A Characterization of Permutation Modules Extending a Theorem of Weiss

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Abstract. Let G be a finite p-group with normal subgroup N. A celebrated theorem of A. Weiss gives a sufficient condition for a $\mathbb{Z}_pG$-lattice to be a permutation module, looking only at its restriction to N and its N-fixed points. In case N has order p, we extend the condition of Weiss to a characterization.

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1 Introduction

Throughout this article let G be a finite p-group. Let A denote either the field $\mathbb{F}_p$ or the p-adic integers $\mathbb{Z}_p$. Let U be an AG-module and N a normal subgroup of G. Denote by $U\downarrow_N$ the *restricted* AN-module, namely U with coefficients restricted to AN. Denote by U^N (resp. U_N) the N-invariants (resp. N-coinvariants) of U – that is, the largest AG submodule (resp. AG quotient module) of U on which N acts trivially. Both U^N and U_N are thus $A[G/N]$-modules.

An *AG-permutation module* is an AG-lattice (that is, a finitely generated AG-module that is free as an A-module) having an A-basis preserved set-wise by the action of G. Permutation modules are extremely well-behaved, and act as a fundamental starting point when trying to understand more general modules for G. However, given a lattice U, it is a surprisingly difficult task to identify whether or not U is a permutation module. The most remarkable detection theorem for $\mathbb{Z}_pG$-permutation modules is due to A. Weiss. In certain circumstances, it identifies a lattice U as being a permutation module by looking only at modules for strictly smaller subgroups:

Theorem 1 ([9, Theorem 2]). *Let U be a $\mathbb{Z}_pG$-lattice and suppose there is a normal subgroup N of G for which*

- *$U\downarrow_N$ is a free $\mathbb{Z}_pN$-module and*
- *U^N is a permutation $\mathbb{Z}_p[G/N]$-module.*

Then U itself is a permutation $\mathbb{Z}_pG$-module.

Weiss' Theorem has important applications to group rings and block theory, and is considered nowadays to be a fundamental theorem of integral representation theory. However, it is not a characterization of permutation modules, because a permutation module need not be free over N. Of course, a permutation $\mathbb{Z}_pG$-module is necessarily a permutation $\mathbb{Z}_pN$-module (a $\mathbb{Z}_p$-basis preserved by G is preserved by N), but there are lattices U for which both the restricted $\mathbb{Z}_pN$-lattice $U\downarrow_N$ and the lattice U^N of N-invariants are permutation modules, but which are not themselves permutation modules (see Section 4).
Our main theorem gives a characterization of permutation $\mathbb{Z}_pG$-modules in terms of modules for a group of order strictly less than $|G|$:

Theorem 2. *Let U be a $\mathbb{Z}_pG$-lattice and let N be a normal subgroup of G of order p. Then U is a permutation module if, and only if*

1. *U^N and U_N are permutation $\mathbb{Z}_p[G/N]$-modules and*
2. *$(U/U^N)_N$ is a permutation $\mathbb{F}_p[G/N]$-module.*

A characterization of this type has until now only been obtained in case G is cyclic [8, Proposition 6.12].
In Section 2 we give the definitions and preliminary observations required for the discussion, in Section 3 we prove Theorem 2, and in Section 4 we provide some illustrative examples.

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2 Preliminaries

Notations introduced here will be used throughout our discussion. In what follows, G is always a finite p-group and N a normal subgroup of G. If ever U is a $\mathbb{Z}_pG$-lattice, we denote by $\overline{U}$ the $\mathbb{F}_pG$-module U/pU. We will at times write $\overline{X}$ when X is a submodule of U – this notation is unambiguous because our submodules will always be $\mathbb{Z}_p$-direct summands of U, so the two possible interpretations, X/pX or $(X+pU)/pU$, will coincide.

Lemma 3. *Let U be a $\mathbb{Z}_pG$-lattice such that $\overline{U}$ is free as an $\mathbb{F}_pG$-module. Then U is free.*

Proof. Let X be an $\mathbb{F}_pG$-basis of $\overline{U}$ and let F be the free $\mathbb{Z}_pG$-module with basis X. The natural map $F \to \overline{U}$ lifts to a homomorphism $\gamma : F \to U$, which is an isomorphism modulo p. By Nakayama's lemma [1, Lemma 1.2.3] γ is surjective, and is thus an isomorphism because U is a lattice. $\square$

Lemma 4. *Let U be a permutation $\mathbb{Z}_pG$-module and let $\overline{U} = X' \oplus Y'$ be an $\mathbb{F}_pG$-module decomposition. There is a $\mathbb{Z}_pG$-module decomposition $U = X \oplus Y$ with $\overline{X} = X'$ and $\overline{Y} = Y'$.*

Proof. Let $a', b' \in \mathrm{End}_{\mathbb{F}_pG}(\overline{U})$ be the projections onto X' and Y', respectively. As U is a permutation module, the natural map $\mathrm{End}(U) \to \mathrm{End}(\overline{U})$ is surjective by [1, Corollary 3.11.4], so that

$$\mathrm{End}(\overline{U}) = \overline{\mathrm{End}(U)}.$$

By [1, Theorem 1.9.4] there is thus a decomposition into orthogonal idempotents $\mathrm{id}_U = a + b$ such that $\overline{a} = a', \overline{b} = b'$. Accordingly, U is a direct sum of $\mathbb{Z}_pG$-modules $X \oplus Y$ with $\overline{X} = X', \overline{Y} = Y'$. $\square$

Lemma 5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathbb{Z}_pG$-lattices. If A and C are permutation modules then the sequence splits.*

Proof. This is a special case of [6, Corollary 6.8]. For the convenience of the reader we sketch a proof, following the argument of [5, Lemma 4.1]. As A, C are permutation modules, using the additivity of Ext, the Mackey decomposition and Shapiro's Lemma, we see that $\mathrm{Ext}^1_{\mathbb{Z}_pG}(C, A)$ is a direct sum of modules of the form

$$\mathrm{Ext}^1_{\mathbb{Z}_pK}(\mathbb{Z}_p, \mathbb{Z}_p[K/L]) \cong H^1(L, \mathbb{Z}_p)$$

for subgroups $L \leqslant K$ of G. But each $H^1(L, \mathbb{Z}_p)$, being isomorphic to group homomorphisms $L \to \mathbb{Z}_p$, is 0. Hence $\mathrm{Ext}^1(C, A) = 0$ and the sequence splits. $\square$

Let U be a $\mathbb{Z}_pG$-lattice and N a normal subgroup of G. The $\mathbb{Z}_p[G/N]$-modules of N-invariants U^N and N-coinvariants U_N are, respectively, the largest submodule and the largest quotient module of U on which N acts trivially. Explicitly

$$U^N = \{u \in U \mid nu = u \ \forall n \in N\}$$
$$U_N = U/I_N U$$

where I_N, the augmentation ideal of $\mathbb{Z}_pN$, is the kernel of the natural map $\mathbb{Z}_pN \to \mathbb{Z}_p$ sending $\sum_{n\in N} \lambda_n n$ to $\sum \lambda_n$. If N is cyclic with generator n_0, then I_N is generated as an ideal by $n_0 - 1$. If ever u is an element of U, we denote by $\widetilde{u}$ its image in U_N. There is a natural map $\psi : U^N \to U_N$ sending u to $\widetilde{u}$, which is injective (because its composition with the map $\varphi : U_N \to U^N$ sending $\widetilde{u}$ to $\sum_{n\in N} nu$ is multiplication by $|N|$) but not usually surjective.

For the rest of the article we will suppose that $N = \langle n_0 \rangle$ has order p. By [4, Theorem 2.6] there are three isomorphism classes of indecomposable $\mathbb{Z}_pN$-lattice, being the trivial module $\mathbb{Z}_p$, the free module $\mathbb{Z}_pN$ and a non-permutation module S of $\mathbb{Z}_p$-rank $p-1$, which can be described in any of the following equivalent ways:

$$S = I_N = \mathbb{Z}_pN/(\mathbb{Z}_pN)^N = \mathbb{Z}_p(\zeta),$$

where the latter is the totally ramified extension of $\mathbb{Z}_p$ by a primitive pth root of unity ζ, on which n_0 acts as multiplication by ζ. Note that S has no non-zero N-fixed points. If V is an indecomposable $\mathbb{Z}_pN$-lattice, then V/V^N is 0 if V is trivial, and is isomorphic to S if V is free or S. In the latter cases, $(V/V^N)_N$ is a non-zero indecomposable N-trivial $\mathbb{Z}_pN$-module on which p acts as 0, and hence it is $\mathbb{F}_p$. It follows that for any $\mathbb{Z}_pG$-lattice U, the module $(U/U^N)_N$ is an $\mathbb{F}_p[G/N]$-module.

Lemma 6. *Let U be a $\mathbb{Z}_pG$-lattice with $U{\downarrow_N}$ a permutation module and choose a decomposition $U{\downarrow_N} = T \oplus F$ with T trivial and F free.*

1. *The $\mathbb{F}_pN$-submodule $\overline{T}$ of $\overline{U_N}$ does not depend on the choice of decomposition and is G-invariant.*
2. *The $\mathbb{F}_pG$-modules $\overline{U_N}/\overline{T}$ and $(U/U^N)_N$ are naturally isomorphic.*

Proof. 1. The homomorphism ψ induces an $\mathbb{F}_pG$-homomorphism $\overline{\psi} : \overline{U^N} \to \overline{U_N}$. The image of $\overline{\psi}$ is an $\mathbb{F}_pG$-submodule X of $\overline{U_N}$. But on the other hand writing $\overline{U^N} = \overline{T} \oplus \overline{F^N}$ the image of $\overline{\psi}$ is $\overline{T}$, because $\overline{T}$ in $\overline{U^N}$ goes isomorphically onto $\overline{T}$ in $\overline{U_N}$, while elements of $\overline{F^N}$ have the form $\sum_{n\in N} nf$ for $f \in \overline{F}$, and hence go to 0 in $\overline{U_N}$. Thus $\overline{T} = X$ is unique and G-invariant.

2. The natural surjection of $\mathbb{Z}_pG$-modules $\rho : U \to U/U^N$ induces a surjection $\overline{\rho_N} : \overline{U_N} \to (U/U^N)_N$. Treating this map as an N-module homomorphism we see that its kernel is exactly $\overline{T}$: $(U/U^N)_N = (F/F^N)_N$, so $\overline{T}$ clearly goes to 0. Furthermore, $\overline{\rho_N}|_{\overline{F_N}}$ is an isomorphism because the modules $\overline{F_N}$ and $(F/F^N)_N$ have the same dimension (the number of $\mathbb{Z}_pN$-summands of F).

□

Remark 1. We observe in passing that the conditions of Weiss' Theorem 1 imply the conditions of Theorem 2: if $U{\downarrow_N}$ is free then $\varphi : U_N \to U^N$ defined before Lemma 6 is an isomorphism so that if U^N is a permutation module then so is U_N, and hence so is $(U/U^N)_N$, being isomorphic by Lemma 6 to $\overline{U_N}/\overline{T} = \overline{U_N}$.

Lemma 7. *Let U be a permutation $\mathbb{Z}_pN$-lattice with maximal trivial summand T. If L' is a complement of $\overline{T}$ in $\overline{U_N}$, there exists a $\mathbb{Z}_pN$-complement F of T in U such that $\overline{F_N} = L'$.*

Proof. Using Lemma 4 (applied with $G = 1$) we obtain a decomposition $U_N = T' \oplus L$ with $\overline{T'} = \overline{T}$ and $\overline{L} = L'$.
We claim that L is a complement to T in U_N: the intersection $T \cap L$ is pure because T, L are and $\mathbb{Z}_p$ is torsion free, but is 0 modulo p and hence is 0. Further $T + L = U_N$ by Nakayama's lemma, because $\overline{T + L} = \overline{U_N}$. This proves the claim.
The submodule L of U_N is the image of a map from $(U/T)_N$ and hence we obtain the diagram

$$\begin{array}{ccc} & & U/T \\ & & \downarrow \\ U & \twoheadrightarrow & U_N = T \oplus L \end{array}$$

Projectivity of U/T yields a $\mathbb{Z}_pN$-module homomorphism $U/T \to U$ whose image is the submodule F we require. □

The reader might wonder why, in Theorem 2, we are not required to make any hypotheses at all on $U{\downarrow_N}$. This is because U_N being a lattice already has strong implications for $U{\downarrow_N}$:

Lemma 8. *Let N be a cyclic group of order p and V a $\mathbb{Z}_pN$-lattice. Then V_N is a lattice if, and only if, V is a permutation module.*

Proof. This is a simple consequence of the classification and discussion of indecomposable $\mathbb{Z}_pN$-lattices given before Lemma 6: the N-coinvariants of the indecomposable trivial and free $\mathbb{Z}_pN$-modules are $\mathbb{Z}_p$-lattices of rank 1, whereas $S_N = \mathbb{F}_p$ is not a lattice. □

The following theorem generalizes [10, Theorem 2.6]

Theorem 9. *Let U be a $\mathbb{Z}_pG$-lattice and suppose that U_N is a permutation module. Let T be a maximal trivial summand of $U{\downarrow_N}$ and suppose that $\overline{T}$ has an $\mathbb{F}_pG$-complement in $\overline{U_N}$. Then U is a permutation module.*

Proof. By Lemma 8, $U{\downarrow_N}$ is a permutation module. Note that $\overline{T}$ is G-invariant in $\overline{U_N}$ by Lemma 6. By Lemma 7 there is a $\mathbb{Z}_pN$-decomposition $U{\downarrow_N} = F \oplus T$ such that the $\mathbb{Z}_pG$-complement to $\overline{T}$ in $\overline{U_N}$ given in the statement is $\overline{F_N}$. By Lemma 4 we obtain a decomposition of $\mathbb{Z}_pG$-modules $U_N = X \oplus Y$, with $\overline{X} = \overline{T}, \overline{Y} = \overline{F_N}$. The natural projection $\gamma : U \to U_N \to X$ now yields a short exact sequence of $\mathbb{Z}_pG$-modules

$$0 \to K \to U \xrightarrow{\gamma} X \to 0,$$

where $K = \mathrm{Ker}(\gamma)$. We have the following commutative diagram, in which the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & U & \xrightarrow{\gamma} & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & U_N & \xrightarrow{\gamma_N} & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{F_N} & \longrightarrow & \overline{U_N} & \longrightarrow & \overline{T} & \longrightarrow & 0 \end{array}$$

The module X, being a summand of the permutation module U_N, is a permutation module by [3, Lemma 0.3]. If we show that K is also a permutation module, then the sequence splits by Lemma 5. For this we use Weiss' Theorem.

- $K{\downarrow_N}$ is free: We have that $\overline{F} \subseteq \overline{K}$, because the diagram above yields by taking the upper row modulo p the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{K} & \longrightarrow & \overline{U} & \xrightarrow{\overline{\gamma}} & \overline{T} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \overline{F_N} & \longrightarrow & \overline{U_N} & \longrightarrow & \overline{T} & \longrightarrow & 0 \end{array}$$

  An element $\overline{f} \in \overline{F} \subseteq \overline{U}$ drops modulo N into $\overline{F}_N = \overline{F_N}$ and hence by exactness of the lower sequence goes to 0 in $\overline{T}$, as claimed. But both $\overline{F}$ and $\overline{K}$ complement $\overline{T}$, and hence they are equal. Thus $\overline{K}{\downarrow_N}$ is free, so by Lemma 3, $K{\downarrow_N}$ is free.

- K_N is a permutation module: since $K{\downarrow_N}, X{\downarrow_N}$ are permutation modules, the sequence splits when restricted to N. Hence applying coinvariants the sequence

$$0 \to K_N \to U_N \xrightarrow{\gamma_N} X \to 0$$

  is exact. Thus K_N is isomorphic to the kernel of the split surjection γ_N and is thus a permutation module, being isomorphic to a summand of U_N.

As K, X are permutation modules, the sequence splits by Lemma 5 and so $U \cong K \oplus X$ is also a permutation module, as required. □

We have already observed that the $\mathbb{Z}_pG$-module U/U^N is, as a $\mathbb{Z}_pN$-module, isomorphic to a direct sum of copies of $\mathbb{Z}_p(\zeta)$ with a fixed generator of N acting as ζ. That is, we may treat U/U^N as an RG-lattice, where $R = \mathbb{Z}_p(\zeta)$. If ever H is a finite p-group, an RH-lattice is said to be *monomial* (or a *generalized permutation* module) if its indecomposable summands are induced from rank 1 lattices for subgroups L of H, whose action from L necessarily comes via a group homomorphism $L \to \langle\zeta\rangle$.

Lemma 10. *If $(U/U^N)_N$ is a permutation $\mathbb{F}_pG$-module, then U/U^N is a monomial RG-module.*

Proof. Taking N-coinvariants corresponds to quotienting out by the maximal ideal of R, and hence this result is [9, Theorem 3]. ☐

A $\mathbb{Z}_pG$-module V is *projective relative to proper subgroups* if, whenever a $\mathbb{Z}_pG$-module homomorphism onto V splits when restricted to any proper subgroup, then it splits. The module V is projective relative to proper subgroups if, and only if, every indecomposable summand of V is a direct summand of a module induced from a proper subgroup of G [2, Lemma 2.2.3]. In particular, a permutation $\mathbb{Z}_pG$-module is projective relative to proper subgroups if, and only if, it has no trivial summands. For later use we also recall that a $\mathbb{Z}_pG$-module V is *projective relative to the subgroup H* if every $\mathbb{Z}_pG$-module homomorphism onto V that splits over H also splits over G. This is the case if, and only if, V is a direct summand of a $\mathbb{Z}_pH$-module induced up to G.

Lemma 11. *Suppose that N is contained in the Frattini subgroup $\Phi(G)$ of G. If $U\downarrow_N$ and $(U/U^N)_N$ are permutation modules, then $(U/U^N)_N$ is projective relative to proper subgroups of G.*

Proof. Being a permutation $\mathbb{F}_pG$-module by hypothesis, we must check that $(U/U^N)_N$ does not possess a trivial summand. By the Krull-Schmidt Theorem and because U/U^N is monomial by Lemma 10, if $(U/U^N)_N$ had a trivial summand then some indecomposable monomial summand of U/U^N would be trivial modulo N. This summand is necessarily of the form $\mathbb{Z}_p(\zeta)$, where G acts on ζ via a group homomorphism $\varphi : G \to \langle\zeta\rangle$. The module $U/U^N = F/F^N$ has no N-fixed points, so that N is not in the kernel of φ. But the kernel has index p, and hence $G = N \times \mathrm{Ker}(\varphi)$, contradicting the hypothesis that $N \leqslant \Phi(G)$. ☐

3 Proof of the main theorem

Proof. (of Theorem 2) To show the forward implication, we may suppose that $U = \mathbb{Z}_p[G/H]$ for some subgroup H of G, in which case U^N and U_N are both isomorphic to $\mathbb{Z}_p[G/HN]$. If N is contained in H then $(U/U^N)_N = 0$ and if not, then $(U/U^N)_N$ is isomorphic to $\mathbb{F}_p[G/HN]$.
In order to show the reverse implication, We work by induction on $|G|$. Note that by Lemma 8, $U\downarrow_N$ is a permutation module. The strategy of the proof is to find an $\mathbb{F}_pG$-complement in $\overline{U_N}$ to the image of a maximal N-trivial summand of U and then apply Theorem 9. There are two cases, which we treat separately. Either $N \leqslant \Phi(G)$ or $G = N \times H$ for some maximal subgroup H of G.

<u>CASE 1:</u> $N \leqslant \Phi(G)$.
Fix a subgroup H of index p in G. Then H contains N since N is in the Frattini subgroup. Note that the conditions of the theorem are satisfied for $U\downarrow_H$, so that $U\downarrow_H$ is a permutation module by the induction hypothesis.

So write $U{\downarrow_H}= T \oplus F$, where the indecomposable summands of T are N-trivial and the indecomposable summands of F are N-free. Lemma 6 tells us that $\overline{T}$ is G-invariant in $\overline{U_N}$ and that the short exact sequence of $\mathbb{F}_pG$-modules

$$0 \to \overline{T} \to \overline{U_N} \to \overline{U_N}/\overline{T} \to 0$$

does not depend on H. Moreover, the sequence splits over H for any choice of H, and hence over every proper subgroup of G. As $\overline{U_N}/\overline{T} \cong (U/U^N)_N$, the sequence thus splits over G by Lemma 11. The result now follows from Theorem 9.

CASE 2: N is a direct factor of G.
If $G = N \times H$, observe that U/U^N can be treated as a $\mathbb{Z}_p(\zeta)H$-module with ζ acting as a fixed generator of N. Since $(U/U^N)_N$ is a permutation module, U/U^N is monomial by Lemma 10.
Let $\rho : U \to U/U^N$ be the canonical projection and let A be an indecomposable G-summand of U/U^N. Then either A is projective relative to a proper subgroup containing N, or $A = \mathbb{Z}_p(\zeta)$. In both cases, treated separately, we will find a G-submodule Z_A of $\overline{U_N}$ mapping isomorphically onto A_N under $\overline{\rho_N}$.

- First suppose that $A = \mathbb{Z}_p(\zeta)\uparrow_{N\times L}^{N\times H}$ for some complement H of N and proper subgroup L of H, and where N acts as ζ on $\mathbb{Z}_p(\zeta)$. The module $U{\downarrow_{N\times L}}$ is a permutation module, so write it as a direct sum $T \oplus F$ with T trivial over N and F free over N. As observed in the proof of Part 2 of Lemma 6, $\overline{\rho_N}$ has kernel $\overline{T}$ and it splits as an $\mathbb{F}_p[N \times L]$-module homomorphism because $U{\downarrow_{N\times L}}$ is a permutation module. As A_N is a summand of $(U/U^N)_N$, the induced map $\overline{U_N} \to A_N$ also splits over $N\times L$ and so, since A_N is projective relative to $N \times L$, it also splits over G. Define Z_A to be the image of this splitting.

- $A = \mathbb{Z}_p(\zeta)$. Let Y be the inverse image of A under the G-homomorphism $\rho : U \to U/U^N$. We claim that $Y{\downarrow_N}$ is a permutation module. If $U{\downarrow_N}= T \oplus F$ with T trivial and F free, then $\rho|_F : F \to U/U^N$ is surjective, and so there is $z \in Y \cap F$ generating an N-free summand of Y and such that $\rho(z)$ generates A. Now, $Y = \langle z\rangle + U^N$ because $Y/U^N \cong A$. Choose any $\mathbb{Z}_p$-complement T_Y of $\langle z\rangle$ (in Y) that is contained inside U^N. The direct sum $Y = \langle z\rangle \oplus T_Y$ is a direct sum of N-modules, the first being N-free and the second being N-trivial. So $Y{\downarrow_N}$ is a permutation module.

  The action of G on A is given by a group homomorphism $G \to \langle\zeta\rangle$, whose kernel, a subgroup S of index p complementing N, acts trivially on A. It follows that the first and third terms in the short exact sequence

  $$0 \to U^N{\downarrow_S} \to Y{\downarrow_S} \xrightarrow{\rho} A{\downarrow_S} \to 0$$

  are permutation modules, so the sequence splits via $\alpha : A{\downarrow_S} \to Y{\downarrow_S}$. If A is generated by a as a G-module, consider $y = \alpha(a)$. The G-module generated by y is equal to the N-module generated by y, because S acts trivially on y. The module $Z = \langle y \rangle$ is not N-trivial, because N doesn't act trivially on $\rho(y) = a$. Also $Z \ncong \mathbb{Z}_p(\zeta)$ because if it were, as Y is a permutation module, Z would be in the kernel of the map $Y \to Y_N$. But $A_N \neq 0$ and so

$$0 \neq \widetilde{a} = \rho_N(\alpha_N(\widetilde{a})) = \rho_N(\widetilde{y}),$$

showing that $\widetilde{y} \neq 0$ in Y_N. Hence Z is N-free and we take $Z_A = \overline{Z_N}$.

So, for each indecomposable summand A in a decomposition of U/U^N we have found an $\mathbb{F}_pG$-submodule Z_A of $\overline{U_N}$ mapping isomorphically onto A. We claim that

$$\overline{U_N} = \overline{T} \oplus \bigoplus_A Z_A.$$

One easily checks that $W := \sum Z_A = \bigoplus Z_A$ (because each $\overline{\rho_N} : Z_A \to A_N$ is injective) and that $W \cap \overline{T} = 0$ (because the kernel of $\overline{\rho_N}$ is $\overline{T}$), so we need only check that $\overline{T} + W = \overline{U_N}$. But this follows because we have a short exact sequence

$$0 \to \overline{T} \to \overline{U_N} \to (U/U^N)_N \to 0$$

and, since $\overline{T} \cap W = 0$ and $W \cong (U/U^N)_N$, we have

$$\dim(\overline{U_N}) = \dim(\overline{T}) + \dim((U/U^N)_N) = \dim(\overline{T}) + \dim(W).$$

Thus $\overline{T} + W = \overline{U_N}$. The result now follows by Theorem 9. □

For certain groups we may remove hypotheses.

Corollary 12. *Let G be a generalized quaternion group (cf. [7, §5.3] for example) and let N be a normal subgroup of order* 2*. The $\mathbb{Z}_2G$-lattice U is a permutation module if, and only if*

1. *U_N is a permutation $\mathbb{Z}_p[G/N]$-module and*
2. *$(U/U^N)_N$ is a permutation $\mathbb{F}_p[G/N]$-module.*

Proof. For such a group, N is contained in $\Phi(H)$ for every non-trivial subgroup H of G. It follows that we only ever use Case 1 of the proof of Theorem 2, which does not require that U^N be a permutation module. □

4 Examples

If G is a cyclic p-group and N a subgroup of G, then [8, Proposition 6.12] shows that a $\mathbb{Z}_pG$-lattice is a permutation module if, and only if, $U\downarrow_N$ and U^N are permutation modules for $\mathbb{Z}_pN$ and $\mathbb{Z}_p[G/N]$ respectively. However, the following examples show that in general we cannot remove either the hypothesis that U^N is a permutation module, nor that U_N is a permutation module, from Theorem 2.

Example 1. Let $G = C_2 \times C_2 = \langle n \rangle \times \langle h \rangle$ and let $N = \langle n \rangle$. Consider the $\mathbb{Z}_2G$-lattice U with $\mathbb{Z}_2$-basis a, b, x and action

$$na = b\,,\; nb = a\,,\; nx = x,$$

$$ha = -a\,,\; hb = -b\,,\; hx = x + a + b.$$

Then $U\downarrow_N$ is isomorphic to $\mathbb{Z}_2N \oplus \mathbb{Z}_2$ and U^N is free as a $\mathbb{Z}_2[G/N]$-module. The module U/U^N has rank 1 with n acting as multiplication by -1, so $(U/U^N)_N = \mathbb{F}_2$ is also a permutation module. But U itself is not a permutation module, as can be seen by observing that U^G has $\mathbb{Z}_2$-rank 1, which is impossible for a $\mathbb{Z}_2G$-permutation module of $\mathbb{Z}_2$-rank 3. Note that in this example U_N is not a permutation module, being isomorphic as a $\mathbb{Z}_2[G/N]$-module to the trivial module with basis $\widetilde{a+x}$ plus the sign module (wherein h acts as multiplication by -1) with basis $\widetilde{a}$.

Example 2. With notation as in Example 1, let U have the same action from N but multiplication

$$ha = a + x\,,\; hb = b + x\,,\; hx = -x.$$

One checks as above that U_N and $(U/U^N)_N$ are permutation modules, while U itself is not. In this case, U^N is not a permutation module.

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