

# A DOLBEAULT–DIRAC SPECTRAL TRIPLE FOR QUANTUM PROJECTIVE SPACE

BISWARUP DAS, RÉAMONN Ó BUACHALLA,  
AND PETR SOMBERG

Received: February 6, 2020

Revised: July 24, 2020

Communicated by Christian Bär

**ABSTRACT.** The notion of a Kähler structure for a differential calculus was recently introduced by the second author as a framework in which to study the noncommutative geometry of quantum flag manifolds. It was subsequently shown that any covariant positive definite Kähler structure has a canonically associated triple satisfying, up to the compact resolvent condition, Connes’ axioms for a spectral triple. In this paper we begin the development of a robust framework in which to investigate the compact resolvent condition, and moreover, the general spectral behaviour of covariant Kähler structures. This framework is then applied to quantum projective space endowed with its Heckenberger–Kolb differential calculus. An even spectral triple with non-trivial associated  $K$ -homology class is produced, directly  $q$ -deforming the Dolbeault–Dirac operator of complex projective space. Finally, the extension of this approach to a certain canonical class of irreducible quantum flag manifolds is discussed in detail.

2020 Mathematics Subject Classification: Primary: 20G42; Secondary: 46L80, 46L87, 58B32, 58B34, 32Q15, 14M15

Keywords and Phrases: Noncommutative differential geometry, quantum groups, quantum flag manifolds, spectral triples, differential calculi, Kähler geometry

## CONTENTS

### 1 INTRODUCTION

1080

2	PRELIMINARIES	1084
2.1	Complex, Hermitian, and Kähler Structures on Differential Calculi . . . . .	1085
2.2	Metrics, Adjoints, and the Hodge Map . . . . .	1087
2.3	CQH-Complex and CQH-Hermitian Spaces . . . . .	1088
2.3.1	Hodge Theory . . . . .	1089
2.4	Dolbeault–Dirac Spectral Triples . . . . .	1091
2.5	Analytic $K$ -Homology and the Anti-Holomorphic Euler Characteristic . . . . .	1093
3	CQH-COMPLEX SPACES OF GELFAND TYPE	1095
3.1	Hodge Decomposition of the Laplacian . . . . .	1095
3.2	CQH-Complex Spaces of Gelfand Type . . . . .	1098
4	DRINFELD–JIMBO QUANTUM GROUPS AND CQH-HERMITIAN SPACES	1100
4.1	Highest and Lowest Weight Vectors . . . . .	1100
4.2	CQH-Complex Spaces and Leibniz Constants . . . . .	1102
4.3	Laplacian Eigenvalues for CQH-Hermitian Spaces of Gelfand Type . . . . .	1105
4.4	Connectedness for Gelfand Type CQH-Complex Spaces . . . . .	1108
5	CQH-COMPLEX SPACES OF ORDER I AND SPECTRAL TRIPLES	1109
5.1	Positivity for Leibniz Constants . . . . .	1109
5.2	CQH-Complex Spaces of Order I . . . . .	1110
5.3	CQH-Hermitian Spaces of Order I and Solidity . . . . .	1112
5.4	Solidity and Dolbeault–Dirac Spectral Triples . . . . .	1114
6	A DOLBEAULT–DIRAC SPECTRAL TRIPLE FOR QUANTUM PROJECTIVE SPACE	1117
6.1	Quantum Projective Space as a Quantum Homogeneous Space	1117
6.1.1	The Heckenberger–Kolb Calculus . . . . .	1118
6.2	An Order I Presentation of the CQH-Complex Space . . . . .	1121
6.3	A Dolbeault–Dirac Pair of Spectral Triples for Quantum Projective Space . . . . .	1126
7	IRREDUCIBLE QUANTUM FLAG MANIFOLDS	1128
7.1	The Quantum Homogeneous Spaces . . . . .	1128
7.2	The Heckenberger–Kolb Calculi and their Kähler Structures . . . . .	1129
7.3	Irreducible Quantum Flag Manifolds of Weak Gelfand Type . . . . .	1130
7.4	Towards Order II CQH-Hermitian Spaces . . . . .	1133
7.4.1	Hodge Decomposition and the $B_{\text{hw}}$ -Action . . . . .	1133
7.4.2	Spherical Generators . . . . .	1133
7.4.3	Leibniz Constants . . . . .	1135
7.4.4	Some Conjectures . . . . .	1136

A	CQGAs, QUANTUM HOMOGENEOUS SPACES, AND FROBENIUS RECIPROCITY	1136
A.1	Hopf Algebras and CQGAs	1137
A.2	CMQGA-Homogeneous Spaces	1137
A.3	Takeuchi’s Equivalence	1138
A.4	Frobenius Reciprocity for Quantum Homogeneous Spaces	1139
B	DRINFELD–JIMBO QUANTUM GROUPS	1140
B.1	Quantum Integers	1140
B.2	Drinfeld–Jimbo Quantised Enveloping Algebras	1141
B.3	Type 1 Representations	1142
B.4	Quantised Coordinate Algebras $\mathcal{O}_q(G)$	1143
B.5	The Hopf Algebra $\mathcal{O}_q(SU_n)$	1144
C	DECOMPOSING $\Omega^{(0,\bullet)}$ INTO IRREDUCIBLE $U_q(\mathfrak{sl}_n)$ -MODULES	1145
C.1	Skew Young Diagrams and Young Tableaux	1145
C.2	The Decomposition	1146

## 1 INTRODUCTION

In Connes’  $K$ -theoretic approach to noncommutative geometry, spectral triples generalise Riemannian spin manifolds and their associated Dirac operators to the noncommutative setting. The question of how to reconcile the theory of spectral triples with Drinfeld–Jimbo quantum groups is one of the major open problems in noncommutative geometry. Since their appearance in the 1980s, quantum groups have attracted serious and significant attention at the highest mathematical levels. In the compact case, the foundations of their noncommutative topology and their noncommutative measure theory have now been firmly established. By contrast, the noncommutative spectral geometry of quantum groups is still very poorly understood. Indeed, despite a large number of important contributions over the last thirty years, there is still no consensus on how to construct a spectral triple for  $\mathcal{O}_q(SU_2)$ , probably the most basic example of a quantum group. These difficulties aside, the prospect of reconciling these two areas still holds great promise for their mutual enrichment. On one hand, it would provide quantum groups with powerful tools from operator algebraic  $K$ -theory and  $K$ -homology. On the other hand, it would provide the theory of spectral triples with a large class of examples of fundamental importance with which to test and guide the future development of the subject.

One of the most important lessons to emerge from the collected efforts to understand the noncommutative geometry of quantum groups is that quantum homogeneous spaces tend to be more amenable to geometric investigation than quantum groups themselves. Philosophically, one can think of the process of forming a quantum homogeneous spaces as quotienting out the most exotic noncommutativity of the quantum group. This produces quantum spaces which are closer to their classical counterparts, and which possess more recognisable

differential structures. The prototypical example here is the *standard Podleś sphere*, the Drinfeld–Jimbo  $q$ -deformation of the Hopf fibration presentation of the 2-sphere  $S^2$ . In contrast to the case of  $\mathcal{O}_q(SU_2)$ , the Podleś sphere admits a canonical spectral triple which directly  $q$ -deforms the classical Dolbeault–Dirac operator of the 2-sphere [13]. Moreover, it is the most widely and consistently accepted example of a spectral triple in the Drinfeld–Jimbo setting. The Podleś sphere also forms a well behaved and motivating example for Majid’s Hopf algebraic theory of braided noncommutative geometry [39]. In particular, it admits an essentially unique differential calculus, the Podleś calculus, to which Majid was able to apply his quantum frame bundle theory to directly  $q$ -deform the classical Kähler geometry of the 2-sphere (cf. [40]).

The Podleś sphere is itself a special example of a large and very beautiful family of quantum homogeneous spaces: the quantum flag manifolds [36, 15]. In one of the outstanding results of the algebraic approach to the noncommutative geometry of quantum groups, Heckenberger and Kolb showed that the quantum flag manifolds of irreducible type admit an essentially unique  $q$ -deformation of their classical Dolbeault double complex. This result forms a far reaching generalisation of the Podleś calculus endowed with Majid’s complex structure, and strongly suggests that the irreducible quantum flag manifolds, or more generally the quantum flag manifolds, have a central role to play in reconciling quantum groups and spectral triples.

The Heckenberger–Kolb classification, however, contains no generalisation of the Kähler geometry of the Podleś sphere (cf. [40]). In a recent paper by the second author, the notion of a noncommutative Kähler structure was introduced to provide a framework in which to do just this. In the quantum homogeneous space case many of the fundamental results of Kähler geometry have been shown to follow from the existence of such a structure. For example, it implies noncommutative generalisations of Lefschetz decomposition, the Lefschetz and Kähler identities, Hodge decomposition, the hard Lefschetz theorem, and the refinement of de Rham cohomology by Dolbeault cohomology. The existence of a Kähler structure was verified for the quantum projective spaces in [50], and it was conjectured that a Kähler structure exists for all the compact quantum Hermitian spaces. Subsequently, for all but a finite number of values of  $q$ , the conjecture was verified for every irreducible quantum flag manifold by Matassa in [41].

The Dolbeault–Dirac operator  $D_{\bar{\partial}} := \bar{\partial} + \bar{\partial}^\dagger$  associated to a Kähler structure is an obvious candidate for a noncommutative Dirac operator. In [12] the authors associated to any covariant positive definite Hermitian structure  $(\Omega^{(\bullet, \bullet)}, \kappa)$ , over a quantum homogeneous space  $B = G^{\text{co}(H)}$ , a Hilbert space  $L^2(\Omega^{(0, \bullet)})$ , carrying a bounded  $*$ -representation  $\rho$  of  $B$ . Moreover,  $D_{\bar{\partial}}$  was shown to act on  $L^2(\Omega^{(0, \bullet)})$  as an essentially self-adjoint operator, with bounded commutators  $[D_{\bar{\partial}}, b]$ , for all  $b \in B$ . Hence, to show that the triple  $(B, L^2(\Omega^{(0, \bullet)}), D_{\bar{\partial}})$  is a spectral triple, it remains to verify the compact resolvent condition. Even in the Drinfeld–Jimbo case, however, it is not clear at present how to conclude the compact resolvent condition from the properties of a general covariant Kähler

structure. (See §7.4.4 for a brief discussion on how this might be achieved.) In the study of classical homogeneous spaces, a difficult problem can often be approached by assuming restrictions on the multiplicities of the  $U(\mathfrak{g})$ -modules appearing in an equivariant geometric structure [29]. Taking inspiration from this approach, we choose to focus on covariant complex structures of *weak Gelfand type*, that is to say, those for which  $\bar{\partial}\Omega^{(0,k)}$  is multiplicity-free as a  $U_q(\mathfrak{g})$ -module. This implies diagonalisability of the Dolbeault–Dirac operator  $D_{\bar{\partial}}$  over irreducible modules, which when combined with the considerable geometric structure of the calculus, allows us to make a number of strong statements about the spectral behaviour of  $D_{\bar{\partial}}$ . In particular, for those covariant complex structures of *Gelfand type*, that is to say, those for which  $\Omega^{(0,k)}$  is multiplicity-free, we produce a sufficient set of routinely verifiable conditions for  $D_{\bar{\partial}}$  to have compact resolvent.

To place our efforts in context, we briefly recall previous spectral calculations for quantum groups, and in particular for quantum flag manifolds. The construction of spectral triples over quantum groups can be very roughly divided into two approaches. The first is isospectral deformation, as exemplified by the work of Neshveyev and Tuset, who constructed isospectral Dirac operators for all the Drinfeld–Jimbo quantum groups [46]. In this approach one takes as an ansatz that the spectrum of the Dirac survives  $q$ -deformation unchanged. A representation of the quantum group is then constructed around this ansatz so as to retain bounded commutators. This approach has the advantage of avoiding the need for spectral calculations, but the disadvantage that the spectral triples produced are too close to the classical case to be completely natural. By contrast, the second approach constructs canonical  $q$ -deformations of the classical spin geometry of a space, and then calculates the spectrum of the resulting  $q$ -deformed Dirac. The prototypical examples here are the Dąbrowski–Sitarz construction of a spectral triple on the Podleś sphere, as discussed above, and Majid’s spectral calculations for his Dolbeault–Dirac operator over the Podleś sphere, as also mentioned above. This is the approach followed in this paper, and just as for the Podleś examples, an unavoidable consequence is a  $q$ -deformation of the classical spectrum.

At around the same time as these works, Krähmer introduced an influential algebraic Dirac operator for the irreducible quantum flag manifolds, which gave a commutator realisation of their Heckenberger–Kolb calculus [31]. A series of papers by Dąbrowski, D’Andrea, and Landi, followed, where spectral triples were constructed for the all quantum projective spaces [10, 9]. This approach used a noncommutative generalisation of the Parthasarathy formula [54] to calculate the Dirac spectrum and hence verify Connes’ axioms. Matassa would subsequently reconstruct this spectral triple [43] in a more formal manner by connecting with the work of Krähmer and Tucker–Simmons [32]. This approach was subsequently extended to the quantum Lagrangian Grassmannian  $\mathcal{O}_q(\mathbf{L}_2)$ , a  $C$ -series irreducible quantum flag manifold [42, 44]. We note that in the quantum setting the Parthasarathy relationship with the Casimir is much more involved than in the classical case. This reflects our poor understanding of

Casimirs in the Drinfeld–Jimbo setting, and the resulting challenges associated with a quantum Casimir approach to spectral calculations.

One of the primary purposes of a spectral triple is to serve as unbounded representatives for the  $K$ -homology classes of a  $C^*$ -algebra. Having an unbounded representative constructed in such a geometric manner has a number of advantages. In particular, it allows us to convert index theory calculations into questions about the Dolbeault cohomology of the complex structure. In [12, Theorem 5.4] the index of the associated  $K$ -homology class has been shown to be equal to the anti-holomorphic Euler characteristic of the calculus. This is calculable using Hodge decomposition in general. In particular, in the non-commutative Fano setting, it follows from the Kodaira vanishing theorem for noncommutative Kähler structures that all cohomologies are concentrated in degree 0 [52]. Hence, the index will be non-zero and the associated  $K$ -homology class non-trivial. This is particularly important given the well-known difficulty of applying Connes’ local index formula in the quantum group setting.

This paper forms part of a series of works investigating the noncommutative geometry of the quantum flag manifolds and their connections with Nichols algebras, Schubert calculus, and non-commutative projective algebraic geometry [2, 51, 58]. It is intended that this paper will serve as a point of contact between these areas and operator algebraic  $K$ -theory. Moreover, in its discussion of order I and order II presentations, the paper can be regarded as a first step towards a systematic extension of the classical results on Harish-Chandra modules (cf. [28], [60]) to the quantum group setting. The typical structure here is a pair  $(\mathfrak{g}, K)$  consisting of a real reductive Lie group  $G$  with complexified Lie algebra  $\mathfrak{g}$ , and a compact subgroup  $K \subset G$ , for which the differential of  $\text{Ad}(K)$  and the restriction  $\text{ad}(\mathfrak{g})|_{\mathfrak{k}}$  are compatible. The representation category  $\mathcal{C}(\mathfrak{g}, K)$  of  $(\mathfrak{g}, K)$ -modules, if an infinitesimal character is specified, is characterized by the existence of finitely irreducible representations of  $K$  termed the collection of *minimal  $K$ -types* (every irreducible  $(\mathfrak{g}, K)$ -module with an infinitesimal character contains one of these  $K$ -types.) As we shall discuss in the present article, this property conveniently carries over to the quantum group setting. Moreover, there is an analogous transfer of the other important related structures such as the Hecke algebra of the pair  $(\mathfrak{g}, K)$ , dualities, and so on, which will be treated elsewhere. Finally, an important issue to be addressed is how the refined spectral analysis of [55] extends to the general quantum projective space setting, and if a local index formula in twisted cyclic cohomology can be produced.

The paper is organised as follows: In §2 we recall the necessary basics of differential calculi, complex structures, and Kähler structures, as well as their interaction with compact quantum matrix group algebras, as originally considered in [12]. We also recall the necessary basics of spectral triples and  $K$ -homology. In §3, we show that the Laplacian  $\Delta_{\bar{\partial}}$  decomposes with respect to Hodge decomposition, allowing us to deduce the spectrum of  $\Delta_{\bar{\partial}}$  from the spectrum of the operator  $\bar{\partial}^\dagger \bar{\partial}$ . Restricting to the covariant case, we then consider multiplicity-free comodules as a framework in which to present complex structures of Gelfand

type. In this case, it is observed that the operator  $\bar{\partial}^\dagger \bar{\partial}$  always diagonalises over irreducible comodules.

In §4, we restrict for the first time to the setting of Drinfeld–Jimbo quantum groups, exploiting the associated highest weight structure of their representation theory. In particular we consider products of a highest weight form  $\omega$ , with powers of a zero form  $z^l$ , for  $l \in \mathbb{N}_0$ . The form  $z^l \omega$  is shown to always be an eigenvector of the Laplacian, and the corresponding eigenvalues  $\mu_l$  are described explicitly. Such sequences  $\{\mu_l\}_{l \in \mathbb{N}_0}$  of eigenvalues form the basis of our approach to the spectrum of the Dolbeault–Dirac operator.

In §5, we abstract the representation theoretic properties of  $\mathbb{C}\mathbb{P}^{n-1}$  and introduce the notion of an *order I* compact quantum homogeneous complex space. We then establish a necessary and sufficient set of conditions (given in terms of the eigenvalue sequences  $\{\mu_l\}_{l \in \mathbb{N}_0}$  discussed above) for such a space to have a Dolbeault–Dirac operator with compact resolvent.

In §6 we examine our motivating example  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ . We begin by recalling the necessary details about its definition as a quantum homogeneous space, its Heckenberger–Kolb calculus, and its covariant complex and Kähler structures. We then construct an order I presentation for the calculus in §6.2. This allows us to apply the general framework of the paper and to prove one of its main results:

**THEOREM 6.21** For quantum projective space  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , endowed with its Heckenberger–Kolb calculus and its unique covariant Kähler structure, a pair of spectral triples is given by

$$\left(\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), L^2(\Omega^{(\bullet,0)}), D_\partial\right), \quad \left(\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), L^2(\Omega^{(0,\bullet)}), D_{\bar{\partial}}\right).$$

In §7 we generalise the notion of Gelfand type to *weak Gelfand type*, and determine which non-exceptional irreducible quantum flag manifolds satisfy the condition. We show that in addition to  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , we have the quantum 2-plane Grassmannians  $\mathcal{O}_q(\text{Gr}_{n,2})$ , the odd- and even-dimensional quantum quadrics  $\mathcal{O}_q(\mathbf{Q}_n)$ . The extension of the framework of the paper to this larger class of examples is then discussed in detail.

We finish with three appendices. In the first we recall the basic definitions of compact quantum group algebras, and quantum homogeneous spaces. In the second we recall basic results about Drinfeld–Jimbo quantised enveloping algebras and their representation theory. In the third we use Frobenius reciprocity for quantum homogeneous spaces, along with some classical branching laws, to derive the decomposition of the anti-holomorphic forms into irreducible  $U_q(\mathfrak{sl}_n)$ -modules.

ACKNOWLEDGEMENTS

The authors would like to thank Marco Matassa, Elmar Wagner, Fredy Díaz García, Andrey Krutov, Karen Strung, Shahn Majid, Simon Brain, Bram Mesland, Branimir Čačić, Adam Rennie, Paolo Saracco, Kenny De Commer,

Matthias Fischmann, Peter Littelmann, Willem A. de Graaf, Jan Šťovíček, and Adam-Christiaan van Roosmalen, for useful discussions during the preparation of this paper. The second author would like to thank IMPAN Wrocław for hosting him in November 2018, and would also like to thank Klaas Landsman and the Institute for Mathematics, Astrophysics and Particle Physics for hosting him in the winter of 2017 and 2018. The second author acknowledges FNRS support through a postdoctoral fellowship within the framework of the MIS Grant "Antipode" grant number F.4502.18. The second and third authors acknowledge support from the Eduard Cech Institute within the framework of the grant GA19-06357S.

## 2 PRELIMINARIES

We recall necessary details about complex, Hermitian, and Kähler structures for differential calculi. We highlight in particular the Dirac and Laplace operators associated to an Hermitian structure, as well as the associated Hodge theory, which plays a central in our spectral calculations. We also introduce the novel notions of CQH-complex, CQH-Hermitian, and CQH-Kähler spaces. These serve as a convenient setting in which to present Dolbeault–Dirac spectral triples in the coming sections.

### 2.1 COMPLEX, HERMITIAN, AND KÄHLER STRUCTURES ON DIFFERENTIAL CALCULI

In this subsection we present the basic definitions and results for complex structures, as introduced in [25] and [2]. (For a presentation using the conventions of this paper see [49].) We also recall the basic definitions and results of Hermitian and Kähler structures, as introduced in [50]. For an excellent presentation of classical complex and Kähler geometry see [23].

Recall that a *differential calculus* is a differential graded algebra  $(\Omega^\bullet \simeq \bigoplus_{k \in \mathbb{N}_0} \Omega^k, d)$  which is generated in degree 0 as a differential graded algebra, which is to say, it is generated as an algebra by the elements  $a, db$ , for  $a, b \in \Omega^0$ . We denote the degree of a homogeneous element  $\omega \in \Omega^\bullet$  by  $|\omega|$ . For a given algebra  $B$ , a differential calculus *over*  $B$  is a differential calculus such that  $B = \Omega^0$ . A differential calculus is said to be of *total degree*  $m \in \mathbb{N}_0$  if  $\Omega^m \neq 0$ , and  $\Omega^k = 0$ , for every  $k > m$ . A *differential \*-calculus* over a \*-algebra  $B$  is a differential calculus over  $B$  such that the \*-map of  $B$  extends to a (necessarily unique) conjugate linear involutive map  $*$  :  $\Omega^\bullet \rightarrow \Omega^\bullet$  satisfying  $d(\omega^*) = (d\omega)^*$ , and

$$(\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*, \quad \text{for all } \omega \in \Omega^k, \nu \in \Omega^l.$$

For  $A$  a Hopf algebra, and  $P$  a left  $A$ -comodule algebra, a differential calculus  $\Omega^\bullet$  over  $P$  is said to be *covariant* if the coaction  $\Delta_L : P \rightarrow A \otimes P$  extends to a (necessarily unique) comodule algebra structure  $\Delta_L : \Omega^\bullet \rightarrow A \otimes \Omega^\bullet$ , with respect to which the differential  $d$  is a left  $A$ -comodule map.

DEFINITION 2.1. A complex structure  $\Omega^{(\bullet, \bullet)}$  for a differential  $*$ -calculus  $(\Omega^\bullet, d)$ , over a  $*$ -algebra  $A$ , is an  $\mathbb{N}_0^2$ -algebra grading  $\bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)}$  for  $\Omega^\bullet$  such that, for all  $(a, b) \in \mathbb{N}_0^2$ :

1.  $\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}$ ,
2.  $*(\Omega^{(a,b)}) = \Omega^{(b,a)}$ ,
3.  $d\Omega^{(a,b)} \subseteq \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)}$ .

We call an element of  $\Omega^{(a,b)}$  an  $(a, b)$ -form. Denoting by  $\text{proj}_{\Omega^{(a+1,b)}}$ , and  $\text{proj}_{\Omega^{(a,b+1)}}$ , the projections from  $\Omega^{a+b+1}$  onto  $\Omega^{(a+1,b)}$ , and  $\Omega^{(a,b+1)}$  respectively, we can define the operators

$$\partial|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a+1,b)}} \circ d, \quad \bar{\partial}|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a,b+1)}} \circ d.$$

Part 3 of the definition of a complex structure then implies the following identities:

$$d = \partial + \bar{\partial}, \quad \bar{\partial} \circ \partial = -\partial \circ \bar{\partial}, \quad \partial^2 = 0, \quad \bar{\partial}^2 = 0.$$

Thus  $(\bigoplus_{(a,b) \in \mathbb{N}^2} \Omega^{(a,b)}, \partial, \bar{\partial})$  is a double complex, which we call the *Dolbeault double complex* of the  $\Omega^{(\bullet, \bullet)}$ . Moreover, it is easily seen that

$$\partial(\omega^*) = (\bar{\partial}\omega)^*, \quad \bar{\partial}(\omega^*) = (\partial\omega)^*, \quad \text{for all } \omega \in \Omega^\bullet. \quad (1)$$

For any complex structure  $\Omega^{(\bullet, \bullet)} = \bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)}$ , a second complex structure, called its *opposite complex structure*, is given by  $\bar{\Omega}^{(\bullet, \bullet)} = \bigoplus_{(a,b) \in \mathbb{N}_0^2} \bar{\Omega}^{(a,b)}$ , where  $\bar{\Omega}^{(a,b)} = \Omega^{(b,a)}$ .

For a left  $A$ -comodule algebra  $P$ , and a covariant differential  $*$ -calculus  $\Omega^\bullet$  over  $P$ , we say that a complex structure for  $\Omega^\bullet$  is *covariant* if  $\Omega^{(a,b)}$  is a left  $A$ -sub-comodule of  $\Omega^\bullet$ , for all  $(a, b) \in \mathbb{N}_0^2$ . A direct consequence of covariance is that the maps  $\partial$  and  $\bar{\partial}$  are left  $A$ -comodule maps.

DEFINITION 2.2. An *Hermitian structure*  $(\Omega^{(\bullet, \bullet)}, \sigma)$  for a differential  $*$ -calculus  $\Omega^\bullet$ , of even total degree  $2n$ , is a pair consisting of a complex structure  $\Omega^{(\bullet, \bullet)}$ , and a central real  $(1, 1)$ -form  $\sigma$ , called the *Hermitian form*, such that, with respect to the *Lefschetz operator*

$$L : \Omega^\bullet \rightarrow \Omega^\bullet, \quad \omega \mapsto \sigma \wedge \omega,$$

isomorphisms are given by

$$L^{n-k} : \Omega^k \rightarrow \Omega^{2n-k}, \quad \text{for all } 1 \leq k < n. \quad (2)$$

For  $L$  the Lefschetz operator of an Hermitian structure, we denote

$$P^{(a,b)} := \begin{cases} \{\alpha \in \Omega^{(a,b)} \mid L^{n-a-b+1}(\alpha) = 0\}, & \text{if } a + b \leq n, \\ 0 & \text{if } a + b > n. \end{cases}$$

Moreover, we denote  $P^k := \bigoplus_{a+b=k} P^{(a,b)}$ , and  $P^\bullet := \bigoplus_{k \in \mathbb{N}_0} P^k$ . An element of  $P^\bullet$  is called a *primitive form*.

PROPOSITION 2.3 (Lefschetz decomposition). *For  $L$  the Lefschetz operator of an Hermitian form, an  $A$ -bicomodule decomposition, called the Lefschetz decomposition, is given by*

$$\Omega^{(a,b)} \simeq \bigoplus_{j \geq 0} L^j(P^{(a-j,b-j)}).$$

Finally, we come to the definition of a Kähler structure, a simple strengthening of the Hermitian structure requirements, but one with profound consequences.

DEFINITION 2.4. A *Kähler structure*  $(\Omega^{(\bullet,\bullet)}, \kappa)$  for a differential  $*$ -calculus is an Hermitian structure such that  $d\kappa = 0$ . We call  $\kappa$  a *Kähler form*.

One of the most important consequences of the Kähler condition (which is not necessarily true for a general Hermitian structure) is the equality, up to a scalar multiple of the three Laplacian operators

$$\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d. \quad (3)$$

Let  $A$  be a Hopf algebra,  $P$  a left  $A$ -comodule algebra, and  $\Omega^\bullet$  a covariant  $*$ -calculus over  $P$ . A *covariant Hermitian structure* for  $\Omega^\bullet$  is an Hermitian structure  $(\Omega^{(\bullet,\bullet)}, \sigma)$  such that  $\Omega^{(\bullet,\bullet)}$  is a covariant complex structure, and such that the Hermitian form  $\sigma$  is left  $A$ -coinvariant, which is to say  $\Delta_L(\sigma) = 1 \otimes \sigma$ . A *covariant Kähler structure* is a covariant Hermitian structure which is also a Kähler structure. Note that in the covariant case, in addition to being a  $P$ -bimodule map and a  $*$ -homomorphism,  $L$  is also a left  $A$ -comodule map.

## 2.2 METRICS, ADJOINTS, AND THE HODGE MAP

In classical Hermitian geometry, the Hodge map of an Hermitian metric is related to the associated Lefschetz decomposition through the well-known Weil formula (see [61, Theorem 1.2] or [23, Proposition 1.2.31]). In the noncommutative setting we take the direct generalisation of the Weil formula for our definition of the Hodge map.

DEFINITION 2.5. The *Hodge map* associated to an Hermitian structure  $(\Omega^{(\bullet,\bullet)}, \sigma)$  is the morphism uniquely defined by

$$*_\sigma(L^j(\omega)) = (-1)^{\frac{k(k+1)}{2}} i^{a-b} \frac{j!}{(n-j-k)!} L^{n-j-k}(\omega), \quad \omega \in P^{(a,b)} \subseteq P^k,$$

Many of the basic properties of the classical Hodge map can now be understood as consequences of the Weil formula. (See [50, §4.3] for a proof.)

LEMMA 2.6. *Let  $\Omega^\bullet$  be a differential  $*$ -calculus, of total degree  $2n$ . For  $(\Omega^{(\bullet,\bullet)}, \sigma)$  a choice of Hermitian structure for  $\Omega^\bullet$  and  $*_\sigma$  the associated Hodge map, it holds that:*

1.  $*_\sigma$  is a  $*$ -map,
2.  $*_\sigma(\Omega^{(a,b)}) = \Omega^{(n-b,n-a)}$ ,
3.  $*_\sigma^2(\omega) = (-1)^k \omega$ , for all  $\omega \in \Omega^k$ ,
4. whenever  $\Omega^\bullet$  is a covariant calculus over a left  $A$ -comodule algebra, and  $(\Omega^{(\bullet,\bullet)}, \sigma)$  is a covariant Hermitian structure, then  $*_\sigma$  is a left  $A$ -comodule map.

Reversing the classical order of construction we now define a metric in terms of the Hodge map.

DEFINITION 2.7. The metric associated to the Hermitian structure  $(\Omega^{(\bullet,\bullet)}, \sigma)$  is the unique map  $g_\sigma : \Omega^\bullet \times \Omega^\bullet \rightarrow A$  for which  $g(\Omega^k \times \Omega^l) = 0$ , for all  $k \neq l$ , and

$$g_\sigma(\omega, \nu) = *_\sigma(*_\sigma(\omega^*) \wedge \nu), \quad \omega, \nu \in \Omega^k.$$

### 2.3 CQH-COMPLEX AND CQH-HERMITIAN SPACES

We now introduce the definitions of the various compact quantum homogeneous spaces which form the framework for our investigation of Dolbeault–Dirac operators. These definitions detail a natural list of compatibility conditions between differential calculi, complex structures, and Hermitian structures on one hand and compact quantum group algebras on the other.

Throughout this subsection, and indeed the rest of the paper,  $A$  and  $H$  will denote Hopf algebras,  $\pi : A \rightarrow H$  a Hopf algebra map, and

$$B := A^{\text{co}(H)} = \{b \in A \mid b_{(1)} \otimes \pi(b_{(2)})\}$$

the associated quantum homogeneous space. See Appendix A for further details on quantum homogeneous spaces.

DEFINITION 2.8. A compact quantum calculus homogeneous space, or simply a CQH-calculus space is a triple

$$\mathbf{B} = (B = A^{\text{co}(H)}, \Omega^\bullet, \text{vol}),$$

comprised of the following elements:

1.  $B = A^{\text{co}(H)}$  a CMQGA-homogeneous space, for which  $A$  is a domain,
2.  $\Omega^\bullet$  a covariant differential  $*$ -calculus over  $B$ , finite-dimensional as an object in  ${}^A_B\text{Mod}_0$ , and of total degree  $m \in \mathbb{N}$ ,
3.  $\text{vol} : \Omega^m \simeq B$  an isomorphism in  ${}^A_B\text{Mod}_0$ , which is also a  $*$ -map, and with respect to which the integral

$$\int := \mathbf{h} \circ \text{vol} : \Omega^m \rightarrow \mathbb{C}$$

is closed, which is to say, satisfies  $\int d\omega = 0$ , for all  $\omega \in \Omega^{m-1}$ .

DEFINITION 2.9. A compact quantum homogeneous complex space, or a CQH-complex space, is a pair  $C = (B, \Omega^{(\bullet, \bullet)})$  where

1.  $B = (B = A^{\text{co}(H)}, \Omega^\bullet, \text{vol})$  is a CQH-calculus space,
2.  $\Omega^{(\bullet, \bullet)}$  is covariant complex structure for  $\Omega^\bullet$ .

The associated opposite CQH-complex space is the pair  $C^{\text{op}} := (B, \overline{\Omega}^{(\bullet, \bullet)})$

In the same spirit, a CQH-Hermitian space is a CQH-calculus space endowed with an Hermitian structure in a natural way. The interaction here, however, is a little more subtle.

DEFINITION 2.10. A compact quantum homogeneous Hermitian space, or alternatively a CQH-Hermitian space, is a triple  $(C, \Omega^{(\bullet, \bullet)}, \sigma)$  consisting of

1.  $C = (B, \Omega^{(\bullet, \bullet)})$  a CQH-complex space,
2.  $(\Omega^{(\bullet, \bullet)}, \sigma)$  a covariant Hermitian structure for the differential  $*$ -calculus  $\Omega^\bullet \in B$ ,
3.  $\text{vol} = *_\sigma|_{\Omega^{2n}} : \Omega^{2n} \rightarrow B$ , for  $2n$  the total degree of the constituent calculus of  $B$ ,
4. the associated metric  $g$  is positive definite, which is to say, for all  $\omega \in \Omega^\bullet$ ,

$$g_\sigma(\omega, \omega) \in B_{>0} := \left\{ \sum_{i=1}^l \lambda_i b_i^* b_i \neq 0 \mid b_i \in B, \lambda_i \in \mathbb{R}_{>0}, l \in \mathbb{N} \right\}.$$

As an immediate consequence of the definition, we get the following useful lemma.

LEMMA 2.11. For any CQH-Hermitian space  $(C, \Omega^{(\bullet, \bullet)}, \sigma)$ , it holds that  $z\omega$  and  $\omega z$  are non-zero, for all non-zero  $z \in B$ , and  $\omega \in \Omega^\bullet$ .

*Proof.* By the definition of  $g_\sigma$ , it holds that  $g_\sigma(\omega, \omega z) = g_\sigma(\omega, \omega)z$ . Since  $B$  is a domain, and  $g_\sigma$  is positive definite,  $g_\sigma(\omega, \omega)z$  is necessarily non-zero, implying that  $\omega z$  is non-zero. The proof that  $z\omega$  is non-zero is entirely analogous.  $\square$

For any CQH-Hermitian space, composing  $\mathbf{h}$  with  $g_\sigma$  gives a sesqui-linear map

$$\langle \cdot, \cdot \rangle : \Omega^{(\bullet, \bullet)} \times \Omega^{(\bullet, \bullet)} \rightarrow \mathbb{C}, \quad (\omega, \nu) \mapsto \mathbf{h} \circ g_\sigma(\omega, \nu).$$

We note that positivity of  $g$ , together with positivity of the Haar state  $\mathbf{h}$ , imply that  $\langle \cdot, \cdot \rangle$  is an inner product. We finish with the obvious extension of these definitions to the Kähler case.

DEFINITION 2.12. A compact quantum Kähler homogeneous space, or a CQH-Kähler space, is a CQH-Hermitian space  $K = (B, \Omega^{(\bullet, \bullet)}, \kappa)$  such that the covariant Hermitian structure  $(\Omega^{(\bullet, \bullet)}, \kappa)$  is a Kähler structure.

2.3.1 HODGE THEORY

We now recall the noncommutative generalisation of Hodge theory associated to any CQH-Hermitian space. Hodge decomposition will play a central role in our calculation of Laplace operator spectra, and the implied equivalence between harmonic forms and cohomology groups allows us to calculate the Dirac operator index in terms of the anti-holomorphic holomorphic Euler characteristic of the calculus, as defined in Definition 2.24.

The exterior derivatives  $d, \partial, \bar{\partial}$  are adjointable with respect to the inner product. Moreover, as established in [50, §5.3.3], their adjoints also are expressible in terms of the Hodge operator:

$$d^\dagger = - *_\sigma \circ d \circ *_\sigma, \quad \partial^\dagger = - *_\sigma \circ \bar{\partial} \circ *_\sigma, \quad \bar{\partial}^\dagger = - *_\sigma \circ \partial \circ *_\sigma.$$

For any Hermitian structure on a differential calculus, generalising the classical situation, we define the  $d$ -,  $\partial$ -, and  $\bar{\partial}$ -Dirac operators to be respectively,

$$D_d := d + d^\dagger, \quad D_\partial := \partial + \partial^\dagger, \quad D_{\bar{\partial}} := \bar{\partial} + \bar{\partial}^\dagger.$$

Moreover, we define the  $d$ -,  $\partial$ -, and  $\bar{\partial}$ -Laplace operators to be

$$\Delta_d := (d + d^\dagger)^2, \quad \Delta_\partial := (\partial + \partial^\dagger)^2, \quad \Delta_{\bar{\partial}} := (\bar{\partial} + \bar{\partial}^\dagger)^2.$$

We introduce  $d$ -harmonic,  $\partial$ -harmonic, and  $\bar{\partial}$ -harmonic forms, respectively, according to

$$\mathcal{H}_d := \ker(\Delta_d), \quad \mathcal{H}_\partial := \ker(\Delta_\partial), \quad \mathcal{H}_{\bar{\partial}} := \ker(\Delta_{\bar{\partial}}).$$

For the case of CQH-Hermitian space, it was shown in [12, Corollary 4.17] that the Dirac and Laplace operators are diagonalisable. Just as in the classical case, this allows us to show ([50, Lemma 6.1]) that the harmonic forms admit the following alternative presentation

$$\mathcal{H}_d = \ker(d) \cap \ker(d^\dagger), \quad \mathcal{H}_\partial = \ker(\partial) \cap \ker(\partial^\dagger), \quad \mathcal{H}_{\bar{\partial}} = \ker(\bar{\partial}) \cap \ker(\bar{\partial}^\dagger). \quad (4)$$

Moreover, as shown in [50, Theorem 6.2], building on earlier work in [35], diagonalisability also allows us to conclude the following noncommutative generalisation of Hodge decomposition for Hermitian manifolds.

**THEOREM 2.13 (Hodge decomposition).** *For a CQH-Hermitian space, direct sum decompositions of  $\Omega^\bullet$ , orthogonal with respect to  $\langle \cdot, \cdot \rangle$ , are given by*

1.  $\Omega^\bullet = \mathcal{H}_d \oplus d\Omega^\bullet \oplus d^\dagger\Omega^\bullet$ ,
2.  $\Omega^\bullet = \mathcal{H}_\partial \oplus \partial\Omega^\bullet \oplus \partial^\dagger\Omega^\bullet$ ,
3.  $\Omega^\bullet = \mathcal{H}_{\bar{\partial}} \oplus \bar{\partial}\Omega^\bullet \oplus \bar{\partial}^\dagger\Omega^\bullet$ .

From Hodge decomposition it is easy to conclude the following equivalence of harmonic forms and cohomologies, see [50, §6.2] for details.

COROLLARY 2.14. *It holds that*

1.  $\ker(d) = \mathcal{H}_d \oplus d\Omega^\bullet$ ,
2.  $\ker(\partial) = \mathcal{H}_\partial \oplus \partial\Omega^\bullet$ ,
3.  $\ker(\bar{\partial}) = \mathcal{H}_{\bar{\partial}} \oplus \bar{\partial}\Omega^\bullet$ ,

and so, we have isomorphisms

$$\mathcal{H}_d^k \rightarrow H_d^k, \quad \mathcal{H}_\partial^{(a,b)} \rightarrow H_\partial^{(a,b)}, \quad \mathcal{H}_{\bar{\partial}}^{(a,b)} \rightarrow H_{\bar{\partial}}^{(a,b)},$$

where  $H_d^k$ ,  $H_\partial^{(a,b)}$ , and  $H_{\bar{\partial}}^{(a,b)}$ , denote the cohomology groups of the complexes  $(\Omega^\bullet, d)$ ,  $(\Omega^\bullet, \partial)$ , and  $(\Omega^\bullet, \bar{\partial})$  respectively.

We finish with an easy but novel observation that directly generalises the classical situation.

LEMMA 2.15. *For any Hermitian structure  $(\Omega^{(\bullet,\bullet)}, \sigma)$ , the Hodge operator restricts to linear isomorphisms*

$$*_\sigma : \bar{\partial}\Omega^\bullet \rightarrow \partial^\dagger\Omega^\bullet, \quad *_\sigma : \partial\Omega^\bullet \rightarrow \bar{\partial}^\dagger\Omega^\bullet.$$

*Proof.* The fact that  $*_\sigma(\bar{\partial}\Omega^\bullet)$  is contained in  $\partial^\dagger\Omega^\bullet$  follows from

$$*_\sigma(\bar{\partial}\omega) = (-1)^k *_\sigma \circ \bar{\partial} \circ *_\sigma(*_\sigma(\omega)) = (-1)^{k+1} \partial^\dagger(*_\sigma(\omega)) \in \partial^\dagger\Omega^\bullet,$$

where  $\omega \in \Omega^k$ . Similarly, the fact that  $*_\sigma(\partial^\dagger\Omega^\bullet)$  is contained in  $\bar{\partial}\Omega^\bullet$  follows from the identity

$$*_\sigma(\partial^\dagger\omega) = -*_\sigma^2 \circ \bar{\partial} \circ *_\sigma(\omega) = (-1)^{2n-k+1} \bar{\partial}(*_\sigma(\omega)) \in \bar{\partial}\Omega^\bullet, \quad \text{for } \omega \in \Omega^k.$$

Thus, since  $*_\sigma : \Omega^\bullet \rightarrow \Omega^\bullet$  is a linear isomorphism, it must restrict to an isomorphism between  $\bar{\partial}\Omega^\bullet$  and  $\partial^\dagger\Omega^\bullet$ . The proof of the second isomorphism is completely analogous, and hence omitted.  $\square$

## 2.4 DOLBEAULT–DIRAC SPECTRAL TRIPLES

In this subsection we recall the definition of a spectral triple, Connes' notion of a noncommutative Riemannian spin manifold [8]. For a presentation of the classical Dolbeault–Dirac operator of an Hermitian manifold as a commutative spectral triple, see [22] or [18]. For a standard reference on the general theory of spectral triples, see [19] or [7]. Motivated by our construction of spectral triples from CQH-Hermitian spaces, we find it convenient to break the definition into two parts.

DEFINITION 2.16. A *bounded-commutator triple*  $(A, \mathcal{H}, D)$ , or simply a *BC-triple*, consists of a unital  $*$ -algebra  $A$ , a separable Hilbert space  $\mathcal{H}$  endowed with a faithful  $*$ -representation  $\rho : A \rightarrow \mathbb{B}(\mathcal{H})$ , and  $D : \text{dom}(D) \rightarrow \mathcal{H}$  a densely-defined self-adjoint operator on  $\mathcal{H}$ , such that

1.  $\rho(a)\text{dom}(D) \subseteq \text{dom}(D)$ , for all  $a \in A$ ,
2.  $[D, \rho(a)]$  is a bounded operator, for all  $a \in A$ .

An *even BC-triple* is a BC-triple  $(A, \mathcal{H}, D)$  together with a  $\mathbb{Z}_2$ -grading  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  of Hilbert spaces, with respect to which  $D$  is a degree 1 operator, and  $\rho(a)$  is a degree 0 operator, for each  $a \in A$ .

It follows the discussions of [12, §4] that every CQH-Hermitian space automatically gives a BC-triple. We now briefly recall the relevant details. Let  $\mathbf{H} = (\mathbf{B}, \Omega^{(\bullet, \bullet)}, \sigma)$  be a CQH-Hermitian space, with constituent quantum homogeneous space  $B = A^{\text{co}(H)}$ . We denote by  $L^2(\Omega^\bullet)$  the Hilbert space completion of  $\Omega^\bullet$  with respect to its inner product  $\langle \cdot, \cdot \rangle$ . By [50, Lemma 5.2.1] the  $\mathbb{N}_0^2$ -grading of the complex structure is orthogonal with respect to  $\langle \cdot, \cdot \rangle$ , hence we have the following decomposition of Hilbert spaces

$$L^2(\Omega^\bullet) = \bigoplus_{(a,b) \in \mathbb{N}_0^2} L^2(\Omega^{(a,b)}). \tag{5}$$

The constituent Hopf algebras of  $\mathbf{H}$  are CMQGAs, in particular, they are finitely generated. This implies that the Hilbert space  $L^2(\Omega^\bullet)$  is separable [12, Lemma 4.3]. Since  $B$  is a unital algebra, a faithful  $*$ -representation  $\rho : B \rightarrow \mathbb{B}(L^2(\Omega^\bullet))$  is given by

$$\rho(b)\omega := b\omega, \quad \text{for } \omega \in \Omega^\bullet, b \in B. \tag{6}$$

It follows from the basic theory of unbounded operators [57, §13] that the Dirac operators  $D_\partial$  and  $D_{\bar{\partial}}$ , as well as the Laplace operators  $\Delta_\partial$  and  $\Delta_{\bar{\partial}}$ , are essentially self-adjoint, see [12, Corollary 4.17] for details. By abuse of notation, we will not distinguish notationally between an operator and its closure. As observed in [12, §7.2], it now follows from [17, Proposition 2.1] that, for  $\text{dom}(D_\partial)$  and  $\text{dom}(D_{\bar{\partial}})$  the domain of the closures of the respective Dirac operators,

$$\rho(b)\text{dom}(D_\partial) \subseteq \text{dom}(D_\partial), \quad \rho(b)\text{dom}(D_{\bar{\partial}}) \subseteq \text{dom}(D_{\bar{\partial}}), \quad \text{for all } b \in B.$$

Moreover, boundedness of the commutators  $[D_\partial, m]$ , and  $[D_{\bar{\partial}}, m]$  follows easily from the Leibniz rule [12, Corollary 4.11]. Collecting these observations together gives us the following proposition.

PROPOSITION 2.17. *For a CQH-Hermitian space  $\mathbf{H} = (\mathbf{B}, \Omega^{(\bullet, \bullet)}, \sigma)$ , with constituent quantum homogeneous space  $B$ , a pair of BC-triples, which we call a Dolbeault–Dirac pair, is given by*

$$\left( B, L^2(\Omega^{(\bullet, 0)}), D_\partial \right), \quad \left( B, L^2(\Omega^{(0, \bullet)}), D_{\bar{\partial}} \right). \tag{7}$$

COROLLARY 2.18. *The Hilbert space decompositions*

$$L^2(\Omega^{(\bullet,0)}) = \bigoplus_{k \in 2\mathbb{N}_0} L^2(\Omega^{(k,0)}) \oplus \bigoplus_{k \in 2\mathbb{N}_0+1} L^2(\Omega^{(k,0)}),$$

$$L^2(\Omega^{(0,\bullet)}) = \bigoplus_{k \in 2\mathbb{N}_0} L^2(\Omega^{(0,k)}) \oplus \bigoplus_{k \in 2\mathbb{N}_0+1} L^2(\Omega^{(0,k)}),$$

define an even structure for the Dolbeault–Dirac pair of BC-triples.

Finally, we come to the definition of a spectral triple, which we present as a BC-triple whose unbounded operator  $D$  has compact resolvent.

DEFINITION 2.19. A *spectral triple* is a BC-triple  $(A, \mathcal{H}, D)$  such that

$$(1 + D^2)^{-1} \in \mathbb{K}(\mathcal{H}),$$

where  $\mathbb{K}(\mathcal{H})$  denotes the compact operators on  $\mathcal{H}$ .

As discussed in the introduction, it is not clear at present how to conclude the compact resolvent condition from the properties of a general CQH-Hermitian space. (See §7.4.4 for a brief discussion on how this might be achieved.) Hence, in our examples we resort to calculating the spectrum explicitly, and directly confirming the appropriate eigenvalue growth. We do, however, know two important general results. Firstly, since the  $*$ -map satisfies  $* \circ \Delta_{\bar{\partial}} = \Delta_{\partial} \circ *$  we have the following lemma.

LEMMA 2.20. *For a CQH-Hermitian space, the operator  $D_{\bar{\partial}} : \Omega^{(0,\bullet)} \rightarrow \Omega^{(0,\bullet)}$  has compact resolvent if and only if the operator  $D_{\partial} : \Omega^{(\bullet,0)} \rightarrow \Omega^{(\bullet,0)}$  has compact resolvent.*

Secondly, we know from [12, Corollary 4.17] that the operators  $D_{\partial}$  and  $D_{\bar{\partial}}$  are diagonalisable on  $L^2(\Omega^{\bullet})$ . Hence, a CQH-Hermitian space gives a spectral triple if and only if the eigenvalues  $\{\mu_n\}_{n \in \mathbb{N}_0}$  of  $\Delta_{\bar{\partial}}$  (with repetitions representing multiplicities) tend to infinity. When dealing with questions of eigenvalue growth, we find the following notation useful.

NOTATION 2.21. For  $T : \text{dom}(T) \rightarrow \mathcal{H}$  a diagonalisable operator on a separable Hilbert space  $\mathcal{H}$ , we write  $\sigma_P(T) \rightarrow \infty$  to denote that the eigenvalues of  $T$  (with repetitions representing multiplicities) tend to infinity.

### 2.5 ANALYTIC $K$ -HOMOLOGY AND THE ANTI-HOLOMORPHIC EULER CHARACTERISTIC

Let us now recall the basics of analytic  $K$ -homology, bearing in mind that one of the main aims of this paper is to establish a connection between  $K$ -theoretic index theory and the Dolbeault cohomology of noncommutative Hermitian structures.

DEFINITION 2.22. Let  $\mathcal{A}$  be a separable unital  $C^*$ -algebra. A *Fredholm module* over  $\mathcal{A}$  is a triple  $(\mathcal{H}, \rho, F)$ , where  $\mathcal{H}$  is a Hilbert space,  $\rho : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$  is a  $*$ -representation, and  $F : \mathcal{H} \rightarrow \mathcal{H}$  a bounded linear operator, such that

$$F^2 - 1, \quad F - F^*, \quad [F, \rho(a)],$$

are all compact operators, for any  $a \in \mathcal{A}$ . An *even Fredholm module* is a Fredholm module  $(\rho, F, \mathcal{H})$  together with a  $\mathbb{Z}_2$ -grading  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  of Hilbert spaces, with respect to which  $F$  is a degree 1 operator, and  $\rho(a)$  is a degree 0 operator, for each  $a \in \mathcal{A}$ .

The *direct sum* of two even Fredholm modules is formed by taking the direct sum of Hilbert spaces, representations, and operators. For  $(\mathcal{H}, \rho, F)$  an even Fredholm module, and  $U : \mathcal{H} \rightarrow \mathcal{H}'$  a degree 0 unitary transformation, the triple  $(\mathcal{H}', U^* \rho U, U^* F U)$  is again a Fredholm module. This defines an equivalence relation on Fredholm modules over  $\mathcal{A}$ , which we call *unitary equivalence*. We say that a norm continuous family of Fredholm modules  $(\rho, \mathcal{H}, F_t)$ , for  $t \in [0, 1]$ , defines an *operator homotopy* between the two Fredholm modules  $(\rho, \mathcal{H}, F_0)$  and  $(\rho, \mathcal{H}, F_1)$ .

DEFINITION 2.23. The *K-homology* group  $K^0(\mathcal{A})$  of a  $C^*$ -algebra  $\mathcal{A}$  is the abelian group with one generator for each unitary equivalence class of even Fredholm modules, subject to the following relations: for any two even Fredholm modules  $x_0, x_1$ ,

1.  $[x_0] = [x_1]$  if there exists an operator homotopy between  $x_0$  and  $x_1$ ,
2.  $[x_0 \oplus x_1] = [x_0] + [x_1]$ , where  $+$  denotes addition in  $K^0(\mathcal{A})$ .

Denoting the Fredholm operator  $F_+ := F|_{\mathcal{H}_+} : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ , a well-defined group homomorphism is given by

$$\text{Index} : K^0(\mathcal{A}) \rightarrow \mathbb{Z}, \quad [F]_K \mapsto \text{Index}(F_+).$$

Spectral triples are important primarily because they provide unbounded representatives for  $K$ -homology classes. For a spectral triple  $(A, \mathcal{H}, D)$ , its *bounded transform* is the operator

$$\mathfrak{b}(D) := \frac{D}{\sqrt{1 + D^2}} \in \mathbb{B}(\mathcal{H}),$$

defined via the functional calculus. Denoting by  $\overline{A}$  the closure of  $\rho(A)$  with respect to the operator topology of  $\mathbb{B}(\mathcal{H})$ , an even Fredholm module over  $\overline{A}$  is given by  $(\mathcal{H}, \rho, \mathfrak{b}(D))$ . (See [6] for details.)

For a classical Hermitian manifold, the index of the Dolbeault–Dirac operator, and the associated index of its  $K$ -homology class, are equal to the holomorphic Euler characteristic of the manifold. This picture extends to the noncommutative setting.

DEFINITION 2.24. Let  $\mathbf{H} := (\mathbb{B}, \Omega^{\bullet, \bullet}, \sigma)$  be a CQH-Hermitian space with constituent differential calculus  $\Omega^\bullet \in \mathbb{B}$  of total degree  $2n$ . The *anti-holomorphic Euler characteristic* of  $\mathbf{H}$  is given by

$$\chi_{\bar{\partial}} := \sum_{k=1}^n (-1)^k \dim_{\mathbb{C}} (H^{(0,k)}).$$

The following proposition now allows us to conclude non-triviality of the  $K$ -homology class of a Dolbeault–Dirac spectral triple from non-vanishing of the Euler characteristic.

PROPOSITION 2.25. [12, Theorem 5.4] For a CQH-Hermitian space  $(\mathbb{B}, \Omega^{\bullet, \bullet}, \sigma)$ , with an associated pair of Dolbeault–Dirac spectral triples, it holds that

$$\text{Index}(\mathfrak{b}(D_{\partial})) = \text{Index}(\mathfrak{b}(D_{\bar{\partial}})) = \chi_{\bar{\partial}}.$$

REMARK 2.26. The calculation of cohomology groups can in general be quite difficult. However, in the Kähler setting there exists a powerful noncommutative generalisation of the Kodaira vanishing theorem [52]. In the more specialised Fano setting (whose definition directly generalises the classical definition) this implies vanishing of all higher cohomologies, implying that the anti-holomorphic Euler characteristic is equal to  $\dim_{\mathbb{C}} (H^{(0,0)})$ . Since  $H^{(0,0)}$  always contains the identity element of  $B$ , this implies that the  $K$ -homology class of  $\mathfrak{b}(D_{\bar{\partial}})$  is always non-zero in the Fano setting.

### 3 CQH-COMPLEX SPACES OF GELFAND TYPE

In this section we begin our examination of the spectral behaviour of the Laplacian operator associated to a CQH-Hermitian space. Our strategy is to exploit the subtle interactions between the Laplace spectrum, Hodge decomposition, and comodule multiplicities. This leads us to focus on a special type of CQH-Hermitian space, Gelfand type spaces, for which the problem is significantly more tractable. The work of this section underlies our investigation of the Drinfeld–Jimbo case in §4.

#### 3.1 HODGE DECOMPOSITION OF THE LAPLACIAN

We decompose the Laplacian with respect to Hodge decomposition and then, in Corollary 3.6, show that  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$  if and only if  $\sigma_P(\bar{\partial}\bar{\partial}^\dagger) \rightarrow \infty$ . We emphasise the fact that nowhere in this subsection do we make any assumption on multiplicities, all results hold for a general CQH-Hermitian space.

LEMMA 3.1. For any CQH-Hermitian space, the Laplacian operator  $\Delta_{\bar{\partial}}$  admits a direct sum decomposition with respect to Hodge decomposition  $\Omega^\bullet = \mathcal{H}^\bullet \oplus \bar{\partial}\Omega^\bullet \oplus \bar{\partial}^\dagger\Omega^\bullet$ , namely

$$\Delta_{\bar{\partial}} = 0 \oplus \bar{\partial}\bar{\partial}^\dagger \oplus \bar{\partial}^\dagger\bar{\partial}.$$

*Proof.* By definition,  $\Delta_{\bar{\partial}}$  restricts to the zero map on the harmonic forms  $\mathcal{H}^\bullet$ . For a non-harmonic  $\omega \in \Omega^\bullet$ , we have

$$\Delta_{\bar{\partial}}(\bar{\partial}\omega) = (\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial})(\bar{\partial}\omega) = \bar{\partial}\bar{\partial}^\dagger(\bar{\partial}\omega) \in \bar{\partial}\Omega^\bullet.$$

Thus we see that  $\bar{\partial}\Omega^\bullet$  is closed under the action of the Laplacian, and moreover, that

$$\Delta_{\bar{\partial}}|_{\bar{\partial}\Omega^\bullet} = \bar{\partial}\bar{\partial}^\dagger.$$

Similarly,  $\bar{\partial}^\dagger\Omega^\bullet$  is closed under the action of  $\Delta_{\bar{\partial}}$ , and  $\Delta_{\bar{\partial}}$  restricts to the operator  $\bar{\partial}^\dagger\bar{\partial}$  on  $\bar{\partial}^\dagger\Omega^\bullet$ . □

LEMMA 3.2. *For any CQH-Hermitian space, it holds that  $[\Delta_{\bar{\partial}}, \bar{\partial}] = [\Delta_{\bar{\partial}}, \bar{\partial}^\dagger] = 0$ .*

*Proof.* Starting with the first commutator, for  $\omega \in \Omega^\bullet$ , we see that

$$[\Delta_{\bar{\partial}}, \bar{\partial}](\omega) = \Delta_{\bar{\partial}} \circ \bar{\partial}(\omega) - \bar{\partial} \circ \Delta_{\bar{\partial}}(\omega) = \bar{\partial}\bar{\partial}^\dagger\bar{\partial}(\omega) - \bar{\partial}\bar{\partial}^\dagger\bar{\partial}(\omega) = 0.$$

Vanishing of the commutator  $[\Delta_{\bar{\partial}}, \bar{\partial}^\dagger]$  is established similarly. □

PROPOSITION 3.3. *For a CQH-Hermitian space, with constituent quantum homogeneous space  $B = A^{\text{co}(H)}$ , left  $A$ -comodule isomorphisms are given by*

1.  $\bar{\partial} : \bar{\partial}^\dagger\Omega^\bullet \rightarrow \bar{\partial}\Omega^\bullet$ ,
2.  $\bar{\partial}^\dagger : \bar{\partial}\Omega^\bullet \rightarrow \bar{\partial}^\dagger\Omega^\bullet$ .

*Proof.* Since by assumption the calculus  $\Omega^\bullet$  and the Hermitian structure  $(\Omega^{(\bullet,\bullet)}, \sigma)$  are covariant, the maps  $\bar{\partial}$  and  $\bar{\partial}^\dagger$  are left  $A$ -comodule maps. Hence, it suffices to show that  $\bar{\partial}$  and  $\bar{\partial}^\dagger$  are linear isomorphisms. By Hodge decomposition  $\bar{\partial}^\dagger\Omega^{(0,\bullet)}$  is orthogonal to the space of harmonic forms, and so, the kernel of the restriction of the Laplacian to  $\bar{\partial}^\dagger\Omega^{(0,\bullet)}$  is trivial. Now Lemma 3.1 tells us that the Laplacian restricts to  $\bar{\partial}^\dagger\bar{\partial}$  on the subspace  $\bar{\partial}^\dagger\Omega^{(0,\bullet)}$ , giving us that

$$\ker\left(\bar{\partial}^\dagger\bar{\partial}|_{\bar{\partial}^\dagger\Omega^{(0,\bullet)}}\right) = 0.$$

Similarly, the kernel of the restriction of  $\bar{\partial}\bar{\partial}^\dagger$  to the subspace  $\bar{\partial}\Omega^{(0,\bullet)}$  is trivial. Since the Laplacian is a self-adjoint operator, its restrictions to the subspaces  $\bar{\partial}\Omega^{(0,\bullet)}$  and  $\bar{\partial}^\dagger\Omega^{(0,\bullet)}$  are diagonalisable. Thus, both operators

$$\bar{\partial}^\dagger\bar{\partial} : \bar{\partial}^\dagger\Omega^\bullet \rightarrow \bar{\partial}^\dagger\Omega^\bullet, \quad \bar{\partial}\bar{\partial}^\dagger : \bar{\partial}\Omega^\bullet \rightarrow \bar{\partial}\Omega^\bullet,$$

are diagonalisable with trivial 0-eigenspaces. From this we see that we can construct explicit inverses for  $\bar{\partial}$  and  $\bar{\partial}^\dagger$ , implying that both maps are isomorphisms as required. □

We now discuss the relationship between the eigenvalues of  $\Delta_{\bar{\partial}}$  and its decomposition with respect to Hodge decomposition. In doing so, we find the following notation useful:

NOTATION 3.4. For  $f : W \rightarrow W$  a linear operator on a vector space  $W$ , we denote by  $E(\mu, f)$  the eigenspace of an eigenvalue  $\mu$  of  $f$ .

COROLLARY 3.5. *The non-zero eigenvalues of the following three operators coincide:*

1.  $\Delta_{\bar{\partial}} : \Omega^{(0, \bullet)} \rightarrow \Omega^{(0, \bullet)}$ ,
2.  $\bar{\partial}^\dagger \bar{\partial} : \bar{\partial}^\dagger \Omega^{(0, \bullet)} \rightarrow \bar{\partial}^\dagger \Omega^{(0, \bullet)}$ ,
3.  $\bar{\partial} \bar{\partial}^\dagger : \bar{\partial} \Omega^{(0, \bullet)} \rightarrow \bar{\partial} \Omega^{(0, \bullet)}$ .

Moreover, for any eigenvalue  $\mu$  of  $\Delta_{\bar{\partial}}$  with finite multiplicity, it holds that

$$\dim_{\mathbb{C}} (E(\mu, \Delta_{\bar{\partial}})) = 2 \dim_{\mathbb{C}} (E(\mu, \bar{\partial}^\dagger \bar{\partial} |_{\bar{\partial}^\dagger \Omega^\bullet})) = 2 \dim_{\mathbb{C}} (E(\mu, \bar{\partial} \bar{\partial}^\dagger |_{\bar{\partial} \Omega^\bullet})). \quad (8)$$

*Proof.* The decomposition of  $\Delta_{\bar{\partial}}$  with respect to Hodge decomposition, as given in Lemma 3.1 above, implies that its non-zero eigenvalues are equal to the union of the eigenvalues of  $\bar{\partial}^\dagger \bar{\partial} |_{\bar{\partial}^\dagger \Omega^\bullet}$  and the eigenvalues of  $\bar{\partial} \bar{\partial}^\dagger |_{\bar{\partial} \Omega^\bullet}$ . Hence, for any eigenvalue  $\mu$ ,

$$E(\mu, \Delta_{\bar{\partial}}) = E(\mu, \bar{\partial}^\dagger \bar{\partial} |_{\bar{\partial}^\dagger \Omega^\bullet}) \oplus E(\mu, \bar{\partial} \bar{\partial}^\dagger |_{\bar{\partial} \Omega^\bullet}).$$

Moreover, Lemma 3.2 and Proposition 3.3 together imply that the set of eigenvalues of the operators  $\bar{\partial}^\dagger \bar{\partial} |_{\bar{\partial}^\dagger \Omega^\bullet}$  and  $\bar{\partial} \bar{\partial}^\dagger |_{\bar{\partial} \Omega^\bullet}$  coincide and have the same multiplicity. The fact that these three operators have a common set of non-zero eigenvalues now follows, as does that identity for multiplicities given in (8).  $\square$

An immediate consequence of the above corollary is now given. In short, it says that verifying the compact resolvent for the Laplacian can be reduced to verifying the condition for either  $\bar{\partial}^\dagger \bar{\partial} |_{\bar{\partial}^\dagger \Omega^\bullet}$  or  $\bar{\partial} \bar{\partial}^\dagger |_{\bar{\partial} \Omega^\bullet}$ .

COROLLARY 3.6. *Assuming finite-dimensional cohomologies, the following three conditions are equivalent:*

1.  $\sigma_P(\Delta_{\bar{\partial}} : \Omega^{(0, \bullet)} \rightarrow \Omega^{(0, \bullet)}) \rightarrow \infty$ ,
2.  $\sigma_P(\bar{\partial} \bar{\partial}^\dagger : \bar{\partial} \Omega^{(0, \bullet)} \rightarrow \bar{\partial} \Omega^{(0, \bullet)}) \rightarrow \infty$ ,
3.  $\sigma_P(\bar{\partial}^\dagger \bar{\partial} : \bar{\partial}^\dagger \Omega^{(0, \bullet)} \rightarrow \bar{\partial}^\dagger \Omega^{(0, \bullet)}) \rightarrow \infty$ .

We finish with an easy observation which follows from the equality of Laplacians in the Kähler case. While not needed elsewhere, the identity is interesting as an alternative description of the action of the Laplacian  $\Delta_{\bar{\partial}}$  on anti-holomorphic forms.

LEMMA 3.7. *For a CQH-Kähler space  $K$ , the operators  $\Delta_{\bar{\partial}}$  and  $\partial^\dagger\partial$  coincide on  $\Omega^{(0,\bullet)}$ .*

*Proof.* From (3) we know that in the Kähler case  $\Delta_{\bar{\partial}} = \Delta_\partial$ . Thus the identity follows from the fact that  $\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$  restricts to  $\partial^\dagger\partial$  on  $\Omega^{(0,\bullet)}$ .  $\square$

### 3.2 CQH-COMPLEX SPACES OF GELFAND TYPE

In this subsection we introduce CQH-complex spaces of Gelfand type. As shown in Lemma 3.11 below, the Laplacians associated to CQH-Hermitian spaces of Gelfand type admit a particularly nice diagonalisation, which makes the calculation of their spectrum significantly more tractable. We begin by introducing graded multiplicity-free comodules as a convenient abstract framework in which to discuss comodule multiplicities for covariant calculi.

DEFINITION 3.8.

1. We say that a left  $A$ -comodule  $C$  is *multiplicity-free* if for any irreducible left  $A$ -comodule  $V$ , it holds that

$$\dim_{\mathbb{C}}(\text{Hom}^A(V, C)) = 1.$$

2. A *graded left  $A$ -comodule*  $P$  is a left  $A$ -comodule, together with an  $\mathbb{N}_0$ -algebra grading  $P = \bigoplus_{m \in \mathbb{N}_0} P_m$ , such that each  $P_m$  is a left  $A$ -sub-comodule of  $P$ , or equivalently, such that the decomposition  $P = \bigoplus_{m \in \mathbb{N}_0} P_m$  is a decomposition in the category  ${}^A\text{Mod}$ .
3. We say that a graded comodule  $P = \bigoplus_{m \in \mathbb{N}_0} P_m$  is *graded multiplicity-free* if the left  $A$ -comodule  $P_m$  is multiplicity-free, for all  $m \in \mathbb{N}_0$ .

An elementary, but very useful, observation about multiplicity-free comodules is presented in the following lemma. The proof is a direct application of Schur’s lemma, and so, is omitted.

LEMMA 3.9. *Let  $P$  be a graded multiplicity-free left  $A$ -comodule,  $\varphi : P \rightarrow P$  a degree 0 left  $A$ -comodule map, and  $V$  an irreducible  $A$ -sub-comodule of  $P$ . Then  $\varphi$  acts on  $V$  as a scalar multiple of the identity. Consequently,  $\varphi$  is diagonalisable on  $P$ .*

We now present the notion of a CQH-complex space of Gelfand type. The definition is given in terms of graded multiplicity-free comodules, and is followed by a direct application of Lemma 3.9.

DEFINITION 3.10. We say that a CQH-complex space  $\mathbf{C} = (\mathbf{B}, \Omega^{(\bullet, \bullet)})$  is of *Gelfand type* if  $\Omega^{(0, \bullet)}$  is a graded multiplicity-free left  $A$ -comodule. We say that a CQH-Hermitian space is of *Gelfand type* if its constituent CQH-complex space is of Gelfand type.

Since the Laplacian is a self-adjoint operator, we already know that it is diagonalisable. However, since it is a degree 0 operator, Corollary 3.9 implies the following stronger result.

LEMMA 3.11. *For a CQH-Hermitian space of Gelfand type, the Laplacian  $\Delta_{\bar{\partial}}$  acts on every irreducible left  $A$ -sub-comodule of  $\Omega^{(0, \bullet)}$  as a scalar multiple of the identity.*

We now use the various symmetries of the Dolbeault double complex to find four equivalent formulations of Gelfand type. We begin with the symmetry induced by the  $*$ -map of the calculus.

LEMMA 3.12. *For  $\mathbf{C} = (\mathbf{B}, \Omega^{(\bullet, \bullet)})$  a CQH-complex space, the following conditions are equivalent:*

1.  $\mathbf{C}$  is of Gelfand type,
2.  $\mathbf{C}^{\text{op}}$ , the opposite CQH-complex space, is of Gelfand type,
3. the graded left  $A$ -comodule  $\Omega^{(\bullet, 0)}$  is graded multiplicity-free.

*Proof.*

- 1  $\Leftrightarrow$  3 The image of an irreducible  $A$ -sub-comodule of  $\Omega^{\bullet}$  under the  $*$ -map is again an irreducible  $A$ -sub-comodule. Thus if  $\Omega^{(0, k)} \simeq \bigoplus_{\alpha} \Omega_{\alpha}^{(0, k)}$  denotes a decomposition of  $\Omega^{(0, k)}$  into irreducible  $A$ -sub-comodules, then a decomposition of  $\Omega^{(k, 0)}$  into irreducible  $A$ -sub-comodules is given by

$$\Omega^{(k, 0)} \simeq \bigoplus_{\alpha} (\Omega_{\alpha}^{(0, k)})^*.$$

Thus we see that  $\Omega^{(k, 0)}$  is multiplicity-free if and only if  $\Omega^{(0, k)}$  is multiplicity-free, which is to say  $\mathbf{C}$  is of Gelfand type if and only if  $\mathbf{C}^{\text{op}}$  is of Gelfand type.

- 2  $\Leftrightarrow$  3 The equivalence of 2 and 3 follows directly from the definition of opposite complex structure and the definition of Gelfand type.  $\square$

If we additionally assume the existence of a covariant Hermitian structure, then the symmetries induced by the Hodge map imply two additional equivalent formulations of Gelfand type. Note that in this case the Gelfand condition can be verified on any of the outer edges of the Hodge diamond.

LEMMA 3.13. For  $\mathbf{H} = (\mathbf{B}, \Omega^{(\bullet, \bullet)}, \sigma)$  a CQH-Hermitian space, the following conditions are equivalent:

1.  $\mathbf{H}$  is of Gelfand type,
2. the graded left  $A$ -comodule  $\Omega^{(\bullet, n)}$  is graded multiplicity-free,
3. the graded left  $A$ -comodule  $\Omega^{(n, \bullet)}$  is graded multiplicity-free.

*Proof.*

- 1  $\Leftrightarrow$  2: By Lemma 2.6, the Hodge map  $*_\sigma$  associated to  $\sigma$  restricts to a left  $A$ -comodule isomorphism  $*_\sigma : \Omega^{(0, \bullet)} \simeq \Omega^{(\bullet, n)}$ . Thus  $\Omega^{(\bullet, n)}$  is graded multiplicity-free if and only if  $\Omega^{(0, \bullet)}$  is graded multiplicity-free, which is to say, if and only if  $\mathbf{H}$  is of Gelfand type.
- 1  $\Leftrightarrow$  3: By Lemma 7.7 above,  $\mathbf{H}$  is of Gelfand type, if and only if  $\Omega^{(\bullet, 0)}$  is graded multiplicity-free. Moreover, using the Hodge map, just as above, we see that  $\Omega^{(n, \bullet)}$  is graded multiplicity-free if and only if  $\Omega^{(0, \bullet)}$  is graded multiplicity-free, giving the required equivalence.  $\square$

#### 4 DRINFELD–JIMBO QUANTUM GROUPS AND CQH-HERMITIAN SPACES

From now on we restrict to the case of Drinfeld–Jimbo quantised enveloping algebras  $U_q(\mathfrak{g})$  and their quantised coordinate algebras  $\mathcal{O}_q(G)$ , as presented in Appendix B. This allows us to exploit the associated highest weight structure on the category  $U_q(\mathfrak{g})$ -modules of type 1, leading to the construction of canonical sequences of eigenvalues in  $\sigma_P(\Delta_{\bar{\theta}})$ . In the next section, under suitable assumptions, we decompose  $\sigma_P(\Delta_{\bar{\theta}})$  into unions of such sequences allowing us to give sufficient and necessary conditions for  $\sigma_P(\Delta_{\bar{\theta}}) \rightarrow \infty$ . Note that throughout this section, we assume the notation and conventions presented in Appendix B.

##### 4.1 HIGHEST AND LOWEST WEIGHT VECTORS

In this subsection we examine the behaviour of highest weight vectors in an  $\mathcal{O}_q(G)$ -comodule algebra  $P$ . We show that the space of highest weight elements is a multiplicative submonoid of  $P$ . Moreover, we show that in the graded multiplicity-free case, any pair of highest weight vectors commute up to a scalar. The algebra of anti-holomorphic forms  $\Omega^{(0, \bullet)}$  of a covariant complex structure is then presented as a motivating example.

DEFINITION 4.1. For any  $\mathcal{Z} \in \mathcal{O}_q(G)\text{Mod}$ , with respect to the  $U_q(\mathfrak{g})$ -action induced by the dual pairing  $U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \rightarrow \mathbb{C}$ , we denote

$$\begin{aligned} \mathcal{Z}_{\text{hw}} &:= \{f \in \mathcal{Z} \mid f \text{ is a highest weight vector of } \mathcal{Z}\}, \\ \mathcal{Z}_{\text{lw}} &:= \{f \in \mathcal{Z} \mid f \text{ is a lowest weight vector of } \mathcal{Z}\}. \end{aligned}$$

LEMMA 4.2. *For a left  $\mathcal{O}_q(G)$ -comodule algebra  $P$ , which is to say a monoid object in the category  ${}^{\mathcal{O}_q(G)}\text{Mod}$ , it holds that*

1.  $\text{wt}(ab) = \text{wt}(ba) = \text{wt}(a) + \text{wt}(b)$ , for all  $a, b \in P$ ,
2. *the multiplication of  $P$  restricts to the structure of a monoid on  $P_{\text{hw}}$  and  $P_{\text{lw}}$ .*

*Proof.* Let  $a, b \in P_{\text{hw}}$ . For  $k = 1, \dots, r$ ,

$$E_k \triangleright (ab) = (K_k \triangleright a)(E_k \triangleright b) + (E_k \triangleright a)(1 \triangleright b) = 0.$$

Moreover,  $K_k \triangleright (ab) = (K_k \triangleright a)(K_k \triangleright b) = q^{(\text{wt}(a), \alpha_k) + (\text{wt}(b), \alpha_k)} ab$ . Thus  $ab \in P_{\text{hw}}$ . To show that we have a monoid it remains to show that the unit of  $P$  is contained in  $P_{\text{hw}}$ . This follows directly from the properties of a dual pairing since  $X \triangleright 1 = \varepsilon(X)1$ .  $\square$

In this subsection we discuss graded multiplicity-free comodules for the Drinfeld–Jimbo quantum groups. This allows us to produce a collection of identities describing proportionality relations between certain highest weight vectors. For the special case of CQH-Hermitian spaces, these identities will be our main tool for calculating the spectrum of a Dolbeault–Dirac operator. The proof of the following lemma is elementary and hence omitted.

LEMMA 4.3. *Let  $\mathcal{A} \simeq \bigoplus_{i \in \mathbb{N}_0} \mathcal{A}_i$  be a graded multiplicity-free  $\mathcal{O}_q(G)$ -comodule, and  $a, b \in \mathcal{A}_{\text{hw}}$  such that*

1.  $|a| = |b|$ ,
2.  $\text{wt}(a) = \text{wt}(b)$ ,
3.  $b \neq 0$ .

*Then there exists a uniquely defined scalar  $C \in \mathbb{C}$  such that  $a = Cb$ .*

DEFINITION 4.4. A *graded  $A$ -comodule algebra* is a graded  $A$ -comodule  $P = \bigoplus_{i \in \mathbb{N}_0} P_i$ , which is also an  $A$ -comodule algebra, such that the grading and multiplication of  $P$  combine to give it the structure of a graded algebra.

The following result now follows immediately from Lemma 4.3.

COROLLARY 4.5. *Let  $P$  be a graded multiplicity-free  $A$ -comodule algebra. If  $c, d \in P_{\text{hw}}$  such that  $dc \neq 0$ , then there exists a unique scalar  $C \in \mathbb{C}$  such that*

$$cd = C dc.$$

EXAMPLE 4.6. The simplest example of a left  $\mathcal{O}_q(G)$ -comodule algebra is a quantum homogeneous space  $B = \mathcal{O}_q(G)^{\text{co}(H)}$ . As discussed in §7, the highest weight monoid is always finitely generated as a monoid. However, as we will see in §6 and §7 the number of generators has strong consequences for the complexity of the spectrum of the Dolbeault–Dirac operators constructed to a CQH-Hermitian spaces over  $B$ .

EXAMPLE 4.7. Each of the following objects are left  $\mathcal{O}_q(G)$ -comodule algebras

$$\Omega^\bullet, \quad \Omega^{(0,\bullet)}, \quad \Omega^{(\bullet,0)}.$$

Lemma 4.2 implies that the respective multiplications restrict to monoid structures on the sets

$$\Omega_{\text{hw}}^\bullet, \quad \Omega_{\text{hw}}^{(0,\bullet)}, \quad \Omega_{\text{hw}}^{(\bullet,0)}.$$

Note that by restriction of the monoid structure of  $\Omega_{\text{hw}}^\bullet$ , we have the following monoid actions

$$\Omega_{\text{hw}}^{(\bullet,0)} \times \Omega_{\text{hw}}^{(\bullet,n)} \rightarrow \Omega_{\text{hw}}^{(\bullet,n)} \quad \Omega_{\text{hw}}^{(0,\bullet)} \times \Omega_{\text{hw}}^{(n,\bullet)} \rightarrow \Omega_{\text{hw}}^{(n,\bullet)}.$$

Moreover, restricting to highest weight forms of degree 0, we have the following monoid action

$$B_{\text{hw}} \times \Omega_{\text{hw}}^{(a,b)} \rightarrow \Omega_{\text{hw}}^{(a,b)}, \quad \text{for all } (a, b) \in \mathbb{N}_0^2.$$

REMARK 4.8. The previous examples can be extended to a more formal general setting using the language of relative Hopf modules over comodule algebras. Let  $A$  be a Hopf algebra, and  $(P, \Delta_P)$  a left  $A$ -comodule algebra. A *relative Hopf  $P$ -module algebra*  $N$  is a left  $A$ -comodule  $(N, \Delta_N)$ , which is also a module over the algebra  $P$ , satisfying the compatibility condition

$$\Delta_N(pn) = \Delta_P(p)\Delta_N(n), \quad \text{for all } p \in P, n \in N.$$

Alternatively, considering  $P$  as a monoid object in the category  ${}^A\text{Mod}$ , a relative Hopf  $P$ -module algebra is just a module object over  $P$  in the category  ${}^A\text{Mod}$ . It is instructive to observe that any object in  ${}^{\mathcal{O}_q(G)}_{\mathcal{B}}\text{Mod}$  is a relative Hopf  $B$ -module algebra. Following the same argument as in Lemma 4.2, one can now establish the following result, formalising the actions appearing in Example 4.7.

LEMMA 4.9. *For  $P$  a left  $\mathcal{O}_q(G)$ -comodule algebra, and  $N$  a relative Hopf  $P$ -module algebra, the action of  $P$  on  $N$  restricts to the structure of a  $P_{\text{hw}}$ -space*

$$P_{\text{hw}} \times N_{\text{hw}} \rightarrow N_{\text{hw}}, \quad (p, n) \mapsto pn.$$

## 4.2 CQH-COMPLEX SPACES AND LEIBNIZ CONSTANTS

In this subsection we apply the general results of the previous subsection to the CQH-complex spaces of Gelfand type. As a result we identify a collection of constants, which we call Leibniz constants, intrinsic to the structure of the calculus. First however, we prove a useful result relating highest weight vectors and the  $*$ -map of a covariant  $*$ -calculus.

LEMMA 4.10. *For a CQH-complex space  $C = (B, \Omega^{(\bullet,\bullet)})$ , the  $*$ -map of the constituent calculus  $\Omega^\bullet \in B$  restricts to a bijection between  $\Omega_{\text{hw}}^\bullet$  and  $\Omega_{\text{lw}}^\bullet$ . Moreover, the bijection is an anti-monoid map.*

*Proof.* For any  $\omega \in \Omega_{\text{hw}}^\bullet$ , we have, for  $k = 1, \dots, \text{rank}(\mathfrak{g})$ ,

$$\begin{aligned} F_k \triangleright \omega^* &= \langle S(F_k), \omega_{(-1)}^* \rangle \omega_{(0)}^* = \overline{\langle S^2(F_k)^*, \omega_{(-1)} \rangle} \omega_{(0)}^* \\ &= (\langle S(S^{-3}(F_k^*)), \omega_{(-1)} \rangle \omega_{(0)})^* \\ &= (S^{-3}(F_k^*) \triangleright \omega)^*. \end{aligned}$$

A direct calculation confirms that  $S^{-3}(F_k^*) = -q^{-2}K_k^{-2}E_k$ . Thus, since  $\omega$  is by assumption a highest weight vector, we must have that  $F_k \triangleright \omega^* = 0$ . Analogously, it can be shown that  $K_k \triangleright \omega^* = q^{-\text{wt}_k(\omega)}\omega^*$ . Hence,  $\omega$  is a lowest weight element of  $\Omega^\bullet$ . The proof that  $*$  sends lowest weight forms to highest weight forms is analogous. Thus since the  $*$ -map is an involution, it must induce a bijection between highest and lowest weight forms. Moreover, since the  $*$ -map is an anti-algebra map, it restricts to an anti-monoid map between  $\Omega_{\text{hw}}^\bullet$  and  $\Omega_{\text{lw}}^\bullet$ .  $\square$

We now restrict to the Gelfand case, beginning with the following lemma which is a direct consequence of Corollary 4.5.

LEMMA 4.11. *Let  $H = (B, \Omega^{\bullet, \bullet})$  be a CQH-complex space of Gelfand type, and  $\omega, \nu \in \Omega_{\text{hw}}^{(0, \bullet)}$  such that  $\nu \wedge \omega \neq 0$ . Then there exists a uniquely defined scalar  $C$  satisfying*

$$\omega \wedge \nu = C \nu \wedge \omega.$$

A very special, but very important, case of this result are the Leibniz commutation constants presented in the corollary below. These constants play a central part in our later description of the spectrum of the Dolbeault–Dirac operator associated to a CQH-space Hermitian space.

COROLLARY 4.12. *For every non-harmonic element  $z \in B_{\text{hw}}$ , there exist non-zero constants  $\lambda_z, \zeta_z \in \mathbb{C}$ , uniquely defined by*

$$(\partial z)z = \lambda_z z \partial z, \quad (\bar{\partial} z)z = \zeta_z z \bar{\partial} z.$$

*We call  $\lambda_z$ , and  $\zeta_z$ , the holomorphic Leibniz constant, and anti-holomorphic Leibniz constant, of  $z$  respectively.*

*Proof.* By Lemma 2.11, the product  $z\bar{\partial}z$  is non-zero. Since the complex structure is of Gelfand type by assumption, Lemma 4.11 implies the existence and uniqueness of the constants  $\lambda_z$  and  $\zeta_z$ .  $\square$

COROLLARY 4.13. *For any  $z \in B_{\text{hw}}$ , with Leibniz constant  $\lambda_z$ , and  $l \in \mathbb{N}_0$ , it holds that*

1.  $\partial z^l = (l)_{\lambda_z} z^{l-1} \partial z,$
2.  $\bar{\partial} z^l = (l)_{\zeta_z} z^{l-1} \bar{\partial} z,$

where  $(l)_{\lambda_z}$  and  $(l)_{\zeta_z}$  are quantum integers, as presented in Appendix B.1.

*Proof.* Let us assume that the required identity holds for some  $l > 1$ . Then

$$\begin{aligned} \partial z^{l+1} &= (\partial z)z^l + z\partial z^l \\ &= \lambda_z^l z^l \partial z + (l)_{\lambda_z} z^l \partial z \\ &= (\lambda_z^l + (l)_{\lambda_z}) z^l \partial z \\ &= (l+1)_{\lambda_z} z^l \partial z. \end{aligned}$$

The required formula now follows by an inductive argument. The formula for  $\bar{\partial} z^l$  is established analogously.  $\square$

In what follows, it proves very useful to have a simple relationship between the holomorphic and anti-holomorphic Leibniz constants. While it is not clear that such a relation exists in general, the assumption of a certain type of self-conjugacy on zero forms is enough to imply an inverse relation between  $\lambda_z$  and  $\zeta_z$ .

DEFINITION 4.14. We say that a quantum homogeneous space  $B = \mathcal{O}_q(G)^{\text{co}(H)}$  is *self-conjugate* if every irreducible sub-comodule  $V \subseteq B$  is a *\*-closed subspace*, which is to say  $V = \{v^* \mid v \in V\}$ .

The following technical lemma serves as a useful means of checking \*-invariance of an irreducible submodule  $V$  in terms of the highest weight vectors of  $V$ .

LEMMA 4.15. For a quantum homogeneous space  $B = \mathcal{O}_q(G)^{\text{co}(H)}$ , and an element  $z \in B_{\text{hw}}$ , the irreducible sub-comodule  $U_q(\mathfrak{g})z$  is a \*-closed subspace if and only if  $z^* \in U_q(\mathfrak{g})z$ .

*Proof.* Since  $z$  is a highest weight vector of the irreducible comodule  $U_q(\mathfrak{g})z$ , for every  $v \in U_q(\mathfrak{g})z$ , there exists an  $X \in U_q(\mathfrak{g})$ , such that  $X \triangleright z = v$ . Note next that

$$v^* = (X \triangleright z)^* = (\langle S(X), z_{(1)} \rangle z_{(2)})^* = \overline{\langle S(X), z_{(1)} \rangle} z_{(2)}^*.$$

Recalling now that we have a dual pairing of Hopf \*-algebras, we see that

$$\begin{aligned} \overline{\langle S(X), z_{(1)} \rangle} z_{(2)}^* &= \langle S^2(X)^*, z_{(1)}^* \rangle z_{(2)}^* \\ &= \langle S^{-2}(X^*), z_{(1)}^* \rangle z_{(2)}^* \\ &= \langle S(S^{-3}(X^*)), z_{(1)}^* \rangle z_{(2)}^* \\ &= S^{-3}(X^*) \triangleright z^*. \end{aligned}$$

Thus if we assume that  $z^* \in U_q(\mathfrak{g})z$ , then we necessarily have  $v^* \in U_q(\mathfrak{g})z$ , for all elements  $v \in U_q(\mathfrak{g})z$ , implying that  $U_q(\mathfrak{g})z$  is \*-closed. The opposite implication is obvious.  $\square$

We finish by showing that the assumption of self-conjugacy does indeed imply an inverse relation between the Leibniz constants  $\lambda_z$  and  $\zeta_z$ .

PROPOSITION 4.16. *Let  $H$  be a self-conjugate CQH-Hermitian space of Gelfand type. For any  $z \in B_{hw}$  with real Leibniz constants, it holds that*

$$(\bar{\partial}z)z = \lambda_z^{-1}z\bar{\partial}z, \quad \text{or equivalently,} \quad \zeta_z = \lambda_z^{-1}.$$

*Proof.* Applying the  $*$ -map to the identity  $(\partial z)z = \lambda_z z \partial z$  gives us the new identity

$$\lambda_z^{-1}z^*\bar{\partial}z^* = (\bar{\partial}z^*)z^*. \tag{9}$$

Lemma 4.10, combined with our assumption that  $H$  is self-conjugate, implies that  $z^*$  is a lowest weight vector of the irreducible module  $U_q(\mathfrak{g})z$ . Thus there exists an  $X \in U_q(\mathfrak{g})$  such that  $X \triangleright z^* = z$ . Note next that

$$X^2 \triangleright (z^*\bar{\partial}z^*) = (X \triangleright z^*)\bar{\partial}(X \triangleright z^*) = z\bar{\partial}z,$$

and that analogously,  $X^2 \triangleright ((\bar{\partial}z^*)z^*) = (\bar{\partial}z)z$ . Thus applying  $X^2$  to both sides of (9) gives the required identity  $(\bar{\partial}z)z = \lambda_z^{-1}z\bar{\partial}z$ .  $\square$

### 4.3 LAPLACIAN EIGENVALUES FOR CQH-HERMITIAN SPACES OF GELFAND TYPE

In the final subsection of this section we show that for a CQH-Hermitian spaces of Gelfand type, Hodge decomposition is a decomposition of  $B_{hw}$ -spaces. Combining this result with the Hodge decomposition of the Laplacian, we compute the eigenvalues of a general sequence of eigenvectors of the form  $z^l\omega$ , where  $z \in B_{hw}$ , and  $\omega \in \bar{\partial}\Omega_{hw}^{(0,\bullet)}$ . In the next section, our strategy is to decompose the point spectrum of the Laplacian  $\sigma_P(\Delta_{\bar{\partial}})$  into a finite union of such sequences and to use this to conclude that, under sufficient assumptions,  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$ . As discussed in §7, the general ideas of this section can be extended to the more general weak Gelfand setting with sufficient care.

Note that in this subsection we make heavy use of the quantum integer notation as presented in Appendix B.1.

LEMMA 4.17. *Let  $H = (\mathbf{M}, \Omega^{(\bullet,\bullet)}, \sigma)$  be a CQH-Hermitian space of Gelfand type, and  $z \in B_{hw}$ . Then, for every  $\omega \in \bar{\partial}\Omega_{hw}^{(0,\bullet)}$ , there exists unique scalars  $A_{z,\omega}, B_{z,\omega} \in \mathbb{C}$ , such that*

1.  $\partial z \wedge *_\sigma(\omega) = A_{z,\omega} z(\partial \circ *_\sigma(\omega))$ ,
2.  $\bar{\partial}z \wedge \bar{\partial}^\dagger \omega = B_{z,\omega} z(\bar{\partial} \bar{\partial}^\dagger(\omega))$ .

*Proof.* 1. Note first that both sides of the identity are highest weight vectors. Next we see that

$$\begin{aligned} K_i \triangleright (\partial z \wedge *_\sigma(\omega)) &= \partial(K_i \triangleright z) \wedge *_\sigma(K_i \triangleright \omega) \\ &= q^{(\text{wt}(z), \alpha_i) + (\text{wt}(\omega), \alpha_i)} \partial z \wedge *_\sigma(\omega). \end{aligned}$$

Analogously, we have that  $K_i \triangleright (z(\partial \circ *_\sigma(\omega))) = q^{(\text{wt}(z), \alpha_i) + (\text{wt}(\omega), \alpha_i)} z(\partial \circ *_\sigma(\omega))$ , meaning that both forms are highest weight vectors of the same weight. Moreover, both forms are homogeneous elements, of the same degree, of the graded comodule  $\Omega^{(\bullet, n)}$ . Now  $\Omega^{(\bullet, n)}$  is multiplicity-free by Lemma 3.13. Thus if  $z(\partial \circ *_\sigma(\omega)) \neq 0$ , then the existence of the required constant would follow from Lemma 4.3. By Lemma 2.15 the form  $*_\sigma(\omega)$  is contained in  $\partial^\dagger \Omega^\bullet$ , and so,  $\bar{\partial} \circ *_\sigma(\omega) \neq 0$ . Thus it follows from Lemma 2.11 that the product  $z(\partial \circ *_\sigma(\omega)) \neq 0$  as required.

2. The proof is analogous to the proof of 1, and so, is omitted. □

We now use the existence of the constant  $A_{z, \omega}$  presented in the above lemma to establish an identity needed for the proofs of Proposition 4.19 and Theorem 4.20.

COROLLARY 4.18. *It holds that*

$$\bar{\partial}^\dagger(z^l \omega) = (A_{z, \omega}(l)_{\lambda_z} + 1) z^l \bar{\partial}^\dagger(\omega).$$

*Proof.* From the identity  $\bar{\partial}^\dagger = - *_\sigma \circ \partial \circ *_\sigma$ , we have that

$$\begin{aligned} \bar{\partial}^\dagger(z^l \omega) &= - *_\sigma \circ \partial \circ *_\sigma(z^l \omega) \\ &= - *_\sigma \circ \partial(z^l *_\sigma(\omega)) \\ &= - *_\sigma \left( (l)_{\lambda_z} z^{l-1} \partial z \wedge *_\sigma(\omega) + z^l (\partial \circ *_\sigma(\omega)) \right) \\ &= - (l)_{\lambda_z} z^{l-1} *_\sigma(\partial z \wedge *_\sigma(\omega)) - z^l (*_\sigma \circ \partial \circ *_\sigma(\omega)). \end{aligned}$$

Recalling Lemma 4.17, we see that there exists a scalar  $A_{z, \omega}$  such that

$$\begin{aligned} \bar{\partial}^\dagger(z^l \omega) &= - (l)_{\lambda_z} z^l A_{z, \omega} (*_\sigma \circ \partial \circ *_\sigma(\omega)) + z^l \bar{\partial}^\dagger(\omega) \\ &= (A_{z, \omega}(l)_{\lambda_z} + 1) z^l \bar{\partial}^\dagger(\omega), \end{aligned}$$

which gives us the claimed identity. □

With these results in hand we are now ready to show that Hodge decomposition implies a decomposition of highest weight forms into  $B_{\text{hw}}$ -subspaces.

PROPOSITION 4.19. *For any CQH-complex space of Gelfand type, with constituent quantum homogeneous space  $B$ , the spaces  $\bar{\partial} \Omega_{\text{hw}}^{(0, \bullet)}$  and  $\bar{\partial}^\dagger \Omega_{\text{hw}}^{(0, \bullet)}$  are closed under the action of the monoid  $B_{\text{hw}}$ .*

*Proof.* Consider elements  $\omega \in \bar{\partial} \Omega^{(0, k)}$ , and  $z \in B_{\text{hw}}$ . Since Hodge decomposition is a decomposition of left  $\mathcal{O}_q(G)$ -comodules, either  $U_q(\mathfrak{g})z^l \omega \subseteq \bar{\partial} \Omega^{(0, k)}$ , or  $U_q(\mathfrak{g})z^l \omega \subseteq \bar{\partial}^\dagger \Omega^{(0, k)}$ , for all  $l \in \mathbb{N}_0$ . In particular either  $z^l \omega \in \bar{\partial} \Omega^{(0, k)}$ , or  $z^l \omega \in \bar{\partial}^\dagger \Omega^{(0, k)}$ . We observe that

$$\bar{\partial}(z^l \omega) = \bar{\partial}(z^l) \wedge \omega + z^l \bar{\partial} \omega = (l)_{\lambda_z^{-1}} z^{l-1} \bar{\partial} z \wedge \omega = (l)_{\lambda_z^{-1}} z^{l-1} \bar{\partial}(z \omega).$$

Thus  $\bar{\partial}(z^l\omega) = 0$  if and only if  $\bar{\partial}(z\omega) = 0$ . This means that  $z\omega \in \bar{\partial}\Omega^{(0,\bullet)}$  if and only if  $z^l\omega \in \bar{\partial}\Omega^{(0,\bullet)}$ , for all  $l \in \mathbb{N}_0$ .

Now, for  $l = 1$ , this is zero if and only if  $B_{z,\omega} = -1$ . In that case, for  $l > 1$ , recalling that  $q \in \mathbb{R}$  (and hence not a complex root of unity), we have

$$\bar{\partial}^\dagger(z^l\omega) = (- (l)_{\lambda_z} + 1)z^l \bar{\partial}^\dagger(\omega) \neq 0.$$

However, this contradicts our earlier observation that  $z^l\omega \in \bar{\partial}\Omega^{(0,\bullet)}$  if and only if  $z\omega$  is contained in  $\bar{\partial}\Omega^{(0,\bullet)}$ . Hence, we can conclude that  $\bar{\partial}\Omega_{\text{hw}}^{(0,\bullet)}$  is closed under the action of  $B_{\text{hw}}$ . The proof that  $\bar{\partial}^\dagger\Omega_{\text{hw}}^{(0,\bullet)}$  is closed under the action of  $B_{\text{hw}}$  is analogous.  $\square$

We now use this lemma to construct an explicit sequence of eigenvalues starting from an element  $z \in B_{\text{hw}}$  and a form  $\omega \in \bar{\partial}\Omega_{\text{hw}}^{(0,k)}$ . In the next section, we introduce an approach to verifying the compact resolvent condition based around such sequences of eigenvalues. The eigenvalues are presented in terms of quantum  $\lambda_z$ -integers, and quantum  $\lambda_z^{-1}$ -integers, where as usual  $\lambda_z$  is the Leibniz constant of  $z$ . In the case of quantum projective space, as presented in §6, we see that eigenvalues of its Dolbeault–Dirac operator are exactly of this form, with the quantum  $\lambda_z$ -integers  $q$ -deforming the integer eigenvalues of the classical operator.

**THEOREM 4.20.** *Let  $\mathbf{H} := (\mathbf{B}, \Omega^{(\bullet,\bullet)}, \sigma)$  be a CQH-Hermitian space of Gelfand type. For any form  $\omega \in \bar{\partial}\Omega_{\text{hw}}^{(0,k)}$ , and  $z \in B_{\text{hw}}$ , it holds that*

1.  $z^l\omega$  is an eigenvector of  $\Delta_{\bar{\partial}}$ , for all  $l \in \mathbb{N}_0$ ,
2. denoting by  $\mu_\omega$  the eigenvalue of  $\omega$ , it holds that

$$\Delta_{\bar{\partial}}(z^l\omega) = \left( A_{z,\omega}(l)_{\lambda_z} + 1 \right) \left( B_{z,\omega}(l)_{\lambda_z^{-1}} + 1 \right) \mu_\omega z^l\omega.$$

*Proof.* By Corollary 4.13 we have that

$$\bar{\partial}(z^l \bar{\partial}^\dagger\omega) = \bar{\partial}(z^l) \wedge \bar{\partial}^\dagger\omega + z^l(\bar{\partial}\bar{\partial}^\dagger\omega) = (l)_{\lambda_z^{-1}}z^{l-1}\bar{\partial}z \wedge \bar{\partial}^\dagger\omega + z^l(\bar{\partial}\bar{\partial}^\dagger\omega).$$

Lemma 4.17 implies that there exists a uniquely defined scalar  $B_{z,\omega}$  such that

$$\begin{aligned} (l)_{\lambda_z^{-1}}z^{l-1}\bar{\partial}z \wedge (\bar{\partial}^\dagger\omega) + z^l \bar{\partial}\bar{\partial}^\dagger\omega &= B_{z,\omega}(l)_{\lambda_z^{-1}}z^l \bar{\partial}\bar{\partial}^\dagger\omega + z^l \bar{\partial}\bar{\partial}^\dagger\omega \\ &= (B_{z,\omega}(l)_{\lambda_z^{-1}} + 1)z^l \bar{\partial}\bar{\partial}^\dagger\omega. \end{aligned}$$

From Proposition 4.19 above, we know that  $z^l\omega \in \bar{\partial}\Omega^{(0,\bullet)}$ . Moreover by Hodge decomposition of the Laplacian, we know that  $\Delta_{\bar{\partial}}$  restricts to  $\bar{\partial}\bar{\partial}^\dagger$  on  $\bar{\partial}\Omega^{(0,\bullet)}$ .

Combining these facts with Corollary 4.18, we now see that

$$\begin{aligned} \Delta_{\bar{\partial}}(z^l\omega) &= \bar{\partial}\bar{\partial}^\dagger(z^l\omega) \\ &= (1 + A_{z,\omega}(l)_{\lambda_z})\bar{\partial}(z^l\bar{\partial}^\dagger\omega) \\ &= (1 + A_{z,\omega}(l)_{\lambda_z})(1 + B_{z,\omega}(l)_{\lambda_z^{-1}})z^l\bar{\partial}\bar{\partial}^\dagger(\omega) \\ &= (1 + A_{z,\omega}(l)_{\lambda_z})(1 + B_{z,\omega}(l)_{\lambda_z^{-1}})z^l\Delta_{\bar{\partial}}(\omega) \\ &= (1 + A_{z,\omega}(l)_{\lambda_z})(1 + B_{z,\omega}(l)_{\lambda_z^{-1}})\mu_\omega z^l\omega, \end{aligned}$$

establishing the required identity. □

#### 4.4 CONNECTEDNESS FOR GELFAND TYPE CQH-COMPLEX SPACES

We finish with a discussion of connectedness in the Gelfand type setting, proving that it is equivalent to finite-dimensionality of the zeroth cohomology group  $H^0$ . This is an interesting, and useful, application of the notion of Gelfand type, especially given the difficulty of demonstrating connectedness in general. We begin by recalling the standard definition of connectedness for a differential calculus.

DEFINITION 4.21. We say that a differential calculus  $(\Omega^\bullet, d)$  is *connected* if

$$H^0 = \ker(d : \Omega^0 \rightarrow \Omega^1) = \mathbb{C}1.$$

It is important to note that if  $\Omega^\bullet$  is endowed with a complex structure  $\Omega^{(\bullet,\bullet)}$ , then an elementary application of the  $*$ -map demonstrates that the calculus is connected if and only if

$$\ker(\partial : \Omega^{(0,0)} \rightarrow \Omega^{(1,0)}) = \ker(\bar{\partial} : \Omega^{(0,0)} \rightarrow \Omega^{(0,1)}) = \mathbb{C}1.$$

Note that the following lemma does not rely on our discussions above. What is used is no more than a multiplicity-free assumption for forms of degree 0 (as implied by Gelfand type) and the assumption that we are working with Drinfeld–Jimbo quantum groups.

LEMMA 4.22. *Let  $C = (B, \Omega^{(\bullet,\bullet)})$  be a CQH-complex space for which  $\Omega^0$  is multiplicity-free as a left  $A$ -comodule. Then the following are equivalent:*

1. *The constituent different calculus  $\Omega^\bullet \in B$  is connected,*
2.  $\dim_{\mathbb{C}}(H^0) < \infty$ .

*Proof.* Assume that the constituent calculus  $\Omega^\bullet \in B$  is not connected, which is to say, assume that  $B$  contains a  $\bar{\partial}$ -closed element  $y$  which is not a scalar multiple of the identity. Denote by  $y = \sum_k y_k$  the decomposition of  $y$  into summands which are homogeneous with respect to the decomposition of  $B$  into irreducible  $U_q(\mathfrak{g})$ -modules. Since  $\Omega^0$  is multiplicity-free by assumption,

and  $\bar{\partial}$  is a left  $U_q(\mathfrak{g})$ -module map,  $\bar{\partial}y = 0$  if and only if  $\bar{\partial}y_k = 0$ , for all  $k$ . Hence, we can assume, without loss of generality, that  $y$  is homogeneous with respect to the decomposition of  $B$  into irreducibles. By Schur's lemma every element in the irreducible module containing  $y$  must be  $\bar{\partial}$ -closed. Thus we can assume, without loss of generality, that  $y$  is a weight vector with non-zero weight. Since weights are additive  $K_i \triangleright y^l = q^{l(\text{wt}(y), \alpha_i)} y^l$ , for any  $l \in \mathbb{N}_0$ , and  $i = 1, \dots, \text{rank}(\mathfrak{g})$ . Thus the weights of the elements  $y^l$  are distinct, for each  $l$ . In particular, the set  $\{y^l \mid l \in \mathbb{N}_0\}$  is linearly independent, and so, infinite-dimensional. Since  $\bar{\partial}y = 0$ , the Leibniz rule implies that  $\bar{\partial}(y^l) = 0$ , for all  $l \in \mathbb{N}_0$ . This means that the space of harmonic elements is infinite-dimensional, and so, by Hodge decomposition  $H^0$  is infinite-dimensional. The proof in the other direction is trivial, meaning that we have established the required equivalence.  $\square$

## 5 CQH-COMPLEX SPACES OF ORDER I AND SPECTRAL TRIPLES

In this section we introduce the notion of CQH-complex space of order I, which can be viewed as an abstraction of the essential representation theoretic properties of the space of holomorphic forms of complex projective space. Necessary and sufficient conditions are then produced for a CQH-Hermitian space to give a Dolbeault–Dirac pair of spectral triples, under the assumption that its underlying CQH-complex space is of order I.

It is proposed in Conjecture 7.11 that the only irreducible quantum flag manifolds of order I are the quantum projective spaces  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ . We formalise its properties for three principal reasons. Firstly, the abstract picture helps to clarify and elucidate the processes at work for quantum projective space. Secondly, it sets the stage for our subsequent investigation of the compact quantum Hermitian spaces of weak Gelfand type, highlighting the subtle but significant changes that occur when passing to this more general setting. Finally, it is hoped that new examples will arise from non-Drinfeld–Jimbo quantisations of  $U_q(\mathfrak{g})$ . In fact, it is important to note that the only essential feature of  $U_q(\mathfrak{g})$  used in this paper is the preservation under  $q$ -deformation of the highest weight structure of the category of  $U(\mathfrak{g})$ -modules.

Note that in this subsection we make heavy use of the quantum integer notation as presented in Appendix B.1.

### 5.1 POSITIVITY FOR LEIBNIZ CONSTANTS

In this subsection we give sufficient conditions for real Leibniz constants to be positive. As well as being an interesting observation in its own right, positivity must hold for any CQH-Hermitian space satisfying  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$ , as we will see in §5.4.

LEMMA 5.1. *Let  $H = (B, \Omega^{(\bullet, \bullet)}, \sigma)$  be a self-conjugate CQH-Hermitian space of Gelfand type, with constituent quantum homogeneous space  $B$ . For any*

$z \in B_{\text{hw}}$ , and  $l \in \mathbb{N}_0$ ,

$$\Delta_{\bar{\partial}}(z^l \bar{\partial} z) = (A_{z, \bar{\partial} z}(l)_{\lambda_z} + 1)(l + 1)_{\lambda_z^{-1}} \mu_z z^l, \tag{10}$$

where  $\mu_z$  is the  $\Delta_{\bar{\partial}}$ -eigenvalue of  $z$ , and as usual  $\lambda_z$  is the Leibniz constant of  $z$ .

*Proof.* For  $\mu_z$  the  $\Delta_{\bar{\partial}}$ -eigenvalue of  $z$ ,

$$\bar{\partial} z \wedge \bar{\partial}^\dagger \bar{\partial} z = \bar{\partial} z \wedge (\mu_z z) = \mu_z \lambda_z^{-1} z \bar{\partial} z = \lambda_z^{-1} z \bar{\partial}(\mu_z z) = \lambda_z^{-1} z \bar{\partial} \bar{\partial}^\dagger \bar{\partial} z.$$

Thus we see that  $B_{z, \bar{\partial} z} = \lambda_z^{-1}$ . Equation (10) now follows from Theorem 4.20.  $\square$

PROPOSITION 5.2. *Let  $H = (B, \Omega^{(\bullet, \bullet)}, \sigma)$  be a connected self-conjugate CQH-Hermitian space of Gelfand type. For any non-harmonic  $z \in B_{\text{hw}}$ , with real Leibniz constant  $\lambda_z$ , it holds that*

1.  $\lambda_z \notin [-1, 0)$ ,
2. if  $A_{z, \bar{\partial} z} \neq 0$ , then  $\lambda_z \in \mathbb{R}_{>0}$ .

*Proof.*

1. Assume that  $-1 < \lambda_z < 0$ . As  $l \rightarrow \infty$ , the sign of the scalar

$$(l + 1)_{\lambda_z^{-1}} = \frac{1 - \lambda_z^{-(l+1)}}{1 - \lambda_z^{-1}}$$

alternates, and its absolute value goes to infinity. Moreover,  $(l)_{\lambda_z}$  is positive for all  $l \in \mathbb{N}_0$ , implying that  $A_{z, \bar{\partial} z}(l)_{\lambda_z} + 1$  will eventually have a constant sign. Thus there exist values of  $l$  for which the eigenvalue in (10) is negative. However, this contradicts the fact that  $\Delta_{\bar{\partial}}$  is a positive operator, forcing us to conclude that  $\lambda_z \notin (-1, 0)$ . Moreover, if  $\lambda_z = -1$ , then  $[2]_{\lambda_z} = 0$ , implying that  $z^2$  is harmonic, contradicting our assumption that  $H$  is connected. Thus we must have that  $\lambda_z \notin [-1, 0)$ .

2. For  $\lambda_z < -1$ , assuming that  $A_{z, \bar{\partial} z} \neq 0$  allows one to produce a negative eigenvalue for  $\Delta_{\bar{\partial}}$ , just as above. Since this again contradicts the positivity of  $\Delta_{\bar{\partial}}$ , we are forced to conclude that  $-1 < \lambda_z$ . Taken together with the fact that  $\lambda_z \notin [-1, 0)$ , this means that  $\lambda_z \in \mathbb{R}_{>0}$ .  $\square$

## 5.2 CQH-COMPLEX SPACES OF ORDER I

In this subsection we introduce the notion of a CQH-complex space of order I. This collects the properties of Gelfand type and self-conjugacy together with the existence of a ladder presentation, a particularly convenient form for the decomposition of the exact anti-holomorphic forms into irreducibles. When

reading the definition below, it is important to bear Proposition 4.19 in mind, in particular the fact that  $\bar{\partial}\Omega^{(0,\bullet)}$  is closed under the action of the monoid  $B_{\text{hw}}$ . The definition is an abstraction of the  $U_q(\mathfrak{sl}_n)$ -module structure of the anti-holomorphic forms of complex projective form, and in particular of the  $K$ -types appearing in the decomposition into irreducibles. (See §1 and [60] for a more detailed discussion of Vogan’s minimal  $K$ -types.)

DEFINITION 5.3. Let  $C = (B, \Omega^{(\bullet,\bullet)})$  be a CQH-complex space. A ladder presentation for  $C$  is a pair  $(z, \Theta)$ , where  $z \in B_{\text{hw}}$ , and  $\Theta \subseteq \bar{\partial}\Omega_{\text{hw}}^{(0,\bullet)}$  is a finite subset of homogeneous forms (that is  $\Theta = \cup_k \Theta_k$ , where  $\Theta_k := \Theta \cap \bar{\partial}\Omega^{(0,k)}$ ) satisfying

$$\bar{\partial}\Omega^{(0,k)} \simeq \bigoplus_{\omega \in \Theta_k} \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{g}) z^l \omega. \tag{11}$$

A ladder presentation  $(z, \Theta)$  is said to be *real* if the Lefschetz constant of  $z$  is a real number, which is to say, if  $\lambda_z \in \mathbb{R}$ . Moreover,  $(z, \Theta)$  is said to be *positive* if the Lefschetz constant of  $z$  is a positive real number, which is to say, if  $\lambda_z \in \mathbb{R}_{>0}$ .

It is instructive to note that in the Hermitian case we have the following implication of the existence of a ladder presentation.

LEMMA 5.4. Let  $H = (B, \Omega^{(\bullet,\bullet)}, \sigma)$  be a CQH-Hermitian space of Gelfand type. If  $(z, \Theta)$  is a ladder presentation of  $(B, \Omega^{(\bullet,\bullet)})$ , then

$$\bar{\partial}^\dagger \Omega^{(0,k)} \simeq \bigoplus_{\omega \in \Theta_k} \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{g}) z^l \bar{\partial}^\dagger \omega, \quad \text{for all } k.$$

*Proof.* Operating by  $\bar{\partial}^\dagger$  on the decomposition (11) associated to  $(z, \Theta)$  gives

$$\bar{\partial}^\dagger \bar{\partial}\Omega^{(0,k)} \simeq \bigoplus_{\omega \in \Theta_k} \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{g}) \bar{\partial}^\dagger(z^l \omega), \quad \text{for all } k.$$

By Hodge decomposition

$$\begin{aligned} \bar{\partial}^\dagger \bar{\partial}\Omega^{(0,k)} &= \bar{\partial}^\dagger \bar{\partial} \left( \bar{\partial}\Omega^{(0,k-1)} \oplus \bar{\partial}^\dagger \Omega^{(0,k+1)} \oplus \mathcal{H}^{(0,k)} \right) \\ &= \bar{\partial}^\dagger \bar{\partial} \left( \bar{\partial}^\dagger \Omega^{(0,k+1)} \right) \\ &= \bar{\partial}^\dagger \Omega^{(0,k)}, \end{aligned}$$

where in the last identity we have used the fact that  $\bar{\partial}^\dagger \bar{\partial} : \bar{\partial}^\dagger \Omega^{(0,\bullet)} \rightarrow \bar{\partial}^\dagger \Omega^{(0,\bullet)}$  is an isomorphism, as established in Proposition 3.3. Moreover, since  $H$  is assumed to be of Gelfand type, and  $\bar{\partial}^\dagger(z^l \omega)$  and  $z^l \bar{\partial}^\dagger \omega$  are clearly highest weight elements of the same degree, it follows from Lemma 4.3 that they are linearly proportional. Thus

$$\bar{\partial}^\dagger \Omega^{(0,k)} \simeq \bigoplus_{\omega \in \Theta_k} \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{g}) z^l \bar{\partial}^\dagger \omega,$$

as claimed. □

Finally, we come to the definition of CQH-complex space of order I. As stated above, this can be viewed as a noncommutative generalisation of projective space. More explicitly, it can be viewed as a noncommutative generalisation of the properties of  $\mathbb{C}\mathbb{P}^{n-1}$  considered as the homogeneous space  $SU_n/U_{n-1} \simeq U_n/(U_{n-1} \times U_1)$ .

DEFINITION 5.5. A CQH-complex space  $C$  is said to be of *order I* if it is

1. self-conjugate,
2. of Gelfand type,
3. admits a real ladder presentation.

Note that if  $C$  is of order I, properties 1 and 2 of the definition together imply that every highest weight space contains an element of the form  $z^l\omega$ , for some  $l \in \mathbb{N}_0$ , and some  $\omega \in \Theta$ . The notion of an order II presentation, which deals with CQH-spaces of weak Gelfand type, is discussed in §7 along with the motivating quantum flag manifolds.

### 5.3 CQH-HERMITIAN SPACES OF ORDER I AND SOLIDITY

In this subsection we introduce a crucial property, which we call solidity, for those CQH-Hermitian spaces whose underlying CQH-complex spaces are of order I. In the next subsection, it will be shown that solidity is equivalent to the Laplace operator having eigenvalues tending to infinity.

DEFINITION 5.6. Let  $H = (B, \Omega^{(\bullet, \bullet)}, \sigma)$  be a CQH-Hermitian space.

1. We say that  $H$  is *solid*<sup>0</sup> if  $(B, \Omega^{(\bullet, \bullet)})$  admits a real ladder presentation  $(z, \Theta)$  and satisfies

$$(a) \lambda_z = 1, \quad (b) A_{z, \omega} \neq 0, \quad \text{or} \quad B_{z, \omega} \neq 0. \tag{12}$$

2. We say that  $H$  is *solid*<sup>+</sup> if  $(B, \Omega^{(\bullet, \bullet)})$  admits a real ladder presentation  $(z, \Theta)$  and satisfies

$$(a) 1 < \lambda_z, \quad (b) A_{z, \omega} \neq 0, \quad (c) B_{z, \omega} \neq \lambda_z - 1. \tag{13}$$

3. We say that  $H$  is *solid*<sup>-</sup> if  $(B, \Omega^{(\bullet, \bullet)})$  admits a real ladder presentation  $(z, \Theta)$  and satisfies

$$(a) 0 < \lambda_z < 1, \quad (b) A_{z, \omega} \neq \lambda_z - 1, \quad (c) B_{z, \omega} \neq 0. \tag{14}$$

4. We say that  $H$  is *solid* if it is either *solid*<sup>0</sup>, *solid*<sup>+</sup>, or *solid*<sup>-</sup>.

In practice it can be difficult to directly verify condition (b) in (12) and conditions (b) and (c) in (13) and (14). The following two lemmas give more convenient reformulations of condition (b) in (13) and condition (b) in (14). As is easily confirmed, the proof extends to analogous reformulations of condition (c) in (13) and condition (c) in (14).

LEMMA 5.7. *Let  $H$  be a CQH-Hermitian space of order  $I$ , and  $(z, \Theta)$  a ladder presentation of  $H$ . For any  $\omega \in \Theta$ , the following are equivalent:*

1.  $A_{z,\omega} \neq 0$ ,
2.  $\partial z \wedge \omega$  is a non-primitive form.

*Proof.* From the defining formula of the Hodge map, and centrality of the Hermitian form, for any  $\omega \in \Omega^{(0,k)}$ , we have

$$\begin{aligned} \partial z \wedge *_\sigma(\omega) &= (-1)^{\frac{k(k+1)}{2}} i^{-k} \frac{1}{(n-k)!} \partial z \wedge L^{n-k}(\omega) \\ &= (-1)^{\frac{k(k+1)}{2}} i^{-k} \frac{1}{(n-k)!} \partial z \wedge \sigma^{n-k} \wedge \omega \\ &= (-1)^{\frac{k(k+1)}{2}} i^{-k} \frac{1}{(n-k)!} \sigma^{n-k} \wedge \partial z \wedge \omega \\ &= (-1)^{\frac{k(k+1)}{2}} i^{-k} \frac{1}{(n-k)!} L^{n-k}(\partial z \wedge \omega). \end{aligned}$$

Thus we see that  $\partial z \wedge *_\sigma(\omega) \neq 0$  if and only if  $L^{n-(k+1)+1}(\partial z \wedge \omega) \neq 0$ , which is to say, if and only if  $\partial z \wedge \omega$  is a non-primitive form.  $\square$

LEMMA 5.8. *Let  $H$  be a CQH-Hermitian space of order  $I$ , and  $(z, \Theta)$  a real ladder presentation of  $H$ . If  $A_{z,\omega} \neq 0$ , then for any  $\omega \in \Theta$ , the following are equivalent:*

1.  $A_{z,\omega} \neq \lambda_z - 1$ ,
2.  $\omega \wedge \partial z$  is a non-primitive form.

*Proof.* By Lemma 2.11, both  $*_\sigma(\omega)z$  and  $z(\partial \circ *_\sigma(\omega))$  are non-zero forms. Since both forms have the same weight and degree, Lemma 3.13 and Lemma 4.3 imply the existence of scalars  $C, C' \in \mathbb{C}$  such that

$$z *_\sigma(\omega) = C *_\sigma(\omega)z, \quad (\partial \circ *_\sigma(\omega))z = C'z(\partial \circ *_\sigma(\omega)). \quad (15)$$

Combining these two identities with Lemma 4.17 we see that

$$\begin{aligned} \partial z \wedge z *_\sigma(\omega) &= C(\partial z \wedge *_\sigma(\omega))z \\ &= CA_{z,\omega}z(\partial \circ *_\sigma(\omega))z \\ &= CC'A_{z,\omega}z^2(\partial \circ *_\sigma(\omega)). \end{aligned}$$

Moreover,

$$\partial z \wedge z *_\sigma(\omega) = \lambda_z z(\partial z \wedge *_\sigma(\omega)) = \lambda_z A_{z,\omega} z^2(\partial \circ *_\sigma(\omega)).$$

Since  $A_{z,\omega} \neq 0$  by assumption, comparing the two expressions for  $\partial z \wedge z *_\sigma(\omega)$  gives us that

$$CC' = \lambda_z. \quad (16)$$

Returning now to (15), we apply  $\partial$  to the first identity to obtain

$$\partial z \wedge *_\sigma(\omega) + z(\partial \circ *_\sigma(\omega)) = C(\partial \circ *_\sigma(\omega))z + (-1)^{|\ast_\sigma(\omega)|} C *_\sigma(\omega) \wedge \partial z.$$

Now by (15) and (16) it holds that  $C(\partial \circ *_\sigma(\omega))z = CC'z(\partial \circ *_\sigma(\omega)) = \lambda_z z(\partial \circ *_\sigma(\omega))$ . Hence

$$\partial z \wedge *_\sigma(\omega) = (\lambda_z - 1)z(\partial \circ *_\sigma(\omega)) + (-1)^{|\ast_\sigma(\omega)|} C *_\sigma(\omega) \wedge \partial z.$$

Thus we see that  $*_\sigma(\omega) \wedge \partial z = 0$  if and only if  $\partial z \wedge *_\sigma(\omega) = (\lambda_z - 1)z(\partial \circ *_\sigma(\omega))$ . Following the same argument as given in Lemma 5.7 above, it can be shown that the form  $*_\sigma(\omega) \wedge \partial z$  is non-zero if and only if the form  $\omega \wedge \partial z$  is non-primitive. The claimed equivalence now follows.  $\square$

#### 5.4 SOLIDITY AND DOLBEAULT–DIRAC SPECTRAL TRIPLES

In this subsection, we show that for a CQH-Hermitian space of order  $I$ , the eigenvalues of its Laplacian tend to infinity if and only if it is solid. Combining this equivalence with Proposition 2.17 we can give necessary and sufficient conditions for a CQH-Hermitian space to give a Dolbeault–Dirac pair of spectral triples. For the convenience of the reader we break the proof into two parts.

LEMMA 5.9. *The Laplacian of a CQH-Hermitian space  $H$  of order  $I$  has eigenvalues tending to infinity only if  $H$  has finite-dimensional anti-holomorphic cohomologies and all ladder presentations are solid.*

*Proof.* If the eigenvalues of  $\Delta_{\bar{\partial}}$  tend to infinity then by definition no eigenvalue can have infinite multiplicity, and in particular 0 cannot have infinite multiplicity. By Hodge decomposition this is equivalent to the complex structure having finite-dimensional anti-holomorphic cohomologies, giving us one part of the implication.

Next we show that  $\lambda_z \geq 0$ . Since we have just shown that  $\dim_{\mathbb{C}}(H^{(0,0)}) < \infty$ , Proposition 4.22 implies that the calculus is connected. Thus Proposition 5.2 tells us that either  $\lambda_z > 0$ , or  $\lambda_z < -1$  and  $A_{z, \bar{\partial}z} = 0$ . In the latter case, Lemma 5.1 implies that

$$\Delta_{\bar{\partial}}(z^l \bar{\partial}z) = (l + 1)_{\lambda_z} z^l \bar{\partial}z.$$

Since  $(l + 1)_{\lambda_z}$  converges as  $l \rightarrow \infty$ , in this case we cannot have that  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$ . Thus we are forced to conclude that  $\lambda_z$  is positive.

For a general ladder presentation  $(z, \Theta)$ , we now have three possibilities:  $0 < \lambda_z < 1$ ,  $\lambda_z = 1$ , or  $1 < \lambda_z$ .

1. Let us assume that  $\lambda_z = 1$ . If  $A_{z, \omega}$  and  $B_{z, \omega}$  are both zero, for some  $\omega \in \Theta$ , then it follows directly from Theorem 4.20 that  $\mu_\omega$  is an eigenvalue of  $\Delta_{\bar{\partial}}$  with infinite-dimensional multiplicity. Thus if  $\lambda_z = 1$ , then  $(z, \Theta)$  must be solid<sup>0</sup>.

2. Let us assume that  $1 < \lambda_z$ , and show that if condition (b) or (c) of the definition of solid<sup>+</sup> does not hold for  $H$ , then  $\sigma_P(\Delta_{\bar{\partial}})$  has a limit point. We

start with condition (b), which is to say, let us assume that  $A_{z,\omega} = 0$ , for some  $\omega \in \Theta$ . It follows directly from Theorem 4.20 that, in this case,

$$\Delta_{\bar{\partial}}(z^l\omega) = (1 + B_{z,\omega}(l)_{\lambda_z^{-1}})z^l\omega, \quad \text{for all } l \in \mathbb{N}_0.$$

Looking now at the limit of this sequence of eigenvalues, we see that

$$\begin{aligned} \lim_{l \rightarrow \infty} (1 + B_{z,\omega}(l)_{\lambda_z^{-1}}) &= 1 + B_{z,\omega} \lim_{l \rightarrow \infty} ((l)_{\lambda_z^{-1}}) \\ &= 1 + B_{z,\omega} \lim_{l \rightarrow \infty} \left( \frac{1 - \lambda_z^{-l}}{1 - \lambda_z^{-1}} \right) \\ &= 1 + \frac{B_{z,\omega}}{1 - \lambda_z^{-1}}. \end{aligned}$$

Thus  $\sigma_P(\Delta_{\bar{\partial}})$  has a limit point in  $\mathbb{R}$ .

Let us next assume that condition (c) of the definition of  $\text{solid}^+$  does not hold, which is to say, let us assume that  $B_{z,\omega} = \lambda_z^{-1} - 1$ , for some  $\omega \in \Theta$ . The limit of the eigenvalues of  $z^l\omega$ , as  $l \rightarrow \infty$ , is given by

$$\begin{aligned} &\lim_{l \rightarrow \infty} \left( (1 + A_{z,\omega}(l)_{\lambda_z}) (1 + B_{z,\omega}(l)_{\lambda_z^{-1}}) \right) \\ &= \lim_{l \rightarrow \infty} \left[ (1 + A_{z,\omega}(l)_{\lambda_z}) \left( 1 - (1 - \lambda_z^{-1}) \frac{1 - \lambda_z^{-l}}{1 - \lambda_z^{-1}} \right) \right] \\ &= \lim_{l \rightarrow \infty} \left[ \left( 1 + A_{z,\omega} \frac{1 - \lambda_z^l}{1 - \lambda_z^{-1}} \right) \lambda_z^{-l} \right] \\ &= \lim_{l \rightarrow \infty} \left( \lambda_z^{-l} + A_{z,\omega} \frac{\lambda_z^{-l} - 1}{1 - \lambda_z^{-1}} \right) \\ &= \frac{A_{z,\omega}}{\lambda_z^{-1} - 1}. \end{aligned}$$

Thus the point spectrum of  $\Delta_{\bar{\partial}}$  again has a limit point in  $\mathbb{R}$ .

Taking these two results together we see that if either of the requirements for a  $\text{solid}^+$  fail to hold for  $(z, \Theta)$ , then the eigenvalues of  $\Delta_{\bar{\partial}}$  do not tend to infinity. Thus we are forced to conclude that  $(z, \Theta)$  is  $\text{solid}^+$ .

3. Finally, for the case of  $0 < \lambda_z < 1$ , an argument analogous to that in 2 verifies that if  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$ , then  $(z, \Theta)$  must be  $\text{solid}^-$ .  $\square$

**THEOREM 5.10.** *Let  $H$  be a CQH-Hermitian structure of order  $I$ . Then the following are equivalent:*

1.  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$ ,
2.  $H$  is solid and has finite-dimensional anti-holomorphic cohomologies.

*Proof.* By Lemma 5.9 above, we need only show that 2 implies 1. Since the complex structure is of order  $I$  by assumption, it is in particular of Gelfand type. Thus by Lemma 3.9, the Laplacian  $\Delta_{\bar{\partial}} : \partial\Omega^{(0,k)} \rightarrow \partial\Omega^{(0,k)}$  acts on any

irreducible  $U_q(\mathfrak{g})$ -submodule  $V \subseteq \bar{\partial}\Omega^{(0,k)}$  as a scalar multiple of the identity. This scalar can be determined by letting  $\Delta_{\bar{\partial}}$  act on any element of  $V$ . In particular, it can be determined by letting  $\Delta_{\bar{\partial}}$  act on a highest weight vector of  $V$ . Since we are assuming that  $\mathbf{H}$  admits a ladder presentation  $(z, \Theta)$ , every highest weight space must contain an element of the form  $z^l\omega$ , for some  $l \in \mathbb{N}_0$ , and some  $\omega \in \Theta$ . Thus, if the eigenvalues of  $z^l\omega$  tend to infinity, for each  $\omega \in \Theta$ , then we know that the eigenvalues of  $\Delta_{\bar{\partial}} : \bar{\partial}\Omega^{(0,\bullet)} \rightarrow \bar{\partial}\Omega^{(0,\bullet)}$  tend to infinity, and have finite multiplicity. Since by assumption we have finite-dimensional anti-holomorphic cohomologies, Corollary 3.6 says that this is sufficient to imply that  $\sigma_P(\Delta_{\bar{\partial}}) \rightarrow \infty$ .

Let us now break the rest of the proof into the three cases where  $(z, \Theta)$  is either  $\text{solid}^0$ ,  $\text{solid}^+$ , or  $\text{solid}^-$ .

1. We first assume that  $(z, \Theta)$  is  $\text{solid}^0$ , and recall for convenience the identity from Theorem 4.20 above:

$$\Delta_{\bar{\partial}}(z^l\omega) = (1 + A_{z,\omega}(l)_{\lambda_z})(1 + B_{z,\omega}(l)_{\lambda_z^{-1}})z^l\omega.$$

Looking at the first factor, we see that since  $A_{z,\omega} \neq 0$ ,

$$\lim_{l \rightarrow \infty} (1 + A_{z,\omega}(l)_{\lambda_z}) = 1 + A_{z,\omega} \lim_{l \rightarrow \infty} ((l)_{\lambda_z}) = \infty.$$

Similarly, the second factor tends to infinity, as  $l \rightarrow \infty$ . Thus the eigenvalues of  $z^l\omega$  tend to infinity, with finite multiplicity, for all  $\omega \in \Theta$ .

2. Let us now assume that  $(z, \Theta)$  is  $\text{solid}^+$ . Just as in the  $\text{solid}^0$  case, the first factor tends to infinity. Looking next at the limit of the second factor, we see

$$\lim_{l \rightarrow \infty} (1 + B_{z,\omega}(l)_{\lambda_z^{-1}}) = 1 + B_{z,\omega} \lim_{l \rightarrow \infty} \left( \frac{1 - \lambda_z^{-(l+1)}}{1 - \lambda_z^{-1}} \right) = 1 + \frac{B_{z,\omega}}{1 - \lambda_z^{-1}}.$$

Since we are assuming that  $B_{z,\omega} \neq \lambda_z^{-1} - 1$ , we see that the second factor does not approach zero. Combining these two observations, we see that the eigenvalues of  $z^l\omega$  go to infinity, with finite multiplicity, as  $l$  goes to infinity, for all  $\omega \in \Theta$ .

3. An analogous argument proves that the existence of a  $\text{solid}^-$  ladder presentation implies that the eigenvalues of  $z^l\omega$  go to infinity, with finite multiplicity, as  $l$  goes to infinity.  $\square$

Combining the results of this section we arrive at the following theorem, which will be used in the next section to construct spectral triples for  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ .

**THEOREM 5.11.** *Let  $\mathbf{H} = (B, \Omega^{(\bullet,\bullet)}, \sigma)$  be a CQH-Hermitian space of order  $I$ , with constituent quantum homogeneous space  $B$ . Then a Dirac–Dolbeault pair of spectral triples is given by*

$$\left( B, L^2(\Omega^{(0,\bullet)}), D_{\bar{\partial}} \right), \quad \left( B, L^2(\Omega^{(\bullet,0)}), D_{\partial} \right),$$

if and only if  $H$  is solid and has finite-dimensional anti-holomorphic cohomologies.

6 A DOLBEAULT–DIRAC SPECTRAL TRIPLE FOR QUANTUM PROJECTIVE SPACE

We are now ready to apply the general framework developed in the previous sections to our motivating example  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  [15, 45]. We begin by recalling the necessary basics about  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  and its Heckenberger–Kolb calculus. We then construct an order I presentation of the associated CQH-space. This allows us to verify the compact resolvent condition by demonstrating solidity, and hence produce a Dolbeault–Dirac pair of spectral triples for  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  with non-trivial  $K$ -homology class.

6.1 QUANTUM PROJECTIVE SPACE AS A QUANTUM HOMOGENEOUS SPACE

Consider the Hopf subalgebra  $U_q(\mathfrak{t}_{n-1}) \subseteq U_q(\mathfrak{sl}_n)$  generated by the elements

$$K_i, E_j, F_j \quad \text{for } i = 1, \dots, n - 1, \text{ and } j = 1, \dots, n - 2.$$

Note that  $U_q(\mathfrak{sl}_{n-1})$  canonically embeds into  $U_q(\mathfrak{t}_{n-1})$  as a subalgebra. For any irreducible  $U_q(\mathfrak{sl}_{n-1})$ -module  $V_\mu$ , and any  $m \in \mathbb{Z}$ , we see that there exists an irreducible representation of  $U_q(\mathfrak{t}_{n-1})$  on  $V_\mu$ , extending the action of  $U_q(\mathfrak{sl}_{n-1})$ , uniquely determined by

$$K_{n-1} \triangleright v = q^m v, \quad \text{for } v \text{ a highest weight vector in } V_\mu.$$

We denote this irreducible representation by  $V_\mu(m)$ , and call  $m$  the *weight* of  $K_{n-1}$ . (See [34, §4.4] for a more detailed discussion of the representation theory of  $U_q(\mathfrak{t}_{n-1})$ .)

As standard, we use the superscript  $\circ$  to denote the Hopf dual of a Hopf algebra. Dual to the Hopf algebra embedding  $\iota : U_q(\mathfrak{t}_{n-1}) \hookrightarrow U_q(\mathfrak{sl}_n)$ , we have the Hopf algebra map  $\iota^\circ : U_q(\mathfrak{sl}_n)^\circ \rightarrow U_q(\mathfrak{t}_{n-1})^\circ$ . By construction  $\mathcal{O}_q(SU_n) \subseteq U_q(\mathfrak{sl}_n)^\circ$ , and so we can consider the restriction map

$$\pi := \iota^\circ|_{\mathcal{O}_q(SU_n)} : \mathcal{O}_q(SU_n) \rightarrow U_q(\mathfrak{t}_{n-1})^\circ,$$

as well as the Hopf subalgebra  $\mathcal{O}_q(U_{n-1}) := \pi(\mathcal{O}_q(SU_n))$ . *Quantum projective space*  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  is the quantum homogeneous space associated to the surjective Hopf  $*$ -algebra map  $\pi : \mathcal{O}_q(SU_n) \rightarrow \mathcal{O}_q(U_{n-1})$ , which is to say, it is the space of coinvariants

$$\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}) = \mathcal{O}_q(SU_n)^{\text{co}(\mathcal{O}_q(U_{n-1}))}.$$

A standard set of generators for  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  is given by

$$\left\{ z_{ij} := u_n^i S(u_j^n) \mid i, j = 1, \dots, n \right\}.$$

6.1.1 THE HECKENBERGER–KOLB CALCULUS

We present the calculus in two steps, beginning with Heckenberger and Kolb’s classification of first-order differential calculi over  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , and then discussing the maximal prolongation of the direct sum of the two calculi identified. First, we recall that a *differential map* between two differential calculi  $(\Omega^\bullet, d)$  and  $(\Gamma^\bullet, \delta)$ , is a degree 0 algebra map  $\varphi : \Omega^\bullet \rightarrow \Gamma^\bullet$  such that  $\varphi \circ d = \delta \circ \varphi$ . Moreover, a *first-order differential calculus* is a differential calculus of total degree 1. We say that a first-order differential calculus is *irreducible* if  $\Omega^1$  is irreducible as a bimodule over  $\Omega^0$ .

**THEOREM 6.1.** [20, Theorem 7.2] *There exist exactly two non-isomorphic irreducible, left-covariant, finite-dimensional, first-order differential calculi over  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ .*

We denote these two calculi by  $\Omega^{(1,0)}$  and  $\Omega^{(0,1)}$ . Moreover, we denote their direct sum by  $\Omega^1 := \Omega^{(1,0)} \oplus \Omega^{(0,1)}$  and call it the *Heckenberger–Kolb first-order differential calculus* of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ . For a proof of the following lemma see [48, Lemma 5.2].

**LEMMA 6.2.** *Bases of  $\Phi(\Omega^{(1,0)})$  and  $\Phi(\Omega^{(0,1)})$  are given respectively by*

$$\left\{ e_a^+ := [\partial z_{an}] \mid a = 1, \dots, n-1 \right\}, \quad \left\{ e_a^- := [\bar{\partial} z_{na}] \mid i = 1, \dots, n-1 \right\}.$$

Moreover, it holds that

$$[\partial z_{nn}] = [\bar{\partial} z_{nn}] = [\partial z_{ab}] = [\bar{\partial} z_{ab}] = 0, \quad \text{for all } a, b = 1, \dots, n-1.$$

We now present the action of the generators of  $U_q(\mathfrak{sl}_{n-1})$  on the basis elements of  $V^{(0,k)}$ . The proof is a direct calculation in terms of the dual pairing (31) and the definition of the action (27).

**LEMMA 6.3.** *The only non-zero actions of the generators of  $U_q(\mathfrak{sl}_{n-1})$  on the basis elements of  $V^{(0,1)}$  are given by*

$$E_k \triangleright e_k^- = q^2 e_{k+1}^-, \quad K_k \triangleright e_a^- = q^{-\delta_{k,n-1} - \delta_{ka} + \delta_{k,a-1}} e_a^-, \quad F_k \triangleright e_{k+1}^- = q^{-2} e_k^-.$$

*Proof.* The first identity is established by the calculation

$$E_k \triangleright e_a^- = E_k \triangleright [\bar{\partial} z_{na}] = [\bar{\partial} (E_k \triangleright z_{na})] = q^2 \delta_{ka} [\bar{\partial} z_{n,k+1}] = q^2 \delta_{ka} e_{k+1}^-.$$

The other two identities are established similarly. □

We say that a differential calculus over an algebra  $A$ , of total degree greater than or equal to 2, *extends* a first-order calculus if there exists a *differential injection*, which is to say, an injective differential map, from the first-order calculus to the differential calculus. Any first-order calculus admits an extension  $(\Omega^\bullet, d)$  which is *maximal* in the sense that, for any other extension  $(\Gamma^\bullet, \delta)$ , there exists a unique surjective differential map  $\varphi : \Omega^\bullet \rightarrow \Gamma^\bullet$ . We call this extension, which

is necessarily unique up to differential isomorphism, the *maximal prolongation* of the first-order calculus. The maximal prolongation of a covariant calculus is again covariant. (See [49, §2.5] for a more detailed discussion in the notation of this paper.)

We denote the maximal prolongation of the Heckenberger–Kolb first-order calculus by  $\Omega^\bullet$ , and call it the *Heckenberger–Kolb differential calculus*. Note that since  $\Omega^\bullet$  is covariant, it is a monoid object in  ${}_{\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})} \mathcal{O}_q(SU_n) \text{Mod}_0$ . Consequently, since Takeuchi’s equivalence is a monoidal equivalence,  $\Phi(\Omega^\bullet)$  is a monoid object in  ${}_{\mathcal{O}_q(U_{n-1})} \text{Mod}$ . We denote the multiplication in  $\Phi(\Omega^\bullet)$  by  $\wedge$ . We call a subset  $I \subseteq \{1, \dots, n-1\}$  *ordered* if  $i_1 < \dots < i_k$ . For any two ordered subsets  $I, J \subseteq \{1, \dots, n-1\}$ , we denote

$$e_I^+ \wedge e_J^- := e_{i_1}^+ \wedge \dots \wedge e_{i_k}^+ \wedge e_{j_1}^- \wedge \dots \wedge e_{j_l}^-,$$

where  $J = \{j_1, \dots, j_l\}$ . We now collect some basic facts about  $\Phi(\Omega^\bullet)$  as an algebra. Theorem 6.4.1 was established in [20, §3.3]. For a proof of Theorem 6.4.2 and Theorem 6.4.3 see [49, Proposition 5.8].

**THEOREM 6.4.** *For  $\Omega^\bullet$  the Heckenberger–Kolb calculus of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ :*

1.  $\Phi(\Omega^k)$  has dimension  $\binom{2n-2}{k}$ ,
2. a basis of  $\Phi(\Omega^k)$  is given by those elements  $e_I^+ \wedge e_J^-$  such that  $|I| + |J| = k$ ,
3. a full set of relations for  $\Phi(\Omega^{(0,\bullet)})$  is given by

$$e_j^- \wedge e_i^- + q^{-1} e_i^- \wedge e_j^-, \quad e_i^- \wedge e_i^-,$$

for  $i, j = 1, \dots, n-1$ , with  $i < j$ .

We now use the given basis of  $\Phi(\Omega^\bullet)$  to define a complex structure for the calculus. Denote by  $V^{(a,b)}$  the subspace of  $\Phi(\Omega^\bullet)$  spanned by those basis elements of the form  $e_I^+ \wedge e_J^-$  for which  $|I| = a, |J| = b$ . We see immediately that

$$\dim_{\mathbb{C}} (V^{(a,b)}) = \binom{n-1}{a} \binom{n-1}{b},$$

and that

$$\Phi(\Omega^k) \simeq \bigoplus_{a+b=k} V^{(a,b)}, \quad \text{for all } k.$$

The following proposition is implied by the presentation of [21, §3.3.4]. (Alternatively, see [49, §6, §7] for a direct proof in the notation of this paper.)

**THEOREM 6.5.** *For  $\Omega^\bullet$  the Heckenberger–Kolb calculus over  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , there is a unique covariant complex structure  $\Omega^{(\bullet,\bullet)}$  on  $\Omega^\bullet$  such that*

$$\Phi(\Omega^{(a,b)}) = V^{(a,b)}.$$

As shown in [50, Lemma 4.17] there exists a left  $\mathcal{O}_q(SU_n)$ -coinvariant form  $\kappa \in \Omega^{(1,1)}$  uniquely defined by

$$[\kappa] = i \sum_{a=1}^{n-1} e_a^- \wedge e_a^+ = i \sum_{a=1}^{n-1} (K_a^a \triangleright e_a^+) \wedge (K_a^{-a} \triangleright e_a^-). \tag{17}$$

This  $(1, 1)$ -form is a direct  $q$ -deformation of the fundamental form of the classical Fubini–Study Kähler metric on complex projective space. As the following theorem shows, much of the associated Kähler geometry survives intact under  $q$ -deformation, see [50, §4.5] for details.

**THEOREM 6.6.** *Let  $\Omega^\bullet$  be the Heckenberger–Kolb calculus of quantum projective space  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ .*

1. *The pair  $(\Omega^{(\bullet, \bullet)}, \kappa)$  is a covariant Hermitian structure for  $\Omega^\bullet$ .*
2. *Up to real scalar multiple of  $\kappa$ , it is the unique covariant Hermitian structure for the calculus  $\Omega^\bullet$ .*
3. *The pair  $(\Omega^{(\bullet, \bullet)}, \kappa)$  is a Kähler structure for  $\Omega^\bullet$ .*
4. *There exists an open interval  $I \subseteq \mathbb{R}$ , containing 1, such that the associated metric  $g_\kappa$  is positive definite, for all  $q \in I$ .*

Thus we see that, for every  $q \in I$ , quantum projective space  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , endowed with the Heckenberger–Kolb calculus, is a CQH-Kähler space with an associated pair of Dolbeault–Dirac BC-triples.

We finish with some results on the anti-holomorphic cohomology of the Heckenberger–Kolb calculus of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ . Taken together, these two results determine the anti-holomorphic Euler characteristic of the calculus.

**THEOREM 6.7** ([26] Corollary 4.2). *The Heckenberger–Kolb calculus  $\Omega_q^\bullet(\mathbb{C}\mathbb{P}^{n-1})$  is connected, which is to say  $\mathcal{H}^0 = \mathbb{C}1$ .*

**THEOREM 6.8** ([11] Proposition 7.2). *For the unique covariant complex structure  $\Omega^{(\bullet, \bullet)}$  of the Heckenberger–Kolb calculus  $\Omega_q^\bullet(\mathbb{C}\mathbb{P}^{n-1})$ , it holds that*

$$H^{(0,k)} = 0, \quad \text{for all } k = 1, \dots, n - 1.$$

**COROLLARY 6.9.** *The anti-holomorphic Euler characteristic of  $\Omega^{(\bullet, \bullet)}$  is given by*

$$\chi_{\bar{\partial}}(\Omega^\bullet) = \sum_{i=0}^{n-1} (-1)^i \dim_{\mathbb{C}}(H^{(0,i)}) = \dim_{\mathbb{C}}(H^{(0,0)}) = 1.$$

In Corollary 6.17 below, it is observed that the vanishing of higher cohomologies can alternatively be concluded from our given ladder presentation of the complex structure of  $\Omega_q^\bullet(\mathbb{C}\mathbb{P}^{n-1})$ .

REMARK 6.10. Most of the results of this subsection have been extended to the more general setting of the quantum Grassmannians (see §7 for a definition of quantum Grassmannians). The existence of a noncommutative Kähler structure was established in [34, 41]. Connectedness of the calculus was extended to the case of the quantum Grassmannians in [51]. The vanishing of higher cohomologies for the quantum Grassmannians was established in [52] using a noncommutative generalisation of the Kodaira vanishing theorem, formulated within the framework of noncommutative Kähler and Fano structures. Thus the anti-holomorphic Euler characteristic of the quantum Grassmannian is again 1.

6.2 AN ORDER I PRESENTATION OF THE CQH-COMPLEX SPACE

In this subsection we introduce a distinguished family of highest weight forms  $\nu_k \in \bar{\partial}\Omega^{(0,k)}$ , for  $k = 1, \dots, n-2$ , and use them to produce a ladder presentation (recall Definition 5.3) of the complex structure in Theorem 6.15. From this we then conclude in Corollary 6.16 that  $\mathcal{O}_q(\mathbb{CP}^{n-1})$  is an order I CQH-complex space. We begin by identifying a highest weight element of  $\mathcal{O}_q(\mathbb{CP}^{n-1})$ , and then calculate its associated Leibniz constants.

LEMMA 6.11. *The element  $z_{1n}$  is a highest weight vector of  $\mathcal{O}_q(\mathbb{CP}^{n-1})_{\text{hw}}$ , with weight*

$$\text{wt}(z_{1n}) = \varpi_1 + \varpi_{n-1}.$$

Moreover, it holds that

$$(\partial z_{1n})z_{1n} = q^2 z_{1n} \partial z_{1n}, \quad (\bar{\partial} z_{1n})z_{1n} = q^{-2} z_{1n} \bar{\partial} z_{1n}. \tag{18}$$

*Proof.* The fact that  $z_{1n} \in \mathcal{O}_q(\mathbb{CP}^{n-1})_{\text{hw}}$ , with the given weight, is a direct consequence of the definition of the action  $\triangleright$  and the dual pairing, as presented in Appendix B.4 and Appendix B.5. To establish (18) we use the unit  $U : \Omega^{(1,0)} \simeq \mathcal{O}_q(SU_n) \square_{\mathcal{O}_q(U_{n-1})} \Phi(\Omega^{(1,0)})$  of Takeuchi’s equivalence. Note first that

$$\begin{aligned} U\left((\partial z_{1n})z_{1n}\right) &= U(\partial z_{1n})z_{1n} \\ &= \text{proj}_{\Omega^{(1,0)}} \left( \left( \sum_{a,b=1}^n u_a^1 S(u_n^b) \square_{\mathcal{O}_q(U_{n-1})} [\partial z_{ab}] \right) z_{1n} \right) \\ &= \sum_{a=1}^{n-1} u_a^1 S(u_n^a) z_{1n} \square_{\mathcal{O}_q(U_{n-1})} [\partial z_{aa}]. \end{aligned}$$

From the defining relations of  $\mathcal{O}_q(SU_n)$  given in Appendix B.5, it is easy to conclude the identity

$$u_a^1 S(u_n^a) z_{1n} = q^2 z_{1n} u_a^1 S(u_n^a), \quad \text{for all } a \neq n. \tag{19}$$

Hence, as claimed,

$$U\left((\partial z_{1n})z_{1n}\right) = q^2 z_{1n} \sum_{a=1}^{n-1} u_a^1 S(u_n^n) \square_{\mathcal{O}_q(U_{n-1})} [\partial z_{an}] = q^2 U\left(z_{1n} \partial z_{1n}\right).$$

The second identity comes from Lemma 4.16 and the fact that  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  is self-conjugate, as established in Corollary C.6. Alternatively, it can be calculated directly just as for the first identity.  $\square$

In the  $q$ -deformed setting it is no longer guaranteed that  $\omega \wedge \omega = 0$ , for all forms  $\omega$ . However, as the following lemma shows, this identity does hold true for forms of type  $\bar{\partial}z_{1j}$ .

COROLLARY 6.12. *It holds that*

1.  $\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1,j-1} = -q^{-1} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1j}$ , for all  $j = 3, \dots, n$ ,
2.  $\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1j} = 0$ , for all  $j = 2, \dots, n$ .

*Proof.* By Corollary 4.13

$$0 = \bar{\partial}^2(z_{1n}^2) = (2)_{q^{-2}} \bar{\partial}(z_{1n} \bar{\partial}z_{1n}) = (2)_{q^{-2}} \bar{\partial}z_{1n} \wedge \bar{\partial}z_{1n}.$$

Assuming now that  $\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1j} = 0$ , for some  $j \geq 3$ , we see that

$$\begin{aligned} 0 &= F_{j-1} \triangleright (\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1j}) \\ &= \bar{\partial}z_{1j} \wedge (F_{j-1} \triangleright \bar{\partial}z_{1j}) + (F_{j-1} \triangleright \bar{\partial}z_{1j}) \wedge (K_{j-1}^{-1} \triangleright \bar{\partial}z_{1j}) \\ &= \bar{\partial}z_{1j} \wedge \bar{\partial}(F_{j-1} \triangleright z_{1j}) + \bar{\partial}(F_{j-1} \triangleright z_{1j}) \wedge \bar{\partial}(K_{j-1}^{-1} \triangleright z_{1j}) \\ &= q^{-2} \bar{\partial}z_{1j} \wedge \bar{\partial}z_{1,j-1} + q^{-3} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1j}. \end{aligned}$$

Thus whenever  $\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1j} = 0$ , we necessarily have that

$$\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1,j-1} = -q^{-1} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1j}.$$

Assume next that

$$\bar{\partial}z_{1j} \wedge \bar{\partial}z_{1,j-1} = -q^{-1} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1j}. \tag{20}$$

Operating by  $F_{j-1}$  gives us that

$$\begin{aligned} 0 &= F_{j-1} \triangleright \left( \bar{\partial}z_{1j} \wedge \bar{\partial}z_{1,j-1} + q^{-1} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1j} \right) \\ &= q^{-1} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1,j-1} + q^{-3} \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1,j-1} \\ &= (q^{-1} + q^{-3}) \bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1,j-1}. \end{aligned}$$

Thus we see that whenever (20) holds, we necessarily have that  $\bar{\partial}z_{1,j-1} \wedge \bar{\partial}z_{1,j-1} = 0$ . The corollary now follows by an inductive argument.  $\square$

LEMMA 6.13. For  $k = 0, \dots, n - 2$ , a highest weight vector is given by

$$\nu_k = \sum_{l=0}^k (-q)^l z_{1,n-l} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1,n-l}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \in \Omega^{(0,k)}, \quad (21)$$

where  $\widehat{\bar{\partial} z_{1,n-l}}$  denotes that the factor  $\bar{\partial} z_{1,n-l}$  has been omitted. Moreover, it holds that

$$\text{wt}(\nu_k) = (k + 1)\varpi_1 + \varpi_{n-k-1}.$$

*Proof.* Since it is clear that  $\nu_k$  is a weight vector, we need only show that  $E_i \triangleright \nu_k = 0$ , for all  $i = 1, \dots, n - 1$ . Note first that, for  $1 \leq i \leq n - k - 1$ , we must have  $E_i \triangleright \nu_k = 0$ . Next, for any  $i = n - k, \dots, n - 1$ , Lemma 6.3 and (28) imply that

$$E_i \triangleright \left( \sum_{l=0}^{n-i-2} (-q)^l z_{1,n-l} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1,n-l}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \right)$$

is equal to the following sum

$$\begin{aligned} & \sum_{l=0}^{n-i-2} (-q)^l z_{1,n-l} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1,n-l}} \wedge \cdots \\ & \qquad \qquad \qquad \cdots \wedge \bar{\partial} (K_i \triangleright z_{1,i+1}) \wedge \bar{\partial} (E_i \triangleright z_{1i}) \wedge \cdots \wedge \bar{\partial} z_{1,n-k}. \end{aligned}$$

Lemma 6.3 implies that this sum is equal to

$$\begin{aligned} & q^3 \sum_{l=0}^{n-i-2} (-q)^l z_{1,n-l} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1,n-l}} \wedge \cdots \\ & \qquad \qquad \qquad \cdots \wedge \bar{\partial} z_{1,i+1} \wedge \bar{\partial} z_{1,i+1} \wedge \cdots \wedge \bar{\partial} z_{1,n-k}, \end{aligned}$$

which by Corollary 6.12 is equal to zero. Similarly, it holds that

$$E_i \triangleright \left( \sum_{l=n-i+1}^k (-q)^l z_{1,n-l} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1,n-l}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \right) = 0.$$

Hence we see that

$$\begin{aligned} E_i \triangleright \nu_k &= E_i \triangleright \left( (-q)^{n-i-1} z_{1,i+1} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1,i+1}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \right. \\ & \qquad \qquad \qquad \left. + (-q)^{n-i} z_{1i} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1i}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \right) \\ &= (-1)^{n-i-1} q^{n-i+2} z_{1,i+1} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1i}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \\ & \qquad \qquad \qquad + (-1)^{n-i} q^{n-i+2} z_{1,i+1} \bar{\partial} z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial} z_{1i}} \wedge \cdots \wedge \bar{\partial} z_{1,n-k} \\ &= 0. \end{aligned}$$

Thus  $E_i \triangleright \nu_k = 0$ , for all  $i = 1, \dots, n - 1$ . Finally, as a direct examination confirms,  $\nu_k$  is a weight vector of weight  $(k + 1)\varpi_1 + \varpi_{n-k-1}$ , and so,  $\nu_k$  is a highest weight vector as claimed.  $\square$

COROLLARY 6.14. *For the form  $\bar{\partial}\nu_k \in \bar{\partial}\Omega_{\text{hw}}^{(0,k)}$ , it holds that:*

1.  $\bar{\partial}\nu_k = (k + 1)_{q^2} \bar{\partial}z_{1n} \wedge \cdots \wedge \bar{\partial}z_{1,n-k}$ ,
2.  $\bar{\partial}\nu_k \neq 0$ ,
3.  $\nu_k \in \bar{\partial}^\dagger \Omega^{(0,k+1)}$ , for  $k = 0, \dots, n - 1$ .

*Proof.*

1. By the commutation relations of Corollary 6.12, we see that

$$\begin{aligned} \bar{\partial}\nu_k &= \sum_{l=0}^k (-q)^l \bar{\partial}z_{1,n-l} \wedge \bar{\partial}z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial}z_{1,n-l}} \wedge \cdots \wedge \bar{\partial}z_{1,n-k} \\ &= \sum_{l=0}^k q^{2l} \bar{\partial}z_{1n} \wedge \cdots \wedge \bar{\partial}z_{1,n-k} \\ &= (k + 1)_{q^2} \bar{\partial}z_{1n} \wedge \cdots \wedge \bar{\partial}z_{1,n-k}. \end{aligned}$$

2. If the coset of  $\bar{\partial}\nu_k$  in  $\Phi(\Omega^\bullet)$  were non-zero, then it is clear that  $\bar{\partial}\nu_k$  would have to be non-zero. Unfortunately, by Lemma 6.2 we have that

$$[\bar{\partial}\nu_k] = (k + 1)_{q^2} [\bar{\partial}z_{1n}] \wedge \cdots \wedge [\bar{\partial}z_{1,n-k}] = 0.$$

On the other hand, the coset of  $\bar{\partial}z_{n1} \wedge \cdots \wedge \bar{\partial}z_{n,n-k}$  in  $\Phi(\Omega^\bullet)$  is non-zero, as we see from

$$[\bar{\partial}z_{n1} \wedge \cdots \wedge \bar{\partial}z_{n,n-k}] = [\bar{\partial}z_{n1}] \wedge \cdots \wedge [\bar{\partial}z_{n,n-k}] = e_1^- \wedge \cdots \wedge e_{k+1}^-.$$

Hence  $\bar{\partial}z_{n1} \wedge \cdots \wedge \bar{\partial}z_{n,n-k} \neq 0$ . Now, as a direct calculation confirms, there exists a non-zero  $\gamma \in \mathbb{R}$ , such that

$$(F_{n-1}^{k+2} \cdots F_{n-k-1}^{k+2} F_{n-k-2}^{k+1} \cdots F_1^{k+1}) \triangleright \bar{\partial}\nu_k = \gamma \bar{\partial}z_{n1} \wedge \cdots \wedge \bar{\partial}z_{n,n-k}.$$

Thus  $\bar{\partial}z_{1n} \wedge \cdots \wedge \bar{\partial}z_{1,n-k}$  must be non-zero as claimed.

3. Since  $\nu_k$  is a highest weight vector, and Hodge decomposition is a decomposition of left  $U_q(\mathfrak{sl}_n)$ -modules, the fact that the complex structure is of Gelfand type implies that  $\nu_k$  is either  $\bar{\partial}$ -exact,  $\bar{\partial}$ -coexact, or harmonic. Since we have just shown that  $\bar{\partial}\nu_k \neq 0$ , we must have that  $\nu_k$  is  $\bar{\partial}$ -coexact as claimed.  $\square$

With these results in hand we are now ready to establish the main result of this subsection, an order I presentation of  $C = (\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), \Omega^\bullet, *_\sigma, \Omega^{\bullet,\bullet})$ .

**THEOREM 6.15.** Denoting  $\Theta := \{\bar{\partial}\nu_k \mid k = 0, \dots, n - 2\}$ , the pair  $(z_{1n}, \Theta)$  is a real ladder presentation for  $C = (\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), \Omega^\bullet, *_\sigma, \Omega^{(\bullet, \bullet)})$ . Moreover, it holds that

$$\mathcal{H}^{(0,0)} = \mathbb{C}1, \quad \mathcal{H}^{(0,m)} = 0, \quad \text{for } m = 1, \dots, n - 1. \quad (22)$$

*Proof.* Example 4.6, together with Lemma 6.11 and Lemma 6.13, imply that  $z_{1n}^l \nu_k$  and  $z_{1n}^l \bar{\partial}\nu_k$  are highest weight vectors, for all  $l \in \mathbb{N}_0$ , and  $k = 0, \dots, n - 1$ . Moreover, it follows from Lemma 6.11 and Lemma 6.13 that

$$\text{wt}(z_{1n}^l \nu_k) = \text{wt}(z_{1n}^l \bar{\partial}\nu_k) = (l + k + 1)\varpi_1 + \varpi_{n-k-1} + l\varpi_{n-1}.$$

Comparing this with the list of highest weights appearing in the decomposition of  $\Omega^{(0,k)}$  in Lemma C.5, and recalling that  $\Omega^{(0,k)}$  is multiplicity-free, we see that every non-trivial highest weight space of  $\Omega^{(0,k)}$  contains an element of the form  $z_{1n}^l \bar{\partial}\nu_{k-1}$ , or  $z_{1n}^l \nu_k$ , for some  $l \in \mathbb{N}_0$ . Hence

$$\begin{aligned} \Omega^{(0,0)} &= \mathbb{C}1 \oplus \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{sl}_n)(z_{1n}^{l+1}), \\ \Omega^{(0,k)} &= \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{sl}_n)(z_{1n}^l \bar{\partial}\nu_{k-1}) \oplus \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{sl}_n)(z_{1n}^l \nu_k). \end{aligned}$$

Proposition 4.19 tells us that  $\bar{\partial}\Omega_{\text{hw}}^{(0,k-1)}$  and  $\bar{\partial}^\dagger\Omega_{\text{hw}}^{(0,k+1)}$  are closed under the action of the monoid  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})_{\text{hw}}$ . Thus  $z_{1n}^l \bar{\partial}\nu_{k-1} \in \bar{\partial}\Omega^{(0,k-1)}$  and  $z_{1n}^l \nu_k \in \bar{\partial}^\dagger\Omega^{(0,k+1)}$ . As a direct consequence

$$\bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{g})(z_{1n}^l \bar{\partial}\nu_{k-1}) = \bar{\partial}\Omega^{(0,k-1)}, \quad \bigoplus_{l \in \mathbb{N}_0} U_q(\mathfrak{g})(z_{1n}^l \nu_k) = \bar{\partial}^\dagger\Omega^{(0,k+1)}.$$

Thus we see that the anti-holomorphic harmonic forms are exactly as claimed in (22). Finally, we see that  $(z_{1n}, \Theta)$  is a ladder presentation of the calculus, which since the Leibniz constants of  $z_{1n}$  are  $q^2$  and  $q^{-2}$ , is a real ladder presentation.  $\square$

Combining this result with Lemma C.5 and Corollary C.6 now gives us the following corollary.

**COROLLARY 6.16.** The CQH-complex space  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  is order I.

We finish by observing that Proposition 6.8 can be concluded directly from Theorem 6.15. Explicitly, since the space of harmonic forms  $\mathcal{H}^{(0,k)}$  is trivial, for all  $k > 0$ , the bijection between harmonic forms and cohomology classes presented in Corollary 2.14 gives us the following result.

**COROLLARY 6.17.** For all  $k > 0$ , the anti-holomorphic cohomology groups  $H^{(0,k)}$  are trivial.

6.3 A DOLBEAULT–DIRAC PAIR OF SPECTRAL TRIPLES FOR QUANTUM PROJECTIVE SPACE

In this subsection we verify solidity for the unique covariant Kähler structure of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ . We then conclude that the Kähler structure gives a Dolbeault–Dirac pair of spectral triples.

LEMMA 6.18. *It holds that*

$$1. A_{z, \bar{\partial}\nu_k} \neq 0, \quad 2. A_{z, \bar{\partial}\nu_k} \neq q^2 - 1, \quad \text{for all } k = 0, \dots, n - 2.$$

*Proof.*

1. By Lemma 5.7, non-vanishing of  $A_{z, \bar{\partial}\nu_k}$  is equivalent to  $\partial z_{n1} \wedge \bar{\partial}\nu_k$  being a non-primitive form. As usual, we would like to demonstrate this by considering the coset of  $\partial z_{n1} \wedge \bar{\partial}\nu_k$  in  $\Phi(\Omega^\bullet)$ . Unfortunately, this coset is trivial. On the other hand, since the Lefschetz map  $L$  is a left  $U_q(\mathfrak{sl}_n)$ -module map,  $\partial z_{n1} \wedge \bar{\partial}\nu_k$  is primitive if and only if  $X \triangleright (\partial z_{n1} \wedge \bar{\partial}\nu_k)$  is primitive, for any  $X \in U_q(\mathfrak{sl}_n)$ . If we now fix

$$X := F_{n-1}^{k+2} \cdots F_1^{k+2},$$

a routine calculation confirms that

$$[X \triangleright (\partial z_{n1} \wedge \bar{\partial}\nu_k)] = \gamma e_{n-k-1}^+ \wedge e_{n-1}^- \wedge \cdots \wedge e_{n-k-1}^-,$$

for a certain non-zero scalar  $\gamma \in \mathbb{R}$ . Note next that

$$\left[ L^{(n-1)-(k+2)+1} (X \triangleright (\partial z_{n1} \wedge \bar{\partial}\nu_k)) \right]$$

is given explicitly by

$$\gamma [\kappa^{n-k-2}] \wedge e_{n-k-1}^+ \wedge e_{n-1}^- \wedge \cdots \wedge e_{n-k-1}^-. \tag{23}$$

It was shown in [50, Lemma 4.18] that  $[\kappa]^{n-k-2}$  is a non-zero scalar multiple of

$$\sum_{I \in O(n-k-2)} e_I^+ \wedge e_I^-,$$

where summation is over all ordered subsets of cardinality  $n - k - 2$ . This implies that (23) is equal to a non-zero scalar multiple of

$$e_1^+ \wedge \cdots \wedge e_{n-k-1}^+ \wedge e_1^- \wedge \cdots \wedge e_{n-1}^-.$$

Thus we see that  $X \triangleright (\partial z_{n1} \wedge \bar{\partial}\nu_k)$  is non-primitive, implying that  $\bar{\partial} z_{n1} \wedge \bar{\partial}\nu_k$  is non-primitive. It now follows from Lemma 5.7 that  $A_{z, \bar{\partial}\nu_k} \neq 0$  as claimed.

2. Using an analogous argument to the one above, it can be confirmed that  $\bar{\partial}\nu_k \wedge \partial z_{1n}$  is non-primitive. Lemma 5.8 then implies that  $A_{z, \bar{\partial}\nu_k} \neq q^2 - 1$  as claimed.  $\square$

LEMMA 6.19. *It holds that*

$$B_{z, \bar{\partial}\nu_k} = q^2(k + 1)_{q^2}^{-1}, \quad \text{for all } k = 0, \dots, n - 2.$$

*Proof.* Following the same argument as Lemma 6.11, one can establish the identity

$$\bar{\partial}z_{1n}z_{1j} = q^{-1}z_{1j}\bar{\partial}z_{1n}, \quad \text{for all } j = 2, \dots, n - 1.$$

It now follows from Corollary 6.12 that

$$\begin{aligned} \bar{\partial}z_{1n} \wedge \nu_k &= \sum_{l=0}^k (-q)^l \bar{\partial}z_{1n} \wedge z_{1, n-l} \bar{\partial}z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial}z_{1, n-l}} \wedge \cdots \wedge \bar{\partial}z_{1, n-k} \\ &= q^2 z_{1n} \bar{\partial}z_{1n} \wedge \bar{\partial}z_{1, n-1} \wedge \cdots \wedge \bar{\partial}z_{1, n-k} \\ &\quad + q^{-1} \sum_{l=1}^k (-q)^l z_{1, n-l} \wedge \bar{\partial}z_{1n} \wedge \bar{\partial}z_{1n} \wedge \cdots \wedge \widehat{\bar{\partial}z_{1, n-l}} \wedge \cdots \wedge \bar{\partial}z_{1, n-k} \\ &= q^2(k + 1)_{q^2}^{-1} z_{1n} \bar{\partial}\nu_k. \end{aligned}$$

Denoting by  $\mu_{\nu_k}$  the  $\Delta_{\bar{\partial}}$ -eigenvalue of  $\nu_k$ ,

$$\begin{aligned} \bar{\partial}z_{1n} \wedge \bar{\partial}^\dagger \bar{\partial}\nu_k &= \mu_{\nu_k} \bar{\partial}z_{1n} \wedge \nu_k \\ &= q^2(k + 1)_{q^2}^{-1} \mu_{\nu_k} z_{1n} \bar{\partial}\nu_k \\ &= q^2(k + 1)_{q^2}^{-1} z_{1n} \bar{\partial} \bar{\partial}^\dagger (\bar{\partial}\nu_k), \end{aligned}$$

giving us the claimed value of  $B_{z, \bar{\partial}\nu_k}$ .  $\square$

COROLLARY 6.20. *The CQH-Hermitian space  $(\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), \Omega^\bullet, *_\sigma, \Omega^{(\bullet, \bullet)}, \sigma)$  is solid, for all  $q \in \mathbb{R} \setminus \{-1, 0\}$ .*

The following theorem and corollary, the main results of this section, and indeed two of the principal results of the paper, now follow directly from Theorem 5.11, and the fact that  $\Omega^{(\bullet, \bullet)}$  is connected and has vanishing higher-order anti-holomorphic cohomologies (as shown in Lemma 6.17).

THEOREM 6.21. *A Dolbeault–Dirac pair of spectral triples is given by*

$$\left( \mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), L^2(\Omega^{(\bullet, 0)}), D_\partial \right), \quad \left( \mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1}), L^2(\Omega^{(0, \bullet)}), D_{\bar{\partial}} \right).$$

COROLLARY 6.22. *The operators  $D_\partial$  and  $D_{\bar{\partial}}$  both have non-trivial associated K-homology classes. In particular,*

$$\text{Index}(\mathfrak{b}(D_\partial)) = \text{Index}(\mathfrak{b}(D_{\bar{\partial}})) = 1.$$

REMARK 6.23. An important question to ask is whether or not the two spectral triples presented in Theorem 6.21 are unitarily equivalent, or at least have the same  $K$ -homology class. Note that since the  $*$ -map is neither an algebra map, nor a bounded operator (see [47, §1.7] for details), it cannot be used to implement such an equivalence.

## 7 IRREDUCIBLE QUANTUM FLAG MANIFOLDS

In the classical setting, complex projective space is a very special example of a flag manifold. This picture extends directly to the quantum group setting, where  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  is a very special type of quantum flag manifold  $\mathcal{O}_q(G/L_S)$  [15]. Moreover, for the irreducible quantum flag manifolds the Heckenberger–Kolb classification of differential calculi presented above for  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  extends directly, as does the existence of a unique covariant Kähler structure. In this section we carefully recall this material, and introduce a natural weakening of the definition of Gelfand type, which we term *weak Gelfand type*. We classify those non-exceptional irreducible quantum flag manifolds satisfying this new condition, and discuss in detail the extension of the order I framework to this more general setting.

### 7.1 THE QUANTUM HOMOGENEOUS SPACES

Let  $\mathfrak{g}$  be a complex simple Lie algebra of rank  $r$  and  $U_q(\mathfrak{g})$  the corresponding Drinfeld–Jimbo quantised enveloping algebra. For  $S$  a subset of simple roots, consider the Hopf  $*$ -subalgebra

$$U_q(\mathfrak{t}_S) := \langle K_i, E_j, F_j \mid i = 1, \dots, r; j \in S \rangle.$$

From the Hopf  $*$ -algebra embedding  $\iota : U_q(\mathfrak{t}_S) \hookrightarrow U_q(\mathfrak{g})$ , we get the dual Hopf  $*$ -algebra map  $\iota^\circ : U_q(\mathfrak{g})^\circ \rightarrow U_q(\mathfrak{t}_S)^\circ$ . By construction  $\mathcal{O}_q(G) \subseteq U_q(\mathfrak{g})^\circ$ , so we can consider the restriction map

$$\pi_S := \iota^\circ|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \rightarrow U_q(\mathfrak{t}_S)^\circ,$$

and the Hopf  $*$ -subalgebra  $\mathcal{O}_q(L_S) := \pi_S(\mathcal{O}_q(G)) \subseteq U_q(\mathfrak{t}_S)^\circ$ . We call the CMQGA-homogeneous space associated to  $\pi_S : \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(L_S)$  the *quantum flag manifold* corresponding to  $S$ , and denote it by

$$\mathcal{O}_q(G/L_S) := \mathcal{O}_q(G)^{\text{co}(\mathcal{O}_q(L_S))}.$$

We see that the definition of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^n)$ , as given in the previous section, corresponds to the special case of  $S = \{\alpha_1, \dots, \alpha_{n-2}\} \subseteq \Pi(\mathfrak{sl}_n)$ .

If  $S = \{1, \dots, r\} \setminus \{\alpha_i\}$ , where  $\alpha_i$  has coefficient 1 in the highest root of  $\mathfrak{g}$ , then we say that the associated quantum flag manifold is of *irreducible*. In the classical limit of  $q = 1$ , these homogeneous spaces reduce to the family of irreducible flag manifolds, or equivalently the compact Hermitian symmetric

spaces [1]. These algebras are also referred to as the *cominiscule* quantum flag manifolds, again reflecting terminology in the classical setting. Presented below is a useful diagrammatic presentation of the set of simple roots defining the irreducible quantum flag manifolds, along with the names of the various series. See [1] for a more detailed discussion.

$A_n$		$\mathcal{O}_q(\text{Gr}_{r,s})$	quantum Grassmanian
$B_n$		$\mathcal{O}_q(\mathbf{Q}_{2n+1})$	odd quantum quadric
$C_n$		$\mathcal{O}_q(\mathbf{L}_n)$	quantum Lagrangian Grassmannian
$D_n$		$\mathcal{O}_q(\mathbf{Q}_{2n})$	even quantum quadric
$D_n$		$\mathcal{O}_q(\mathbf{S}_n)$	quantum spinor variety
$E_6$		$\mathcal{O}_q(\mathbb{O}\mathbb{P}^2)$	quantum Cayley plane
$E_7$		$\mathcal{O}_q(\mathbf{F})$	quantum Freudenthal variety

### 7.2 THE HECKENBERGER–KOLB CALCULI AND THEIR KÄHLER STRUCTURES

We now recall the extension of Theorem 6.1, the Heckenberger–Kolb classification of first-order differential calculi over  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , to the setting of irreducible quantum flag manifolds.

We call the maximal prolongation of the direct sum of these two calculi the *Heckenberger–Kolb calculus* of  $\mathcal{O}_q(G/L_S)$ , and denote it by  $\Omega_q^\bullet(G/L_S)$ . It was shown in [21] that  $\Omega^\bullet(G/L_S)$  has classical dimension, and admits a left  $\mathcal{O}_q(G)$ -covariant  $\mathbb{N}_0^2$ -grading  $\Omega^{(\bullet,\bullet)}$  satisfying conditions 1 and 3 of the definition of a complex structure. It was later shown in [41] that the  $*$ -algebra structure of  $\mathcal{O}_q(G/L_S)$  extends to the structure of a differential  $*$ -calculus on  $\Omega_q^\bullet(G/L_S)$ , for each irreducible quantum flag manifolds. Moreover, it was shown that the  $\mathbb{N}_0^2$ -grading  $\Omega^{(\bullet,\bullet)}$  is a covariant complex structure with respect to this  $*$ -structure.

Finally, we note that since  $\Omega^{(1,0)}$  and  $\Omega^{(0,1)}$  are non-isomorphic as objects in  ${}_{\mathcal{O}_q(G/L_S)}\mathcal{O}_q(G)\text{-Mod}_0$ , this is the unique covariant complex structure for  $\Omega_q^\bullet(G/L_S)$ .

**THEOREM 7.1.** [20, Theorem 7.2] *For  $S$  a subset of simple roots of irreducible type, there exist exactly two non-isomorphic, irreducible, left-covariant, finite-dimensional, first-order differential calculi of finite dimension over  $\mathcal{O}_q(G/L_S)$ .*

It was observed in [50, §4.5] that, for each  $\Omega_q^\bullet(G/L_S)$ , the space of  $(1, 1)$ -forms contains a left-covariant real form  $\kappa$  which is unique up to real scalar multiple. Moreover, by extending the argument of [50, §4.5] for the case of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , the form  $\kappa$  is easily seen to be a closed central element of  $\Omega^\bullet$ . We now recall a conjecture originally proposed in [50, §4.5].

**CONJECTURE 7.2.** *For every irreducible quantum flag manifold  $\mathcal{O}_q(G/L_S)$ , a covariant Kähler structure for the Heckenberger–Kolb calculus of  $\mathcal{O}_q(G/L_S)$  is given by the pair  $(\Omega^{(\bullet,\bullet)}, \kappa)$ .*

The conjecture was motivated by the Kähler structure of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ , as presented in §6.6, and as originally considered in [50]. Subsequently, for all but a finite number of values of  $q$ , the conjecture was verified for every irreducible quantum flag manifold by Matassa.

**THEOREM 7.3** ([41] Theorem 5.10). *Let  $\Omega_q^\bullet(G/L_S)$  be the Heckenberger–Kolb calculus of the irreducible quantum flag manifold  $\mathcal{O}_q(G/L_S)$ . The pair  $(\Omega^{(\bullet,\bullet)}, \kappa)$  is a covariant Kähler structure for all  $q \in \mathbb{R}_{>0} \setminus F$ , where  $F$  is a finite, possibly empty, subset of  $\mathbb{R}_{>0}$ . Moreover, any element of  $F$  is necessarily non-transcendental.*

Building on this result, a positive definiteness result for the metric was established in [12, Lemma 10.10], allowing us to conclude that we have a CQH-Hermitian space, and hence a BC-triple.

**COROLLARY 7.4.** *For each irreducible quantum flag manifold  $\mathcal{O}_q(G/L_S)$ , there exists an open interval  $I \subseteq \mathbb{R}$  around 1, such that the associated metric  $g_\kappa$  is positive definite, for all  $q \in I$ . Hence,  $\mathcal{O}_q(G/L_S)$ , endowed with its Heckenberger–Kolb calculus, is a CQH-Hermitian space, for all  $q \in I$ , with an associated pair of Dolbeault–Dirac BC-triples.*

### 7.3 IRREDUCIBLE QUANTUM FLAG MANIFOLDS OF WEAK GELFAND TYPE

As is shown below, the only irreducible quantum flag manifold of Gelfand type is  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$ . However, the notion admits a natural generalisation, retaining many of the features of Gelfand type CQH-complex spaces, as we now present.

**DEFINITION 7.5.** We say that a CQH-complex space  $\mathbf{C} = (\mathbf{B}, \Omega^{(\bullet,\bullet)})$  is of *weak Gelfand type* if  $\bar{\partial}\Omega^{(0,\bullet)}$  is a graded multiplicity-free left  $A$ -comodule. We say that a CQH-Hermitian space is of *weak Gelfand type* if its constituent CQH-complex space is of weak Gelfand type.

Note first that all the results of §3.1 and §4.1 hold since neither make any assumptions about multiplicities. Moreover, the proof of Lemma 3.11 generalises directly to the weak Gelfand setting giving us the following lemma.

LEMMA 7.6. *For a CQH-Hermitian space of weak Gelfand type, the Laplacian  $\Delta_{\bar{\partial}}$  acts on every irreducible left  $A$ -sub-comodule of  $\bar{\partial}\Omega^{(0,\bullet)}$  and  $\bar{\partial}^\dagger\Omega^{(0,\bullet)}$  as a scalar multiple of the identity.*

Just as in the Gelfand setting, we can use the various symmetries of the Dolbeault double complex to find a number of equivalent formulations of weak Gelfand type. We omit the proof, which is a straightforward generalisation of the proof in the Gelfand case.

LEMMA 7.7. *Let  $\mathbf{C} = (\mathbf{B}, \Omega^{(\bullet,\bullet)})$  be a quantum homogeneous complex space, then the following conditions are equivalent:*

1.  $\mathbf{C}$  is of weak Gelfand type,
2.  $\mathbf{C}^{\text{op}}$ , the opposite CQH-complex space, is of weak Gelfand type,
3. the graded left  $A$ -comodule  $\partial\Omega^{(\bullet,0)}$  is graded multiplicity-free.

For the case of a quantum homogeneous Hermitian space, the definition of weak Gelfand type admits an additional number of equivalent formulations. We omit the proof of the following lemma, which is an easy consequence of Lemma 2.15, Lemma 3.3, and Lemma 7.7 above.

LEMMA 7.8. *For  $\mathbf{H} = (\mathbf{B}, \Omega^{(\bullet,\bullet)}, \sigma)$  a CQH-Hermitian space, the following are equivalent:*

1.  $\mathbf{H}$  is of Gelfand type,
2. the graded left  $A$ -comodule  $\bar{\partial}^\dagger\Omega^{(0,\bullet)}$  is graded multiplicity-free,
3. the graded left  $A$ -comodule  $\partial^\dagger\Omega^{(\bullet,0)}$  is graded multiplicity-free,
4. the graded left  $A$ -comodule  $\partial\Omega^{(\bullet,n)}$  is graded multiplicity-free,
5. the graded left  $A$ -comodule  $\partial^\dagger\Omega^{(\bullet,n)}$  is graded multiplicity-free,
6. the graded left  $A$ -comodule  $\bar{\partial}\Omega^{(n,\bullet)}$  is graded multiplicity-free,
7. the graded left  $A$ -comodule  $\bar{\partial}^\dagger\Omega^{(n,\bullet)}$  is graded multiplicity-free.

Finally, we note that this lemma implies that the assumptions of Lemma 4.22 hold in the weak Gelfand setting. Hence we again have an equivalence between connectivity and finite-dimensionality of  $H^0$ .

We would now like to identify those irreducible quantum flag manifolds which are of weak Gelfand type. As observed by Stokman and Dijkhuizen [15], the preservation under  $q$ -deformation of the Weyl character formula implies that, for any subset  $S$  of simple roots, the branching rules for the inclusion  $U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g})$  are the same as in the classical case:

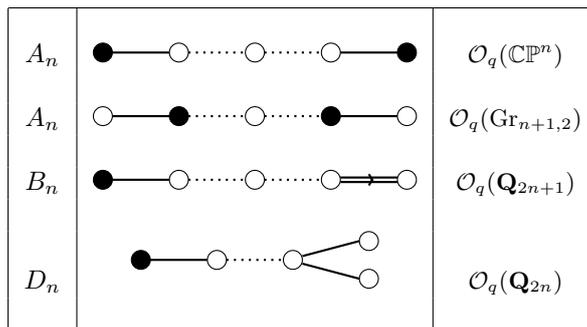
PROPOSITION 7.9 ([15] Proposition 4.2). *Let  $\mu \in \mathcal{P}^+$ . The multiplicity of any irreducible  $U_q(\mathfrak{sl}_S)$ -module in the decomposition of  $V_\mu$  into irreducible  $U_q(\mathfrak{sl}_S)$ -modules is the same as in the classical  $q = 1$  case.*

For any quantum homogeneous space  $B = A^{\text{co}(H)}$ , and any left  $H$ -comodule  $V$ , Frobenius reciprocity tells us that the multiplicities of a right  $A$ -comodule in  $\Psi(V)$  is completely determined by the branching rules for the quantum homogeneous space. Thus we get the following corollary.

COROLLARY 7.10. *An irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus, is of (weak) Gelfand type if and only if the classical compact Hermitian space  $G/L_S$  is of (weak) Gelfand type.*

To the best of our knowledge this question has not been explicitly addressed in the literature. However, there exist numerous suitable formulations of the necessary classical branching laws. (For example, we point the reader to the Young diagram approach of [30], which is very similar in spirit to the presentation of Appendix C.) This allows us to answer the question with relative ease for the non-exceptional cases. To keep the paper to a reasonable length, the technical details of the proof will be given in later work. Pending this, we present the following as a conjecture.

CONJECTURE 7.11. *The non-exceptional compact quantum Hermitian spaces, for which the covariant complex structure of the associated Heckenberger–Kolb calculus is of weak Gelfand type, are precisely those presented in the following two diagrams: The first identifies four countable families:*



The second diagram identifies three isolated examples, arising from low-dimensional redundancies in the table of compact quantum Hermitian spaces given above.

$B_3$		$\mathcal{O}_q(\mathbf{L}_2) \simeq \mathcal{O}_q(\mathbf{Q}_5)$
$D_3$		$\mathcal{O}_q(\mathbf{S}_3) \simeq \mathcal{O}_q(\mathbb{C}\mathbb{P}^3)$
$D_4$		$\mathcal{O}_q(\mathbf{S}_4) \simeq \mathcal{O}_q(\mathbf{Q}_8)$

Of these CQH-complex spaces, only quantum projective space  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^{n-1})$  is of Gelfand type.

For the two exceptional cases, that is, the quantum Cayley plane  $\mathcal{O}_q(\mathbb{O}\mathbb{P}^2)$ , and the quantum Freudenthal variety  $\mathcal{O}_q(\mathbf{F})$ , an explicit presentation of the necessary branching laws does not seem to have appeared in the literature. Hence, it is most likely necessary to derive them from a general framework, such as the Littlemann path model [38, 37]. We postpone such intricacies to later work, and for now satisfy ourselves with Conjecture 7.13, as motivated in §7.4.2.

#### 7.4 TOWARDS ORDER II CQH-HERMITIAN SPACES

In the previous subsection we presented the definition of weak Gelfand type and discussed those results which carry over from the Gelfand setting. In this subsection we detail some of the *new* behaviours that arise in the weak Gelfand setting. We finish with some conjectures, motivated by the discussion of the subsection.

##### 7.4.1 HODGE DECOMPOSITION AND THE $B_{\text{hw}}$ -ACTION

In the weak Gelfand case Proposition 4.19 is no longer guaranteed to hold. Explicitly, for any  $\omega \in \bar{\partial}\Omega_{\text{hw}}^{(0,\bullet)}$ , and any  $z \in \mathcal{O}_q(G/L_S)_{\text{hw}}$ , it is no longer guaranteed that the product  $z\omega$  is contained in  $\bar{\partial}\Omega^{(0,\bullet)}$ . This forces us to consider highest weight vectors of the form  $\Pi_{\bar{\partial}}(z\omega)$ , where we have denoted by  $\Pi_{\bar{\partial}} : \Omega^{(0,\bullet)} \rightarrow \bar{\partial}\Omega^{(0,\bullet)}$  the projection operator associated to Hodge decomposition. Such forms are still eigenvectors of  $\Delta_{\bar{\partial}}$ , with an important observation being that  $\Delta_{\bar{\partial}}(\Pi_{\bar{\partial}}(z\omega)) = \bar{\partial}\bar{\partial}^\dagger(z\omega)$ .

##### 7.4.2 SPHERICAL GENERATORS

By the considerations of §4.1, the weights of the highest weight vectors of  $\mathcal{O}_q(G/L_S)$  form an additive submonoid  $Z_S \subseteq \mathfrak{h}$  under addition. By Proposition 7.9, it is clear that the minimal number of generators of  $Z_S$ , considered as a

monoid, is the same in the classical  $q = 1$  case. A distinguished minimal set of generators for  $Z_S$ , the *spherical weights*, was presented in the classical case by Krämer in [33, Tabelle 1], and reproduced here in the table below.

In this table, the Dynkin nodes for the  $A$ -,  $B$ -,  $C$ -, and  $D$ -types are labelled in increasing order from left to right. For the exceptional cases, we again number from left to right in increasing order, finishing with the side node. Finally, note that for the spherical weights of  $\mathcal{O}_q(\mathbf{S}_{2m})$ , the weight  $2\varpi_{2m-1}$  or  $2\varpi_{2m}$  appears depending on the defining crossed node.

Comparing this table with the proposed classification of Conjecture 7.11, we make the following alternative version of the conjecture.

CONJECTURE 7.12. *The non-exceptional irreducible quantum flag manifolds  $\mathcal{O}_q(G/L_S)$  of Gelfand type are exactly those for which  $Z_S$  is generated by a single element, and those of weak Gelfand type are exactly those for which  $Z_S$  which is generated by two elements.*

Thus to accommodate the irreducible quantum flag manifolds of weak Gelfand type, it is necessary to generalise the order I requirements and consider irreducible  $U_q(\mathfrak{g})$ -modules of the form

$$U_q(\mathfrak{g})\Pi_{\bar{\partial}}(y^{l_1}z^{l_2}\omega), \quad \text{for } y, z \in B, \omega \in \Omega_{\text{hw}}^{(0,k)}, \text{ and } l_1, l_2 \in \mathbb{N}_0.$$

Coming finally to the exceptional cases, we note that, from the table of spherical weights below,  $\mathcal{O}_q(\mathbb{O}\mathbb{P}^2)$  has two spherical generators, while  $\mathcal{O}_q(\mathbf{F})$  has three. Speculating that Conjecture 7.12 extends to the non-exceptional setting, we make the following conjecture.

CONJECTURE 7.13. *It holds that:*

1. *The quantum Cayley plane  $\mathcal{O}_q(\mathbb{O}\mathbb{P}^2)$ , endowed with its Heckenberger–Kolb calculus, is of weak Gelfand type.*
2. *The quantum Freudenthal variety  $\mathcal{O}_q(\mathbf{F})$ , endowed with its Heckenberger–Kolb calculus, is not of weak Gelfand type.*

IRREDUCIBLE QUANTUM FLAG MANIFOLD $\mathcal{O}_q(G/L_S)$	SPHERICAL WEIGHTS OF $\mathcal{O}_q(G/L_S)$
$\mathcal{O}_q(\mathrm{Gr}_{r,s})$	$\varpi_1 + \varpi_{r+s-1}, \varpi_2 + \varpi_{r+s-2}, \dots, \varpi_r + \varpi_s$
$\mathcal{O}_q(\mathbf{Q}_{2n+1})$	$2\varpi_1, \varpi_2$
$\mathcal{O}_q(\mathbf{L}_n)$	$2\varpi_1, 2\varpi_2, \dots, 2\varpi_n$
$\mathcal{O}_q(\mathbf{Q}_{2n})$	$2\varpi_1, \varpi_2$
$\mathcal{O}_q(\mathbf{S}_{2m})$	$\varpi_2, \varpi_4, \dots, \varpi_{2m-2}, 2\varpi_{2m-1}$ or $2\varpi_{2m}$
$\mathcal{O}_q(\mathbf{S}_{2m+1})$	$\varpi_2, \varpi_4, \dots, \varpi_{2m-2}, \varpi_{2m} + \varpi_{2m+1}$
$\mathcal{O}_q(\mathbb{O}\mathbb{P}^2)$	$\varpi_1 + \varpi_5, \varpi_6$
$\mathcal{O}_q(\mathbf{F})$	$2\varpi_1, \varpi_2, \varpi_6$

### 7.4.3 LEIBNIZ CONSTANTS

In the weak Gelfand setting, Leibniz constants are not guaranteed to exist, which is to say Corollary 4.12 does not necessarily hold. We do however have the following lemma, whose proof is a direct generalisation of Corollary 4.12.

LEMMA 7.14. *Let  $\mathbf{C}$  be a CQH-complex space of weak Gelfand type, with constituent quantum homogeneous space  $B_{\mathrm{hw}}$ , and  $z \in B_{\mathrm{hw}}$  a non-harmonic element. Then there exist non-zero constants  $\lambda_z, \zeta_z \in \mathbb{C}$ , uniquely defined by*

$$\Pi_{\partial}((\partial z)z) = \lambda_z \Pi_{\partial}(z\partial z), \quad \Pi_{\bar{\partial}}((\bar{\partial} z)z) = \zeta_z \Pi_{\bar{\partial}}(z\bar{\partial} z),$$

where  $\Pi_{\partial}$ , and  $\Pi_{\bar{\partial}}$ , are the projections from  $\Omega^{\bullet}$  to  $\partial\Omega^{\bullet}$ , and  $\bar{\partial}\Omega^{\bullet}$ , respectively, given by Hodge decomposition.

We now see that Proposition 4.16 extends directly to the weak Gelfand setting.

COROLLARY 7.15. *Let  $\mathbf{C}$  be a CQH-complex space with constituent quantum homogeneous space  $B$ . If  $B$  is self-conjugate, then, for any  $z \in B$  such that  $\lambda_z, \zeta_z \in \mathbb{R}$ , we have  $\zeta_z = \lambda_z^{-1}$ .*

*Proof.* Note first that we have the identity  $* \circ \Pi_{\partial} = \Pi_{\bar{\partial}} \circ *$ . Using this identity, the proof of Proposition 4.16 now carries over directly to the weak Gelfand setting.  $\square$

7.4.4 SOME CONJECTURES

Taking the above novel features into account, it is possible to generalise the order I framework, to an *order II* framework, and include the irreducible quantum flag manifolds of weak Gelfand type. Solidity can be shown to extend to this setting, as well as its equivalence with the compact resolvent of the Dolbeault–Dirac operator. This gives us a robust set of tools with which to construct spectral triples for the irreducible quantum flag manifolds of weak Gelfand type. Preliminary investigations [14] strongly suggest that solidity holds, motivating us to make the following conjecture.

CONJECTURE 7.16. *Let  $\mathcal{O}_q(G/L_S)$  be an irreducible quantum flag manifold of weak Gelfand type, and let  $(\Omega^{(\bullet,\bullet)}, \kappa)$  be its covariant Kähler structure, unique up to real scalar multiple. A Dirac–Dolbeault pair of spectral triples is given by*

$$\left(\mathcal{O}_q(G/L_S), L^2(\Omega^{(\bullet,0)}), D_\partial\right), \quad \left(\mathcal{O}_q(G/L_S), L^2(\Omega^{(0,\bullet)}), D_{\bar{\partial}}\right).$$

*Moreover, the associated  $K$ -homology class of each spectral triple is non-trivial.*

In the above conjecture we restrict to those irreducible quantum flag manifolds of weak Gelfand type, reflecting the fact that the approach of this paper naturally extends to this subfamily of spaces. However, there is no reason to suspect that the compact resolvent condition does not hold for all the irreducible quantum flag manifolds. Verifying the condition in this more general setting requires a more formal approach, based on the classical proof of the compact resolvent condition for general Dirac operators [18, §4]. This is at present being undertaken in the setting of noncommutative Sobolev spaces [3], and so, we are motivated to reformulate a conjecture originally presented in [50].

CONJECTURE 7.17. *For every irreducible quantum flag manifold  $\mathcal{O}_q(G/L_S)$ , a covariant Kähler structure for the Heckenberger–Kolb calculus of  $\mathcal{O}_q(G/L_S)$  is given by the pair  $(\Omega^{(\bullet,\bullet)}, \kappa)$ . Moreover, a Dirac–Dolbeault pair of spectral triples is given by*

$$\left(\mathcal{O}_q(G/L_S), L^2(\Omega^{(\bullet,0)}), D_\partial\right), \quad \left(\mathcal{O}_q(G/L_S), L^2(\Omega^{(0,\bullet)}), D_{\bar{\partial}}\right),$$

*and the associated  $K$ -homology class of each spectral triple is non-trivial.*

A CQGAS, QUANTUM HOMOGENEOUS SPACES, AND FROBENIUS RECIPROCITY

In this appendix we recall the basics of cosemisimple Hopf algebras, compact matrix quantum groups algebras, quantum homogeneous spaces, as well as some natural compatibility requirements between them. We then recall the version of Takeuchi’s categorical equivalence for quantum homogeneous spaces most suited to our purposes. Finally, we present an extension of Frobenius reciprocity to the setting of quantum homogeneous spaces.

## A.1 HOPF ALGEBRAS AND CQGAS

All algebras are assumed to be unital and defined over  $\mathbb{C}$ . All unadorned tensor products are defined over  $\mathbb{C}$ . The symbols  $A$  and  $H$  will denote Hopf algebras with comultiplication  $\Delta$ , counit  $\varepsilon$ , antipode  $S$ , unit  $1$ , and multiplication  $m$ . We use Sweedler notation throughout, and denote  $a^+ := a - \varepsilon(a)1$ , for  $a \in A$ , and  $A^+ = A \cap \ker(\varepsilon)$ .

For any left  $A$ -comodule  $(V, \Delta_L)$ , its space of *matrix elements* is the sub-coalgebra

$$C(V) := \text{span}_{\mathbb{C}}\{(\text{id} \otimes f)\Delta_L(v) \mid f \in \text{Hom}_{\mathbb{C}}(V, \mathbb{C}), v \in V\} \subseteq A.$$

The notion of cosemisimplicity for a Hopf algebra will be essential in the paper and all Hopf algebras will be assumed to be cosemisimple. We present three equivalent formulations of the definition (a proof of their equivalence can be found in [27, §11.2.1]).

DEFINITION A.1. A Hopf algebra  $A$  is called *cosemisimple* if it satisfies the following three equivalent conditions:

1. There is an isomorphism  $A \simeq \bigoplus_{V \in \widehat{A}} C(V)$ , where  $\widehat{A}$  denotes the equivalence classes of left  $A$ -comodules.
2. The abelian category  ${}^A\text{Mod}$  of left  $A$ -comodules is semisimple.
3. There exists a unique linear map  $\mathbf{h} : A \rightarrow \mathbb{C}$ , which we call the *Haar measure*, such that  $\mathbf{h}(1) = 1$ , and

$$(\text{id} \otimes \mathbf{h}) \circ \Delta(a) = \mathbf{h}(a)1, \quad (\mathbf{h} \otimes \text{id}) \circ \Delta(a) = \mathbf{h}(a)1.$$

While the assumption of cosemisimplicity is enough for most of our requirements, when discussing positive definiteness we need the following stronger structure introduced in [16].

DEFINITION A.2. A *compact matrix quantum group algebra*, or a *CMQGA*, is a finitely generated cosemisimple Hopf  $*$ -algebra  $A$  such that  $\mathbf{h}(a^*a) > 0$ , for all non-zero  $a \in A$ .

It is important to note that every CMQGA admits a (not necessarily unique)  $C^*$ -algebraic completion to a compact matrix quantum group in the sense of Woronowicz [63]. Moreover, every compact matrix quantum group contains a dense CMQGA [16].

## A.2 CMQGA-HOMOGENEOUS SPACES

A left  $A$ -comodule algebra  $P$  is a comodule which is also an algebra such that the comodule structure map  $\Delta_L : P \rightarrow A \otimes P$  is an algebra map. Equivalently, it can be defined as a monoid object in  ${}^A\text{Mod}$ , the category of left  $A$ -comodules.

For a left  $A$ -comodule  $V$  with structure map  $\Delta_R$ , we say that an element  $v \in V$  is *coinvariant* if  $\Delta_L(v) = 1 \otimes v$ . We denote the subspace of all  $A$ -coinvariant elements by  ${}^{\text{co}(A)}V$ , and call it the *coinvariant subspace* of the coaction. We use the analogous conventions for right comodules.

DEFINITION A.3. A *homogeneous* right  $H$ -coaction on  $A$  is a coaction of the form  $(\text{id} \otimes \pi) \circ \Delta$ , where  $\pi : A \rightarrow H$  is a surjective Hopf algebra map. A *quantum homogeneous space*  $B := A^{\text{co}(H)}$  is the coinvariant subspace of such a coaction.

As is easily verified, every quantum homogeneous space  $B := A^{\text{co}(H)}$  is a subalgebra of  $A$ . Moreover, the coaction of  $A$  restricts to a left  $A$ -coaction  $\Delta_L : B \rightarrow A \otimes B$  giving it the structure of a left  $A$ -comodule algebra. We finish with a convenient, and natural, definition, which identifies the class of homogeneous spaces we will concern ourselves with in this paper.

DEFINITION A.4. A *CMQGA-homogeneous space* is a quantum homogeneous space given by a Hopf  $*$ -algebra projection  $\pi : A \rightarrow H$ , such that  $A$  and  $H$  are CMQGA's.

### A.3 TAKEUCHI'S EQUIVALENCE

We briefly recall Takeuchi's equivalence [59], or more precisely a special case of the bimodule version of Takeuchi's equivalence. For a more detailed presentation we refer the reader to [59], [20], or [49, 50]. Throughout this subsection,  $A$  and  $H$  will be Hopf algebras,  $\pi : A \rightarrow H$  a Hopf algebra map, and  $B = A^{\text{co}(H)}$  the associated quantum homogeneous space.

DEFINITION A.5. The category  ${}^A_B\text{Mod}_0$  has as objects  $B$ -bimodules  $\mathcal{F}$ , endowed with a left  $A$ -comodule structure  $\Delta_L : \mathcal{F} \rightarrow A \otimes \mathcal{F}$ , such that

1.  $\mathcal{F}B^+ \subseteq B^+\mathcal{F}$
2.  $\Delta_L(bfb') = \Delta_L(b)\Delta_L(f)\Delta_L(b')$ , for all  $b, b' \in B, f \in \mathcal{F}$ .

The morphisms in  ${}^A_B\text{Mod}_0$  are those  $B$ -bimodule homomorphisms which are also homomorphisms of left  $A$ -comodules. The usual bimodule tensor product  $\otimes_B$  can be used to give the category a monoidal structure defined by

$$\Delta_L : \mathcal{F} \otimes_B \mathcal{D} \rightarrow A \otimes \mathcal{F} \otimes_B \mathcal{D}, \quad f \otimes d \mapsto f_{(-1)}d_{(-1)} \otimes f_{(0)} \otimes_B d_{(0)},$$

for  $\mathcal{F}, \mathcal{D} \in {}^A_B\text{Mod}_0$ .

DEFINITION A.6. The category  ${}^H\text{Mod}$  has left  $H$ -comodules as objects, and right  $H$ -comodule maps as morphisms. The usual tensor product of comodules  $\otimes$  now endows the category with a monoidal structure

$$\Delta_L : V \otimes W \rightarrow A \otimes V \otimes W, \quad v \otimes w \mapsto v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}.$$

If  $\mathcal{F} \in {}^A_B\text{Mod}_0$ , then  $\mathcal{F}/(B^+\mathcal{F})$  becomes an object in  ${}^H\text{Mod}$  with the left  $H$ -coaction

$$\Delta_L[f] = \pi(f_{(-1)}) \otimes [f_{(0)}], \quad f \in \mathcal{F}, \quad (24)$$

where  $[f]$  denotes the coset of  $f$  in  $\mathcal{F}/(B^+\mathcal{F})$ . A functor  $\Phi : {}^A_B\text{Mod}_0 \rightarrow {}^H\text{Mod}$  is now defined as follows:  $\Phi(\mathcal{F}) := \mathcal{F}/(B^+\mathcal{F})$ , and if  $g : \mathcal{F} \rightarrow \mathcal{D}$  is a morphism in  ${}^A_B\text{Mod}_0$ , then  $\Phi(g) : \Phi(\mathcal{F}) \rightarrow \Phi(\mathcal{D})$  is the map to which  $g$  descends on  $\Phi(\mathcal{F})$ . If  $V \in {}^H\text{Mod}$ , then the *cotensor product* of  $A$  and  $V$ , defined by

$$A \square_H V := \ker(\Delta_R \otimes \text{id} - \text{id} \otimes \Delta_L : A \otimes V \rightarrow A \otimes H \otimes V),$$

becomes an object in  ${}^A_B\text{Mod}_0$  by defining a  $B$ -bimodule structure

$$b\left(\sum_i a^i \otimes v^i\right) := \sum_i ba^i \otimes v^i, \quad \left(\sum_i a^i \otimes v^i\right)b := \sum_i a^i b \otimes v^i, \quad (25)$$

and a left  $A$ -coaction  $\Delta_L\left(\sum_i a^i \otimes v^i\right) := \sum_i a^i_{(1)} \otimes a^i_{(2)} \otimes v^i$ . We now define a functor  $\Psi : {}^H\text{Mod} \rightarrow {}^A_B\text{Mod}_0$  as follows:  $\Psi(V) := A \square_H V$ , and if  $\gamma$  is a morphism in  ${}^H\text{Mod}$ , then  $\Psi(\gamma) := \text{id} \otimes \gamma$ .

**THEOREM A.7** (Takeuchi’s Equivalence). *An adjoint equivalence of categories between  ${}^A_B\text{Mod}_0$  and  ${}^H\text{Mod}$  is given by the functors  $\Phi$  and  $\Psi$  and the natural isomorphisms*

$$\begin{aligned} C : \Phi \circ \Psi(V) &\rightarrow V, & \left[\sum_i a^i \otimes v^i\right] &\mapsto \sum_i \varepsilon(a^i)v^i, \\ U : \mathcal{F} &\rightarrow \Psi \circ \Phi(\mathcal{F}), & f &\mapsto f_{(-1)} \otimes [f_{(0)}]. \end{aligned}$$

Moreover, the equivalence is a monoidal equivalence with respect to the the natural isomorphism

$$\mu : \Phi(\mathcal{F}) \otimes \Phi(\mathcal{D}) \rightarrow \Phi(\mathcal{F} \otimes_B \mathcal{D}), \quad [f] \otimes [d] \mapsto [f \otimes_B d].$$

We define the *dimension* of an object  $\mathcal{F} \in {}^A_B\text{Mod}_0$  to be the vector space dimension of  $\Phi(\mathcal{F})$ . Note that by cosemisimplicity of  $A$ , the category  ${}^H\text{Mod}$  is semisimple, and so,  ${}^A_B\text{Mod}_0$  must also be semisimple.

#### A.4 FROBENIUS RECIPROCITY FOR QUANTUM HOMOGENEOUS SPACES

In this subsection we present a direct Hopf algebraic generalisation of Frobenius reciprocity for equivariant vector bundles over a homogeneous space. The proof carries over from the classical situation without difficulty, so we will not state it. Moreover, there exists a large number of related formulations of this result in the literature. For example, it is established in the compact quantum group setting in [53], and in the general setting of coring comodules in [5, §22.12].

The result is stated for two Hopf algebras  $A$  and  $H$ , and a surjective Hopf map  $\pi : A \rightarrow H$ . For  $U, W$  two right  $A$ -comodules, we denote by  $\text{Hom}^A(U, W)$  the space of right  $A$ -comodule maps from  $U$  to  $W$ . Moreover, for any right  $H$ -comodule  $V$ , we denote by  $\text{Hom}^H(U, V)$  the space of right  $H$ -comodule maps from  $U$  to  $V$ , with respect to the right  $H$ -comodule structure induced on  $U$  by  $\pi$ .

LEMMA A.8 (Frobenius Reciprocity). *For  $U \in {}^A\text{Mod}$  and  $V \in {}^H\text{Mod}$ , it holds that*

$$\dim_{\mathbb{C}}(\text{Hom}^A(U, A \square_H V)) = \dim_{\mathbb{C}}(\text{Hom}^H(U, V)).$$

## B DRINFELD–JIMBO QUANTUM GROUPS

In this appendix we recall some basic material about semisimple complex Lie algebras  $\mathfrak{g}$  and their associated Drinfeld–Jimbo quantised enveloping algebras  $U_q(\mathfrak{g})$ . We also discuss their type 1 representation theory, along with the associated quantum coordinate algebras  $\mathcal{O}_q(G)$ . Throughout, where basic proofs or details are omitted we refer the reader to [27, §6, §7, §9]. We then specialise to the  $A$ -series quantum coordinate algebra  $\mathcal{O}_q(SU_n)$ . We recall its explicit FRT-presentation and identify it with the abstract quantum coordinate algebra presentation via a dual pairing of Hopf algebras. We finish with some standard material on Young diagrams necessary for an unambiguous presentation of the branching laws of §6.

### B.1 QUANTUM INTEGERS

Quantum integers are ubiquitous in the study of quantum groups. In this paper they play a significant role in describing the spectrum of noncommutative Dolbeault–Dirac operators. There are two different but related formulations for quantum integers, so we take care to clarify our choice of conventions. For  $q \in \mathbb{C}$ , and  $m \in \mathbb{N}$ , the *quantum  $q$ -integer*  $(m)_q$  is the complex number

$$(m)_q := 1 + q + q^2 + \dots + q^{m-1}.$$

Note that when  $q \neq 1$ , we have the identity

$$(m)_q = \frac{1 - q^m}{1 - q}.$$

We contrast this with the alternative (perhaps more standard) version of quantum integers:

$$[l]_q := q^{-m+1} + q^{-m+3} + \dots + q^{m-3} + q^{m-1}.$$

It is instructive to note that the two conventions are related by the identity

$$[m]_q = q^{1-m}(m)_{q^2}.$$

The second version of quantum integers was used in [50], but will *not* be used in this paper. Instead, we adopt the first formulation, it being the one which arises naturally in our spectral calculations, as is most readily evidenced by Corollary 4.13.

B.2 DRINFELD–JIMBO QUANTISED ENVELOPING ALGEBRAS

Let  $\mathfrak{g}$  be a finite-dimensional complex simple Lie algebra of rank  $r$ . We fix a Cartan subalgebra  $\mathfrak{h}$  with corresponding root system  $\Delta \subseteq \mathfrak{h}^*$ , where  $\mathfrak{h}^*$  denotes the linear dual of  $\mathfrak{h}$ . Let  $\Delta^+$  be a choice of positive roots, and let  $\Pi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_r\}$  be the corresponding set of simple roots. Denote by  $(\cdot, \cdot)$  the symmetric bilinear form induced on  $\mathfrak{h}^*$  by the Killing form of  $\mathfrak{g}$ , normalised so that any shortest simple root  $\alpha_i$  satisfies  $(\alpha_i, \alpha_i) = 2$ . The *coroot*  $\alpha_i^\vee$  of a simple root  $\alpha_i$  is defined by

$$\alpha_i^\vee := \frac{\alpha_i}{d_i} = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \quad \text{where } d_i := \frac{(\alpha_i, \alpha_i)}{2}.$$

The Cartan matrix  $(a_{ij})_{ij}$  of  $\mathfrak{g}$  defined by  $a_{ij} := (\alpha_i^\vee, \alpha_j)$ . Let  $q \in \mathbb{R}$  such that  $q \neq -1, 0, 1$ , and denote  $q_i := q^{d_i}$ . The *quantised enveloping algebra*  $U_q(\mathfrak{g})$  is the noncommutative associative algebra generated by the elements  $E_i, F_i$ , and  $K_i, K_i^{-1}$ , for  $i = 1, \dots, r$ , subject to the relations

$$\begin{aligned} K_i E_j &= q_i^{a_{ij}} E_j K_i, & K_i F_j &= q_i^{-a_{ij}} F_j K_i, \\ K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \end{aligned}$$

along with the quantum Serre relations

$$\begin{aligned} \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{q_i} E_i^{1-a_{ij}-s} E_j E_i^s &= 0, & \text{for } i \neq j, \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{q_i} F_i^{1-a_{ij}-s} F_j F_i^s &= 0, & \text{for } i \neq j, \end{aligned}$$

where we have used the  $q$ -binomial coefficient

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{[n]_q!}{[r]_q! [n-r]_q!}.$$

A Hopf algebra structure is defined on  $U_q(\mathfrak{g})$  by setting

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, & \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i \\ S(E_i) &= -E_i K_i^{-1}, & S(F_i) &= -K_i F_i, & S(K_i) &= K_i^{-1}, \\ \varepsilon(E_i) &= 0, & \varepsilon(F_i) &= 0, & \varepsilon(K_i) &= 1. \end{aligned}$$

A Hopf  $*$ -algebra structure, called the *compact real form*, is defined by

$$K_i^* := K_i, \quad E_i^* := K_i F_i, \quad F_i^* := E_i K_i^{-1}.$$

Finally, we denote by  $U_q(\mathfrak{n}_+)$ , and  $U_q(\mathfrak{n}_-)$ , the unital subalgebras of  $U_q(\mathfrak{g})$  generated by the elements  $E_1, \dots, E_r$ , and  $F_1, \dots, F_r$ , respectively.

### B.3 TYPE 1 REPRESENTATIONS

The set of *fundamental weights*  $\{\varpi_1, \dots, \varpi_r\}$  of  $\mathfrak{g}$  is the dual basis of simple coroots  $\{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ , which is to say

$$(\alpha_i^\vee, \varpi_j) = \delta_{ij}, \quad \text{for all } i, j = 1, \dots, r.$$

We denote by  $\mathcal{P}$  the *integral weight lattice* of  $\mathfrak{g}$ , which is to say the  $\mathbb{Z}$ -span of the fundamental weights. Moreover,  $\mathcal{P}^+$  denotes the cone of *dominant integral weights*, which is to say the  $\mathbb{N}_0$ -span of the fundamental weights.

A  $U_q(\mathfrak{g})$ -module  $V$  is irreducible if and only if it is of the form  $U_q(\mathfrak{g})z$ , for  $z$  a highest weight vector. Moreover, the space of highest weight elements of any irreducible module is necessarily one-dimensional.

For each  $\mu \in \mathcal{P}^+$  there exists an irreducible finite-dimensional  $U_q(\mathfrak{g})$ -module  $V_\mu$ , uniquely defined by the existence of a vector  $v_\mu \in V_\mu$ , which we call a *highest weight vector*, satisfying

$$E_i \triangleright v_\mu = 0, \quad K_i \triangleright v_\mu = q^{(\mu, \alpha_i)} v_\mu, \quad \text{for all } i = 1, \dots, r.$$

Moreover,  $v_\mu$  is unique up to scalar multiple. We call any finite direct sum of such  $U_q(\mathfrak{g})$ -representations a *type-1 representation*. In general, a vector  $v \in V_\mu$  is called a *weight vector* of *weight*  $\text{wt}(v) \in \mathcal{P}$  if

$$K_i \triangleright v = q^{(\text{wt}(v), \alpha_i)} v, \quad \text{for all } i = 1, \dots, r. \tag{26}$$

Each type 1 module  $V_\mu$  decomposes into a direct sum of *weight spaces*, which is to say, those subspaces of  $V_\mu$  spanned by weight vectors of any given weight. For any such highest weight vector  $v$ , we find it convenient to denote

$$\text{wt}(v) := \mu, \quad \text{and} \quad \text{wt}_i(v) := \mu_i, \quad \text{where} \quad \mu = \sum_{i=1}^r \mu_i \varpi_i.$$

We denote by  $U_q(\mathfrak{g})\mathbf{type}_1$  the full subcategory of  $U_q(\mathfrak{g})$ -modules whose objects are finite sums of type-1 modules  $V_\mu$ , for  $\mu \in \mathcal{P}^+$ . It is clear that  $U_q(\mathfrak{g})\mathbf{type}_1$  is abelian, semisimple, and equivalent to the category of finite-dimensional representations of  $\mathfrak{g}$ . Moreover, the Weyl character formula remains unchanged under  $q$ -deformation, which is to say, for any  $\mu \in \mathcal{P}^+$ , the dimensions of the weight spaces of the  $U_q(\mathfrak{g})$ -module  $V_\mu$  have the same as the dimensions as for the corresponding classical  $\mathfrak{g}$ -module. Finally, we denote by  $U_q(\mathfrak{g})\mathbf{LF}_1$  the full subcategory of  $U_q(\mathfrak{g})$ -modules whose objects are (not necessarily finite) sums of the type-1 modules. (Note that LF stands for *locally finite*.)

B.4 QUANTISED COORDINATE ALGEBRAS  $\mathcal{O}_q(G)$

Let  $V$  be a finite-dimensional  $U_q(\mathfrak{g})$ -module,  $v \in V$ , and  $f \in V^*$ , the linear dual of  $V$ . Consider the function

$$c_{f,v}^V : U_q(\mathfrak{g}) \rightarrow \mathbb{C}, \quad X \mapsto f(X(v)).$$

The *coordinate ring* of  $V$  is the subspace

$$C(V) := \text{span}_{\mathbb{C}}\{c_{f,v}^V \mid v \in V, f \in V^*\} \subseteq U_q(\mathfrak{g})^*.$$

In fact, we see that  $C(V) \subseteq U_q(\mathfrak{g})^\circ$ , and that a Hopf subalgebra of  $U_q(\mathfrak{g})^\circ$  is given by

$$\mathcal{O}_q(G) := \bigoplus_{V \in \text{type}_1} C(V).$$

We call  $\mathcal{O}_q(G)$  the *type-1 Hopf dual* of  $U_q(\mathfrak{g})$ , or alternatively the *quantum coordinate algebra* of  $G$ , where  $G$  is the unique connected, simply connected, complex algebraic group having  $\mathfrak{g}$  as its complex Lie algebra.

By construction  $\mathcal{O}_q(G)$  is cosemisimple. Moreover,  $\mathcal{O}_q(G)$  is a domain (for a proof see [4, Theorem I.8.9]). Dualising the compact real form of  $U_q(\mathfrak{g})$  gives a Hopf  $*$ -algebra structure on  $\mathcal{O}_q(G)$ , with respect to which it is a CQGA. Since every finite-dimensional irreducible representation of  $U_q(\mathfrak{g})$  is contained in some tensor product of fundamental representations,  $\mathcal{O}_q(G)$  is finitely generated, and hence a CQMGA.

The evaluation pairing  $\mathcal{O}_q(G) \times U_q(\mathfrak{g}) \rightarrow \mathbb{C}$  is by construction a dual pairing of Hopf algebras. In particular it gives us a dual pairing of Hopf  $*$ -algebras, which is to say

$$\langle X^*, f \rangle = \overline{\langle X, S(f)^* \rangle}, \quad \langle X, f^* \rangle = \overline{\langle S(X)^*, f \rangle}, \quad \text{for all } X \in U_q(\mathfrak{g}), f \in \mathcal{O}_q(G).$$

For any left  $\mathcal{O}_q(G)$ -comodule algebra  $(V, \Delta_L)$ , we can define a left  $U_q(\mathfrak{g})$ -module structure on  $V$  according to

$$U_q(\mathfrak{g}) \times V \rightarrow V, \quad (X, v) \mapsto \langle S(X), v_{(-1)} \rangle v_{(0)}. \tag{27}$$

This gives us the equivalence of categories

$$\mathcal{O}_q(G)\text{Mod} \rightarrow {}_{U_q(\mathfrak{g})}\mathbf{LF}_1, \quad (V, \Delta_L) \mapsto (V, \triangleright).$$

We will use this equivalence throughout the paper, tacitly identifying  $\mathcal{O}_q(G)$ -comodules and  $U_q(\mathfrak{g})$ -modules of type-1. For any left  $\mathcal{O}_q(G)$ -comodule algebra  $P$ , a useful result is that, for  $X \in U_q(\mathfrak{g})$ , and  $a, b \in P$ , we have

$$X \triangleright (ab) = (X_{(2)} \triangleright a)(X_{(1)} \triangleright b). \tag{28}$$

B.5 THE HOPF ALGEBRA  $\mathcal{O}_q(SU_n)$

In this subsection, we recall the well-known FRT-presentation of  $\mathcal{O}_q(SU_{n+1})$ , see [56] or [27, §9] for further details.

For  $q \in \mathbb{R} \setminus \{-1, 0\}$ , let  $\mathcal{O}_q(M_n)$  be the unital complex algebra generated by the elements  $u_j^i$ , for  $i, j = 1, \dots, n$  satisfying the relations

$$u_k^i u_k^j = q u_k^j u_k^i, \quad u_i^k u_j^k = q u_j^k u_i^k,$$

whenever  $1 \leq i < j \leq n$ , and  $1 \leq k \leq n$ , and the relations

$$u_l^i u_k^j = u_k^j u_l^i, \quad u_k^i u_l^j = u_l^j u_k^i + (q - q^{-1}) u_l^i u_k^j,$$

whenever  $1 \leq i < j \leq n$  and  $1 \leq k < l \leq n$ . A bialgebra structure on  $\mathcal{O}_q(M_n)$ , with coproduct  $\Delta$  and counit  $\varepsilon$ , is uniquely determined by  $\Delta(u_j^i) := \sum_{k=1}^n u_k^i \otimes u_j^k$  and  $\varepsilon(u_j^i) := \delta_{ij}$ . Let  $\det_n$ , the *quantum determinant*, denote the element

$$\begin{aligned} \det_n &:= \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u_{\sigma(1)}^1 u_{\sigma(2)}^2 \cdots u_{\sigma(n)}^n \\ &= \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u_1^{\sigma(1)} u_2^{\sigma(2)} \cdots u_n^{\sigma(n)}, \end{aligned}$$

with summation taken over all permutations  $\sigma$  of the set  $\{1, \dots, n\}$ , and  $\ell(\sigma)$  the number of inversions in  $\sigma$ . As is well known [27, §9.2.2],  $\det_n$  is a central and grouplike element of  $\mathcal{O}_q(M_n)$ . Consider next the quotient algebra  $\mathcal{O}_q(M_n)/\langle \det_n - 1 \rangle$ , where  $\langle \det_n - 1 \rangle$  denotes the ideal generated by  $\det_n - 1$ . Note that the maps  $\Delta$  and  $\varepsilon$  descend to a well-defined bialgebra structure on the quotient algebra, which in addition can be endowed with a Hopf algebra structure by defining

$$S(u_j^i) := (-q)^{i-j} \sum_{\sigma \in S_{n-1}} (-q)^{\ell(\sigma)} u_{l_1}^{\sigma(k_1)} u_{l_2}^{\sigma(k_2)} \cdots u_{l_{n-1}}^{\sigma(k_{n-1})} \quad (29)$$

$$= (-q)^{i-j} \sum_{\sigma \in S_{n-1}} (-q)^{\ell(\sigma)} u_{\sigma(l_1)}^{k_1} u_{\sigma(l_2)}^{k_2} \cdots u_{\sigma(l_{n-1})}^{k_{n-1}}, \quad (30)$$

where  $\{k_1, \dots, k_{n-1}\} := \{1, \dots, n\} \setminus \{j\}$ , and  $\{l_1, \dots, l_{n-1}\} := \{1, \dots, n\} \setminus \{i\}$  as ordered sets. Finally, a Hopf  $*$ -algebra structure can be defined by setting  $(u_j^i)^* := S(u_j^j)$ . We denote this Hopf  $*$ -algebra by  $\mathcal{O}_q(SU_n)$ , and call it the *quantum special unitary group of degree  $n$* .

We now present a Hopf algebra isomorphism between  $\mathcal{O}_q(SU_n)$  and the type-1 Hopf dual of  $U_q(\mathfrak{sl}_n)$ . A non-degenerate dual pairing of Hopf algebras between  $\mathcal{O}_q(SU_n)$  and  $U_q(\mathfrak{sl}_n)$  is uniquely determined by

$$\langle K_i, u_j^j \rangle = q^{\delta_{i,j-1} - \delta_{ij}}, \quad \langle E_i, u_i^{i+1} \rangle = 1, \quad \langle F_i, u_{i+1}^i \rangle = 1, \quad (31)$$

with all other pairings of generators being zero. This determines a Hopf algebra embedding of  $\mathcal{O}_q(SU_n)$  into  $U_q(\mathfrak{sl}_n)^\circ$ , the image of which is precisely the quantum coordinate algebra of  $U_q(\mathfrak{sl}_n)$ .

C DECOMPOSING  $\Omega^{(0, \bullet)}$  INTO IRREDUCIBLE  $U_q(\mathfrak{sl}_n)$ -MODULES

In this appendix, we use Frobenius reciprocity to decompose  $\Omega^{(0, \bullet)}$  into irreducible left  $U_q(\mathfrak{sl}_n)$ -submodules. As a direct application, we prove that  $\mathcal{O}_q(\mathbb{CP}^{n-1})$ , endowed with its Heckenberger–Kolb calculus, is of Gelfand type and self-conjugate.

Presentations of the irreducible modules occurring in the anti-holomorphic forms have previously appeared in both the classical [24, Proposition 5.2] and quantum group literature [9, Proposition 5.5]. We re-establish the presentation here so as to completely guarantee consistency of conventions, and to give a self-contained exposition of how such presentations are obtained for the benefit of non-experts.

## C.1 SKEW YOUNG DIAGRAMS AND YOUNG TABLEAUX

A *Young diagram* is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. We see that Young diagrams with  $p$  rows are equivalent to *integer partitions of order  $p$* , which is to say  $p$ -tuples

$$\mu = (\mu_1, \dots, \mu_p) \in \mathbb{N}_0^p, \quad \text{such that } \mu_1 \geq \dots \geq \mu_p.$$

We denote by  $F^\mu$  the Young diagram corresponding to an integer partition  $\mu$ . For a partition  $\mu = (\mu_1, \dots, \mu_p)$ , its *conjugate*  $\mu' = (\mu'_1, \dots, \mu'_{\mu_1})$  is the unique partition of order  $\mu_1$  such that  $\mu'_s$  is equal to the number of boxes in the  $s^{\text{th}}$  column of  $F^\mu$ . Note that the Young diagram  $F^{\mu'}$  can be obtained from  $F^\mu$  by reflecting it along its northwest-southeast diagonal.

We can put a partial order on the set of all Young diagrams, or equivalently, a partial order on the set of all integer partitions, by defining  $\mu \succeq \nu$  whenever  $\mu_i \geq \nu_i$ , for all  $i = 1, \dots, p$ , with the possible addition of trailing zeros. For any pair  $\mu \succeq \nu$ , the associated *skew Young diagram*  $F^{\mu \setminus \nu}$  is given by removing from  $F^\mu$  all boxes belonging to the obvious superimposition of  $F^\nu$  on  $F^\mu$ . The *weight* of a skew Young diagram is the number of boxes in the diagram, which is to say,

$$|\mu \setminus \nu| := \sum_{i=1}^p \mu_i - \nu_i \in \mathbb{N}_0.$$

In what follows, we find the following alternative presentation of partitions useful. Denote by  $e_i$ , for  $i = 1, \dots, p$ , the standard generators of the monoid  $\mathbb{N}_0^p$ . The *fundamental* partitions are given by

$$\varpi_k := e_1 + e_2 + \dots + e_k, \quad \text{for } k = 1, \dots, p.$$

Note that any partition  $\mu = (\mu_1, \dots, \mu_p)$  can be expressed as the sum of fundamental partitions:

$$\mu = (\mu_1 - \mu_2)\varpi_1 + (\mu_2 - \mu_3)\varpi_2 + \dots + (\mu_{p-1} - \mu_p)\varpi_{p-1} + \mu_p\varpi_p,$$

A *horizontal strip* is a skew Young diagram where every column contains at most one box. Alternatively, horizontal strips can be presented in terms of pairs of integer partitions  $\mu \succeq \nu$  such that, for  $p$  the order of  $\mu$ ,

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq \nu_p. \tag{32}$$

We denote by  $\text{HSC}(\mu)$  the set of all partitions  $\nu \succeq \mu$  such that  $F^{\mu \setminus \nu}$  is a horizontal strip.

As suggested by the notation, we have an obvious bijection between the irreducible representations of the quantised enveloping algebra  $U_q(\mathfrak{sl}_n)$ , and Young diagrams with  $r = \text{rank}(\mathfrak{g})$  rows. Explicitly, let  $V_\mu$  be an irreducible  $U_q(\mathfrak{g})$ -module, with defining dominant integral weight  $\mu \in \mathcal{P}^+$ . Expressing  $\mu$  in terms of the fundamental weights as  $\mu = \sum_{i=1}^r a_i \varpi_i$ , we have the corresponding partition  $\mu = \sum_{i=1}^r a_i \varpi_i$ , and hence the corresponding Young diagram  $F^\mu$ .

### C.2 THE DECOMPOSITION

Applying Frobenius reciprocity requires an understanding of the branching rules for the inclusion  $U_q(\mathfrak{sl}_{n-1}) \hookrightarrow U_q(\mathfrak{sl}_n)$ . However, as explained in greater detail in §7, the fact that Weyl’s character formula remains unchanged under  $q$ -deformation implies that the branching laws are equivalent to the classical case. The branching laws for the inclusion  $\mathfrak{sl}_{n-1} \hookrightarrow \mathfrak{sl}_n$  admit a well-known formulation originally due to Weyl [62] (see also [24, Proposition 5.1]) which allows us to immediately write down the corresponding  $q$ -deformed branching laws.

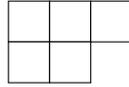
LEMMA C.1. *For any partition  $\mu$ , with corresponding irreducible  $U_q(\mathfrak{sl}_n)$ -module  $V_\mu$ , an isomorphism of  $U_q(\mathfrak{sl}_{n-1})$ -modules is given by*

$$V_\mu \simeq \bigoplus_{\nu \in \text{HSC}(\mu)} V_{\bar{\nu}}(\nu_{n-1} - |\mu \setminus \nu|), \tag{33}$$

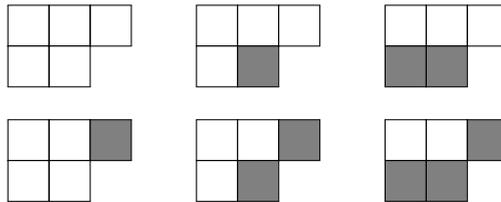
where  $\nu = \sum_{i=1}^{n-1} \nu_i \varpi_i$ , and  $\bar{\nu} := \nu - \nu_{n-1} \varpi_{n-1}$ , for  $\varpi_i$  the fundamental partitions, and where summation is taken over all partitions  $\nu \in \text{HSC}(\mu)$ , for  $\text{HSC}(\mu)$  the set of partitions complementary to the horizontal strips of  $\mu$  (as defined in Appendix C.1). In particular, the decomposition is multiplicity-free for each  $V_\mu$ .

In preparation for the application of Frobenius reciprocity below, we now consider two specific applications of the branching rules for the case of  $U_q(\mathfrak{sl}_3)$ . The presentation here is given in terms of Young diagrams, with the intention of providing the reader with some visual intuition.

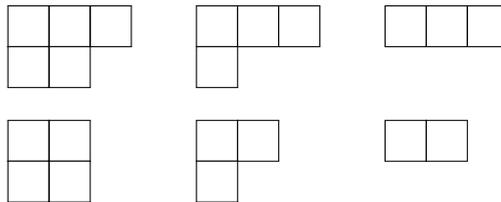
EXAMPLE C.2. We begin with an explicit example of the branching procedure for the inclusion  $U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{sl}_3)$ , corresponding to the quantum homogeneous space  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^2)$ . Consider the partition  $\mu := (3, 2)$  with corresponding Young diagram:



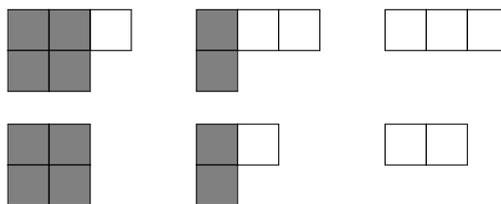
We present the 6 possible partitions  $\nu \in \text{HSC}(\mu)$  in Young diagram form, highlighting those boxes which form horizontal strips:



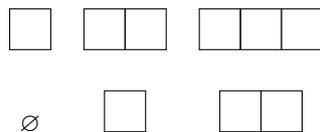
Removing the highlighted boxes, we arrive at the set of Young diagrams corresponding to the partitions in  $\text{HSC}(\mu)$ :



Next we present the Young diagrams of the partitions  $\bar{\nu} = \nu - \nu_2 \varpi_2$ . We do this in two steps, first highlighting the boxes to be removed from  $F^\nu$  in order to arrive at  $F^{\bar{\nu}}$ :



Removing the highlighted columns we arrive at the following list of Young diagrams, where  $\emptyset$  denotes the empty Young diagram:



Thus the decomposition of  $V_\mu$  into irreducible  $U_q(\mathfrak{sl}_2)$ -modules is given by

$$V_\mu \simeq V_{\varpi_1} \oplus V_{2\varpi_1} \oplus V_{3\varpi_1} \oplus \mathbb{C} \oplus V_{\varpi_1} \oplus V_{\varpi_2}. \tag{34}$$

Finally, subtracting the number of boxes removed in the first step, from the number of columns removed in the second step, gives us the weight of  $K_2$ . Explicitly, the decomposition of  $V_\mu$  into irreducible  $U_q(\mathfrak{sl}_2)$ -modules is given by

$$V_\mu \simeq V_{\varpi_1}(2) \oplus V_{2\varpi_1}(0) \oplus V_{3\varpi_1}(-2) \oplus \mathbb{C}(1) \oplus V_{\varpi_1}(-1) \oplus V_{2\varpi_1}(-3). \tag{35}$$

Note that while the decomposition of  $V_\mu$  into  $U_q(\mathfrak{sl}_2)$ -submodules contains multiplicities, the decomposition into  $U_q(\mathfrak{sl}_2)$ -modules is multiplicity-free.

EXAMPLE C.3. As we saw in the previous example, the Young diagram presentation of the branching process can be understood as a combination of two steps:

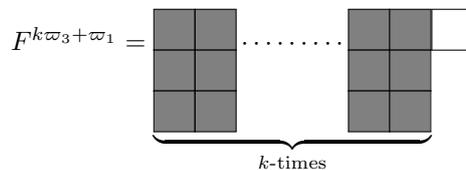
1. Remove from a Young diagram  $F^\mu$  a horizontal strip  $F^{\mu \setminus \nu}$ .
2. Remove all columns of height  $n - 1$ .

Thus finding all possible  $U_q(\mathfrak{sl}_n)$ -modules whose  $U_q(\mathfrak{sl}_{n-1})$ -branching contains a given module  $V$  amounts to finding all possible Young diagrams which can be operated on by steps 1 and 2 to produce the Young diagram corresponding to  $V$ .

Let us apply this process to a concrete example corresponding to the case of  $\mathcal{O}_q(\mathbb{C}\mathbb{P}^3)$ : For the  $U_q(\mathfrak{sl}_3)$ -module  $V_{\varpi_1}$  we will find all possible  $U_q(\mathfrak{sl}_4)$ -modules whose branching contains  $V_{\varpi_1}$  as a decomposition. Recall first that  $V_{\varpi_1}$  has the corresponding Young diagram:



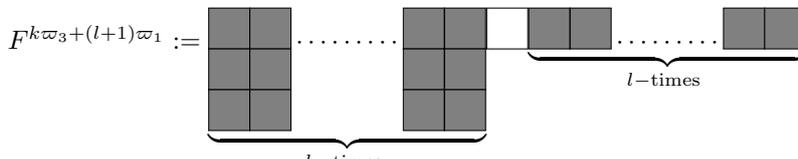
We reverse step 2 by adding an arbitrary number  $k \in \mathbb{N}_0$  of columns of height 3 to obtain the Young diagram:



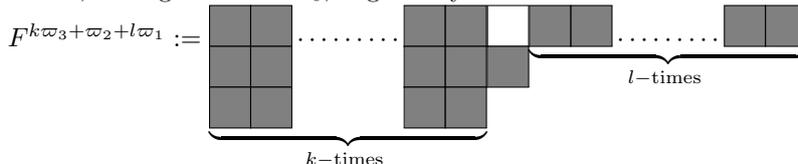
To reverse step 1, we must find all possible Young diagrams  $F^\mu$  such that

$$k\varpi_3 + \varpi_1 \in HSC(\mu).$$

In fact, we see that there exist exactly two such families of Young diagrams. The first, for a general  $l \in \mathbb{N}_0$ , is given by



The second, for a general  $l \in \mathbb{N}_0$ , is given by



To perform this process for  $U_q(\mathfrak{sl}_3)$ -branching, we need to take care of the weight of  $K_3$ . Let us assume that we want to branch to the representation  $V_{\varpi_1}(c)$ , for some  $c \in \mathbb{Z}$ . As we saw in the previous example, the weight of  $K_3$  is precisely the number of boxes removed in step 1 minus the number of columns removed in step 2. Thus we see that the branched representation  $\varpi_3 + (l+1)\varpi_1$  contains  $V_{\varpi_1}(c)$  if and only if  $l - k = c$ , while the branched representation  $\varpi_3 + \varpi_2 + l\varpi_1$  contains  $V_{\varpi_1}(c)$  if and only if  $l - k + 1 = c$ .

Returning now to the general case of  $\mathcal{O}_q(\mathbb{CP}^{n-1})$ . We would now like a complete description of the  $U_q(\mathfrak{sl}_n)$ -modules appearing in the irreducible decomposition of  $\Omega^{(0,k)}$ . First, however, we need to identify the inducing representations  $V^{(0,k)}$  as a  $U_q(\mathfrak{sl}_{n-1})$ -modules. (Note that we present the trivial case of  $V^{(0,0)}$  to highlight the fact that its  $K_{n-1}$ -weight does not follow the general pattern for the higher forms.)

COROLLARY C.4. *We have the following isomorphisms of left  $U_q(\mathfrak{sl}_{n-1})$ -modules*

$$V^{(0,0)} \simeq \mathbb{C}(0), \quad V^{(0,k)} \simeq V_{\varpi_{n-k-1}}(-k-1), \quad \text{for } k = 1, \dots, n-1.$$

*Proof.* Since the isomorphism  $V^{(0,0)} \simeq \mathbb{C}(0)$  is obvious, we can move directly onto the higher forms. Recalling that  $e_i^- \wedge e_i^- = 0$ , for all  $i$ , we see from Lemma 6.3 that

$$E_j \triangleright (e_{n-1}^- \wedge \dots \wedge e_{n-k}^-) = 0, \quad \text{for } j = 1, \dots, n-2.$$

Another application of Lemma 6.3 confirms that

$$K_j \triangleright (e_{n-1}^- \wedge \dots \wedge e_{n-k}^-) = q^{\delta_{j,n-k-1}} e_{n-1}^- \wedge \dots \wedge e_{n-k}^-, \quad \text{for } j = 1, \dots, n-2.$$

Hence  $e_{n-1}^- \wedge \dots \wedge e_{n-k}^-$  is a  $U_q(\mathfrak{sl}_{n-1})$ -highest weight vector of weight  $\varpi_{n-k-1}$ . Recalling that the dimension of universal enveloping algebra modules is unchanged under  $q$ -deformation, we see that

$$\dim_{\mathbb{C}} \left( U_q(\mathfrak{sl}_{n-1}) e_{n-1}^- \wedge \dots \wedge e_{n-k}^- \right) = \binom{n-1}{k}.$$

Recalling from Theorem 6.4 that  $\Phi(\Omega^{(0,k)})$  is also an  $\binom{n-1}{k}$ -dimensional space, we see that

$$U_q(\mathfrak{sl}_{n-1}) e_{n-1}^- \wedge \cdots \wedge e_{n-k}^- = V^{(0,k)}.$$

Thus  $V^{(0,k)}$  is isomorphic as a  $U_q(\mathfrak{sl}_{n-1})$ -module, to  $V_{\varpi_{n-k-1}}$ . Finally, Lemma 6.3 tells us that

$$K_{n-1} \triangleright (e_{n-1}^- \wedge \cdots \wedge e_{n-k}^-) = q^{-k-1} e_{n-1}^- \wedge \cdots \wedge e_{n-k}^-.$$

Hence  $V^{(0,k)}$  is isomorphic, as a  $U_q(\mathfrak{sl}_{n-1})$ -module, to  $V_{\varpi_{n-k-1}}(-k-1)$ , as claimed.  $\square$

LEMMA C.5. *The complex structure  $\Omega^{(\bullet,\bullet)}$  is of Gelfand type. Moreover, an irreducible  $U_q(\mathfrak{sl}_n)$ -module appears in the decomposition of  $\Omega^{(0,k)}$  into irreducibles only if its highest weight is of the form*

1. 0 when  $k = 0$ ,
2.  $(l+k+1)\varpi_1 + \varpi_{n-k-1} + l\varpi_{n-1}$ , for  $l \in \mathbb{N}_0$ , when  $k = 0, \dots, n-2$ ,
3.  $(l+k)\varpi_1 + \varpi_{n-k} + l\varpi_{n-1}$ , for  $l \in \mathbb{N}_0$ , when  $k = 2, \dots, n-1$ .

*Proof.* Lemma C.1 tells us that the decomposition of any  $U_q(\mathfrak{sl}_n)$ -module into  $U_q(\mathfrak{sl}_{n-1})$ -submodules is multiplicity-free. Frobenius reciprocity, as presented in Proposition A.8, now implies that  $\Omega^{(\bullet,\bullet)}$  is of Gelfand type.

Let us now assume that  $k = 1, \dots, n-2$ . By Corollary C.4 above,  $V^{(0,k)}$  is isomorphic to  $V_{\varpi_{n-k-1}}(-k-1)$  as a  $U_q(\mathfrak{sl}_{n-1})$ -module. Lemma C.1 tells us that  $V_{\varpi_{n-k-1}}(-k-1)$  appears as a  $U_q(\mathfrak{sl}_{n-1})$ -submodule, of some  $U_q(\mathfrak{sl}_n)$ -module  $V_\mu$ , if and only if there exists a  $\nu = \sum_{i=1}^{n-1} \nu_i \varpi_i$  in  $\text{HSC}(\mu)$  such that  $\bar{\nu} = \varpi_{n-k-1}$  and  $\nu_{n-1} - |\mu \setminus \nu| = -k-1$ . Note first that a partition  $\nu$  satisfies  $\bar{\nu} = \varpi_{n-k-1}$  if and only if

$$\nu = l\varpi_{n-1} + \varpi_{n-k-1}, \quad \text{for some } l \in \mathbb{N}_0. \tag{36}$$

Next, recall that  $\nu \in \text{HSC}(\mu)$ , for some partition  $\mu$ , if and only if (32) is satisfied. Thus any  $\mu$  must be of the form

$$l\varpi_{n-1} + \varpi_{n-k-1} + a\varpi_1, \quad \text{or} \quad l\varpi_{n-1} + \varpi_{n-k} + a\varpi_1, \quad \text{for some } a \in \mathbb{N}_0. \tag{37}$$

Next we see that the identity  $\nu_{n-1} - |\mu \setminus \nu| = -k-1$  is satisfied in the first case if and only if  $l-a = -k-1$ , and in the second case if  $l-(a+1) = -k-1$ . Thus  $V^{(0,k)}$  appears as a summand in the decomposition of  $V_\mu$  into irreducible  $U_q(\mathfrak{sl}_{n-1})$ -modules if and only if

$$\mu = (l+k+1)\varpi_1 + \varpi_{n-k-1} + l\varpi_{n-1}, \quad \text{or} \quad \mu = (l+k)\varpi_1 + \varpi_{n-k} + l\varpi_{n-1}. \tag{38}$$

Frobenius reciprocity, for the inclusion  $U_q(\mathfrak{sl}_{n-1}) \hookrightarrow U_q(\mathfrak{sl}_n)$ , now tells us that a left  $U_q(\mathfrak{sl}_n)$ -module appears as a submodule of  $\Omega^{(0,k)}$  if and only if its highest weight is of the form (38), as claimed. The proofs for the cases  $k = 0$ , and  $k = n - 1$ , are analogous, and so, we omit them.  $\square$

COROLLARY C.6. *Quantum projective space  $\mathcal{O}_q(\mathbb{CP}^{n-1})$  is self-conjugate.*

*Proof.* From Lemma C.5 above we see that the irreducible submodules of  $\Omega^{(0,0)}$  are distinct in dimension. Since the dimension of an irreducible  $U_q(\mathfrak{sl}_n)$ -module  $V$  is clearly equal to the dimension of its image under the  $*$ -map,  $\mathcal{O}_q(\mathbb{CP}^{n-1})$  must be self-conjugate.  $\square$

#### REFERENCES

- [1] R. J. BASTON AND M. G. EASTWOOD, *The Penrose transform. Its interaction with representation theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1989. Oxford Science Publications.
- [2] E. J. BEGGS AND P. S. SMITH, *Noncommutative complex differential geometry*, J. Geom. Phys., 72 (2013), pp. 7–33.
- [3] E. J. BEGGS AND T. BRZEZIŃSKI, *Noncommutative differential operators, Sobolev spaces and the centre of a category*, J. Pure Appl. Algebra, 218 (2014), pp. 1–17.
- [4] K. A. BROWN AND K. R. GOODEARL, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2002.
- [5] T. BRZEZIŃSKI AND R. WISBAUER, *Corings and comodules*, vol. 309 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003.
- [6] A. L. CAREY AND J. PHILLIPS, *Unbounded Fredholm modules and spectral flow*, Canad. J. Math., 50 (1998), pp. 673–718.
- [7] A. L. CAREY, J. PHILLIPS, AND A. RENNIE, *Spectral triples: examples and index theory*, in Noncommutative geometry and physics: renormalisation, motives, index theory, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2011, pp. 175–265.
- [8] A. CONNES, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.
- [9] F. D’ANDREA AND L. DĄBROWSKI, *Dirac operators on quantum projective spaces*, Comm. Math. Phys., 295 (2010), pp. 731–790.

- [10] F. D’ANDREA, L. DĄBROWSKI, AND G. LANDI, *The noncommutative geometry of the quantum projective plane*, Rev. Math. Phys., 20 (2008), pp. 979–1006.
- [11] F. D’ANDREA AND G. LANDI, *Geometry of quantum projective spaces*, in Noncommutative geometry and Physics, vol. 3, 2013, pp. 373–416.
- [12] B. DAS, R. Ó BUACHALLA, AND P. SOMBERG, *Dolbeault–Dirac spectral triples on quantum homogeneous spaces*. arXiv preprint math.QA/1903.07599.
- [13] L. DĄBROWSKI AND A. SITARZ, *Dirac operator on the standard Podleś quantum sphere*, in Noncommutative geometry and quantum groups (Warsaw, 2001), vol. 61 of Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 49–58.
- [14] F. DÍAZ GARCÍA, R. Ó BUACHALLA, AND E. WAGNER, *A spectral triple for the rank 2 quantum quadric*. In preparation.
- [15] M. S. DIJKHUIZEN AND J. STOKMAN, *Quantized flag manifolds and irreducible  $*$ -representations*, Comm. Math. Phys. (2), 203 (1999), pp. 297–324.
- [16] M. S. DIJKHUIZEN AND T. H. KOORNWINDER, *CQG algebras: a direct algebraic approach to compact quantum groups*, Lett. Math. Phys., 32 (1994), pp. 315–330.
- [17] I. FORSYTH, B. MESLAND, AND A. RENNIE, *Dense domains, symmetric operators and spectral triples*, New York J. Math., 20 (2014), pp. 1001–1020.
- [18] T. FRIEDRICH, *Dirac operators in Riemannian geometry*, vol. 25 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.
- [19] J. M. GRACIA-BONDÍA, J. C. VÁRILLY, AND H. FIGUEROA, *Elements of noncommutative geometry*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2001.
- [20] I. HECKENBERGER AND S. KOLB, *The locally finite part of the dual coalgebra of quantized irreducible flag manifolds*, Proc. London Math. Soc. (3), 89 (2004), pp. 457–484.
- [21] I. HECKENBERGER AND S. KOLB, *De Rham complex for quantized irreducible flag manifolds*, J. Algebra, 305 (2006), pp. 704–741.

- [22] N. HIGSON AND J. ROE, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. Oxford Science Publications.
- [23] D. HUYBRECHTS, *Complex geometry: an introduction*, universitext, Springer-Verlag, 2005.
- [24] A. IKEDA AND Y. TANIGUCHI, *Spectra and eigenforms of the Laplacian on  $S^n$  and  $P^n(\mathbf{C})$* , Osaka J. Math., 15 (1978), pp. 515–546.
- [25] M. KHALKHALI, G. LANDI, AND W. D. VAN SUIJLEKOM, *Holomorphic structures on the quantum projective line*, Int. Math. Res. Not. IMRN, (2011), pp. 851–884.
- [26] M. KHALKHALI AND A. MOATADELRO, *Noncommutative complex geometry of the quantum projective space*, J. Geom. Phys., 61 (2011), pp. 2436–2452.
- [27] A. KLIMYK AND K. SCHMÜDGEN, *Quantum Groups and Their Representations*, Texts and Monographs in Physics, Springer-Verlag, 1997.
- [28] A. W. KNAPP AND D. A. VOGAN, JR., *Cohomological induction and unitary representations*, vol. 45 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1995.
- [29] T. KOBAYASHI, *Multiplicity-free representations and visible actions on complex manifolds*, Publ. Res. Inst. Math. Sci., 41 (2005), pp. 497–549.
- [30] K. KOIKE AND I. TERADA, *Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank*, Adv. Math., 79 (1990), pp. 104–135.
- [31] U. KRÄHMER, *Dirac operators on quantum flag manifolds*, Lett. Math. Phys., 67 (2004), pp. 49–59.
- [32] U. KRÄHMER AND M. TUCKER-SIMMONS, *On the Dolbeault-Dirac operator of quantized symmetric spaces*, Trans. London Math. Soc., 2 (2015), pp. 33–56.
- [33] M. KRÄMER, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math., 38 (1979), pp. 129–153.
- [34] F. DÍAZ GARCÍA, A. KRUTOV, R. Ó BUACHALLA, P. SOMBERG, AND K. R. STRUNG, *Positive line bundles over the irreducible quantum flag manifolds*. arXiv preprint math.QA/1912.08802.

- [35] J. KUSTERMANS, G. J. MURPHY, AND L. TUSET, *Quantum groups, differential calculi and the eigenvalues of the laplacian*, Trans. Amer. Math. Soc., 357 (2005), pp. 4681–4717.
- [36] V. LAKSHMIBAI AND N. RESHETIKHIN, *Quantum deformations of flag and schubert schemes*, C. R. Acad. Sci. Paris, 313 (1991), pp. 121–126.
- [37] P. LITTELMANN, *Paths and root operators in representation theory*, Ann. of Math., 142, pp. 499–525.
- [38] P. LITTELMANN, *A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras*, Invent. Math., 116 (1994), pp. 329–346.
- [39] S. MAJID, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995.
- [40] S. MAJID, *Noncommutative Riemannian and spin geometry of the standard  $q$ -sphere*, Comm. Math. Phys., 256 (2005), pp. 255–285.
- [41] M. MATASSA, *Kähler structures on quantum irreducible flag manifolds*, J. Geom. Phys., 145 (2019), 103477, 16 pp.
- [42] M. MATASSA, *Dolbeault-Dirac operators, quantum Clifford algebras and the Parthasarathy formula*, Adv. Appl. Clifford Algebr., 27 (2017), pp. 1581–1609.
- [43] M. MATASSA, *On the Dolbeault-Dirac operators on quantum projective spaces*, J. Lie Theory, 28 (2018), pp. 211–244.
- [44] M. MATASSA, *The Parthasarathy formula and a spectral triple for the quantum Lagrangian Grassmannian of rank two*, Lett. Math. Phys., 109 (2019), pp. 1703–1734.
- [45] U. MEYER, *Projective quantum spaces*, Lett. Math. Phys., 35 (1995), pp. 91–97.
- [46] S. NESHVEYEV AND L. TUSET, *The Dirac operator on compact quantum groups*, J. Reine Angew. Math., 641 (2010), pp. 1–20.
- [47] S. NESHVEYEV AND L. TUSET, *Compact quantum groups and their representation categories*, vol. 20 of Cours Spécialisés [Specialized Courses], Société Mathématique de France, Paris, 2013.
- [48] R. Ó BUACHALLA, *Quantum bundle description of quantum projective spaces*, Comm. Math. Phys., 316 (2012), pp. 345–373.
- [49] R. Ó BUACHALLA, *Noncommutative complex structures on quantum homogeneous spaces*, J. Geom. Phys., 99 (2016), pp. 154–173.
- [50] R. Ó BUACHALLA, *Noncommutative Kähler structures on quantum homogeneous spaces*, Adv. Math., 322 (2017), pp. 892–939.

- [51] R. Ó BUACHALLA AND C. MROZINSKI, *A Borel–Weil theorem for the quantum Grassmannians*. arXiv preprint math.QA/1611.07969.
- [52] R. Ó BUACHALLA, J. ŠŤOVIČEK, AND A.-C. VAN ROOSMALEN, *A Kodaira vanishing theorem for noncommutative Kähler structures*. arXiv preprint math.QA/1801.08125., 2018.
- [53] A. PAL, *Induced representation and Frobenius reciprocity for compact quantum groups*, Proc. Indian Acad. Sci. Math. Sci., 105 (1995), pp. 157–167.
- [54] R. PARTHASARATHY, *Dirac operator and the discrete series*, Ann. of Math. (2), 96 (1972), pp. 1–30.
- [55] A. RENNIE AND R. SENIOR, *The resolvent cocycle in twisted cyclic cohomology and a local index formula for the Podleś sphere*, J. Noncommut. Geom., 8 (2014), pp. 1–43.
- [56] N. Y. RESHETIKHIN, L. A. TAKHTADZHYAN, AND L. D. FADDEEV, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz, 1 (1989), pp. 178–206.
- [57] W. RUDIN, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, second ed., 1991.
- [58] J. T. STAFFORD, *Noncommutative projective geometry*, in Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 93–103.
- [59] M. TAKEUCHI, *Relative Hopf modules—equivalences and freeness criteria*, J. Algebra, 60 (1979), pp. 452–471.
- [60] D. A. VOGAN, JR., *Representations of real reductive Lie groups*, vol. 15 of Progress in Mathematics, Birkhäuser, Boston, Mass., 1981.
- [61] A. WEIL, *Introduction à l’étude des variétés kählériennes*, no. 1267 in Publications de l’Institut de Mathématique de l’Université de Nancago, VI., Hermann, Paris, 1958.
- [62] H. WEYL, *The classical groups. Their invariants and representations*, Princeton Landmarks in Mathematics, Princeton University Press, 1997.
- [63] S. L. WORONOWICZ, *Compact matrix pseudogroups*, Comm. Math. Phys., 111 (1987), pp. 613–665.

Biswarup Das	Réamonn Ó Buachalla
Laboratory of Advanced	Mathematical Institute
Combinatorics and Network	Charles University
Applications	Sokolovská 83
Department of Applied Mathematics	Prague
Moscow Institute of Physics and	Czech Republic
Technology	and
Moscow	Département de Mathématiques
Russia	Faculté des Sciences
and	Université Libre de Bruxelles
Instytut Matematyczny	Boulevard du Triomphe
Uniwersytet Wrocławski	B-1050 Bruxelles
pl. Grunwaldzki 2/4 50-384	Belgium
Wrocław	reamonnobuachalla@gmail.com
Poland	
biswarup.das@math.uni.wroc.pl	

Petr Somberg  
 Mathematical Institute  
 Charles University  
 Sokolovská 83  
 Prague  
 Czech Republic  
 somberg@karlin.mff.cuni.cz

