Fractional Exponential Decay in the
Forbidden Region for Toeplitz Operators

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Abstract. We prove several results of concentration for eigenfunctions in Toeplitz quantization. Under mild regularity assumptions, we prove that eigenfunctions are $O(\exp(-cN^\delta))$ away from the corresponding level set of the symbol, where N is the inverse semiclassical parameter and $0 < \delta < 1$ depends on the regularity. As an application, we prove a precise bound for the free energy of spin systems at high temperatures, sharpening a result of Lieb.

1   Introduction

Localisation or microlocalisation estimates are central in semiclassical analysis. The most practical context for studying localisation of quantum states is the case of a smooth symbol on a fixed, finite-dimensional manifold. Indeed, in this case one can use the symbolic calculus to prove $O(\hbar^\infty)$ decay in the forbidden region.

How to improve these bounds? One idea is to impose more regularity (e.g. real-analyticity) and try to obtain more precise microlocalisation estimates (see section 3.5 in [24] for the pseudodifferential case and [9] for the Toeplitz case). Recent work developing or using exponential estimates in analytic regularity features magnetic Schrödinger operators [14, 15], the focusing NLS equation [11], resonances of Schrödinger operators [12] and the Steklov problem [13].

In this article, we are interested in localisation estimates in low regularity for *Toeplitz quantization* [21]. Given a compact Kähler manifold (M, ω, J), where ω is a symplectic form with integer periods and J is a complex structure, one can construct a Hermitian complex line bundle (L, h) over M, such that $\mathrm{curv}(h) = 2i\pi\omega$; then the essential ingredient for the quantization is the family of *Szegő projectors* $(S_N)_{N \in \mathbb{N}}$: for every $N \in \mathbb{N}$, S_N is the orthogonal projector from the section space $L^2(M, L^{\otimes N})$ to the subspace of holomorphic sections $H^0(M, L^{\otimes N})$. Then, the Toeplitz operator $T_N(f)$ associated with a function $f : M \to \mathbb{C}$ is the composition of the multiplication by f and the Szegő projector:

$$
\begin{array}{rccc}
T_N(f): & H^0(M, L^{\otimes N}) & \to & H^0(M, L^{\otimes N}) \\
& u & \mapsto & S_N(fu).
\end{array}
$$

One should think of N as an inverse semiclsasical parameter: $N = \hbar^{-1}$. The Toeplitz operator $T_N(f)$ is well-defined, and uniformly bounded in operator norm, as long as $f \in L^\infty$. This fact already hints towards a different behaviour of Toeplitz and Weyl quantization for low-regularity symbols (in Weyl quantization, one must assume some regularity to obtain $L^2 \to L^2$ boundedness). We are now ready to state the first main result of this article.

THEOREM A. *Let (M, ω, J) be a compact, quantizable Kähler manifold. Let $\alpha = \frac{1}{2}$ if (M, ω, J) is $C^{1,1}$ and $\alpha = 1$ if (M, ω, J) is real-analytic.*
Let $f \in L^\infty(M, \mathbb{R})$. For every $\delta > 0$ there exist $C > 0, c > 0, N_0 > 0$ such that, for any $N \geq N_0$, for any $\epsilon > CN^{-\frac{1}{4}+\delta}$, for any normalised $u \in H^0(M, L^{\otimes N})$ and any $\lambda \in \mathbb{R}$ such that

$$
T_N(f)u = \lambda u,
$$

with

$$
W = \{x \in M, \mathrm{dist}(x, \{f \geq \lambda + \epsilon\}) > \epsilon\},
$$

one has

$$
\|u\|^2_{L^2(W)} \leq \frac{C}{\epsilon} \exp(-c(N\epsilon^4)^{\frac{\alpha}{2\alpha+1}}).
$$

In particular, if W is at fixed distance from a sublevel of f (that is, if ϵ does not depend on N), then the mass of u on W is always $O(\exp(-cN^{\frac{1}{4}}))$. This precision is much better than the symbolic calculus even for smooth symbols on smooth manifolds (which only leads to $O(N^{-\infty})$) and, in fact, it is better than the knowledge of the Szegő projector near the diagonal [26, 25, 4, 23], which is also $O(N^{-\infty})$.

In fact, Theorem A, as well as Theorems B and C below, only depend on the off-diagonal decay of the Szegő projector (Proposition 2.2). In particular, equivalents of these Theorems hold on various generalisations of Kähler quantization, as long as this off-diagonal decay holds: spin^c-Dirac quantization [23], or Bochner Laplacians [16, 20]. Semiclassical constructions of Szegő almost projectors, like the one used for almost Kähler quantization (appendix of [3]) are not precise enough here: they are only defined modulo $O(N^{-\infty})$ so the

kernel decay is blurred at this limit. However, all methods used here work in this context, yielding $O(N^{-\infty})$ estimates for low-regularity symbols.

The factor $N\epsilon^4$, or equivalently the condition $\epsilon > CN^{-\frac{1}{4}+}$, does not correspond to usual statements about microlocalisation. Usually, operator calculus works for symbols in mildly exotic classes $S_{\frac{1}{2}-}$, so that one can prove $O_\delta(\hbar^\infty)$ decay at distance $\hbar^{\frac{1}{2}-}$.

The FBI transform (or equivalently, the Bargmann transform) allows to conjugate Toeplitz operators on $\mathbb{C}^n$ with pseudodifferential operators on $\mathbb{R}^{2n}$. Unfortunately, the error terms in this conjugation are usually much larger than the decay rates in Theorem A: indeed, even for C^∞ symbols it is not better than $O(\hbar^\infty)$. Thus one cannot apply Theorem A to pseudodifferential operators. Apart from the case of Gevrey or analytic regularity, the only situation in which one is able prove exponential decay for pseudodifferential operators is Agmon estimates for differential operators [1].

On the Toeplitz side, the quantization of indicator function of sets has raised recent interest [6, 27], in connection with Fermi statistics. We also must mention the work [19], which obtains fractional exponential decay (more precisely, $O(\exp(-cN^{\frac{1}{2}}))$) at finite distance for Toeplitz operators with C^∞ symbols; in fact, the proof of this localisation result only uses $C^{1,1}$ regularity of the symbol. The method used is a weighted estimate for the Kohn Laplacian (or rather, the Bochner Laplacian): one writes S_N as the kernel of an elliptic differential operator, then conjugate with rapidly oscillating weights.

Using the decay properties of the Szegő projector, one can simplify a great deal the method used in [19] and relax the regularity hypotheses. This leads to the following improvement of Theorem A.

THEOREM B. *Let (M, ω, J) be a compact, quantizable Kähler manifold of regularity $C^{1,1}$.*
Let $f \in \mathrm{Lip}(M, \mathbb{R})$. There exist $C > 0, c > 0$ such that, for any $N \in \mathbb{N}$, for any $\epsilon > CN^{-\frac{1}{2}}$, for any normalised $u \in H^0(M, L^{\otimes N})$ and any $\lambda \in \mathbb{R}$ such that

$$T_N(f)u = \lambda u,$$

if

$$W = \{x \in M, \mathrm{dist}(x, \{f \geq \lambda + CN^{-\frac{1}{2}}\}) > \epsilon\},$$

one has

$$\|u\|^2_{L^2(W)} \leq CN^{\frac{1}{2}} \exp(-c\epsilon\sqrt{N}).$$

A byproduct of Theorem B is that the eigenfunction u is $O(N^\infty)$ (in fact, exponentially small) on $\{|f - \lambda| > N^{-\frac{1}{2}+\delta}\}$, for any $\delta > 0$. If λ is a regular value of f, the sharpness of this localisation region cannot be improved: the uncertainty principle forbids quantum states in Toeplitz quantization to be concentrated on a band thinner than $N^{-\frac{1}{2}}$.

A version of Theorem B is used in [19] to study the low-energy spectrum of symbols with more regularity. If $f \in C^{1,1}(M, \mathbb{R})$ and $\min(f) = 0$, then testing

against coherent states shows that the smallest eigenvalue of $T_N(f)$ is of order $\min(\mathrm{Sp}(T_N(f))) = O(N^{-1})$. In this situation, one should expect the corresponding eigenvector u to be concentrated on $\{f \leq N^{-1+\delta}\}$. In the case where $f \in C^\infty$, this can be obtained from the symbolic calculus [5, 8]. Here, we are able to modify the proof of Theorem B, yielding a sharper result.

THEOREM C. *Let (M, ω, J) be a compact, quantizable Kähler manifold of regularity $C^{1,1}$.*
Let $f \in C^{1,1}(M, \mathbb{R})$ with $\min(f) = 0$. For every $\delta > 0$ and every $C_0 > 0$, there exists $C > 0$ and $c > 0$ such that, for any $N \in \mathbb{N}$, for any normalixed $u \in H^0(M, L^{\otimes N})$ and any $\lambda < C_0 N^{-1}$ such that

$$T_N(f)u = \lambda u,$$

one has

$$\|u\|^2_{L^2(\{f \geq N^{-1+\delta}\})} \leq C e^{-cN^{\frac{\delta}{2}}}.$$

A natural set of quantum Hamiltonians which can be written as Toeplitz operators consists in *spin operators*: here, the manifold is $(\mathbb{CP}^1)^d \approx (\mathbb{S}^2)^d$, and the symbol f is a polynomial in the coordinates for the natural immersion into $(\mathbb{R}^3)^d$. Such a symbol is real-analytic, so for fixed d and $N \to +\infty$ this result is weaker than the $O(\exp(-cN))$ decay established in previous work [9]. However, in experimental situations d is much larger than N, which raises the question of uniform (in d) localisation estimates for a reasonable sequence of symbols.
Usual tools for the study of microlocalisation fail in this context. The symbolic calclulus makes sense for fixed d but goes awry as d increases: for instance, the stationary phase lemma typically requires a number of derivatives which grows linearly with d. Theorems B and C rely on the pointwise decay property of the Szegő projector by means of the Schur test. This also fails in large dimension (see Subsection 2.2).
However, the method of proof used in [19] adapts to the limit $d \to +\infty$ quite well. Controlling the various constants yields

THEOREM D. *Let g be a tame spin system (see Definition 6.2). There exists $C > 0$ and $c > 0$ such that, for every $N \in \mathbb{N}$, for every $d \geq d_0(N)$ large enough, for every $u \in H^0((\mathbb{S}^2)^d, L^{\otimes N})$ of norm 1 and $\lambda \in \mathbb{R}$ such that*

$$T_N(g)u = \lambda u,$$

then with

$$U = \{|g - \lambda| < CN^{-\frac{1}{4}} d^{\frac{3}{4}}\}$$

and

$$W = \{x \in (\mathbb{S}^2)^d, \mathrm{dist}(x, U) > CN^{-\frac{1}{2}}\sqrt{d}\},$$

one has

$$\int_W e^{c\sqrt{N}\frac{\mathrm{dist}(x,U)}{\sqrt{d}}} |u(x)|^2 \leq C.$$

The previous result implies fractional exponential decay at distance $C\sqrt{d}$ of U. Localisation estimates can be used to understand, at least at dominant order, the behaviour of the heat operator generated by $T_N(f)$. This heat operator is the complex extension of the wave propagator, restricted to imaginary time. The analysis of this operator is relevant not only with respect to the Egorov theorem, but also because it is believed to be related to geodesics in the space of Kähler metrics on M. Furthermore, in the case of spin systems, the quantity $Z = \mathrm{Tr}(e^{-\beta T_N(f)})$ is called *partition function* at inverse temperature β and is a key element of the understanding of the statistical mechanics of spin systems.

PROPOSITION 1.1. *Let g be a tame spin system. Consider, for $N \in \mathbb{N}$ and $\beta \geq 0$, the quantum free energy*

$$f_Q = -\frac{1}{\beta d}\log(\mathrm{Tr}(\exp(-\beta T_N(g)))).$$

Consider also the normalized classical free energy

$$f_C = -\frac{1}{\beta d}\log\left[\left(\frac{N+1}{\pi}\right)^d \int_{(\mathbb{S}^2)^d} e^{-\beta g}\right].$$

Then there exists $c > 0$ and $C > 0$ such that, uniformly in d and N, uniformly in $\beta \leq cN^{\frac{1}{2}}d^{-1}$, one has

$$|f_C - f_Q| \leq CN^{-\frac{1}{2}}.$$

As for the standard estimate found in [22], Proposition 1.1 is a "Weyl-law" type control: one estimates a quantum quantity, related to the distribution of eigenvalues, using only the volume form on the phase space. Such estimates cannot distinguish between situations where there is a phase space transformation preserving the volume form but not the symplectic form (for instance, between a Heisenberg antiferromagnet and a Heisenberg ferromagnet).

Proposition 1.1 improves [22] by a factor d, but is unsatisfactory because the range of temperatures covered is too restrictive for practical applications; obtaining a result valid for $\beta \leq c(N)$, independent of d, could hopefully be obtained by sharpening the techniques used here, in particular, Conjecture 1 and Proposition 1.1.

This article is organised as follows. In Section 2 we review the properties of the Szegő projector that we will use to prove Theorems A, B and C. In particular, Subsection 2.2 is devoted to an analysis of the case of a product of a large number of spheres.

In Section 3, we prove Theorem A. The method used is a decomposition of M into shells corresponding to the distance to a level set. In Section 4, we derive weighted estimates by simplifying the methods of [19], in order to prove Theorems B and C.

The two last sections of this article are devoted to Theorem D and Proposition 1.1. In Section 5, we review the proof of the weighted estimate in [19], and

we give an explicit dependence of the constants in the objects (the manifold, the weight, and the symbol). In Section 6, we construct a weight adapted to a spin system in large dimension, and conclude the proofs.

2 Rate of decay of the Szegő projector

2.1 General case

One of the essential properties of the Szegő projector is its rapid off-diagonal decay. It is much easier to derive a good off-diagonal decay rate than to study the Szegő projector near the diagonal with a corresponding degree of precision; in fact, save for the case where M is real-analytic, the off-diagonal decay is faster than the precision available on the diagonal.

Proposition 2.1 (Pointwise estimates). *Let M be a compact Kähler quantizable manifold of complex dimension d. For $N \in \mathbb{N}$, let S_N denote the Szegő (or Bergman) projector on M. Then the following is true.*

1. *[10] If the metric of M is $C^{1,1}$, then there exist $C > 0, c > 0$ such that, for any $N \in \mathbb{N}$, for any $(x,y) \in M^2$,*

$$|S_N(x,y)|^2 \leq CN^d \exp(-c\sqrt{N}\operatorname{dist}(x,y)).$$

2. *[2] If the metric of M is real-analytic, then there exist $C > 0$, $c > 0$ such that, for any $N \in \mathbb{N}$, for any $(x,y) \in M^2$,*

$$|S_N(x,y)|^2 \leq CN^d \exp(-cN\operatorname{dist}(x,y)^2).$$

In the previous Proposition, the decay rate of case 1 is essentially sharp (up to a power of $\log(N)$) if the metric of M is C^∞ or less [7]. Case 2 is also sharp: in the easiest examples $M = \mathbb{C}^n$ or $M = \mathbb{CP}^n$, one has exactly $|S_N(x,y)|^2 = CP(N)\exp(-cN(\operatorname{dist}(x,y)^2 + O(\operatorname{dist}(x,y)^4)))$. In the case of s-Gevrey regularity, one can interpolate between cases 1 and 2, obtaining $(N\operatorname{dist}(x,y)^2)^{\frac{s}{2s-1}}$, see [17]; we do not know if this decay rate is sharp.
This pointwise decay immediately leads, via the Schur test, to a decay in terms of operators.

Proposition 2.2 (Operator estimates). *Let M be a compact Kähler quantizable manifold of complex dimension d. For $N \in \mathbb{N}$, let S_N denote the Szegő (or Bergman) projector on M. Then the following is true.*

1. *If the metric of M is $C^{1,1}$, then there exist $C > 0, c > 0$ such that, for any $N \in \mathbb{N}$, for any open sets U, V of M,*

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \mapsto L^2} \leq C \exp(-c\sqrt{N}\operatorname{dist}(U,V)).$$

2. *If the metric of M is real-analytic, then there exist $C > 0$, $c > 0$ such that, for any $N \in \mathbb{N}$, for any open sets U, V of M,*

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \mapsto L^2} \leq C \exp(-cN\operatorname{dist}(U,V)^2).$$

The constant C is not trivial to get rid of. In particular, one gets estimates of the form

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \mapsto L^2} \leq \exp(-c'(N \operatorname{dist}(U,V)^2)^{1 \text{ or } 1/2})$$

only under the condition that $\operatorname{dist}(U,V) \geq C_1 N^{-\frac{1}{2}}$. This remark is of little importance on a fixed Kähler manifold, but as we will see, the constant C blows up with the dimension in the case $M = (\mathbb{S}^2)^d$, at least when using a Schur test.

2.2 Products of spheres

This subsection is devoted to a discussion of Proposition 2.2 in the case $M = (\mathbb{S}^2)^d$. Unfortunately, we are not able to prove a d-independent version of Proposition 2.2 in this context, but we conjecture it is the case, and give a simple proof of a weaker result.

We take the following scaling convention: the area of the sphere is 1. The Szegő kernel on $(\mathbb{S}^2)^d$ is easily obtained from that on $\mathbb{S}^2$: one has

$$|S_{N,d}(x,y)| = (N+1)^d \prod_{x=1}^{d} (x_i \dot{y}_i)^N.$$

For fixed d and $x \neq y$, as $N \to +\infty$ this quantity decays exponentially fast. As d increases, however, this behaviour is destroyed. It makes sense to try to estimate operator norms of the form

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2}$$

where U and V are at positive distance, independently on d. Indeed, in this version of the kernel estimate the factor N^d is not present anymore (see the difference between Propositions 2.1 and 2.2). Moreover, in the proof of Theorem A, we only use Proposition 2.2.

Lemma 2.3. *For $0 \leq \theta \leq \pi/2$ one has*

$$\cos(\theta) \leq \exp(-\theta^2/2).$$

Proof. The two first non-zero terms in a Taylor expansion on both sides coincide, so that

$$\exp(-\theta^2/2) - \cos(\theta) = \sum_{k=2}^{+\infty} (-1)^k \theta^{2k} \left[\frac{1}{2^k k!} - \frac{1}{(2k)!} \right].$$

The claim then follows from the fact that the non-negative sequence

$$\left(\theta^{2k} \left[\frac{1}{2^k k!} - \frac{1}{(2k)!} \right] \right)_{k \geq 2}$$

is non-increasing and the alternating series theorem.

Indeed, the difference between two consecutive terms is

$$\frac{\theta^{2k}}{2^k k!}\left[1 - \frac{\theta}{2(k+1)}\right] - \frac{\theta^{2k}}{(2k)!}\left[1 - \frac{\theta}{(2k+1)(2k+2)}\right].$$

Since $\frac{\theta}{2(k+1)} \leq \frac{\pi}{12}$, the difference between two consecutive terms is larger than

$$\frac{\theta^{2k}}{(2k)!}\left[1 \cdot 3 \cdots (2k-1)\left(1 - \frac{\pi}{12}\right) - 1\right] \geq \frac{\theta^{2k}}{(2k)!}\left[2 - \frac{\pi}{4}\right] \geq 0.$$

$\square$

PROPOSITION 2.4. *Let d, N be positive integers and let $D > 0$. Let U, V be subsets of $(\mathbb{S}^2)^d$ such that $\mathrm{dist}(U, V) = D > 0$.*
Then

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^1 \to L^\infty} \leq \frac{4}{\sqrt{2\pi d}} 4^d \exp(-(N+1)D^2/16).$$

In particular,

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2} \leq \frac{4}{\sqrt{2\pi d}} 4^d \exp(-(N+1)D^2/16).$$

Proof. One has

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^1 \to L^\infty} = \sup_{x \in U} \int_{y \in V} |S_N(x, y)|.$$

Letting $P = [0, \pi]^d$ and $B(0, D)$ denote the Euclidean ball of radius D in $\mathbb{R}^d$, one has

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^1 \to L^\infty} \leq \frac{(N+1)^d}{2^d} \int_{P \backslash B(0,D)} \prod_{j=1}^d \underbrace{\cos(\theta_j/2)^N \sin(\theta_j)}_{=2\cos(\theta_j/2)^{N+1}\sin(\theta_j/2)} \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_d.$$

From Lemma 2.3 and the classic inequality $|\sin(x)| \leq x$, one is left with

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^1 \to L^\infty} \leq \frac{(N+1)^d}{2^d} \int_{P \backslash B(0,D)} e^{-(N+1)\theta^2/8} \left(\prod_{j=1}^d \theta_j\right) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_d.$$

Letting $\widetilde{P} = \{z \in \mathbb{C}, |z| < \pi\}^d$ and $\widetilde{B}(0, D)$ denote the Hilbert ball of radius D

in $\mathbb{C}^d$, one has $\widetilde{P} \subset \widetilde{B}(0, \sqrt{d}\pi)$, so that

$$\int_{P \backslash B(0,D)} e^{-(N+1)\theta^2/8} \left(\prod_{j=1}^{d} \theta_j \right) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_d$$

$$= \frac{1}{(2\pi)^d} \int_{\widetilde{P} \backslash \widetilde{B}(0,D)} e^{-(N+1)|z|^2/8} \mathrm{d}z_1 \mathrm{d}\overline{z_1} \ldots \mathrm{d}z_d \mathrm{d}\overline{z_d}$$

$$\leq \frac{1}{(2\pi)^d} \int_{\widetilde{B}(0,\sqrt{d}\pi) \backslash \widetilde{B}(0,D)} e^{-(N+1)|z|^2/8} \mathrm{d}z_1 \mathrm{d}\overline{z_1} \ldots \mathrm{d}z_d \mathrm{d}\overline{z_d}$$

$$= \frac{\omega_{2d-1}}{2(2\pi)^d} \int_{D^2}^{d\pi^2} e^{-(N+1)u/8} u^{d-1} \mathrm{d}u.$$

Here $\omega_{2d-1} = \frac{2\pi^d}{(d-1)!}$ is the volume of the unit sphere in dimension $2d-1$. The Stirling formula yields

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^1 \to L^\infty} \leq \frac{(N+1)^d}{\sqrt{2\pi d}} \int_{D^2}^{d\pi^2} e^{-(N+1)u/8} \left(\frac{eu}{4(d-1)} \right)^{d-1} \mathrm{d}u$$

$$= \frac{1}{\sqrt{2\pi d}} \int_{D^2(N+1)}^{d\pi^2(N+1)} e^{-x/8} \left(\frac{ex}{4(d-1)} \right)^{d-1} \mathrm{d}x.$$

The quantity to be integrated is equal to

$$e^{-x/16} \left(e^{-\frac{x}{16(d-1)}} \frac{ex}{4(d-1)} \right)^{d-1} \leq 4^{d-1} e^{-x/16}.$$

In particular, one has

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^1 \to L^\infty} \leq \frac{4}{\sqrt{2\pi d}} 4^d e^{-(N+1)D^2/16},$$

hence the claim. $\square$

Using the Schur test to estimate $\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2}$ seems rather weak. Indeed, an easy bound is

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2} \leq 1.$$

Theorem 2.4 beats this easy bound when $d \geq 3$ under the condition

$$D \geq 5 \sqrt{\frac{d-1}{N+1}}.$$

In particular, one has

PROPOSITION 2.5. *If $d \geq 3$, if $D \geq 10\sqrt{\frac{d}{N+1}}$ and if U, V are two open sets of $(\mathbb{S}^2)^d$ at distance D, then*

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2} \leq \exp\left(-\frac{1}{21}(N+1)D^2 \right).$$

We will rely heavily on Proposition 2.5 later on.

Using the Schur test to estimate $\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2}$ is very crude. We conclude this section with the following conjecture.

CONJECTURE 1. *There exists a universal constant $c > 0$ such that, for any integers d, N, for any open sets U, V in $(\mathbb{S}^2)^d$, one has*

$$\|\mathbb{1}_U S_N \mathbb{1}_V\|_{L^2 \to L^2} \leq \exp(-cN \operatorname{dist}(U,V)^2).$$

This conjecture is at least true if U is a ball around one point, and V is the complement of a larger ball around that same point. If we want to prove Theorem A in the context of a large product of spheres, one would need to apply this conjecture to distances much shorter than $\sqrt{d}N$.
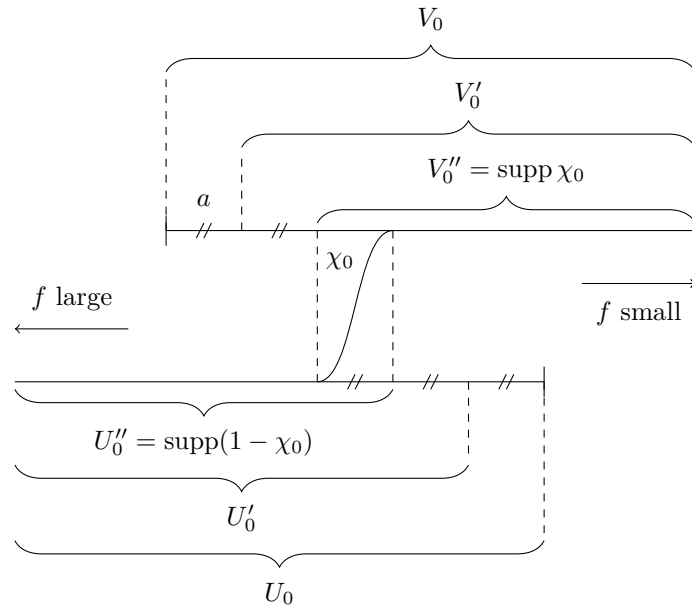
3   FRACTIONAL DECAY OF EIGENFUNCTIONS WITHOUT REGULARITY

In this section we prove Theorem A.

Let f, u, λ be as above. Let us fix $U_0 = \{f \geq \lambda + \epsilon\}$. Let (see also picture below)

$$a = \epsilon^{\frac{2}{1+2\alpha}} N^{-\frac{\alpha}{2\alpha+1}}$$
$$U_0' = \{x \in U_0, \operatorname{dist}(x, \partial U_0) > a\}$$
$$U_0'' = \{x \in U_0, \operatorname{dist}(x, \partial U_0) > 2a\}$$
$$V_0'' = M \setminus \{x \in U_0, \operatorname{dist}(x, \partial U_0) \geq 3a\}$$
$$V_0' = M \setminus \{x \in U_0, \operatorname{dist}(x, \partial U_0) \geq 4a\}$$
$$V_0 = M \setminus \{x \in U_0, \operatorname{dist}(x, \partial U_0) \geq 5a\}.$$

Note that $a = o(\epsilon)$ and $N^{-\frac{1}{2}} = o(a)$ as $N \to +\infty$. We also let $\chi_0 \in C^\infty(M, [0,1])$ be such that $\operatorname{supp}(\chi_0) \subset V_0''$ and $\operatorname{supp}(1 - \chi_0) \subset U_0''$.

Now

$$0 = \langle u, (f - \lambda)u \rangle$$
$$= \langle \chi_0 u, (f - \lambda)\chi_0 u \rangle + 2\langle (1 - \chi_0)u, (f - \lambda)\chi_0 u \rangle$$
$$+ \langle (1 - \chi_0)u, (f - \lambda)(1 - \chi_0)u \rangle.$$

Since $f - \lambda \geq \epsilon$ on the support of $1 - \chi_0$, one has

$$\langle (1 - \chi_0)u, (f - \lambda)(1 - \chi_0)u \rangle \geq \epsilon \|u\|_{L^2(U_0'')}^2.$$

Moreover,

$$\langle (1 - \chi_0)u, (f - \lambda)\chi_0 u \rangle \leq \max_{U_0'' \cap V_0''}(f - \lambda)\|u\|_{L^2(U_0'' \cap V_0'')}^2.$$

It remains to bound

$$\langle \chi_0 u, (f - \lambda)\chi_0 u \rangle = \langle (f - \lambda)u, \chi_0^2 u \rangle$$

from above. To this end, observe that $S_N(f - \lambda)u = 0$, and it remains to estimate

$$\langle (f - \lambda)u, (1 - S_N)\chi_0^2 u \rangle$$
$$= \iint_{x \in M, y \in M} (f(x) - \lambda)\overline{u(x)}S_N(x, y)\left[\chi_0^2(x)u(x) - \chi_0^2(y)u(y)\right]\mathrm{d}y\mathrm{d}x.$$

We first examine this integral restricted to $x \in U_0' \cap V_0'$, that is,

$$A = \langle (f - \lambda)u, \mathbb{1}_{U_0' \cap V_0'}[1 - S_N]\chi_0^2 u \rangle$$

which we decompose as

$$
\begin{aligned}
A &= A_1 + A_2 \\
A_1 &= \langle (f - \lambda)u, \mathbb{1}_{U_0' \cap V_0'}[1 - S_N]\mathbb{1}_{M \setminus (U_0 \cap V_0)}\chi_0^2 u \rangle \\
A_2 &= \langle (f - \lambda)u, \mathbb{1}_{U_0' \cap V_0'}[1 - S_N]\mathbb{1}_{U_0 \cap V_0}\chi_0^2 u \rangle.
\end{aligned}
$$

Since $\mathrm{dist}(U_0' \cap V_0', M \setminus (U_0 \cap V_0)) \geq a$, one can apply Proposition 2.2, so that

$$|A_1| \leq C \max_{U_0' \cap V_0'}(f - \lambda) \exp(-c(Na^2)^\alpha)\|u\|_{L^2(U_0' \cap V_0')}.$$

Moreover, one has

$$|A_2| \leq \max_{U_0' \cap V_0'}(f - \lambda)\|u\|^2_{L^2(U_0 \cap V_0)}.$$

Now we consider the integral restricted to $x \in M \setminus (U_0' \cap V_0')$, that is

$$B = \langle \mathbb{1}_{M \setminus (U_0' \cap V_0')}(f - \lambda)u, [1 - S_N]\chi_0^2 u \rangle.$$

One has, since $\chi_0 = 0$ on $M \setminus V_0''$,

$$
\begin{aligned}
B &= B_1 + B_2 + B_3 \\
B_1 &= \langle \mathbb{1}_{M \setminus (U_0' \cap V_0')}(f - \lambda)u, [1 - S_N]\chi_0^2 \mathbb{1}_{U_0'' \cap V_0''} \rangle \\
B_2 &= \langle \mathbb{1}_{M \setminus V_0'}(f - \lambda)u, [1 - S_N]\chi_0^2 \mathbb{1}_{M \setminus U_0''} \rangle \\
B_3 &= \langle \mathbb{1}_{M \setminus U_0'}(f - \lambda)u, [1 - S_N]\chi_0^2 \mathbb{1}_{M \setminus U_0''} \rangle.
\end{aligned}
$$

From Proposition 2.2 one has

$$
\begin{aligned}
|B_1| &\leq C \max_M(f - \lambda) \exp(-c(Na^2)^\alpha)\|u\|_{L^2(U_0'' \cap V_0'')} \\
|B_2| &\leq C \max_M(f - \lambda) \exp(-c(4Na^2)^\alpha)\|u\|_{L^2(M \setminus V_0')}.
\end{aligned}
$$

Moreover $\chi_0^2 = 1$ on $M \setminus U_0''$, so that, since $S_N u = u$,

$$
\begin{aligned}
|B_3| &= \langle \mathbb{1}_{M \setminus U_0'}(f - \lambda)u, [1 - S_N]\chi_0^2 \mathbb{1}_{M \setminus U_0''} \rangle \\
&= -\langle \mathbb{1}_{M \setminus U_0'}(f - \lambda)u, [1 - S_N]\chi_0^2 \mathbb{1}_{U_0''} \rangle.
\end{aligned}
$$

Then again

$$|B_3| \leq C \max_{M \setminus U_0'}(f - \lambda) \exp(-c(N^2 a)^\alpha)\|u\|_{L^2(U_0'')}.$$

To conclude, from

$$0 = \langle \chi_0 u, (f-\lambda)\chi_0 u \rangle + 2\langle (1-\chi_0)u, (f-\lambda)\chi_0 u \rangle + \langle (1-\chi_0)u, (f-\lambda)(1-\chi_0)u \rangle,$$

we obtain the inequality

$$
\begin{aligned}
c_0 \|u\|_{L^2(V_0'')}^2 &\leq \max_{U_0'' \cap V_0''}(f-\lambda)\|u\|_{L^2(U_0'' \cap V_0'')}^2 \\
&\quad + C \max_{U_0' \cap V_0'}(f-\lambda)\exp(-c(Na^2)^\alpha)\|u\|_{L^2(U_0' \cap V_0')} \\
&\quad + \max_{U_0' \cap V_0'}(f-\lambda)\|u\|_{L^2(U_0 \cap V_0)}^2 \\
&\quad + C \max_M (f-\lambda)\exp(-c(Na^2)^\alpha)\|u\|_{L^2(U_0'' \cap V_0'')} \\
&\quad + C \max_M (f-\lambda)\exp(-c(4Na^2)^\alpha)\|u\|_{L^2(M \setminus V_0')} \\
&\quad + C \max_{M \setminus U_0'} \exp(-c(N^2 a)^\alpha)\|u\|_{L^2(U_0'')},
\end{aligned}
$$

which we simplify into

$$
\begin{aligned}
\left(\epsilon - 4C \max_M |f| e^{-c(N^2 a)^\alpha}\right) \|u\|_{L^2(U_0'')}^2 \\
\leq 2C \max_M (f-\lambda)\|u\|_{L^2(U_0 \cap V_0)}\left(\|u\|_{L^2(U_0 \cap V_0)} + e^{-c(Na^2)^\alpha}\right).
\end{aligned}
$$

Since $N^2 a = N\epsilon^4 \geq N^\delta$, let us restrict ourselves to N large enough (depending on δ) so that

$$4C \max_M |f| e^{-c(N^2 a)^\alpha} \leq \epsilon/2.$$

In conclusion, one has the following dichotomy.

- Either $\|u\|_{L^2(U_0 \cap V_0)} \leq e^{-c(Na^2)^\alpha}$, in which case

$$\|u\|_{L^2(U_0'')}^2 \leq 4\frac{C}{\epsilon}e^{-c(Na^2)^\alpha}.$$

- Or $\|u\|_{L^2(U_0 \cap V_0)} \geq e^{-c(Na^2)^\alpha}$, so that

$$\|u\|_{L^2(U_0'')}^2 \leq 4\frac{C}{\epsilon}\|u\|_{L^2(U_0 \cap V_0)}^2.$$

In the second case, one proceeds to an induction, letting

$$U_1 = int(M \setminus V_0)$$

where $int(E)$ is the interior of the set E. One has then

$$U_1 = \{x \in M, \operatorname{dist}(x, U_0) > 5a\}$$

$$\|u\|^2_{L^2(U_1)} \leq \frac{4\dfrac{C}{\epsilon}}{1 + 4\dfrac{C}{\epsilon}}\|u\|^2_{L^2(U_0)}.$$

We proceed in the induction, considering sets U_k, V_k, and so on, until one of these conditions is satisfied: $k = \frac{\epsilon}{6a}$ or

$$\|u\|^2_{L^2(U_k)} \leq 4\frac{C}{\epsilon}e^{-c(Na^2)^\alpha}.$$

If we have reached $k = \frac{\epsilon}{6a}$, then U_k is the set of points at distance at least $\frac{5}{6}\epsilon + O(a)$ of U_0, and

$$\|u\|^2_{L^2(U_k)} \leq \left[\frac{4\dfrac{C}{\epsilon}}{1 + 4\dfrac{C}{\epsilon}}\right]^k \leq \exp(-c\epsilon k).$$

In the other case, the last iteration U_k contains the set of points at distance ϵ of U_0, and is such that

$$\|u\|^2_{L^2(U_k)} \leq 4\frac{C}{\epsilon}e^{-c(Na^2)^\alpha}.$$

Now since $k = \frac{\epsilon}{5a}$, one has

$$\epsilon k \approx c(Na^2)^\alpha \approx N^{\frac{\alpha}{2\alpha+1}},$$

where $\approx$ means "up to some constant".
This concludes the proof.

4 Decay of eigenfunctions for Lipschitz symbols

In this section we prove Theorems B and C. They respectively follow from the two following weighted estimates:

PROPOSITION 4.1. *Let M be a $C^{1,1}$ Kähler manifold. Let $\rho \in \operatorname{Lip}(M, \mathbb{R})$.*
There exist two constants $c > 0$ and $C > 0$ such that, for every $\alpha \in \mathbb{R}$ with $|\alpha| < c$, for every $f \in \operatorname{Lip}(M, \mathbb{R})$ with Lipschitz constant K, for any $N \in \mathbb{N}$, for any $u \in H^0(M, L^{\otimes N})$ such that $T_N(f)u = \lambda u$ for some $\lambda \in \mathbb{R}$, one has

$$\int_M e^{2\alpha\sqrt{N}\rho(x)}\left(f(x) - \lambda - CK|\alpha|N^{-\frac{1}{2}}\right)|u(x)|^2\mathrm{d}x \leq 0.$$

Moreover, the constants c and C only depend on the Lipschitz constant of ρ.

Proposition 4.2. *Let M be a $C^{1,1}$ Kähler manifold, and let $f \in C^{1,1}(M, \mathbb{R})$ with $\min(f) = 0$. Let $u \in H^0(M, L^{\otimes N})$ be such that $T_N(f)u = \lambda u$, with $\lambda = O(N^{-1})$. For $k \in \mathbb{N}$ and $\epsilon > 0$, define*

$$g_k^\epsilon = \begin{cases} f - \lambda & \text{if } k = 0 \\ \max(f - \lambda, N^{-1 + \frac{1}{2^k} + 2\epsilon}) & \text{otherwise.} \end{cases}$$

If, for some $k \geq 0$, for all $\epsilon > 0$, there exists $C_k > 0$ and $c_k > 0$ such that, for all $|\alpha| < c_k$, one has

$$\int_M e^{2\alpha\sqrt{N}\sqrt{f(x)}} \left(g_k^\epsilon(x) - C_k N^{-1 + \frac{1}{2^{k+1}} + \epsilon} \right) |u(x)|^2 \mathrm{d}x \leq 0,$$

then for all $\epsilon > 0$ there exists $C_{k+1} > 0$ and $c_{k+1} > 0$ such that, for all $|\alpha| < c_{k+1}$, one has

$$\int_M e^{2\alpha\sqrt{N}\sqrt{f(x)}} \left(g_{k+1}^\epsilon(x) - C_{k+1} N^{-1 + \frac{1}{2^{k+2}} + \epsilon} \right) |u(x)|^2 \mathrm{d}x \leq 0.$$

We postpone the proof of these estimates, and first use them to prove Theorems B and C.

Proof of Theorem B.

Letting M, f, u, λ be as in Proposition 4.1, we choose c, C corresponding to the Lipschitz constant 1; indeed we will choose $\rho = \mathrm{dist}(\cdot, U)$ where U will be defined later.

Now, for every $|\alpha| < c$, one has, by Proposition 4.1

$$0 \geq \int_M e^{2\alpha\sqrt{N}\rho(x)} \left(f(x) - \lambda - CK|\alpha|N^{-\frac{1}{2}} \right) |u(x)|^2 \mathrm{d}x.$$

Let us decompose this integral in two pieces, corresponding to the sign of $f - \lambda - CK|\alpha|N^{-\frac{1}{2}}$: with $\lambda_1 = \lambda + CK|\alpha|N^{-\frac{1}{2}}$ and $\lambda_2 = \lambda + 2CK|\alpha|N^{-\frac{1}{2}}$, one has

$$\begin{aligned} 0 \geq &\int_{\{f \geq \lambda_2\}} e^{2\alpha\sqrt{N}\rho(x)} \left(f(x) - \lambda_1 \right) |u(x)|^2 \mathrm{d}x \\ &+ \int_{\{\lambda_1 \leq f \leq \lambda_2\}} e^{2\alpha\sqrt{N}\rho(x)} \left(f(x) - \lambda_1 \right) |u(x)|^2 \mathrm{d}x \\ &+ \int_{f \leq \lambda_1} e^{2\alpha\sqrt{N}\rho(x)} \left(f(x) - \lambda_1 \right) |u(x)|^2 \mathrm{d}x. \end{aligned}$$

The second contribution is positive, and one can remove it; with

$$\rho = \mathrm{dist}(\cdot, \{f \leq \lambda_1\}),$$

this yields

$$CK|\alpha|N^{-\frac{1}{2}} \int_{\{f \geq \lambda_2\}} e^{2\alpha\sqrt{N}\operatorname{dist}(x,\{f\leq\lambda_1\})}|u(x)|^2 dx$$

$$\leq (\lambda_1 - \min(f)) \int_{\{f\leq\lambda_1\}} |u(x)|^2 dx.$$

To conclude the proof of Theorem B, we let $\alpha = \frac{c}{2}$; then for $\epsilon > CN^{-\frac{1}{2}}$, with

$$W = \{x \in M, \operatorname{dist}(x, \{f \geq \lambda_2\}) > \epsilon\},$$

on W one has $\operatorname{dist}(\cdot, \{f \leq \lambda_1\}) > \epsilon$, so that

$$\int_W |u|^2 \leq e^{-c\epsilon\sqrt{N}} \int_{\{f\geq\lambda_2\}} e^{2\alpha\sqrt{N}\operatorname{dist}(x,\{f\leq\lambda_1\})}|u(x)|^2 dx \leq CN^{\frac{1}{2}}e^{-c\epsilon\sqrt{N}}\|u\|_{L^2}^2.$$

This concludes the proof.

PROOF OF THEOREM C. Let $f \in C^{1,1}(M, \mathbb{R})$ with $\min(f) = 0$. It is well-known that $\sqrt{f}$ is Lipschitz-continuous. In particular, the initialisation of the induction in Proposition 4.2 is given by Proposition 4.1, and thus, for all $k \in \mathbb{N}$, for all $\epsilon > 0$, one has

$$\int_M e^{2\alpha\sqrt{N}\sqrt{f(x)}}\left(g_k^\epsilon(x) - C_k N^{-1+\frac{1}{2^{k+1}}+\epsilon}\right)|u(x)|^2 dx \leq 0,$$

for $|\alpha_k| < c_k(\epsilon)$.

Let $\delta > 0$; for some k large enough and for some $\epsilon > 0$ one has $\delta = \frac{1}{2^{k+1}} + \epsilon$. We now proceed as in the proof of Theorem B: let $\lambda_1 = \lambda + C_k N^{-1+\delta}$ and $\lambda_2 = \lambda + 2C_k N^{-1+\delta}$. Then

$$0 \geq \int_{\{f\geq\lambda_2\}} e^{2\alpha\sqrt{N}\sqrt{f}}(g_k^\epsilon - \lambda_1 + \lambda)|u(x)|^2 dx$$

$$+ \int_{\{f\leq\lambda_1\}} e^{2\alpha\sqrt{N}\sqrt{f}}(g_k^\epsilon - \lambda_1 + \lambda)|u(x)|^2 dx.$$

In particular,

$$C_k N^{-1+\delta} e^{2\alpha\sqrt{2C_k}N^{\frac{\delta}{2}}} \int_{\{f\geq\lambda_2\}} |u|^2 \leq C_k N^{-1+\delta} e^{2\alpha\sqrt{C_k}N^{\frac{\delta}{2}}} \int_{\{f\leq\lambda_1\}} |u|^2,$$

so that, finally,

$$\int_{\{f\geq\lambda+C_k N^{-1+\delta}\}} |u|^2 \leq e^{-2(\sqrt{2}-1)\alpha\sqrt{C_k}N^{\frac{\delta}{2}}}.$$

The proof of Propositions 4.1 and 4.2 rely on the following commutator estimates.

LEMMA 4.3. *Let M be a $C^{1,1}$ Kähler manifold. Let $\rho \in \mathrm{Lip}(M, \mathbb{R})$.*
There exist two constants $c > 0$ and $C > 0$ such that, for any $\alpha \in \mathbb{R}$ with $|\alpha| < c$, for any $f \in \mathrm{Lip}(M, \mathbb{R})$, if K denotes the Lipschitz constant of f, one has

$$\| \exp(-\alpha\sqrt{N}\rho)[S_N, \exp(2\alpha\sqrt{N}\rho)] \exp(-\alpha\sqrt{N}\rho)\|_{L^2 \to L^2} \leq C|\alpha|$$

$$\| \exp(\alpha\sqrt{N}\rho)[f, S_N] \exp(-\alpha\sqrt{N}\rho)\|_{L^2 \to L^2} \leq CKN^{-\frac{1}{2}}.$$

Moreover the constants c, C depend only on the Lipschitz constant of ρ.

Proof. We first prove the second bound; the first bound is a consequence of the second one.

Recall from Proposition 2.1 that the kernel of S_N is bounded everywhere: there exists $C_0 > 0, c_0 > 0$ such that for all $(x, y) \in M \times M$, for all $N \in \mathbb{N}$, one has

$$|S_N(x, y)| \leq CN^d \exp(-c\sqrt{N}\,\mathrm{dist}(x, y)).$$

Here d denotes again the dimension of M.
The kernel of $\exp(\alpha\sqrt{N}\rho)[f, S_N]\exp(-\alpha\sqrt{N}\rho)$ is

$$(x, y) \mapsto S_N(x, y)(f(x) - f(y))\exp(\alpha\sqrt{N}(\rho(x) - \rho(y))).$$

Let L denote the Lipschitz contant of ρ; then the kernel above is everywhere bounded by

$$(x, y) \mapsto C_0 K\,\mathrm{dist}(x, y)N^d \exp((-c_0 + \alpha L)\,\mathrm{dist}(x, y)\sqrt{N}).$$

Let $c = \frac{c_0}{L}$. For $|\alpha| < c$, the Schur norm of this kernel is smaller than

$$C_0 K \sup_{y \in M} \int_{x \in M} \mathrm{dist}(x, y)\exp\left(-\frac{c_0}{2}\,\mathrm{dist}(x, y)\sqrt{N}\right) \leq C_1 KN^{-\frac{1}{2}}.$$

For the first bound, we proceed by differentiation with respect to α. The statement clearly holds for $\alpha = 0$, in which case $[S_N, 1] = 0$. With

$$T(a) = \exp(-\alpha\sqrt{N}\rho)[S_N, \exp(2\alpha\sqrt{N}\rho)]\exp(-\alpha\sqrt{N}\rho),$$

one has

$$T'(a) =$$
$$\sqrt{N}\left[\exp(-\alpha\sqrt{N}\rho)[S_N, \rho]\exp(\alpha\sqrt{N}\rho) - \exp(\alpha\sqrt{N}\rho)[S_N, \rho]\exp(-\alpha\sqrt{N}\rho)\right].$$

We can now apply the second bound (with $f = \rho$); as long as $|\alpha| < c$, one has

$$\|T'(a)\|_{L^2 \to L^2} \leq 2C_1 L\sqrt{N}.$$

This concludes the proof. □

Proof of Proposition 4.1.

Without loss of generality, one can assume $\lambda = 0$ by replacing f with $f - \lambda$. As in [19], since $S_N f u = 0$, one can write

$$\langle e^{\alpha \sqrt{N} \rho} f u, e^{\alpha \sqrt{N} \rho} u \rangle = \langle [S_N, e^{2\alpha \sqrt{N} \rho}] f u, u \rangle$$
$$= \langle [S_N, e^{2\alpha \sqrt{N} \rho}][f, S_N] u, u \rangle.$$

To use Lemma 4.3, we need to introduce a few supplementary exponential factors:

$$\langle [S_N, e^{2\alpha \sqrt{N} \rho}][f, S_N] u, u \rangle$$
$$= \langle e^{-\alpha \sqrt{N} \rho}[S_N, e^{2\alpha \sqrt{N} \rho}] e^{-\alpha \sqrt{N} \rho} e^{\alpha \sqrt{N} \rho}[f, S_N] e^{-\alpha \sqrt{N} \rho} e^{2\alpha \sqrt{N} \rho} u \rangle.$$

Hence, if K denotes the Lipschitz constant of f one has, by Lemma 4.3

$$\langle [S_N, e^{2\alpha \sqrt{N} \rho}][f, S_N] u, u \rangle \leq C^2 |\alpha| K N^{-\frac{1}{2}} \| e^{\alpha \sqrt{N} \rho} u \|_{L^2}^2.$$

This concludes the proof.

Proof of Proposition 4.2.

Let us modify the proof of Proposition 4.1 in this context where u is an eigenfunction of $T_N(g_{k+1}^\epsilon)$ only up to some error, given by the induction hypothesis. Let ρ be a Lipschitz function. Then, for all α', one has

$$\langle e^{\alpha' \sqrt{N} \rho} g_{k+1} u, e^{\alpha' \sqrt{N} \rho} u \rangle - \langle [S_N, e^{2\alpha' \sqrt{N} \rho}][g_{k+1}^\epsilon, S_N] u, u \rangle$$
$$= -\langle e^{\alpha' \sqrt{N} \rho} S_N(g_{k+1} u), e^{\alpha' \sqrt{N} \rho} u \rangle.$$

If $f \in C^{1,1}$ is non-negative then $\sqrt{f}$ is Lipschitz-continuous; we now fix $\rho = \sqrt{f}$. For $|\alpha'|$ small enough, we want to estimate

$$e^{2\alpha' \sqrt{N} \sqrt{f}} S_N(g_{k+1}^\epsilon u) = \left(e^{2\alpha' \sqrt{N} \sqrt{f}} S_N e^{-2\alpha' \sqrt{N} \sqrt{f}} \right) e^{2\alpha' \sqrt{N} \sqrt{f}} (g_{k+1}^\epsilon - f + \lambda) u.$$

The operator $e^{2\alpha' \sqrt{N} \sqrt{f}} S_N e^{-2\alpha' \sqrt{N} \sqrt{f}}$ is bounded independently of N from L^2 to L^2 if $|\alpha'|$ is small enough. Moreover, $g_{k+1}^\epsilon - f + \lambda$ is supported on

$$\{ f \geq N^{-1 + \frac{1}{2^{k+1}} + 2\epsilon} \},$$

so that

$$\int_M e^{4\alpha' \sqrt{N} \sqrt{f}} |g_{k+1}^\epsilon - f + \lambda|^2 |u|^2 \leq C \int_{\{ f \geq \lambda + N^{-1 + \frac{1}{2^{k+1}} + 2\epsilon} \}} e^{4\alpha' \sqrt{N} \sqrt{f}} |u|^2.$$

Let $\alpha > \frac{c_k(\epsilon)}{2}$ (so that the weighted estimate of the induction is satisfied). Then, on $\{ f \geq \lambda + N^{-1 + \frac{1}{2^{k+1}} + 2\epsilon} \}$, one has, for $|\alpha'|$ small enough, for some $c > 0$,

$$e^{4\alpha' \sqrt{N} \sqrt{f}} \leq e^{-cN^{\frac{1}{2^{k+2}} + \epsilon}} e^{2\alpha \sqrt{N} \sqrt{f}},$$

so that

$$\int_{\{f \geq \lambda + N^{-1+\frac{1}{2^{k+1}}+2\epsilon}\}} e^{4\alpha'\sqrt{N}\sqrt{f}}|u|^2$$
$$\leq e^{-cN^{\frac{1}{2^{k+2}}+\epsilon}} \int_{\{f \geq \lambda + N^{-1+\frac{1}{2^{k+1}}+2\epsilon}\}} e^{2\alpha\sqrt{N}\sqrt{f}}|u|^2.$$

By hypothesis, one has

$$\int_M e^{2\alpha\sqrt{N}\sqrt{f}(x)} \left(g_k^\epsilon(x) - C_k N^{-1+\frac{1}{2^{k+1}}+\epsilon} \right) |u(x)|^2 \mathrm{d}x \leq 0.$$

In particular,

$$0 \geq CN^{-1+\frac{1}{2^{k+1}}+\epsilon} \int_{\{f \geq \lambda + 2CN^{-1+\frac{1}{2^{k+1}}+\epsilon}\}} e^{2\alpha\sqrt{N}\sqrt{f}}|u|^2$$
$$- C|\alpha|N^{-1+\frac{1}{2^{k+1}}+\epsilon} e^{C|\alpha|N^{\frac{1}{2^{k+2}}+\frac{\epsilon}{2}}} \int_{\{f \leq \lambda + N^{-1+\frac{1}{2^{k+1}}+\epsilon}\}} |u|^2,$$

so that

$$\int_{\{f \geq \lambda + 2CN^{-1+\frac{1}{2^{k+1}}+\epsilon}\}} e^{2\alpha\sqrt{N}\sqrt{f}}|u|^2 \leq Ce^{C|\alpha|N^{\frac{1}{2^{k+2}}+\frac{\epsilon}{2}}}.$$

Hence, for some $c' > 0$, one has

$$\|e^{2\alpha'\sqrt{N}\sqrt{f}}S_N(gu)\|_{L^2}^2 \leq Ce^{-cN^{\frac{1}{2^{k+2}}+\epsilon}} e^{C|\alpha|N^{\frac{1}{2^{k+2}}+\frac{\epsilon}{2}}} \leq Ce^{-c'N^{\frac{1}{2^{k+2}}+\epsilon}},$$

so that

$$\left| \langle e^{\alpha'\sqrt{N}\rho}gu, e^{\alpha'\sqrt{N}\rho}u \rangle - \langle [S_N, e^{2\alpha'\sqrt{N}\rho}][g, S_N]u, u \rangle \right| \leq Ce^{-c'N^{\frac{1}{2^{k+2}}+\epsilon}}.$$

We can now, up to this error, reproduce the end of the proof of Proposition 4.1. Since the Lipschitz constant of g_k^ϵ is $N^{-\frac{1}{2}+\frac{1}{2^{k+2}}+\epsilon}$, this yields

$$\int_M e^{2\alpha'\sqrt{N}\sqrt{f}(x)} \left(g(x) - \lambda - C|\alpha|N^{-1+\frac{1}{2^{k+2}}\epsilon} \right) |u(x)|^2 \leq 0.$$

This concludes the proof.

5 WEIGHTED ESTIMATES: UNIFORMITY IN THE DIMENSION

Kordyukov [19] has proposed a method for obtaining weighted estimates for eigenfunctions of Toeplitz operators, based on the ellipticity of the Hodge Laplacian (thus generalizing results on the off-diagonal decay of the Szegő projector).

In this section we revisit the proof of Theorem 1.3 in [19], while making the dependency on the geometry more explicit.

Let M be a quantizable Kähler manifold of complex dimension d, with L its prequantum bundle. If ∇^N is the Levi-Civita holomorphic connection on $L^{\otimes N}$, then $H^0(M, L^{\otimes N})$ is the kernel of

$$\Box_N = (\nabla^N)^* \nabla^N - \pi \dim(M) N.$$

Let $\rho \in C^2(M, \mathbb{R})$ and $\alpha \in \mathbb{R}$. Conjugating $\Box_N$ with $e^{\alpha\sqrt{N}\rho}$ yields

$$\Box_{N;\alpha} = \exp(\alpha\sqrt{N}\rho)\Box_N \exp(-\alpha\sqrt{N}\rho) = \Box_N + \alpha A_N + \alpha^2 B_N,$$

where

$$A_N = \sqrt{N}(\Delta\rho + 2\nabla\rho \cdot \nabla^N) \tag{1}$$

$$B_N = -N\|\nabla\rho\|^2. \tag{2}$$

Here, ∇ is the Riemannian gradient.

In this section, we consider an integrable Kähler manifold of the form $M = M_0^{d'}$, and obtain estimates with explicit dependence on d'. Throughout the section, the constants appearing are, unless otherwise noted, independent on d', that is, independent on the dimension $d = \dim(M) = \dim(M_0) \times d'$. The main illustration of the results of this section is presented in Section 6, where $M = \mathbb{CP}^1$, so that d' will simply be equal to d.

If M is a product of manifolds $M = (M_0)^{d'}$, then there holds a uniform bound on the spectral gap of $\Box_N$.

PROPOSITION 5.1. *Let M_0 be a compact, quantizable Kähler manifold of regularity $C^{1,1}$. There exists $C_0 > 0$, $\mu > 0$ such that the following is true.*
Let $d' \in \mathbb{N}$ and let $M = M_0^{d'}$. For $N \in \mathbb{N}$, we let $\Box_N$ be the Hodge Laplacian over M with semiclassical parameter $\frac{1}{N}$. Then for any $\lambda \in \mathbb{C}$ such that $|\lambda| = \mu_0$, one has

$$\left\|\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|^2_{L^2(M)\to L^2(M,L^{\otimes N})}$$
$$+ \frac{1}{\sqrt{N}}\left\|\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|^2_{L^2(M)\to \dot{H}_1(M,L^{\otimes N})} \leq C_0,$$

where the $\dot{H}_1$ quasinorm on sections of $L^{\otimes N}$ is defined as

$$\|u\|^2_{\dot{H}_1(M,L^{\otimes N})} = \int_M \|\nabla^N u(x)\|^2_{\ell^2(TX\otimes L^{\otimes N})} \mathrm{d}\,Vol(x).$$

Proof. The claim is true for $d' = 1$, where it follows from the usual Hörmander-Kohn estimate [18]. Indeed, in this case $\Box_{N,M_0}$ is a self-adjoint operator on $L^2(M_0, L_0^{\otimes N})$ and this estimate implies that

$$\sigma(\Box_{N,M_0}) \subset \{0\} \cap [CN, +\infty)$$

for some $C > 0$.

If $(u_j)_{j\in\mathbb{N}}$ is an orthonormal basis of eigenfunctions of $\Box_{N,M_0}$, with eigenvalues $(\mu_j)_{j\in\mathbb{N}}$, then the eigenfunctions of $\Box_N$ are tensor products of the u_j's (acting on different variables), since

$$\Box_N = \sum_{j=1}^{d'} I^{\otimes j-1} \otimes \Box_{N,M_0} \otimes I^{d'-j};$$

moreover the eigenvalues of $\Box_N$ are the sums of d' eigenvalues of $\Box_{N,M_0}$. In particular, the spectral gap on $\Box_{N,M_0}$ propagates to $\Box_N$, leading to

$$\left\|\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|^2_{L^2(M)\to L^2(M,L^{\otimes N})} \leq \frac{C_0}{2}.$$

for $|\lambda| = \frac{1}{2C}$ and $C_0 = \frac{4}{C}$.

Moreover, the family $(u_j)_{j\in\mathbb{N}}$ is also orthogonal for the $\dot{H}_1$ product, since

$$\langle u_j, u_k\rangle_{\dot{H}_1} = \langle \nabla^N u_j, \nabla^N u_k\rangle_{L^2} = \mu_k \langle u_j, u_k\rangle_{L^2}.$$

Thus the estimate on the operator norm $L^2 \to \dot{H}_1$ also propagates from M_0 to M, which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By the usual resolvent identity, this leads to a spectral gap on $\Box_{N;\alpha}$ for $|\alpha|$ small.

PROPOSITION 5.2. *In the situation of Proposition 5.1, let $\rho \in \mathrm{Lip}(M,\mathbb{R})$. For all α such that*

$$|\alpha| \leq \min\left[\|\nabla\rho\|_{L^\infty}^{-1}, \frac{1}{2C_0}\left(N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty} + 3\|\nabla\rho\|_{L^\infty}\right)^{-1}\right],$$

one has

$$\left\|\left(\lambda - \frac{1}{N}\Box_{N;\alpha}\right)^{-1}\right\|^2_{L^2(M)\to L^2(M,L^{\otimes N})}$$

$$+ \frac{1}{\sqrt{N}}\left\|\left(\lambda - \frac{1}{N}\Box_{N;\alpha}\right)^{-1}\right\|^2_{L^2(M)\to \dot{H}_1(M,L^{\otimes N})} \leq 2C_0.$$

Proof. One has

$$\left(\lambda - \frac{1}{N}\Box_{N;\alpha}\right)^{-1} - \left(\lambda - \frac{1}{N}\Box_N\right)^{-1}$$
$$= \frac{1}{N}\left(\lambda - \frac{1}{N}\Box_{N;\alpha}\right)^{-1}\left(\alpha A_N + \alpha^2 B_N\right)\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}.$$

Here A_N and B_N are given by (1). Writing $A_N = A_{N,0} + A_{N,1}\cdot\nabla^N$ where $A_{N,0}, A_{N,1}$ are respectively $\Delta\rho$ and $2\nabla\rho$, one has

$$\frac{1}{N}\left\|\alpha A_{N,0} + \alpha^2 B_N\right\|_{L^2\to L^2} \le |\alpha|N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty} + \alpha^2\|\nabla\rho\|_{L^\infty}^2$$

and

$$\frac{1}{N}\|A_{N,1}\cdot\nabla^N\|_{\dot{H}_1\to L^2} \le 2|\alpha|N^{-\frac{1}{2}}\|\nabla\rho\|_{L^\infty}.$$

In particular, by Proposition 5.1,

$$\left\|\left(\alpha A_N + \alpha^2 B_N\right)\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|_{L^2\to L^2}$$
$$\le C_0|\alpha|\left(N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty} + \|\nabla\rho\|_{L^\infty}\left(2 + |\alpha|\|\nabla\rho\|_{L^\infty}\right)\right)$$

so that, if

$$|\alpha| \le \min\left[\|\nabla\rho\|_{L^\infty}^{-1}, \frac{1}{2C_0}\left(N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty} + 3\|\nabla\rho\|_{L^\infty}\right)^{-1}\right]$$

then

$$2 + |\alpha|\|\nabla\rho\|_{L^\infty} \le 3.$$

In particular,

$$\left\|\left(\alpha A_N + \alpha^2 B_N\right)\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|_{L^2\to L^2} \le \frac{1}{2}.$$

Hence, the operator $I - \left(\alpha A_N + \alpha^2 B_N\right)\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}$ is invertible on L^2, with operator norm bounded by 2, so that the resolvent identity yields

$$\left\|\left(\lambda - \frac{1}{N}\Box_{N;\alpha}\right)^{-1}\right\|_{L^2\to L^2} \le 2\left\|\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|_{L^2\to L^2}$$
$$\left\|\left(\lambda - \frac{1}{N}\Box_{N;\alpha}\right)^{-1}\right\|_{L^2\to \dot{H}_1} \le 2\left\|\left(\lambda - \frac{1}{N}\Box_N\right)^{-1}\right\|_{L^2\to \dot{H}_1}.$$

One can then conclude from Proposition 5.1. $\Box$

REMARK 5.3. Proposition 5.2 can be used to obtain off-diagonal exponential estimates for the kernel of the Szegő projector. For fixed d' and ρ, $|\alpha|$ is bounded by a constant, which limits this method to a decay of the form $\exp(-\sqrt{N}\,\mathrm{dist}(x,y))$.

As d' increases, using a similar construction as in Subsection 6.1, this method is able to yield, at best, a decay of the form

$$\|\mathbb{1}_U S_N \mathbb{1}_V\| \le C \exp(-c_1 N d^{-\frac{1}{2}}\,\mathrm{dist}(U,V)),$$

which is too weak for our purpose; in particular, the more elementary estimate of Proposition 2.4 beats this estimate on most of $M \times M$.

Following [19] we then obtain a dimension-independent version of Lemma 4.3.

LEMMA 5.4. *In the situation of Proposition 5.2, there exists $C_1(M_0)$ such that*

$$\| \exp(-\alpha\sqrt{N}\rho)[S_N, \exp(2\alpha\sqrt{N}\rho)] \exp(-\alpha\sqrt{N}\rho)\|_{L^2 \to L^2}$$
$$\le C_1|\alpha| \left[N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty} + \|\nabla\rho\|_{L^\infty} \right].$$

Moreover, for every $f \in C^2(M,\mathbb{R})$, one has

$$\| \exp(\alpha\sqrt{N}\rho)[f, S_N] \exp(-\alpha\sqrt{N}\rho)\|_{L^2 \to L^2}$$
$$\le C_1 \left[N^{-1}\|\Delta f\|_{L^\infty} + N^{-\frac{1}{2}}\|\nabla f\|_{L^\infty} (1 + \|\nabla\rho\|_{L^\infty}) \right].$$

Proof. By Proposition 5.1 and the spectral gap property, the Szegő kernel is given by the following integral:

$$S_N = \frac{1}{2i\pi} \oint_{|\lambda|=\mu_0} \left(\lambda - \frac{1}{N}\square_N \right)^{-1}.$$

In particular, one has

$$\exp(-\alpha\sqrt{N}\rho)S_N \exp(\alpha\sqrt{N}\rho) - \exp(\alpha\sqrt{N}\rho)S_N \exp(-\alpha\sqrt{N}\rho)$$
$$= \frac{1}{2i\pi} \oint_{|\lambda|=\mu_0} \left[\left(\lambda - \frac{1}{N}\Delta_{N;\alpha} \right)^{-1} - \left(\lambda - \frac{1}{N}\Delta_{N;-\alpha} \right)^{-1} \right]$$
$$= \frac{2\alpha}{N} \frac{1}{2i\pi} \oint_{|\lambda|=\mu_0} \left(\lambda - \frac{1}{N}\Delta_{N;\alpha} \right)^{-1} A_N \left(\lambda - \frac{1}{N}\Delta_{N;\alpha} \right)^{-1}.$$

By Proposition 5.2 and the expression of A_N given in (1), we obtain the desired control.

For the second estimate, we need to commute f with $\Box_N$ and $\Box_{N,\alpha}$. From the computations

$$[f, \Box_N] = \Delta f + 2\nabla f \cdot \nabla^{L^P}$$
$$[f, A_N] = -2\sqrt{N}\langle\nabla f, \nabla\rho\rangle$$
$$[f, B_N] = 0,$$

and Proposition 5.2 one has

$$\left\|\left[f, \left(\lambda - \frac{1}{N}\Box_{N,\alpha}\right)^{-1}\right]\right\|_{L^2 \to L^2}$$
$$= \left\|\left(\lambda - \frac{1}{N}\Box_{N,\alpha}\right)^{-1}\left[f, \frac{1}{N}\Delta_{N,\alpha}\right]\left(\lambda - \frac{1}{N}\Box_{N,\alpha}\right)^{-1}\right\|_{L^2 \to L^2}$$
$$\leq C\left(N^{-1}\|\Delta f\|_{L^\infty} + N^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}\|\nabla\rho\|_{L^\infty} + N^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}\right).$$

Now

$$\exp(\alpha\sqrt{N}\rho)[f, S_N]\exp(-\alpha\sqrt{N}\rho) = [f, \exp(\alpha\sqrt{N}\rho)S_N\exp(-\alpha\sqrt{N}\rho)]$$
$$= \frac{1}{2i\pi}\int_{|\lambda|=\mu_0}\left[f, \left(1 - \frac{1}{N}\Box_{N,\alpha}\right)^{-1}\right],$$

which concludes the proof. $\Box$

In the case of a quantum spin system, f is a finite sum of eigenfunctions of Δ, in which case the commutator is smaller.

LEMMA 5.5. *Under the hypotheses of Lemma 5.4, if $\Delta f = -\mu f$, then there exists $C_2(\mu, M_0)$ such that*

$$\|\exp(\alpha\sqrt{N}\rho)[f, S_N]\exp(-\alpha\sqrt{N}\rho)\|_{L^2 \to L^2} \leq CN^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}(1 + \|\nabla\rho\|_{L^\infty}).$$

Proof. The proof proceeds as previously, isolating $\Delta f = -\mu f$ in $[f, \Box_{N,\alpha}]$. A

first application of the resolvent formula yields

$$
\int_{|\lambda|=\mu_0}\left[f,\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-1}\right]=
$$

$$
\int_{|\lambda|=\mu_0}\underbrace{\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-1}\left(\alpha N^{-\frac{1}{2}}\nabla f\cdot\nabla\rho+N^{-1}\nabla f\cdot\nabla^N\right)}_{\|\cdot\|_{L^2\to L^2}\le CN^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}(1+\|\nabla\rho\|_{L^\infty})}\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-1}
$$

$$
+\mu N^{-1}f\underbrace{\int_{|\lambda|=\mu_0}\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-2}}_{=0}
$$

$$
+\mu N^{-1}\int_{|\lambda|=\mu_0}\left[\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-1},f\right]\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-1}.
$$

By induction,

$$
\left\|\frac{1}{2i\pi}\int_{|\lambda|=\mu_0}\left[f,\left(\lambda-\frac{1}{N}\square_{N,\alpha}\right)^{-1}\right]\right\|_{L^2\to L^2}
$$

$$
\le CN^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}(1+\|\nabla\rho\|_{L^\infty})\sum_{k=0}^{+\infty}(\mu N^{-1})^k.
$$

This concludes the proof. □

We are now in position to prove a weighted estimate on eigenfunctions.

PROPOSITION 5.6. *Let M_0 be a compact Kähler manifold. There exists $C_3>0$ such that, for every $N\ge 1$, for every $f\in C^2(M_0^{d'},\mathbb{R})$ and every $\rho\in \mathrm{Lip}(M_0^{d'})$, if S_N denotes the Szegő kernel on $M_0^{d'}$ and if $\lambda\in\mathbb{R}, u\in H_0(M_0^{d'},L^{\otimes N})$ are such that $S_N(fu)=\lambda u$, then*

$$
\int_{M_0^{d'}}e^{2\alpha\sqrt{N}\rho(x)}(f(x)-\lambda-C(f,\rho)|\alpha|)|u(x)|^2\mathrm{d}\,Vol(x)\le 0,
$$

where

$$
C(f,\rho)=C_3\left[N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty}+\|\nabla\rho\|_{L^\infty}\right]
$$

$$
\times\left[N^{-1}\|\Delta f\|_{L^\infty}+N^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}(1+\|\nabla\rho\|_{L^\infty})\right].
$$

If f is a sum of eigenfunctions of $-\Delta$ on $M_0^{d'}$, with frequencies bounded by μ independently on d', then one can choose

$$
C(f,\rho)=C_3(\mu)\left[N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty}+\|\nabla\rho\|_{L^\infty}\right]\left[N^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}(1+\|\nabla\rho\|_{L^\infty})\right].
$$

Proof. Up to replacing f with $f - \lambda$, one has $\lambda = 0$.
As in [19] one has

$$\langle \exp(\alpha\sqrt{N}\rho)(f - \lambda)u, \exp(\alpha\sqrt{N}\rho)u\rangle$$
$$= \langle [S_N, \exp(2\alpha\sqrt{N}\rho)]fu, u\rangle$$
$$= \langle [S_N, \exp(2\alpha\sqrt{N}\rho)]S_N fu, u\rangle + \langle [S_N, \exp(2\alpha\sqrt{N}\rho)](1 - S_N)fu, u\rangle$$
$$= \langle [S_N, \exp(2\alpha\sqrt{N}\rho)][f, S_N]u, u\rangle.$$

We write

$$\langle [S_N, \exp(2\alpha\sqrt{N}\rho)][f, S_N]u, u\rangle$$
$$= \langle e^{-\alpha\sqrt{N}\rho}[S_N, e^{2\alpha\sqrt{N}\rho}]e^{-\alpha\sqrt{N}\rho}e^{\alpha\sqrt{N}\rho}[f, S_N]e^{-\alpha\sqrt{N}\rho}e^{\alpha\sqrt{N}\rho}u, e^{\alpha\sqrt{N}\rho}u\rangle$$

so that, by Lemma 5.4,

$$\left| \langle [S_N, \exp(2\alpha\sqrt{N}\rho)][f, S_N]u, u\rangle \right| \leq$$
$$C|\alpha| \left[N^{-1}\|\Delta f\|_{L^\infty} + N^{-\frac{1}{2}}\|\nabla f\|_{L^\infty}(1 + \|\nabla\rho\|_{L^\infty}) \right]$$
$$\times \left[N^{-\frac{1}{2}}\|\Delta\rho\|_{L^\infty} + \|\nabla\rho\|_{L^\infty} \right] \|\exp(\alpha\sqrt{N}\rho)u\|_2^2.$$

This concludes the proof in the general case. If f is a sum of eigenfunctions of $-\Delta$, then one can remove the factor $N^{-1}\|\Delta f\|_{L^\infty}$ by Lemma 5.5. This concludes the proof. $\square$

6 Case study: spin systems

In this section, we study Proposition 5.6 in the particular case of spin systems, where $M_0 = S^2 = \mathbb{CP}^1$. Since $\dim(M_0) = 1$, we drop the notation d' and use directly the dimension d in our estimates instead.

6.1 Construction of the weight

Let us construct a weight ρ adapted to Proposition 5.6.
Let $U \subset M = (\mathbb{S}^2)^d$ be an open set. Let $\rho_0 : M \to \mathbb{R}$ be as follows:

$$\rho_0 : x \mapsto \begin{cases} 0 & \text{if } \mathrm{dist}(x, U) \leq c_0\sqrt{d} \\ \mathrm{dist}(x, U) - c_0\sqrt{d} & \text{otherwise.} \end{cases}$$

Let also $\chi : \mathbb{R} \to \mathbb{R}$ be as follows:

$$\chi : x \mapsto \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We will inject in Proposition 5.6 the following function:

$$\rho : x \mapsto \left[\int_{y \in M} \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) \mathrm{d}y \right]^{-1} \int_{y \in M} \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) \rho_0(y) \mathrm{d}y.$$

Note that ρ is supported on $\{\operatorname{dist}(x, U) \geq \frac{c_0}{2} \sqrt{d}\}$ and is greater than $\frac{1}{2} \operatorname{dist}(x, U)$ on $\{\operatorname{dist}(x, U) \geq 3 c_0 \sqrt{d}\}$.

PROPOSITION 6.1. *The following controls hold independently on c_0 and d:*

$$\|\nabla \rho\|_{L^\infty} \leq 1$$

$$\|\Delta \rho\|_{L^\infty} \leq \frac{16 \sqrt{d}}{c_0}.$$

Proof. Let $x_0, x_1 \in M$. There exists $u \in (\mathfrak{so}_3)^d$, of norm 1, such that $\exp(\operatorname{dist}(x_0, x_1) u) x_0 = x_1$. From the definition of ρ and the invariance of the integral kernel under $(SO_3)^d$, one has

$$\rho(x_0) - \rho(x_1) = \frac{\int_{y \in M} \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) \left[\rho_0(y) - \rho_0 \left(e^{\operatorname{dist}(x_0, x_1) u} y \right) \right] \mathrm{d}y}{\int_{y \in M} \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) \mathrm{d}y}$$

From there, since ρ_0 is 1-Lipschitz, ρ is 1-Lipschitz.
To estimate $\Delta \rho$, let us bound

$$\int_{y \in M} \left| \nabla_x \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) \right| \mathrm{d}y,$$

where the norm of the gradient is the ℓ^2 norm.
First, one has almost everywhere

$$\nabla_x \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) = \frac{8}{c_0^2 d} \operatorname{dist}(x, y) \mathbb{1}_{d(x,y) \leq \frac{c_0 \sqrt{d}}{2}} \gamma,$$

where γ is the derivative at 0 of the unique unit speed geodesic from y to x with minimal length.
In particular, on the complement of

$$\left\{ d(x, y) \in \left[\frac{c_0 \sqrt{d}}{2} \left(\sqrt{1 + \frac{1}{4d^2}} - \frac{1}{2d} \right), \frac{c_0 \sqrt{d}}{2} \right] \right\},$$

one has

$$\left\| \nabla_x \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right) \right\| \leq \frac{4 \sqrt{d}}{c_0} \chi \left(\frac{2 \operatorname{dist}(y, x)}{c_0 \sqrt{d}} \right)$$

so that

$$\int_{B\left(x,\frac{c_0\sqrt{d}}{2}\left(\sqrt{1+\frac{1}{4d^2}}-\frac{1}{2d}\right)\right)} \left\| \nabla_x \chi \left(\frac{2\operatorname{dist}(y,x)}{c_0\sqrt{d}} \right) \right\| \mathrm{d}y$$

$$\leq \frac{4\sqrt{d}}{c_0} \int_M \chi \left(\frac{2\operatorname{dist}(y,x)}{c_0\sqrt{d}} \right) \mathrm{d}y.$$

To estimate the integral on the complement, we introduce $f : \mathbb{R}_*^+ \to \mathbb{R}_*^+$ as the ratio of the area of spheres on M versus $\mathbb{C}^d$:

$$f : r \mapsto \frac{Vol_{2d-1}(S_M(x,r))}{Vol_{2d-1}(S_{\mathbb{C}^d}(0,r))}.$$

An essential property of f is that it is decreasing. Indeed,

$$f(r) = \frac{1}{Vol_{2d-1}(S_{\mathbb{C}^d}(0,1))} \int_{S_{\mathbb{C}^d}(0,1)} \prod_{i=1}^d \mathbb{1}_{|z_i|<\frac{\pi}{r}} \frac{\sin(r|z_i|)}{r|z_i|} \mathrm{d}z\mathrm{d}\overline{z}$$

where the quantity to be integrated decreases with respect to r.
Now

$$\int_{B\left(x,\frac{c_0\sqrt{d}}{2}\right)\setminus B\left(x,\frac{c_0\sqrt{d}}{2}\left(\sqrt{1+\frac{1}{4d^2}}-\frac{1}{2d}\right)\right)} \left\| \nabla_x \chi \left(\frac{2\operatorname{dist}(y,x)}{c_0\sqrt{d}} \right) \right\| \mathrm{d}y$$

$$= \operatorname{Vol}_{2d-1}(S^{2n-1}) \int_{\frac{c_0\sqrt{d}}{2}\left(\sqrt{1+\frac{1}{4d^2}}-\frac{1}{2d}\right)}^{\frac{c_0\sqrt{d}}{2}} r^{2d-1} f(r) \frac{8}{c_0^2 d} r \mathrm{d}r.$$

For r in the integration range, $f(r) \leq f\left(r\left(1-\frac{1}{2d}\right)\right)$ since f is decreasing; moreover, for all $d \in \mathbb{N}$,

$$\frac{r^{2d}}{\left(r\left(1-\frac{1}{2d}\right)\right)^{2d}} = \left(1-\frac{1}{2d}\right)^{-2d} \leq 4.$$

Hence,

$$\int_{B\left(x,\frac{c_0\sqrt{d}}{2}\right)\setminus B\left(x,\frac{c_0\sqrt{d}}{2}\left(\sqrt{1+\frac{1}{4d^2}}-\frac{1}{2d}\right)\right)} \left\| \nabla_x \chi \left(\frac{2\operatorname{dist}(y,x)}{c_0\sqrt{d}} \right) \right\| \mathrm{d}y$$

$$\leq 4\operatorname{Vol}_{2d-1}(S^{2n-1}) \int_{\frac{c_0\sqrt{d}}{2}\left(\sqrt{1+\frac{1}{4d^2}}-\frac{1}{2d}\right)\left(1-\frac{1}{2d}\right)}^{\frac{c_0\sqrt{d}}{2}\left(1-\frac{1}{2d}\right)} r^{2d} f(r) \frac{8}{c_0^2 d} \mathrm{d}r.$$

Since

$$1 - \frac{1}{2d} \leq \sqrt{1 + \frac{1}{4d^2}} - \frac{1}{2d},$$

one is left with part of the integral controlled previously:

$$\mathrm{Vol}_{2d-1}(S^{2n-1}) \int_{\frac{c_0\sqrt{d}}{2}\left(\sqrt{1+\frac{1}{4d^2}}-\frac{1}{2d}\right)\left(1-\frac{1}{2d}\right)}^{\frac{c_0\sqrt{d}}{2}\left(1-\frac{1}{2d}\right)} r^{2d} f(r) \frac{8}{c_0^2 d} \mathrm{d}r$$

$$\leq \frac{4\sqrt{d}}{c_0} \int_M \chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right) \mathrm{d}y.$$

Thus, one has the following control:

$$\left(\int_M \chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right) \mathrm{d}y\right)^{-1} \int_M \left\|\nabla_x\chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right)\right\| \mathrm{d}y \leq \frac{16}{c_0}\sqrt{d}.$$

Let $x \in M$. Without loss of generality, $x = (1,\ldots,1)$ is the North pole. Let $(X_i)_{1\leq i\leq d}$ and $(Y_i)_{1\leq i\leq d}$ be the vector fields on M corresponding to unit speed rotation around the X or Y axis on the i-th sphere. Then

$$\Delta\rho(x) = \left(\int_M \chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right) \mathrm{d}y\right)^{-1} \sum_{i=1}^d \int_M \partial_{X_i}\chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right) \partial_{X_i}\rho_0(y)$$

$$+ \partial_{Y_i}\chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right) \partial_{Y_i}\rho_0(y).$$

The semidefinite scalar product induced by (X_i) and (Y_i) is everywhere controlled by the usual one: for all $u,v \in TM$ with same base point,

$$\left|\sum_{i=1}^d \langle X_i, u\rangle\langle X_i, v\rangle + \langle Y_i, u\rangle\langle Y_i, v\rangle\right| \leq |\langle u,v\rangle|.$$

In particular, since $\|\nabla\rho_0\| \leq 1$, one has

$$|\Delta\rho(x)| \leq \left(\int_M \chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right) \mathrm{d}y\right)^{-1} \int_M \left\|\nabla_x\chi\left(\frac{2\,\mathrm{dist}(y,x)}{c_0\sqrt{d}}\right)\right\| \mathrm{d}y,$$

which concludes the proof. $\qquad\square$

6.2 IMPLEMENTING THE WEIGHTED ESTIMATES

To begin with, let us define the class of symbols, called *tame spin systems*, with which we will work.

DEFINITION 6.2. Let $G = (V,E)$ be a graph with $|V| = d$ vertices. Suppose that the valence at each site is bounded by v. Assign to each edge $e \in E$ a colour among k elements; one can decompose $E = E_1 \sqcup E_2 \sqcup \ldots \sqcup E_k$ into the disjoint union of the sets of edges of a prescribed colour. Now, for each colour

j, let $w_j : M_0 \times M_0 \to \mathbb{R}$ be a C^2 function, where $M_0 = (\mathbb{S}^2)^{m_0}$ is a product of spheres; suppose that w_j is a finite sum of eigenfunctions of the Laplace operator.

Then the following function g is a TAME SPIN SYSTEM on $(M_0)^G = \{(x_a), a \in V\}$:

$$g : x \mapsto \sum_{j=1}^{k} \sum_{(a,b) \in E_j} w_j(x_a, x_b).$$

This very broad class of functions contains any finite-range spin system on a lattice, quasi-crystal, or random graph with bounded valence, with any reasonable boundary condition. Examples of spin systems not satisfying the control above are

- The boundary condition "all spins at the boundary are identical", except for spin chains

- Infinite range interactions (with sufficiently slow decay)

- Mean field theories

- Random interactions (if the strength of the interaction is not bounded).

Since this section is concerned with the $d \to +\infty$ limit, we will consider d-dependent families of tame spin systems. Without risk of confusion, we will call "tame spin system" a family of tame spin systems where, with the notations of Definition 6.2, the objects $m_0, v, k, (w_j)_{1 \le j \le k}$ are fixed.

The following property follows immediately from the definition.

PROPOSITION 6.3. *Let g be a tame spin system. There exists C such that, for every d, one has*

$$\|g\|_{L^\infty} \le Cd$$

$$\|\nabla g\|_{L^\infty} \le C\sqrt{d}$$

$$\|\Delta g\|_{L^\infty} \le Cd.$$

We will not apply Proposition 5.6 to a tame spin system g itself, but to the N-dependent symbol f which is such that

$$T_N(f) = d^{-1} T_N(g - \lambda)^2,$$

where λ is the eigenvalue to be studied.

The properties of f depend on the symbol calculus on $\mathbb{S}^2$.

PROPOSITION 6.4. *Uniformly in N and d, one has*

$$\|\nabla f\|_\infty \le C\sqrt{d}$$

$$\|\Delta f\|_\infty \le Cd$$

$$f = d^{-1}(u - \lambda)^2 + O(N^{-1}).$$

Proof. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the Berezin transform, defined as follows: for $f \in C^\infty(M, \mathbb{R})$, the operator $T_N(f)$ has an integral kernel; we let

$$\mathcal{B}_N f : x \mapsto \frac{\pi}{N+1} T_N(f)(x, x).$$

The Berezin transform is related to the symbol product ([4], Proposition 6). It admits an expansion in negative powers of N:

$$\mathcal{B}_N = \mathrm{I} + \sum_{k=1}^{+\infty} N^{-k} B_k + O(N^{-\infty}),$$

where B_k is a differential operator of order $2k$.

The operator $\mathcal{B}_N$ commutes with the $SO(3)$ action on $\mathbb{S}^2$ (since the Szegő kernel is invariant by this action). In particular, there exist coefficients $(c_{\ell,k})_{\ell \leq k}$ such that, for every k,

$$B_k = \sum_{\ell=0}^{k} c_{\ell,k} \Delta^\ell.$$

Moreover, one has $\mathcal{B}_N(1) = 1$ by definition, so that $c_{0,k} = 0$ for all $k \geq 1$. In other terms, for some differential operators C_k one can write

$$\mathcal{B}_N = \mathrm{I} + \sum_{k=1}^{+\infty} N^{-k} C_k \Delta + O(N^{-\infty}).$$

The symbolic product is then a polarisation of the Berezin transform: a monomial term in $\mathcal{B}_N$ of the form Δ^ℓ leads to a term in the symbol product of a and b of the form $\partial^\ell a \overline{\partial}^\ell b$.

The Berezin transform on $(\mathbb{S}^2)^d$ is the tensor product of the Berezin transform on each sphere. In particular, one has

$$df = (g - \lambda)^2 + \sum_{\substack{\mathcal{J} \subset \{1,\dots,d\} \\ |\mathcal{J}| \geq 1}} \prod_{j \in \mathcal{J}} \left(\sum_{k=1}^{+\infty} N^{-k} \widetilde{C_{k;j}} \right) \left(\partial^{\mathcal{J}}(g - \lambda), \overline{\partial}^{\mathcal{J}}(g - \lambda) \right).$$

Here, $\widetilde{C_{k;j}}$ denotes the polarisation of C_k acting on the j-th coordinate (holomorphic derivatives act on the first function, antiholomorphic derivatives on the second function). We, crucially, use the fact that the Berezin transform, and the symbol calculus, lead to absolutely converging sums for spin systems. If g is a tame spin system, then for any $j_0 \in \{1, \dots, d\}$ the number of $\mathcal{J} \subset \{1, \dots, d\}$ such that $j_0 \in \mathcal{J}$ and $\partial^{\mathcal{J}} g \neq 0$ is bounded independently on j_0 and d. Using the notations of Definition 6.2, an upper bound is $2^{m_0 v} - 1$. In particular, uniformly in j_0 and d,

$$\sum_{\substack{\mathcal{J} \subset \{1,\dots,d\} \\ j_0 \subset \mathcal{J}}} \prod_{j \in \mathcal{J}} \left(\sum_{j=1}^{+\infty} N^{-k} \widetilde{C_{k;j}} \right) \left(\partial^{\mathcal{J}}(g - \lambda), \overline{\partial}^{\mathcal{J}}(g - \lambda) \right) = O(N^{-1}).$$

In fact, $N(df - (g - \lambda)^2)$ is again a tame spin system (with classical dependence on N) and satisfies the same type of bounds as g, as in Proposition 6.3. This yields the desired bounds on ∇f and Δf. $\qquad\square$

Proof of Theorem D

Let ρ be constructed as in Section 6.1 (we will define U and c_0 later) and let f be as above. The spectral gap condition of Proposition 5.2 amounts (for d large enough) to

$$|\alpha| \le c_3 N^{\frac{1}{2}} d^{-\frac{1}{2}} c_0$$

and the constant in Proposition 5.6 is controlled by

$$C(f, \rho) \le C_3(\mu) N^{-1} d c_0^{-1}.$$

In particular, one has

$$\int_M e^{c_4 \sqrt{N} \alpha \rho} \left(f - C(\mu) N^{-\frac{1}{2}} d^{\frac{1}{2}} \right) |u|^2 \le 0.$$

Let now $\lambda_1 = C(\mu) N^{-\frac{1}{2}} \sqrt{d}$ and $\lambda_2 = 2 C(\mu) N^{-\frac{1}{2}} \sqrt{d}$. There holds

$$(h - C c_3 (\lambda + N^{-\frac{1}{2}} d)) \ge C(\mu) N^{-\frac{1}{2}} \sqrt{d}$$

on $\{h \ge \lambda_2\}$.
We now choose

$$U = \{f \le \lambda_1\}$$
$$c_0 = N^{-\frac{1}{2}}$$

and α large enough. Then, decomposing the integral yields, for some $c_4 > 0$,

$$0 \ge C(\mu) N^{-\frac{1}{2}} \sqrt{d} \int_V e^{c_4 \sqrt{N} \operatorname{dist}(x, U) d^{-\frac{1}{2}}} |u|^2(x) \mathrm{d}x$$
$$- C(\mu) N^{-\frac{1}{2}} \sqrt{d} \int_U |u|^2(x) \mathrm{d}x,$$

where

$$V = \left\{ x \in M, \operatorname{dist}(x, U) \ge 3 N^{-\frac{1}{2}} \sqrt{d} \right\} \cap \{f \ge 2 C(\mu) N^{-\frac{1}{2}} \sqrt{d}\}.$$

Since $f = \frac{1}{d}(g - \lambda)^2 + O(N^{-1})$, one has

$$\{f(x) \ge \lambda_1\} \subset \{|g(x) - \lambda| \ge C N^{-\frac{1}{4}} d^{\frac{3}{4}}\}.$$

This yields Theorem D.

REMARK 6.5. The window $|g-\lambda| \geq CN^{-\frac{1}{4}}d^{\frac{3}{4}}$ seems larger than what Theorem B allows for: by applying Proposition 5.6 directly to g, we would have obtained $|g - \lambda| \geq CN^{-\frac{1}{2}}d^{\frac{1}{2}}$. However, since even the lowest eigenvalue of $T_N(g)$ is typically of order $N^{-1}d$ if $\min(g) = 0$, this constant appears in the lower bound for the negative part of the weighted integral; this would yield an estimate of the form

$$\int_W e^{c_4\sqrt{N}\,\mathrm{dist}(x,U)d^{-\frac{1}{2}}}|u|^2(x)\mathrm{d}x \leq CN^{-\frac{1}{2}}\sqrt{d},$$

which, as d increases, is no better than the trivial estimate

$$\int_W e^{c_4\sqrt{N}\,\mathrm{dist}(x,U)d^{-\frac{1}{2}}}|u|^2(x)\mathrm{d}x \leq Ce^{C\sqrt{N}}.$$

REMARK 6.6. Letting c_0 be a small constant rather than $N^{-\frac{1}{2}}$ in the proof of Theorem D, we obtain the following variant:

$$\int_W e^{c_4 N\,\mathrm{dist}(x,U)d^{-\frac{1}{2}}}|u(x)|^2\mathrm{d}x \leq C,$$

where

$$W = \{x \in (\mathbb{S}^2)^d, \mathrm{dist}(x,U) \geq Cc_0\sqrt{d}\}.$$

6.3   THE PARTITION FUNCTION

To conclude, in this subsection we use Theorem D to prove Proposition 1.1. We let g be a tame spin system and $\beta \leq cN^{\frac{1}{2}}d^{-1}$. Let $(u_k)_{1\leq k\leq(N+1)^d}$ be a spectral basis for $T_N(g)$ and $(\lambda_k)_{1\leq k\leq(N+1)^d}$ be the associated family of eigenfunctions. Then

$$Tr(e^{-\beta T_N(g)}) = \sum_{k=1}^{(N+1)^d} e^{-\beta\lambda_k}.$$

We wish to compare

$$\langle u_k, e^{-\beta g}u_k\rangle$$

and

$$\langle u_k, e^{-\beta T_N(g)}u_k\rangle = e^{-\beta\lambda_k},$$

for $\beta \leq cN^{\frac{1}{2}}d^{-1}$.
Following Theorem D, let

$$W = \left\{x \in (\mathbb{S}^2)^d, \mathrm{dist}(x, \{|g - \lambda_k| < C_0N^{-\frac{1}{4}}d^{\frac{3}{4}}\}) > KC_0N^{-\frac{1}{2}}\sqrt{d}\right\}.$$

Here C_0 is the constant C in Theorem D, and K is an integer large enough. Since $\|\nabla g\| < C(w,v)\sqrt{d}$ by Proposition 6.3, for $x \in (\mathbb{S}^2)^d\backslash\Omega$ one has $|g-\lambda_k| < CN^{-\frac{1}{2}}d$. In particular, for some $C > 0$ independent of N and d, one has

$$e^{-\beta\lambda_k}e^{-CN^{-\frac{1}{2}}\beta d}\int_{W^c}|u(x)|^2 \leq \int_{W^c}e^{-\beta g}|u(x)|^2 \leq e^{-\beta\lambda_k}e^{CN^{-\frac{1}{2}}\beta d}\int_{W^c}|u(x)|^2.$$

A first application of Theorem D yields

$$\int_W |u(x)|^2 \leq C_0 e^{-c_0 K},$$

so that, in particular, for K large enough

$$\frac{1}{2} \leq \int_{W^c} |u(x)|^2 \leq 1.$$

One can then simplify the previous inequality into

$$e^{-\beta\lambda_k} e^{-C'N^{-\frac{1}{2}}\beta d} \leq \int_{W^c} e^{-\beta g} |u(x)|^2 \leq e^{-\beta\lambda_k} e^{CN^{-\frac{1}{2}}\beta d},$$

for some $C' > C$.

It remains to give an upper bound on

$$\int_W e^{-\beta g} |u|^2.$$

To this end, we observe that, on W,

$$-\beta(g - \lambda_k) \leq \beta C N^{-\frac{1}{4}} d^{\frac{3}{4}} + \mathrm{dist}(x, \{|g - \lambda| < CN^{-\frac{1}{4}} d^{\frac{3}{4}}\})\beta\|\nabla g\|.$$

By Proposition 6.3, the bound on β and the definition of W, this simplifies into

$$-\beta(g - \lambda_k) \leq C(N^{-\frac{1}{2}}\beta d^{-1})N^{\frac{1}{2}}d^{-\frac{1}{2}}\,\mathrm{dist}(x, \{|g - \lambda| < CN^{-\frac{1}{4}} d^{\frac{3}{4}}\})$$

$$\leq c_0 N^{-\frac{1}{2}} d^{-\frac{1}{2}}\,\mathrm{dist}(x, \{|g - \lambda| < CN^{-\frac{1}{4}} d^{\frac{3}{4}}\})$$

if $\beta < cN^{\frac{1}{2}}d^{-1}$ with c small enough. Here c_0 is the exponential constant in Theorem D.

Hence, by Theorem D,

$$\int_W e^{-\beta g} |u_k|^2 \leq e^{-\beta\lambda_k} \leq e^{-\beta\lambda_k}(e^{C'N^{-\frac{1}{2}}\beta d} - 1).$$

To conclude,

$$\frac{1}{2}e^{-\beta\lambda_k} e^{-C'N^{-\frac{1}{2}}\beta d} \leq \int e^{-\beta g}|u_k|^2 \leq e^{-\beta\lambda_k} e^{C'N^{-\frac{1}{2}}\beta d}.$$

Summing this estimate over k yields:

$$Tr(\exp(-\beta T_N(g)))e^{-C'N^{-\frac{1}{2}}\beta d} \leq \int e^{-\beta g}\left(\sum_k |u_k|^2\right)$$

$$\leq Tr(\exp(-\beta T_N(g)))e^{C'N^{-\frac{1}{2}}\beta d}.$$

Since the u_k's form an orthonormal basis of the Hilbert space, one has, for every $x \in M$,

$$\sum_k |u_k|^2 = \left(\frac{N+1}{\pi}\right)^d.$$

Up to this factor, the quantum partition function $Tr(\exp(-\beta T_N(g)))$ is, approximately, given by the classical partition function $\int_M e^{-\beta g}$. This concludes the proof.

REMARK 6.7. The multiplicative error term $e^{C'N^{\delta-\epsilon}}$ is better than the outcome of the method used by Lieb [22]. In this method, one bounds the quantum partition function by a term of the form

$$\left(\frac{N+1+2C}{\pi}\right)^d \int_M e^{-\beta g},$$

where C is the maximal order of the spin polynomial at one given site (the order of $S_{x,j}S_{x,j+1}$ is 1, but the order of $S_{x,j}S_{y,j}$ is 2). The error is then

$$\left(\frac{N+1+2C}{N+1}\right)^d = O(\exp(CdN^{-1})).$$

REMARK 6.8. In this method, the upper bound on β is driven by the bound in the weighted estimate in Theorem D. Following Remark 6.6, on the improved range $\beta \leq cNd^{-1}$, we obtain the weaker estimate

$$|f_Q(N,\beta) - f_C(N,\beta)| \leq C.$$

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