A Finiteness Theorem for Special Unitary Groups of Quaternionic Skew-Hermitian Forms with Good Reduction

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Abstract. Given a field K equipped with a set of discrete valuations V, we develop a general theory to relate reduction properties of skew-hermitian forms over a quaternion K-algebra Q to quadratic forms over the function field $K(Q)$ obtained via Morita equivalence. Using this we show that if (K, V) satisfies certain conditions, then the number of K-isomorphism classes of the universal coverings of the special unitary groups of quaternionic skew-hermitian forms that have good reduction at all valuations in V is finite and bounded by a value that depends on size of a quotient of the Picard group of V and the size of the kernel and cokernel of residue maps in Galois cohomology of K with finite coefficients. As a corollary we prove a conjecture of Chernousov, Rapinchuk, Rapinchuk for groups of this type.

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1 Introduction

The concept of good reduction of elliptic curves is well studied in the literature and can be characterized by unramified points of finite order. It is also well understood in the more general setting of abelian varieties ([ST68], [Fal83]). In the case of linear algebraic groups, the study of good reduction basically started with Harder ([Har67]) with focus on number fields. This was followed by more progress in this direction due to many authors. It was not until very recently, that this topic was approached in a more general setting by Chernousov, Rapinchuk, Rapinchuk [CRR19] where they consider arbitrary finitely generated fields and provide results for two dimensional global fields as well as for some fields that are not finitely generated.

Given a discrete valuation v on a field K, let K_v, $\mathcal{O}_v$, and $K^{(v)}$ denote respectively the the completion of K, its valuation ring and the residue field. Assume that all the fields under consideration have characteristic $\neq 2$. Let G be an absolutely almost simple linear algebraic group defined over K. Then G is said to have *good reduction at v* if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ (see Definition 2.7, Exp. XIX, [DG70] for the definition of reductive group schemes) with generic fiber $\mathcal{G} \otimes_{\mathcal{O}_v} K_v$ isomorphic to $G \otimes_K K_v$. In a recent paper, Chernousov, Rapinchuk, Rapinchuk ([CRR19]) asked the following question:

"Given an absolutely almost simple simply connected algebraic K-group G, can one equip K with a set of discrete valuations V such that the set of K-isomorphism classes of K-forms of G having good reduction at all $v \in V$ is finite?"

An affirmative answer to the question has important implications such as properness of local-global map in Galois cohomology, finiteness of genus of an algebraic group (see [CRR13], [CRR19] for the definition of genus and its relation to good reduction) and in eigenvalue rigidity problems. A detailed explanation of why this question is important and interesting can be found in [CRR16a] and [CRR19].

It is well known that the answer to the question is affirmative over number fields where V can be chosen to be the set containing almost all non-archimedean places([Gro96], [Con15], [JL15]). In the general setting of arbitrary fields, when studying good reduction of classical algebraic groups one is inevitably led to analyzing the underlying sesquilinear form that defines the group. In particular, reduction properties of the special unitary groups $SU(h)$ where h is a non-degenerate hermitian/skew-hermitian form over a division algebra with involution (Types A, C, D) and the spinor groups $Spin(q)$ of non-degenerate quadratic forms over K (Types B, D) is related to reduction properties of the underlying forms h and q respectively. Thus the above question on finiteness of number of isomorphism classes of K-forms of G with good reduction at a set of valuations is reduced to asking if the number of similarity classes of forms associated to the type of G that have good reduction at these valuations is finite. In the case of spinor groups $Spin(q)$, since the underlying form is quadratic, one can use Milnor isomorphism to map its Witt class to Galois cohomology class and take advantage of powerful cohomological tools to prove such finiteness theorems. This is the methodology adopted in [CRR19] to prove finiteness results for the class of spinor groups $Spin(q)$ defined over $K = k(C)$, function field of a smooth geometrically integral curve C over a field k of characteristic $\neq 2$ that satisfies condition (F'_2) (See §4.1 for the definition, properties and examples of fields of type (F'_m), examples include local fields and higher dimensional local fields) and V is the set of valuations corresponding to closed points of C. In the case of special unitary groups $SU(h)$ where underlying form h is hermitian over a quadratic extension of K or hermitian over a quaternion division algebra with center K, one applies Jacobson's theorem ([Jac40]) to associate h to a quadratic form q_h of higher rank over K so that reduction properties can be be studied via ramification of q_h using cohomological methods as before. This yields required finiteness results for the special unitary groups $SU(h)$ for h as above (see §8 in [CRR19]). Based on the above evidence Chernousov, Rapinchuk, Rapinchuk conjectured that with mild assumptions such finiteness results must hold for any absolutely almost simple simply connected algebraic K-group. We restate the conjecture below:

Conjecture 1.1. *(Conjecture 7.3 in [CRR19]) : Let $K = k(C)$ be the function field of a smooth affine geometrically integral curve over a field k and let V be the set of discrete valuations associated with the closed points of C. Furthermore, let G be an absolutely almost simple simply connected algebraic K-group and let m be the order of the automorphism group of its root system. Assume that char k is prime to m and that k satisfies (F'_m). Then the set of K-isomorphism classes of K-forms of G that have good reduction at all $v \in V$ is finite.*

An important missing link to prove the conjecture is the case of the universal covering of the special unitary groups $SU(h)$ of skew-hermitian form h over quaternion Q over K. See Remark 8.6 in [CRR19]. In this paper, we provide answer to this missing link:

Main Theorem. *Let $K = k(C)$ be the function field of a geometrically integral curve over a field k of characteristic $\neq 2$ that satisfies (F'_2) and let V be the set of discrete valuations on K corresponding to closed points of C. Then the number of K-isomorphism classes of the universal coverings of special unitary groups $SU_m(h)$ of non-degenerate m-dimensional skew-hermitian forms h over some quaternion K-algebra with the canonical involution, having good reduction at all places in V is finite.*

This is obtained as a consequence of a general theory that we briefly outline below without getting into technical details. Let h be skew-hermitian form over a quaternion division algebra Q with center K and let $K(Q)$ denote the function field of the Severi-Brauer variety associated to Q. Then the form $h_{K(Q)} :=$ $h \otimes_K K(Q)$ can be reduced to a quadratic form $q_{h_{K(Q)}}$ via Morita equivalence. We give a method to extend a given valuation v on K to a valuation $\tilde{v}$ on $K(Q)$ in such a way that if h has good reduction at v then the quadratic form $q_{h_{K(Q)}}$ is unramified at $\tilde{v}$ (see §4.2 for the notion of ramification and good reduction of forms). We then use cohomological methods to show finiteness results for certain unramified Galois cohomology classes over $K(Q)$. This allows us to derive an upper bound on the number of K-isomorphism classes of special unitary groups of quaternionic skew-hermitian forms with good reduction at all places in V thereby proving Conjecture 1.1 for groups of this type.

2 Notations

All the fields we consider here have characteristic $\neq 2$. For a discrete valuation v on a field F, let $\mathcal{O}_v$ (or $\mathcal{O}_F$) denote the valuation ring in F when the underlying field F (resp. the underlying valuation) is clear. Let F_v denote the completion of F with respect to v and let $F^{(v)}$ denote its residue field. The Brauer group of F is denoted by $Br(F)$ and its n-torsion subgroup by ${}_nBr(K)$. For a quaternion algebra Q over F, we denote its Brauer class by $[Q]$. The involution on Q is the canonical (symplectic) involution. Let $Q_v := Q \otimes_F F_v$. For a ring R, let R^* denote its group of units. All the forms considered in this paper are finite dimensional and non-degenerate. The Witt ring of F is denoted by $W(F)$ and $\mathcal{I}(F)$ denotes its fundamental ideal. For a quadratic form q over F, let $[q]$ denote its class in the Witt ring $W(F)$.

3 Outline of the proof of the Main Theorem

Let K be a field equipped with a discrete valuation v where *char* $K^{(v)} \neq 2$. Let Q be a quaternion division algebra over K. Assume that Q is unramified at v (see §4.2 for the notion of unramified quaternion). Let $K(Q)$ denote the function field of the Severi-Brauer variety associated to Q. Let h be a non-degenerate skew-hermitian form over Q and let $q_{h_{K(Q)}}$ be the quadratic form associated to $h_{K(Q)} := h \otimes_K K(Q)$ via Morita equivalence (see §5). We refer the reader to §4.2 for the notion of ramification and good reduction of forms.

Theorem 3.1. *There exists a valuation $\tilde{v}$ on $K(Q)$ extending v such that if h has good reduction at v, then $q_{h_{K(Q)}}$ is unramified at $\tilde{v}$.*

Proof. See §6. □

We use the above result to prove finiteness statements as claimed. Recall the following notations and facts from [CRR19]. Let V be a set of discrete valuations on a field K that satisfies the following conditions.

(A) For any $a \in K^*$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite.

(B) *char* $K^{(v)} \neq 2$ $\forall v \in V$.

As noted in [CRR19], condition (A) is satisfied for a divisoral set of valuations V on a finitely generated K. Due to conditions (A) and (B), for l a power of 2, we have residue maps in Galois cohomology (See §4.2 for a discussion on this)

$$r_l^i : H^i(K, \mu_l^{\otimes i-1}) \to \bigoplus_{v \in V} H^{i-1}(K^{(v)}, \mu_l^{\otimes i-2}) \tag{3.1}$$

where μ_l denotes l-th roots of unity. For $l = 2$, this is just

$$r_2^i : H^i(K, \mu_2) \to \bigoplus_{v \in V} H^{i-1}(K^{(v)}, \mu_2) \tag{3.2}$$

Let $H^i(K, \mu_l^{\otimes i-1})_V$ and $H^i(K, \mu_l^{\otimes i-1})^V$ denote respectively the $Ker\ r_l^i$ and $Coker\ r_l^i$. Let $Pic(V)$ denote the Picard group of V (see §2 in [CRR19] for the definition). The main theorem is a consequence of the following.

Theorem 3.2. *Let V be a set of discrete valuations on K satisfying conditions (A) and (B). Suppose the following holds:*

1. *the quotient $Pic(V)/2Pic(V)$ is finite and*
2. *the cohomology groups $H^2(K, \mu_2)_V$, $H^i(K, \mu_2)^V$ and $H^i(K, \mu_4^{\otimes i-1})_V$ are finite for all $i = 1, 2, \cdots l := [log_2\ 2n] + 1$.*

Then the number of K-isomorphism classes of special unitary groups of n-dimensional skew-hermitian forms $SU_n(h)$, where h is skew-hermitian over some quaternion (not necessarily division) algebra with the canonical involution, having good reduction at all $v \in V$ is finite and bounded above by

$$|H^2(K, \mu_2)_V| \cdot |Pic(V)/2Pic(V)| \cdot \prod_{i=1}^{l} |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V| \quad (3.3)$$

Proof. See §7. □

A situation where the hypothesis of Theorem 3.2 holds is the following.

Proposition 3.3. *Let $K = k(C)$ be the function field of a geometrically integral curve C over a field k of characteristic $\neq 2$ that satisfies (F_2') (see §4.1) and let V be the set of discrete valuations on K corresponding to closed points of C. Then (K, V) satisfies the hypothesis of Theorem 3.2.*

Proof. Conditions (A) and (B) are easily seen to be satisfied. The finiteness of $Pic(V)/2Pic(V)$ is shown in the proof of Theorem 1.4 in [CRR19]. We will now show finiteness of $H^2(K, \mu_2)_V$, $H^i(K, \mu_2)^V$ and $H^i(K, \mu_4^{\otimes i-1})_V$. For any l coprime to $char\ K^{(v)}$, the Bloch-Ogus spectral sequence and Kato complexes yield a long exact sequence in cohomology (see §4 in [Rap19])

$$\begin{aligned} \cdots \to H^i_{\acute{e}t}(C, \mu_l^{\otimes i-1}) \to H^i(K, \mu_l^{\otimes i-1}) \xrightarrow{r_l^i} & \bigoplus_{v \in V} H^{i-1}(K^{(v)}, \mu_l^{\otimes i-2}) \\ & \to H^{i+1}_{\acute{e}t}(C, \mu_l^{\otimes i-1}) \to \cdots \end{aligned}$$

By Lemma 4.1 and Corollary 3.2 in [Rap19], $H^i_{\acute{e}t}(C, \mu_l^{\otimes j})$ is finite for all $i \geq 0$ and all j. This proves the claim. □

The Main Theorem now follows from Proposition 3.3 and Theorem 3.2.

The paper is organized as follows. In §4 we briefly recall the necessary results from the literature that will lay foundation for the rest of the paper. In §4.2, we define the notion of good reduction of skew-hermitian forms and relate it to good reduction of the universal covering of special unitary groups of skew-hermitian forms. In §5 we use Morita theory to reduce skew-hermitian forms over quaternions to quadratic forms and give an explicit example of the correspondence which will be used later. Next in §6 we describe a method to extend valuations from a discrete valued field to the function field of Severi-Brauer variety associated to a quaternion algebra over the field and discuss the properties of this extension. Finally, in §7, we prove Theorem 3.2.

4 Preliminaries

4.1 Fields of type (F'_m)

The notion of fields of type (F'_m) is introduced in [Rap19]. Recall from [Rap19] that for m prime to *char* k, a field k is said to be *of type* (F'_m) if for every finite separable extension L/k, the quotient $L^*/(L^*)^m$ is finite (here L^* is the multiplicative group of units). This notion generalizes Serre's condition (F) ([Ser02]) and is useful for many applications. It is shown that over fields of type (F'_m), certain Galois cohomology groups are finite (see Theorem 1.1 in [Rap19]), which is useful for computations of unramified cohomologies (Proposition 4.2 in [Rap19]). Examples of such fields include finite fields, local fields and higher dimensional local fields such as $\mathbb{Q}_p((t_1))\cdots((t_n))$(note that the last two are not finitely generated). See also Example 2.9 in [Rap19]. We now make the following observation (I thank P. Deligne to remark about this).

Lemma 4.1. *Let k be a field and let m be prime to its characteristic. Then k is of type (F'_m) if and only if it is of type (F'_p) for every prime p dividing m.*

Proof. By definition, it is clear that if k is of type (F'_m), it is of type (F'_p) for every prime p dividing m. We will prove the other direction by induction on the number of primes dividing m. Let r be the number of primes dividing m. Consider the case $r = 1$. Assume that k is of type (F'_p). Let L be a finite separable extension of k. For every $j \geq 1$ we have exact sequences of groups

$$0 \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{p^j}} \to \frac{(L^*)}{(L^*)^{p^j}} \to \frac{(L^*)}{(L^*)^{p^{j-1}}} \to 0$$
$$x \mapsto x$$

$$0 \to \frac{\mu_{p^{j-1}}(L^*)}{\mu_{p^{j-1}}(L^*) \cap (L^*)^p} \to \frac{(L^*)}{(L^*)^p} \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{p^j}} \to 0$$
$$x \mapsto x^{p^{j-1}}$$

where $\mu_n(L^*)$ denotes n-th roots of unity in L^*. By hypothesis, these sequences imply that $\frac{(L^*)}{(L^*)^{p^j}}$ is finite if $\frac{(L^*)}{(L^*)^{p^{j-1}}}$ is finite. Thus by induction on j we conclude that k is of type (F'_{p^j}). Therefore the the statement of the lemma is true for $r=1$. Assume that m is arbitrary and k is of type (F'_p) for every prime p dividing m. Let $m=p^j n$ where n is coprime to p and $j\geq 1$. We have the following exact sequences

$$0 \to \frac{(L^*)^{p^j}}{(L^*)^m} \to \frac{(L^*)}{(L^*)^m} \to \frac{(L^*)}{(L^*)^{p^j}} \to 0$$
$$x \mapsto x$$

$$0 \to \frac{\mu_{p^j}(L^*)}{\mu_{p^j}(L^*)\cap (L^*)^n} \to \frac{(L^*)}{(L^*)^n} \to \frac{(L^*)^{p^j}}{(L^*)^m} \to 0$$
$$x \mapsto x^{p^j}$$

By induction hypothesis on r, $\frac{(L^*)}{(L^*)^n}$ is finite and by the case $r=1$, $\frac{(L^*)}{(L^*)^{p^j}}$ is finite. Therefore we conclude that $\frac{(L^*)}{(L^*)^m}$ is finite. This proves that k is of type (F'_m). $\square$

This settles the query raised in the statement below Conjecture 7.3 in [CRR19].

4.2 Residue maps and ramification

Let K be a field with discrete valuation v and let l be prime to *char* $K^{(v)}$. Recall from Chapter II in [GMS03] that for every integer j we have residue maps in Galois cohomology

$$r^j_{l,v} : H^i(K, \mu_l^{\otimes j}) \to H^{i-1}(K^{(v)}, \mu_l^{\otimes j-1}),\ i\geq 1 \tag{4.1}$$

where μ_l is the group of l-th roots of unity in the separable closure of $K^{(v)}$ and $\mu_l^{\otimes j}$ is the j-th Tate twist of μ_l as described in [GMS03] (Chapter II, §7.8). An element of $H^i(K,\mu_l^{\otimes j})$ is said to be *unramified at* v if is in the kernel of the above residue map. Now assume that *char* $K^{(v)}\neq 2$ and $l=2$. Then we simply have

$$r_{2,v} : H^i(K,\mu_2) \to H^{i-1}(K^{(v)},\mu_2)$$

Let π be a uniformizer of K_v. Let $W(K)$ denote the Witt ring of K. Recall from [Lam05], Chapter VI, §1 that we have residue homomorphisms of groups

$$\partial_{i,v} : W(K) \xrightarrow{Res_{K_v/K}} W(K_v) \to W(K^{(v)})$$

The residue homomorphisms can be described as follows. Let q be a quadratic form over K and let [q] denote its class in $W(K)$. Suppose $Res_{K_v/K}([q]) =< u_1, u_2, \cdots u_m, \pi u_{m+1}\cdots \pi u_n >, u_i \in \mathcal{O}^*_{K_v}$. Then

$$\partial_{1,v}([q]) =< \overline{u}_1, \cdots, \overline{u}_m >$$
$$\partial_{2,v}([q]) =< \overline{u}_{m+1}, \cdots, \overline{u}_n >$$

Here $\overline{u_i}$ denotes the image of u_i in $K^{(v)}$. When $K = K_v$, let $W_0(K_v)$ denote the kernel of $\partial_{2,v}$. It is the subring of $W(K_v)$ generated by classes $< u >, u \in \mathcal{O}^*_{K_v}$. Then we have a split exact sequence (see §5 in [Mil70])

$$0 \to W_0(K_v) \to W(K_v) \xrightarrow{\partial_{2,v}} W(K^{(v)}) \to 0$$

Recall now that due to Voevodsky's proof of the Milnor conjecture ([OVV07], [Voe03]), for any field F with characteristic $\neq 2$ we have natural isomorphisms

$$e_n : \mathcal{I}(F)^n/\mathcal{I}(F)^{n+1} \to H^n(F, \mu_2) \tag{4.2}$$

where $\mathcal{I}(F)$ denotes the fundamental ideal in $W(F)$. Moreover, the isomorphisms e_n commute with the respective residue homomorphisms (see Satz 4.11 in [Ara75]), that is for $[q] \in \mathcal{I}(K)^n$, we have

$$e_{n-1}(\partial_{2,v}([q])) = r_{2,v}(e_n([q]))$$

Definition. *We say that q is unramified at v if $[q] \in W_0(K_v)$ (There is a slight abuse of notation here to make it look tidy, what we really mean is $Res_{K_v/K}([q]) \in W_0(K_v)$).*

Let us denote the kernel of the map (see §10 in [Sal99] and [Ser02] Chapter II, Appendix)

$${}_2Br(K) \simeq H^2(K, \mu_2) \xrightarrow{r_v} H^1(K^{(v)}, \mu_2) \tag{4.3}$$

by ${}_2Br(K)_v$. For a quaternion Q over K, if $[Q]$ is in the kernel of the above map, we say that Q is *unramified* at v. Let $Q_v := Q \otimes_K K_v$. Then Q_v is either split i.e, a matrix algebra or is a quaternion division algebra over K_v. Suppose Q_v is not split. Since K_v is Henselian, one can extend the valuation v on K_v to a (necessarily unique) valuation on Q_v (Corollary 2.2 in [Wad02]), which by abuse of notation is also denoted by v. The extended valuation on Q_v is given by (equation (2.7) in [Wad02])

$$v(a) = \frac{1}{2} v(Nrd(a)) \ \forall a \in Q^*_v \tag{4.4}$$

where Nrd denotes the reduced norm on Q_v. Let

$$A_v = \{a \in Q^*_v \mid v(a) \geq 0\} \cup 0$$

be the valuation ring of Q_v. Note that its group of units is given by

$$A^*_v = \{a \in Q^*_v \mid v(a) = 0\}$$

If $Q_v \simeq M_2(K_v)$ is split, we set $A_v := M_2(\mathcal{O}_v)$. By Theorem 10.3 in [Sal99] and Theorem 3.2 in [Wad02], if Q_v is unramified at v, then A_v is an Azumaya algebra over $\mathcal{O}_v$ and $Q_v \simeq A_v \otimes_{\mathcal{O}_{K_v}} K_v$. Since $[A_v : \mathcal{O}_{K_v}] = [Q_v : K_v] = 4$, A_v

is quaternionic and has representation given by $A_v = (d_v, t_v)$ for some $d_v, t_v \in \mathcal{O}^*_{K_v}$. (See Theorem 3.2 in [Wad02] and Example 2.4 (ii) and Proposition 2.5 in [JW90]).
Recall now the following exact sequence (see Prop 7.7 in [GMS03], §3 in [Wad02] and Theorem 2, §3 Chapter XII in [Ser79])

$$0 \to H^2(K^{(v)}, \mu_2) \xrightarrow{s} H^2(K_v, \mu_2) \xrightarrow{r} H^1(K^{(v)}, \mu_2) \to 0 \tag{4.5}$$

where r is the residue map and s is the canonical map resulting from $\mathcal{G}_{K_v} \to \mathcal{G}_{K^{(v)}}$ (Here $\mathcal{G}_F$ denotes the absolute Galois group of a field F). This yields an isomorphism (see equation (3.7) in [Wad02])

$$\begin{aligned} s^{-1} : ker(r) := {}_2Br(K_v)_v &\xrightarrow{\simeq} {}_2Br(K^{(v)}) \\ Q_v &\mapsto Q^{(v)} \end{aligned}$$

where $Q^{(v)}$ is the *residue quaternion algebra* given by $(\overline{d}_v, \overline{t}_v)$ (Here $\overline{d}_v, \overline{t}_v \in K^{(v)}$ are the residues obtained by taking the quotients of d_v, t_v modulo the maximal ideal in $\mathcal{O}_{K_v}$).

Definition. *Let (h, Q) be a non-degenerate skew-hermitian form over Q and let $(h_v, Q_v) := (h, Q) \otimes_K K_v$ be the form over Q_v obtained via base change. We say that (h, Q) has good reduction at v if h_v is obtained via base change from a non-degenerate skew-hermitian form $\tilde{h}$ over the Azumaya algebra A_v i.e, $(h_v, Q_v) \simeq (\tilde{h}, A_v) \otimes_{\mathcal{O}_{K_v}} K_v$.*

Remark 4.2. *By Theorem 10.3 in [Sal99], if (h, Q) has good reduction at v then Q is unramified at v.*

Remark 4.3. *The universal covering of $SU(h, Q)$ has good reduction at v if and only if the form $(\alpha_v h_v, Q_v)$ has good reduction at v for some $\alpha_v \in K^*_v$. (The "if" direction is clear. For the "only if" direction use the classification from [Sri20] and the equivalence between Azumaya algebras with involutions and hermitian spaces from §2.2 in [Bek13a])*

5 Reduction to Quadratic Forms via Morita Equivalence

As before, let Q denote a (not necessarily division) quaternion algebra with center K.

5.1 General theory

The general theory of Morita equivalence for Hermitian modules can be found in Knus' book [Knu91] (See Chapter 1, §9). In particular, by Morita theory, a non-degenerate skew- hermitian form of rank n over Q gives rise to a non-degenerate quadratic form of rank $2n$ over K whenever Q is split. In this case, let us denote the quadratic form associated to the skew-hermitian from h by q_h. By the properties of Morita equivalence, h is determined by q_h and moreover, two such skew-hermitian forms are isometric if and only if the associated quadratic forms are isometric. So whenever Q is split the skew-hermitian forms over Q can be completely studied by studying the associated quadratic forms over K. For an explicit description of Morita equivalence in this case see [Sch85], p. 361-362.

Let h be a skew-hermitian form over a non-split Q. A generic way to split Q is by extending the base field to the function field of the associated Severi-Brauer variety. Let $K(Q)$ denote the function field of the Severi-Brauer variety associated to Q. Now $Q_{K(Q)}$ is isomorphic to the matrix algebra $M_2(K(Q))$ with involution given by

$$M \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

Since $Q_{K(Q)}$ is split, the skew-hermitian form $h_{K(Q)} := h \otimes_K K(Q)$ can be reduced to a quadratic form $q_{h_{K(Q)}}$ via Morita equivalence. This reduction has nice properties due to the following result from [PSS01].

Proposition 5.1. *(Proposition 3.3 in [PSS01]) Let $W^{-1}(Q)$ denote the Witt group of skew-hermitian forms over Q. With the notations as above, the canonical homomorphism*

$$W^{-1}(Q) \to W^{-1}(Q \otimes_K K(Q))$$

is injective.

This means that h is hyperbolic if and only if $h_{K(Q)}$ is hyperbolic if and only if, by Morita equivalence, $q_{h_{K(Q)}}$ is hyperbolic. This philosophy of understanding the skew-hermitan form h over Q by studying the quadratic form $q_{h_{K(Q)}}$ is cleverly employed in Berhuy's paper [Ber07] to find cohomological invariants. We will be using this philosophy to study reduction properties of these forms.

5.2 An explicit example

Example 5.2. As before let $Q = (d, t)$ be a quaternionic (not necessarily division) algebra over K with basis $< 1, i, j, ij \mid i^2 = d, j^2 = t, ij = -ji >$. Then the Severi-Brauer variety of Q has function field $K(Q)$ given by the fraction field of $K[x, y]/(dx^2 + ty^2 - 1)$. An explicit splitting of Q over $K(Q)$

is given by the following.

$$\begin{aligned} Q \otimes_K K(Q) &\xrightarrow{\simeq} M_2(K(Q)) \\ i &\mapsto \begin{bmatrix} dx & -y \\ -dty & -dx \end{bmatrix} \\ j &\mapsto \begin{bmatrix} ty & x \\ dtx & -ty \end{bmatrix} \\ ij &\mapsto \begin{bmatrix} 0 & 1 \\ -dt & 0 \end{bmatrix} \end{aligned}$$

Let h be a non-degenerate skew-hermitian form over Q of rank n. Then it is well-known that h has a diagonal matrix representation over Q (see for example §6 in [Lew82]). By abuse of notation, let us denote the matrix also by h. Since h is skew-hermitian, the diagonal entries are pure quaternions. Let

$$h \simeq \bigoplus_{l=1}^{n} a_l i + b_l j + c_l ij, \qquad a_l, b_l, c_l \in K$$

Then,

$$h_{K(Q)} \simeq \bigoplus_{l=1}^{n} a_l \begin{bmatrix} dx & -y \\ -dty & -dx \end{bmatrix} + b_l \begin{bmatrix} ty & x \\ dtx & -ty \end{bmatrix} + c_l \begin{bmatrix} 0 & 1 \\ -dt & 0 \end{bmatrix}$$

Now we use the explicit description of Morita equivalence from [Sch85], p. 361-362 to conclude that the quadratic form associated to $h_{K(Q)}$ has matrix given by (again by abuse of notation)

$$q_{h_{K(Q)}} \simeq \bigoplus_{l=1}^{n} a_l \begin{bmatrix} -dty & -dx \\ -dx & y \end{bmatrix} + b_l \begin{bmatrix} dtx & -ty \\ -ty & -x \end{bmatrix} + c_l \begin{bmatrix} -dt & 0 \\ 0 & -1 \end{bmatrix}$$

Let $N_l = Nrd_Q(a_l i + b_l j + c_l ij) \in K$ denote the reduced norm of the quaternion $(a_l i + b_l j + c_l ij)$ in Q. Then diagonalizing the above matrix yields

$$q_{h_{K(Q)}} \simeq \bigoplus_{l=1}^{n} \begin{bmatrix} (a_l y - b_l x - c_l) & 0 \\ 0 & -(a_l y - b_l x - c_l) N_l \end{bmatrix}$$

We will be using this matrix representation of $q_{h_{K(Q)}}$ later.

Remark 5.3. *From the above description of Morita equivalence, it is clear that for $\lambda \in K^*$,*

$$q_{(\lambda h)_{K(Q)}} = \lambda q_{h_{K(Q)}}$$

Proposition 5.4. *Let h be a non-degenerate skew-hermitian form over Q. If h has good reduction at v then Q is unramified at v and* *(i) q_{h_v} is unramified at v if Q_v is split or*

(ii) h_v has a diagonal matrix representation with diagonal entries taking values in A_v^* if Q_v is not split.

Proof. The case when Q_v is split is clear by Morita theory. When Q_v is not split, the claim follows by observing that A_v has no zero divisors and hence any non-degenerate skew-hermitian form over A_v has a diagonal representation with units along the diagonal (see Proposition 6.8 and Proposition 3.2 in [Bek13b]). □

6 Extension of valuation from K to $K(Q)$

In this section assume that Q is a quaternion *division* algebra over K unramified at v. We first extend the valuation v from K_v to $K_v(Q_v)$. There are two cases:

(i) Q_v is not split. Then as already discussed in §4.2, $Q_v \simeq A_v \otimes_{\mathcal{O}_{K_v}} K_v$ where $A_v = (d_v, t_v)$ is a quaternionic Azumaya algebra over $\mathcal{O}_{K_v}$ with $d_v, t_v \in \mathcal{O}^*_{K_v}$ (hence $v(d_v) = v(t_v) = 0$). We extend the valuation v on K_v to $K_v(Q_v)$ as follows. First we extend the valuation v from K_v to the valuation v' on $K_v[x]$ by

$$v'(\sum_{l=0}^{m} a_l x^l) = min\{v(a_0), \cdots, v(a_m)\} \tag{6.1}$$

and then extend to $K_v(x)$ by

$$v'(\frac{f}{g}) = v'(f) - v'(g) \tag{6.2}$$

This is indeed a valuation on $K_v(x)$ with residue field $K^{(v)}(\overline{x})$, where $\overline{x}$ is the residue of x at v' and ramification index $e_{v'/v} = 1$ (see Example 2.3.3 in [FJ08]).
Now $K_v(Q_v) = K_v(x)[y]/(d_v x^2 + t_v y^2 - 1)$ is a quadratic extension of $K_v(x)$. Let $\tilde{v}$ denote any valuation on $K_v(Q_v)$ extending the one on $K(x)$ given above (One can always extend valuations to a larger field by Theorem 4.1 in [Lan02]).

(ii) Q_v is split. In this case $K_v(Q_v) \simeq K_v(x)$ where x is transcendental. In this case we extend the valuation on $K_v(Q_v)$ using (6.1) and (6.2).

Proposition 6.1. *The valuation in $\tilde{v}$ on $K_v(Q_v)$ defined above has the following properties.*

(i) The residue field of the valuation $\tilde{v}$, denoted by $K_v(Q_v)^{(\tilde{v})}$ is isomorphic to $K^{(v)}(Q^{(v)})$, the function field of the Severi-Brauer variety associated to the residue division algebra $Q^{(v)}$ over $K^{(v)}$.

(ii) *The ramification index* $e_{\tilde{v}/v} = 1$.

(iii) *The valuation* $\tilde{v}$ *is the unique one extending* v'. *In particular, for* $\alpha \in K_v(Q_v)$

$$\tilde{v}(\alpha) = \frac{1}{2}v'(Norm_{K_v(Q_v)/K_v(x)}(\alpha))$$

(iv) *For* $a_i \in \mathcal{O}_{K_v}$, $\tilde{v}(a_1 y + a_2 x + a_3) = 0$ *if and only if* $min\{v(a_i)\} = 0$.

Proof. All of the above claims are clear when Q_v splits. So assume that Q_v is not split. To prove (i), note that $\tilde{v}(y^2) = v'(\frac{1}{t_v} - \frac{d_v}{t_v}x^2) = 0$ since d_v, t_v are units in $\mathcal{O}_{K_v}$. Hence $\tilde{v}(y) = 0$ and $y \in \mathcal{O}^*_{\tilde{v}}$. Let $\overline{y}, \overline{x}, \overline{d_v}, \overline{t_v}$ denote the corresponding residues at $\tilde{v}$. Then we have an embedding,

$$F := K^{(v)}(\overline{x})[\overline{y}]/(\overline{d}_v\overline{x}^2 + \overline{t}_v\overline{y}^2 - 1) \hookrightarrow K_v(Q_v)^{(\tilde{v})}$$

Now $(\overline{d}_v\overline{x}^2 + \overline{t}_v\overline{y}^2 - 1)$ is the conic corresponding to the residue quaternion algebra $Q^{(v)}$. So $F \simeq K^{(v)}(Q^{(v)})$. Moreover $Q^{(v)}$ is not split because Q is unramified at v and due to injectivity of s in the exact sequence of (4.5) in §4.2. Therefore the conic $(\overline{d}_v\overline{x}^2 + \overline{t}_v\overline{y}^2 - 1)$ is not hyperbolic over $K^{(v)}(\overline{x})$ and hence $[K^{(v)}(Q^{(v)}) : K^{(v)}(\overline{x})] = 2$. But since $[K_v(Q_v)^{(\tilde{v})} : K^{(v)}(\overline{x})] \leq [K_v(Q_v) : K_v(x)] = 2$, we conclude that $K_v(Q_v)^{(\tilde{v})}$ is isomorphic to $K^{(v)}(Q^{(v)})$.
We now prove (ii) and (iii). Let g be the number of distinct valuations on $K_v(Q_v)$ extending v'. From the above argument we see that $f_{\tilde{v}/v'} = [K_v(Q_v)^{(\tilde{v})} : K^{(v)}(\overline{x})] = 2$. This implies $e_{\tilde{v}/v'} = 1$ and $g = 1$ from the equality

$$[K_v(Q_v) : K_v(x)] = e_{\tilde{v}/v'} f_{\tilde{v}/v'} g$$

Also as mentioned before $e_{v'/v} = 1$ (from Example 2.3.3 in [FJ08]). This proves that $e_{\tilde{v}/v} = 1$. From the fact the Galois group $G(K_v(Q_v)/K_v(x))$ acts transitively on the extensions on the valuation v' on $K_v(Q_v)$ (Exercise 8, Chapter 2 in [FJ08]) and $g = 1$, we get (iii).
Note that by (iii) we have

$$\begin{aligned}
\tilde{v}(a_1 y + a_2 x + a_3) &= \frac{1}{2}v'(Norm_{K_v(Q_v)/K_v(x)}(a_1 y + a_2 x + a_3)) \\
&= \frac{1}{2}v'(-a_1^2(\frac{1}{t_v} - \frac{d_v}{t_v}x^2) + a_2^2 x^2 + a_3^2 + 2a_2 a_3 x) \\
&= \frac{1}{2}v'((a_2^2 + \frac{d_v}{t_v}a_1^2)x^2 + 2a_2 a_3 x + (a_3^2 - \frac{1}{t_v}a_1^2))
\end{aligned}$$

From this (iv) easily follows. □

By abuse of notation, let $\tilde{v}$ also denote the valuation on $K(Q)$ obtained by restriction via the embedding $K(Q) \hookrightarrow K(Q) \otimes_K K_v \simeq K_v(Q_v)$.
We now prove Theorem 3.1 that relates good reduction of a skew-hermitian

form h over Q with center K to the ramification of the associated quadratic form $q_{h_{K(Q)}}$.

Proof of Theorem 3.1: Given a discrete valuation v on K, let $\tilde{v}$ be the valuation on $K(Q)$ described as above. By hypothesis, h has good reduction at v. So by Remark 4.2, Q is necessarily unramified at v. We need to show that $q_{h_{K(Q)}}$ is unramified $\tilde{v}$. There are two cases:

(i) Q_v is split. Since $K(Q)_{\tilde{v}}$ contains $K_v(Q_v)$ as a subfield, it suffices to show that $q_{h_{K_v(Q_v)}}$ is unramified at $\tilde{v}$. But by functoriality of Morita equivalence, we have $q_{h_{K_v(Q_v)}} = q_{h_v} \otimes_{K_v} K_v(Q_v)$. Together with Proposition 5.4(i), we get that $q_{h_{K(Q)}}$ is unramified $\tilde{v}$.

(ii) Q_v is not split. With notation as in §4.2, $Q_v = A_v \otimes_{\mathcal{O}_{K_v}} K_v$, where A_v, the valuation ring of Q_v, is a quaternionic Azumaya algebra over $\mathcal{O}_{K_v}$ given by $A_v =< 1, i, j, ij \mid i^2 = d_v, j^2 = t_v, ij = -ji, d_v, t_v \in A_v^* >$. By Proposition 5.4(ii), h_v has a matrix representation that is diagonal with diagonal entries taking values in A_v^*. Consider one such representation

$$h_v \simeq \bigoplus_{l=1}^{n} a_l i + b_l j + c_l ij$$

where $a_l i+b_l j+c_l ij \in A_v^*$. Let $N_l = Nrd_Q(a_l i+b_l j+c_l ij)$ be the reduced norm. Then by (4.4),

$$\begin{aligned} 0 &= v(a_l i + b_l j + c_l ij) \\ &= \frac{1}{2} v(N_l) \\ &= \frac{1}{2} v(-a_l^2 d_v - b_l^2 t_v + c_l^2 d_v t_v) \end{aligned}$$

This implies that for each l that $min\{v(a_l), v(b_l), v(c_l)\} = 0$. Now recall that by Example 5.2 in §5.2,

$$q_{h_{K_v(Q_v)}} \simeq \bigoplus_{l=1}^{n} \begin{bmatrix} (a_l y - b_l x - c_l) & 0 \\ 0 & -(a_l y - b_l x - c_l)N_l \end{bmatrix}$$

We are now done by Proposition 6.1(iv).

7 Proof of Theorem 3.2

We begin with an easy lemma. We stick to the notations as before.

Lemma 7.1. *Let Q be unramified at v. Then we have*

$$r_2^i([Q] \cup H^{i-2}(K, \mu_2)) \subseteq [Q^{(v)}] \cup H^{i-3}(K^{(v)}, \mu_2)$$

where r_2^i is the residue map given by (3.2) in §3.

Proof. Since Q is unramified at v, $r_2^i([Q]) = 0$. Therefore by the exact sequence (see Proposition 7.7 in [GMS03])

$$0 \to H^i(K^{(v)}, \mu_2) \xrightarrow{s_2^i} H^i(K_v, \mu_2) \xrightarrow{r_2^i} H^{i-1}(K^{(v)}, \mu_2) \to 0$$

we have that $[Q_v]$ is uniquely identified as the image of the residue quaternion algebra $[Q^{(v)}]$ over $K^{(v)}$ under s_2^i. The result now follows from Chapter II §7, Exercise 7.12 in [GMS03]. □

Now let us recall some of the facts discussed in §1.2.2 of Berhuy's paper [Ber07]. To simplify notations, let $F(Q)$ be the function field of the Severi-Brauer variety associated to a quaternion algebra Q over an arbitrary field F. Now consider the valuations W on $F(Q)$ arising from closed points on the conic defined by Q. Then for every $l \geq 1$, the kernel of the corresponding residue map

$$r_l^i : H^i(F(Q), \mu_l^{\otimes i-1}) \to \bigoplus_{w \in W} H^{i-1}(F(Q)^{(w)}, \mu_l^{\otimes i-2})$$

is the unramified cohomology group with respect to the valuations in W denoted by $H^i_{nr}(F(Q), \mu_l^{\otimes i-1})$. We now let

$$\mathbb{Q}/\mathbb{Z}(i-1) = \lim_{\to} \mu_l^{\otimes i-1}$$

where the limit is taken over all the integers prime to the characteristic of F. Then $H^i(F(Q), \mathbb{Q}/\mathbb{Z}(i-1))$ is the direct limit of the groups $H^i(F(Q), \mu_l^{\otimes i-1})$ with respect to the maps

$$H^i(F(Q), \mu_m^{\otimes i-1}) \to H^i(F(Q), \mu_n^{\otimes i-1}), m|n$$

The corresponding residue maps are compatible to each other yielding the unramified cohomology $H^i_{nr}(F(Q), \mathbb{Q}/\mathbb{Z}(i-1))$. Moreover for any field E, $H^i(E, \mu_{2^m}^{\otimes i-1})$ is identified with the 2^m torsion subgroup of $H^i(E, \mathbb{Q}/\mathbb{Z}(i-1))$ and hence the canonical map of change of coefficients

$$H^i(E, \mu_{2^m}^{\otimes i-1}) \hookrightarrow H^i(E, \mu_{2^n}^{\otimes i-1}) \quad m|n \tag{7.1}$$

is injective. We now recall the following results from [Ber07].

Proposition 7.2. *(Proposition 7 and Proposition 9 in [Ber07]) Let h be a skew-hermitian form over a quaternionic F-algebra Q. Then*

(i) $e_i([q_{h_{F(Q)}}]) \in H^i_{nr}(F(Q), \mu_2)$

(ii) For $i \geq 1$, the restriction map yields an isomorphism

$$Res_{F(Q)/F} : H^i(F, \mathbb{Q}/\mathbb{Z}(i-1)/([Q] \cup H^{i-2}(F, \mu_2)) \simeq H^i_{nr}(F(Q), \mathbb{Q}/\mathbb{Z}(i-1))$$

where $[Q] \cup H^{i-2}(F, \mu_2)$ is viewed as a subgroup of $H^i(F, \mathbb{Q}/\mathbb{Z}(i-1))$ (If $i = 1$, $[Q] \cup H^{i-2}(F, \mu_2) = 0$). In particular, the inverse image denoted by S, of 2-torsion subgroup of the unramified cohomology group $H^i_{nr}(F(Q), \mu_2)$ under $Res_{F(Q)/F}$ is a subgroup of $H^i(F, \mu_4^{\otimes i-1})/([Q] \cup H^{i-2}(F, \mu_2))$. Thus we have an isomorphism $\jmath$ obtained by restricting $Res_{F(Q)/F}$ to S,

$$\jmath : S \xrightarrow{\simeq} H^i_{nr}(F(Q), \mu_2)$$

Notation: Let $\tilde{V}$ denote the collection of valuations $\tilde{v}$ on $K(Q)$ obtained by extending the valuations $v \in V$ on K as described in §6. To simplify notations, from now on let $L = K(Q)$.

We now prove that the hypothesis of Theorem 3.2 implies the following finiteness theorem.

Theorem 7.3. *Let (K, V) satisfy the hypothesis of Theorem 3.2. Then for each $i \geq 1$, the kernel of the residue map*

$$R^i_2 : H^i_{nr}(L, \mu_2) \to \bigoplus_{\tilde{v} \in \tilde{V}} H^{i-1}(L^{(\tilde{v})}, \mu_2)$$

denoted by $H^i_{nr}(L, \mu_2)_{\tilde{V}}$, is finite and is bounded by

$$|H^i_{nr}(L, \mu_2)_{\tilde{V}}| \leq |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|$$

Proof. For each $v \in V$, the following diagram with arrows representing the natural restriction maps commutes by the functoriality.

$$\begin{tikzcd}
H^i(K, \mu_4^{\otimes i-1}) \arrow[r] \arrow[d] & H^i(K_v, \mu_4^{\otimes i-1}) \arrow[d] \\
H^i(L, \mu_4^{\otimes i-1}) \arrow[r] & H^i(L_{\tilde{v}}, \mu_4^{\otimes i-1})
\end{tikzcd}$$

Moreover by Chapter II, §8, Proposition 8.2 in [GMS03], the functoriality of restriction yields

$$\begin{tikzcd}
H^i(K_v, \mu_4^{\otimes i-1}) \arrow[r] \arrow[d] & H^{i-1}(K^{(v)}, \mu_4^{\otimes i-2}) \arrow[d, "e_{\tilde{v}/v} = 1"] \\
H^i(L_{\tilde{v}}, \mu_4^{\otimes i-1}) \arrow[r] & H^{i-1}(L^{(\tilde{v})}, \mu_4^{\otimes i-2})
\end{tikzcd}$$

where the horizontal arrows represent the residue maps. Combining the above commutative diagrams for each $v \in V$, together with Proposition 7.2, Lemma 7.1 and (7.1), we get that the following diagram commutes where $\imath$ is injective and $\jmath$ is an isomorphism.

$$\begin{tikzcd}
S \arrow[r, "\imath"] \arrow[dr, "\jmath"] & \frac{H^i(K,\mu_4^{\otimes i-1})}{([Q]\cup H^{i-2}(K,\mu_2))} \arrow[r, "r_4^i"] & \bigoplus_{v\in V} \frac{H^{i-1}(K^{(v)},\mu_4^{\otimes i-2})}{([Q^{(v)}]\cup H^{i-3}(K^{(v)},\mu_2))} \arrow[d, "e_{\tilde{v}/v} = 1"] \\
& H^i_{nr}(L, \mu_2) \arrow[r, "R_2^i"'] & \bigoplus_{\tilde{v}\in V} H^{i-1}(L^{(\tilde{v})}, \mu_2)
\end{tikzcd}$$

Now $L_{\tilde{v}}$ is also the completion of $K_v(Q_v)$ at $\tilde{v}$. So by Proposition 6.1(ii), for the extension $L_{\tilde{v}}/K_v$, we have ramification index $e_{\tilde{v}/v} = 1$ and the residue field $L^{(\tilde{v})} \simeq K^{(v)}(Q^{(v)})$. Hence the right vertical map is injective by Proposition 7.2. By the hypothesis on (K, V), it is easy to see that $Ker\ r_4^i$ is finite and

$$|H^i_{nr}(L, \mu_2)_{\tilde{V}}| = |Ker\ R_2^i| \leq |Ker\ r_4^i| = |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|$$

$\square$

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2 :
Since the unramified 2-torsion Brauer group with respect to V, ${}_2Br(K)_V$ is isomorphic to $H^2(K,\mu_2)_V$ which is finite by hypothesis, there are only finitely many quaternion algebras unramified at all $v \in V$. So it suffices to show that for a fixed unramified Q over K, the number of K-isomorphism classes of the universal covering of the special unitary groups $SU_n(h,Q)$ of n-dimensional skew-hermitian forms h over Q that have good reduction at all $v \in V$is upper bounded by

$$|Pic(V)/2Pic(V)| \cdot \prod_{i=1}^{l} |H^i(K,\mu_2)^V| \cdot |H^i(K,\mu_4^{\otimes i-1})_V|$$

We have two cases.

(i) Q is a split quaternion. In this case $SU(h,Q) \simeq SO_{2n}(q_h,K)$. Then by the proof of Theorem 2.1 in [CRR19], we conclude that the number of K-isomorphism classes of $SO_{2n}(q_h,K)$ that have good reduction at all $v \in V$ is finite and bounded above by

$$\begin{aligned}|Pic(V)/2Pic(V)| \cdot \prod_{i=1}^{l} |H^i(K,\mu_2)_V| \\ \leq |Pic(V)/2Pic(V)| \cdot \prod_{i=1}^{l} |H^i(K,\mu_2)^V| \cdot |H^i(K,\mu_4^{\otimes i-1})_V|\end{aligned}$$

The above inequality is due to (7.1).

(ii) Q is a quaternionic division algebra unramified at all $v \in V$. The idea of the proof in this case is to go back and forth between h and $q_{h_{K(Q)}}$ and using arguments similar to the one in [CRR19].

Notation: In order to avoid notational complexity, we will be simplifying some notations as follows.

- For a skew-hermitian form h over a quaternion algebra Q with center a field F, we will make a slight abuse of notation and write q_h instead of $q_{h_{F(Q)}}$ for the quadratic form corresponding to $h_{F(Q)}$ obtained via Morita theory.
- For a field F and a class $[q] \in \mathcal{I}(F)^m$, we denote by $< q > \in H^m(F,\mu_2)$, its image under the natural map

$$\mathcal{I}(F)^m \to \mathcal{I}(F)^m/\mathcal{I}(F)^{m+1} \xrightarrow{\simeq} H^m(F,\mu_2)$$

where the first map is the natural projection and the second one is the Milnor isomorphism as mentioned in (4.2).

As before let $L := K(Q)$. Let $\{h_i\}_{i\in I}$ denote a family of n-dimensional non-degenerate skew-hermitian forms over Q such that

- for each $i \in I$, the universal covering of $G_i = SU_n(h_i, Q)$ has good reduction at all $v \in V$ and
- for $i, j \in I, i \neq j$, the forms h_i and h_j are not similar i.e., $h_i \not\cong \lambda h_j, \lambda \in K^*$.

It suffices to show that

$$|I| \leq \prod_{i=0}^{l} d_i$$

where $d_0 = Pic(V)/2Pic(V)$ and for $1 \leq i \leq l = [log_2\ 2n] + 1$,

$$d_i = |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|$$

Note that by Remark 4.3, the above conditions imply that for each $i \in I$ and any $v \in V$, there exists $\lambda_v^{(i)} \in K_v^*$ such that the form $\lambda_v^{(i)} h_i$ over Q_v is has good reduction at v. Also because of condition (A) on K, we can assume that $\lambda_v^{(i)} = 1$ for almost all $v \in V$. Recall by Lemma 2.2 in [CRR19] that there is a natural isomorphism

$$Pic(V)/2Pic(V) \simeq \mathbb{I}(K,V)/\mathbb{I}(K,V)^2\mathbb{I}_0(K,V)K^* \tag{7.2}$$

where

$$\mathbb{I}(K,V) = \left\{(x_v) \in \prod_{v\in V} K_v^* \mid x_v \in \mathcal{O}_v^* \text{ for almost all } v \in V\right\}$$

is the *group of idèles* and

$$\mathbb{I}_0(K,V) = \prod_{v\in V} \mathcal{O}_v^*$$

is the *subgroup of integral idèles.*

So $\lambda^{(i)} := (\lambda_v^{(i)})_{v\in V} \in \mathbb{I}(K,V)$. Since d_0 is finite by hypothesis, using (7.2), we conclude that there exists a subset $J_0 \subseteq I$ of size $\geq I/d_0$ (if I is infinite so is J_0) such that all $\lambda^{(i)}, i \in J_0$ have the same image in $\mathbb{I}(K,V)/\mathbb{I}(K,V)^2\mathbb{I}_0(K,V)K^*$. Fix $j_0 \in J_0$. For any $j \in J_0$, we can write

$$\lambda^{(j)} = \lambda^{(j_0)}(\alpha^{(j)})^2\beta^{(j)}\delta^{(j)}$$

with $\alpha^{(j)} \in \mathbb{I}(K,V)$, $\beta^{(j)} \in \mathbb{I}_0(K,V)$ and $\delta^{(j)} \in K^*$. Then set

$$H_j = \delta^{(j)}h_j$$
$$\Lambda_j = (\delta^{(j)})^{-1}\lambda^{(j)} = \lambda^{(j_0)}(\alpha^{(j)})^2\beta^{(j)}$$

It is easy to see that for $j \neq j'$, $H_j \ncong H_{j'}$ and hence $q_{H_j} \ncong q_{H_{j'}}$ as quadratic forms over L (see §5.1). Moreover, $\Lambda_v^{(j)} H_j = \lambda_v^{(j)} h_j$ and hence $\Lambda_v^{(j)} H_j$ has good reduction at v. Therefore by Theorem 3.1

$$q_{\Lambda_v^{(j)} H_j} \in W_0(L_{\tilde{v}}),\ j \in J_0$$

(See §4.2 for the definition of $W_0(L_{\tilde{v}})$). Also note that

$$\begin{aligned} q(j, \tilde{v}) &:= \Lambda_v^{(j_0)}(q_{H_j} \perp -q_{H_{j_0}}) \\ &= \Lambda_v^{(j_0)} \cdot (\Lambda_v^{(j)})^{-1} \cdot \Lambda_v^{(j)} q_{H_j} \perp -\Lambda_v^{(j_0)} q_{H_{j_0}} \\ &= (\alpha^{(j)})^{-2}(\beta^{(j)})^{-1} q_{\Lambda_v^{(j)} H_j} \perp -q_{\Lambda_v^{(j_0)} H_{j_0}} && \text{(by Remark 5.3)} \\ &\cong (\beta^{(j)})^{-1} q_{\Lambda_v^{(j)} H_j} \perp -q_{\Lambda_v^{(j_0)} H_{j_0}} \end{aligned}$$

As $(\beta^{(j)})^{-1} \in \mathcal{O}_v^*$, we see that for every $v \in V$

$$[q(j, \tilde{v})] = \Lambda_v^{(j_0)}([q_{H_j}] \perp -[q_{H_{j_0}}]) \in W_0(L_{\tilde{v}}) \cap \mathcal{I}(L_v)$$

Now by Lemma 3.3 in [CRR19] and Proposition 7.2, we get

$$< q_{H_j} > - < q_{H_{j_0}} > \in H^1_{nr}(L, \mu_2)_{\tilde{V}} \leq d_1$$

The last inequality follows from Theorem 7.3. Therefore we can find a subset $J_1 \subseteq J_0$ of size $\geq |J_0|/d_1 \geq |I|/d_0 d_1$ such that for $j \in J_1$, the classes $[q_{H_j}] - [q_{H_{j_0}}] \in \mathcal{I}(L)$ all have the same image in $H^1(L, \mu_2)$. Fix $j_1 \in J_1$. Then for any $j \in J_1$, we have

$$\begin{aligned} < q_{H_j} > - < q_{H_{j_1}} > &= (< q_{H_j} > - < q_{H_{j_0}} >) - (< q_{H_{j_1}} > - < q_{H_{j_0}} >) \\ &= 0 \in H^1(L, \mu_2) \end{aligned}$$

Hence $[q_{H_j}] - [q_{H_{j_1}}] \in \mathcal{I}(L)^2$. Moreover

$$\Lambda^{(j)} = \Lambda^{(j_1)}(\overline{\alpha}^{(j)})^2(\overline{\beta}^{(j)}), \qquad \overline{\alpha}^{(j)} \in \mathbb{I}(K, V), \overline{\beta}^{(j)} \in \mathbb{I}_0(K, V)$$

Then as before we conclude that for every $\tilde{v} \in \tilde{V}$

$$\Lambda_v^{(j_1)}([q_{H_j}] \perp -[q_{H_{j_1}}]) \in W_0(L_{\tilde{v}}) \cap \mathcal{I}(L_{\tilde{v}})^2$$

Again by using Lemma 3.3 in [CRR19], Proposition 7.2 and Theorem 7.3 as before, we get

$$< q_{H_j} > - < q_{H_{j_1}} > \in H^2_{nr}(L, \mu_2)_{\tilde{V}} \leq d_2$$

So there exists a subset $J_2 \subseteq J_1$ of size $\geq |I|/d_1 d_1 d_2$ such that that for each $j \in J_2$, the classes $[q_{H_j}] - [q_{H_{j_1}}]$ have the same image in $H^2(L, \mu_2)$. Fixing $j_2 \in J_2$, we have

$$[q_{H_j}] - [q_{H_{j_2}}] \in \mathcal{I}(L)^3 \qquad j \in J_2$$

Proceeding inductively we get a nested chain of subsets

$$I \supseteq J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_l$$

such that for any $m = 1, 2, \cdots l$,

- $|J_m| \geq |I|/d_0 d_1 \cdots d_m$ and
- for $j \in J_m$, we have

$$[q_{H_j}] - [q_{H_{j_m}}] \in \mathcal{I}(L)^{m+1} \qquad j \in J_m$$

But by a theorem of Arason and Pfister ([AP71], also see [Lam05], Chapter X , Hauptsatz 5.1), the dimension of any positive dimensional anisotropic form in $\mathcal{I}(K)^{l+1}$ is $\geq 2^{l+1} > 2^{log_2 2n+1} = 4n$. Thus $[q_{H_j}] - [q_{H_{j_l}}] \in \mathcal{I}(L)^{l+1}$ implies that $q_{H_j} \cong q_{H_{j_l}}, \forall j \in J_l$. But as seen before the forms q_{H_j} are pairwise inequivalent. Hence we conclude that $|J_l| = 1$ and

$$|I| \leq |Pic(V)/2Pic(V)| \cdot \prod_{i=1}^{l} |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|.$$

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