slope α is a non-trivial abelian uniserial category whose connected components form stable tubes; the tubular family $\mathbf{t}_\alpha$ is parametrized again by a weighted noncommutative regular projective curve $\mathbb{X}_\alpha$ over k which satisfies $\chi'_{orb}(\mathbb{X}_\alpha) = 0$ and is derived-equivalent to $\mathbb{X}$.* $\square$

We therefore write

$$\mathcal{H} = \bigvee_{\alpha \in \widehat{\mathbb{Q}}} \mathbf{t}_\alpha.$$

Note that the component $\mathbf{t}_\infty$ coincides with the sheaves of finite length.

Quasicoherent sheaves with real slope

We recall that by [45, 1] the notion of slope is extended to all quasicoherent sheaves using the following partitions of $\vec{\mathcal{H}}$. For $w \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ we define

$$\mathbf{p}_w = \bigcup_{\alpha < w} \mathbf{t}_\alpha \qquad \mathbf{q}_w = \bigcup_{w < \beta} \mathbf{t}_\beta,$$

where $\alpha, \beta \in \widehat{\mathbb{Q}}$. We then have a partition of $\mathcal{H}$ as $\mathcal{H} = \mathbf{p}_w \vee \mathbf{t}_w \vee \mathbf{q}_w$ if w is rational, and $\mathcal{H} = \mathbf{p}_w \vee \mathbf{q}_w$ if w is irrational. We define the *sheaves of slope w* to be those contained in $\mathcal{M}(w) = \mathcal{B}_w \cap \mathcal{C}_w$ where

$$\mathcal{C}_w = \mathbf{q}_w^{\perp_0} = {}^{\perp_1}\mathbf{q}_w \quad \text{and} \quad \mathcal{B}_w = {}^{\perp_0}\mathbf{p}_w = \mathbf{p}_w^{\perp_1}.$$

For coherent sheaves this definition of slope is equivalent to the former one given as fraction of degree and rank, and for irrational w there are only non-coherent sheaves in $\mathcal{M}(w)$.

The following fundamental statement can be found in [45, Thm. 13.1], [1, Thm. 7.6].

Theorem 8.2 (Reiten-Ringel).   *(1) $\mathrm{Hom}(\mathcal{M}(w'), \mathcal{M}(w)) = 0$ for $w < w'$.*

(2) Every indecomposable sheaf has a well-defined slope $w \in \widehat{\mathbb{R}}$. $\square$

Cotilting sheaves that have a slope

Theorem 8.3. *Let $w \in \widehat{\mathbb{R}}$.*

(1) There is a large cotilting sheaf $\mathbf{W}_w$ of slope w and with cotilting class $\mathcal{C}_w$.

(2) If w is irrational, then $\mathbf{W}_w$ is, up to equivalence, the unique cotilting sheaf of slope w.

Proof. (1) The class $\mathrm{add}(\mathbf{p}_w) \subseteq \mathcal{H}$ is torsionfree and generates $\mathcal{H}$. Thus there is, by Theorem 3.7, a cotilting sheaf $\mathbf{W}_w$ with $\mathrm{Cogen}(\mathbf{W}_w) = \varinjlim(\mathbf{p}_w) = \mathcal{C}_w$.

Moreover, by the cotilting property clearly $\mathbf{W}_w \in \mathcal{C}_w^{\perp_1}$, which is a subclass of $\mathcal{B}_w$. Since in $\mathcal{H}$ no cotilting object has a slope (cf. [1, Rem. 7.7]), $\mathbf{W}_w$ is large.
(2) Let C be cotilting of irrational slope w. Since $C \in \mathcal{B}_w$, we have $\mathbf{p}_w \subseteq {}^{\perp_1}\mathbf{W}_w \subseteq {}^{\perp_1}C$, and thus $\mathbf{p}_w \subseteq {}^{\perp_1}C \cap \mathcal{H}$. Since ${}^{\perp_1}C \subseteq \mathcal{C}_w$, and since w is irrational, we obtain ${}^{\perp_1}C \cap \mathcal{H} \subseteq \mathbf{p}_w$. We obtain ${}^{\perp_1}C \cap \mathcal{H} = \mathbf{p}_w = {}^{\perp_1}\mathbf{W}_w \cap \mathcal{H}$. From Theorem 3.7 the result follows.   □

EXAMPLE 8.4. For $w = \infty$ the Reiten-Ringel sheaf $\mathbf{W} := \mathbf{W}_\infty$, given as the direct sum of the generic $\mathcal{K}$ and all the Prüfer sheaves, is cotilting with class ${}^{\perp_1}\mathbf{W} = \mathcal{C}_\infty = \widetilde{\mathcal{H}}$. Since $\mathcal{K}$ belongs to $\mathrm{Prod}(S[\infty])$ for any Prüfer sheaf, also $\mathbf{W}/\mathcal{K}$ is cotilting and equivalent to $\mathbf{W}$.

INTERVAL CATEGORIES

Let $w \in \widehat{\mathbb{R}}$. We denote by $\mathcal{H}\langle w \rangle$ the full subcategory of $\mathcal{D}^b(\mathcal{H})$ defined as

$$\bigvee_{\beta > w} \mathbf{t}_\beta[-1] \vee \bigvee_{\gamma \leq w} \mathbf{t}_\gamma.$$

The category $\mathcal{H}\langle w \rangle$ is a ([−1]-shifted) HRS-tilt of $\mathcal{H}$ in $\mathcal{D}^b(\mathcal{H})$ with respect to the split torsion pair $(\mathcal{T}_w, \mathcal{F}_w)$ in $\mathcal{H}$ given by $\mathcal{T}_w = \bigvee_{\beta > w} \mathbf{t}_\beta$ and $\mathcal{F}_w = \bigvee_{\gamma \leq w} \mathbf{t}_\gamma$ and hence is abelian, see [16, I. Thm. 3.3] and [37, Prop. 2.2]. Moreover $\mathcal{D}^b(\mathcal{H}\langle w \rangle) = \mathcal{D}^b(\mathcal{H})$, and $\mathcal{H}\langle w \rangle$ is hereditary abelian, satisfying Serre duality. If $w = \alpha \in \widehat{\mathbb{Q}}$, then by [30, Prop. 8.1.6], [31, Thm. 9.7] we have that $\mathcal{H}\langle \alpha \rangle = \mathrm{coh}\,\mathbb{X}_\alpha$ for some curve $\mathbb{X}_\alpha$ with $\chi'_{orb}(\mathbb{X}_\alpha) = 0$. If k is algebraically closed, then $\mathbb{X}_\alpha$ is always isomorphic to $\mathbb{X}$. The rank function on $\mathcal{H}\langle \alpha \rangle$ defines a linear form $\mathrm{rk}_\alpha \colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$.

LEMMA 8.5 (Reiten-Ringel [45]). *For every $w \in \widehat{\mathbb{R}}$ the pair $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$ is a torsion pair. If $w \in \widehat{\mathbb{Q}}$, then the torsion pair splits.*   □

Let $\alpha \in \widehat{\mathbb{Q}}$. By $\vec{\mathcal{H}}\langle \alpha \rangle = \mathcal{G}_\alpha[-1]$ we denote the [−1]-shifted heart of the t-structure in $\mathcal{D}(\vec{\mathcal{H}})$ induced by the torsion pair $({}^{\perp_0}\mathbf{W}_\alpha, {}^{\perp_1}\mathbf{W}_\alpha) = (\mathrm{Gen}\,\mathbf{q}_\alpha, \mathcal{C}_\alpha)$. We have $\vec{\mathcal{H}}\langle \alpha \rangle = \mathrm{Qcoh}\,\mathbb{X}_\alpha$, cf. Theorem 8.1. If $X \in \vec{\mathcal{H}}$ has a rational slope α, then clearly $X \in \vec{\mathcal{H}} \cap \vec{\mathcal{H}}\langle \alpha \rangle$ where the intersection is formed in $\mathcal{D}(\vec{\mathcal{H}}) = \mathcal{D}(\vec{\mathcal{H}}\langle \alpha \rangle)$; in $\vec{\mathcal{H}}\langle \alpha \rangle$ then X has slope ∞. In particular, $\vec{\mathcal{H}}\langle \alpha \rangle$ is locally noetherian with $\mathrm{fp}(\vec{\mathcal{H}}\langle \alpha \rangle) = \mathcal{H}\langle \alpha \rangle$.

Let w be irrational. Similarly, we denote by $\vec{\mathcal{H}}\langle w \rangle = \mathcal{G}_w[-1]$ the [−1]-shifted heart of the t-structure in $\mathcal{D}(\vec{\mathcal{H}})$ associated with the cotilting torsion pair $({}^{\perp_0}\mathbf{W}_w, {}^{\perp_1}\mathbf{W}_w) = (\mathcal{Q}_w = {}^{\perp_0}\mathcal{C}_w, \mathcal{C}_w)$. We note that $\mathcal{Q}_w = \mathrm{Gen}\,\mathbf{q}_w$, which follows, arguing in $\vec{\mathcal{H}}$, with the same arguments as in [45, Lem. 1.3+1.4] replacing "finite length" by "noetherian". In $\mathcal{H}$ this induces the (splitting) torsion pair $(\mathcal{Q}_w \cap \mathcal{H}, \mathcal{C}_w \cap \mathcal{H})$, and we have $\mathcal{Q}_w \cap \mathcal{H} = \mathrm{Gen}(\mathbf{q}_w) \cap \mathcal{H} = \mathrm{add}\,\mathbf{q}_w = \bigvee_{\beta > w} \mathbf{t}_\beta$ and $\mathcal{C}_w \cap \mathcal{H} = \mathrm{Cogen}\,\mathbf{W}_w \cap \mathcal{H} = \bigvee_{\gamma < w} \mathbf{t}_\gamma$. Since $\vec{\mathcal{H}}$ and $\mathcal{H}$ are hereditary,

$(\mathcal{C}_w, \mathcal{Q}_w[-1])$ and $(\mathcal{C}_w \cap \mathcal{H}, (\mathcal{Q}_w \cap \mathcal{H})[-1])$ are splitting torsion pairs in, respectively, $\vec{\mathcal{H}}\langle w \rangle$ and $\mathcal{H}\langle w \rangle$. Moreover, $\mathcal{C}_w = \varinjlim(\mathbf{p}_w)$ and $\mathcal{Q}_w = \varinjlim(\mathbf{q}_w) \subseteq \mathcal{B}_w$. The situation is illustrated in Figure 8.1.

Let $w \in \widehat{\mathbb{R}}$. A sequence $\eta \colon 0 \to E' \xrightarrow{u} E \xrightarrow{v} E'' \to 0$ with objects E', E, E'' in $\vec{\mathcal{H}} \cap \vec{\mathcal{H}}\langle w \rangle$ is exact in $\vec{\mathcal{H}}$ if and only if it is exact in $\vec{\mathcal{H}}\langle w \rangle$; indeed, both conditions are equivalent to $E' \xrightarrow{u} E \xrightarrow{v} E'' \xrightarrow{\eta} E'[1]$ being a triangle in $\mathcal{D}(\vec{\mathcal{H}})$. (Cf. [57, Prop. 2.2].)
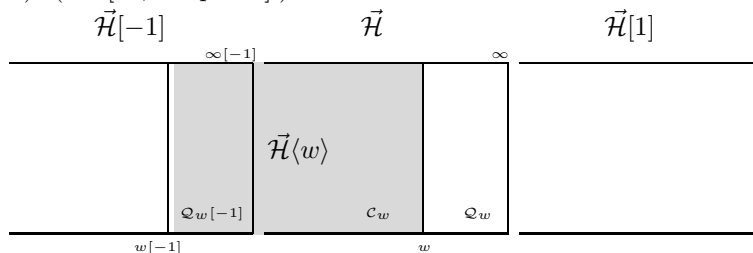

Figure 8.1: The interval category $\vec{\mathcal{H}}\langle w \rangle$

For irrational w we will show several properties of $\vec{\mathcal{H}}\langle w \rangle$ in Section 9.

Injective Cogenerators

PROPOSITION 8.6. *Let $w \in \widehat{\mathbb{R}}$. The cotilting sheaf $\mathbf{W}_w$ yields an injective cogenerator of $\vec{\mathcal{H}}\langle w \rangle$.*

Proof. This follows from [10, Prop. 4.4]. □

REMARK 8.7. Let $w \in \widehat{\mathbb{R}}$. We can choose $\mathbf{W}_w$ to be a *minimal* injective cogenerator in $\vec{\mathcal{H}}\langle w \rangle$, and we will do so in the following. Then $\mathbf{W}_w$ is discrete and its indecomposable summands correspond, up to isomorphism, bijectively to the simple objects in $\vec{\mathcal{H}}\langle w \rangle$.

PROPOSITION 8.8. *Let $w \in \mathbb{R} \setminus \mathbb{Q}$. The sheaves of slope w are the pure subsheaves of products of copies of $\mathbf{W}_w$.*

Proof. Let E be a sheaf of slope w. Since $\mathbf{W}_w$ is an injective cogenerator in $\vec{\mathcal{H}}\langle w \rangle$ there is a short exact sequence $0 \to E \to C_0 \to C_1 \to 0$ in $\vec{\mathcal{H}}\langle w \rangle$ with $C_0 \in \mathrm{Prod}(\mathbf{W}_w)$. This sequence is pure-exact in $\vec{\mathcal{H}}$: let $F \in \mathcal{H}$ be coherent, without loss of generality, indecomposable. We have to show that the sequence stays exact under $\mathrm{Hom}(F, -)$. Thus we can also assume that $\mathrm{Hom}(F, C_1) \neq 0$. Since w is irrational, this means $F \in \mathbf{p}_w$. Now $\mathrm{Ext}^1(F, E) = 0$ since $E \in \mathcal{B}_w = \mathbf{p}_w^{\perp_1}$. □

COROLLARY 8.9. *Let $w \in \mathbb{R} \setminus \mathbb{Q}$. The class of pure-injective sheaves of slope w is given by $\mathrm{Prod}(\mathbf{W}_w)$.* □

Rational slope

Let $\alpha \in \widehat{\mathbb{Q}}$ and $M \in \vec{\mathcal{H}}$. Since the torsion pair from Lemma 8.5 splits, we have $M = M' \oplus M''$ with $M' \in \mathcal{C}_\alpha$ and $M'' \in \operatorname{Gen} \mathbf{q}_\alpha$.

Lemma 8.10. *Let $M \in \vec{\mathcal{H}}$ be indecomposable of slope $\alpha \in \widehat{\mathbb{Q}}$. Then M is pure-injective in $\vec{\mathcal{H}}$ if and only if M is pure-injective considered as an object in $\vec{\mathcal{H}}\langle\alpha\rangle$.*

Proof. The class $\mathcal{M}(\alpha) = \mathcal{C}_\alpha \cap \mathcal{B}_\alpha$ is definable, both in $\vec{\mathcal{H}}$ and in $\vec{\mathcal{H}}\langle\alpha\rangle$, in particular closed under forming products in $\vec{\mathcal{H}}$ and $\vec{\mathcal{H}}\langle\alpha\rangle$. For every set I, forming the product M^I in $\vec{\mathcal{H}}$ is the same as forming the product M^I in $\vec{\mathcal{H}}\langle\alpha\rangle$. Indeed, consider the product M^I in $\vec{\mathcal{H}}\langle\alpha\rangle$ with projections $p_i \colon M^I \to M$ $(i \in I)$. Let $X \in \vec{\mathcal{H}}$ and $f \in \operatorname{Hom}(X, M)$. Write $X = X' \oplus X''$ as above, so that $X' \in \vec{\mathcal{H}}\langle\alpha\rangle$ and $\operatorname{Hom}(X'', M) = 0$. By the universal property of the product M^I in $\vec{\mathcal{H}}\langle\alpha\rangle$ there is a unique $\bar{f} \in \operatorname{Hom}(X', M^I)$ with $p_i \circ \bar{f} = f$ for all i. This property is then trivially extended to X, which shows, that M^I with the projections p_i is also the product in $\vec{\mathcal{H}}$. The converse direction is similar. The claim now follows with the Jensen-Lenzing criterion Proposition 2.4. $\square$

Indecomposable pure-injective sheaves

We obtain the following version for sheaves of [2, Thm. 6.7].

Theorem 8.11. *The following is a complete list of the indecomposable pure-injective sheaves in $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X}$:*

   (1) The indecomposable coherent sheaves.

   (2) For every $\alpha \in \widehat{\mathbb{Q}}$ the generic, the Prüfer and the adic sheaves of slope α.

   (3) For every irrational w the indecomposable objects of $\operatorname{Prod}(\mathbf{W}_w)$.

Proof. We recall that each indecomposable object has a slope. Because of Corollary 8.9 we only need to consider slopes α in $\widehat{\mathbb{Q}}$, and by the preceding lemma we can restrict even to the case $\alpha = \infty$. Let now M be indecomposable of slope ∞. Then we can apply Theorem 5.11. $\square$

Every large cotilting sheaf has a slope

Lemma 8.12. *Let C be cotilting and T be a corresponding tilting sheaf (of finite type): $C = \Gamma(T)$. Let $w \in \widehat{\mathbb{R}}$. Then*

$$C \text{ has slope } w \quad \Leftrightarrow \quad T \text{ has slope } w.$$

Proof. We show the following:

   (1) $C \in \mathcal{B}_w \quad \Leftrightarrow \quad T \in \mathcal{B}_w$.

(2) $C \in \mathcal{C}_w \quad \Leftrightarrow \quad T \in \mathcal{C}_w$.

To this end let $({}^{\perp_0}C, {}^{\perp_1}C)$ and $(T^{\perp_1}, T^{\perp_0})$ be the corresponding cotilting, resp. tilting, torsion pairs. Moreover, let $\mathcal{F} = {}^{\perp_1}C \cap \mathcal{H} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H} = \mathscr{S}$ be the corresponding "small" torsionfree/resolving class. We have $T^{\perp_1} = \mathscr{S}^{\perp_1}$ and ${}^{\perp_0}C = {}^{\perp_0}\vec{\mathcal{F}}$. We remark that $\tau(\mathbf{p}_w) = \mathbf{p}_w$ and $\tau(\mathbf{q}_w) = \mathbf{q}_w$.

(1) We have $C \in \mathcal{B}_w = {}^{\perp_0}\mathbf{p}_w$ iff $\mathbf{p}_w \subseteq C^{\perp_0}$ iff $\mathbf{p}_w \subseteq {}^{\perp_1}C$ iff $\mathbf{p}_w \subseteq \mathcal{F} = \mathscr{S}$ iff $\mathbf{p}_w \subseteq (T^{\perp_1})^{\perp_0}$ iff $T \in {}^{\perp_0}\mathbf{p}_w = \mathcal{B}_w$.

(2) We have $C \in \mathcal{C}_w = \mathbf{q}_w{}^{\perp_0}$ iff $\mathbf{q}_w \subseteq {}^{\perp_0}C = {}^{\perp_0}\vec{\mathcal{F}}$ iff $\mathbf{q}_w \subseteq {}^{\perp_0}\mathcal{F}$ iff $\mathbf{q}_w \subseteq \mathscr{S}^{\perp_1} = T^{\perp_1}$ iff $T \in {}^{\perp_1}\mathbf{q}_w = \mathbf{q}_w{}^{\perp_0} = \mathcal{C}_w$. $\qquad\square$

The main result of this section is the following, which follows from the lemma and the corresponding result for large tilting sheaves [1, Thm. 8.5 + 9.1].

Theorem 8.13. *For every large cotilting sheaf C in $\vec{\mathcal{H}}$, there is $w \in \widehat{\mathbb{R}}$ such that C has slope w.* $\qquad\square$

Example 8.14. For every $w \in \widehat{\mathbb{R}}$ denote by $\mathbf{L}_w$ the tilting sheaf in $\vec{\mathcal{H}}$ with tilting class $\mathcal{B}_w$, cf. [1]. Then $\Gamma(\mathbf{L}_w) = \mathbf{W}_w$.

Reduction from rational slope to slope ∞

Lemma 8.15. *Let $\alpha \in \widehat{\mathbb{Q}}$. For an object C in $\vec{\mathcal{H}}$ the following conditions are equivalent:*

(1) C is a cotilting sheaf in $\vec{\mathcal{H}}$ of slope α;

(2) C is a cotilting sheaf in $\vec{\mathcal{H}}\langle\alpha\rangle$ of slope ∞.

Proof. Clearly, by changing the roles of $\vec{\mathcal{H}}$ and $\vec{\mathcal{H}}\langle\alpha\rangle$, it suffices to show (1)$\Rightarrow$(2). Assuming (1) we show (CS1), (CS2) w.r.t. $\vec{\mathcal{H}}\langle\alpha\rangle$ and that ${}^{\perp_1}C \cap \mathcal{H}\langle\alpha\rangle$, formed in $\vec{\mathcal{H}}\langle\alpha\rangle$, generates. For (CS1) it suffices to remark that forming the product C^I in $\vec{\mathcal{H}}$ and $\vec{\mathcal{H}}\langle\alpha\rangle$ yields the same; this follows from [10, Cor. 2.13]. For (CS2) let $X \in \vec{\mathcal{H}}\langle\alpha\rangle$ such that $\mathrm{Hom}_{\vec{\mathcal{H}}\langle\alpha\rangle}(X, C) = 0 = \mathrm{Ext}^1_{\vec{\mathcal{H}}\langle\alpha\rangle}(X, C)$. Since the "cut" at $\mathbf{t}_\infty[-1]$ defines by Lemma 8.5 a splitting torsion pair $(\mathcal{T}_\infty, \mathcal{F}_\infty)$ in $\vec{\mathcal{H}}\langle\alpha\rangle$, we can write $X = X' \oplus X''$ with $X' \in \mathcal{T}_\infty$, that is, lying in $\vec{\mathcal{H}}$, and $X'' \in \mathcal{F}_\infty$, that is, lying in $\vec{\mathcal{H}}[-1]$. Using (CS2) w.r.t. $\vec{\mathcal{H}}$ (for C) and $\vec{\mathcal{H}}[-1]$ (for $C[-1]$), we conclude $X' = 0 = X''$, and hence $X = 0$. Moreover, the same splitting property shows that all objects from $\mathcal{F}_\infty$ belong to $\mathrm{Ker}\,\mathrm{Ext}^1_{\vec{\mathcal{H}}\langle\alpha\rangle}(-, C)$. This concludes the proof that C is cotilting in $\vec{\mathcal{H}}\langle\alpha\rangle$. $\qquad\square$

Let B_α be a sheaf of slope α that becomes a branch sheaf of finite length in $\vec{\mathcal{H}}\langle\alpha\rangle$. Then we call B_α a *branch sheaf of slope* α. Note that the direct summands of B_α are contained in a subcategory $\mathcal{W}_\alpha$ that becomes a wing in $\vec{\mathcal{H}}\langle\alpha\rangle$. We call $\mathcal{W}_\alpha$ a *wing of slope* α and we adopt all of the appropriate notation and terminology suggested by Section 6.7.

We conclude this chapter by summarizing our results on large cotilting sheaves in the tubular and the elliptic cases.

Theorem 8.16. *Every large cotilting sheaf C (minimal, without loss of generality) in $\vec{\mathcal{H}}$ has a slope $w \in \widehat{\mathbb{R}}$, and for irrational w we have $C \cong \mathbf{W}_w$. Let α be rational or infinite.*

(1) *Let $V_\alpha \subseteq \mathbb{X}_\alpha$ and B_α be a branch sheaf of slope α. There is a unique minimal cotilting sheaf $C = C_+ \oplus C_0$ of slope α whose torsion part is given by*

$$C_0 = B_\alpha \oplus \bigoplus_{x \in V_\alpha} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty],$$

where the non-empty sets $\mathcal{R}_x \subseteq \{0, \dots, p(x)-1\}$ are uniquely determined by B_α as in 6.7.

(2) *Every cotilting sheaf of slope α is, up to equivalence, as in (1).*

(3) *The indecomposable summands of the torsionfree part C_+ of C are the following:*

   - *the adic sheaves $\tau^\ell S_y[-\infty]$ with $y \in \mathbb{X}_\alpha \setminus V_\alpha$ and ℓ such that $\tau^\ell S_y \notin \tau \mathcal{W}$ for any wing $\mathcal{W}$ associated with an exterior branch part of B_α; if $V_\alpha \neq \emptyset$ then C_+ is the pure-injective envelope of these adic sheaves;*
   - *if $V_\alpha = \emptyset$, additionally the generic sheaf of slope α.*

(4) *If $V_\alpha = \mathbb{X}_\alpha$ and $\mathcal{R}_x = \{0, \dots, p(x)-1\}$ for all x, then $C \cong \mathbf{W}_\alpha$.* □

9   Additional results related to irrational slopes

We continue to assume that the orbifold Euler characteristic $\chi'_{orb}(\mathbb{X})$ is zero. Throughout, we let w be irrational.

Our understanding of $\mathrm{Prod}(\mathbf{W}_w)$, the class of pure-injectives in $\vec{\mathcal{H}}$ of slope w, is still quite small. The natural home of the object $\mathbf{W}_w$ is the category $\vec{\mathcal{H}}\langle w \rangle$, of which it is an injective cogenerator. One should regard this Grothendieck category as a geometrical object (in the sense of noncommutative algebraic geometry, cf. the introductions in [55, 1.2] or [51, Ch. III]), where the points are given by the simple objects, or equivalently, by the indecomposable summands of $\mathbf{W}_w$. Some of the statements in the following proposition were already stated in [1, Rem. 7.5] without proofs; part (2) was obtained in discussions with H. Lenzing.

Proposition 9.1. *The following holds.*

(1) *$\vec{\mathcal{H}}\langle w \rangle$ is a locally coherent Grothendieck category with $\mathrm{coh}(\vec{\mathcal{H}}\langle w \rangle) = \mathrm{fp}(\vec{\mathcal{H}}\langle w \rangle) = \mathcal{H}\langle w \rangle$.*

(2) *$\mathcal{H}\langle w \rangle$ does not contain any simple object.*

(3) *Every non-zero object in $\mathcal{H}\langle w \rangle$ is not noetherian, and thus $\vec{\mathcal{H}}\langle w \rangle$ is not locally noetherian.*

(4) There exist simple objects in $\vec{\mathcal{H}}\langle w \rangle$.

Proof. (1) This follows from Theorem 3.12.

(2) We assume that there is a simple object S in $\mathcal{H}\langle w \rangle$. Then there is a rational $\alpha < w$ such that $S \in \mathbf{t}_\alpha$. We choose a rational β with $\alpha < \beta < w$. The sheaf category $\mathcal{H}\langle \beta \rangle$ defines a rank function rk_β, which is additive on short exact sequences in particular in $\mathcal{H}\langle w \rangle \cap \mathcal{H}\langle \alpha \rangle$ and τ-invariant. Moreover, $\mathrm{rk}_\beta(F) > 0$ for every indecomposable F in $\mathcal{H}\langle w \rangle \cap \mathcal{H}\langle \alpha \rangle$. We choose F such that $\mathrm{rk}_\beta(F)$ is minimal, and moreover with $F \in \mathbf{t}_\gamma$ such that $\gamma < \alpha$. By [31, Thm. 13.8] we may assume that $\mathrm{Hom}_{\mathcal{H}\langle w \rangle}(F, S) \neq 0$. Since S is simple, there is a short exact sequence $0 \to U \to F \to S \to 0$ in $\mathcal{H}\langle w \rangle$, and by the choice of F we get $\mathrm{rk}_\beta(U) = 0$, that is, $U = 0$. Hence we get an isomorphism $F \cong S$, which gives a contradiction since F and S have different slopes.

(3) Since a non-zero noetherian object has a maximal subobject, this follows directly from (2).

(4) Let E be a non-zero, finitely generated object in $\vec{\mathcal{H}}\langle w \rangle$ (for instance, $E \neq 0$ finitely presented). Then it contains a maximal subobject, and the quotient is simple. (Thus one might expect that there are even "many" simple objects in $\vec{\mathcal{H}}\langle w \rangle$.) $\qquad\square$

COROLLARY 9.2. *For $w \in \widehat{\mathbb{R}}$, the cotilting sheaf $\mathbf{W}_w$ is Σ-pure-injective if and only if $w \in \widehat{\mathbb{Q}}$.*

Proof. This follows from [54, Prop. V.4.3] and the fact that the category $\vec{\mathcal{H}}\langle w \rangle$ is locally noetherian if and only if $w \in \widehat{\mathbb{Q}}$. $\qquad\square$

PROPOSITION 9.3. *The class of injective objects in $\vec{\mathcal{H}}\langle w \rangle$ is given by $\mathrm{Prod}(\mathbf{W}_w)$, where Prod can be formed either in $\vec{\mathcal{H}}\langle w \rangle$ or in $\vec{\mathcal{H}}$. Each injective object and each simple object in $\vec{\mathcal{H}}\langle w \rangle$ has "internal" slope w, that is, belongs to ${}^{\perp_0}\mathcal{H}\langle w \rangle$, this class of objects formed in $\vec{\mathcal{H}}\langle w \rangle$.*

Proof. The statement on forming Prod follows from [10, Cor. 2.13]. Every injective object Q in $\vec{\mathcal{H}}\langle w \rangle$ is a direct summand of a power $\mathbf{W}_w{}^I$ of $\mathbf{W}_w$ (for some set I). Since ${}^{\perp_0}\mathcal{H}\langle w \rangle$ is closed under products, which follows by the same arguments as in [45, Prop. 13.5], we conclude that $Q \in {}^{\perp_0}\mathcal{H}\langle w \rangle$.

Let S be a simple object in $\vec{\mathcal{H}}\langle w \rangle$. If $S \notin {}^{\perp_0}\mathcal{H}\langle w \rangle$. Then there is a monomorphism $S \to F$ for an object $F \in \mathcal{H}\langle w \rangle$. Since F is coherent and S finitely generated, we obtain $S \in \mathcal{H}\langle w \rangle$, and S is simple in $\mathcal{H}\langle w \rangle$. This yields a contradiction by Proposition 9.1. $\qquad\square$

REMARK 9.4. The statement in the preceding proposition on simple objects in $\vec{\mathcal{H}}\langle w \rangle$ is also shown in [44, Thm. 8.2.3] with completely different methods. Moreover, based on ideas by J. Šťovíček, in that thesis a simple object in $\vec{\mathcal{H}}\langle w \rangle$ is constructed in an explicit way as a direct limit of finitely presented objects of rational slopes.

QUESTION 9.5. Is $\vec{\mathcal{H}}\langle w \rangle$ hereditary?

This interesting question is still open. We know that $\mathcal{H}\langle w \rangle$ is hereditary, and by considering the derived category we see that at least $\operatorname{Ext}^i_{\mathcal{H}\langle w \rangle}(-,-) = 0$ for $i \geq 3$. Moreover, if S is a simple object (or any object of slope w) in $\vec{\mathcal{H}}\langle w \rangle$, then $\mathrm{E}(S)/S$ is injective. For heredity we would need that every factor object of an injective object is injective. At least $\vec{\mathcal{H}}\langle w \rangle$ is *semihereditary*[1] in the following sense:

PROPOSITION 9.6. *In $\vec{\mathcal{H}}\langle w \rangle$ each of the following equivalent conditions holds true.*

   (1) Each factor object of an fp-injective object is fp-injective.

   (2) Each factor object of an injective object is fp-injective.

   (3) For all $X, Y \in \vec{\mathcal{H}}\langle w \rangle$, with X finitely presented, $\operatorname{Ext}^2_{\mathcal{H}\langle w \rangle}(X,Y) = 0$.

Moreover, the fp-injective objects coincide with the objects of slope w in $\vec{\mathcal{H}}\langle w \rangle$, and they form a definable subcategory of $\vec{\mathcal{H}}\langle w \rangle$.

Proof. The equivalence of the conditions follows from standard arguments by applying $\operatorname{Hom}_{\vec{\mathcal{H}}\langle w \rangle}(X,-)$ with X finitely presented to a short exact sequence of the form $0 \to Y \to Q \to Q/Y \to 0$ with Q injective or fp-injective.

In $\vec{\mathcal{H}}\langle w \rangle$ we have (generalised) Serre duality, that is, $\mathrm{D}\operatorname{Ext}^1_{\mathcal{H}\langle w \rangle}(X,Y) = \operatorname{Hom}_{\vec{\mathcal{H}}\langle w \rangle}(Y, \tau X)$, where $X, Y \in \vec{\mathcal{H}}\langle w \rangle$ with X finitely presented. Indeed, this follows from Lemma 2.10, applied to the derived category $\mathcal{D}(\vec{\mathcal{H}}\langle w \rangle) = \mathcal{D}(\vec{\mathcal{H}})$ and using that $\vec{\mathcal{H}}$ is locally noetherian and the compact objects are given by $\mathcal{D}^b(\mathcal{H})$, cf. 2.8.

By this we see $^{\perp_0}\mathcal{H}\langle w \rangle = \{Y \in \vec{\mathcal{H}}\langle w \rangle \mid \operatorname{Hom}_{\vec{\mathcal{H}}\langle w \rangle}(Y, \mathcal{H}\langle w \rangle) = 0\} = \{Y \in \vec{\mathcal{H}}\langle w \rangle \mid \operatorname{Ext}^1_{\mathcal{H}\langle w \rangle}(\mathcal{H}\langle w \rangle, Y) = 0\} = \mathcal{H}\langle w \rangle^{\perp_1}$, and hence the objects of slope w coincide with the fp-injective objects in $\vec{\mathcal{H}}\langle w \rangle$. Since every factor object of an injective has slope w, by Proposition 9.3, it is fp-injective. Thus (2) holds.

Moreover, it follows as in Proposition 5.4 that the class of objects in $\vec{\mathcal{H}}\langle w \rangle$ of slope w is definable. □

Since $\vec{\mathcal{H}}\langle w \rangle$ is not locally noetherian, there are fp-injective objects which are not injective ([23, Prop. A.11]).

We discuss several equivalent formulations of Question 9.5.

LEMMA 9.7 (Reiten-Ringel). *Let w be irrational, $\beta_1 > \beta_2 > \cdots > w$ a sequence of rational numbers converging to w and $Q_i \in \operatorname{Add}(\mathbf{t}_{\beta_i})$ (for $i = 1, 2, \dots$). Then in $\vec{\mathcal{H}}$ we have $\prod Q_i / \bigoplus Q_i \in \mathcal{M}(w)$.*

Proof. This is a slightly more general version of "The First Construction" in [45, 13.4]; the proof therein still holds. □

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[1] For the corresponding ring-theoretic notion we refer to [39].

Proposition 9.8. *Let w be irrational. The following are equivalent:*

(1) *The abelian category $\vec{\mathcal{H}}\langle w \rangle$ is hereditary.*

(2) *The torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ in $\vec{\mathcal{H}}$ splits.*

(3) $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(\mathrm{Prod}(\mathbf{W}_w), \mathrm{Add}(\mathbf{q}_w)) = 0$; *in the second argument, one can restrict to coproducts of objects in $\mathbf{q}_w$ whose slopes converge to w.*

(4) *For each sequence $Q_i \in \mathrm{Add}(\mathbf{t}_{\beta_i})$ with $\beta_1 > \beta_2 > \cdots > w$ converging to w the canonical monomorphism $\bigoplus Q_i \to \prod Q_i$ splits.*

(5) *In the category $\vec{\mathcal{H}}\langle w \rangle$ the following holds: for all objects $X \in \vec{\mathcal{H}}\langle w \rangle$ of the form $X = \bigoplus X_i$ with $X_i \in \mathrm{Add}(\mathbf{t}_{\gamma_i})$, the γ_i converging to $w[-1]$, and for each monomorphism f from X to an injective object in $\vec{\mathcal{H}}\langle w \rangle$, and for any object Y of slope w we have $\mathrm{Ext}^1_{\vec{\mathcal{H}}\langle w \rangle}(Y, \mathrm{Coker}\, f) = 0$ (or equivalently, $\mathrm{Ext}^2_{\vec{\mathcal{H}}\langle w \rangle}(Y, X) = 0$).*

Proof. (1)$\Rightarrow$(2) Let $\vec{\mathcal{H}}\langle w \rangle$ be hereditary. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $\vec{\mathcal{H}}$ with $X' \in \mathcal{Q}_w$ and $X'' \in \mathcal{C}_w$. Then $V = X'[-1] \in \mathcal{Q}_w[-1] \subseteq \vec{\mathcal{H}}\langle w \rangle$, and this yields $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(X'', X') = \mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(X'', X'[1]) = \mathrm{Ext}^2_{\vec{\mathcal{H}}\langle w \rangle}(X'', V) = 0$. Hence $(\mathcal{Q}_w, \mathcal{C}_w)$ splits.

(2)$\Rightarrow$(1) Conversely, we assume that $(\mathcal{Q}_w, \mathcal{C}_w)$ splits. Let $X, Y \in \vec{\mathcal{H}}\langle w \rangle$. Since $(\mathcal{C}_w, \mathcal{Q}_w[-1])$ is a torsion pair in $\vec{\mathcal{H}}\langle w \rangle$, for showing $\mathrm{Ext}^2_{\vec{\mathcal{H}}\langle w \rangle}(X, Y) = 0$ it is sufficient to assume $X, Y \in \mathcal{C}_w \cup \mathcal{Q}_w[-1]$. Since $\mathrm{Ext}^2_{\vec{\mathcal{H}}\langle w \rangle}(X, Y) = \mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(X, Y[2])$ and $\vec{\mathcal{H}}$ is hereditary, the only crucial case is when $X \in \mathcal{C}_w$ and $Y \in \mathcal{Q}_w[-1]$. But then $Y[1] \in \mathcal{Q}_w$ and $\mathrm{Hom}_{\mathcal{D}(\vec{\mathcal{H}})}(X, Y[2]) = \mathrm{Ext}^1_{\vec{\mathcal{H}}}(X, Y[1]) = 0$ since $(\mathcal{Q}_w, \mathcal{C}_w)$ splits.

(2)$\Leftrightarrow$(3) is easy to show since $\vec{\mathcal{H}}$ is hereditary, and moreover $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(\mathcal{C}_w, \mathcal{B}_\beta) = \mathrm{Ext}^2_{\vec{\mathcal{H}}\langle \beta \rangle}(\mathcal{C}_w, \mathcal{B}_\beta[-1]) = 0$ for every rational $\beta > w$, because $\vec{\mathcal{H}}\langle \beta \rangle$ is hereditary.

(3)$\Rightarrow$(4) Follows directly with Lemma 9.7.

(4)$\Rightarrow$(3) Let $\eta \colon 0 \to \bigoplus_i Q_i \to E \to \mathbf{W}_w{}^J \to 0$ be a short exact sequence with Q_i as in (4). We have to show that η splits. Since each $Q_i \in \mathrm{Add}(\mathbf{t}_{\beta_i}) \subseteq \mathcal{B}_{\beta_i}$ with $\beta_i > w$ we have $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(\mathbf{W}_w{}^J, Q_i) = 0$ for each i, hence we obtain $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(\mathbf{W}_w{}^J, \prod Q_i) = 0$ by [10, Cor. A.2]. Since by (4) the coproduct $\bigoplus Q_i$ is a direct summand of the product $\prod Q_i$, we obtain $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(\mathbf{W}_w{}^J, \bigoplus Q_i) = 0$, and thus η splits.

(1)$\Rightarrow$(5) This is clear.

(5)$\Rightarrow$(1) We recall that there is the splitting torsion pair $(\mathcal{C}_w, \mathcal{Q}_w[-1])$ in $\vec{\mathcal{H}}\langle w \rangle$, and since $\vec{\mathcal{H}}\langle w \rangle$ is locally coherent with $\mathrm{fp}(\vec{\mathcal{H}}\langle w \rangle) = \mathcal{H}\langle w \rangle$, the class $\mathcal{Q}_w[-1]$ is generated in $\vec{\mathcal{H}}\langle w \rangle$ by $\mathbf{q}_w[-1]$. We have to show that for any short exact sequence $0 \to X \to \mathbf{W}_w{}^I \to B \to 0$ in $\vec{\mathcal{H}}\langle w \rangle$, the cokernel object B is injective, that is, $\mathrm{Ext}^2_{\vec{\mathcal{H}}\langle w \rangle}(Y, X) = 0$ holds for each $Y \in \vec{\mathcal{H}}\langle w \rangle$. Since $\mathrm{Ext}^3_{\vec{\mathcal{H}}\langle w \rangle}(-, -) = 0$,

we obtain that for X it is sufficient to assume that it is a coproduct of objects in $\mathbf{q}_w[-1]$. Moreover, since for every rational β we have $\operatorname{Hom}_{\mathcal{D}(\vec{\mathcal{H}}\langle\beta\rangle)}(M, N) = 0$ for all $M \in \vec{\mathcal{H}}\langle\beta\rangle[-1]$ and $N \in \vec{\mathcal{H}}\langle\beta\rangle[1]$ (which follows from heredity of $\vec{\mathcal{H}}\langle\beta\rangle$), we deduce that we can, moreover, assume X to be of the form as in (5), and to test injectivity with objects Y of slope w. □

The equivalence of (1) and (2) also follows from [57, Thm. 5.2].

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1078