On Unramified Brauer Groups of Torsors over Tori

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Abstract. In this paper we introduce a method to obtain algebraic information using arithmetic one in the study of tori and their principal homogeneous spaces. In particular, using some results of the authors with Ting-Yu Lee, we determine the unramified Brauer groups of some norm one tori, and their torsors.

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1 Introduction

In this paper, we use arithmetic information to obtain algebraic ones. Let G be a finite group, and let M be a G-lattice. Let ℓ'/ℓ a finite *unramified* extension of number fields with Galois group G; such an extension exists by [F 62]. Let T be an ℓ-torus with character group M. We have the following isomorphisms

$$(*) \qquad Ш^2_{\mathrm{cycl}}(G,M) \simeq Ш^2(\ell,M) \simeq Ш^1(\ell,T)^*;$$

the first isomorphism is Proposition 4.1, the second one follows from Poitou-Tate duality (see §2 and §3 for the notation). In the following, we combine $(*)$ with arithmetic results as well as some theorems of Colliot-Thélène and Sansuc; we illustrate the results with the following example (see §12):

Example. Let k be a field, and let $L = K_1 \times \cdots \times K_n$, where $K_1, \dots, K_n$ are cyclic extensions of k of prime degree p. Let $\mathrm{N}_{L/k} : L \to k$ denote the norm map, and let $T_{L/k} = R^{(1)}_{L/k}(\mathbf{G}_m)$ be the k-torus defined by

$$1 \to T_{L/k} \to R_{L/k}(\mathbf{G}_m) \xrightarrow{\mathrm{N}_{L/k}} \mathbf{G}_m \to 1.$$

Let k'/k be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \mathrm{Gal}(k'/k)$. Set $T = T_{L/k}$, let T^c be a smooth compactification of T, and let $\mathrm{Br}(T^c)$ be its Brauer group.

Let $a \in k^\times$, and let X be the affine k-variety determined by the equation $\mathrm{N}_{L/k}(t) = a$; X is a torsor under $T_{L/k}$. Let X^c be a smooth compactification of X. We denote by $\mathrm{Br}(X^c)$ the Brauer group of X^c.

Theorem. (a) *If $G \not\simeq C_p \times C_p$, then*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) = \mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0.$$

(b) *If $G \simeq C_p \times C_p$, then*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq \mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}.$$

This is proved in Theorem 12.1 using $(*)$ as well as some (arithmetic) results of [BLP 19].

Further, we also give generators for the group $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k))$ (see Theorem 12.2), in the spirit of Colliot-Thélène's results for biquadratic extensions (see [CT 14], §4).

The starting point is the following key observation of Jean-Louis Colliot-Thélène :

Proposition. *Let G be a finite group, and let M be a G-lattice. If for all number fields k and every k-torus T with character group isomorphic to the Galois module M via a surjection $\Gamma_k \to G$ one can show that $Ш^1(k,T) = 0$, then $Ш^2_{\mathrm{cycl}}(G,M) = 0$, and every such k-torus T has weak approximation.*

Since every finite group is the Galois group of some unramified extension of number fields, we may realize the purely algebraic group $Ш^2_{\mathrm{cycl}}(G,M)$ as the Tate-Shafarewich group of a torus over a number field; this is summarized in $(*)$. If the module satisfies the hypothesis of the proposition, then $Ш^2_{\mathrm{cycl}}(G,M) = 0$, and weak approximation follows from an exact sequence due to Voskresenskii (see 3.4). As we will see, the hypotheses of the proposition can be weakened (see Corollary 5.3).

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2 Algebraic preliminaries

Let k be a field, let k_s be a separable closure of k and set $\Gamma_k = \mathrm{Gal}(k_s/k)$. We fix once and for all this separable closure k_s, and all separable extensions of k that will appear in the paper will be contained in k_s. We use standard notation in Galois cohomology; in particular, if M is a discrete Γ_k-module and i is an integer ≥ 0, we set $H^i(k, M) = H^i(\Gamma_k, M)$. A Γ_k*-lattice* will be a torsion free $\mathbf{Z}$-module of finite rank on which Γ_k acts continuously.

Lemma 2.1. *Let M be a Γ_k-lattice, and let k'/k be a finite Galois extension with Galois group G such that $\Gamma_{k'}$ acts trivially on M. Then the natural map $H^2(G, M) \to H^2(k, M)$ has trivial kernel.*

Proof. Since M is isomorphic to the trivial $\Gamma_{k'}$-module $\mathbf{Z}^n$ for some n, we have $H^1(k', M) = 0$. Hence the exact sequence of groups $0 \to \Gamma_{k'} \to \Gamma_k \to G \to 0$ induces an exact sequence in Galois cohomology (cf. [S 79], page 118, Prop. 5)

$$(*) \qquad 0 \to H^2(G, M) \to H^2(k, M) \to H^2(k', M)^G.$$

Therefore the map $H^2(G, M) \to H^2(k, M)$ has trivial kernel, as claimed.

Let G be a finite group. A *G-lattice* is by definition a $\mathbf{Z}$-torsion free $\mathbf{Z}[G]$-module of finite rank. For a k-torus T, we denote by $\hat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$ its character group; it is a Γ_k-lattice.

Proposition 2.2. *Let M be a G-lattice. Let $\eta : \Gamma_k \to G$ be a continuous surjective homomorphism. There exists a k-torus T such that $\hat{T}$ is isomorphic to the G-lattice M, regarded as Γ_k-lattice through η.*

Proof. See [Bo 91], Chapter III, 8.12.
If $g \in G$, we denote by $\langle g \rangle$ the cyclic subgroup of G generated by g. Let M be a G-lattice. The cyclic Tate-Shafarevich group $Ш^2_{\mathrm{cycl}}(G, M)$ is the group

$$Ш^2_{cycl}(G, M) = \mathrm{Ker}[H^2(G, M) \to \prod_{g \in G} H^2(\langle g \rangle, M)].$$

We recall a result of Colliot-Thélène and Sansuc:

Theorem 2.3. *Let G be a finite group, let T be a k-torus, and assume that the character group of T is a G-lattice via a surjection $\Gamma_k \to G$. Let T^c be a smooth compactification of T. We have $\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq Ш^2_{\mathrm{cycl}}(G, \hat{T})$.*

Proof. See [CTS 87], Theorem 9.5 (ii). In [CTS 87] , the hypothesis $\mathrm{char}(k) = 0$ is only used to ensure the existence of a a smooth compactification of T; since this is now known in any characteristic (see [CTHSk 05]), the result holds in general.

Let Y be a smooth projective, geometrically connected k-variety, and set $\overline{Y} = Y \times_k k_s$. We have the following spectral sequence

$$H^p(k, H^q(\overline{Y}, \mathbf{G}_m)) \implies H^{p+q}(Y, \mathbf{G}_m)$$

giving the exact sequence

$$(**) \qquad H^2(k, \mathbf{G}_m) \to \mathrm{Ker}[H^2_{\mathrm{et}}(Y, \mathbf{G}_m) \to H^2_{\mathrm{et}}(\overline{Y}, \mathbf{G}_m)] \to H^1(k, \mathrm{Pic}(\overline{Y})) \to$$
$$\to \mathrm{Ker}[H^3_{\mathrm{et}}(k, \mathbf{G}_m) \to H^3_{\mathrm{et}}(Y, \mathbf{G}_m)].$$

We refer to [CTHSk 03], Section 2 for the following theorem.

Theorem 2.4. *Let Y be a smooth projective variety defined over k with $\overline{Y}$ birational to the projective space. Then there is an injection*

$$\mathrm{Br}(Y)/\mathrm{Im}(\mathrm{Br}(k)) \to H^1(k, \mathrm{Pic}(\overline{Y})).$$

If further $Y(k) \neq \emptyset$, the above injection yields an isomorphism $\mathrm{Br}(Y)/\mathrm{Br}(k) \simeq H^1(k, \mathrm{Pic}(\overline{Y}))$.

Proof. Since $\overline{Y}$ is birational to the projective space, $H^2_{\mathrm{et}}(\overline{Y}, \mathbf{G}_m) = \mathrm{Br}(\overline{Y}) = 0$ (cf. [CTSk 19], Theorem 5.1.3, Corollary 5.2.6) and we have an injection

$$\mathrm{Br}(Y)/\mathrm{Im}(\mathrm{Br}(k)) \to H^1(k, \mathrm{Pic}(\overline{Y})).$$

If further $Y(k) \neq \emptyset$, $\mathrm{Ker}[H^i(k, \mathbf{G}_m) \to H^i_{\mathrm{et}}(Y, \mathbf{G}_m)] = 0$ for all i, so that the injection above becomes an isomorphism

$$\mathrm{Br}(Y)/\mathrm{Br}(k) \simeq H^1(k, \mathrm{Pic}(\overline{Y})).$$

3 Arithmetic preliminaries

Let k be a global field, and let Ω_k be the set of all places of k; if $v \in \Omega_k$, we denote by k_v the completion of k at v.

For any k-torus T, set $Ш^i(k, T) = \mathrm{Ker}(H^i(k, T) \to \prod_{v \in \Omega_k} H^i(k_v, T))$. If M is a Γ_k-module, set $Ш^i(k, M) = \mathrm{Ker}(H^i(k, M) \to \prod_{v \in \Omega_k} H^i(k_v, M))$, and let $Ш^i_\omega(k, M)$ be the set of $x \in H^i(k, M)$ that map to 0 in $H^i(k_v, M)$ for almost all $v \in \Omega_k$. Recall that by Poitou-Tate duality, we have an isomorphism of finite groups $Ш^2(k, \hat{T}) \simeq Ш^1(k, T)^*$, where $Ш^1(k, T)^* = \mathrm{Hom}(Ш^1(k, T), \mathbf{Q}/\mathbf{Z})$ denotes the dual of $Ш^1(k, T)$. We denote by G_M the kernel of the map $\Gamma_k \to \mathrm{Aut}(M)$; set $k_M = (k_s)^{G_M}$, and let $G = \Gamma_k / G_M$. If $v \in \Omega_k$, we denote by G_v the decomposition group of a prime w lying over v in the extension k_M/k. For various w over v, the groups G_v's are conjugate and are subgroups of G. Let $Ш^i_{\mathrm{cycl}}(k, M)$ be the set of $x \in H^i(k, M)$ that map to 0 in $H^i(k_v, M)$ for all $v \in \Omega_k$ such that G_v is cyclic. The following lemmas are well-known:

Lemma 3.1. *Let $\mathbf{Z}$ be the trivial Γ_k-module. Then $Ш^2_\omega(k, \mathbf{Z}) = 0$. In particular $Ш^2(k, \mathbf{Z}) = 0$.*

Proof. Let L/k be a field extension. The trivial Γ_L-module $\mathbf{Q}$ is uniquely divisible, hence $H^i(L,\mathbf{Q})=0$ for all $i \geq 1$. Hence the connecting map for the exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ yields an isomorphism $H^1(L,\mathbf{Q}/\mathbf{Z}) \simeq H^2(L,\mathbf{Z})$. Thus $Ш^1_\omega(k,\mathbf{Q}/\mathbf{Z}) \simeq Ш^2_\omega(k,\mathbf{Z})$. Since $Ш^1_\omega(k,\mathbf{Q}/\mathbf{Z})$ classifies cyclic extensions of k (together with a generator of the Galois group) which are locally almost everywhere split, by Chebotarev density theorem, $Ш^1_\omega(k,\mathbf{Q}/\mathbf{Z})=0$. In particular $Ш^2(k,\mathbf{Z})=0$.

Lemma 3.2. *Let M be a Γ_k-module and G_M the kernel of the map $\Gamma_k \to \mathrm{Aut}(M)$. Let $G=\Gamma_k/G_M$. The image of the homomorphism $H^2(G,M) \to H^2(k,M)$ contains $Ш^2_\omega(k,M)$.*

Proof. Let $x \in Ш^2_\omega(k,M)$. Let $k_M=(k_s)^{G_M}$. Since M becomes isomorphic to $\mathbf{Z}^n$ over k_M, by Lemma 3.1 x restricts to zero in $H^2(k_M,M)$. Hence from the exact sequence $(*)$, x belongs to the image of $H^2(G,M) \to H^2(k,M)$.

Lemma 3.3. *Let M be a Γ_k-module, let G_M be the kernel of the map $\Gamma_k \to \mathrm{Aut}(M)$ and let that $\Gamma_k/G_M=G$. Then $Ш^2_\omega(k,M)=Ш^2_{\mathrm{cycl}}(k,M)$.*

Proof. If $v \in \Omega_k$ is unramified in k_M/k, then G_v is cyclic. Hence $Ш^2_{\mathrm{cycl}}(k,M) \subset Ш^2_\omega(k,M)$. We show that $Ш^2_\omega(k,M) \subset Ш^2_{\mathrm{cycl}}(k,M)$. Let $x \in Ш^2_\omega(k,M)$. By Lemma 3.2, there is $y \in H^2(G,M)$ mapping to $x \in H^2(k,M)$. Let $v \in \Omega_k$ be such that G_v is cyclic. By Chebotarev's density theorem, there exist infinitely many $w \in \Omega_k$ such that G_w is conjugate to G_v. Pick $w \in \Omega_k$ such that x maps to zero in $H^2(k_w,M)$ and there is a $g \in G$ such that $gG_wg^{-1}=G_v$. The map $\psi_g : M \to M$ given by $m \to gm$ is $\mathrm{Int}(g)$ semilinear and induces an isomorphism $H^2(\psi_g) : H^2(G,M) \to H^2(G,M)$ which is the identity [HS 71] Proposition 16.2. Further, $H^2(\psi_g)$ restricts to an isomorphism $H^2(G_w,M) \to H^2(G_v,M)$. Thus the restriction of y in $H^2(G_w,M)$ being zero, its restriction to $H^2(G_v,M)$ is zero and therefore its image in $H^2(k_v,M)$ is zero. But this coincides with the image of x in $H^2(k_v,M)$. Thus x maps to zero in $H^2(k_v,M)$. This is true for every v with G_v is cyclic so that $x \in Ш^2_{\mathrm{cycl}}(k,M)$.

Let $\iota : T(k) \to \prod_{v\in\Omega_k} T(k_v)$ be the diagonal embedding, and let $A(T)$ be the quotient of $\prod_{v\in\Omega_k} T(k_v)$ by the closure of the image of ι; the group $A(T)$ is the obstruction to weak approximation on T. Set $Ш(T)=Ш^1(k,T)$; this is the obstruction to the Hasse principle for torsors under T.

The following is a reformulation of a result of Voskresenskii:

Proposition 3.4. *Let G be a finite group, let T be a k-torus, and assume that the character group of T is a G-lattice via a surjection $\Gamma_k \to G$. We have an exact sequence*

$$0 \to A(T) \to Ш^2_{\mathrm{cycl}}(G,\hat{T})^* \to Ш(T) \to 0.$$

Proof. Let T^c be a smooth compactification of T; by [San 81], Theorem 9.5. (M) we have the exact sequence

$$0 \to A(T) \to \mathrm{Br}_a(T^c)^* \to Ш(T) \to 0,$$

Note that since T_{k_s} is split and hence rational, we have $\mathrm{Br}(T^c_{k_s}) = 0$ (see [CTSk 19], Corollary 5.2.6). By Proposition 2.3 we have $\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq Ш^2_{\mathrm{cycl}}(G, \hat{T})$, hence we get the exact sequence

$$0 \to A(T) \to Ш^2_{\mathrm{cycl}}(G, \hat{T})^* \to Ш(T) \to 0.$$

4 The group $Ш^2_{cycl}(G, M)$

Let G be a finite group.

Proposition 4.1. *Let ℓ'/ℓ be a finite Galois extension of global fields with Galois group G which is unramified at all the finite places. Let M be a G-lattice regarded as a Γ_ℓ-module via the surjection $\Gamma_\ell \to G$. Then we have*

$$Ш^2_{\mathrm{cycl}}(G, M) \simeq Ш^2(\ell, M).$$

This proposition is an immediate consequence of Proposition 4.2 below :

Proposition 4.2. *Let ℓ'/ℓ be a finite Galois extension of global fields with Galois group G. Assume that all the decomposition groups of ℓ'/ℓ are cyclic. Let M be a G-lattice, regarded as Γ_ℓ-lattice through the surjection $\Gamma_\ell \to G$. Then we have*

$$Ш^2_{\mathrm{cycl}}(G, M) \simeq Ш^2(\ell, M).$$

Proposition 4.2 follows from Proposition 4.3 below. We use the notation of the previous section.

Proposition 4.3. *Let ℓ be a global field, let M be a Γ_ℓ-module, and assume that $\Gamma_\ell/G_M \simeq G$. Then we have*

$$Ш^2_{\mathrm{cycl}}(G, M) \simeq Ш^2_{\mathrm{cycl}}(\ell, M).$$

Proof of Proposition 4.3. Set $\ell' = \ell_M$; note that ℓ'/ℓ is a Galois extension with group G. The homomorphism $f : H^2(G, M) \to H^2(\ell, M)$ induced by the surjection $\Gamma_\ell \to G$ is injective by Lemma 2.1. By Lemmas 3.2 and 3.3, the image of f contains $Ш^2\mathrm{cycl}(\ell, M)$. We next prove that the restriction of f to $Ш^2_{\mathrm{cycl}}(G, M)$ maps it into $Ш^2_{\mathrm{cycl}}(\ell, M)$. Let $x \in Ш^2_{\mathrm{cycl}}(G, M)$. Let $v \in \Omega_\ell$ such that the decomposition group G_v of ℓ'/ℓ at v is a cyclic subgroup of G. Then the restriction of $x \in H^2(G, M)$ to $H^2(G_v, M)$ is zero. The composite $H^2(G, M) \to H^2(\ell, M) \to H^2(\ell_v, M)$ factors as $H^2(G, M) \to H^2(G_v, M) \to H^2(\ell_v, M)$. Hence $f(x)$ maps to zero in $H^2(\ell_v, M)$. Thus $f(x) \in Ш^2_{\mathrm{cycl}}(\ell, M)$.

Clearly $Ш^2_{\rm cycl}(G,M) \to Ш^2_{\rm cycl}(\ell,M)$ is injective. We prove that this map is surjective. Let $y \in Ш^2_{\rm cycl}(\ell,M)$ and let $x \in H^2(G,M)$ be such that $f(x)=y$. Let $g \in G$. By Chebotarev's density theorem, there is a finite place $v \in \Omega_\ell$ such that $G_v = \langle g \rangle$. We claim that the restriction of x to $H^2(\langle g \rangle, M) = H^2(G_v,M)$ maps to zero in $H^2(\ell_v,M)$. In fact this image is the same as the restriction of y to $H^2(\ell_v,M)$. Since $y \in Ш^2_{\rm cycl}(\ell,M)$ and G_v is cyclic, the image of y in $H^2(\ell_v,M)$ is zero. By lemma 2.1, the map $H^2(G_v,M) \to H^2(\ell_v,M)$ is injective. It follows that the restriction of x to $H^2(G_v,M)$ is zero and hence x belongs to $Ш^2_{\rm cycl}(G,M)$. This completes the proof of the proposition.

Proof of Proposition 4.2. Since all the decomposition groups of ℓ'/ℓ are cyclic, we have $Ш^2(\ell,M) = Ш^2_{\rm cycl}(\ell,M)$. By Proposition 4.3, we have $Ш^2_{cycl}(G,M) \simeq Ш^2_{\rm cycl}(\ell,M)$, hence $Ш^2_{cycl}(G,M) \simeq Ш^2(\ell,M)$, as claimed.

Proof of Proposition 4.1. Since ℓ'/ℓ is unramified at all the finite places, all the decomposition groups are cyclic; we conclude by applying Proposition 4.2.

Corollary 4.4. *Let ℓ be a global field, let M be a Γ_ℓ-module, and assume that $\Gamma_\ell/G_M \simeq G$. Then we have Theorem 9.3 (b)*

$$Ш^2_{\rm cycl}(G,M) \simeq Ш^2_\omega(\ell,M).$$

Proof. This follows from Proposition 4.3 and Lemma 3.3.

Corollary 4.5. *Let M be a G-lattice. Let ℓ'/ℓ be as in Proposition 4.2, and let T be an ℓ-torus with character group M. We have*

$$Ш^2_{\rm cycl}(G,M) \simeq Ш^2(\ell,M) = Ш^2(\ell,\hat{T}) \simeq Ш^1(\ell,T)^*.$$

Proof. This follows from Proposition 4.2, and from Poitou-Tate duality.

In the following sections, we also need a result of Fröhlich :

Proposition 4.6. (Fröhlich) *There exists a Galois extension of number fields with Galois group G that is unramified at all the finite places.*

Proof. This is the main result of [F 62].

In the next sections we use Corollary 4.5 together with 4.6 to realize the purely algebraic group $Ш^2_{\rm cycl}(G,M)$ as the Tate-Shafarevich group of a torus over a number field. This makes it possible to apply arithmetic results to obtain algebraic ones.

5 Vanishing results

The aim of this section is to apply the results of §4 and, under an additional hypothesis (condition (C) below) prove some vanishing theorems. Let G be a finite group.

Definition 5.1. Let M be a G-lattice. We say that M *satisfies condition* (C) if there exists a Galois extension ℓ'/ℓ of number fields with Galois group G such that all the decomposition groups of ℓ'/ℓ are cyclic, and such that the ℓ-torus S associated to the Galois lattice M with the Galois group Γ_ℓ acting through the quotient group G has the property $Ш^1(\ell, S) = 0$.

Corollary 5.2. *Let M be a G-lattice satisfying condition* (C)*. Then*

$$Ш^2_{\text{cycl}}(G, M) = 0.$$

Proof. By Poitou-Tate duality, we have $Ш^1(\ell, S) \simeq Ш^2(\ell, \hat{S})^* = Ш^2(\ell, M)^*$. Corollary 4.5 implies that the groups $Ш^2(\ell, \hat{S}) = Ш^2(\ell, M)$ and $Ш^2_{cycl}(G, M)$ are isomorphic. Hence $Ш^1(\ell, S)$ is dual to the group $Ш^2_{cycl}(G, M)$. Since $Ш^1(\ell, S) = 0$, we have $Ш^2_{cycl}(G, \hat{S}) = Ш^2_{cycl}(G, M) = 0$.

Corollary 5.3. *Let k be a global field with a surjection $\Gamma_k \to G$, and let T be a k-torus; assume that the character group of T is a G-lattice satisfying condition* (C)*. Then*

$$Ш^2_\omega(k, \hat{T}) = 0,$$

and Hasse principle and weak approximation hold for torsors under T.

Proof. By Proposition 3.4, we have the exact sequence

$$0 \to A(T) \to Ш^2_{\text{cycl}}(G, \hat{T})^* \to Ш(T) \to 0,$$

We have $Ш^2_{\text{cycl}}(G, \hat{T})^* = 0$ by Corollary 5.2, hence $A(T) = Ш(T) = 0$. By Corollary 4.4, we have $Ш_\omega(k, \hat{T}) = 0$; weak approximation holds for T, and Hasse principle holds for torsors under T.

This implies Colliot-Thélène's observation cited in the introduction :

Corollary 5.4. *Let G be a finite group, and let M be a G-lattice. If for all number fields k and every k-torus T of character group isomorphic to the Galois module M via a surjection of $\Gamma_k \to G$, one can show that $Ш^1(k, T) = 0$, then $Ш^2_{\text{cycl}}(G, M) = 0$, and every such k-torus T satisfies weak approximation.*

6 Unramified Brauer groups

Let k be a field, T a k-torus and X a torsor under T. Let T^c be a smooth equivariant compactification of T; such a compactification exists in any characteristic (see [CTHSk 05]). Then, the contracted product $X^c = X \times^T T^c$ is a smooth compactification of X. Note that $\overline{T}^c \simeq \overline{X}^c$ is birational to the projective space; further $T^c(k) \neq \emptyset$. We therefore have exact sequences, by Theorem 2.4,

$$\text{Br}(k) \to \text{Br}(X^c) \to H^1(k, \text{Pic}(\overline{X}^c))$$

and

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}(T^c) \to H^1(k, \mathrm{Pic}(\overline{T}^c)) \to 0.$$

By [CTHSk 03, Lemma 2.1], $H^1(k, \mathrm{Pic}(\overline{X}^c)) \simeq H^1(k, \mathrm{Pic}(\overline{T}^c))$, we have an injection

$$\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) \to \mathrm{Br}(T^c)/\mathrm{Br}(k).$$

The aim of this section and the following ones is to use the results of the previous sections to obtain information on the quotients $\mathrm{Br}(T^c)/\mathrm{Br}(k)$ and $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k))$. We start with some vanishing results,

Proposition 6.1. *Let G be a finite group. Assume that the character group of T is a G-lattice satisfying condition* (C). *Then* $\mathrm{Br}(T^c)/\mathrm{Br}(k) = 0$. *If X is a torsor over T, we have* $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0$.

Proof. By Corollary 5.2, we have $Ш^2_{\mathrm{cycl}}(G, \hat{T}) = 0$. On the other hand, by Theorem 2.3, $\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq Ш^2_{\mathrm{cycl}}(G, \hat{T})$), therefore $\mathrm{Br}(T^c)/\mathrm{Br}(k) = 0$. As we saw above, $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k))$ injects into $\mathrm{Br}(T^c)/\mathrm{Br}(k)$, hence this implies that $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0$.

7 Norm equations

Let k be a field, and let L be an étale k-algebra of finite rank (in other words, a product of a finite number of separable extensions of k). Let $T_{L/k} = R^{(1)}_{L/k}(\mathbf{G}_m)$ be the k-torus defined by

$$1 \to T_{L/k} \to R_{L/k}(\mathbf{G}_m) \xrightarrow{\mathrm{N}_{L/k}} \mathbf{G}_m \to 1.$$

Let $a \in k^\times$. Let X be the affine k-variety associated to the *norm equation*

$$\mathrm{N}_{L/k}(t) = a.$$

The variety X is a torsor under $T_{L/k}$; let X^c be a smooth compactification of X.

In the later sections, we need the following result

Theorem 7.1. *Suppose $L = K \times E$ with K/k a cyclic extension and E an étale k-algebra of finite rank. Then*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq \mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)).$$

Proof. Since $K \otimes_k L \simeq K \times L'$ for some étale algebra L' over K, the variety X_K is isomorphic to $R_{L'/K}(G_m)$ and hence is K-rational. Since K/k is cyclic, by [CT 14, Proposition 1.1], the map $H^1(k, \mathrm{Pic}\overline{X}^c) \to H^3(k, \mathbf{G}_m)$ in the sequence $(**)$ is zero and one has an isomorphism (cf. Theorem 2.4) $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) \simeq H^1(k, \mathrm{Pic}\overline{X}^c)$. We also have an isomorphism (cf. Theorem 2.4)

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq H^1(k, \mathrm{Pic}\overline{T}^c).$$

By [CTHSk 03, Lemma 2.1], we have an isomorphism

$$H^1(k, \mathrm{Pic}\overline{X}^c) \simeq H^1(k, \mathrm{Pic}\overline{T}^c).$$

This leads to an isomorphism

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq \mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)).$$

8 Norm equations - first examples

The aim of this section and the next ones is to give some examples of étale algebras for which we apply the results of the previous sections, obtaining information about the unramified Brauer group, Hasse principle and weak approximation (in the case where k is a global field) for the variety X. We keep the notation of the previous section.

The first examples concern étale algebras that are products of two fields, finite extensions of the ground field k.

Products of two fields

We start by introducing some notation that will be used in the two examples of this section. Let $L = K_1 \times K_2$, where K_1 and K_2 are finite extensions of k. Let k'/k be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \mathrm{Gal}(k'/k)$; let $M = \hat{T}_{L/k}$ be the character G-lattice of $T_{L/k}$. For $i = 1, 2$, let H_i be the subgroup of G such that $K_i = (k')^{H_i}$.We have the exact sequence of G-modules

$$0 \to \mathbf{Z} \to \mathbf{Z}[G/H_1] \oplus \mathbf{Z}[G/H_2] \to M \to 0.$$

Let ℓ'/ℓ be an unramified extension of number fields with Galois group G.

Some consequences of Hrlimann's theorem

The first example is based on a result of Hürlimann [H 84]. With the notation above, we assume that K_1/k is a cyclic extension.

Theorem 8.1. *We have* $Ш^2_{\mathrm{cycl}}(G, M) = 0$.

Proof. Set $\ell_i = (\ell')^{H_i}$ for $i = 1,2$. Let S be the norm torus corresponding to the étale ℓ-algebra $\ell_1 \times \ell_2$. Hürlimann's result [H 84], Proposition 3.3 implies that $Ш^1(\ell, S) = 0$ (in [H 84], the extension K_2/k is supposed to be Galois, but this is not necessary; see [BLP 19], Proposition 4.1 for a different proof of the general case). We have $\hat{S} \simeq M$ by construction, hence by Proposition 4.1 we have $Ш^2_{\mathrm{cycl}}(G, M) = 0$.

Remark 8.2. Theorem 8.1 was proved by Sansuc (unpublished) by algebraic methods. His proof is rather involved; the proof presented here, passing from arithmetic to algebra, is simpler, since the proof of the arithmetic result in [BLP 19], Proposition 4.1 is quite short.

Theorem 8.3. *We have* $\mathrm{Br}(T^c_{L/k})/\mathrm{Br}(k) = 0$, *and* $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0$.

Proof. By Theorem 8.1 and Proposition 6.1, we have $\mathrm{Br}(T^c_{L/k})/\mathrm{Br}(k) = 0$ and $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0$.

Linearly disjoint Galois extensions

This example is based on a result of Pollio and Rapinchuk, [PR 13]. Let $L = K_1 \times K_2$, where K_1 and K_2 are finite extensions of k such that the Galois closures of K_1 and K_2 are linearly disjoint.

Theorem 8.4. *We have* $\mathrm{Br}(T^c_{L/k})/\mathrm{Br}(k) = 0$, *and* $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0$.

Proof. Let K'_1 and K'_2 be the Galois closures of K_1, respectively K_2, and let H'_1, H'_2 be the subgroups of G such that $K'_i = (k')^{H'_i}$, for $i = 1,2$. By hypothesis, the extensions K'_1 and K'_2 are linearly disjoint, hence $G = H'_1.H'_2$.

Recall that ℓ'/ℓ is an unramified extension of number fields with Galois group G. Set $\ell_i = (\ell')^{H_i}$ and $\ell'_i = (\ell')^{H'_i}$ for $i = 1,2$. Since H'_i is a normal subgroup of G for $i = 1,2$, the extensions ℓ'_i/ℓ are Galois, and $|H'_i| = [\ell' : \ell'_i]$. Since $G = H'_1.H'_2$, the fields ℓ'_i are linearly disjoint.

Let S be the norm torus corresponding to the étale ℓ-algebra $\ell_1 \times \ell_2$; the main theorem of [PR 13] implies that $Ш^1(\ell, S) = 0$. We have $\hat{S} \simeq M$ by construction, hence by Corollary 5.2 we have $Ш^2_{\mathrm{cycl}}(G, M) = 0$. By Proposition 6.1, we have $\mathrm{Br}(T^c_{L/k})/\mathrm{Br}(k) = 0$ and $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0$.

9 Products of cyclic extensions of prime power degree - statement of results and notation

The proofs of the results of this section will be given in §11. Let p be a prime number. If K/k is a cyclic extension of degree a power of p, we denote by $(K)_{\mathrm{prim}}$ the unique subfield of K of degree p over k; if $E = \prod_{i \in I} K_i$, where K_i/k is a cyclic extension of degree a power of p for all $i \in I$, set $E_{\mathrm{prim}} = \prod_{i \in I} (K_i)_{\mathrm{prim}}$.

Let L be a product of n cyclic extensions of degrees powers of p. With the notation of the previous section, set $T = T_{L/k}$. Let $a \in k^\times$, and X be the affine k-variety associated to the norm equation $\mathrm{N}_{L/k}(t) = a$; let X^c be a smooth compactification of X.

Let $T_{\mathrm{prim}} = T_{L_{\mathrm{prim}}/k}$. We denote by X_{prim} the affine k-variety associated to $\mathrm{N}_{L_{\mathrm{prim}}/k}(t) = a$, and by X^c_{prim} a smooth compactification of X_{prim}.

Theorem 9.1. *We have*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) = 0 \iff \mathrm{Br}(T^c_{\mathrm{prim}})/\mathrm{Br}(k) = 0.$$

Let k'/k be a Galois extension of minimal degree splitting T, and let $G = \mathrm{Gal}(k'/k)$; similarly, let k'_{prim}/k be a Galois extension of minimal degree splitting T_{prim}, and let $G_{\mathrm{prim}} = \mathrm{Gal}(k'_{\mathrm{prim}}/k)$.

Theorem 9.2. *We have*

$$Ш^2_{\mathrm{cycl}}(G, \hat{T}) = 0 \iff Ш^2_{\mathrm{cycl}}(G_{\mathrm{prim}}, \hat{T}_{\mathrm{prim}}) = 0.$$

Products of at least $p+2$ pairwise disjoint cyclic extensions

With the notation above, we now consider the case where $n \geq p + 2$.

Theorem 9.3. *Assume that L is a product of at least $p+2$ pairwise disjoint cyclic extensions of degrees powers of p. Then we have*

(a)
$$\mathrm{Br}(T^c)/\mathrm{Br}(k) = 0.$$

(b)
$$\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0.$$

(c) *Suppose that k is a global field. Then $Ш^2_{\omega}(k, \hat{T}_{L/k}) = 0$, and Hasse principle and weak approximation hold for X.*

Theorem 9.4. *Assume that $n \geq p + 2$. Then we have*

$$Ш^2_{\mathrm{cycl}}(G, \hat{T}) = 0.$$

At least one cyclic factor of degree p

In the next results, we assume that L has at least one factor of degree p.

Theorem 9.5. *Assume that L is a product of n pairwise disjoint cyclic extensions of degrees powers of p, and that at least one of these is of degree p. Then we have*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq \mathrm{Br}(T^c_{\mathrm{prim}})/\mathrm{Br}(k).$$

Theorem 9.6. *Assume that L is a product of n pairwise disjoint cyclic extensions of degrees powers of p, and that at least one of these is of degree p. Then we have*

$$Ш^2_{\mathrm{cycl}}(G,\hat{T}) \simeq Ш^2_{\mathrm{cycl}}(G_{\mathrm{prim}},\hat{T}_{\mathrm{prim}}).$$

Products of cyclic extensions of degree p

Finally, we determine $\mathrm{Br}(T^c_{\mathrm{prim}})/\mathrm{Br}(k)$. Let us denote by C_p the cyclic group of order p.

Theorem 9.7. (a) *If $G_{\mathrm{prim}} \not\simeq C_p \times C_p$, then*

$$\mathrm{Br}(T^c_{\mathrm{prim}})/\mathrm{Br}(k) = \mathrm{Br}(X^c_{\mathrm{prim}})/\mathrm{Im}(\mathrm{Br}(k)) = 0.$$

(b) *If $G_{\mathrm{prim}} \simeq C_p \times C_p$, then*

$$\mathrm{Br}(T^c_{\mathrm{prim}})/\mathrm{Br}(k) \simeq \mathrm{Br}(X^c_{\mathrm{prim}})/\mathrm{Im}(\mathrm{Br}(k)) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}.$$

Theorem 9.8. (a) *If $G_{\mathrm{prim}} \not\simeq C_p \times C_p$, then $Ш^2_{\mathrm{cycl}}(G_{\mathrm{prim}},\hat{T}_{\mathrm{prim}}) = 0$.*

(b) *If $G_{\mathrm{prim}} \simeq C_p \times C_p$, then $Ш^2_{\mathrm{cycl}}(G_{\mathrm{prim}},\hat{T}_{\mathrm{prim}}) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$.*

In the case where k is a global field, we obtain the following corollaries

Corollary 9.9. *Suppose that k is a global field. Assume that L is a product of n pairwise disjoint cyclic extensions of degrees powers of p, and that at least one of these is of degree p.*

(a) *If $G_{\mathrm{prim}} \not\simeq C_p \times C_p$, then Hasse principle and weak approximation hold for T.*

(b) *If $G_{\mathrm{prim}} \simeq C_p \times C_p$, then either Hasse principle holds for torsors over T (and weak approximation for T fails), or weak approximation holds for T (and Hasse principle for torsors over T fails).*

Proof. Part (a) follows from Theorems 9.8 (a), 9.6 and 3.4. To prove part (b), we apply Theorems 9.8 (b), 9.6, 3.4, as well as [BLP 19], Theorem 8.3 and Corollary 5.17.

Corollary 9.10. *Assume that k is a global field, and that L is a product of n distinct cyclic extensions of degree p. If $G \not\simeq C_p \times C_p$, then*

$$Ш_\omega(k,\hat{T}_{L/k}) = 0.$$

If $G \simeq C_p \times C_p$, then

$$Ш_\omega(k,\hat{T}_{L/k}) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}.$$

Proof. This follows from Theorem 9.8 and Lemma 4.4.

Remark 9.11. Corollary 9.10 was also proved by Macedo, see [Ma 19], Theorem 4.9 and Corollary 4.10, with different methods (namely, using a generalization of the approach of Drakokhrust and Platonov).

10 Products of cyclic extensions of prime power degree - global fields

We recall some results from [BLP 19]; these will be used in the next section to prove the results of §9. Let k be a global field. We start by recalling some notation from [BLP 19]. If L is an étale algebra of finite rank over k having at least one factor that is a cyclic extension of k, the paper [BLP 19] introduces a finite abelian group $Ш(L)$ (see [BLP 19], §5) and proves (see [BLP 19], Corollary 5.17) that $Ш(L)^* \simeq Ш^1(k, T_{L/k})$; equivalently, by Poitou-Tate duality, we have $Ш(L) \simeq Ш^2(k, \hat{T}_{L/k})$.

Let p be a prime number.

Proposition 10.1. *Let L be a product of cyclic extensions of degrees powers of p. The group $Ш^1(k, T_{L_{\mathrm{prim}}/k})^*$ injects into $Ш^1(k, T_{L/k})^*$.*

Proof. This follows from [BLP 19], Lemma 8.7.

Theorem 10.2. *Let L be a product of cyclic extensions of degrees powers of p. Then we have*

$$Ш^1(k, T_{L/k}) = 0 \iff Ш^1(k, T_{L_{\mathrm{prim}}/k}) = 0.$$

Proof. This is an immediate consequence of [BLP 19], Theorem 8.1.

Proposition 10.3. *Let L be a product of n distinct cyclic extensions of degrees powers of p, and assume that at least one of the extensions is of degree p. Then $Ш^1(k, T_{L/k})$ is a finite abelian group of type $(p, ..., p)$ of order at most p^{n-2}.*

Proof. Let us write L as a product $L = K \times K'$, where K is a cyclic extension of k of degree p, and K' is a product of $n-1$ cyclic extensions of degrees powers of p. In [BLP 19], 5.1, we construct a finite abelian group $Ш(L) = Ш(K, K')$ such that when K is cyclic of order p, the group $Ш(K, K')$ is of type $(p, ..., p)$ of order at most p^{n-2}. It is shown in [BLP 19], 5.3 that the group $Ш(K, K')$ does not depend on the decomposition of L as $K \times K'$, where K is a cyclic extension of k, and that $Ш(K, K')^* \simeq Ш^1(k, T_{L/k})$ (see [BLP 19], Corollary 5.17). Hence $Ш^1(k, T_{L/k})$ is a finite abelian group of type $(p, ..., p)$ of order at most p^{n-2}, as claimed.

Theorem 10.4. *Let $L = K_1 \times \cdots \times K_n$, where K_i are distinct cyclic extensions of degree p of k.*

(*a*) *If $n \leq 2$, or $n \geq p+2$, or $3 \leq n \leq p+1$ and $K_1, \ldots, K_n$ are not contained in some field extension of k of degree p^2 having all local degrees $\leq p$, then*

$$Ш^1(k, T_{L/k}) = 0.$$

(*b*) *Assume that $3 \leq n \leq p+1$ and that the fields $K_1, \ldots, K_n$ are contained in some field extension of k of degree p^2 having all local degrees $\leq p$, then*

$$Ш^1(k, T_{L/k}) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}.$$

Proof. This follows from [BLP 19], Theorem 8.3, Proposition 8.5 and Corollary 5.17.

Remark. More generally, one can treat the case where the K_i's are field extensions of degree p, with at least one of them cyclic (see [BLP 19], Proposition 8.5).

Theorem 10.5. *Let L be a product of n pairwise disjoint cyclic extensions of degrees powers of p, and assume that at least one of the extensions is of degree p. Then we have*

$$Ш^1(k, T_{L/k}) \simeq Ш^1(k, T_{L_{\mathrm{prim}}/k}).$$

Proof. By Theorem 10.4, we have either $Ш^1(k, T_{L_{\mathrm{prim}}/k}) = 0$ or $Ш^1(k, T_{L_{\mathrm{prim}}/k}) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$. If $Ш^1(k, T_{L_{\mathrm{prim}}/k}) = 0$, then by Theorem 10.2, we have $Ш^1(k, T_{L/k}) = 0$. Assume now that $Ш^1(k, T_{L_{\mathrm{prim}}/k}) \neq 0$; then by Theorem 10.4 we have $Ш^1(k, T_{L_{\mathrm{prim}}/k}) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$, therefore the order of $Ш^1(k, T_{L_{\mathrm{prim}}/k})$ is equal to p^{n-2}. By Proposition 10.1, this implies that the order of $Ш^1(k, T_{L/k})$ is at least p^{n-2}. On the other hand, since at least one of the factors of L is of order p, Proposition 10.3 implies that $Ш^1(k, T_{L/k})$ is a finite abelian group of type $(p, \ldots, p)$ of order at most p^{n-2}. Hence the order of $Ш^1(k, T_{L/k})$ is equal to p^{n-2}, and this completes the proof of the Theorem.

11 Products of cyclic extensions of prime power degree - proofs

We keep the notation of §9. In particular, p is a prime number, L is a product of n cyclic extensions of degrees powers of p, k'/k is a Galois extension of minimal degree splitting $T = T_{L/k}$, and $G = \mathrm{Gal}(k'/k)$. Let $M = \hat{T}$ be the G-lattice of characters of T. Let us write $L = \prod_{i\in I} K_i$. Since k' splits T, k' also splits $R_{L/k}(G_m)$ and it follows that k' contains all the factors K_i of L. Let H_i be the subgroup of G such that $K_i = (k')^{H_i}$. We have the exact sequence of G-modules

$$0 \to \mathbf{Z} \to \underset{i\in I}{\oplus} \mathbf{Z}[G/H_i] \to M \to 0.$$

Recall that k'_{prim}/k is a Galois extension of minimal degree splitting T_{prim}, and that $G_{\mathrm{prim}} = \mathrm{Gal}(k'_{\mathrm{prim}}/k)$. Let M_{prim} be the G_{prim}-lattice of characters of T_{prim}. Let $(H_i)_{\mathrm{prim}}$ be the subgroup of G_{prim} such that $(K_i)_{\mathrm{prim}} = (k'_{\mathrm{prim}})^{(H_i)_{\mathrm{prim}}}$. We have the exact sequence of G_{prim}-modules

$$0 \to \mathbf{Z} \to \underset{i\in I}{\oplus} \mathbf{Z}[G_{\mathrm{prim}}/(H_i)_{\mathrm{prim}}] \to M_{\mathrm{prim}} \to 0.$$

Set $\Gamma_i = (G/H_i)/(G_{\mathrm{prim}}/(H_i)_{\mathrm{prim}})$, and note that $(K_i)_{\mathrm{prim}} = K_i^{\Gamma_i}$.

Let ℓ'/ℓ be an extension of number fields with Galois group G which is unramified at all the finite places. For all $i \in I$, let L_i be the fixed field of H_i in ℓ', and set $E = \prod_{i \in I} L_i$. The character lattice of the torus $T_{E/\ell}$ is isomorphic to the G-lattice M. We have $(L_i)_{\rm prim} = L_i^{\Gamma_i}$. Note that $\text{Ш}^2_{\rm cycl}(G,M) \simeq \text{Ш}^2(\ell, \hat{T}_{E/\ell})$ and that $\text{Ш}^2_{\rm cycl}(G_{\rm prim}, M_{\rm prim}) \simeq \text{Ш}^2(\ell, \hat{T}_{E_{\rm prim}/\ell})$.

Proof of Theorem 9.2. By Theorem 10.2, we have

$$\text{Ш}^1(\ell, T_{E/\ell}) = 0 \iff \text{Ш}^1(\ell, T_{E_{\rm prim}/\ell}) = 0,$$

therefore, since $\hat{T}_{E/\ell} \simeq M$ and $\hat{T}_{E_{\rm prim}/\ell} \simeq M_{\rm prim}$, we have

$$\text{Ш}^2(\ell, M) = 0 \iff \text{Ш}^2(\ell, M_{\rm prim}) = 0.$$

By Proposition 4.1, we have

$$\text{Ш}^2_{cycl}(G,M) \simeq \text{Ш}^2(\ell, M)$$

and

$$\text{Ш}^2_{cycl}(G_{\rm prim}, M_{\rm prim}) \simeq \text{Ш}^2(\ell, M_{\rm prim}).$$

Since $M = \hat{T}$ and $M_{\rm prim} = \hat{T}_{\rm prim}$, we obtain

$$\text{Ш}^2_{\rm cycl}(G, \hat{T}) = 0 \iff \text{Ш}^2_{\rm cycl}(G_{\rm prim}, \hat{T}_{\rm prim}) = 0,$$

as claimed.

Proof of Theorem 9.1. Theorems 9.2 and 9.1 are equivalent by Theorem 2.3.

From now on, we assume that the extensions K_i/k are pairwise disjoint which implies the same property for the extensions L_i/l.

Proof of Theorem 9.8 Note that $G_{\rm prim}$ is an elementary abelian p-group, with $|G_{\rm prim}| = p$ if $n = 1$, and $|G_{\rm prim}| \geq p^2$ if $n \geq 2$. We may assume that $n \geq 2$.

Assume first that $G_{\rm prim} \not\simeq C_p \times C_p$. Then all the factors of the étale ℓ-algebra E are not contained in a field extension of degree p^2 of ℓ, therefore Theorem 10.4 implies that $\text{Ш}^1(\ell, T) = 0$, and by Corollary 5.2 this implies that $\text{Ш}^2_{cycl}(G_{\rm prim}, M) = 0$.

Assume now that $G_{\rm prim} \simeq C_p \times C_p$. Then, ℓ' is a degree p^2 extension of l containing all the factors of E. Since the extensions L_i are pairwise disjoint, $n \leq p + 1$. Since ℓ'/ℓ is unramified at all the finite places, the local degrees are $\leq p$. By Theorem 10.4, this implies that $\text{Ш}^1(\ell, T) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$. Hence $\text{Ш}^2(\ell, \hat{T})) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$. Since the G-lattices $\hat{T}$ and M are isomorphic, by Corollary 4.5 we have $\text{Ш}^2_{\rm cycl}(G,M) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$.

Proof of Theorem 9.7. Both (a) and (b) follow directly from Theorems 2.3, 7.1 and 9.8.

Proof of Theorem 9.4. By Theorem 9.8, we have $Ш^2_{\mathrm{cycl}}(G_{\mathrm{prim}}, \hat{T}_{\mathrm{prim}}) = 0$, and by Theorem 9.2 this implies that $Ш^2_{\mathrm{cycl}}(G, \hat{T}) = 0$.

Proof of Theorem 9.3. Theorems 9.4 and 9.3 (a) are equivalent by Theorem 2.3. Theorem 9.3 (b) follows Theorem 7.1. Theorem 9.3 (c) follows from Corollary 4.4 and Proposition 3.4.

Proof of Theorem 9.6. By Theorem 10.5, we have

$$Ш^1(\ell, T_{E/\ell})^* \simeq Ш^1(\ell, T_{E_{\mathrm{prim}}/\ell})^*,$$

and by Poitou-Tate duality this implies that $Ш^2(\ell, \hat{T}_{E/\ell}) \simeq Ш^2(\ell, \hat{T}_{E_{\mathrm{prim}}/\ell})$. Since $\hat{T}_{E/\ell} \simeq M \simeq \hat{T}$, and $\hat{T}_{E_{\mathrm{prim}}/\ell} \simeq M_{\mathrm{prim}} \simeq \hat{T}_{\mathrm{prim}}$, applying Corollary 4.5, we obtain $Ш^2_{\mathrm{cycl}}(G_{\mathrm{prim}}, \hat{T}_{\mathrm{prim}}) \simeq Ш^2_{\mathrm{cycl}}(G, \hat{T})$, as claimed.

Proof of Theorem 9.5. Theorems 9.6 and 9.5 are equivalent by Theorem 2.3.

12 Unramified Brauer groups and products of cyclic extensions

Let p be a prime number, and let $L = K_1 \times \cdots \times K_n$, where $K_1, \ldots, K_n$ are distinct cyclic extensions of k of degree p. Let $T_{L/k} = R^{(1)}_{L/k}(\mathbf{G}_m)$ be the k-torus defined by

$$1 \to T_{L/k} \to R_{L/k}(\mathbf{G}_m) \xrightarrow{\mathrm{N}_{L/k}} \mathbf{G}_m.$$

Let k'/k be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \mathrm{Gal}(k'/k)$. Let M be the G-lattice of characters of $T_{L/k}$.

Let $a \in k^\times$, and let X be the affine k-variety determined by the equation $\mathrm{N}_{L/k}(t) = a$, and note that X is a torsor under $T = T_{L/k}$. Let X^c be a smooth compactification of X.

The following result is an immediate consequence of Theorems 9.7 and 7.1:

Theorem 12.1. (a) *If $G \not\simeq C_p \times C_p$, then*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) = \mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) = 0.$$

(b) *If $G \simeq C_p \times C_p$, then*

$$\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq \mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k)) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}.$$

Assuming that $\mathrm{char}(k) \neq p$, we obtain a more precise result, namely we give generators for the group $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k))$.

Set $I = \{1, \ldots, n\}$. We consider the norm polynomials $\mathrm{N}_{K_i/k}(t_i)$ for $i \in I$ as elements of $k(X)$. For all $i \in I$, set $N_i = \mathrm{N}_{K_i/k}(t_i)$ and let σ_i be a generator of $\mathrm{Gal}(K_i/k)$. Let $\tilde{K}_n \in H^1(k, \mathbf{Z}/p\mathbf{Z})$ be the element associated to the pair (K_n, σ_n). We identify $H^1(k(X^c), \mu_p)$ with $k(X^c)^*/k(X^c)^{*p}$ via the Kummer isomorphism and regard $[N_i] \in H^1(k(X^c), \mu_p)$. The variety X is defined by $N_1 N_2 \ldots N_n = a$ in the affine space $k[t_i, 1 \leq i \leq n]$. Let $(N_i, \tilde{K}_n)$ denote the class of the cyclic algebra of degree p over $k(X^c)$ associated to $[N_i] \in H^1(k(X^c), \mu_p)$ and $\tilde{K}_n \in H^1(k(X^c), \mathbf{Z}/p\mathbf{Z})$.

Theorem 12.2. *Suppose that* $\mathrm{char}(k) \neq p$, *and that* $G \simeq C_p \times C_p$. *Then* $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k))$ *is generated by the* $n-2$ *linearly independent elements*

$$(N_i, \tilde{K}_n), \quad i = 1, \ldots, n-2.$$

in $\mathrm{Br}(k(X^c))$.

We begin with the following lemma:

Lemma 12.3. *Let* K/k *be a cyclic extension of degree* n *with* $(n, \mathrm{char}(k)) = 1$. *Let* σ *be a generator of* $\mathrm{Gal}(K/k)$ *and let* A *be the cyclic algebra over* k *defined by* $((K, \sigma), c)$ *for some* $c \in k^\times$. *Let* X *be the variety* $\mathrm{N}_{K/k}(t) = c$. *Then the kernel of* $\mathrm{Br}(k) \to \mathrm{Br}(k(X))$ *is generated by the class of* A.

Proof. Let Y_A be the Severi-Brauer variety of A. Since A is split by $k(Y_A)$, the element c is a norm from the extension $Kk(Y_A)/k(Y_A)$. Thus X has a rational point over $k(Y_A)$ and the map $\mathrm{Br}(k(Y_A)) \to \mathrm{Br}(k(Y_A)(X))$ has trivial kernel. We have,

$$\mathrm{Ker}(\mathrm{Br}(k) \to \mathrm{Br}(k(X))) \subset \mathrm{Ker}(\mathrm{Br}(k) \to \mathrm{Br}(k(X)(Y_A))) =$$

$$= \mathrm{Ker}(\mathrm{Br}(k) \to \mathrm{Br}(k(Y_A))) = \langle [A] \rangle$$

by a theorem of Amitsur [GS 06], Theorem 5.4.1. Since $[A]$ is zero in $\mathrm{Br}(k(X))$, it follows that $\mathrm{Ker}(\mathrm{Br}(k) \to \mathrm{Br}(k(X)) = \langle [A] \rangle$.

In the following proof of Theorem 12.2, we use the fact that the Brauer group of X^c is the unramified Brauer group of X^c (cf [CT 14], Section 2), namely the subgroup of $\mathrm{Br}(k(X^c))$ consisting of all elements which are unramified at all discrete valuations of $k(X^c)$. This is a consequence of the purity results of Cesnavius [C 19], Theorem 1.2.

Proof of Theorem 12.2. The strategy of the proof is the following. We first show that the algebras $(N_i, \tilde{K}_n), \quad i = 1, \ldots, n-2$ belong to $\mathrm{Br}(X^c)$. By Theorem 12.1 (b), we know that $\mathrm{Br}(T^c)/\mathrm{Br}(k) \simeq (\mathbf{Z}/p\mathbf{Z})^{n-2}$. Moreover, $\mathrm{Br}(X^c)/\mathrm{Im}(\mathrm{Br}(k))$ injects into $\mathrm{Br}(T^c)/\mathrm{Br}(k)$ (see §6). We next show that the elements $(N_i, \tilde{K}_n), \quad i = 1, \ldots, n-2.$ are linearly independent over $\mathbf{Z}/p\mathbf{Z}$, and this yields the desired result.

Identifying G with $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$, let $\sigma(i,j) \in G$ correspond to $(i,j) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$, $0 \leq i \leq p-1$, $0 \leq j \leq p-1$. We assume $K_1 = (k')^{\sigma(0,1)}$ and

$K_n = (k')^{\sigma(1,0)}$. Pick a generator σ_1 of $\mathrm{Gal}(K_1/k)$ and let $\tilde{K}_1 = [(K_1, \sigma_1)] \in H^1(k, \mathbf{Z}/p\mathbf{Z})$. For each $j \in I$, one can choose a generator σ_j of $\mathrm{Gal}(K_j/k)$ such that for $j \geq 2$, we have $\tilde{K}_j = i_j \tilde{K}_1 + \tilde{K}_n$ for some i_j with $1 \leq i_j \leq p-1$. In fact for $j \geq 2$, we have $K_j = (k')^{\sigma(1,j)}$, and σ_j is determined by σ_1 and σ_n.

Since $N_i \in N_{K_i k(X^c)/k(X^c)}(K_i k(X^c))$, we get $(N_i, \tilde{K}_i)_{k(X^c)} = 0$. Hence we have $(N_1, \tilde{K}_j) = (N_1, i_j \tilde{K}_1 + \tilde{K}_n) = (N_1, \tilde{K}_n)$ for every $j \in I$, $j \geq 2$.
We show that the cyclic algebras $(N_i, \tilde{K}_n)$ are unramified on X^c for all $i \in I$, and that $(N_1, \tilde{K}_n), \dots, (N_{n-2}, \tilde{K}_n)$ are linearly independent in $\mathrm{Br}(X^c)/\mathrm{Br}(k)$.

Let R be a discrete valuation ring containing the field k, with fraction field of R equal to $k(X^c)$. We prove that the algebras $(N_i, \tilde{K}_n)$ are unramified with respect to the valuation v_R. Let us denote by κ the residue field of R, and let $\partial_R : \mathrm{Br}(k(X^c)) \to H^1(\kappa, \mathbf{Q}/\mathbf{Z})$ be the residue map.

Let $\overline{[K_i]}$ denote the image of $\tilde{K}_i$ in $H^1(\kappa, \mathbf{Z}/p\mathbf{Z})$. We have (see for instance [GS 06] Lemma 6.8.4 and construction 6.8.5),

$$\partial_R(N_i, \tilde{K}_n) = \overline{[K_n]}^{v_R(N_i)}.$$

If $K_n \subset \kappa$ we have $\overline{[\tilde{K}_n]} = 0$ and $\partial_R(N_i, \tilde{K}_n) = 0$ for $1 \leq i \leq n-2$.. Suppose that $\partial_R(N_1, \tilde{K}_n) \neq 0$. Then K_n is not contained in κ. In this case, $K_n \kappa$ is a degree p cyclic extension of κ. The extension $k(X^c)K_n/k(X^c)$ is cyclic of degree p, and has residual degree p, hence is unramified at R. Further $N_n \in k(X^c)$ is a norm from the extension $k(X^c)K_n/k(X^c)$. Hence the valuation $v_R(N_n)$ is divisible by p. Since $(N_1, \tilde{K}_j) = (N_1, \tilde{K}_n)$ for $j \geq 2$, we have $\partial_R(N_1, \tilde{K}_j) \neq 0$, and K_j is not contained in κ. Repeating the above argument, we see that p divides $v_R(N_j)$ for all $j \geq 2$. Since $a = N_1 \dots N_n \in k$, $v_R(a) = 0$, p divides $v_R(N_j)$ for $2 \leq j \leq n$, and it follows that p divides $v_R(N_1)$. This implies that $\partial_R(N_1, \tilde{K}_n) = (\overline{K}_n)^{v_R(N_1)} = 0$, contradicting the assumption that $\partial_R(N_1, \tilde{K}_n) \neq 0$. This implies that $\partial_R(N_1, \tilde{K}_n) = 0$. A similar argument, interchanging 1 and i, with $i \leq n-1$, gives that $\partial_R(N_i, \tilde{K}_n) = 0$. Hence the elements $(N_i, \tilde{K}_n)$ are unramified at R for every discrete valuation ring R with field of fractions $k(X^c)$. By purity for $\mathrm{Br}(X^c)$, we have $(N_i, \tilde{K}_n) \in \mathrm{Br}(X^c)$.

Let us check that the algebras $(N_1, \tilde{K}_n), \dots, (N_{n-2}, \tilde{K}_n)$ are linearly independent in $\mathrm{Br}(k(X^c))/\mathrm{Br}(k)$. Let us project X to the d-dimensional affine space, where $d = (n-1)p$, corresponding to the coordinates involving the first $n-1$ norm polynomials. Let M be the function field of this affine space; we have $k \subset M \subset k(X^c)$. Note that $N_1, \dots, N_{n-1} \in M$. We have $(N_i, \tilde{K}_n) \in \mathrm{Br}(M)$ for all $i = 1, \dots, n-1$.

We want to show that the algebras $(N_1, \tilde{K}_n), \dots, (N_{n-2}, \tilde{K}_n)$ are linearly independent in $\mathrm{Br}(k(X^c))/\mathrm{Br}(k)$. If not, then there exist $r_1, \dots, r_{n-2} \in \mathbf{Z}$ not all zero with $0 \leq r_i \leq p-1$ such that $\sum_{i=1,\dots,n-2} r_i (N_i, \tilde{K}_n)_{k(X^c)} = \alpha$ for some $\alpha \in \mathrm{Br}(k)$.

The kernel of the natural homomorphism $\mathrm{Br}(M) \to \mathrm{Br}(k(X^c))$ is generated by the class of the algebra $(a^{-1}N_1 \ldots N_{n-1}, \tilde{K}_n)$, by Lemma 12.3 applied to $c = aN_1^{-1}.N_2^{-1}.....N_{n-1}^{-1}$.
Hence there exists $s \in \mathbf{Z}$ with $0 \le s \le p-1$ such that

$$\sum_{i=1,\ldots,n-2} r_i(N_i, \tilde{K}_n)_M - \alpha = s(a^{-1}N_1 \ldots N_{n-1}, \tilde{K}_n)_M$$

in $\mathrm{Br}(M)$. The polynomials N_i are irreducible (see [F 53], Theorem 2). Take residue on both sides at the valuation v_{N_i} corresponding to the irreducible polynomial N_i for all $i = 1, \ldots, n-2$. The residue of the left side is $r_i\overline{[K_n]}$, and the right side is $s\overline{[K_n]}$.

Claim. $\overline{[K_n]} \neq 0$ in the residue field $\kappa(v_{N_i})$ for all $i = 1, \ldots, n-2$.

Assume that the claim holds. Then we have $r_i = s \neq 0$ for all $i = 1, \ldots, n-2$, and we get $s(a^{-1}N_{n-1}, \tilde{K}_n) = -\alpha$. With respect to the valuation $v_{N_{n-1}}$, we have the residue $\partial(a^{-1}N_{n-1}, \tilde{K}_n) = \overline{[K_n]} \neq 0$ by the claim, and this leads to a contradiction since $\partial(\alpha) = 0$, α being an element of k, and $s \neq 0$.

It remains to prove the claim. Let us show that K_n is not contained in $\kappa(v_{N_i})$ for all $i = 1, \ldots, n-2$. Let M_i be the function field of the k-variety determined by the polynomial N_i. Since $\kappa(v_{N_i})$ is a purely transcendental extension of M_i obtained by adjoining the coordinates involved in the polynomials $\{N_j, j \neq i, j \le n-1\}$, it suffices to show that K_n is not contained in M_i. Suppose that K_n is a subfield of M_i. Let us base change to K_i : the field K_iK_n is a subfield of K_iM_i. After base change to K_i, the polynomial N_i is transformed to the product $X_1 \ldots X_p$ for some variables $X_1, \ldots, X_p$. Hence K_iM_i is a product of rational function fields over K_i; therefore it cannot contain K_n, thereby leading to contradiction.

13 Some consequences for semi-global fields

Let K be a complete discrete valued field with valuation ring O and residue field κ. Let X/K be a normal projective, geometically integral curve over K and let $F = K(X)$. We call F a *semi-global field*. In [CTPS 16], §2.3, certain higher reciprocity obstructions were constructed to study the failure of the Hasse principle for varieties over F with respect to discrete valuations of F centered on a regular proper model $\mathcal{X}/O$ of the curve X/K. It was also proved in [CTPS 16], Example 2.6, that these obstructions do not suffice to detect the failure of the Hasse principle on principal homogeneous spaces under tori defined over F. Using an example of a multinorm torus constructed in [Su 19], Corollary 7.12, we give another example where the obstructions constructed in [CTPS 16] do not suffice to detect failure of Hasse principle.

We recall the following construction from [Su 19], Corollary 7.12. Let $K = \mathbf{C}((t))$ and $F = \mathbf{C}((t))(X)[Y]/(XY(X+Y-1)(X-2)-t)$. Let $L_1 =$

$F((XY)^{1/n}, (Y(X+Y-1))^{1/n})$ and $L_2 = F((XY\theta_1)^{1/n}, (Y(X+Y-1)\theta_2)^{1/n})$, where $\theta_1 = (X-2)/(X-2+XY(X+Y-1))$ and $\theta_2 = (Y-2)/(Y-2+XY(X+Y-1)$. Then L_1 and L_2 are Galois extensions of F which are linearly disjoint over F (c.f [Su 19], Corollary 7.12). Let $T = \mathrm{R}^1_{L_1\times L_2/F}(\mathbf{G_m})$ be the associated norm one torus. It is proved in [Su 19], Corollary 7.12, there is a principal homogeneous space under T which fails the Hasse principle with respect to all discrete valuations of F. The proof invokes R-equivalence of tori to prove that Hasse principle fails in the patching setting of Harbater-Hartmann-Krashen. The second step is to prove that failures of Hasse principle in the patching setting implies the failure of the Hasse principle with respect to all discrete valuations of F. Since the cohomological dimension of F is 2, the only obstruction of [CTPS 16] in this case is the one coming from the Brauer group of X^c. In view of Theorem 8.4, we have $\mathrm{Br}(X^c)/\mathrm{Br}(k) = 0$, and the obstruction vanishes. However, Hasse principle fails for X.

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