Nonarchimedean Analytic Cyclic Homology

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Abstract. Let V be a complete discrete valuation ring with fraction field F of characteristic zero and with residue field $\mathbb{F}$. We introduce analytic cyclic homology of complete torsion-free bornological algebras over V. We prove that it is homotopy invariant, stable, invariant under certain nilpotent extensions, and satisfies excision. We use these properties to compute it for tensor products with dagger completions of Leavitt path algebras. If R is a smooth commutative V-algebra of relative dimension 1, then we identify the analytic cyclic homology of its dagger completion with Berthelot's rigid cohomology of $R \otimes_V \mathbb{F}$.

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1 Introduction

Analytic cyclic homology of complete bornological algebras over $\mathbb{R}$ and $\mathbb{C}$ was introduced in [16] as a bivariant generalisation from Banach to bornological algebras of the entire cyclic cohomology defined by Connes [5] and further studied by Khalkhali [14]. It was shown to be stable under tensoring with algebras of nuclear operators and invariant under differentiable homotopies and under analytically nilpotent extensions and to satisfy excision for semi-split extensions [18].
Let V be a complete discrete valuation ring whose fraction field F has characteristic zero. Let π be a uniformiser and let $\mathbb{F} := V/\pi V$ be the residue field. In

this article, we define and study an analytic cyclic homology theory for complete, torsion-free bornological V-algebras (see Section 2 for the definitions of these terms). For example, if R is a torsion-free, finitely generated, commutative V-algebra, then its Monsky–Washnitzer dagger completion $R^\dagger$ introduced in [20] is such a complete bornological algebra (see [7, 19]).

We prove that analytic cyclic homology is invariant under dagger homotopies and under certain nilpotent extensions, that it is matrix stable, and that it satisfies excision for semi-split extensions. We use these properties to compute the analytic cyclic homology for dagger completed Leavitt and Cohn path algebras of countable graphs. For finite graphs, we also compute the analytic cyclic homology for tensor products with such algebras. In particular, it follows that the analytic cyclic homology of the completed tensor product of R with $V[t, t^{-1}]^\dagger$ is isomorphic to the direct sum $\mathrm{HA}_*(R) \oplus \mathrm{HA}_*(R)[1]$, where HA_* denotes analytic cyclic homology. This is a variant of the fundamental theorem in algebraic K-theory.

We also compute $\mathrm{HA}_*(R^\dagger)$ for a smooth, commutative V-algebra R of relative dimension 1. Namely, it is isomorphic to the de Rham cohomology of $R^\dagger$. If $\mathbb{F}$ has finite characteristic, then this agrees with Berthelot's rigid cohomology of $R \otimes \mathbb{F}$ (see [7]). Partial results that we have for smooth, commutative V-algebras of higher dimension have not been included because we have not been able to prove that analytic and periodic cyclic homology coincide in this generality.

Monsky–Washnitzer cohomology and Berthelot's rigid cohomology are defined for varieties in finite characteristic by lifting them to characteristic zero. In order to define analogous theories for noncommutative $\mathbb{F}$-algebras, it is natural to replace de Rham cohomology by cyclic homology. Indeed, in [7], Berthelot's rigid cohomology for commutative $\mathbb{F}$-algebras is linked to the *periodic* cyclic homology of suitable dagger completed commutative V-algebras. When we allow noncommutative algebras, however, then the dagger completion process forces us to replace periodic cyclic homology by the analytic cyclic homology that is studied here.

In work in progress, we are going to use the theory defined in this article in order to define an analytic cyclic homology theory for algebras over the residue field $\mathbb{F}$. We want to prove $\mathrm{HA}_*(A) \cong \mathrm{HA}_*(R^\dagger)$ whenever R is a torsion-free V-algebra and $A \cong R/\pi R$ is its reduction to an $\mathbb{F}$-algebra. The crucial point is that this should not depend on the choice of R – and this is where we need analytic instead of periodic cyclic homology.

All theorems in this paper require the fraction field F to have characteristic 0. This is needed for the homotopy invariance of analytic cyclic homology because the proof involves integration of polynomials. Our proofs of the excision theorem and of matrix stability use characteristic 0 indirectly because they are based on homotopy invariance. Variants of periodic cyclic homology such as (negative) cyclic homology are not homotopy invariant. This is why our methods do not apply to these theories.

Several groups of authors have recently been studying cohomology theories for varieties in finite characteristic with different approaches. We mention, in particular, the work of Petrov and Vologodsky [21] that uses topological cyclic homology.

This paper is organised as follows. Some notational conventions used throughout the article are reviewed at the end of this introduction.

In Section 2, we recall some basic notions from bornological analysis and from the Cuntz–Quillen approach to cyclic homology theories. In particular, we introduce dagger completions relative to an ideal (Section 2.2) and review the appropriate notions of extension of bornological modules, noncommutative differential forms, tensor algebra, and X-complex for bornological algebras.

Section 3 introduces the analytic cyclic pro-complex $\mathbb{HA}(R)$ of a complete, torsion-free bornological algebra R. It is defined as the X-complex of the scalar extension $\mathcal{T}R \otimes_V F$ of a certain projective system $\mathcal{T}R$ of complete bornological V-algebras functorially associated to R. Hence, by definition, $\mathbb{HA}(R)$ is a pro-supercomplex (that is, a projective system of $\mathbb{Z}/2$-graded chain complexes) of complete bornological vector spaces over F. The analytic cyclic homology of R is defined as the homology of the homotopy limit of $\mathbb{HA}(R)$,

$$\mathrm{HA}_*(R) := H_*(\operatorname{holim} \mathbb{HA}(R)).$$

By definition, this is a $\mathbb{Z}/2$-graded bornological vector space over F.

The results about excision, homotopy invariance and matrix stability in this article are all about $\mathbb{HA}$ as a functor to the homotopy category of chain complexes of projective systems of complete bornological F-vector spaces. Here "homotopy category" means that we take chain homotopy classes of chain maps as arrows. It seems, however, that we must pass to a suitable derived category for results that compare $\mathbb{HA}$ for two different liftings of an algebra over the residue field $\mathbb{F}$. We do not discuss here which weak equivalences must be inverted in order to make the theory well defined for algebras over $\mathbb{F}$.

Section 4 is concerned with analytic nilpotence. Analytically nilpotent pro-algebras and analytically nilpotent extensions of algebras and pro-algebras are introduced. A pro-algebra R is called analytically quasi-free if every semi-split, analytically nilpotent extension of R splits. In particular, the analytic tensor pro-algebra $\mathcal{T}R$ (see Definition 4.4.1) is analytically quasi-free and is part of a semi-split, analytically nilpotent extension

$$\mathcal{J}R \rightarrowtail \mathcal{T}R \twoheadrightarrow R.$$

We define dagger homotopy of (pro-)algebra homomorphisms using the dagger completion $V[t]^\dagger$, and we show that any semi-split analytically nilpotent extension $N \rightarrowtail E \twoheadrightarrow R$ with analytically quasi-free E is dagger homotopy equivalent to the extension above. We use this and the invariance of the X-complex under dagger homotopies to show that $\mathbb{HA}$ is invariant under dagger homotopies. This implies that $\mathbb{HA}$ is invariant under analytically nilpotent extensions and that $\mathbb{HA}(R)$ is homotopy equivalent to $X(R \otimes F)$ if R is analytically quasi-free.

Section 5 is devoted to the proof of the Excision Theorem, which says that if

$$K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$$

is a semi-split pro-algebra extension, then there is a natural exact triangle

$$\mathbb{HA}(K) \xrightarrow{i_*} \mathbb{HA}(E) \xrightarrow{p_*} \mathbb{HA}(Q) \xrightarrow{\delta} \mathbb{HA}(K)[-1].$$

Applying the homotopy projective limit and taking homology, this implies a natural 6-term exact sequence

$$\begin{array}{ccccc} \mathrm{HA}_0(K) & \xrightarrow{i_*} & \mathrm{HA}_0(E) & \xrightarrow{p_*} & \mathrm{HA}_0(Q) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ \mathrm{HA}_1(Q) & \xleftarrow[p_*]{} & \mathrm{HA}_1(E) & \xleftarrow[i_*]{} & \mathrm{HA}_1(K). \end{array}$$

The proof of the excision theorem follows the structure of its archimedean version in [17, 18], and adapts it to the present case.
The stability of $\mathbb{HA}$ under matricial embeddings is proved in Section 6. Any pair X, Y of torsion-free bornological V-modules with a surjective bounded linear map $\langle\cdot,\cdot\rangle: Y \otimes X \to V$ gives rise to a dagger algebra $\overline{\mathcal{M}}(X, Y)$ with underlying bornological V-module $X \overline{\otimes} Y$. We show in Proposition 6.2 that $\mathbb{HA}$ is invariant under tensoring with $\overline{\mathcal{M}}(X, Y)$. For example, the algebra of finite matrices $\mathbb{M}_n$ with $n \le \infty$ and the algebra of matrices with entries going to zero at infinity are of the form $\overline{\mathcal{M}}(X, Y)$ for suitable X and Y. Thus $\mathbb{HA}$ is invariant under tensoring with such algebras. This implies that $\mathbb{HA}$ for unital algebras is functorial for certain bimodules and invariant under Morita equivalence (see Section 7).
Section 8 is concerned with Leavitt path algebras. For a directed graph E with finitely many vertices and a complete bornological algebra R, Theorems 8.1 and 8.3 compute $\mathbb{HA}(R \overline{\otimes} L(E)^\dagger)$ in terms of $\mathbb{HA}(R)$ and a matrix N_E related to the incidence matrix of E:

$$\mathbb{HA}(R \overline{\otimes} L(E)^\dagger) \simeq (\operatorname{coker}(N_E) \oplus \ker(N_E)[1]) \otimes \mathbb{HA}(R).$$

For trivial R, the homotopy equivalence

$$\mathbb{HA}(L(E)^\dagger) \simeq (\operatorname{coker}(N_E) \oplus \ker(N_E)[1])$$

is shown also for graphs with countably many vertices. If E is the graph with one vertex and one loop, it follows that $\mathbb{HA}$ satisfies a version of Bass' fundamental theorem:

$$\mathbb{HA}(R \overline{\otimes} V[t, t^{-1}]^\dagger) \simeq \mathbb{HA}(R) \oplus \mathbb{HA}(R)[-1].$$

We also compute $\mathbb{HA}(R \overline{\otimes} C(E)^\dagger)$ for the Cohn path algebra of E if E has finitely many vertices, and $\mathbb{HA}(C(E)^\dagger)$ if E has countably many vertices.
In Section 9 we show that if R is smooth commutative of relative dimension one, then the analytic cyclic homology of its dagger completion is the same as the rigid cohomology of its reduction modulo π (see Theorem 9.2.9). That is,

$$\mathrm{HA}_n(R^\dagger) \cong H^n_{\mathrm{rig}}(R/\pi R)$$

for $n = 0, 1$. We outline the idea of the proof. By [7], $H^n_{\mathrm{rig}}(R/\pi R)$ is isomorphic to the periodic cyclic homology of $R^\dagger \otimes F$. And by Corollary 4.7.2, HA and $\mathrm{HP}(\cdot \otimes F)$ agree on analytically quasi-free bornological V-algebras. It is well known that a smooth algebra R of relative dimension 1 is quasi-free in the sense that any square-zero extension of R splits or, equivalently, that the bimodule $\Omega^1(R)$ of noncommutative differential 1-forms admits a connection. We show in Theorem 9.1.9 that if R is a torsion-free, complete bornological algebra and ∇ is a connection on $\Omega^1(R)$ that satisfies an extra condition, then $R^\dagger$ is analytically quasi-free. We prove that a smooth commutative algebra of relative dimension 1 with the fine bornology admits such a connection (see Lemma 9.2.3).

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1.1 Some notation

Throughout this article, we shall use the following notation. Let $\mathbb{N}^*$ be the set of nonzero natural numbers. Let V be a complete discrete valuation ring, $\pi \in V$ a uniformiser, $\mathbb{F}$ the residue field $V/(\pi)$ of V, and F the fraction field of V. While our definitions work in complete generality, our homotopy invariance, stability and excision theorems only work if F has characteristic zero. All tensor products $\otimes$ are taken over V. By convention, algebras are allowed to be non-unital throughout this article. An *ideal* in a possibly non-unital V-algebra means a two-sided ideal that is also a V-submodule.

2 Preparations

2.1 Bornologies

As in [7], bornological V-algebras play a crucial role. We first recall some basic terminology about bornologies from [7, 19].

Definition 2.1.1. A *bornology* on a set S is a set $\mathcal{B}$ of subsets, called *bounded subsets*, such that finite unions and subsets of bounded subsets are bounded and finite subsets are bounded. A *bornological set* is a set with a bornology.

Definition 2.1.2. A map $f\colon S_1 \to S_2$ between bornological sets is *bounded* if it maps bounded subsets to bounded subsets. It is a *bornological embedding* if it is injective and $T \subseteq S_1$ is bounded if and only if $f(T) \subseteq S_2$ is bounded. It is a *bornological quotient map* if it is bounded and any bounded subset $T \subseteq S_2$ is the image of a bounded subset of S_1.

Definition 2.1.3. A *bornological V-module* is a V-module R with a bornology such that any bounded subset is contained in a bounded V-submodule or, equivalently, the V-submodule generated by a bounded subset is again bounded. A *bornological V-algebra* is a bornological V-module R with a multiplication $R \times R \to R$ that is bounded in the sense that $S \cdot T$ is bounded if $S, T \subseteq R$ are bounded.

Definition 2.1.4. A bornological V-module is *complete* if any bounded subset is contained in a bounded V-submodule that is π-adically complete. The *completion* $\overline{M}$ of a bornological V-module M is a complete bornological V-module with a bounded map $M \to \overline{M}$ that is universal in the sense that any bounded map from M to a complete bornological V-module factors uniquely through it (see [7, Definition 2.14]).

Example 2.1.5. Let M be a V-module. The *fine bornology* on M consists of those subsets of M that are contained in a finitely generated V-submodule. It is the smallest V-module bornology on M. It is the only bornology on M if M itself is finitely generated. If R is a V-algebra, then the fine bornology makes it a bornological V-algebra. The fine bornology is automatically complete.
We always equip the fraction field F with the fine bornology.

Definition 2.1.6. Let M_1 and M_2 be bornological V-modules. The *tensor product bornology* on the V-module $M_1 \otimes M_2$ consists of all subsets that are contained in $S_1 \otimes S_2$ for bounded V-submodules $S_j \subseteq M_j$ for $j = 1, 2$. The *complete bornological tensor product* $M_1 \overline{\otimes} M_2$ is defined as the bornological completion of $M_1 \otimes M_2$ with the tensor product bornology.

The universal property of tensor products easily implies the following:

Proposition 2.1.7. *Let M_1, M_2 and N be bornological V-modules. Bounded V-linear maps $M_1 \otimes M_2 \to N$ are in natural bijection with bounded V-bilinear maps $M_1 \times M_2 \to N$.*

Corollary 2.1.8. *Let M_1, M_2 and N be complete bornological V-modules. Bounded V-linear maps $M_1 \overline{\otimes} M_2 \to N$ are in natural bijection with bounded V-bilinear maps $M_1 \times M_2 \to N$.*

Example 2.1.9. Continuing Example 2.1.5, let M_1 be a V-module with the fine bornology and let M_2 be a complete bornological V-module. Then the tensor product bornology on $M_1 \otimes M_2$ is already complete because the tensor product of a π-adically complete V-module with a finitely generated V-module is complete. Thus $M_1 \overline{\otimes} M_2 = M_1 \otimes M_2$ in this case. This applies, in particular, if $M_1 = F$. If both M_1 and M_2 carry the fine bornology, then the tensor product bornology on $M_1 \overline{\otimes} M_2 = M_1 \otimes M_2$ is the fine bornology as well.

DEFINITION 2.1.10 ([19, Definition 4.1]). A bornological V-module is (*bornologically*) *torsion-free* if multiplication by π is a bornological embedding.

Remark 2.1.11. Let M be a bornological V-module. If $S \subseteq M$, then define

$$\pi^{-1}S := \{x \in M : \pi \cdot x \in S\}.$$

This depends on M and not just on S. By definition, M is torsion-free if and only if multiplication by π is injective and $\pi^{-1}S$ is bounded for all bounded subsets $S \subseteq M$.

PROPOSITION 2.1.12 ([19, Proposition 4.3]). *A bornological V-module M is torsion-free if and only if the canonical map $M \to M \otimes F$ is a bornological embedding.*

EXAMPLE 2.1.13. A V-module M with the fine bornology is torsion-free if and only if M is torsion-free in the usual sense.

DEFINITION 2.1.14. Let M be any bornological V-module and define $M_{\mathrm{tf}} \subseteq M \otimes F$ as the image of the canonical map $M \to M \otimes F$, equipped with the restriction of the bornology of $M \otimes F$.

PROPOSITION 2.1.15 ([19, Proposition 4.4]). *The canonical map $M \to M_{\mathrm{tf}}$ is the universal map from M to a torsion-free bornological V-module.*

DEFINITION 2.1.16. A bornological V-algebra R is *semi-dagger* if any bounded subset $S \subseteq R$ is contained in a bounded V-submodule $T \subseteq R$ with $\pi \cdot T \cdot T \subseteq T$ (see [19, Proposition 3.4]). Let R with the bornology $\mathcal{B}$ be a bornological V-algebra. There is a smallest semi-dagger bornology on R that contains $\mathcal{B}$. It is denoted $\mathcal{B}_{\mathrm{lg}}$ and called the *linear growth bornology* on R; we write R_{lg} for R with the linear growth bornology (see [19, Definition 3.5 and Lemma 3.6]).

DEFINITION 2.1.17. A *dagger algebra* is a bornological V-algebra that is complete, (bornologically) torsion-free, and semi-dagger. The *dagger completion* of a bornological V-algebra R is a dagger algebra $R^\dagger$ with a bounded V-algebra homomorphism $R \to R^\dagger$ that is universal in the sense that any bounded homomorphism from R to a dagger algebra factors uniquely through it.

THEOREM 2.1.18 ([19, Theorem 5.3]). *If R is already torsion-free, then $R^\dagger$ is the completion of R_{lg}. In general, it is the completion of $(R_{\mathrm{tf}})_{\mathrm{lg}}$.*

EXAMPLE 2.1.19. The dagger completion $R^\dagger$ of a torsion-free, finitely generated, commutative V-algebra is usually defined as the weak completion of R by Monsky and Washnitzer [20]. This agrees with our definition of $R^\dagger$ by [7, Theorem 3.2.1]: the dagger completion of R with the fine bornology is naturally isomorphic to the weak completion of R, equipped with a canonical bornology.

PROPOSITION 2.1.20 ([7, Proposition 3.1.25]). *Let A and B be torsion-free, complete bornological algebras. Then $(A \otimes B)_{\mathrm{lg}} \cong A_{\mathrm{lg}} \otimes B_{\mathrm{lg}}$ and $(A \otimes B)^\dagger \cong A^\dagger \mathbin{\overline{\otimes}} B^\dagger$.*

Corollary 2.1.21. *A completed tensor product of two dagger algebras is again a dagger algebra.*

Proof. A completed tensor product is complete by definition. It remains semi-dagger by Proposition 2.1.20, and torsion-free by [19, Proposition 4.12]. ☐

2.2 Relative dagger completions

We shall define analytic cyclic homology for torsion-free, complete bornological V-algebras R that need not be dagger algebras. This uses a variant of the linear growth bornology relative to an ideal.

Let R be a V-algebra and let M and N be V-submodules of R. Let $MN \subseteq R$ be the V-submodule generated by all products xy with $x \in M$ and $y \in N$. Let

$$M^\diamond := \sum_{i=0}^{\infty} \pi^i M^{i+1}. \tag{2.2.1}$$

A subset of R has linear growth if and only if it is contained in $M^\diamond$ for some bounded V-submodule M of R; with the present definitions, this is [19, Lemma 3.6].

Lemma 2.2.2. *Let R be a V-algebra and let $M, N \subseteq R$ be V-submodules. Then*

(1) $M^\diamond + N^\diamond \subseteq (M + N)^\diamond$;

(2) $M \cdot N^\diamond \subseteq (M \cdot N + N)^\diamond$ *and* $N^\diamond \cdot M \subseteq (N \cdot M + N)^\diamond$;

(3) $\pi \cdot M^\diamond \cdot M^\diamond \subseteq M^\diamond$;

(4) $M^\diamond \cdot N^\diamond \subseteq (M + N + MN)^\diamond$;

(5) $(M^\diamond)^\diamond = M^\diamond$.

Proof. The definition of $M^\diamond$ immediately implies (1).

The following computation shows (4):

$$M^\diamond \cdot N^\diamond = \sum_{i,j=0}^{\infty} \pi^{i+j} M^i (MN) N^j \subseteq \sum_{i,j=0}^{\infty} \pi^{i+j} (M + N + MN)^{i+j+1}$$

Similar calculations give (2).

And (3) follows from $\pi \cdot \pi^i M^{i+1} \cdot \pi^j M^{j+1} = \pi^{i+j+1} M^{(i+j+1)+1}$ for all $i, j \in \mathbb{N}$. Then $\pi^i (M^\diamond)^{i+1} \subseteq M^\diamond$ follows by induction on i. This implies (5). ☐

Definition 2.2.3. Let R be a bornological V-algebra and $I \triangleleft R$ an ideal. Let $\mathcal{B}_{\mathrm{lg}(I)}$ be the set of all subsets of R that are contained in $M + N^\diamond$ for bounded V-submodules $M \subseteq R$ and $N \subseteq I$. This is a bornology on R, called the *linear growth bornology relative to I*. Let $R_{\mathrm{lg}(I)}$ be R with this bornology.

Example 2.2.4. By definition, $\mathcal{B}_{\mathrm{lg}(0)} = \mathcal{B}$ and $\mathcal{B}_{\mathrm{lg}(R)}$ is the usual linear growth bornology on R. So $R_{\mathrm{lg}(0)} = R$ and $R_{\mathrm{lg}(R)} = R_{\mathrm{lg}}$.

LEMMA 2.2.5. *The bornology $\mathcal{B}_{\mathrm{lg}(I)}$ is an algebra bornology, and its restriction to I is semi-dagger. Let S be a bornological V-algebra. A homomorphism $f: R \to S$ is bounded for the bornology $\mathcal{B}_{\mathrm{lg}(I)}$ if and only if $f(N)$ has linear growth in S for all bounded subsets $N \subseteq I$ and $f(M)$ is bounded in S for all bounded subsets $M \subseteq R$.*

Proof. Since I is an ideal, Lemma 2.2.2 implies that $\mathcal{B}_{\mathrm{lg}(I)}$ makes R a bornological V-algebra. And a subset of I belongs to $\mathcal{B}_{\mathrm{lg}(I)}$ if and only if it is contained in $N^\diamond$ for some bounded V-submodule $N \subseteq I$. The restriction of $\mathcal{B}_{\mathrm{lg}(I)}$ to I is semi-dagger by Lemma 2.2.2. If M and N are as in Definition 2.2.3, then $f(M + N^\diamond) = f(M) + f(N)^\diamond$. This is bounded in S if and only if $f(M)$ is bounded and $f(N)$ has linear growth. □

LEMMA 2.2.6. *Let R be a bornological algebra and let I and J be ideals in R with $I \subseteq J$ and $R/I = (R/I)_{\mathrm{lg}(J/I)}$. Then $R_{\mathrm{lg}(J)} = R_{\mathrm{lg}(I)}$. In particular, if R/I is semi-dagger, then $R_{\mathrm{lg}(I)} = R_{\mathrm{lg}}$.*

Proof. By Lemma 2.2.5, the bornology $\mathcal{B}_{\mathrm{lg}(J)}$ on R is the smallest one that contains the given bornology and makes J semi-dagger, and similarly for I. And the assumption $R/I = (R/I)_{\mathrm{lg}(J/I)}$ says that $J/I \subseteq R/I$ is semi-dagger in the quotient bornology on R/I. This is the same as the quotient bornology induced by $\mathcal{B}_{\mathrm{lg}(I)}$. [19, Theorem 3.7] says that an extension of semi-dagger algebras remains semi-dagger. This theorem applied to the extension $I \rightarrowtail J \twoheadrightarrow J/I$, equipped with the restrictions of the bornology $\mathcal{B}_{\mathrm{lg}(I)}$ on I and J and the resulting quotient bornology on J/I shows that J is semi-dagger also in the bornology $\mathcal{B}_{\mathrm{lg}(I)}$. Then $\mathcal{B}_{\mathrm{lg}(J)} \subseteq \mathcal{B}_{\mathrm{lg}(I)}$. And $\mathcal{B}_{\mathrm{lg}(I)} \subseteq \mathcal{B}_{\mathrm{lg}(J)}$ is trivial. □

LEMMA 2.2.7. *Let R be a bornological algebra and $I \triangleleft R$ an ideal. If R is torsion-free, then so is $R_{\mathrm{lg}(I)}$.*

Proof. Let $S \subseteq \pi R$ be a bounded subset in $R_{\mathrm{lg}(I)}$. By definition, there are bounded submodules $M \subseteq R$ and $N \subseteq I$ with $S \subseteq M + N^\diamond$. And

$$M + N^\diamond = M + N + \sum_{i=1}^{\infty} \pi^i N^{i+1} = M + N + \pi \cdot \left(\sum_{i=0}^{\infty} \pi^i N^{i+2} \right).$$

Since $\pi^i N^{i+2} \subseteq \pi^i (N + N^2)^{i+1}$ for all $i \geq 0$, the subset $\sum_{i=0}^{\infty} \pi^i N^{i+2}$ belongs to $\mathcal{B}_{\mathrm{lg}(I)}$. Since $M + N$ is bounded in R and R is torsion-free, $\pi^{-1} \cdot (M + N)$ is bounded. Then

$$\pi^{-1} S \subseteq \pi^{-1}(M + N^\diamond) \subseteq \pi^{-1}(M + N) + \sum_{i=0}^{\infty} \pi^i N^{i+2} \in \mathcal{B}_{\mathrm{lg}(I)}. \qquad \square$$

DEFINITION 2.2.8. Let R be a torsion-free bornological algebra and $I \triangleleft R$ an ideal. The *dagger completion of R relative to I* is the completion

$$(R, I)^\dagger := \overline{R_{\mathrm{lg}(I)}}.$$

We shall never apply (relative) dagger completions when R is not already bornologically torsion-free. In general, the correct definition of the relative dagger completion of (R, I) would be $(R_{\mathrm{tf}}, I_{\mathrm{tf}})^{\dagger}$, where I_{tf} is identified with its image in R_{tf} (compare Theorem 2.1.18).

Proposition 2.2.9. *Let R and S be torsion-free bornological V-algebras, $I \subseteq R$ an ideal, and $f\colon R \to S$ a bounded algebra homomorphism. Assume S to be complete. There is a bounded algebra homomorphism $(R,I)^{\dagger} \to S$ extending f – necessarily unique – if and only if $f(M)$ has linear growth for each bounded V-submodule M of I.*

Proof. Use Lemma 2.2.5 and the universal property of the completion. □

We know no analogue of Proposition 2.1.20 for relative dagger completions.

2.3 Extensions of bornological modules

An *extension* of V-modules is a diagram of V-modules

$$K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q \tag{2.3.1}$$

that is algebraically exact and such that i is a bornological embedding and p is a bornological quotient map. Equivalently, i is a kernel of p and p is a cokernel of i in the additive category of bornological V-modules. The following elementary lemma says that this category is *quasi-abelian* (see [24]):

Lemma 2.3.2. *Let (2.3.1) be an extension of bornological V-modules and let $f\colon K \to K'$ and $g\colon Q'' \to Q$ be bounded V-module maps. The pushout of i, f and the pullback of p, g exist and are part of morphisms of extensions*

$$\begin{array}{ccccc} K & \overset{i}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & Q \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{f}} & & \| \\ K' & \overset{i'}{\rightarrowtail} & E' & \overset{p'}{\twoheadrightarrow} & Q, \end{array} \qquad \begin{array}{ccccc} K & \overset{i''}{\rightarrowtail} & E'' & \overset{p''}{\twoheadrightarrow} & Q'' \\ \| & & \downarrow{\scriptstyle \hat{g}} & & \downarrow{\scriptstyle g} \\ K & \overset{i}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & Q. \end{array}$$

Here

$$E' := \frac{K' \oplus E}{\{(f(k), -i(k)) : k \in K\}}, \qquad E'' := \{(e, q'') \in E \times Q'' : p(e) = g(q'')\},$$

equipped with the quotient and the subspace bornology, respectively, and $\hat{f}(e) = [(0,e)]$, $i'(k') = [(k',0)]$, $p'[(k',e)] = p(e)$, $\hat{g}(e,q'') = e$, $p''(e,q'') = q''$, and $i''(k) = (i(k),0)$ for $e \in E$, $k' \in K'$, $q'' \in Q''$, $k \in K$.

The following proposition is an analogue of Lemma 2.2.6 for completions, describing a situation when a partial completion relative to a submodule is equal to the completion.

Proposition 2.3.3. *Assume Q in an extension (2.3.1) of bornological V-modules to be complete and bornologically torsion-free. Form the pushout diagram*

$$\begin{array}{ccccc} K & \overset{i}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & Q \\ \downarrow{\scriptstyle \mathrm{can}_K} & & \downarrow{\scriptstyle \gamma} & & \| \\ \overline{K} & \overset{i'}{\rightarrowtail} & E' & \overset{p'}{\twoheadrightarrow} & Q. \end{array}$$

Then there is a unique isomorphism $\varphi: E' \xrightarrow{\cong} \overline{E}$ such that $\varphi \circ \gamma$ is the canonical map $E \to \overline{E}$.

Proof. The bottom row is an extension by Lemma 2.3.2. Then E' is complete by [19, Theorem 2.3]. The maps $\mathrm{can}_E: E \to \overline{E}$ and $\overline{i}: \overline{K} \to \overline{E}$ induce a bounded V-module map $\varphi: E' \to \overline{E}$ by the universal property of pushouts. Since E' is complete, the universal property of $\overline{E}$ gives a unique map $\psi: \overline{E} \to E'$ with $\psi \circ \mathrm{can}_E = \gamma$. Then $\varphi \circ \psi \circ \mathrm{can}_E = \varphi \circ \gamma = \mathrm{can}_E$. This implies $\varphi \circ \psi = \mathrm{id}_{\overline{E}}$. Next, $\psi \circ \overline{i} \circ \mathrm{can}_K = \gamma \circ i = i' \circ \mathrm{can}_K$ implies $\psi \circ \overline{i} = i'$, and then $\psi \circ \varphi \circ i' = \psi \circ \overline{i} = i'$ and $\psi \circ \varphi \circ \gamma = \psi \circ \mathrm{can}_E = \gamma$ imply $\psi \circ \varphi = \mathrm{id}_{E'}$. So φ is an isomorphism. □

2.4 Injective maps between completions

Unlike in the archimedean case, all Banach spaces over F have a simple structure. This implies that they all satisfy a variant of Grothendieck's Approximation Property. This is Proposition 2.4.5, and it will be useful to describe completions of tensor products.

Definition 2.4.1. Let D be a set. Let $\mathrm{C}_0(D,V)$ be the V-module of all functions $f: D \to V$ such that for each $\delta > 0$ there is a finite subset $S \subseteq D$ with $|f(x)| < \delta$ for all $x \in D \setminus S$. Define $\mathrm{C}_0(D,F)$ similarly. Equip $\mathrm{C}_0(D,V)$ and $\mathrm{C}_0(D,F)$ with the supremum norm.

Theorem 2.4.2. *Any π-adically complete, torsion-free V-module M is isomorphic to $\mathrm{C}_0(D,V)$ for some set D.*

Proof. The map $M \to M \otimes F$ is an embedding because M is torsion-free. Define the *gauge norm* on $F \cdot M$ by

$$\|x\| := \inf\{|\pi|^j : \pi^{-j} \cdot x \in M\}.$$

It is a nonarchimedean norm and makes $F \cdot M$ a Banach F-vector space with unit ball M. It takes values in $\{|\pi|^n : n \in \mathbb{Z}\} \cup \{0\}$ by construction. Hence there is a set D and an isometric isomorphism $FM \cong \mathrm{C}_0(D,F)$ (see [23, Remark 10.2]). It maps M isomorphically onto the unit ball $\mathrm{C}_0(D,V)$ of $\mathrm{C}_0(D,F)$. □

Corollary 2.4.3. *Any complete, torsion-free bornological V-module W is isomorphic to the colimit of an inductive system of complete V-modules of the form $(\mathrm{C}_0(D_n,V), f_{n,m})_{n,m \in S}$ with a directed set $(S, \leq)$, sets D_n for $n \in S$, and injective, bounded V-linear maps $f_{n,m}: \mathrm{C}_0(D_m,V) \hookrightarrow \mathrm{C}_0(D_n,V)$ for $n,m \in S$, $n \geq m$.*

Proof. The complete V-submodules of W form a directed set under inclusion. By [7, Proposition 2.10], this is an inductive system with injective structure maps and with colimit W. Each complete V-submodule of W is π-adically complete and torsion-free. Then it is isomorphic to $\mathrm{C}_0(D,V)$ for some set D by Theorem 2.4.2. □

Lemma 2.4.4. *Let $f:\mathrm{C}_0(D_1,V)\hookrightarrow \mathrm{C}_0(D_2,V)$ and $g:\mathrm{C}_0(D_3,V)\hookrightarrow \mathrm{C}_0(D_4,V)$ be injective, bounded V-linear maps. Then the induced bounded map*

$$f\mathbin{\widehat{\otimes}} g:\mathrm{C}_0(D_1,V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(D_3,V)\to \mathrm{C}_0(D_2,V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(D_4,V)$$

is injective as well. And here $\mathrm{C}_0(D_m,V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(D_n,V)\cong \mathrm{C}_0(D_m\times D_n,V)$.

Proof. The universal property of the complete bornological tensor product implies that $\mathrm{C}_0(D_1,V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(D_3,V)\cong \mathrm{C}_0(D_1\times D_3,V)$ for all sets D_1 and D_3. Define $\mathrm{C}_0(D_1,\mathrm{C}_0(D_3,V))$ to be the space of all functions $f:D_1\to \mathrm{C}_0(D_3,V)$ for which the gauge norm $\|f\|$ vanishes at ∞. There is a canonical isomorphism

$$\mathrm{C}_0(D_1\times D_3,V)\xrightarrow{\cong}\mathrm{C}_0(D_1,\mathrm{C}_0(D_3,V)),\qquad f\mapsto (s\mapsto f(s,\cdot)).$$

Similarly, $\mathrm{C}_0(D_1\times D_3,V)\cong \mathrm{C}_0(D_3,\mathrm{C}_0(D_1,V))$. Now we factorise the map $f\mathbin{\overset{\frown}{\otimes}} g$ as

$$\begin{aligned}\mathrm{C}_0(D_1,V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(D_3,V)&\cong \mathrm{C}_0(D_1\times D_3,V)\cong \mathrm{C}_0(D_1,\mathrm{C}_0(D_3,V))\\ &\overset{g_*}{\hookrightarrow}\mathrm{C}_0(D_1,\mathrm{C}_0(D_4,V))\cong \mathrm{C}_0(D_4,\mathrm{C}_0(D_1,V))\\ &\overset{f_*}{\hookrightarrow}\mathrm{C}_0(D_4,\mathrm{C}_0(D_2,V))\cong \mathrm{C}_0(D_2\times D_4,V)\cong \mathrm{C}_0(D_2,V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(D_4,V);\end{aligned}$$

here the maps f_* and g_* are injective because f and g are injective. □

Proposition 2.4.5. *Let M_1, W_1, M_2 and W_2 be complete, torsion-free bornological V-modules and let $\varphi_j:M_j\hookrightarrow W_j$ for $j=1,2$ be injective bounded V-module maps. Then $\varphi_1\mathbin{\overset{\frown}{\otimes}}\varphi_2:M_1\mathbin{\overset{\frown}{\otimes}}M_2\to W_1\mathbin{\overset{\frown}{\otimes}}W_2$ is injective.*

Proof. Write W_1 and W_2 as inductive limits as in Corollary 2.4.3. Then $W_1\otimes W_2$ is naturally isomorphic to the inductive limit of the inductive system defined by the maps

$$f_{1,n_1,m_1}\otimes f_{2,n_2,m_2}:\mathrm{C}_0(D_{n_1},V)\otimes \mathrm{C}_0(J_{n_2},V)\to \mathrm{C}_0(D_{m_1},V)\otimes \mathrm{C}_0(J_{m_2},V),$$

and $W_1\mathbin{\overset{\frown}{\otimes}}W_2$ is naturally isomorphic to the inductive limit of the inductive system defined by the maps

$$f_{1,n_1,m_1}\mathbin{\widehat{\otimes}} f_{2,n_2,m_2}:\mathrm{C}_0(D_{n_1},V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(J_{n_2},V)\to \mathrm{C}_0(D_{m_1},V)\mathbin{\widehat{\otimes}}\mathrm{C}_0(J_{m_2},V).$$

All these bounded maps are injective by Lemma 2.4.4. Therefore, the tensor product is isomorphic to an ordinary union of these V-modules, equipped with the bornology cofinally generated by these V-submodules. The tensor products $M_1\otimes M_2$ and $M_1\mathbin{\overset{\frown}{\otimes}}M_2$ are described similarly, and the maps φ_1 and φ_2 are described by injective maps between the entries of the appropriate inductive systems. Then Lemma 2.4.4 shows that $\varphi_1\mathbin{\overset{\frown}{\otimes}}\varphi_2$ is injective. □

2.5 The bimodule of differential 1-forms

We are going to define the (complete) bimodule $\overline{\Omega}^1(A)$ of *noncommutative differential* 1*-forms* over a complete bornological V-algebra A.
For a unital algebra in the usual sense, $\Omega^1(A)$ is defined in [10, Section 1] as $A \otimes (A/\mathbb{C}\cdot 1)$ with a certain bimodule structure. Its elements are denoted by $a\,\mathrm{d}b$ with $a \in A$, $b \in A/\mathbb{C}\cdot 1$. We shall use the version for non-unital algebras, which uses the unitalisation A^+ instead of A. This is $A^+ := A \oplus V$ with the multiplication

$$(x,\lambda)\cdot(y,\mu) := (xy + \mu x + \lambda y, \lambda\mu)$$

for $x, y \in A$, $\lambda, \mu \in V$. So $(0,1)$ is the unit element in A^+, which we denote simply by 1. The inclusion map $A \to A^+$ is the universal bounded homomorphism from A to a unital bornological algebra.
It is clear from the definition that the map $\Omega^1(A) \to A^+ \otimes A^+$, $a\,\mathrm{d}b \mapsto a\otimes b - ab\otimes 1$, is an isomorphism onto the kernel of the multiplication map $A^+ \otimes A^+ \to A^+$. In [18, Appendix A.3], $\Omega^1(A)$ is defined as this kernel when A is an algebra in an additive monoidal category. This definition applies in our setting, using the tensor product $\overline{\otimes}$. By definition, $\Omega^1(A) \subseteq A^+ \mathbin{\overline{\otimes}} A^+$ is a complete bornological A-bimodule. The map

$$\mathrm{d}\colon A \to \overline{\Omega}^1(A), \qquad \mathrm{d}(x) := 1 \otimes x - x \otimes 1,$$

is the universal bounded derivation into a complete A-bimodule, that is, any bounded derivation $\partial\colon A \to M$ into a complete A-bimodule factors uniquely through d. Namely, there is a unique bounded bimodule homomorphism $\overline{\Omega}^1(A) \to M$, $a_0\,\mathrm{d}a_1 \mapsto a_0 \cdot \partial(a_1)$. This factorisation exists because there are bornological isomorphisms

$$\begin{aligned} A^+ \mathbin{\overline{\otimes}} A &\to \overline{\Omega}^1(A), &\qquad x\otimes y &\mapsto x\,\mathrm{d}y,\\ A \mathbin{\overline{\otimes}} A^+ &\to \overline{\Omega}^1(A), &\qquad x\otimes y &\mapsto (\mathrm{d}x)\cdot y = \mathrm{d}(x\cdot y) - x\,\mathrm{d}y. \end{aligned}$$

The first one is left and the second one right A-linear.
We now relate $\overline{\Omega}^1(A)$ to sections of semi-split, square-zero extensions of A (see [18, Theorem A.53] or [10, Proposition 3.3]). Let M be a complete bornological A-bimodule. Give $A \oplus M$ the multiplication

$$(a_1, m_1)\cdot(a_2, m_2) := (a_1\cdot a_2, a_1\cdot m_2 + m_1\cdot a_2).$$

The inclusion $M \rightarrowtail A \oplus M$ and the projection $A \oplus M \twoheadrightarrow A$ form a square-zero extension that splits by the inclusion homomorphism $A \hookrightarrow A \oplus M$.

Lemma 2.5.1. *Let A be a complete bornological algebra and let M be a complete bornological A-bimodule. There is a natural bijection between bounded bimodule homomorphisms $\overline{\Omega}^1(A) \to M$ and bounded V-algebra homomorphisms $A \to A \oplus M$ that split the extension $M \rightarrowtail A \oplus M \twoheadrightarrow A$.*

Proof. Any bounded linear section $s: A \to A \oplus M$ has the form $a \mapsto (a, \partial(m))$ for a bounded linear map $\partial: A \to M$. And s is multiplicative if and only if ∂ is a derivation. Bounded bimodule maps $\overline{\Omega}^1(A) \to M$ are in bijection with bounded derivations. □

We shall also apply the definition and the lemma above to incomplete bornological algebras, where we define $\Omega^1(A)$ by leaving out the completions in the construction above. And we shall use a variant of $\Omega^1(A)$ for projective systems of algebras. In general, the definition and the lemma above carry over to algebras in any additive monoidal category.

2.6 Tensor algebras and noncommutative differential forms

We describe the tensor algebra of a bornological V-module and the algebra of differential forms over a bornological algebra and relate the two. All this goes back to Cuntz and Quillen [10]. Their constructions make sense in any additive monoidal category with countable direct sums, and we specialise this generalisation of their constructions to bornological V-modules and to complete bornological V-modules. We shall mainly use the incomplete versions below because we are going to modify tensor algebras further before completing them. Let W be a bornological V-module. Equip $W^{\otimes n}$ for $n \geq 1$ with the tensor product bornology and $\mathsf{T}W := \bigoplus_{n\geq 1} W^{\otimes n}$ with the direct sum bornology; that is, a subset M of $\mathsf{T}W$ is bounded if and only if it is contained in the image of $\bigoplus_{j=1}^{n} N^{\otimes j}$ for some $n \geq 1$ and some bounded submodule $N \subseteq W$. The multiplication $\mathsf{T}W \times \mathsf{T}W \to \mathsf{T}W$ defined by

$$(x_1 \otimes \cdots \otimes x_n) \cdot (x_{n+1} \otimes \cdots \otimes x_{n+m}) := x_1 \otimes \cdots \otimes x_{n+m}$$

makes $\mathsf{T}W$ a bornological algebra, called the *tensor algebra* of W. Let $\sigma_W: W \to \mathsf{T}W$ be the inclusion of the first summand. It is a bounded V-module homomorphism, but not an algebra homomorphism.

Lemma 2.6.1. *The map $\sigma_W: W \to \mathsf{T}W$ is the universal bounded V-module map from W to a bornological algebra. That is, $\mathsf{T}W$ is a bornological V-algebra and if $f: W \to S$ is a bounded V-module map to a bornological V-algebra S, then there is a unique bounded algebra homomorphism $f^\#: \mathsf{T}W \to S$ with $f^\# \circ \sigma_W = f$.*

Proof. The multiplication above is well defined and bounded by the universal property of the bornological tensor product. Let $f: W \to S$ be a bounded V-module map. Then there is a unique bounded V-module map $f^\#: \mathsf{T}W \to S$ with

$$f^\#(x_1 \otimes \cdots \otimes x_n) := f(x_1) \cdots f(x_n)$$

for all $x_1, \ldots, x_n \in W$. This is a bounded algebra homomorphism. And it is the unique one with $f^\# \circ \sigma_W = f$. □

Let W be a complete bornological V-module. The completion of $\mathsf{T}W$ is

$$\overline{\mathsf{T}}W \coloneqq \bigoplus_{n\geq 1} W^{\overline{\otimes} n},$$

the direct sum of the completed tensor products, equipped with the direct sum bornology. By the universal property of completions, the canonical arrow $\overline{\sigma_W}: W \to \overline{\mathsf{T}}W$ is the universal bounded V-module map from W to a complete bornological algebra. That is, $\overline{\mathsf{T}}W$ is a complete bornological V-algebra and if $f: W \to S$ is a bounded V-module map to a complete bornological V-algebra S, then there is a unique bounded algebra homomorphism $f^{\#}: \overline{\mathsf{T}}W \to S$ with $f^{\#} \circ \overline{\sigma_W} = f$.

Remark 2.6.2. If W is torsion-free, then so is $\mathsf{T}W$. If W is complete and torsion-free, then so is $\overline{\mathsf{T}}W$. This uses [19, Theorem 4.6 and Proposition 4.12] and that completeness and torsion-freeness are hereditary for direct sums.

Let R be a bornological V-algebra. Then so is $\mathsf{T}R$. The identity map on R induces a bounded homomorphism $p \coloneqq \mathrm{id}_R^{\#}: \mathsf{T}R \to R$ by Lemma 2.6.1. Let

$$\mathsf{J}R \coloneqq \ker(p: \mathsf{T}R \twoheadrightarrow R). \tag{2.6.3}$$

This is a closed two-sided ideal in $\mathsf{T}R$. The inclusion $\mathsf{J}R \rightarrowtail \mathsf{T}R$ and the projection $p: \mathsf{T}R \twoheadrightarrow R$ form an extension of bornological V-algebras, which splits by the bounded V-module map $\sigma_R: R \to \mathsf{T}R$. Similarly, if R is a complete bornological V-algebra, then there is an extension of complete bornological V-algebras

$$\overline{\mathsf{J}}R \rightarrowtail \overline{\mathsf{T}}R \twoheadrightarrow R$$

that splits by the bounded V-module map $\overline{\sigma_R}$.

We are going to rewrite the tensor algebra using the Fedosov product on the algebra of noncommutative differential forms, following Cuntz and Quillen [10]. This alternative picture is important because it allows to describe the ideal $\mathsf{J}R$ and the tube algebras that we shall need. It is sketched in [18, Appendix A.3–4] why all this continues to work for algebras in additive monoidal categories. This observation goes back further to [9].

Let $\Omega^0 R \coloneqq R$ and, for $n \geq 1$, let $\Omega^n R \coloneqq R^+ \otimes R^{\otimes n}$, equipped with the tensor product bornology. That is, a submodule $N \subseteq \Omega^n R$ is bounded if and only if there is a bounded submodule $M \subseteq R$ such that N is contained in the image of $\Omega^n M = M^+ \otimes M^{\otimes n}$. Let $\Omega R \coloneqq \bigoplus_{n\geq 0} \Omega^n R$, equipped with the direct sum bornology. We interpret an element $x_0 \otimes x_1 \otimes \dots \otimes x_n \in \Omega^n R$ as a noncommutative differential form $x_0\, \mathrm{d}x_1 \dots \mathrm{d}x_n$. There is a unique structure of differential graded algebra on ΩR whose multiplication restricts to the given multiplication on $R = \Omega^0 R$ and whose differential satisfies

$$\mathrm{d}(x_0\, \mathrm{d}x_1 \dots \mathrm{d}x_n) \coloneqq 1 \cdot \mathrm{d}x_0\, \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Namely, the (graded) Leibniz rule dictates that

$$x_0\,\mathrm{d}x_1\dots\mathrm{d}x_n\cdot x_{n+1}\,\mathrm{d}x_{n+2}\dots\mathrm{d}x_{n+m} \\ \coloneqq \sum_{j=0}^{n}(-1)^{n-j}x_0\,\mathrm{d}x_1\dots\mathrm{d}(x_j\cdot x_{j+1})\dots\mathrm{d}x_{n+m}.$$

The *Fedosov product* on a differential graded algebra such as ΩR is defined by

$$\xi\odot\eta \coloneqq \xi\eta-(-1)^{i\cdot j}d(\xi)d(\eta)\qquad\text{for } \xi\in\Omega^i R,\ \eta\in\Omega^j R. \tag{2.6.4}$$

If $p,q\geq 0$ and $M,N\subseteq R$ are bounded V-submodules, then

$$\Omega^p M\odot\Omega^q N\subseteq\Omega^{p+q}(M+N+MN+M^2)\oplus\Omega^{p+q+2}(M+N). \tag{2.6.5}$$

Hence $(\Omega R,\odot)$ is a bornological algebra. Its completion $\overline{\Omega} R$ is the bornological direct sum $\bigoplus_{n\geq 0}\overline{\Omega}^n R$ of the completed differential forms. Let $\Omega^{\mathrm{ev}}R\subseteq\Omega R$ be the bornological subalgebra of differential forms of even degree. In the following, we always equip $\Omega^{\mathrm{ev}}R$ with the Fedosov product.
The inclusion map $R=\Omega^0 R\hookrightarrow\Omega^{\mathrm{ev}}R$ induces a bounded homomorphism

$$\mathsf{T}R\to\Omega^{\mathrm{ev}}R,\qquad x_1\otimes\dots\otimes x_n\mapsto x_1\odot\dots\odot x_n, \tag{2.6.6}$$

by Lemma 2.6.1, which is, in fact, a bornological isomorphism. To understand why, let $f: R\to S$ be a V-module map. Its *curvature* is the V-module map

$$\omega_f: R\otimes R\to S,\qquad \omega_f(x,y)=f(x\cdot y)-f(x)\cdot f(y).$$

It is bounded if f is. The composite of the induced homomorphism $f^{\#}:\mathsf{T}R\to S$ with the inverse of the map in (2.6.6) must be given by the formula

$$f^{\#}(x_0\,\mathrm{d}x_1\dots\mathrm{d}x_{2n})=f(x_0)\cdot\omega_f(x_1,x_2)\cdots\omega_f(x_{2n-1},x_{2n}) \tag{2.6.7}$$

because the inclusion map $R\to\Omega^{\mathrm{ev}}R$ has the curvature $(x,y)\mapsto x\cdot y-x\odot y=\mathrm{d}x\,\mathrm{d}y$. Indeed, this defines a bounded homomorphism $f^{\#}:\Omega^{\mathrm{ev}}R\to S$. So $\Omega^{\mathrm{ev}}R$ enjoys the same universal property as $\mathsf{T}R$. Then the map in (2.6.6) is a bornological isomorphism.
The map $p:\mathsf{T}R\to R$ corresponds to the map $p:\Omega^{\mathrm{ev}}R\to R$ that vanishes on $\Omega^{2n}R$ for $n\geq 1$ and is the identity on $\Omega^0 R=R$. Therefore, the isomorphism $\mathsf{T}R\cong\Omega^{\mathrm{ev}}R$ maps $\mathsf{J}R$ onto $\bigoplus_{n\geq 1}\Omega^{2n}R$. Then it follows by induction that the isomorphism maps the ideal $\mathsf{J}R^m$ onto $\bigoplus_{n\geq m}\Omega^{2n}R$. This simple description of all the powers $\mathsf{J}R^m$ is the main point of rewriting the tensor algebra using the Fedosov product on the even-degree differential forms.

Remark 2.6.8. The map $\mathsf{J}R^{\otimes m}\to\mathsf{J}R^m$ splits by the bounded V-module map

$$a_0\,\mathrm{d}a_1\dots\mathrm{d}a_{2(m+n)}\mapsto a_0\,\mathrm{d}a_1\,\mathrm{d}a_2\otimes\mathrm{d}a_3\,\mathrm{d}a_4\otimes\dots\otimes\mathrm{d}a_{2m-3}\,\mathrm{d}a_{2m-2}\otimes\mathrm{d}a_{2m-1}\dots\mathrm{d}a_{2n}.$$

Thus $\mathsf{J}R^{\otimes m}\to\mathsf{J}R^m$ is a quotient map, and the same is true upon completion.

2.7 The X-complex

The X-complex introduced by Cuntz and Quillen in [11] is an important ingredient in their approach to cyclic homology theories. It is defined for algebras in additive monoidal categories (see also [18, Appendix A.6]). We shall specialise this definition to the additive monoidal category of complete bornological algebras over F or V.

Let $\overline{\Omega}^1(S)/[,]$ be the commutator quotient of $\overline{\Omega}^1(S)$, that is, the quotient of $\overline{\Omega}^1(S)$ by the closure of the image of

$$S \mathbin{\overline{\otimes}} \overline{\Omega}^1(S) \to \overline{\Omega}^1(S), \qquad x \otimes \omega \mapsto x \cdot \omega - \omega \cdot x.$$

With the quotient bornology, this is a complete bornological V-module (see [19, Theorem 2.3]). The closure comes in because we take a cokernel in the category of complete bornological V-modules, which forces us to make the quotient separated.

Let $q: \overline{\Omega}^1(S) \to \overline{\Omega}^1(S)/[,]$ be the quotient map. There is a unique bounded linear map $b: \overline{\Omega}^1(S) \to S$ that satisfies $b(x\,\mathrm{d}y) = x \cdot y - y \cdot x$. It descends to a bounded linear map $\tilde{b}: \overline{\Omega}^1(S)/[,] \to S$. The X-complex of S is the following $\mathbb{Z}/2$-graded chain complex of complete bornological V-modules:

$$X(S) := \Big(S \underset{\tilde{b}}{\overset{q\circ \mathrm{d}}{\rightleftarrows}} \overline{\Omega}^1(S)/[\cdot,\cdot] \Big).$$

We briefly call $\mathbb{Z}/2$-graded chain complexes *supercomplexes*. If S is a complete bornological F-algebra, then $X(S)$ is a supercomplex of complete bornological F-vector spaces.

3 Definition of analytic cyclic homology

Let A be a torsion-free, complete bornological V-algebra. We are going to define the analytic cyclic homology of A. The idea is to make a universal "analytically nilpotent" extension of A and then take the X-complex of that, tensored with F to ensure its homotopy invariance. (The concept of analytic nilpotence will be introduced later in Section 4.3.) The starting point is the tensor algebra extension, which is the universal extension with a bounded linear section. To make the kernel of this extension nilpotent mod π, we pass to a tube algebra. Then we dagger complete this kernel to make it analytically nilpotent. The tube algebra construction produces a projective system of algebras. Tensoring with F and taking the X-complex, we thus get a projective systems of chain complexes. We could define analytic cyclic homology as an invariant in a suitable derived category of such chain complexes. Our main theorems hold in that setting. We prefer, however, to define it as an ordinary F-vector space. Therefore, we also apply the homotopy projective limit and take homology in the very end.

Now we go through the construction in small steps. In the *first step*, let

$$R := \mathsf{T}A, \qquad I := \mathsf{J}A,$$

be the tensor algebra over A and the kernel of the canonical homomorphism $\mathsf{T}A \twoheadrightarrow A$.

The *second step* enlarges R to a projective system of tube algebras relative to powers of the ideal I:

Definition 3.1. Let R be a torsion-free bornological V-algebra and I an ideal in R. Let I^j for $j \in \mathbb{N}^*$ denote the V-linear span of products $x_1 \cdots x_j$ with $x_1, \dots, x_j \in I$. The *tube algebra of* $I^l \lhd R$ for $l \in \mathbb{N}^*$ is

$$\mathcal{U}(R, I^l) := \sum_{j=0}^{\infty} \pi^{-j} I^{l \cdot j} \subseteq R \otimes F$$

with the subspace bornology; this is indeed a V-subalgebra of $R \otimes F$. If $l \geq j$, then $\mathcal{U}(R, I^l) \subseteq \mathcal{U}(R, I^j)$ is a bornological subalgebra. Let $\mathcal{U}(R, I^\infty)$ be the projective system of bornological V-algebras $(\mathcal{U}(R, I^l))_{l \in \mathbb{N}^*}$.

Since $\mathcal{U}(R, I^l)$ is defined as a bornological submodule of an F-vector space, it is bornologically torsion-free. And the inclusion $R \hookrightarrow \mathcal{U}(R, I^l)$ induces a bornological isomorphism $\mathcal{U}(R, I^l) \otimes F \cong R \otimes F$.

Remark 3.2. In [7, Definition 3.1.19], the tube algebra $\mathcal{U}(R, I^l)$ of a bornological V-algebra is equipped with a different bornology, namely, the bornology that is generated by subsets bounded in R and subsets of the form $\pi^{-1} M^l$ for bounded subsets $M \subseteq I$. This makes no difference if R carries the fine bornology. For general R, however, the two bornologies on the tube algebra need not be the same. It is easy to check that both bornologies induce the same bornology on $\mathcal{U}(R, I^l) \otimes F \cong R \otimes F$. Thus the two bornologies coincide if and only if the bornology defined in [7] is bornologically torsion-free. This concept is introduced only later in [19]. The more complicated bornology defined in [7] gives the tube algebra the expected universal property for bornological algebras that are torsion-free as algebras, but not bornologically torsion-free.

The *third step* equips $\mathcal{U}(R, I^l)$ for $l \in \mathbb{N}^*$ with the linear growth bornology relative to the ideal $\mathcal{U}(I, I^l)$. This gives a projective system of bornological algebras

$$\mathcal{U}(R, I^\infty)_{\mathrm{lg}(\mathcal{U}(I, I^\infty))} = \big(\mathcal{U}(R, I^l)_{\mathrm{lg}(\mathcal{U}(I, I^l))}\big)_{l \in \mathbb{N}^*}$$

because the inclusion homomorphism $\mathcal{U}(R, I^{l+1}) \hookrightarrow \mathcal{U}(R, I^l)$ maps $\mathcal{U}(I, I^{l+1})$ to $\mathcal{U}(I, I^l)$. All these bornological algebras are torsion-free by Lemma 2.2.7.

The *fourth step* applies the completion functor. By [19, Theorem 4.6], this gives a projective system of complete, torsion-free bornological V-algebras

$$(\mathcal{U}(R, I^\infty), \mathcal{U}(I, I^\infty))^\dagger = \big((\mathcal{U}(R, I^l), \mathcal{U}(I, I^l))^\dagger\big)_{l \in \mathbb{N}^*}.$$

The *fifth step* is to tensor with F. This gives a projective system of complete bornological F-algebras

$$(\mathcal{U}(R,I^\infty),\mathcal{U}(I,I^\infty))^\dagger \otimes F := ((\mathcal{U}(R,I^l),\mathcal{U}(I,I^l))^\dagger \otimes F)_{l\in\mathbb{N}^*}.$$

The *sixth step* is to take the X-complex. Being natural, it extends to a functor from projective systems of complete bornological algebras to projective systems of supercomplexes. In particular, the canonical maps $\mathcal{U}(R,I^{l+1}) \to \mathcal{U}(R,I^l)$ induce bounded chain maps

$$\sigma_l\colon X\big((\mathcal{U}(R,I^{l+1}),\mathcal{U}(I,I^{l+1}))^\dagger \otimes F\big) \to X\big((\mathcal{U}(R,I^l),\mathcal{U}(I,I^l))^\dagger \otimes F\big).$$

These define a projective system of supercomplexes of complete bornological F-vector spaces, which we denote by

$$\mathbb{HA}(A) := X\big((\mathcal{U}(R,I^\infty),\mathcal{U}(I,I^\infty))^\dagger \otimes F\big).$$

The *seventh step* takes the *homotopy projective limit* $\operatorname{holim}\mathbb{HA}(A)$. Explicitly, this is the mapping cone of the chain map

$$\prod_{l\in\mathbb{N}^*} X\big((\mathcal{U}(R,I^l),\mathcal{U}(I,I^l))^\dagger \otimes F\big) \to \prod_{l\in\mathbb{N}^*} X\big((\mathcal{U}(R,I^l),\mathcal{U}(I,I^l))^\dagger \otimes F\big),$$
$$(x_l) \mapsto \big(x_l - \sigma_l(x_{l+1})\big)_{l\in\mathbb{N}^*}.$$

It is a supercomplex of complete bornological F-vector spaces.
The final, *eighth step* takes its homology:

Definition 3.3. The *analytic cyclic homology* $\mathrm{HA}_*(A)$ of a complete, torsion-free bornological -algebra A for $* \in \mathbb{Z}/2$ is the homology of $\operatorname{holim}\mathbb{HA}(A)$, that is, the quotient of the kernel of the differential by the image of the differential. We do not take the closure of the image, so that this quotient need not be bornologically separated. For this reason, we prefer to forget the induced bornology on $\mathrm{HA}_*(A)$.

3.1 Bivariant analytic cyclic homology

Besides the analytic cyclic homology functor HA_*, we also have the functor $\mathbb{HA}$ taking values in suitable homotopy categories of chain complexes of projective systems of bornological V-modules. This functor contains more information. In particular, it yields a bivariant analytic cyclic homology theory by letting $\mathrm{HA}_*(A_1,A_2)$ be the set of morphisms $\mathbb{HA}(A_1) \to \mathbb{HA}(A_2)$. Cuntz and Quillen use the same idea in [11] to extend periodic cyclic homology to a bivariant theory. The actual definition of $\mathrm{HA}_*(A_1,A_2)$ depends on the choice of the target category, however, and this is somewhat flexible. We do not pick any choice in this article, but only point out two natural options.
The analytic cyclic homology computations in this paper often prove a chain homotopy equivalence $\mathbb{HA}(A) \simeq \mathbb{HA}(B)$, as supercomplexes of projective systems

of bornological V-modules. These are equivalences in the homotopy category of supercomplexes, where homotopy is understood simply as chain homotopy. In all cases where we compute $\mathrm{HA}_*(A)$ in this paper, we actually prove that $\mathbb{HA}(A)$ is chain homotopy equivalent to a supercomplex with zero boundary map, so that it contains no more information than the bornological F-vector space $\mathrm{HA}_*(A)$. Homotopy projective limits are sufficiently compatible with chain homotopies to preserve chain homotopy equivalence; and this implies an isomorphism on homology.

A larger class of weak equivalences is used in [9] to define a homotopy category of chain complexes of projective systems. A good aspect of this construction is that it clarifies the role of the homotopy projective limit: this just replaces a given complex by one that is weakly equivalent to it and fibrant in a suitable sense, so that the arrows to it in the homotopy category are the same as chain homotopy classes of chain maps. Thus $\mathrm{HA}_*(A)$ is isomorphic to the space of arrows from the trivial supercomplex V to $\mathbb{HA}(A)$ in the homotopy category of [9]. We will see later that $\mathbb{HA}(V)$ is chain homotopy equivalent to the trivial supercomplex F (see Corollary 4.7.3). So the homotopy category of [9] is such that the bivariant analytic cyclic homology group $\mathrm{HA}_*(V, A)$ simplifies to $\mathrm{HA}_*(A)$.

4 Analytic nilpotence and analytically quasi-free resolutions

Cuntz and Quillen described the periodic cyclic homology of an algebra A as the homology of the X-complex of a certain projective system built from the tensor algebra $\mathsf{T}A$ of A. This approach to periodic cyclic homology is the key to proving that it satisfies excision. The Cuntz–Quillen approach is carried over to more analytic versions of periodic cyclic homology in [18]. Our proof of excision for HA_* in Section 5 will follow the pattern in [18]. In this section, we explain how HA_* as defined above fits into this framework.

4.1 Pro-Algebras

An important idea in [18] is that an analytic variant of periodic cyclic homology is defined by a suitable notion of "analytic nilpotence". This leads to an analytic tensor algebra of an algebra A, which is universal among analytically nilpotent extensions of A. It also leads to the concept of analytically quasi-free algebras. The theory is set up so that any two analytically quasi-free, analytically nilpotent extensions of a given algebra are homotopy equivalent. In characteristic 0, this implies that their X-complexes are chain homotopy equivalent. Thus the X-complex of the analytic tensor algebra is chain homotopy equivalent to the X-complex of any analytically quasi-free resolution of A. In this discussion, "algebras" are always more complex objects – such as projective systems of algebras or bornological algebras – because there is no suitable concept of analytic nilpotence for mere algebras without extra structure. For the analytic cyclic homology defined above, the appropriate type of algebra is a projective system of torsion-free, complete bornological V-algebras. For brevity, we call torsion-free, complete bornological V-algebras *algebras* and projective systems of them *pro-algebras*.

A pro-algebra is given by a directed set $(N, \le)$, algebras A_n for $n \in N$, and bounded algebra homomorphisms $\alpha_{m,n}: A_n \to A_m$ for $m, n \in N$ with $n \ge m$ that satisfy $\alpha_{m,m} = \mathrm{id}_{A_m}$ for all $m \in N$ and $\alpha_{m,n} \circ \alpha_{n,p} = \alpha_{m,p}$ for all $m, n, p \in N$ with $p \ge n \ge m$. The morphism set between two pro-algebras is

$$\mathrm{Hom}\big((A_l)_{l\in L}, (B_n)_{n\in N}\big) := \varprojlim_n \varinjlim_l \mathrm{Hom}(A_l, B_n).$$

We shall only need pro-algebras $(A_n)_{n\in N}$ where N is countable. Restricting to a cofinal increasing sequence in N gives an isomorphic pro-algebra with $N = \mathbb{N}$. Then the maps $\alpha_{m,n}$ are uniquely determined by $\alpha_{n,n+1}: A_{n+1} \to A_n$ for $n \in \mathbb{N}$. An algebra A is also a pro-algebra by taking $A_n = A$ and $\alpha_{n,n+1} := \mathrm{id}_A$ for all $n \in \mathbb{N}$. Such projective systems are called *constant*. For a pro-algebra $A = (A_n, \alpha_{m,n})$, there are canonical morphisms $A \to \mathrm{const}(A_n)$ for all $n \in N$.

The analytic tensor algebra of a torsion-free algebra A is the torsion-free pro-algebra $(\mathcal{U}(\mathsf{T}A, \mathsf{J}A^\infty), \mathcal{U}(\mathsf{J}A, \mathsf{J}A^\infty))^\dagger$ in the above definition of analytic cyclic homology. This comes with a canonical homomorphism to A, whose kernel is the pro-algebra $(\mathcal{U}(\mathsf{J}A, \mathsf{J}A^\infty))^\dagger$. This projective system of complete, torsion-free bornological algebras has two important extra properties: it is semi-dagger – hence dagger – and nilpotent mod π – this concept will be defined below. A pro-algebra with these two properties is called analytically nilpotent. The tube algebra construction and the relative dagger completion in the construction of the analytic tensor algebra are the universal way to make a pro-algebra extension with an analytically nilpotent kernel.

Any functor from algebras to algebras extends canonically to an endofunctor on the category of pro-algebras by applying it entrywise. The definition of analytic cyclic homology already used this extension to pro-algebras for completions and tensor products with F. The constructions of $\mathsf{T}A$ and $\mathsf{J}A$ for algebras are also functors and thus extend to pro-algebras. So is the tensor product bifunctor $- \overleftrightarrow{\otimes} -$, which extends to pro-algebras by

$$(A_n, \alpha_{m,n})_{m,n\in N} \overleftrightarrow{\otimes} (B_n, \beta_{m,n})_{m,n\in N'} \\ := (A_{n_1} \overleftrightarrow{\otimes} B_{n_2}, \alpha_{m_1,n_1} \overleftrightarrow{\otimes} \beta_{m_2,n_2})_{m_1,n_1\in N, m_2,n_2\in N'}.$$

In particular, we may tensor a pro-algebra with an algebra such as $V[t]^\dagger$, viewed as a constant pro-algebra.

DEFINITION 4.1.1. An *elementary dagger homotopy* between two morphisms of pro-algebras $f_0, f_1: A \rightrightarrows B$ is a morphism of pro-algebras $f: A \to B \overleftrightarrow{\otimes} V[t]^\dagger$ that satisfies $(\mathrm{id}_A \otimes \mathrm{ev}_t) \circ f = f_t$ for $t = 0, 1$. We call f_0, f_1 *elementary dagger homotopic* if there is such a homotopy. *Dagger homotopy* is the equivalence relation generated by elementary dagger homotopy.

4.2 The universal property of the tube algebra construction

First, we generalise the construction of tube algebras to pro-algebras. Actually, in this subsection, we drop the completeness assumption for algebras because tube algebras are usually incomplete. So “algebras” are torsion-free bornological algebras and pro-algebras are projective systems of such algebras until the end of this subsection.

An *ideal in a pro-algebra* $A = (A_n, \alpha_{m,n})_{i\in N}$ is a family of ideals $I_n \lhd A_n$ with $\alpha_{m,n}(I_n) \subseteq I_m$ for all $n, m \in N$ with $n \geq m$; then $\alpha_{m,n}$ induces homomorphisms $\mathcal{U}(A_n, I_n^l) \to \mathcal{U}(A_m, I_m^l)$ for all $l \in \mathbb{N}^*$, which intertwine the inclusion maps $\mathcal{U}(A_n, I_n^l) \hookrightarrow \mathcal{U}(A_n, I_n^j)$ for $l \geq j$. These homomorphisms form a pro-algebra

$$\mathcal{U}(A, I^\infty) \coloneqq \big(\mathcal{U}(A_n, I_n^l)\big)_{n\in N, l\in\mathbb{N}^*}.$$

If $l \in \mathbb{N}^*$, then $\mathcal{U}(A, I^l) \coloneqq \big(\mathcal{U}(A_n, I_n^l)\big)_{n\in N}$ is a pro-algebra. The pro-algebra $\mathcal{U}(A, I^l)$ for $l \in \mathbb{N}^* \cup \{\infty\}$ contains $\mathcal{U}(I, I^l)$ as an ideal. Since $A_n \subseteq \mathcal{U}(A_n, I_n^l)$ for all $n \in N$, $l \in \mathbb{N}^*$, the inclusion maps define a pro-algebra homomorphism $\iota_{A,I}: A \to \mathcal{U}(A, I^\infty)$.

Remark 4.2.1. The notion of ideal above suffices for our purposes and is convenient to define the tube algebra quickly. It has the problem of not being invariant under isomorphism of pro-algebras. A better definition would be to define an ideal to be the kernel of a pro-algebra homomorphism. It is, however, possible to switch to isomorphic pro-algebras to make a pro-algebra homomorphism into a homomorphism of diagrams. And then the kernel becomes a family of ideals as above. This allows to extend the construction of tube algebras to ideals in the more general sense.

Definition 4.2.2. A pro-algebra $(A_n, \alpha_{m,n})_{n\in N}$ is *nilpotent mod* π if, for each $m \in N$, there are $n \in N_{\geq m}$ and $l \in \mathbb{N}^*$ such that $\alpha_{m,n}(A_n^l) \subseteq \pi A_m$; here A_n^l denotes the V-submodule generated by all products $x_1 \cdots x_l$ of l factors in A_n.

Remark 4.2.3. Let $A = (A_n, \alpha_{m,n})_{m,n\in N}$ be a pro-algebra. Let $A/(\pi)$ be the projective system of $\mathbb{F}$-algebras formed by the quotients $A_n/(\pi)$ with the homomorphisms induced by $\alpha_{m,n}$. By definition, A is nilpotent mod π if and only if $A/(\pi)$ has the following property: for each $n \in N$ there are $m \in N$ and $l \in \mathbb{N}^*$ such that the l-fold multiplication map $(A_m/(\pi))^{\otimes l} \to A_n/(\pi)$ is zero. This is equivalent to the definition that a projective system of $\mathbb{F}$-algebras is pro-nilpotent in [18, Definition 4.3].

Proposition 4.2.4. *Let A and B be pro-algebras and let I and J be ideals in A and B, respectively. Let $\varphi: A \to B$ be a pro-algebra morphism that restricts to a pro-algebra morphism $I \to J$. Let $\iota_{A,I}: A \to \mathcal{U}(A, I^\infty)$ denote the canonical pro-algebra morphism.*

(1) *The pro-algebra $\mathcal{U}(I, I^\infty)$ is nilpotent mod π.*

(2) *If J is nilpotent mod π, then there is a unique morphism $\bar{\varphi}: \mathcal{U}(A, I^\infty) \to B$ with $\bar{\varphi} \circ \iota_{A,I} = \varphi$. It restricts to a morphism $\mathcal{U}(I, I^\infty) \to J$.*

(3) *There is a unique morphism* $\varphi_*:\mathcal{U}(A,I^\infty)\to\mathcal{U}(B,J^\infty)$ *with* $\varphi_*\circ\iota_{A,I} = \iota_{B,J}\circ\varphi$*. It restricts to a morphism* $\mathcal{U}(I,I^\infty)\to\mathcal{U}(J,J^\infty)$.

Proof. Write $A = (A_n,\alpha_{m,n})_{n\in N}$, $I = (I_n)_{n\in N}$ with ideals I_n in A_n with $\alpha_{m,n}(I_n)\subseteq I_m$ and $B = (B_n,\beta_{m,n})_{n\in N'}$, $J = (J_n)_{n\in N'}$ with ideals J_n in B_n with $\beta_{m,n}(J_n)\subseteq J_m$. The tube algebra $\mathcal{U}(A,I^\infty)$ is the projective limit of the tube algebras $\mathcal{U}(A_n,I_n^\infty)$ in the category of pro-algebras.
Being nilpotent mod π is hereditary for projective limits. So it suffices to prove (1) when A is a constant pro-algebra. Fix $n\in\mathbb{N}^*$ and let $m = 2n$, $l = n$. Then

$$\mathcal{U}(I,I^m)^l = \mathcal{U}(I,I^{2n})^n = \left(I + \sum_{j=1}^\infty \pi^{-j} I^{2nj}\right)^n \subseteq I^n + \sum_{j=1}^\infty \pi^{-j} I^{2nj} \tag{4.2.5}$$

because $\sum_{j=1}^\infty \pi^{-j} I^{2nj}$ is an ideal in $\mathcal{U}(A,I^{2n})$. Since $\pi^{-1}I^n$ and $\pi^{-2j}I^{2nj}$ are contained in $\mathcal{U}(I,I^n)$, all summands on the right hand side of (4.2.5) are contained in $\pi\cdot\mathcal{U}(I,I^n)$. Thus $\mathcal{U}(I,I^\infty)$ is nilpotent mod π.
We prove statement (2). The morphism $\varphi: A\to B$ is described by a coherent family of V-algebra homomorphisms $\varphi_n: A_{\psi(n)}\to B_n$ for all $n\in N'$. Each B_n is torsion-free by our definition of "algebra". Then the homomorphism φ_n is determined by $\varphi_n\otimes\mathrm{id}_F: A_{\psi(n)}\otimes F\to B_n\otimes F$. By construction, $\mathcal{U}(A_\nu,I^m)\otimes F = A_\nu\otimes F$ for all $\nu\in N$, $m\in\mathbb{N}^*$. Thus a factorisation of φ through $\mathcal{U}(A,I^\infty)$ is unique if it exists.
Fix $n\in N'$. Since J is nilpotent mod π, there are $m\in N'_{\ge n}$ and $l\in\mathbb{N}^*$ with $\beta_{n,m}(J_m^l)\subseteq\pi\cdot J_n$. Since φ is coherent, there is $\nu\in N_{\ge\psi(m)}$ with $\beta_{n,m}\circ\varphi_m\circ\alpha_{\psi(m),\nu} = \varphi_n\circ\alpha_{n,\nu}$. Since φ restricts to a morphism $I\to J$, we may also arrange that $\varphi_m\circ\alpha_{\psi(m),\nu}(I_\nu)\subseteq J_m$ by increasing ν if necessary. Hence

$$\varphi_n\circ\alpha_{n,\nu}(I_\nu^l) = \beta_{n,m}\circ\varphi_m\circ\alpha_{\psi(m),\nu}(I_\nu^l)\subseteq\beta_{n,m}(J_m^l)\subseteq\pi\cdot J_n.$$

Thus the homomorphism $(\varphi_n\circ\alpha_{n,\nu})\otimes\mathrm{id}_F: A_\nu\otimes F\to B_n\otimes F$ maps the tube algebra $\mathcal{U}(A_\nu,I_\nu^l)\subseteq A_\nu\otimes F$ into $B_n\subseteq B_n\otimes F$ and $\mathcal{U}(I_\nu,I_\nu^l)\subseteq A_\nu\otimes F$ into $J_n\subseteq B_n\otimes F$. This gives a homomorphism $\bar\varphi_n:\mathcal{U}(A_\nu,I_\nu^l)\to B_n$ with $\bar\varphi_n\circ\iota_{A_\nu,I_\nu^l} = \varphi_n\circ\alpha_{n,\nu}$. Since $\mathcal{U}(A_\nu,I^m)\subseteq A_\nu\otimes F$, the homomorphisms $\bar\varphi_n$ inherit the coherence property of a pro-algebra morphism from the maps φ_n.
We prove statement (3) of the proposition. We compose $\varphi: A\to B$ with the canonical map $B\to\mathcal{U}(B,J^\infty)$ to get a morphism $A\to\mathcal{U}(B,J^\infty)$. It restricts to a morphism $I\to J\to\mathcal{U}(J,J^\infty)$. The ideal $\mathcal{U}(J,J^\infty)$ in $\mathcal{U}(B,J^\infty)$ is nilpotent mod π by (1). So (2) shows that our morphism extends uniquely to a morphism $\mathcal{U}(A,I^\infty)\to\mathcal{U}(B,J^\infty)$ that maps $\mathcal{U}(I,I^\infty)$ to $\mathcal{U}(J,J^\infty)$. ∎

We summarise the tube algebra construction in category-theoretic language. Let $\mathfrak{Pro}$ be the category whose objects are pairs (A,I), where A is a pro-algebra and I is an ideal in A and whose morphisms are pro-algebra morphisms that restrict to a morphism between the ideals. The pairs (A,I) where I is nilpotent mod π form a subcategory $\mathfrak{Pro}_{\mathrm{nil}}$ in $\mathfrak{Pro}$. The first two statements in

Proposition 4.2.4 say that the canonical arrow $(A,I) \to (\mathcal{U}(A,I^\infty),\mathcal{U}(I,I^\infty))$ is a universal arrow from (A,I) to an object in $\mathfrak{Pro}_{\mathrm{nil}}$. Thus $\mathfrak{Pro}_{\mathrm{nil}}$ is a reflective subcategory in $\mathfrak{Pro}$ and the reflector acts on objects by $(A,I) \mapsto (\mathcal{U}(A,I^\infty),\mathcal{U}(I,I^\infty))$. Its functoriality is Proposition 4.2.4.(3). If I is already nilpotent mod π, then it follows that the identity map on A extends uniquely to an isomorphism of pro-algebras $\mathcal{U}(A,I^\infty) \cong A$.

The inheritance properties of nilpotence mod π proven in the following proposition are needed by the analytic cyclic homology machinery in [18].

Proposition 4.2.6. *The class of nilpotent mod π pro-algebras has the following properties:*

- *Let $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$ be an extension of pro-algebras. If A and C are nilpotent mod π, then so is B, and vice versa.*
- *A pro-subalgebra $D \subseteq B$ is nilpotent mod π if B is so and B/D is isomorphic to a projective system of torsion-free bornological V-modules.*
- *Being nilpotent mod π is hereditary for projective limits.*
- *A tensor product $A \overleftrightarrow{\otimes} B$ is nilpotent mod π if A or B is nilpotent mod π.*

Proof. Remark 4.2.3 translates all these statements to statements about the class of pro-nilpotent projective systems of $\mathbb{F}$-algebras. In this way, the statements follow from [18, Theorem 4.4]. We briefly explain direct proofs for the first two claims. The claims about projective limits and tensor products are easy and left to the reader.

As in [18], we may write any extension of pro-algebras $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$ as a projective system of extensions $A_n \overset{i_n}{\rightarrowtail} B_n \overset{p_n}{\twoheadrightarrow} C_n$, with morphisms of extensions

$$\begin{array}{ccccc} A_n & \overset{i_n}{\rightarrowtail} & B_n & \overset{p_n}{\twoheadrightarrow} & C_n \\ \downarrow{\scriptstyle \alpha_{m,n}} & & \downarrow{\scriptstyle \beta_{m,n}} & & \downarrow{\scriptstyle \gamma_{m,n}} \\ A_m & \overset{i_m}{\rightarrowtail} & B_m & \overset{p_m}{\twoheadrightarrow} & C_m \end{array}$$

for $n \geq m$ as structure maps (this construction is also explained during the proof of Proposition 4.3.13 below). Assume that A and C are nilpotent mod π. Pick $m \in N$. There are $n_1 \in N_{\geq m}$ and $j_1 \in \mathbb{N}^*$ so that $\alpha_{m,n_1}(A_{n_1}^{j_1}) \subseteq \pi \cdot A_m$. And there are $n_2 \in N_{\geq n_1}$ and $j_2 \in \mathbb{N}^*$ so that $\gamma_{n_1,n_2}(C_{n_2}^{j_2}) \subseteq \pi \cdot C_{n_1}$. Then $p_{n_1}(\beta_{n_1,n_2}(B_{n_2}^{j_2})) \subseteq \pi \cdot C_{n_1}$. This implies $\beta_{n_1,n_2}(B_{n_2}^{j_2}) \subseteq \pi \cdot B_{n_1} + i_{n_1}(A_{n_1})$. Then

$$\begin{aligned} \beta_{m,n_2}(B_{n_2}^{j_1 \cdot j_2}) \subseteq \beta_{m,n_1}(\pi \cdot B_{n_1} + i_{n_1}(A_{n_1}))^{j_1} &\subseteq \pi \cdot B_m + i_m(\alpha_{m,n_1}(A_{n_1}^{j_1})) \\ &\subseteq \pi \cdot B_m + i_m(\pi A_m) \subseteq \pi \cdot B_m. \end{aligned}$$

So B is nilpotent mod π. Conversely, if B is nilpotent mod π, then C is nilpotent mod π because $p_m(B_m) = C_m$ and $p_m(\pi \cdot B_m) = \pi \cdot C_m$. The claim that A is nilpotent mod π if B is follows from the claim about pro-subalgebras.

Given a pro-subalgebra $D \subseteq B$, we may write $B = (B_n, \beta_{m,n})_{n\in N}$ and $D = (D_n, \delta_{m,n})_{n\in N}$ so that $D_n \subseteq B_n$ for all $n \in N$ and $\delta_{m,n} = \beta_{m,n}|_{D_n}: D_n \to D_m$ for all $m, n \in N$ with $m \le n$. Let $m \in N$. Since B/D is isomorphic to a projective system of torsion-free bornological V-modules, there is $n \in N_{\ge m}$ so that the structure map $B_n/D_n \to B_m/D_m$ kills all elements $x \in B_n/D_n$ with $\pi \cdot x = 0$. Equivalently, if $x \in B_n$ satisfies $\pi \cdot x \in D_n$, then $\beta_{m,n}(x) \in D_m$. Thus $\beta_{m,n}(\pi \cdot B_n \cap D_n) \subseteq \pi \cdot D_m$. If B is nilpotent mod π, then there are $l \in N_{\ge n}$ and $j \in \mathbb{N}^*$ with $\beta_{n,l}(B_l^j) \subseteq \pi \cdot B_n$. Hence

$$\delta_{m,l}(D_l^j) \subseteq \delta_{m,n}(\delta_{n,l}(D_l^j)) \subseteq \beta_{m,n}(\pi \cdot B_n \cap D_n) \subseteq \pi \cdot D_m.$$

Thus D is nilpotent mod π. □

4.3 Analytically nilpotent pro-algebras

From now on, "algebra" means a complete, torsion-free bornological algebra.

Definition 4.3.1. A pro-algebra J is *analytically nilpotent* if it is isomorphic to a pro-dagger algebra and nilpotent mod π. It is *square-zero* if its multiplication map is 0. An extension of pro-algebras $J \rightarrowtail E \twoheadrightarrow A$ is *analytically nilpotent* or *square-zero* if J is analytically nilpotent or square-zero, respectively.

In an analytically nilpotent pro-algebra, any power series $\sum c_n x^n$ for an "element" $x \in J$ and a bounded sequence $(c_n)_{n\in\mathbb{N}}$ in V may be evaluated (see the proof of Proposition 4.3.6 for the precise meaning of this in a pro-algebra). This uses nilpotence mod π in order to reduce to sequences whose valuation grows linearly, and being a pro-dagger algebra to ensure that such series converge.

Definition 4.3.2. A *pro-linear map* between two pro-algebras is a morphism of projective systems of bornological V-modules between them; so pro-linear maps need not be multiplicative. An extension of pro-algebras $J \rightarrowtail E \twoheadrightarrow A$ is *semi-split* if it splits by a pro-linear map.

Definition 4.3.3. A pro-algebra A is *analytically quasi-free* if any semi-split analytically nilpotent extension $J \rightarrowtail E \twoheadrightarrow A$ splits by a pro-algebra homomorphism $A \to E$. It is *quasi-free* if any semi-split square-zero extension $J \rightarrowtail E \twoheadrightarrow A$ splits by a pro-algebra homomorphism $A \to E$.

The following lemma gives an equivalent reformulation of the last definition:

Lemma 4.3.4. *A pro-algebra A is analytically quasi-free if and only if, for any semi-split analytically nilpotent extension $J \rightarrowtail E \twoheadrightarrow B$, any homomorphism $f: A \to B$ lifts to a homomorphism $A \to E$. A pro-algebra A is quasi-free if and only if, for any semi-split square-zero extension $J \rightarrowtail E \twoheadrightarrow B$, any homomorphism $f: A \to B$ lifts to a homomorphism $A \to E$.*

Proof. We may pull the given extension back to a semi-split extension $J \rightarrowtail \hat{E} \twoheadrightarrow A$, such that a section $A \to \hat{E}$ is equivalent to a lifting of f. □

Remark 4.3.5. A pro-algebra is square-zero if and only if it is isomorphic to a projective system of torsion-free complete bornological V-modules, each equipped with the zero map as multiplication. Then it is analytically nilpotent. As a consequence, analytically quasi-free algebras are quasi-free.

Proposition 4.3.6. *The base ring V viewed as a constant pro-algebra is analytically quasi-free.*

Proof. The proof follows [11, Section 12]. This idea is, in fact, much older, see [15, Section 3.6]. Let $J \rightarrowtail E \overset{p}{\twoheadrightarrow} Q$ be a semi-split, analytically nilpotent extension of pro-algebras. Analytic quasi-freeness of V is equivalent to the assertion that any idempotent in Q lifts to an idempotent in E. Here by an idempotent in a pro-algebra $A = (A_n)_n$, we mean a collection $a = (a_n)_n$ of idempotents $a_n \in A_n$. Each $a_n \in A_n$ is equivalent to a homomorphism $V \to A_n$. Let $\dot{e} = (\dot{e}_n)_n \in Q$ be an idempotent and let $e \in E$ be the image of $\dot{e}$ under a pro-linear section for $p\colon E \twoheadrightarrow Q$. Let $x := e - e^2 \in J$. We use an Ansatz by Cuntz and Quillen to find an idempotent $\hat{e} \in E$ with $e - \hat{e} \in J$. Namely, we assume $\hat{e} = e + (2e-1)\varphi(x)$ for some power series $\varphi \in t\mathbb{Z}[[t]]$. We compute

$$\hat{e}^2 - \hat{e} = (\varphi(x)^2 + \varphi(x))(1-4x) - x.$$

So $\hat{e}^2 = \hat{e}$ if and only if $\varphi(x)^2 + \varphi(x) = \frac{x}{1-4x}$. This is solved by $\varphi(x) := \sum_{n=1}^{\infty} \binom{2n-1}{n} x^n$. Write $J = (J_l)_{l\in N}$. We show that the power series $\sum_{n=1}^{\infty} \binom{2n-1}{n} x_l^n$ converges in J_l for each $l \in N$.
As J is nilpotent mod π, there are $m \geq l$ and $j \in \mathbb{N}$ so that the multiplication and the structure map send J_m^j to πJ_l. Since x_l^j is the image of x_m^j, it follows that $x_l^j \in \pi J_l$. Since J is semi-dagger, the set $\{\pi^{-\lfloor k/(2j)\rfloor} x_l^k : k \in \mathbb{N}\}$ is bounded in J_l. Since J is complete, the series $\sum_{n=1}^{\infty} \binom{2n-1}{n} x_l^n$ converges. These series for $l \in N$ define an element $\varphi(x)$ of J. And then $\hat{e} := e + (2e-1)\varphi(x)$ is the desired idempotent lifting of e. ◻

Proposition 4.3.7. *An algebra A is analytically quasi-free if and only if its unitalisation A^+ is analytically quasi-free.*

Proof. Proposition 4.3.6 implies this as in the proof of [18, Proposition 5.53]. ◻

Proposition 4.3.8. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of unital, analytically quasi-free pro-algebras. Then $\bigoplus_{n\in\mathbb{N}} A_n$ is analytically quasi-free.*

Proof. The proof of [18, Proposition 5.53] carries over to this context. ◻

Corollary 4.3.9. *The direct sum $\bigoplus_{n\in\mathbb{N}} V$ is analytically quasi-free.*

Proposition 4.3.10. *Let $J_i \rightarrowtail E_i \twoheadrightarrow A_i$ for $i = 1, 2$ be semi-split, analytically nilpotent extensions of pro-algebras. Assume that E_1 is analytically quasi-free.*

(1) *Any pro-algebra morphism $f\colon A_1 \to A_2$ lifts to a morphism of extensions*

$$\begin{array}{ccccc} J_1 & \rightarrowtail & E_1 & \xrightarrow{q_1} & A_1 \\ \downarrow & & \downarrow \hat{f} & & \downarrow f \\ J_2 & \rightarrowtail & E_2 & \xrightarrow{q_2} & A_2 \end{array}$$

This lifting is unique up to dagger homotopy.

(2) *Let $\hat{f},\hat{g}\colon E_1 \rightrightarrows E_2$ be pro-algebra homomorphisms that lift pro-algebra homomorphisms $f,g\colon A_1 \rightrightarrows A_2$. Then an elementary dagger homotopy $h\colon A_1 \to A_2 \overleftrightarrow{\otimes} V[t]^\dagger$ between f and g lifts to an elementary dagger homotopy $\hat{h}\colon E_1 \to E_2 \overleftrightarrow{\otimes} V[t]^\dagger$ between $\hat{f}$ and $\hat{g}$.*

(3) *Any elementary dagger homotopy $A_1 \to A_2 \overleftrightarrow{\otimes} V[t]^\dagger$ lifts to an elementary dagger homotopy $E_1 \to E_2 \overleftrightarrow{\otimes} V[t]^\dagger$.*

Proof. Let $f\colon A_1 \to A_2$ be a pro-algebra homomorphism. Since E_1 is analytically quasi-free and the extension $J_2 \rightarrowtail E_2 \twoheadrightarrow A_2$ is semi-split and analytically nilpotent, the homomorphism $f \circ q_1$ lifts to a homomorphism $\hat{f}\colon E_1 \to E_2$. Since $q_2 \circ \hat{f} = f \circ q_1$ vanishes on J_1, $\hat{f}$ restricts to a homomorphism $J_1 \to J_2$. Thus $\hat{f}$ gives a morphism of extensions.

The uniqueness claim in (1) follows from (2) by taking $f = g$. And (3) follows from (1) and (2). So it remains to prove (2). Assume that we are in the situation of (2). Let $\mathrm{ev}_0, \mathrm{ev}_1\colon A_2 \overleftrightarrow{\otimes} V[t]^\dagger \rightrightarrows A_2$ and $\mathrm{ev}_0, \mathrm{ev}_1\colon E_2 \overleftrightarrow{\otimes} V[t]^\dagger \rightrightarrows E_2$ denote the evaluation homomorphisms. Form the pull-back pro-algebra

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & E_2 \oplus E_2 \\ \downarrow & & \downarrow q_2 \oplus q_2 \\ A_2 \overleftrightarrow{\otimes} V[t]^\dagger & \xrightarrow{(\mathrm{ev}_0\ \mathrm{ev}_1)} & A_2 \oplus A_2. \end{array}$$

The universal property of the pull back gives pro-algebra homomorphisms

$$q := (\mathrm{ev}_0, \mathrm{ev}_1, q_2 \overleftrightarrow{\otimes} \mathrm{id}_{V[t]^\dagger})_*\colon E_2 \overleftrightarrow{\otimes} V[t]^\dagger \to \mathcal{E},$$
$$(\hat{f}, \hat{g}, h \circ q_1)_*\colon E_1 \to \mathcal{E},$$

because $\hat{f}$ and $\hat{g}$ lift $\mathrm{ev}_t \circ h$ for $t = 0, 1$, respectively. Let

$$V[t]^\dagger_0 := \{\varphi \in V[t]^\dagger : \varphi(0) = 0,\ \varphi(1) = 0\}.$$

We claim that q is part of a semi-split extension of pro-algebras

$$J_2 \overleftrightarrow{\otimes} V[t]^\dagger_0 \rightarrowtail E_2 \overleftrightarrow{\otimes} V[t]^\dagger \twoheadrightarrow \mathcal{E}. \tag{4.3.11}$$

To see this, we forget multiplications and treat everything as a projective system of bornological V-modules. In this category, a pro-linear section $s\colon A_2 \to E_2$

for the semi-split extension $J_2 \to E_2 \to A_2$ gives a direct sum decomposition $E_2 \cong J_2 \oplus A_2$. And $V[t]^\dagger \cong V[t]_0^\dagger \oplus V \oplus V$, where the latter two summands are, say, spanned by the functions $1-t$ and t. This induces decompositions

$$A_2 \overline{\otimes} V[t]^\dagger \cong (A_2 \overline{\otimes} V[t]_0^\dagger) \oplus A_2 \oplus A_2, \qquad E_2 \overline{\otimes} V[t]^\dagger \cong (E_2 \overline{\otimes} V[t]_0^\dagger) \oplus E_2 \oplus E_2,$$

such that $(\mathrm{ev}_0, \mathrm{ev}_1)$ is the projection to the second and third summand both for A_2 and E_2. These direct sum decompositions imply

$$E_2 \overline{\otimes} V[t]^\dagger \cong (J_2 \overline{\otimes} V[t]_0^\dagger) \oplus (A_2 \overline{\otimes} V[t]_0^\dagger) \oplus E_2 \oplus E_2 \cong (J_2 \overline{\otimes} V[t]_0^\dagger) \oplus \mathcal{E}.$$

And this proves the claim.
Corollary 2.1.21 and Proposition 4.2.6 imply that the tensor product $J_2 \overline{\otimes} V[t]_0^\dagger$ is analytically nilpotent. Since E_1 is analytically quasi-free, the homomorphism $(\hat{f}, \hat{g}, h \circ q_1)$ lifts to a homomorphism $\hat{h}: E_1 \to E_2 \overline{\otimes} V[t]^\dagger$ in the extension (4.3.11). This finishes the proof of (2). □

Corollary 4.3.12. *Any two analytically quasi-free, analytically nilpotent extensions of a pro-algebra are dagger homotopy equivalent.*

Proof. By Proposition 4.3.10, there are morphisms of extensions in both directions which lift the identity map on A and whose composite maps are dagger homotopic to the identity maps. □

Proposition 4.3.13. *Let $A \rightarrowtail E \twoheadrightarrow B$ be an extension of pro-algebras. If A and B are isomorphic to projective systems of dagger algebras, then so is E. If A and B are analytically nilpotent, then so is E.*

Proof. Being nilpotent mod π is hereditary for pro-algebra extensions by Proposition 4.2.6. Hence the second statement follows from the first one. Its proof has several steps. First, we rewrite the given extension of pro-algebras as a projective limit of a projective system of algebra extensions. Similar ideas in a less specialised setting also appear in [4, Appendix].
Write E and B as projective systems of (torsion-free, complete bornological) algebras $(E_n, \gamma_{n,m})$ and $(B_n, \beta_{n,m})$ that are indexed by directed sets N_E and N_B, respectively. By assumption, B is isomorphic to a projective system of dagger algebras. We assume that we have picked this representative above, that is, each B_n is a dagger algebra. We describe the pro-algebra morphism $E \to B$ by a coherent family of bounded homomorphisms $\varphi_n: E_{m(n)} \to B_n$ for all $n \in N_B$. Let $N := \{(m,n) \in N_E \times N_B : m \geq m(n)\}$. Define a partial order on N by $(m_1, n_1) \geq (m_2, n_2)$ if $m_1 \geq m_2$, $n_1 \geq n_2$, $m_1 \geq m(n_2)$, and $\beta_{n_2,n_1} \circ \varphi_{n_1} \circ \gamma_{m(n_1),m_1} = \varphi_{n_2} \circ \gamma_{m(n_2),m_1}$. This partially ordered set is directed because N_B and N_E are directed and the maps φ_n for $n \in N$ form a morphism of projective systems. The objects E_m and B_n for $(m,n) \in N$ and the maps γ_{m_1,m_2} and β_{n_1,n_2} for $m_1 \geq m_2$ and $n_1 \geq n_2$ form projective systems E' and B' of bornological algebras. They are isomorphic to E and B, respectively. The homomorphisms

$$\varphi'_{(m,n)} := \varphi_n \circ \gamma_{m(n),m}: E'_{(m,n)} = E_m \to B_n = B'_{(m,n)}$$

for $(m,n) \in N$ are coherent in the strong sense that

$$\beta'_{(m_1,n_1),(m_2,n_2)} \circ \varphi'_{(m_2,n_2)} = \varphi'_{(m_1,n_1)} \circ \gamma'_{(m_1,n_1),(m_2,n_2)}$$

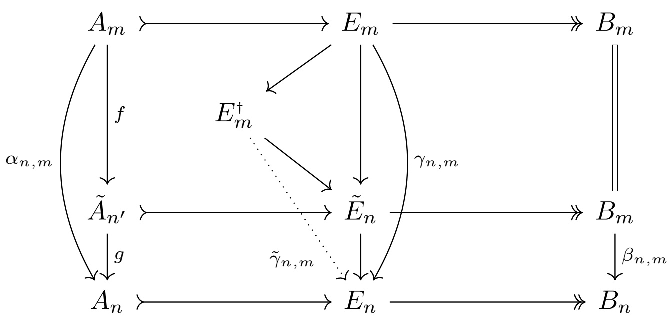
for all $(m_1,n_1),(m_2,n_2) \in N$ with $(m_1,n_1) \le (m_2,n_2)$. Here γ' and β' denote the structure maps of the projective systems E' and B', respectively. By construction, each B'_n is a dagger algebra.
By assumption, the inclusion $A \to E$ is the kernel of the morphism $E \to B$. This is isomorphic to the kernel of $\varphi'\colon E' \to B'$. So A is isomorphic to the projective system A' formed by the closed ideals $A'_n := \ker \varphi_n \subseteq E'_n$ for $n \in N$ with the structure maps $\alpha'_{n_1,n_2} = \gamma'_{n_1,n_2}|_{A_{n_2}}$ for $n_1, n_2 \in N$ with $n_1 \le n_2$; and the canonical morphism $A' \to E'$ is the strongly coherent family of inclusion maps $A'_n \hookrightarrow E'_n$ for $n \in N$. Each A'_n is complete and torsion-free because E'_n and B'_n are (see [19, Theorem 2.3 and Lemma 4.2]).
The quotients E'_n/A'_n with the structure maps $\dot{\gamma}'_{n,m}$ induced by $\gamma'_{n,m}$ form a projective system of complete bornological algebras, which is the cokernel for the inclusion $A' \hookrightarrow E'$. The map φ'_n for $n \in N$ descends to an injective, bounded homomorphism $\varrho_n\colon E'_n/A'_n \to B'_n$. The pro-algebra morphism $\varrho = (\varrho_n)_{n\in N}$ is an isomorphism because $E \to B$ is assumed to be another cokernel for the map $A \to E$. Next, we modify our projective systems so that these become equalities; this replaces the quotients E'_n/A'_n by dagger algebras. The inverse of ϱ is given by a choice of $m(n) \in N$ for $n \in N$ and bounded homomorphisms $\psi_n\colon B'_{m(n)} \to E'_n/A'_n$. Increasing $m(n)$ if necesessary, we may arrange that $\varrho_n \circ \psi_n = \beta'_{n,m(n)}\colon B'_{m(n)} \to B'_n$ and $\psi_n \circ \varrho_{m(n)} = \dot{\gamma}'_{n,m(n)}\colon E'_{m(n)}/A'_{m(n)} \to E'_n/A'_n$. Let $N' := \{(m,n) \in N \times N : m \ge m(n)\}$. For $(m,n) \in N'$, pull the extension $A'_n \rightarrowtail E'_n \twoheadrightarrow E'_n/A'_n$ back along ψ_n as in Lemma 2.3.2. This gives a diagram of extensions of bornological V-modules

$$\begin{array}{ccccc} A''_{(m,n)} & \rightarrowtail & E''_{(m,n)} & \twoheadrightarrow & B''_{(m,n)} \\ \Vert & & \downarrow & & \downarrow{\scriptstyle \psi_n} \\ A'_n & \rightarrowtail & E'_n & \twoheadrightarrow & E'_n/A'_n \end{array}$$

with $A''_{(m,n)} = A'_n$ and $B''_{(m,n)} = B'_m$. The latter is a dagger algebra because it is equal to B_m for suitable $m \in N_B$ depending on $n \in N'$. There is a unique bornological algebra structure on $E''_{(m,n)}$ for which all maps in this diagram are homomorphisms. We claim that $E''_{(m,n)}$ is complete. First, A'_n is closed in E'_n because B'_n is separated. Then E'_n/A'_n is separated (see [19, Lemma 2.1]). Then $E''_{(m,n)}$ is closed in $B'_m \oplus E'_n$. And then $E''_{(m,n)}$ is complete by [19, Theorem 2.3]. As above, there is a partial order on N' that makes it a directed set and such that $A''_n \rightarrowtail E''_n \twoheadrightarrow B''_n$ becomes a projective system of algebra extensions. This projective system is isomorphic to $A' \rightarrowtail E' \twoheadrightarrow E'/A'$ because it is the pullback along the pro-algebra isomorphism $B' \xrightarrow{\simeq} E'/A'$. Thus it is isomorphic to the original extension $A \rightarrowtail E \twoheadrightarrow B$. We have now replaced this pro-algebra extension by a projective system of algebra extensions where the quotients B''_n are dagger algebras.

To simplify notation, we remove the primes now and assume that our pro-algebra extension already comes to us as a projective system of algebra extensions $A_n \rightarrowtail E_n \twoheadrightarrow B_n$, where A_n and E_n are torsion-free, complete bornological algebras and B_n are dagger algebras for all $n \in N$. The dagger completions $E_n^\dagger$ for $n \in N$ form a projective system of dagger algebras, and the canonical maps $E_n \to E_n^\dagger$ form a pro-algebra morphism. We claim that this pro-algebra morphism is an isomorphism. Equivalently, for each $n \in N$ there are $m \in N$ with $m \geq n$ and a bounded homomorphism $\tilde{\gamma}_{n,m}\colon E_m^\dagger \to E_n$ such that the composite map $E_m \to E_m^\dagger \to E_n$ is $\gamma_{n,m}$; then the other composite map $E_m^\dagger \to E_n \to E_n^\dagger$ is the map on the dagger completions induced by $\gamma_{n,m}$, and these two equalities of compositions say that we are dealing with morphisms of pro-algebras inverse to each other.

Fix $n \in N$. We are going to build the following commuting diagram, where the dashed arrow is the desired map $\tilde{\gamma}_{n,m}$:



By assumption, A is isomorphic to a projective system of dagger algebras $(\tilde{A}_{n'})_{n' \in N'}$. Therefore, there are $m \in N$, $n' \in N'$, and maps $f\colon A_m \to \tilde{A}_{n'}$ and $g\colon \tilde{A}_{n'} \to A_n$ such that $m \geq n$ and $g \circ f = \alpha_{n,m}\colon A_m \to A_n$. Let $\tilde{E}_n$ be the pushout bornological V-module of the maps $A_m \to E_m$ and $A_m \to \tilde{A}_{n'}$. This fits in an extension of bornological V-modules $\tilde{A}_{n'} \rightarrowtail \tilde{E}_n \twoheadrightarrow B_m$ by Lemma 2.3.2. Since $\tilde{A}_{n'}$ and B_m are torsion-free and complete, $\tilde{E}_n$ is complete by [19, Theorem 2.3]. Since $\tilde{A}_{n'}$ is semi-dagger, the canonical map $E_m \to \tilde{E}_n$ remains bounded when we give E_m the linear growth bornology relative to the ideal A'_m. This bornology is equal to the absolute linear growth bornology on E_m by Lemma 2.2.6 because $B_m = E_m/A_m$ is a dagger algebra. Since $\tilde{E}_n$ is complete, the map $E_m \to \tilde{E}_n$ extends to a bounded V-module homomorphism $E_m^\dagger \to \tilde{E}_n$. By construction, the map $\gamma_{n,m}\colon E_m \to E_n$ agrees on A_m with the composite map

$$A_m \xrightarrow{f} \tilde{A}_{n'} \xrightarrow{g} A_n \to E_n.$$

Then the universal property of pushouts gives an induced bounded V-module homomorphism $\tau\colon \tilde{E}_n \to E_n$. Let $\tilde{\gamma}_{n,m}\colon E_m^\dagger \to E_n$ be the composite of the bounded V-module homomorphisms $E_m^\dagger \to \tilde{E}_n$ and $\tilde{E}_n \to E_n$ defined above.

The composite map $E_m \to E_m^\dagger \to E_n$ is $\gamma_{n,m}$ by construction. This finishes the proof that E_n is isomorphic to a projective system of dagger algebras. □

4.4 The analytic tensor algebra

Let R be a constant pro-algebra. The definitions of $\mathbb{HA}(R)$ and $\mathrm{HA}_*(R)$ use a certain pro-algebra $\mathcal{T}R$ defined by completing the tensor algebra $\mathsf{T}R$. We call $\mathcal{T}R$ the analytic tensor algebra of R. We show that there is a semi-split analytically nilpotent extension $\mathcal{J}R \rightarrowtail \mathcal{T}R \twoheadrightarrow R$ and that $\mathcal{T}R$ is analytically quasi-free. Since it is not more difficult, we extend the construction of the analytic tensor algebra to pro-algebras right away.

Definition 4.4.1. Let $R = (R_n, \alpha_{m,n})_{m,n\in N}$ be a pro-algebra. Extending the tensor algebra construction to pro-algebras gives a natural semi-split pro-algebra extension

$$\mathsf{J}R \rightarrowtail \mathsf{T}R \twoheadrightarrow R$$

with $\mathsf{T}R = (\mathsf{T}R_n)_{n\in N}$ and $\mathsf{J}R = (\mathsf{J}R_n)_{n\in N}$. For each $n \in N$, we form the tube algebras $\mathcal{U}(\mathsf{T}R_n, (\mathsf{J}R_n)^l)$ with the ideals $\mathcal{U}(\mathsf{J}R_n, (\mathsf{J}R_n)^l)$, and their relative dagger completions $(\mathcal{U}(\mathsf{T}R_n, (\mathsf{J}R_n)^l), \mathcal{U}(\mathsf{J}R_n, (\mathsf{J}R_n)^l))^\dagger$. These form a pro-algebra indexed by the product set $N \times \mathbb{N}$, which we call the *analytic tensor algebra* of R and denote by $\mathcal{T}R$.

Lemma 4.4.2. *The canonical homomorphism $p: \mathsf{T}R \to R$ extends uniquely to a pro-algebra homomorphism $\tilde{p}: \mathcal{T}R \to R$. The composite σ_{an} of the pro-linear map $\sigma_R: R \to \mathsf{T}R$ and the canonical homomorphism $\mathsf{T}R \to \mathcal{T}R$ is a section for $\tilde{p}$.*

Proof. Fix $n \in N$ and $l \in \mathbb{N}^*$. The canonical homomorphism $\mathsf{T}R_n \to R_n$ vanishes on $\mathsf{J}R_n$. Then it extends uniquely to the tube algebra $\mathcal{U}(\mathsf{T}R_n, (\mathsf{J}R_n)^l)$ by Proposition 4.2.4. This extension vanishes on $\mathcal{U}(\mathsf{J}R_n, (\mathsf{J}R_n)^l)$. Then it remains bounded for the linear growth bornology relative to this ideal and extends uniquely to a homomorphism on the relative dagger completion. These maps for all n and l form a morphism of pro-algebras $\tilde{p}: \mathcal{T}R \to R$. The canonical maps $\sigma_{R_n}: R_n \to \mathsf{T}R_n$ form a pro-linear section for $p: \mathsf{T}R \to R$. Composing with the canonical map $\mathsf{T}R \to \mathcal{T}R$ gives a section for $\tilde{p}$. □

Definition 4.4.3. Let $\mathcal{J}R$ be the kernel of $\tilde{p}: \mathcal{T}R \twoheadrightarrow R$.

Lemma 4.4.2 implies that there is a semi-split extension of pro-algebras

$$\mathcal{J}R \rightarrowtail \mathcal{T}R \underset{\sigma_R}{\overset{\tilde{p}}{\twoheadrightarrow}} R.$$

Proposition 4.4.4. *The pro-algebra $\mathcal{J}R$ is analytically nilpotent.*

Proof. It suffices to prove this when R is a constant pro-algebra. Let $m \in \mathbb{N}^*$. The linear growth bornology on $\mathcal{U}(\mathsf{T}R, (\mathsf{J}R)^m)$ relative to $\mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m)$

restricts to the "absolute" linear growth bornology on $\mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^m)$ by Lemma 2.2.5. The tensor algebra is bornologically torsion-free by Remark 2.6.2. Then so is $\mathcal{U}(\mathsf{T}R,(\mathsf{J}R)^m)$ by the definition of the bornology on the tube algebra. Then the relative linear growth bornology on it is torsion-free by Lemma 2.2.7, and this property is preserved by completions (see [19, Theorem 4.6]). Hence, the completion of $\mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^m)$ in the linear growth bornology is a dagger algebra. Then $\mathcal{J}R$ is a pro-dagger algebra. And $\mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^\infty)$ is nilpotent mod π by Proposition 4.2.4. This remains unaffected when we equip the tube algebras with the linear growth bornology and complete. □

Remark 4.4.5. Let $R = (R_n, \alpha_{m,n})_{m,n\in N}$ be a projective system of dagger algebras. Since $\mathcal{U}(\mathsf{T}R,(\mathsf{J}R)^l) \big/ \mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^l) \cong R$ is semi-dagger, the linear growth bornology on $\mathcal{U}(\mathsf{T}R,(\mathsf{J}R)^l)$ is equal to the linear growth bornology relative to $\mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^l)$ by Lemma 2.2.6. Hence $\mathcal{T}R$ is also equal to the "absolute" dagger completion,

$$\mathcal{T}R \cong \mathcal{U}(\mathsf{T}R,(\mathsf{J}R)^\infty)^\dagger.$$

Proposition 4.4.6. *The analytic tensor algebra $\mathcal{T}R$ is analytically quasi-free and quasi-free. The bimodule $\overleftrightarrow{\Omega}^1(\mathcal{T}R)$ is isomorphic to the free bimodule on R, that is,*

$$(\mathcal{T}R)^+ \overleftrightarrow{\otimes} R \overleftrightarrow{\otimes} (\mathcal{T}R)^+ \cong \overleftrightarrow{\Omega}^1(\mathcal{T}R); \tag{4.4.7}$$

the isomorphism is the map $\omega \otimes x \otimes \eta \mapsto \omega \cdot (\mathrm{d}\sigma_R(x)) \cdot \eta$. And the following maps are isomorphisms of left or right $\mathcal{T}R$-modules, respectively:

$$(\mathcal{T}R)^+ \overleftrightarrow{\otimes} R \xrightarrow{\simeq} \mathcal{T}R, \qquad \omega \otimes x \mapsto \omega \odot \sigma_R(x),$$
$$R \overleftrightarrow{\otimes} (\mathcal{T}R)^+ \xrightarrow{\simeq} \mathcal{T}R, \qquad x \otimes \omega \mapsto \sigma_R(x) \odot \omega.$$

Proof. Let $J \rightarrowtail E \overset{q}{\twoheadrightarrow} \mathcal{T}R$ be a semi-split, analytically nilpotent pro-algebra extension. Pull it back along the inclusion $\mathcal{J}R \hookrightarrow \mathcal{T}R$ to a pro-algebra extension $J \rightarrowtail K \twoheadrightarrow \mathcal{J}R$ and identify K with an ideal in E. Since J and $\mathcal{J}R$ are analytically nilpotent, so is K by Proposition 4.3.13. Let $s\colon \mathcal{T}R \to E$ be a pro-linear section and let $\sigma_R\colon R \to \mathcal{T}R$ be the canonical pro-linear section. The pro-linear map $s \circ \sigma_R$ induces a pro-algebra homomorphism $(s\circ\sigma_R)^\#\colon \mathsf{T}R \to E$ by Lemma 2.6.1. It satisfies $q \circ (s\circ\sigma_R)^\# = \sigma_R^\#$, and $\sigma_R^\#\colon \mathsf{T}R \to \mathcal{T}R$ is the canonical homomorphism because $\sigma_R^\#$ and the inclusion map agree on the image of R in $\mathsf{T}R$. In particular, $(s\circ\sigma_R)^\#$ maps $\mathsf{J}R$ into $K \lhd E$. Since K is nilpotent mod π, Proposition 4.2.4 shows that $(s\circ\sigma_R)^\#$ extends to the tube algebra $\mathcal{U}(\mathsf{T}R,(\mathsf{J}R)^\infty)$, in such a way that $\mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^\infty)$ is mapped to K. Since K is a pro-dagger algebra, the criterion in Proposition 2.2.9 shows that the morphism $\mathcal{U}(\mathsf{T}R,(\mathsf{J}R)^\infty) \to E$ extends uniquely to the dagger completion relative to $\mathcal{U}(\mathsf{J}R,(\mathsf{J}R)^\infty)$. This gives a pro-algebra morphism $\mathcal{T}R \to E$ that is a section for the extension $J \rightarrowtail E \overset{q}{\twoheadrightarrow} \mathcal{T}R$. So $\mathcal{T}R$ is analytically quasi-free.

If $h\colon R \to E$ is any pro-linear map with $q\circ h = \sigma_R$, then the argument above shows that $h^\#\colon \mathsf{T}R \to E$ extends uniquely to a pro-algebra morphism $\mathcal{T}R \to E$ that is

a section for the extension. Conversely, any multiplicative section $g:\mathcal{T}R \to E$ is of this form for $h := g \circ \sigma_R$. Thus the multiplicative sections for the extension $J \rightarrowtail E \overset{q}{\twoheadrightarrow} \mathcal{T}R$ are in bijection with pro-linear maps $R \to E$ with $q \circ h = \sigma_R$. Any such pro-linear map is equal to $s \circ \sigma_R + h_0$ for a unique pro-linear map $h_0: R \to J$. So multiplicative sections for our extension are in bijection with pro-linear maps $R \to J$. Combined with Lemma 2.5.1, we get a natural bijection for all $\mathcal{T}R$-bimodules M between pro-bimodule homomorphisms $\overline{\Omega}^1(\mathcal{T}R) \to M$ and pro-linear maps $R \to M$. Thus $\overline{\Omega}^1(\mathcal{T}R)$ is isomorphic to the free complete bimodule on R, which is $(\mathcal{T}R)^+ \mathbin{\overline{\otimes}} R \mathbin{\overline{\otimes}} (\mathcal{T}R)^+$. And this isomorphism is indeed induced by the map $\omega \otimes x \otimes \eta \mapsto \omega \cdot (\mathrm{d}\sigma_R(x)) \cdot \eta$.
Now let M be a left $\mathcal{T}R$-module. Turn M into a $\mathcal{T}R$-bimodule by taking the zero map as right module structure. Then a bimodule derivation $\mathcal{T}R \to M$ is just a left module map. Therefore, left module homomorphisms $\mathcal{T}R \to M$ are in bijection with pro-linear maps $R \to M$. Thus the map

$$(\mathcal{T}R)^+ \mathbin{\overline{\otimes}} R, \qquad \omega \otimes x \mapsto \omega \odot \sigma_R(x),$$

is an isomorphism of left $\mathcal{T}R$-modules. Here we have written $\odot$ for the multiplication in $\mathcal{T}R$ because we will later use these formulas when $\mathcal{T}R$ is identified with $\Omega^{\mathrm{ev}}R$ with the Fedosov product. A similar argument works for right modules. □

We now describe the analytic tensor algebra and its bornology more concretely. For this, we assume that R is a torsion-free, complete bornological algebra. A projective system $(R_n)_{n\in N}$ is treated by applying the following discussion to R_n for each $n \in N$. We identify $\mathsf{T}R$ with $\Omega^{\mathrm{ev}}R$ with the Fedosov product as in Section 2.6. Recall that the isomorphism $\mathsf{T}R \cong \Omega^{\mathrm{ev}}R$ maps the ideal $\mathsf{J}R^m$ onto $\bigoplus_{n\ge m} \Omega^{2n}R$. Thus $\mathcal{U}(\mathsf{T}R, (\mathsf{J}R)^m)$ is spanned by $\pi^{-j}\Omega^{2n}R$ with $n \ge m\cdot j$. And $\mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m)$ is spanned by $\pi^{-j}\Omega^{2n}R$ with $n \ge m\cdot j$ and $n \ge 1$. Equivalently,

$$\mathcal{U}(\mathsf{T}R, (\mathsf{J}R)^m) = \sum_{n=0}^{\infty} \pi^{-\lfloor n/m \rfloor}\Omega^{2n}R, \qquad \mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m) = \sum_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor}\Omega^{2n}R. \tag{4.4.8}$$

The following lemma estimates the growth of Fedosov products in ΩR. We define

$$M^{(n)} := \sum_{i=1}^{n} M^i. \tag{4.4.9}$$

Lemma 4.4.10. *Let R be an algebra and let $M \subseteq R$ be a submodule. Let $i_0, \dots, i_n \ge 1$ and $i := i_0 + \dots + i_n$. Then*

$$\Omega^{i_0}M \odot \dots \odot \Omega^{i_n}M \subseteq \bigoplus_{j=0}^{n} \Omega^{i+2j}(M^{(3)}).$$

Proof. As in the proof of [18, Theorem 5.11], we show the more precise estimate

$$\Omega^{i_0}M \odot \dots \odot \Omega^{i_n}M \subseteq \bigoplus_{j=0}^{n} (M^{(2)})^+ \,\mathrm{d}(M^{(3)})^{i+2j} \tag{4.4.11}$$

by induction on n. This is trivial for $n = 0$. The induction step uses (2.6.5) and

$$\Omega^i M \odot (M^{(2)})^+ \subseteq (M^{(2)})^+ \mathrm{d}(M^{(3)})^i + (\mathrm{d}M)^{i+1}\, \mathrm{d}(M^{(2)}). \qquad \square$$

Proposition 4.4.12. *Let R be a torsion-free bornological algebra and $m \geq 1$. If $M \subseteq R$ is bounded, $\alpha \in \mathbb{Q} \cap (0, 1/m)$, and $f \in \mathbb{N}_0$, then define*

$$D_m(M, \alpha, f) \coloneqq \bigoplus_{n=0}^{\infty} \pi^{-\lfloor \min\{n/m, \alpha \cdot n + f\}\rfloor} \Omega^{2n} M. \tag{4.4.13}$$

These are V-submodules of $\mathcal{U}(\mathsf{T}R, (\mathsf{J}R)^m)$ that cofinally generate its linear growth bornology relative to the ideal $\mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m)$.

Proof. Let $M \subseteq R$ be bounded, $\alpha \in \mathbb{Q} \cap (0, 1/m)$, and $f \in \mathbb{N}_0$. Equation (4.4.8) implies $D_m(M, \alpha, f) \subseteq \mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$. Our first goal is to show that $D_m(M, \alpha, f)$ has linear growth relative to $\mathcal{U}(\mathsf{J}R, \mathsf{J}R^m)$. Let $e \geq 1$. We claim that

$$M^+ \cdot \left(\sum_{n=1}^{em} \pi^{-\lfloor n/m \rfloor} (\mathrm{d}M\, \mathrm{d}M)^n\right)^{\diamond} = \bigoplus_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor + \lceil \frac{n}{em} \rceil - 1} \Omega^{2n} M. \tag{4.4.14}$$

By definition, the left hand side is spanned by Fedosov products

$$\pi^{j-1-\lfloor i_1/m\rfloor - \dots - \lfloor i_j/m \rfloor} M^+ \odot (\mathrm{d}M\mathrm{d}M)^{i_1} \odot \dots \odot (\mathrm{d}M\mathrm{d}M)^{i_j} \\ = \pi^{j-1-\lfloor i_0/m\rfloor - \dots - \lfloor i_j/m \rfloor} \Omega^{2(i_1+\dots+i_j)}(M)$$

for $j \geq 1$ and $1 \leq i_1, \dots, i_j \leq em$. These contribute to $\Omega^{2n} M$ if $i_1 + \dots + i_j = n$. For fixed n and j, the sum of floors $\lfloor i_1/m \rfloor + \dots + \lfloor i_j/m \rfloor$ is maximal if all but one of the i_j are divisible by m, and then it becomes $\lfloor n/m \rfloor$. For fixed n, the term $j - 1 - \lfloor n/m \rfloor$ becomes minimal if j is minimal. Equivalently, we choose $i_j = em$ for all but one j, and then $j = \lceil n/em \rceil$. This finishes the proof of (4.4.14).

The submodule in (4.4.14) is one of the generators of the linear growth bornology relative to $\mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m)$. For fixed $\alpha < 1/m$ and f as above, there is $e \in \mathbb{N}^*$ with $1/m - 1/(em) > \alpha$. Then there is $k \in \mathbb{N}$ with

$$\lfloor n/m \rfloor - \left\lceil \frac{n}{em} \right\rceil + 1 \geq \lfloor \min\{n/m, \alpha \cdot n + f\} \rfloor$$

for $n > k$. Then

$$D_m(M, \alpha, f) \subseteq \sum_{n=0}^{k} \pi^{-\lfloor \min\{n/m, \alpha \cdot n + f\}\rfloor} \Omega^{2n} M + M^+ \cdot \left(\sum_{n=1}^{em} \pi^{-\lfloor n/m \rfloor} (\mathrm{d}M\, \mathrm{d}M)^n\right)^{\diamond}.$$

The first, finite sum is already bounded in $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$. As a result, $D_m(M, \alpha, f)$ has linear growth relative to $\mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m)$.

Now let S be any V-submodule of $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$ that has linear growth relative to $\mathcal{U}(\mathsf{J}R, (\mathsf{J}R)^m)$. We claim that S is contained in $D_m(M, \alpha, f)$ for suitable M, α, f. By definition of the relative linear growth bornology, there are $k, e \in \mathbb{N}$

and a bounded submodule $M \subseteq R$ such that S is contained in the sum of $\sum_{n=0}^{k} \pi^{-\lfloor n/m \rfloor}\Omega^{2n}M$ and $\big(\sum_{i=1}^{em} \pi^{-\lfloor \frac{i}{m} \rfloor}\Omega^{2i}M\big)^{\diamond}$. The latter is spanned by Fedosov products

$$\pi^{j-1-\lfloor \frac{i_1}{m} \rfloor - \dots - \lfloor \frac{i_j}{m} \rfloor}\Omega^{2i_1}M \odot \dots \odot \Omega^{2i_j}M$$

with $j \in \mathbb{N}^*$, $1 \le i_1, \dots, i_j \le em$. By Lemma 4.4.10, $\Omega^{2i_1}M \odot \dots \odot \Omega^{2i_j}M$ is contained in the sum of $\Omega^{2n}(M^{(3)})$, where n lies between $i := \sum_{k=1}^{j} i_k$ and $i+j$. As above, the sum of the floors $\lfloor i_k/m \rfloor$ for fixed i is maximal if all but one i_k are divisible by m, and then it is $\lfloor i/m \rfloor$. The constraints $i_k \le em$ are equivalent to the constraint $i \le j \cdot em$. So S is contained in the sum of $\pi^{j-1-\lfloor i/m \rfloor}\Omega^{2n}(M^{(3)})$ with $i \le n \le i+j$ and $i \le j \cdot em$. For fixed n, j, the exponent $j-1-\lfloor i/m \rfloor$ is minimal if i is maximal. So we may assume that i is the minimum of n and jem. Then the optimal choice for j is the minimal one, which is $\lceil n/(em) \rceil$ if $i = n$ and $j = \lceil n/(em+1) \rceil$ if $i = jem$. The resulting exponents of π become $\lceil n/(em) \rceil - 1 - \lfloor n/m \rfloor$ in the first case and $\lceil n/(em+1) \rceil - 1 - \lceil n/(em+1) \rceil \cdot e$ in the second. If $\alpha > 1/m - 1/(em)$ and n is large enough, then both terms are greater or equal $-\lfloor \alpha n \rfloor$. Choosing f big enough, we may arrange that both are greater or equal $-\lfloor \min\{n/m, \alpha n + f\} \rfloor$ for all $n \in \mathbb{N}$. Then $S \subseteq D_m(M^{(3)}, \alpha, f)$. □

Corollary 4.4.15. *For $m \in \mathbb{N}^*$, let $\mathcal{B}_m$ be the bornology on $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$ that contains a subset if and only if it is contained in $\bigoplus_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor}\Omega^{2n}M$ for some bounded V-submodule $M \subseteq R$. This bornology makes $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$ a torsion-free bornological algebra. The projective system of bornological algebras $(\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m), \mathcal{B}_m)_{m \in \mathbb{N}^*}$ is isomorphic to the projective system formed by $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$ with the linear growth bornology relative to $\mathcal{U}(\mathsf{J}R, \mathsf{J}R^m)$.*

Proof. The Fedosov product is bounded for the bornology $\mathcal{B}_m$ by Lemma 4.4.10. The subsets $D_m(M, \alpha, f)$ in (4.4.13) are clearly in $\mathcal{B}_m$. Conversely,

$$\bigoplus_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m+1} \rfloor}\Omega^{2n}M = D_m\big(M, \tfrac{1}{m+1}, 0\big).$$

Thus any subset in $\mathcal{B}_{m+1}$ is mapped to a subset of $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$ with linear growth relative to $\mathcal{U}(\mathsf{J}R, \mathsf{J}R^m)$. The asserted isomorphism of projective systems follows. □

Now we can describe the completion $\mathcal{T}R$. Recall that $\overleftrightarrow{\Omega}^n R$ denotes the completion $R^+ \overleftrightarrow{\otimes} R^{\overleftrightarrow{\otimes} n}$ of $\Omega^n R = R^+ \otimes R^{\otimes n}$. For $m \in \mathbb{N}^*$ and a bounded V-submodule $M \subseteq R$, the canonical map $\overleftrightarrow{\Omega}^{2n}M \to \overleftrightarrow{\Omega}^{2n}R$ is injective by Proposition 2.4.5. Then we may view $\prod_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor}\overleftrightarrow{\Omega}^{2n}M$ as a V-submodule of $\prod_{n=0}^{\infty} \overleftrightarrow{\Omega}^{2n}R \otimes F$. Let $\overleftrightarrow{\Omega}^{\mathrm{ev}}(R)_m$ be the union of $\prod_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor}\overleftrightarrow{\Omega}^{2n}M$ for all bounded V-submodules $M \subseteq R$, with the bornology where a subset is bounded if and only if it is contained in $\prod_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor}\overleftrightarrow{\Omega}^{2n}M$ for some bounded V-submodules $M \subseteq R$. These form a decreasing sequence of subalgebras with bounded inclusion maps $\overleftrightarrow{\Omega}^{\mathrm{ev}}(R)_{m+1} \hookrightarrow \overleftrightarrow{\Omega}^{\mathrm{ev}}(R)_m$.

Proposition 4.4.16. *If R is a torsion-free, complete bornological algebra, then $\mathcal{T}R$ is naturally isomorphic to the projective system of complete bornological algebras* $(\overline{\Omega}^{\mathrm{ev}}(R)_m)_{m\in\mathbb{N}^*}$.

Proof. We shall use the explicit description of the relative linear growth bornology in Proposition 4.4.12. Each $\pi^{-\lfloor n/m\rfloor}\Omega^{2n}R$ is a direct summand of $\mathcal{U}(\mathsf{T}R, \mathsf{J}R^m)$, and the projection is bounded in the linear growth bornology relative to $\mathcal{U}(\mathsf{J}R, \mathsf{J}R^m)$. This gives us maps from the completed tube to $\pi^{-\lfloor n/m\rfloor}\overline{\Omega}^{2n}R$ for all $n\in\mathbb{N}$. It is easy to see that the π-adic completion of $D_m(M,\alpha,f)$ is isomorphic to the subspace of $\prod_{n=0}^{\infty}\pi^{-\lfloor n/m\rfloor}\overline{\Omega}^{2n}M$ consisting of all $(\omega_n)_{n\in\mathbb{N}}$ for which there is a sequence $(h_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ with $\lim h_j=\infty$ and $\omega_n\in\pi^{-\lfloor\min\{n/m,\alpha\cdot n+f\}\rfloor+h_n}\overline{\Omega}^{2n}M$ for all $n\in\mathbb{N}$. Any such subset is bounded in $\overline{\Omega}^{\mathrm{ev}}(R)_m$. Conversely, any bounded subset in $\overline{\Omega}^{\mathrm{ev}}(R)_{m+1}$ is contained in a subset of this form with $f=0$ and $m<1/\alpha<m+1$. Therefore, the projective system formed by the relative dagger completions $\big(\mathcal{U}(\mathsf{T}R,\mathsf{J}R^m),\mathcal{U}(\mathsf{J}R,\mathsf{J}R^m)\big)^{\dagger}$ is isomorphic to the projective system $(\overline{\Omega}^{\mathrm{ev}}(R)_m)_{m\in\mathbb{N}^*}$. □

4.5 Pro-Linear maps with nilpotent curvature

Let R and S be pro-algebras. We are going to describe pro-algebra homomorphisms $\mathcal{T}R\to S$ through a certain class of pro-linear maps $R\to S$, namely, those with analytically nilpotent curvature. This follows rather easily from the concrete description of the relative linear growth bornology on the tensor algebra above. The main issue is to define analytically nilpotent curvature. We begin with the analogue of nilpotent curvature mod π.

Definition 4.5.1. Let $X=(X_{n'})_{n'\in N'}$ be a bornological pro-module, $S=(S_n)_{n\in N}$ a pro-algebra, and $\omega\colon X\to S$ a pro-linear map. We call ω *nilpotent mod* π if, for each $n\in N$, there is $m\in\mathbb{N}^*$ such that the composite map

$$X^{\otimes m}\xrightarrow{\omega^{\otimes m}} S_m^{\otimes m}\xrightarrow{\text{mult}} S_m\to S_m/\pi S_m \tag{4.5.2}$$

is zero; here mult denotes the m-fold multiplication map of S.

Let $\omega\colon X\to S$ be nilpotent mod π and represent ω by a coherent family of bounded V-module maps $\omega_n\colon X_{r(n)}\to S_n$ with $r(n)\in N'$ for $n\in N$. For $n\in N$ and $n'\in N'$ with $n'\ge r(n)$, let $\omega_{n,n'}\colon X_{n'}\to S_n$ be the composite map $X_{n'}\to X_{r(n)}\to S_n$. Let $n\in N$ and choose m so that the map in (4.5.2) vanishes. Then there is $n'\in N'$ with $n'\ge r(n)$ such that the composite map $X_{n'}^{\otimes m}\to S_n^{\otimes m}\to S_n\to S_n/\pi S_n$ vanishes. That is, $\omega_{n,n'}(x_1)\cdots\omega_{n,n'}(x_m)\in\pi\cdot S_n$ for all $x_1,\dots,x_m\in X_{n'}$. Let $M\subseteq X_{n'}$ be bounded. Since $\omega_{n,n'}$ is bounded and S_n is torsion-free, it follows that $\omega_{n,n'}(M)^m\subseteq\pi S_n$ and that $\pi^{-1}\cdot\omega_{n,n'}(M)^m\subseteq S_n$ is bounded. Then

$$\omega_{n,n'}(M)_e := \sum_{j=1}^{em}\pi^{-\lfloor j/m\rfloor}\omega_{n,n'}(M)^j \tag{4.5.3}$$

is bounded for every $e\ge 1$.

Definition 4.5.4. Let $X = (X_m)_{m\in N'}$ be a bornological pro-module, $S = (S_n)_{n\in N}$ a pro-algebra, and $\omega{:}\,X \to S$ a pro-linear map. Represent ω by a coherent family of bounded V-module maps $\omega_{n,n'}{:}\,X_{n'} \to S_n$ as above. The map ω is called *analytically nilpotent* if, for every n, there are $m \in \mathbb{N}^*$ and $n' \in N'$ with $n' \geq r(n)$ such that for any bounded subset $M \subseteq X_{n'}$, the subset

$$\sum_{j=0}^{\infty} \pi^{-\lfloor j/m \rfloor} \omega_{n,n'}(M)^j \subseteq S_n \otimes F$$

is bounded in S_n.

Proposition 4.5.5. *Let R and S be pro-algebras and $f{:}\,R \to S$ a pro-linear map. Let $\omega{:}\,R\otimes R \to S$, $x\otimes y \mapsto f(x\cdot y) - f(x)\cdot f(y)$, be its curvature. There is a pro-algebra homomorphism $f^{\#}{:}\,\mathcal{T}R \to S$ with $f = f^{\#}\sigma_R = f$ if and only if ω is analytically nilpotent.*

Proof. Write $R = (R_{n'})_{n'\in N'}$ and $S = (S_n)_{n\in N}$ as projective systems of algebras. Identify $\mathcal{T}R$ with the completion of the projective system of bornological algebras $T := (\mathcal{U}(\mathsf{T}R_{n'}, \mathsf{J}R_{n'}^m), \mathcal{B}_m)_{n'\in N', m\in\mathbb{N}^*}$ with the bornologies $\mathcal{B}_m$ in Corollary 4.4.15. Since S is complete, any homomorphism of projective systems of bornological algebras $T \to S$ extends uniquely to $\mathcal{T}R$. Since S is torsion-free, such a homomorphism $T \to S$ is determined by its restriction to $\mathsf{T}R$. Then there is a unique pro-linear map $f{:}\,R \to S$ such that the homomorphism is $f^{\#}{:}\,\mathsf{T}R \to S$ as in (2.6.7). Corollary 4.4.15 shows that $f^{\#}$ extends to a homomorphism $T \to S$ if and only if f has analytically nilpotent curvature. ☐

Corollary 4.5.6. *Let $f{:}\,R \to S$, $g{:}\,S \to T$ be pro-linear maps and let U be a projective system of dagger algebras. If f and g have analytically nilpotent curvature, then so do $g\circ f$ and $f \overleftrightarrow{\otimes} U{:}\,R \overleftrightarrow{\otimes} U \to S \overleftrightarrow{\otimes} U$.*

Proof. The assertion about $g \circ f$ follows as in the proof of [18, Theorem 5.23], using [19, Theorems 3.7 and 4.5]. Since f has analytically nilpotent curvature, there is a homomorphism $f^{\#}{:}\,\mathcal{T}R \to S$ with $f^{\#} \circ \sigma_R = f$. The extension

$$(\mathcal{J}R) \overleftrightarrow{\otimes} U \rightarrowtail (\mathcal{T}R) \overleftrightarrow{\otimes} U \twoheadrightarrow R \overleftrightarrow{\otimes} U$$

is analytically nilpotent by Proposition 4.2.6 because $(\mathcal{J}R) \overleftrightarrow{\otimes} U$ is nilpotent mod π. And $(\mathcal{J}R) \overleftrightarrow{\otimes} U$ is a pro-dagger algebra by the extension of Corollary 2.1.21 to projective systems. The pro-linear section $\sigma_R \overleftrightarrow{\otimes} U$ induces a homomorphism $\mathcal{T}(R \overleftrightarrow{\otimes} U) \to (\mathcal{T}R) \overleftrightarrow{\otimes} U$. When composed with $f^{\#}$, it gives a homomorphism $\mathcal{T}(R \overleftrightarrow{\otimes} U) \to S$ that extends $f \overleftrightarrow{\otimes} U$. Thus $f \overleftrightarrow{\otimes} U$ has analytically nilpotent curvature. ☐

4.6 Homotopy invariance of the X-complex

In this section, we assume that the field F has characteristic 0. This is needed to prove that homotopic homomorphisms defined on a quasi-free algebra induce chain homotopic maps between the X-complexes. If we understand homotopy to mean "polynomial homotopy", then this is already shown by Cuntz and Quillen (see [11, Sections 7–8]). In our context, the proof for polynomial homotopies still works for dagger homotopies. The corresponding statement for the B,b-bicomplexes is [7, Proposition 4.3.3]. For quasi-free algebras, the canonical projection from the B,b-bicomplex to the X-complex is a chain homotopy equivalence. This implies the following:

Proposition 4.6.1. *Let R and S be projective systems of complete bornological F-algebras. Let $f,g\colon R \rightrightarrows S$ be two homomorphisms that are dagger homotopic. Assume that F has characteristic 0 and that R is quasi-free. Then the induced chain maps $X(f),X(g)\colon X(R) \rightrightarrows X(S)$ are chain homotopic.*

Proof. It suffices to treat an elementary dagger homotopy. Define

$$\eta_n\colon \Omega^n(S\otimes V[t])\otimes F \to \Omega^{n-1}(S)\otimes F,$$
$$a_0\,\mathrm{d}a_1\dots\mathrm{d}a_n \mapsto \int_0^1 a_0(t)\frac{\partial a_1(t)}{\partial t}\mathrm{d}a_2(t)\dots\mathrm{d}a_n(t)\,\mathrm{d}t,$$

for $n=1,2$. Here integration and differentiation are defined formally by rescaling the coefficients of polynomials $a_i \in S\otimes F[t]$. We claim that η_n extends to a bounded linear map $\eta_n\colon \Omega^n(S\otimes V[t]^\dagger)\otimes F \to \Omega^{n-1}(S)\otimes F$. To see this, let $T := S\otimes V[t]_{\mathrm{lg}}$. Then $\Omega^n(T)\cong T^+\otimes T^{\otimes n}\cong T^{\otimes n}\oplus T^{\otimes n+1}$. So it suffices to show that η_n is bounded on $T^{\otimes n}\otimes F\cong S^{\otimes n}\otimes V[t]_{\mathrm{lg}}^{\otimes n}\otimes F$. This follows if the map

$$V[t]_{\mathrm{lg}}^{\otimes n+1}\otimes F\to F,$$
$$a_0\otimes a_1\otimes\cdots\otimes a_n \mapsto \int_0^1 a_0(t)\frac{\partial a_1(t)}{\partial t}\cdot a_2(t)\cdots a_n(t)\,\mathrm{d}t$$

is bounded. The formal differentiation on $V[t]_{\mathrm{lg}}$ is clearly bounded. And $V[t]_{\mathrm{lg}}$ is a bornological algebra. So this happens if and only if the integration map

$$V[t]_{\mathrm{lg}}\otimes F\to F,\qquad a(t)=\sum_{l=0}^\infty c_l t^l \mapsto \sum_{l=0}^\infty \frac{c_l}{l+1}$$

is bounded. If $\mathbb{F}$ has characteristic 0, then $l+1$ is invertible in V for all $l\in\mathbb{N}$. If $\mathbb{F}$ has finite characteristic p, then the valuation of $l+1$ grows at most logarithmically. In any case, this is dominated by the linear growth of the exponents of π for a subset of linear growth in $V[t]$. Thus the integration map above is bounded. And then so are the maps η_n. We still write η_n for their unique bounded extensions to the completions.

Let $\eta_0=0$. Then $[\eta,b]=0$. Therefore, $\eta_2(b(\overline{\Omega}^3(S\mathbin{\overline{\otimes}}V[t]^\dagger)))\subseteq b(\overline{\Omega}^2(S))$. Let $X^{(2)}$ be the truncated $B+b$-complex defined in [18, Definition A.122]. We get a map $\eta\colon X^{(2)}(S\mathbin{\overline{\otimes}}V[t]^\dagger)\to X(S)$. Let $\xi_2\colon X^{(2)}(S\mathbin{\overline{\otimes}}V[t]^\dagger)\to X(S\mathbin{\overline{\otimes}}V[t]^\dagger)$ be the canonical projection. Then

$$[\eta,B+b]=(X(\mathrm{ev}_1)-X(\mathrm{ev}_0))\circ\xi\colon X^{(2)}(S\mathbin{\overline{\otimes}}V[t]^\dagger)\to X(S).$$

Now let $H: R \to S \otimes V[t]^\dagger$ be an elementary dagger homotopy between f and g. Then $\eta \circ X^{(2)}(H): X^{(2)}(R) \to X(S)$ is a chain homotopy between $X(f)\circ\xi_2$ and $X(g)\circ\xi_2$, where $\xi_2: X^{(2)}(R) \to X(R)$ is the canonical projection. Since R is analytically quasi-free, it is in particular quasi-free. So ξ_2 is a chain homotopy equivalence. Let $\alpha: X(R) \to X^{(2)}(R)$ be the homotopy inverse of ξ_2. Then $\eta\circ\alpha$ is the desired chain homotopy between $X(f)$ and $X(g)$. □

Theorem 4.6.2. *Let A and B be pro-algebras. If two homomorphisms $f_0, f_1: A \rightrightarrows B$ are dagger homotopic, then they induce homotopic chain maps $\mathbb{HA}(A) \to \mathbb{HA}(B)$. And then* $\mathrm{HA}_*(f_0) = \mathrm{HA}_*(f_1)$.

Proof. The homomorphisms $\mathcal{T}f_0, \mathcal{T}f_1: \mathcal{T}A \rightrightarrows \mathcal{T}B$ lift f_0 and f_1. Since $\mathcal{T}A$ is analytically quasi-free and $\mathcal{J}B$ is analytically nilpotent, Proposition 4.3.10 provides a dagger homotopy between $\mathcal{T}f_0$ and $\mathcal{T}f_1$. Then the chain maps $X(\mathcal{T}A \otimes F) \rightrightarrows X(\mathcal{T}B \otimes F)$ induced by f_0 and f_1 are chain homotopic by Proposition 4.6.1. This remains so on the homotopy projective limits. And then f_0 and f_1 induce the same map on the homology of the homotopy projective limits. That is, $\mathrm{HA}_*(f_0) = \mathrm{HA}_*(f_1)$. □

4.7 Invariance under analytically nilpotent extensions

We continue to assume that F has characteristic 0.

Theorem 4.7.1. *Let $J \rightarrowtail E \overset{p}{\twoheadrightarrow} A$ be a semi-split, analytically nilpotent extension of pro-algebras. Then p induces a chain homotopy equivalence $\mathbb{HA}(E) \simeq \mathbb{HA}(A)$, and $\mathbb{HA}(J)$ is contractible. So $\mathrm{HA}_*(E) \cong \mathrm{HA}_*(A)$ and $\mathrm{HA}_*(J) = 0$. If E is analytically quasi-free, then $\mathbb{HA}(A)$ is chain homotopy equivalent to $X(E\otimes F)$ and $\mathrm{HA}_*(A)$ is isomorphic to the homology of the homotopy projective limit of $X(E \otimes F)$.*

Proof. The composite map $\mathcal{T}E \twoheadrightarrow E \twoheadrightarrow A$ is a pro-algebra homomorphism with a pro-linear section. Its kernel K is an extension of $\mathcal{J}E$ by J and hence analytically nilpotent by Proposition 4.3.13. Both $\mathcal{T}E$ and $\mathcal{T}A$ are analytically quasi-free by Proposition 4.4.6. Proposition 4.3.10 applied to the extensions $K \rightarrowtail \mathcal{T}E \twoheadrightarrow A$ and $\mathcal{J}A \rightarrowtail \mathcal{T}A \twoheadrightarrow A$ shows that $\mathcal{T}A$ and $\mathcal{T}E$ are dagger homotopy equivalent. This together with Proposition 4.6.1 implies that $\mathbb{HA}(A) = X(\mathcal{T}A\otimes F)$ and $\mathbb{HA}(E) = X(\mathcal{T}E\otimes F)$ are homotopy equivalent. This remains so for their homotopy projective limits. So $\mathrm{HA}_*(E) \cong \mathrm{HA}_*(A)$. More precisely, the isomorphism is the map induced by the quotient map $E \twoheadrightarrow A$.
Since J and $\mathcal{J}J$ are analytically nilpotent, so is $\mathcal{T}J$ by Proposition 4.3.13. Since $\mathcal{T}J$ is analytically quasi-free, Proposition 4.3.10 may be applied to the extensions $\mathcal{T}J = \mathcal{T}J \to 0$ and $0 = 0 = 0$ of 0. Thus $\mathcal{T}J$ is dagger homotopy equivalent to 0. Then $\mathbb{HA}(J) \simeq 0$ and $\mathrm{HA}_*(J) \cong 0$.
Now assume E to be analytically quasi-free. Then Proposition 4.3.10 shows that the extensions of A by $\mathcal{T}A$ and E are dagger homotopy equivalent. Then $X(E) \otimes F$ is homotopy equivalent to $X(\mathcal{T}A) \otimes F$. Then $\mathbb{HA}(A)$ is homotopy

equivalent to the homotopy projective limit of the projective system of chain complexes $X(E) \otimes F$. □

Corollary 4.7.2. *Let A be an analytically quasi-free algebra. Then $\mathbb{HA}(A)$ is chain homotopy equivalent to $X(A \otimes F)$ and $\mathrm{HA}_*(A)$ is isomorphic to the homology of $X(A \otimes F)$.*

Proof. By Theorem 4.7.1, $\mathbb{HA}(A)$ is homotopy equivalent to $X(A \otimes F)$. Then $\mathrm{HA}_*(A)$ is isomorphic to the homology of $\operatorname{holim} X(A \otimes F)$. Since $X(A \otimes F)$ is a constant projective system, it is chain homotopy equivalent to its homotopy projective limit. So we simply get the ordinary homology of $X(A \otimes F)$. □

Corollary 4.7.3. *$\mathbb{HA}(V)$ is homotopy equivalent to F with zero boundary map.*

Proof. The algebra V is analytically quasi-free by Proposition 4.3.6. Then $\mathbb{HA}(V) \simeq X(V)$ by Corollary 4.7.2. A small calculation shows that any element of $\Omega^1(V)$ is a commutator. So $X(V)$ is F with zero boundary map. □

5 Excision

The goal of this section is to prove the following excision theorem for analytic cyclic homology:

Theorem 5.1. *Let $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ be a semi-split extension of pro-algebras with a pro-linear section $s: Q \to E$. Then there is a natural exact triangle*

$$\mathbb{HA}(K) \xrightarrow{i_*} \mathbb{HA}(E) \xrightarrow{p_*} \mathbb{HA}(Q) \xrightarrow{\delta} \mathbb{HA}(K)[-1]$$

in the homotopy category of chain complexes of projective systems of bornological V-modules. Thus there is a natural long exact sequence

$$\begin{array}{ccccc} \mathrm{HA}_0(K) & \xrightarrow{i_*} & \mathrm{HA}_0(E) & \xrightarrow{p_*} & \mathrm{HA}_0(Q) \\ \delta\uparrow & & & & \downarrow\delta \\ \mathrm{HA}_1(Q) & \xleftarrow[p_*]{} & \mathrm{HA}_1(E) & \xleftarrow[i_*]{} & \mathrm{HA}_1(K). \end{array}$$

Here the arrows in the “homotopy category” are chain homotopy classes of chain maps. This homotopy category is triangulated over any additive category, with triangles coming from mapping cones of chain maps.
The proof will take up the rest of this section. It follows [17, 18]. We use the left ideal $\mathcal{L}$ in $\mathcal{T}E$ generated by K and prove chain homotopy equivalences $X(\mathcal{T}K) \simeq X(\mathcal{L})$ and $X(\mathcal{L}) \simeq X(\mathcal{T}E : \mathcal{T}Q)$ as chain complexes in the additive category of projective systems of bornological V-modules. First, the pro-linear section s yields two bounded maps $s_L, s_R: \Omega^{\mathrm{ev}}Q \rightrightarrows \Omega^{\mathrm{ev}}E$ defined by

$$s_L(q_0\, \mathrm{d}q_1 \dots \mathrm{d}q_{2n}) := s(q_0)\, \mathrm{d}s(q_1) \dots\, \mathrm{d}s(q_{2n}),$$
$$s_R(\mathrm{d}q_1 \dots \mathrm{d}q_{2n}\, q_{2n+1}) := \mathrm{d}s(q_1) \dots \mathrm{d}s(q_{2n})\, s(q_{2n+1})$$

for all $q_0, q_{2n+1} \in Q^+$ and $q_i \in Q$ for $1 \le i \le 2n$. Let $m \in \mathbb{N}^*$. Both s_L and s_R map $\mathsf{J}Q^{mj}$ to $\mathsf{J}E^{mj}$ for all $j \in \mathbb{N}$ by (4.4.8). Thus they induce bounded linear maps on the tubes, from $\mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)$ to $\mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)$. Both are sections for the canonical projection $\mathcal{U}(\mathsf{T}E, \mathsf{J}E^m) \to \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)$. These sections remain bounded for the linear growth bornologies relative to $\mathcal{U}(\mathsf{J}E, \mathsf{J}E^m)$ and $\mathcal{U}(\mathsf{J}Q, \mathsf{J}Q^m)$ by Proposition 4.4.12. Thus they extend to bounded V-module maps on the completions. These maps for all $m \in \mathbb{N}^*$ form two pro-linear sections for $\mathcal{T}p: \mathcal{T}E \to \mathcal{T}Q$. They induce two sections for the canonical chain map $X(\mathcal{T}p): X(\mathcal{T}E) \to X(\mathcal{T}Q)$. Let

$$X(\mathcal{T}E : \mathcal{T}Q) := \ker\big(X(\mathcal{T}p): X(\mathcal{T}E) \to X(\mathcal{T}Q)\big).$$

There is a semi-split extension of chain complexes

$$X(\mathcal{T}E : \mathcal{T}Q) \rightarrowtail \mathcal{T}E \to \mathcal{T}Q.$$

Since $X(\mathcal{T}p) \circ X(\mathcal{T}i) = X(\mathcal{T}(p \circ i)) = 0$, the chain map $X(\mathcal{T}i)$ factors through a chain map $X(\mathcal{T}K) \to X(\mathcal{T}E : \mathcal{T}Q)$. We are going to prove that the latter is a chain homotopy equivalence. Then the homotopy projective limit of $X(\mathcal{T}K)$ is homotopy equivalent to that of $X(\mathcal{T}E : \mathcal{T}Q)$. And the latter fits into a semi-split extension of chain complexes with the homotopy projective limits of $X(\mathcal{T}E)$ and $X(\mathcal{T}Q)$. As a result, Theorem 5.1 follows if the inclusion map $X(\mathcal{T}K) \to X(\mathcal{T}E : \mathcal{T}Q)$ is a chain homotopy equivalence.

Our construction of the chain homotopy equivalence will, in principle, be explicit and natural, using only the multiplication maps in our pro-algebras and the pro-linear sections s_L and s_R above. Therefore, we assume for simplicity from now on that we are dealing with an extension of (complete, torsion-free bornological) algebras $K \rightarrowtail E \twoheadrightarrow Q$. In general, we may rewrite the semi-split extension above as a projective system of semi-split algebra extensions $K_n \rightarrowtail E_n \twoheadrightarrow Q_n$ with compatible bounded linear sections; this uses arguments as in the proof of Proposition 4.3.13. To simplify notation, we write down the proof below only for a semi-split algebra extension. The chain maps and homotopies that we are going to build for the extensions $K_n \rightarrowtail E_n \twoheadrightarrow Q_n$ form morphisms of projective systems. So the same proof works for a semi-split extension of pro-algebras.

5.1 The pro-algebra $\mathcal{L}$

In the following, we identify $\mathsf{T}E$ with $\Omega^{\mathrm{ev}}E$ and E with $\Omega^0(E) \subseteq \Omega^{\mathrm{ev}}E$. So the map $\sigma_E: E \to \mathcal{T}E$ disappears from our notation. Proposition 4.4.6 gives an isomorphism of left $\mathcal{T}E$-modules

$$(\mathcal{T}E)^+ \overleftarrow{\otimes} E \xrightarrow{\simeq} \mathcal{T}E, \qquad \omega \otimes x \mapsto \omega \odot x. \tag{5.1.1}$$

Explicitly, the inverse of this isomorphism is given by

$$\omega \, \mathrm{d}e_{2n-1} \, \mathrm{d}e_{2n} \mapsto \omega \otimes (e_{2n-1} \cdot e_{2n}) - (\omega \odot e_{2n-1}) \otimes e_{2n}. \tag{5.1.2}$$

These two maps also define an isomorphism for the purely algebraic tensor algebras:

$$(\mathsf{T}E)^+ \otimes E \xrightarrow{\cong} \mathsf{T}E, \qquad \omega \otimes e \mapsto \omega \odot e. \tag{5.1.3}$$

Variants of this isomorphism and the following ones were proven already in [18, Section 4.3.2]. Let $L \subseteq \mathsf{T}E$ be the left ideal generated by K. The bounded linear section $s: Q \to E$ yields an isomorphism of bornological V-modules $E \cong K \oplus Q$. Then (5.1.3) implies an isomorphism

$$(\mathsf{T}E)^+ \otimes K \xrightarrow{\cong} L, \qquad \omega \otimes k \mapsto \omega \odot k. \tag{5.1.4}$$

The explicit formula for the isomorphism in (5.1.2) and its inverse imply

$$L = K \oplus \bigoplus_{n\ge 1} \Omega^{2n-1}(E)\,\mathrm{d}K$$

as in the proof of [18, Lemma 4.55]. Let $I := \ker(\mathsf{T}p: \mathsf{T}E \twoheadrightarrow \mathsf{T}Q)$. This is part of semi-split extensions

$$I \rightarrowtail \mathsf{T}E \overset{\mathsf{T}p}{\underset{s_L}{\twoheadrightarrow}} \mathsf{T}Q \qquad I \rightarrowtail (\mathsf{T}E)^+ \overset{(\mathsf{T}p)^+}{\underset{s_L}{\twoheadrightarrow}} (\mathsf{T}Q)^+. \tag{5.1.5}$$

Lemma 5.1.6. *The following maps are isomorphisms:*

$$\Psi: L^+ \otimes (\mathsf{T}Q)^+ \xrightarrow{\cong} (\mathsf{T}E)^+, \qquad l \otimes \eta \mapsto l \odot s_L(\eta), \tag{5.1.7}$$

$$L \otimes (\mathsf{T}Q)^+ \xrightarrow{\cong} I, \qquad l \otimes \eta \mapsto l \odot s_L(\eta), \tag{5.1.8}$$

$$(\mathsf{T}E)^+ \otimes K \otimes (\mathsf{T}Q)^+ \xrightarrow{\cong} I, \qquad \omega \otimes k \otimes \eta \mapsto \omega \odot k \odot s_L(\eta), \tag{5.1.9}$$

$$(\mathsf{T}Q)^+ \otimes K \otimes (\mathsf{T}E)^+ \xrightarrow{\cong} I, \qquad \eta \otimes k \otimes \omega \mapsto s_R(\eta) \odot k \odot \omega, \tag{5.1.10}$$

$$(\mathsf{T}Q)^+ \otimes K \otimes L^+ \xrightarrow{\cong} L, \qquad \eta \otimes k \otimes l \mapsto s_R(\eta) \odot k \odot l. \tag{5.1.11}$$

Proof. The computations in [18, Section 4.3.1] show this. We briefly sketch them. The isomorphisms (5.1.7) and (5.1.8) are equivalent because of the semi-split extension (5.1.5). And (5.1.8) and (5.1.9) are equivalent because of the isomorphism (5.1.4). The isomorphisms (5.1.9) and (5.1.10) imply each other by taking opposite algebras because this reverses the order of multiplication and exchanges s_L and s_R. And (5.1.10) implies (5.1.11) by substituting $(\mathsf{T}E)^+ \cong L^+ \otimes (\mathsf{T}Q)^+$ and $I \cong L \otimes (\mathsf{T}Q)^+$ in (5.1.10) and then cancelling the factor $(\mathsf{T}Q)^+$ on both sides.

So it suffices to prove that Ψ is an isomorphism. We describe its inverse Ψ^{-1}. Split a differential form $e_0\,\mathrm{d}e_1 \dots \mathrm{d}e_{2n} \in \Omega^{2n}E$ so that each coefficient e_j belongs either to K or $s(Q)$, or is 1 in case of e_0; this is possible because of the direct sum decomposition $E \cong K \oplus s(Q)$; write $k_i := e_i$ or $q_i := s^{-1}(e_i)$ accordingly. If no e_i belongs to K, then

$$\Psi^{-1}\big(s(q_0)\,\mathrm{d}s(q_1) \dots \mathrm{d}s(q_{2n})\big) = 1 \otimes q_0\,\mathrm{d}q_1 \dots \mathrm{d}q_{2n}.$$

Otherwise, there is a largest $i \le 2n$ with $e_i \in K$. If $i = 0$, then

$$\Psi^{-1}\big(k_0 \,\mathrm{d}s(q_1)\dots\mathrm{d}s(q_{2n})\big) = k_0 \otimes \mathrm{d}q_1\dots\mathrm{d}q_{2n}.$$

If i is even and non-zero, then

$$\Psi^{-1}\big(e_0 \,\mathrm{d}e_1\dots\mathrm{d}e_{i-1}\,\mathrm{d}k_i\,\mathrm{d}s(q_{i+1})\dots\mathrm{d}s(q_{2n})\big) = e_0\,\mathrm{d}e_1\dots\mathrm{d}e_{i-1}\,\mathrm{d}k_i{\otimes}\mathrm{d}q_{i+1}\dots\mathrm{d}q_{2n}.$$

If i is odd, then

$$\begin{aligned}\Psi^{-1}\big(e_0 \,\mathrm{d}e_1\dots\mathrm{d}e_{i-1}\,\mathrm{d}k_i\,\mathrm{d}s(q_{i+1})\dots\mathrm{d}s(q_{2n})\big)&\\ = e_0\,\mathrm{d}e_1\dots\mathrm{d}e_{i-1}\odot(k_i\cdot s(q_{i+1}))\otimes\mathrm{d}q_{i+2}\dots\mathrm{d}q_{2n}&\\ - e_0\,\mathrm{d}e_1\dots\mathrm{d}e_{i-1}\odot k_i\otimes q_{i+1}\,\mathrm{d}q_{i+2}\dots\mathrm{d}q_{2n}&.\end{aligned}$$

A direct computation using $\mathrm{d}k_i\,\mathrm{d}s(q_{i+1}) = k_i\cdot s(q_{i+1}) - k_i\odot s(q_{i+1})$ shows that

$$\Psi\circ\Psi^{-1}(e_0\,\mathrm{d}e_1\dots\,\mathrm{d}e_{2n}) = e_0\,\mathrm{d}e_1\dots\,\mathrm{d}e_{2n}$$

for all $e_0 \in \{1\}\cup K\cup s(Q)$, $e_1,\dots,e_n \in K\cup s(Q)$. Then one shows that the map Ψ^{-1} is surjective: its image contains all elements of the form $1\otimes\eta$ for $\eta\in(\mathsf{T}Q)^+$ and $\omega\otimes\mathrm{d}q_1\dots\mathrm{d}q_{2n}$ with $\omega\in L^+$ by the first two cases with no i or even i, respectively. And modulo a term of this form, the image of Ψ^{-1} contains all $\omega\odot k\otimes q_0\,\mathrm{d}q_1\dots\mathrm{d}q_{2n}$ with $\omega\in(\mathsf{T}E)^+$, $k\in K$ because of the formula in the case of odd i. This exhausts $L^+{\otimes}(\mathsf{T}Q)^+$ because of the isomorphism (5.1.4). ☐

We are going to pass to the analytic tensor algebras and describe "analytic" analogues of $L, I \subseteq \mathsf{T}E$ and of the isomorphisms and semi-split extensions above. For $m\in\mathbb{N}^*$, let

$$\begin{aligned}I_{(m)} &:= \ker\big(\mathcal{U}(\mathsf{T}E,\mathsf{J}E^m)\to\mathcal{U}(\mathsf{T}Q,\mathsf{J}Q^m)\big),\\ L_{(m)} &:= K\oplus\bigoplus_{n\ge1}\pi^{-\lfloor n/m\rfloor}\cdot\Omega^{2n-1}(E)\,\mathrm{d}K.\end{aligned}$$

It is easy to see that $I_{(m)}$ is a two-sided and $L_{(m)}$ a left ideal in $\mathcal{U}(\mathsf{T}E,\mathsf{J}E^\infty)$. In particular, both are V-algebras in their own right. Inspection shows that

$$I_{(m)} = \mathcal{U}(\mathsf{T}E,\mathsf{J}E^m)\cap(I\otimes F),\qquad L_{(m)} = \mathcal{U}(\mathsf{T}E,\mathsf{J}E^m)\cap(L\otimes F) \tag{5.1.12}$$

as V-submodules of $\mathsf{T}E\otimes F$. The maps in the projective system $\mathcal{U}(\mathsf{T}E,\mathsf{J}E^\infty)$ make $(I_{(m)})_{m\in\mathbb{N}^*}$ and $(L_{(m)})_{m\in\mathbb{N}^*}$ projective systems by restriction. We equip each $\mathcal{U}(\mathsf{T}E,\mathsf{J}E^m)$ with the bornology $\mathcal{B}_m$ described in Corollary 4.4.15; using the linear growth bornology instead would slightly complicate the estimates below. We give $I_{(m)}$ and $L_{(m)}$ the subspace bornologies. So the bornology on $L_{(m)}$ is cofinally generated by

$$(M\cap K)\oplus\bigoplus_{n=1}^{\infty}\pi^{-\lfloor n/m\rfloor}\Omega^{2n-1}M\,\mathrm{d}(M\cap K) \tag{5.1.13}$$

for bounded V-submodules $M \subseteq E$. Let $\mathcal{I} := \big(\overline{I_{(m)}}\big)_{m\in\mathbb{N}^*}$ and $\mathcal{L} := \big(\overline{L_{(m)}}\big)_{m\in\mathbb{N}^*}$ be the projective systems formed by the completions.

Since $\mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)$ is a subalgebra of $\mathsf{T}E \otimes F$ and the maps in (5.1.3), (5.1.4) and (5.1.7)–(5.1.11) only involve Fedosov products and the maps s_L and s_R, (5.1.12) implies that these maps still exist and are bounded if $\mathsf{T}E, \mathsf{T}Q, I, L$ are replaced by $\mathcal{U}(\mathsf{T}E, \mathsf{J}E^m), \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m), I_{(m)}, L_{(m)}$, respectively, each equipped with the bornologies specified above. The inverse maps for these isomorphisms are slightly more complicated, however: they may shift the index m in the projective system:

Lemma 5.1.14. *The inverses to the isomorphisms above extend to bounded maps*

$$\begin{aligned}
\mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m+1}) &\to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)^+ \otimes E,\\
L_{(m+1)} &\to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)^+ \otimes K,\\
\mathcal{U}(\mathsf{T}E, \mathsf{J}E^{2m})^+ &\to L_{(m)}^+ \otimes \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)^+,\\
I_{(2m)} &\to L_{(m)} \otimes \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)^+,\\
I_{(2m)} &\to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)^+ \otimes K \otimes \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)^+,\\
I_{(2m)} &\to \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)^+ \otimes K \otimes \mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)^+,\\
L_{(2m)} &\to \mathcal{U}(\mathsf{T}Q, \mathsf{J}Q^m)^+ \otimes K \otimes L_{(m)}^+.
\end{aligned}$$

Proof. Our explicit formula for the first map shows that it reduces the total degree of a differential form by at most 2. Since $\frac{n+1}{m+1} \le \frac{n}{m}$ for all $n \ge m$ and $\lfloor \frac{n+1}{m+1} \rfloor = \lfloor \frac{n}{m} \rfloor = 0$ if $n < m$, it follows that it defines a map $\mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m+1}) \to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^m)^+ \otimes E$ that is bounded for the bornologies described in Corollary 4.4.15. The second map is a restriction of the first map, so that it is covered by the same argument.

Our explicit formula for the third map shows that it maps a differential form of degree $2n$ to a sum of tensor products involving differential forms of degree $2j$ and $2(n-j-1)$ or $2(n-j)$; in the first case, $j < n$ and the differential form in L is already explicitly written as $\omega \odot k$, so that the isomorphism $L \to (\mathsf{T}E)^+ \otimes K$ does not reduce the degree any further. This shows that the same degree estimate applies to the fourth map in the lemma. The fifth map differs from that only by taking opposite algebras, and the sixth map is a restriction of the fifth one. This is why the following estimates cover all these maps at the same time.

That these maps are well defined between the relevant tube algebras amounts to the estimate $\lfloor n/2m \rfloor \le \lfloor j/m \rfloor + \lfloor (n-j-1)/m \rfloor$ for all $n \in \mathbb{N}$, $0 \le j < n$. This is trivial for $n < 2m$, so that we assume $n \ge 2m$. For fixed n, the right hand side is minimal if $j = m-1$. And then the needed estimate simplifies to $\lfloor n/2m \rfloor \le \lfloor (n-m)/m \rfloor$. This is true for $2m \le n < 4m$. Since adding $2m$ to n increases $\lfloor n/2m \rfloor$ by 1 and $\lfloor (n-m)/m \rfloor$ by 2, the inequality follows for all $n \in \mathbb{N}$. Now it follows that the maps in the lemma are well defined and bounded for the bornologies described in Corollary 4.4.15. ∎

The composite maps

$$\mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m+1})^+ \otimes E \to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m+1}) \to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m})^+ \otimes E,$$
$$\mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m+1}) \to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m})^+ \otimes E \to \mathcal{U}(\mathsf{T}E, \mathsf{J}E^{m})^+$$

are the structure maps in our projective systems because they extend the identity maps on $(\mathsf{T}E)^+ \otimes E$ and $\mathsf{T}E$, respectively. Thus these two families of maps for $m \in \mathbb{N}^*$ are isomorphisms of projective systems of bornological V-modules that are inverse to each other. This remains so when we complete, giving an isomorphism $(\mathcal{T}E)^+ \overleftarrow{\otimes} E \xrightarrow{\simeq} \mathcal{T}E$. The same argument applies to the other isomorphisms above. Summing up, we get the following isomorphisms of projective systems of bornological V-modules:

$$(\mathcal{T}E)^+ \overleftarrow{\otimes} E \xrightarrow{\simeq} \mathcal{T}E, \qquad \omega \otimes e \mapsto \omega \odot e, \tag{5.1.15}$$
$$(\mathcal{T}E)^+ \overleftarrow{\otimes} K \xrightarrow{\simeq} \mathcal{L}, \qquad \omega \otimes k \mapsto \omega \odot k, \tag{5.1.16}$$
$$\mathcal{L}^+ \overleftarrow{\otimes} (\mathcal{T}Q)^+ \xrightarrow{\simeq} (\mathcal{T}E)^+, \qquad l \otimes \eta \mapsto l \odot s_{\mathcal{L}}(\eta), \tag{5.1.17}$$
$$\mathcal{L} \overleftarrow{\otimes} (\mathcal{T}Q)^+ \xrightarrow{\simeq} \mathcal{I}, \qquad l \otimes \eta \mapsto l \odot s_L(\eta), \tag{5.1.18}$$
$$(\mathcal{T}E)^+ \overleftarrow{\otimes} K \overleftarrow{\otimes} (\mathcal{T}Q)^+ \xrightarrow{\simeq} \mathcal{I}, \qquad \omega \otimes k \otimes \eta \mapsto \omega \odot k \odot s_L(\eta), \tag{5.1.19}$$
$$(\mathcal{T}Q)^+ \overleftarrow{\otimes} K \overleftarrow{\otimes} (\mathcal{T}E)^+ \xrightarrow{\simeq} \mathcal{I}, \qquad \eta \otimes k \otimes \omega \mapsto s_R(\eta) \odot k \odot \omega, \tag{5.1.20}$$
$$(\mathcal{T}Q)^+ \overleftarrow{\otimes} K \overleftarrow{\otimes} \mathcal{L}^+ \xrightarrow{\simeq} \mathcal{L}, \qquad \eta \otimes k \otimes l \mapsto s_R(\eta) \odot k \odot l. \tag{5.1.21}$$

In addition, there are semi-split extensions

$$\mathcal{I} \rightarrowtail \mathcal{T}E \xrightarrow{\mathcal{T}p} \mathcal{T}Q \quad (s_L) \qquad \mathcal{I} \rightarrowtail (\mathcal{T}E)^+ \xrightarrow{(\mathcal{T}p)^+} (\mathcal{T}Q)^+ \quad (s_L). \tag{5.1.22}$$

Here (5.1.15) is the same as (5.1.1). So it follows already from the analytic nilpotence machinery in Section 4. And (5.1.15) easily implies (5.1.16). The isomorphisms (5.1.18)–(5.1.21) follow from (5.1.15)–(5.1.17) and the semi-split extension (5.1.22) as in the proof of Lemma 5.1.6. It seems, however, that the existence of the maps $s_L, s_R: \mathcal{T}Q \rightrightarrows \mathcal{T}E$ and (5.1.17) do not follow from the machinery in Section 4 and must be checked by hand.

Theorem 5.1.23. *The chain map $X(\mathcal{L}) \to X(\mathcal{T}E : \mathcal{T}Q)$ induced by the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}E$ is a chain homotopy equivalence.*

Proof. The proofs of [18, Theorems 4.66 and 4.67] carry over literallly to our analytic tensor algebras, using the isomorphisms (5.1.15)–(5.1.21) and the semi-split extension (5.1.22). We merely have to replace the symbols $\otimes$, $A := \overleftarrow{\mathsf{T}}E$, $\overleftarrow{\mathsf{T}}Q$, $\overleftarrow{L}$, $\overleftarrow{I}$ and $\overleftarrow{G}$ in that proof by $\overleftarrow{\otimes}$, $\mathcal{T}E$, $\mathcal{T}Q$, $\mathcal{L}$, $\mathcal{I}$ and $(\mathcal{T}Q)^+ \overleftarrow{\otimes} K$, respectively; and $\overleftarrow{\Omega}^{\mathrm{ev}}E$ and $\overleftarrow{\Omega}^{\mathrm{odd}}E$ in [18] become $\mathcal{T}E$ and $(\mathcal{T}E)^+ \overleftarrow{\otimes} E$, respectively,

with the latter identified with differential forms of odd degree. [18, Theorem 5.80] is a similar translation exercise for the analytic cyclic homology theory for bornological algebras over the complex numbers, and the situation in this article is quite similar.
We briefly sketch the main idea of the proof. Proposition 4.4.6 and the definition of $\overline{\Omega}^1(\mathcal{T}E)$ imply that there is a semi-split free $\mathcal{T}E$-bimodule resolution

$$\overline{\Omega}^1(\mathcal{T}E) \rightarrowtail (\mathcal{T}E)^+ \overline{\otimes} (\mathcal{T}E)^+ \twoheadrightarrow (\mathcal{T}E)^+$$

with a natural pro-linear section $(\mathcal{T}E)^+ \to (\mathcal{T}E)^+ \overline{\otimes} (\mathcal{T}E)^+$, $x \mapsto 1 \otimes x$. Let

$$P_0 \coloneqq \mathcal{L}^+ \overline{\otimes} \mathcal{L}^+ + (\mathcal{T}E)^+ \overline{\otimes} \mathcal{L} \subseteq (\mathcal{T}E)^+ \overline{\otimes} (\mathcal{T}E)^+,$$
$$P_1 \coloneqq (\mathcal{T}E)^+ D\mathcal{L} \subseteq \overline{\Omega}^1(\mathcal{T}E)^+.$$

This together with $\mathcal{L}^+ \subseteq (\mathcal{T}E)^+$ gives a subcomplex of the resolution above. The standard section above yields a contracting homotopy for it, making it a resolution. The bimodules P_0 and P_1 are free; this is where the isomorphisms above enter. So $P_1 \rightarrowtail P_0 \twoheadrightarrow \mathcal{L}^+$ is a free $\mathcal{L}$-bimodule resolution. Then $\mathcal{L}$ is quasi-free, and the X-complex computes its periodic cyclic homology. And the commutator quotient complex $P_1/[\mathcal{L}, P_1] \to P_0/[\mathcal{L}, P_0]$ computes the Hochschild homology of $\mathcal{L}$. These commutator quotients are computed explicitly and shown to compute the relative Hochschild homology for the quotient map $\mathcal{T}E \twoheadrightarrow \mathcal{T}Q$. And then the isomorphism on Hochschild homology implies an isomorphism in cyclic homology and thus periodic cyclic homology. □

5.2 Analytic quasi-freeness of $\mathcal{L}$

The proof of the excision theorem is completed by the following theorem:

Theorem 5.2.1. *The pro-algebra $\mathcal{L}$ is analytically quasi-free and there is a semi-split, analytically nilpotent extension $\mathcal{J}E \cap \mathcal{L} \rightarrowtail \mathcal{L} \twoheadrightarrow K$.*

This theorem and Theorem 4.7.1 imply that $\mathbb{HA}(K)$ is chain homotopy equivalent to the X-complex of $\mathcal{L}$. Theorem 5.1.23 identifies this with the relative X-complex $X(\mathcal{T}E : \mathcal{T}Q)$. And this yields the excision theorem. So it only remains to prove Theorem 5.2.1.
The canonical projection $\mathcal{T}E \twoheadrightarrow E$ restricts to a semi-split projection $\mathcal{L} \twoheadrightarrow K$. Its kernel $\mathcal{J}E \cap \mathcal{L} \subseteq \mathcal{J}E$ is a projective system of closed subalgebras. These are complete and torsion-free by [19, Theorem 2.3 and Lemma 4.2]; and subalgebras also clearly inherit the property of being semi-dagger. So $\mathcal{J}E \cap \mathcal{L}$ is a projective system of dagger algebras. Proposition 4.2.6 implies that it is again nilpotent mod π because $\mathcal{J}E/(\mathcal{J}E \cap \mathcal{L})$ is torsion-free.
The proof of Theorem 5.1.23 already shows that $\mathcal{L}$ is quasi-free. We need it to be analytically quasi-free, however. This is the main difficulty in Theorem 5.2.1. The proof of this uses the same ideas as the proof of the corresponding statement for analytic cyclic homology for bornological algebras over $\mathbb{C}$ in [18]. First,

we define a homomorphism $\upsilon\colon L \to \mathsf{T}L$ for the purely algebraic version L of $\mathcal{L}$. Then we show that this homomorphism extends uniquely to a homomorphism of pro-algebras $\mathcal{L} \to \mathcal{TL}$ that is a section for the canonical projection $\mathcal{TL} \twoheadrightarrow \mathcal{L}$.
We need some notation for elements of $\mathsf{T}L$ and a certain grading on $\mathsf{T}L$. Elements of $\mathsf{T}L$ are sums of differential forms $l_0\,\mathrm{D}l_1\dots\mathrm{D}l_{2n}$ with $l_0 \in L^+$, $l_1,\dots,l_{2n} \in L$. We write $\circledcirc$ for the Fedosov product in $\Omega^{\mathrm{ev}}L$ to distinguish it from the Fedosov product $\odot$ in L and the resulting usual multiplication on ΩL. Call an element of $\mathsf{T}L$ *elementary* if it is of the form $l_0\,\mathrm{D}l_1\dots\mathrm{D}l_{2n}$ with $l_j = e_{j,0}\,\mathrm{d}e_{j,1}\dots\mathrm{d}e_{j,2i_j}$ for $0 \le j \le 2n$, and $e_{j,k} \in K \cup s(Q)$ for all occurring indices j,k, except that we allow $e_{j,0} = 1$ if $i_j \ge 1$ and $l_0 = 1$ if $i_0 = 0$; here $e_{j,2i_j} \in K$ because $l_j \in L$. Any element of $\mathsf{T}L$ is a finite linear combination of such elementary elements. The *entries* of an elementary element ξ are the elements $e_{j,l} \in E$; its *internal degree* is $\deg_i(\xi) = \sum_{j=0}^{2n} 2i_j$; its *external degree* is $\deg_e(\xi) = 6n$ if $l_0 \in L$ and $\deg_e = 6n-4$ if $l_0 = 1$, and the total degree $\deg_t(\xi)$ is the sum of these two degrees; this particular total degree already appears in the proof of [18, Lemma 5.102].
The definition of υ is based on the isomorphism $L \cong (\mathsf{T}E)^+ \otimes K$ in (5.1.4). The restriction of υ to $K = (\Omega^0\mathsf{T}E \cap L) \subseteq L$ is the obvious inclusion of K into $\mathsf{T}L$. We extend this map to L using a homomorphism from $\mathsf{T}E$ to the algebra of V-module homomorphisms $\mathsf{T}L \to \mathsf{T}L$. Such a homomorphism is equivalent to a linear map $E \to \mathrm{Hom}(\mathsf{T}L,\mathsf{T}L)$, which is, in turn, equivalent to a V-bilinear map $E \times \mathsf{T}L \to \mathsf{T}L$, which we denote as an operation $(e,\xi) \mapsto e \triangleright \xi$ for $e \in E$, $\xi \in \mathsf{T}L$. As in [18], we first define the map $\nabla\colon L \to \Omega^1(L)$ by $\nabla(s_R(\xi) \odot k \odot l) := s_R(\xi) \odot k\,\mathrm{D}l$ for all $\xi \in (\mathsf{T}Q)^+$, $k \in K$, $l \in L^+$, with the understanding that $\mathrm{D}1 = 0$ if l is the unit element of L; this uses the inverse of the isomorphism (5.1.11). Then we let

$$\begin{aligned} e \triangleright x_0\,\mathrm{D}x_1\dots\mathrm{D}x_{2n} &= e \odot x_0\,\mathrm{D}x_1\dots\mathrm{D}x_{2n} - \mathrm{D}\nabla(e \odot x_0)\,\mathrm{D}x_1\dots\mathrm{D}x_{2n},\\ e \triangleright \mathrm{D}x_1\,\mathrm{D}x_2\dots\mathrm{D}x_{2n} &= \nabla(e \odot x_1)\,\mathrm{D}x_2\dots\mathrm{D}x_{2n}. \end{aligned}$$

The curvature of the corresponding map $E \to \mathrm{Hom}(\mathsf{T}L,\mathsf{T}L)$ acts by the operation $\omega_\triangleright(e_1,e_2)\xi := (e_1 \cdot e_2) \triangleright \xi - e_1 \triangleright (e_2 \triangleright \xi)$. It is computed in [18, Equation (5.91)]:

$$\begin{aligned} \omega_\triangleright(e_1,e_2)l_0\,\mathrm{D}l_1\dots\mathrm{D}l_{2n} &= (\mathrm{d}e_1\mathrm{d}e_2 \odot l_0)\,\mathrm{D}l_1\dots\mathrm{D}l_{2n}\\ &\qquad + \nabla\big(e_1 \odot \nabla(e_2 \odot l_0)\big)\,\mathrm{D}l_1\dots\mathrm{D}l_{2n}\\ &\qquad - \mathrm{D}\nabla(\mathrm{d}e_1\mathrm{d}e_2 \odot l_0)\,\mathrm{D}l_1\dots\mathrm{D}l_{2n},\\ \omega_\triangleright(e_1,e_2)\mathrm{D}l_1\dots\mathrm{D}l_{2n} &= \nabla\big((e_1 \cdot e_2) \odot l_1\big)\,\mathrm{D}l_2\dots\mathrm{D}l_{2n}\\ &\qquad - e_1 \odot \nabla(e_2 \odot l_1)\,\mathrm{D}l_2\dots\mathrm{D}l_{2n}\\ &\qquad + \mathrm{D}\nabla\big(e_1 \odot \nabla(e_2 \odot l_1)\big)\,\mathrm{D}l_2\dots\mathrm{D}l_{2n}. \end{aligned}$$

Finally, we define

$$\upsilon(e_0\,\mathrm{d}e_1\dots\mathrm{d}e_{2n} \odot k) := e_0 \triangleright \big(\omega_\triangleright(e_1,e_2) \circ \dots \circ \omega_\triangleright(e_{2n-1},e_{2n})\big)(k).$$

Lemma 5.2.2. *The map* $\upsilon: L \to \mathsf{T}L$ *is an algebra homomorphism, and* $p \circ \upsilon = \mathrm{id}_L$ *for the canonical projection* $p: \mathsf{T}L \to L$.
If $l \in \Omega^{2n-1}(E)\,\mathrm{d}K \subseteq L$ *has degree* $2n$*, then* $\upsilon(l)$ *is a sum of elementary elements of* $\mathsf{T}L$ *with total degree at least* $2n$.
Let $M \subseteq E$ *be a bounded* V*-submodule. There is a bounded subset* $M' \subseteq E$ *such that if* $e_0\,\mathrm{d}e_1 \dots \mathrm{d}e_{2n} \in \Omega^{2n} M \cap L$*, then* $\upsilon(e_0\,\mathrm{d}e_1 \dots \mathrm{d}e_{2n})$ *is a sum of elementary elements of* $\mathsf{T}L$ *with entries in* M'.

Proof. As shown in [18] or in [17], the left action $\triangleright$ is by left multipliers, that is, $e \triangleright (\xi \circledcirc \tau) = (e \triangleright \xi) \circledcirc \tau$ for all $e \in E$, $\xi, \tau \in \mathsf{T}L$. And $k \triangleright \xi = k \circledcirc \xi$ for all $k \in K$. This implies that υ is a homomorphism.
A short computation shows that each summand in the formula for $\omega_\triangleright(e_1, e_2)$ increases the total degree defined above by at least 2; this is already shown in the proof of [18, Lemma 5.102]. By induction on n, it follows that υ maps $\Omega^{2n} L$ into the subgroup spanned by elementary elements of $\mathsf{T}L$ with total degree at least $2n$.
Given a bounded subset $M \subseteq E$, the proof of [18, Lemma 5.92] provides a bounded subset $M' \subseteq E$ such that $\upsilon(e_0\,\mathrm{d}e_1 \dots \mathrm{d}e_{2n} \odot k)$ is a sum of elementary elements of $\mathsf{T}L$ with entries in M'. □

The homomorphism υ induces an F-algebra homomorphism $L \otimes F \to \mathsf{T}L \otimes F$. Recall that

$$L_{(m)} := K \oplus \bigoplus_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor} \Omega^{2n-1}(E)\,\mathrm{d}K$$

for $m \in \mathbb{N}^*$. These are V-subalgebras of $L \otimes F$ that satisfy $L_{(n)} \subseteq L_{(m)}$ if $n \geq m$. Each $L_{(m)}$ is equipped with the bornology cofinally generated by the submodules in (5.1.13).
Let $(\mathsf{T}L)_{(m)} \subseteq \mathsf{T}L \otimes F$ be the subgroup generated by $\pi^{-\lfloor d/m \rfloor}\xi$ for elementary elements ξ of total degree d. These are V-subalgebras of $\mathsf{T}L \otimes F$ that satisfy $(\mathsf{T}L)_{(n)} \subseteq (\mathsf{T}L)_{(m)}$ if $n \geq m$. If $M \subseteq E$ is a bounded V-submodule, then let $D^{\mathsf{T}}_m(M) \subseteq (\mathsf{T}L)_{(m)}$ be the subgroup generated by $\pi^{-\lfloor d/m \rfloor}\xi$ for elementary elements ξ of total degree d. We give $(\mathsf{T}L)_{(m)}$ the bornology that is cofinally generated by these V-submodules. This bornology is the analogue of the bornology in Corollary 4.4.15. It is torsion-free and makes the multiplication in $(\mathsf{T}L)_{(m)}$ and the inclusion maps $(\mathsf{T}L)_{(n)} \hookrightarrow (\mathsf{T}L)_{(m)}$ for $n \geq m$ bounded. So we have turned $\big((\mathsf{T}L)_{(m)}\big)_{m \in \mathbb{N}^*}$ into a projective system of torsion-free bornological algebras.
The second paragraph in Lemma 5.2.2 says that the extension $L \otimes F \to \mathsf{T}L \otimes F$ of υ maps $L_{(m)}$ to $(\mathsf{T}L)_{(m)}$ for each $m \in \mathbb{N}^*$. And the third paragraph says that this homomorphism is bounded. Thus υ is a homomorphism of projective systems of bornological algebras. By Corollary 4.4.15, $\mathcal{L}$ is isomorphic to the projective system of the completions $\overline{L_{(m)}}$ for $m \in \mathbb{N}^*$, with the bornologies described above.

Lemma 5.2.3. *The embedding* $\mathsf{T}L \hookrightarrow \mathcal{T}L$ *extends to an isomorphism of projective systems from the projective system of completions* $\overline{(\mathsf{T}L)_{(m)}}$ *for* $m \in \mathbb{N}^*$ *to* $\mathcal{T}\mathcal{L}$.

Proof. For a bounded V-submodule $M \subseteq E$, let $M_K := M \cap K$ and let $\overleftrightarrow{\Omega}^0_{\mathcal{L}}(M) := M_K$ and $\overleftrightarrow{\Omega}^{2k}_{\mathcal{L}}(M) := \overleftrightarrow{\Omega}^{2k-1}(M) \mathbin{\overleftrightarrow{\otimes}} M_K$ for $k > 1$. A proof like that for Proposition 4.4.16 shows that the completion of $(\mathsf{T}L)_{(m)}$ is the union of the products

$$\prod_{j\geq 0, i_0,\dots,i_{2j}\geq 0}^{\infty} \pi^{-\lfloor (6j+2i_0+\dots+2i_{2j})/m\rfloor} \overleftrightarrow{\Omega}^{2i_0}_{\mathcal{L}}(M) \mathbin{\overleftrightarrow{\otimes}} \overleftrightarrow{\Omega}^{2i_1}_{\mathcal{L}}(M) \mathbin{\overleftrightarrow{\otimes}} \dots \mathbin{\overleftrightarrow{\otimes}} \overleftrightarrow{\Omega}^{2i_{2j}}_{\mathcal{L}}(M)$$
$$\times \prod_{j\geq 0, i_1,\dots,i_{2j}\geq 0}^{\infty} \pi^{-\lfloor (6j-4+2i_0+\dots+2i_{2j})/m\rfloor} \overleftrightarrow{\Omega}^{2i_1}_{\mathcal{L}}(M) \mathbin{\overleftrightarrow{\otimes}} \overleftrightarrow{\Omega}^{2i_2}_{\mathcal{L}}(M) \mathbin{\overleftrightarrow{\otimes}} \dots \mathbin{\overleftrightarrow{\otimes}} \overleftrightarrow{\Omega}^{2i_{2j}}_{\mathcal{L}}(M) \tag{5.2.4}$$

taken over all bounded V-submodules $M \subseteq E$; elementary tensors in a factor of the first product correspond to differential forms $l_0 \,\mathrm{D}l_1 \dots \mathrm{D}l_{2j}$ with $l_0, \dots, l_{2j} \in \mathcal{L}$ and $\deg(l_j) = 2i_j$, whereas those for the second product correspond to differential forms $\mathrm{D}l_1 \dots \mathrm{D}l_{2j}$. The exponent of π is the total degree defined above.

Proposition 4.4.16 describes $\mathcal{T}E$. The pro-subalgebra $\mathcal{L}$ is described similarly, by also asking for the last entry of all differential forms to belong to K. Then a second application of Proposition 4.4.16 describes $\mathcal{T}\mathcal{L}$. The result is very similar to the projective system above. The only difference is that the exponent of π in the bornology is replaced by $h := \lfloor j/k\rfloor + \sum_{l=0}^{2j} \lfloor i_l/m\rfloor$ for each factor in (5.2.4), for some parameters $k, m \in \mathbb{N}^*$. Here we may take $k = m$ because this gives a cofinal subset. So it remains to prove linear estimates between these two notions of "degree". In one direction, this is the trivial estimate

$$\left\lfloor \frac{j}{m} \right\rfloor + \sum_{l=0}^{2j} \left\lfloor \frac{i_l}{m} \right\rfloor \leq \left\lfloor \frac{j}{m} + \sum_{l=0}^{2j} \frac{i_l}{m} \right\rfloor \leq \left\lfloor \frac{1}{m}\left(6j + \sum_{l=0}^{2j} 2i_l\right)\right\rfloor$$

for $j \geq 0$ and a similar estimate with $6j - 4 = 4(j-1) + 2j$ instead of $6j$ for $j \geq 1$. In the other direction, we distinguish two cases. Let $i := \sum i_l$. If $i < 4j \cdot m$, then $6j + 2i < j \cdot (6 + 8m)$ and we estimate

$$\left\lfloor \frac{j}{m} \right\rfloor + \sum_{l=0}^{2j} \left\lfloor \frac{i_l}{m} \right\rfloor \geq \left\lfloor \frac{j}{m} \right\rfloor \geq \left\lfloor \frac{6j + 2i}{(6 + 8m) \cdot m} \right\rfloor.$$

The other case is $i \geq 4j \cdot m$. Each floor operation changes a number by at most 1, and $6j + 2i \leq \frac{3}{2m} i + 2i \leq 4i$. So

$$\left\lfloor \frac{j}{m} \right\rfloor + \sum_{l=0}^{2j} \left\lfloor \frac{i_l}{m} \right\rfloor \geq \frac{i}{m} - 2j \geq \frac{i}{2m} \geq \left\lfloor \frac{6j + 2i}{8m} \right\rfloor.$$

As a result, υ defines a pro-algebra homomorphism $\mathcal{L} \to \mathcal{T}\mathcal{L}$. Then $\mathcal{L}$ is analytically quasi-free. This ends the proof of the excision theorem. ∎

6 Stability for algebras of matrices

A *matricial pair* consists of two torsion-free bornological modules X and Y and a surjective linear map $\langle\cdot,\cdot\rangle\colon Y\otimes X\to V$. Any such map is bounded. A *homomorphism* from (X,Y) to another matricial pair (W,Z) is a pair $f=(f_1,f_2)$ of bounded linear homomorphisms $f_1\colon X\to W$, $f_2\colon Y\to Z$ such that $\langle f_2(y), f_1(x)\rangle=\langle y,x\rangle$ for all $x\in X$ and $y\in Y$. An *elementary homotopy* is a pair $H=(H_1,H_2)$ of bounded linear maps, where $H_1\colon X\to W[t]$ and $H_2\colon Y\to Z$ or $H_1\colon X\to W$ and $H_2\colon Y\to Z[t]$, such that the following diagram commutes:

$$\begin{array}{ccc} Y\otimes X & \xrightarrow{H_2\otimes H_1} & Z\otimes W[t] \\ \downarrow{\scriptstyle\langle\,,\,\rangle} & & \downarrow{\scriptstyle\langle\,,\,\rangle\otimes \mathrm{id}} \\ V & \xrightarrow{\text{inc}} & V[t] \end{array}$$

Let (X,Y) be a matricial pair. Let $\mathcal{M}=\mathcal{M}(X,Y)$ be $X\otimes Y$ with the product

$$(x_1\otimes y_1)(x_2\otimes y_2)=\langle y_1, x_2\rangle x_1\otimes y_2.$$

This product is associative and bounded, and it even makes $\mathcal{M}$ a semi-dagger algebra. The bornological algebra $\mathcal{M}$ is also torsion-free by [19, Proposition 4.12]. Thus the completion $\overline{\mathcal{M}}$ is a dagger algebra and $\overline{\mathcal{M}}=\mathcal{M}^\dagger$.
Homomorphisms and homotopies of matricial pairs induce homomorphisms and homotopies of the corresponding algebras. Any pair $(\xi,\eta)\in X\times Y$ with $\langle\eta,\xi\rangle=1$ yields a bounded algebra homomorphism

$$\iota=\iota_{\xi,\eta}\colon V\to\mathcal{M},\qquad \iota(1)=\xi\otimes\eta.$$

We shall also write ι for the composite of the map above with the completion map $\mathcal{M}\to\overline{\mathcal{M}}=\mathcal{M}^\dagger$. If R is a torsion-free bornological algebra, then $R\mathbin{\overline{\otimes}}\mathcal{M}^\dagger$ is torsion-free by [19, Theorem 4.6 and Propositions 14.11 and 14.12]. Define

$$\iota_R\coloneqq \mathrm{id}_R\otimes\iota\colon R\to R\mathbin{\overline{\otimes}}\mathcal{M}^\dagger. \tag{6.1}$$

Proposition 6.2. *Let R be a complete, torsion-free bornological algebra. Then the map ι_R induces a chain homotopy equivalence $\mathbb{HA}(R)\simeq\mathbb{HA}(R\mathbin{\overline{\otimes}}\mathcal{M}^\dagger)$ and an isomorphism $\mathrm{HA}_*(R)\cong\mathrm{HA}_*(R\mathbin{\overline{\otimes}}\mathcal{M}^\dagger)$.*

Proof. Corollary 4.5.6 yields a natural pro-algebra homomorphism

$$\mathcal{T}(R\mathbin{\overline{\otimes}}\mathcal{M}^\dagger)\to\mathcal{T}(R)\mathbin{\overline{\otimes}}\mathcal{M}^\dagger$$

covering the identity of $R\mathbin{\overline{\otimes}}\mathcal{M}^\dagger$. And any elementary homotopy between matricial pairs (X,Y) and (W,Z) yields an elementary dagger homotopy $\mathcal{M}(X,Y)^\dagger\to\mathcal{M}(Z,W)^\dagger\mathbin{\overline{\otimes}}V[t]^\dagger$. The X-complex is invariant under dagger homotopies by Proposition 4.6.1. Taking all this into account, the argument of the proof of [18, Theorem 5.65] now applies verbatim and proves the proposition. ◻

Let Λ be a set. We now describe increasingly complicated algebras of matrices indexed by the set Λ.

Example 6.3. Let $V^{(\Lambda)}$ be the V-module of finitely supported functions $\Lambda \to V$. This is the free V-module on Λ. The characteristic functions of the singletons $\{\chi_\lambda : \lambda \in \Lambda\}$ form a module basis. The algebra $\mathcal{M}(V^{(\Lambda)}, V^{(\Lambda)})$ associated to the bilinear form $\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda,\mu}$ is the algebra M_Λ of finitely supported $\Lambda \times \Lambda$-matrices with the fine bornology. It is already a dagger algebra. Proposition 6.2 implies $\mathbb{HA}(R) \cong \mathbb{HA}(M_\Lambda \otimes R)$ for all R.

Example 6.4. Define $V^{(\Lambda)}$ as in Example 6.3. Its π-adic completion is the Banach module $c_0(\Lambda) := c_0(\Lambda, V)$ with the supremum norm. The bilinear form in Example 6.3 extends to $c_0(\Lambda)$. The π-adic completion of $\mathcal{M}(c_0(\Lambda), c_0(\Lambda))$ is isomorphic to the Banach V-algebra $M^0_\Lambda \cong c_0(\Lambda \times \Lambda)$ of matrices indexed by $\Lambda \times \Lambda$ with entries in V that go to zero at infinity. The Banach V-modules above become bornological by declaring all subsets to be bounded. The completions and tensor products as Banach V-modules and as bornological V-modules are the same. Proposition 6.2 implies $\mathbb{HA}(R) \cong \mathbb{HA}(M^0_\Lambda \overline{\otimes} R)$ for all R.

Example 6.5. Let $\ell : \Lambda \to \mathbb{N}$ be a proper function, that is, for each $n \in \mathbb{N}$ the set of $x \in \Lambda$ with $\ell(x) \le n$ is finite. Define $V^{(\Lambda)}$ as in Example 6.3 and give it the bornology that is cofinally generated by the V-submodules

$$S_m := \sum_{\lambda \in \Lambda} \pi^{\lfloor \ell(\lambda)/m \rfloor} \chi_\lambda$$

for $m \in \mathbb{N}^*$. The bilinear form in Example 6.3 remains bounded for this bornology on $V^{(\Lambda)}$. So $\mathcal{M}(V^{(\Lambda)}, V^{(\Lambda)})$ with the tensor product bornology from the above bornology is a bornological algebra as well. It is torsion-free and semi-dagger. So its dagger completion is the same as its completion. We denote it by M^ℓ_Λ. It is isomorphic to the algebra of infinite matrices $(c_{x,y})_{x,y \in \Lambda}$ for which there is $m \in \mathbb{N}^*$ such that $c_{x,y} \in \pi^{\lfloor (\ell(x)+\ell(y))/m \rfloor}$ for all $x, y \in \Lambda$; this is the same as asking for $\lim \left| c_{x,y} \pi^{-\lfloor (\ell(x)+\ell(y))/m \rfloor} \right| = 0$ because ℓ is proper. It makes no difference to replace the exponent of π by $\lfloor \ell(x)/m \rfloor + \lfloor \ell(y)/m \rfloor$ or $\lfloor \max\{\ell(x), \ell(y)\}/m \rfloor$ because we may vary m. Proposition 6.2 implies $\mathbb{HA}(R) \cong \mathbb{HA}(M^\ell_\Lambda \overline{\otimes} R)$ for all R.

The following completed matrix algebras will be needed in Section 8.

Example 6.6. Let Λ be a set with a filtration by a directed set I. That is, there are subsets $\Lambda_S \subseteq \Lambda$ for $S \in I$ with $\Lambda_S \subseteq \Lambda_T$ for $S \le T$ and $\Lambda = \bigcup_{S \in I} \Lambda_S$. Let $\ell : \Lambda \to \mathbb{N}$ be a function whose restriction to Λ_S is proper for each $S \in I$. For $S \in I$, form the matrix algebra $M^\ell_{\Lambda_S}$ as in Example 6.5. These algebras for $S \in I$ form an inductive system. Let $\varinjlim M^\ell_{\Lambda_S}$ be its bornological inductive limit. This bornological algebra is also associated to a matricial pair, namely, the pair based on $\varinjlim V^{(\Lambda_S)}$, where each $V^{(\Lambda_S)}$ carries the bornology described in Example 6.5. Proposition 6.2 implies $\mathbb{HA}(R) \cong \mathbb{HA}(\varinjlim M^\ell_{\Lambda_S} \overline{\otimes} R)$ for all R.

7 Morita functoriality

In this section, we show that analytic cyclic homology is functorial for certain bimodules. Let A and B be unital, torsion-free, complete bornological V-algebras and let P be an A-B-bimodule. Assume P to be finitely generated and projective as a right B-module. Then there are $n \in \mathbb{N}$ and an idempotent matrix $e \in \mathbb{M}_n(B)$ such that $P \cong eB^n$. The left action of A on P induces a V-algebra homomorphism

$$\iota_A: A \to \mathrm{End}_B(P) \cong e\mathbb{M}_n(B)e \subseteq \mathbb{M}_n(B).$$

Proposition 6.2 describes a chain homotopy equivalence $\mathbb{HA}(B) \cong \mathbb{HA}(\mathbb{M}_n(B))$ for any $n \in \mathbb{N}_{\geq 1}$. Composing this with the map induced by ι_A gives a chain map

$$\mathbb{HA}(P): \mathbb{HA}(A) \to \mathbb{HA}(B).$$

This induces maps $\mathrm{HA}_*(P): \mathrm{HA}_*(A) \to \mathrm{HA}_*(B)$ for $* = 0, 1$.

Lemma 7.1. *The homotopy class of $\mathbb{HA}(P)$ only depends on the isomorphism class of P. That is, if $P \cong e \cdot B^n \cong f \cdot B^m$ for $n, m \in \mathbb{N}$ and idempotent $e \in \mathbb{M}_n(B)$, $f \in \mathbb{M}_m(B)$, then the resulting chain maps $\mathbb{HA}(A) \to \mathbb{HA}(B)$ are chain homotopic. If $A = B = P$, then $\mathbb{HA}(P)$ is homotopic to the identity chain map.*

Proof. The chain homotopy equivalence $\mathbb{HA}(B) \cong \mathbb{HA}(\mathbb{M}_n(B))$ in Proposition 6.2 is induced by the corner embedding $\iota_n: B \to \mathbb{M}_n(B)$, $b \mapsto b \cdot E_{11}$. If $k \geq n$, then the inclusion $j_{kn}: \mathbb{M}_n(B) \to \mathbb{M}_k(B)$ that appends zeros on the right and at the bottom satisfies $j_{kn} \circ \iota_n = \iota_k$. Hence there is a commuting diagram of chain homotopy equivalences

$$\begin{array}{ccc} \mathbb{HA}(B) & \xrightarrow{\mathbb{HA}(\iota_n)} & \mathbb{HA}(\mathbb{M}_n(B)) \\ & \searrow^{\mathbb{HA}(\iota_k)} & \downarrow{\scriptstyle \mathbb{HA}(j_{kn})} \\ & & \mathbb{HA}(\mathbb{M}_k(B)). \end{array}$$

Therefore, the maps $\mathbb{HA}(A) \to \mathbb{HA}(B)$ remain unchanged when we replace e and f by $j_{kn}(e) \in \mathbb{M}_k(B)$ and $j_{km}(f) \in \mathbb{M}_k(B)$ for $k \geq n, m$. This allows us to reduce to the case $n = m$. And then we may still choose $k = 2n = 2m$ to create extra room.

Since $fB^m \cong eB^m$, the idempotent matrices e and f are *Murray–von-Neumann equivalent*. That is, there are matrices $v, w \in \mathbb{M}_m(B)$ with

$$e = vw, \qquad f = wv, \qquad vwv = v, \qquad wvw = w.$$

Let $\iota_A^e, \iota_A^f: A \rightrightarrows \mathbb{M}_m(B)$ be the two homomorphisms defined above using the idempotent elements e and f, respectively. By construction, $\iota_A^e(a) = v\iota_A^f(a)w$

and $\iota_A^f(a) = w\iota_A^e(a)v$ for all $a \in A$. It is well known that $j_{2m,m}(e)$ and $j_{2m,m}(f)$ are homotopic. We recall the elementary proof. Let

$$v_t := tv + (1-t), \qquad w_t := tw + (1-t)$$

in $\mathbb{M}_m(B[t])$ and let

$$u_t := \begin{pmatrix} v_t & v_t w_t - 1 \\ 1 - w_t v_t & 2w_t - w_t v_t w_t \end{pmatrix}, \qquad u_t^{-1} := \begin{pmatrix} 2w_t - w_t v_t w_t & 1 - w_t v_t \\ v_t w_t - 1 & v_t \end{pmatrix}.$$

Easy computations show that the latter two elements of $\mathbb{M}_{2m}(B[t])$ satisfy $u_0 = 1$ and $u_t u_t^{-1} = 1 = u_t^{-1} u_t$. And

$$u_1 \begin{pmatrix} \iota_A^f(a) & 0 \\ 0 & 0 \end{pmatrix} u_1^{-1} = \begin{pmatrix} \iota_A^e(a) & 0 \\ 0 & 0 \end{pmatrix}$$

for all $a \in A$. Therefore, conjugation by u_t defines a polynomial homotopy between the homomorphisms ι_A^e and ι_A^f. Since $\mathbb{HA}$ is homotopy invariant by Theorem 4.6.2, it follows that the chain maps $\mathbb{HA}(A) \to \mathbb{HA}(\mathbb{M}_{2m}(B))$ induced by ι_A^e and ι_A^f are chain homotopic.
To prove the last claim about $\mathbb{HA}(A)$ for the identity bimodule A, use $m = 1$ and $e = 1$. Then $\iota_A \colon A \to \mathbb{M}_m(A)$ is the identity map. ☐

Lemma 7.2. *Let A, B, C be unital, torsion-free, complete bornological V-algebras and let P be an A-B-bimodule and Q a B-C-bimodule. Assume P and Q to be finitely generated and projective as right modules. Then $P \otimes_B Q$ is finitely generated and projective as a right module, and $\mathbb{HA}(P \otimes_B Q)$ is chain homotopic to $\mathbb{HA}(P) \circ \mathbb{HA}(Q)$.*

Proof. By assumption, there are $m, n \in \mathbb{N}$ and idempotent matrices $e \in \mathbb{M}_m(B)$ and $f \in \mathbb{M}_n(C)$ with $P \cong eB^m$ and $Q \cong fC^n$. Then

$$P \otimes_B Q \cong (e \cdot B^m) \otimes_B (f \cdot C^n) \cong (e \otimes_B 1) \cdot (B^m \otimes_B (f \cdot C^n)) = (e \otimes_B 1) \cdot (f \cdot C^n)^m.$$

This identifies $P \otimes_B Q$ with the image of $\mathbb{M}_m(\iota_B)(e) \in \mathbb{M}_{m \cdot n}(C)$, where $\iota_B \colon B \to \mathbb{M}_n(C)$ is the homomorphism associated to f and $\mathbb{M}_m(\iota_B) \colon \mathbb{M}_m(B) \to \mathbb{M}_{m \cdot n}(C)$ is the induced homomorphism on matrices. Inspection shows that the homomorphism $A \to \mathbb{M}_{m \cdot n}(C)$ defined by realising $P \otimes_B Q$ in this way is equal to the composite homomorphism $\mathbb{M}_m(\iota_B) \circ \iota_A \colon A \to \mathbb{M}_m(B) \to \mathbb{M}_{m \cdot n}(C)$. This implies the claim because the chain homotopy equivalences $\mathbb{HA}(B) \cong \mathbb{HA}(\mathbb{M}_m(B))$ are natural. ☐

Theorem 7.3. *Let A and B be unital, torsion-free, complete bornological V-algebras. A Morita equivalence between them induces a chain homotopy equivalence $\mathbb{HA}(A) \simeq \mathbb{HA}(B)$ and isomorphisms $\mathrm{HA}_*(A) \cong \mathrm{HA}_*(B)$ for $* = 0, 1$.*

Proof. The Morita equivalence is given by bimodules P and Q over A, B and B, A with $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$. It is well known that the equivalence bimodules P and Q are finitely generated and projective as right modules. Hence they induce well defined chain maps $\mathbb{HA}(A) \leftrightarrow \mathbb{HA}(B)$ by the construction above. These are inverse up to chain homotopy by Lemma 7.2. This homotopy equivalence implies isomorphisms $\mathrm{HA}_*(A) \cong \mathrm{HA}_*(B)$ for $* = 0, 1$ on analytic cyclic homology. □

When dealing with non-unital algebras, Morita theory gets more difficult. In particular, we know less about the bimodules involved in a Morita equivalence. The issue is to impose the right assumptions on an A, B-bimodule so that there are a matricial pair as in Section 6, an idempotent double centraliser e of $B \overline{\otimes} \overline{\mathcal{M}}$, and an algebra homomorphism $A \to e(B \overline{\otimes} \overline{\mathcal{M}})e$. We do not discuss sufficient conditions on bimodules that allow to associate such data to them.

8 Leavitt path algebras

Our next goal is to compute the analytic cyclic homology for tensor products with Leavitt and Cohn path algebras of directed graphs and their dagger completions. A *directed graph* E consists of a set E^0 of vertices and a set E^1 of edges together with source and range maps $s, r: E^1 \to E^0$. A vertex $v \in E^0$ is *regular* if $0 < |s^{-1}(\{v\})| < \infty$. Let $\mathrm{reg}(E) \subseteq E^0$ be the subset of regular vertices. Define

$$N_E: E^0 \times \mathrm{reg}(E) \to \mathbb{Z}, \qquad (v, w) \mapsto \delta_{v,w} - |s^{-1}(\{w\}) \cap r^{-1}(\{v\})|.$$

Let $L(E)$ and $C(E)$ be the Leavitt and Cohn path algebras over V, as defined in [1, Definitions 1.2.3 and 1.2.5]. We consider them as bornological algebras with the fine bornology. The following theorem follows from the results in [8] and the formal properties of analytic cyclic homology:

Theorem 8.1. *Assume* $\mathrm{char}\, F = 0$*. Let R be a complete bornological algebra. Let E be a graph with countably many vertices. Then*

$$\mathbb{HA}(R \otimes C(E)) \simeq \mathbb{HA}(R \otimes V^{(E^0)}), \qquad \mathbb{HA}(C(E)) \simeq V^{(E^0)},$$
$$\mathbb{HA}(L(E)) \simeq \mathrm{coker}(N_E) \oplus \ker(N_E)[1],$$

If E^0 is finite, then

$$\mathbb{HA}(R \otimes C(E)) \simeq \bigoplus_{v \in E^0} \mathbb{HA}(R),$$
$$\mathbb{HA}(R \otimes L(E)) \simeq (\mathrm{coker}(N_E) \oplus \ker(N_E)[1]) \otimes \mathbb{HA}(R).$$

Proof. We define a functor H from the category of V-algebras to the triangulated category of pro-supercomplexes by giving A the fine bornology and taking $\mathbb{HA}(R \otimes A)$. The functor H is homotopy invariant for polynomial (and even dagger) homotopies by Theorem 4.6.2, stable for algebras of finite matrices over any set Λ by Proposition 6.2 applied to Example 6.3, and exact on semi-split extensions by Theorem 5.1. Theorem 5.1 also implies that $\mathbb{HA}$ is finitely additive. It is not countably additive in general, but Corollary 4.3.9 shows that it is countably additive on the ground ring V. Now [8, Theorem 4.2] proves a homotopy equivalence

$$\mathbb{HA}(R \otimes C(E)) \simeq \mathbb{HA}(R \otimes V^{(E^0)}).$$

If E^0 is finite, then this is homotopy equivalent to $\mathbb{HA}(R) \otimes V^{(E^0)} = \bigoplus_{v \in E^0} \mathbb{HA}(R)$ by finite additivity. And if $R = V$, then Corollary 4.3.9 identifies $\mathbb{HA}(V^{(E^0)}) \simeq V^{(E^0)}$.
[8, Proposition 5.2] yields a distinguished triangle of pro-supercomplexes

$$\mathbb{HA}(R \otimes V^{(\mathrm{reg}(E))}) \xrightarrow{f} \mathbb{HA}(R \otimes V^{(E^0)}) \to \mathbb{HA}(R \otimes L(E)) \to \mathbb{HA}(R \otimes V^{(\mathrm{reg}(E))})$$

and partly describes the map f. If $R = V$ and E^0 is countable, then Corollary 4.3.9 identifies $\mathbb{HA}(V^{(E^0)}) \simeq V^{(E^0)}$ and $\mathbb{HA}(V^{\mathrm{reg}(E)}) \simeq V^{\mathrm{reg}(E)}$, and the information about the map f in [8, Proposition 5.2] shows that it multiplies vectors with the matrix N_E. If E^0 is finite, then $\mathbb{HA}$ is E^0-additive because of excision. Then [8, Theorem 5.4] gives a distinguished triangle

$$\mathbb{HA}(R) \otimes F^{\mathrm{reg}(E)} \xrightarrow{\mathrm{id} \otimes N_E} \mathbb{HA}(R) \otimes F^{E^0} \to \mathbb{HA}(R \otimes L(E)) \to \cdots.$$

Since $\mathrm{char}(F) = 0$, there are invertible matrices x, y with entries in F such that $x N_E y$ is a diagonal matrix with only zeros and ones in the diagonal. We may replace the map N_E or $\mathrm{id} \otimes N_E$ above by $\mathrm{id} \otimes (x N_E y)$. Then the formulas for $\mathbb{HA}(L(E))$ in general and for $\mathbb{HA}(R \otimes L(E))$ for finite E^0 follow. □

COROLLARY 8.2. $\mathbb{HA}(R \otimes V[t, t^{-1}])$ *is chain homotopy equivalent to* $\mathbb{HA}(R) \oplus \mathbb{HA}(R)[1]$ *and* $\mathrm{HA}_*(R \otimes V[t, t^{-1}]) \cong \mathrm{HA}_*(R) \oplus \mathrm{HA}_*(R)[1]$.

Proof. Apply Theorem 8.1 to the graph with one vertex and one loop. □

The following theorem says that Theorem 8.1 remains true for the dagger completions $C(E)^\dagger$ and $L(E)^\dagger$ of $C(E)$ and $L(E)$:

THEOREM 8.3. *Let R be a complete bornological algebra and let E be a graph. Then*

$$\mathbb{HA}(R \otimes C(E)) \simeq \mathbb{HA}(R \overline{\otimes} C(E)^\dagger), \qquad \mathbb{HA}(R \otimes L(E)) \simeq \mathbb{HA}(R \overline{\otimes} L(E)^\dagger).$$

So Theorem 8.1 also computes $\mathbb{HA}(R \overline{\otimes} C(E)^\dagger)$ and $\mathbb{HA}(R \overline{\otimes} L(E)^\dagger)$ – assuming E^0 to be countable or finite or $R = V$ for the different cases.

COROLLARY 8.4 (Fundamental Theorem). $\mathbb{HA}(R \overline{\otimes} V[t, t^{-1}]^\dagger)$ *is chain homotopy equivalent to* $\mathbb{HA}(R) \oplus \mathbb{HA}(R)[1]$ *and* $\mathrm{HA}_*(R \overline{\otimes} V[t, t^{-1}]^\dagger) \cong \mathrm{HA}_*(R) \oplus \mathrm{HA}_*(R)[1]$.

Proof. Combine Theorem 8.3 and Corollary 8.2. □

We are going to prove Theorem 8.3 by showing that the proofs in [8] continue to work when we suitably complete all algebras that occur there. We must be careful, however, because the dagger completion is *not* an exact functor. We first recall some basic facts that are used in [8]. These will be used to describe the dagger completions $C(E)^\dagger$ and $L(E)^\dagger$.
By definition, $L(E)$ has the same generators as $C(E)$ and more relations. This provides a quotient map $p: C(E) \twoheadrightarrow L(E)$. Let $K(E) \subseteq C(E)$ be its kernel.

Lemma 8.5. *There is a semi-split extension of V-algebras*

$$K(E) \rightarrowtail C(E) \twoheadrightarrow L(E).$$

Proof. Let $\mathcal{P}$ be the set of finite paths in E. For $v \in \operatorname{reg}(E)$, choose $e_v \in s^{-1}(\{v\})$. Let

$$\mathcal{B} := \{\alpha\beta^* : \alpha, \beta \in \mathcal{P},\ r(\alpha) = r(\beta)\},$$
$$\mathcal{B}' := \mathcal{B} \setminus \{\alpha e_v e_v^* \beta^* : v \in \operatorname{reg}(E),\ \alpha, \beta \in \mathcal{P},\ r(\alpha) = r(\beta) = v\}.$$

By [1, Propositions 1.5.6 and 1.5.11], $\mathcal{B}$ is a basis of $C(E)$ and $\mathcal{B}'$ is a basis of $L(E)$. Let $\sigma: L(E) \to C(E)$ be the linear map that sends each element of $\mathcal{B}'$ to itself. This is a section for the quotient map $p: C(E) \to L(E)$. □

Next we describe $K(E)$ as in [1, Proposition 1.5.11]. Let $v \in \operatorname{reg}(E)$. Define

$$q_v := v - \sum_{s(e)=v} ee^*.$$

Let $\mathcal{P}_v \subseteq \mathcal{P}$ be the set of all paths with $r(\alpha) = v$. Let $V^{(\mathcal{P}_v)}$ be the free V-module on the set $\mathcal{P}_v$ and let $\mathcal{M}_{\mathcal{P}_v}$ be the algebra of finite matrices indexed by $\mathcal{P}_v$ as in Example 6.3. The map

$$\bigoplus_{v \in \operatorname{reg}(E)} \mathcal{M}_{\mathcal{P}_v} \to K(E), \qquad \alpha \otimes \beta \mapsto \alpha q_v \beta^*,$$

is a V-algebra isomorphism by [1, Proposition 1.5.11]. Each $\mathcal{M}_{\mathcal{P}_v}$ with the fine bornology is a dagger algebra because it is a union of finite-dimensional subalgebras. Thus $K(E)$ is a dagger algebra as well. In contrast, $C(E)$ and $L(E)$ with the fine bornology are not semi-dagger. The restriction to $K(E)$ of the linear growth bornology of $C(E)$ is *not* just the fine bornology: this is visible in the special case where $C(E)$ is the Toeplitz algebra and $L(E) = V[t, t^{-1}]$.
We are going to describe the linear growth bornology on $C(E)$. Let $\mathcal{F}$ be the set of all finite subsets $S \subseteq E^0 \cup E^1$ such that

$$e \in S \cap E^1 \text{ and } s(e) \in \operatorname{reg}(E) \Rightarrow \{s(e)\} \cup s^{-1}(s(e)) \subseteq S.$$

Let $S^{(\infty)}$ for $S \in \mathcal{F}$ be the set of all paths that consist only of edges in S. Let $|\alpha|$ be the length of a path $\alpha \in \mathcal{P}$. For $n \in \mathbb{N}$, let

$$S_n := \{\alpha\beta^* : \alpha, \beta \in S^{(\infty)},\ |\alpha| + |\beta| \le n\} \subseteq \mathcal{B}.$$

This is an increasing filtration on the basis $\mathcal{B}$ of $C(E)$.

Lemma 8.6. *A subset of $C(E)$ has linear growth if and only if there are $S \in \mathcal{F}$ and $m \in \mathbb{N}^*$ such that it is contained in the V-linear span of $\bigcup_{n\in\mathbb{N}} \pi^{\lfloor n/m\rfloor} S_n$.*

Proof. It is easy to see that the V-linear span of $\bigcup_{n\in\mathbb{N}} \pi^{\lfloor n/m\rfloor} S_n$ in $C(E)$ has linear growth. Conversely, we claim that any subset of linear growth is contained in one of this form. Every finite subset of $E^0 \cup E^1$ is contained in an element of $\mathcal{F}$. It follows that, for every finitely generated submodule $M \subseteq C(E)$, there are $S \in \mathcal{F}$ and $m \geq 1$ such that M is contained in the V-submodule generated by S_m. Then M^j is contained in the V-submodule generated by S_{mj} for all $j \in \mathbb{N}^*$. Thus $M^\diamond$ is contained in the V-submodule generated by $\pi^{j-1} S_{mj}$ for all $j \in \mathbb{N}^*$. This is the V-linear span of $\bigcup_{n\in\mathbb{N}^*} \pi^{\lceil n/m\rceil -1} S_n$. Letting m vary, we may replace $\lceil n/m\rceil - 1$ by $\lfloor n/m\rfloor$. ☐

Constructing linear growth bornologies commutes with taking quotients. So a subset of $L(E)$ has linear growth if and only if it is the image of a subset of linear growth in $C(E)$. Next we show that the section $\sigma\colon L(E) \to C(E)$ is bounded for the linear growth bornologies, and we describe the restriction to $K(E)$ of the linear growth bornology on $C(E)$:

Lemma 8.7. *Give $V^{(\mathcal{P}_v)} \subseteq V^{(\mathcal{P})}$ the bornology where a subset is bounded if and only if it is contained in the linear span of $\{\pi^{\lfloor |\alpha|/m\rfloor}\alpha : \alpha \in S^{(\infty)}\}$ for some $S \in \mathcal{F}$ and some $m \in \mathbb{N}^*$. Equip the matrix algebra $\mathcal{M}_{\mathcal{P}_v} = V^{(\mathcal{P}_v\times\mathcal{P}_v)}$ with the resulting tensor product bornology and the multiplication defined by the obvious bilinear pairing as in Section 6, and give $\bigoplus_{v\in\operatorname{reg}(E)} \mathcal{M}_{\mathcal{P}_v}$ the direct sum bornology. There is a semi-split extension of bornological algebras*

$$\bigoplus_{v\in\operatorname{reg}(E)} \mathcal{M}_{\mathcal{P}_v} \overset{i}{\rightarrowtail} C(E)_{\mathrm{lg}} \overset{p}{\twoheadrightarrow} L(E)_{\mathrm{lg}}, \qquad \sigma\colon L(E)_{\mathrm{lg}} \dashrightarrow C(E)_{\mathrm{lg}}.$$

Proof. Let $S \in \mathcal{F}$. We claim that $\sigma\circ p$ maps the linear span of S_n into itself. If $\alpha\beta^* \in \mathcal{B}'$, then $\sigma\circ p(\alpha\beta^*) = \alpha\beta^*$. If $\alpha\beta^* \notin \mathcal{B}'$, then $\alpha = \alpha_0 e_v$, $\beta = \beta_0 e_v$ for some $v \in \operatorname{reg}(E)$, $\alpha_0, \beta_0 \in \mathcal{P}_v$. And then

$$p(\alpha\beta^*) = p(\alpha_0\beta_0^*) - \sum_{s(e)=v, e\neq e_v} p(\alpha_0 e e^* \beta_0).$$

Since $\alpha_0\beta_0^*$ is shorter than $\alpha\beta^*$ and $\alpha_0 e e^*\beta_0 \in \mathcal{B}'$ for $e \in E^1$ with $s(e) = v$ and $e \neq e_v$, an induction over $|\alpha| + |\beta|$ shows that $\sigma\circ p(\alpha\beta^*)$ is always a V-linear combination of shorter words; in addition, all edges in these words are again contained in S because $S \in \mathcal{F}$. This proves the claim. Now Lemma 8.6 implies that $\sigma\circ p$ preserves linear growth of subsets. Equivalently, σ is a bounded map $L(E)_{\mathrm{lg}} \to C(E)_{\mathrm{lg}}$. Then a subset of $K(E)$ has linear growth in $C(E)$ if and only if it is of the form $(\mathrm{id} - \sigma\circ p)(M)$ for a V-submodule $M \subseteq C(E)$ that has linear growth. The projection $\mathrm{id} - \sigma\circ p$ kills $\alpha\beta^* \in \mathcal{B}'$. Thus we may disregard these generators when we describe the restriction to $K(E)$ of the linear growth bornology on $C(E)$. Instead of applying $\mathrm{id} - \sigma\circ p$ to the remaining basis vectors

$\alpha e_v e_v^* \beta^*$ for $r(\alpha) = r(\beta) = v \in \operatorname{reg}(E)$, we may also apply it to $\alpha e_v e_v^* \beta^* - \alpha\beta^*$ because $\alpha\beta^*$ is a shorter basis vector that involves the same edges. And

$$\begin{aligned}(\mathrm{id} - \sigma \circ p)(\alpha e_v e_v^* \beta^* - \alpha\beta^*) &= \alpha e_v e_v^* \beta^* - \alpha\beta^* + \sigma\Bigg(\sum_{s(e)=v,\ e\neq e_v} p(\alpha e e^* \beta^*)\Bigg)\\ &= -\alpha\beta^* + \sum_{s(e)=v} \alpha e e^* \beta^* = -\alpha q_v \beta^*.\end{aligned}$$

Now Lemma 8.6 implies that a subset of $K(E)$ has linear growth in $C(E)$ if and only if there are $S \in \mathcal{F}$ and $m \in \mathbb{N}^*$ so that it belongs to the V-linear span of $\pi^{\lfloor n/m \rfloor}\alpha q_v \beta^*$ with $v \in \operatorname{reg}(E)$, $\alpha, \beta \in \mathcal{P}_v \cap S^{(\infty)}$, and $|\alpha| + |\beta| + 2 \le n$. Under the isomorphism $\bigoplus_{v\in\operatorname{reg}(E)} \mathcal{M}_{\mathcal{P}_v} \cong K(E)$, this becomes equal to the bornology on $\bigoplus_{v\in\operatorname{reg}(E)} \mathcal{M}_{\mathcal{P}_v}$ specified in the statement of the lemma. □

The semi-split extension in Lemma 8.7 implies a similar semi-split extension involving the dagger completions $C(E)^\dagger$, $L(E)^\dagger$ and the completion of $\bigoplus_{v\in\operatorname{reg}(E)} \mathcal{M}_{\mathcal{P}_v}$ for the bornology specified in Lemma 8.7.
Now Theorem 8.3 is proven by showing that all homomorphisms and quasi-homomorphisms that are used in [8] remain bounded and all homotopies among them remain dagger homotopies when we give all algebras that occur the suitable "linear growth" bornology, defined using the lengths of paths to define linear growth. This is because all maps in [8] are described by explicit formulas in terms of paths, which change the length only by finite amounts. We have put linear growth in quotation marks because the correct bornologies on the ideals $K(E)$ and $\hat{K}(E)$ in [8] are restrictions of linear growth bornologies on larger algebras as in Lemma 8.7. These bornological algebras are special cases of Example 6.6, and so $\mathbb{HA}$ is stable for such matrix algebras. The bornology on $K(E)$ in Lemma 8.7 actually deserves to be called a "linear growth bornology". But the relevant length function is specified by hand and not by the length of products as in Definition 2.1.16.

9 Filtered Noetherian rings and analytic quasi-freeness

In Section 9.1, we develop a criterion for a quasi-free algebra to be analytically quasi-free. It uses a connection with a growth condition, called finite degree. In Section 9.2, we show that the criterion from Section 9.1 applies to dagger completions of smooth, commutative V-algebras of relative dimension 1. And we show that any smooth curve over $\mathbb{F}$ lifts to such a V-algebra.

9.1 Finite-degree connections

Recall that a complete bornological V-algebra R is called quasi-free if all its square-zero extensions split. This is equivalent to the existence of a *connection* on $\overline{\Omega}^1(R)$, that is, a linear map $\nabla: \overline{\Omega}^1(R) \to \overline{\Omega}^2(R)$ satisfying

$$\nabla(a\omega) = a\nabla(\omega) \quad \text{and} \quad \nabla(\omega a) = \nabla(\omega)a + \omega\,\mathrm{d}a,$$

for all $a \in R$ and $\omega \in \overline{\Omega}^1(R)$. And this is, in turn, equivalent to $\overline{\Omega}^1(R)$ being projective for extensions of complete bornological R-bimodules with a bounded V-linear section. The above claims go back to Cuntz and Quillen [10] for algebras without extra structure. They also hold for algebras in additive monoidal categories and hence for complete bornological V-algebras (see, for instance, [18]). A related result is Proposition 4.4.6.

We are going to prove that a quasi-free algebra R is analytically quasi-free if $\overline{\Omega}^1(R)$ has a connection whose growth is controlled in a certain way. This uses increasing filtrations. An (increasing) *filtration* on a V-module M is an increasing sequence of V-submodules $(\mathcal{F}_n M)_{n\in\mathbb{N}}$ with $\bigcup \mathcal{F}_n M = M$. For a V-algebra R, we require, in addition, that $\mathcal{F}_n R \cdot \mathcal{F}_m R \subseteq \mathcal{F}_{n+m} R$ for all $n, m \in \mathbb{N}$. And for a module M over a V-algebra R with a fixed filtration $(\mathcal{F}_n R)_{n\in\mathbb{N}}$, we require, in addition, that $\mathcal{F}_n R \cdot \mathcal{F}_m M \subseteq \mathcal{F}_{n+m} M$ for all $n, m \in \mathbb{N}$. Then we speak of a *filtered algebra* and a *filtered module*, respectively.

Definition 9.1.1. A map $f : M \to N$ between filtered V-modules has *finite degree* if there is $a \in \mathbb{N}$ – the degree – such that $f(\mathcal{F}_n M) \subseteq \mathcal{F}_{n+a}(N)$ for all $n \in \mathbb{N}$. Two filtrations $(\mathcal{F}_n M)_n$ and $(\mathcal{F}'_n M)_n$ on a filtered V-module M are *shift equivalent* if there is $a \in \mathbb{N}$ such that $\mathcal{F}_n M \subseteq \mathcal{F}'_{n+a} M$ and $\mathcal{F}'_n M \subseteq \mathcal{F}_{n+a} M$ for all $n \in \mathbb{N}$.

Example 9.1.2. Let R be a torsion-free bornological V-algebra. Define $M^{(j)}$ for a complete bounded submodule $M \subseteq R$ and $j \geq 0$ as in (4.4.9). Put

$$\mathcal{F}_r^M \overline{\Omega}^j R := \sum_{i_0+\dots+i_j \leq r} M^{(i_0)}\, \mathrm{d}M^{(i_1)} \dots \mathrm{d}M^{(i_j)} \oplus \sum_{i_1+\dots+i_j \leq r} \mathrm{d}M^{(i_1)} \dots \mathrm{d}M^{(i_j)} \tag{9.1.3}$$

for $r \in \mathbb{N}$. This is an increasing filtration on the differential j-forms of the subalgebra $M^{(\infty)} \subseteq R$ generated by M.

The following lemma relates such filtrations to the linear growth bornology:

Lemma 9.1.4. *Let R be a torsion-free bornological algebra, $M \subseteq R$ a bounded V-submodule and $n \geq 0$. Then*

$$\sum_{i\geq 0} \pi^i \mathcal{F}_{i+n}^M \overline{\Omega}^n R \subseteq \overline{\Omega}^n(M^\diamond) \subseteq \sum_{i\geq 0} \pi^i \mathcal{F}_{i+n+1}^M \overline{\Omega}^n R.$$

Proof. We compute

$$\begin{aligned}\overline{\Omega}^n(M^\diamond) &= M^\diamond\, \mathrm{d}(M^\diamond)^n \oplus \mathrm{d}(M^\diamond)^n \\ &= \sum_{i\geq 0} \pi^i \Bigg(\sum_{i_0+\dots+i_n = i} M^{(i_0+1)}\, \mathrm{d}M^{(i_1+1)} \dots \mathrm{d}M^{(i_n+1)} \\ &\qquad \oplus \sum_{i_1+\dots+i_n=i} \mathrm{d}M^{(i_1+1)} \dots \mathrm{d}M^{(i_n+1)} \Bigg).\end{aligned}$$

□

Lemma 9.1.5. *Let $M \subseteq R$ be a bounded submodule, $r, b \geq 1$ and $s \geq 0$. Then*

$$\mathcal{F}_r^M \overline{\Omega}^s R \subseteq \mathcal{F}_{\lceil r/b \rceil + s}^{M^{(b)}} \overline{\Omega}^s R \subseteq \mathcal{F}_{r+b(s+1)}^M \overline{\Omega}^s R.$$

Proof. Straightforward. ☐

Lemma 9.1.6. *Let X and Y be torsion-free bornological modules. Let (f_n) be a sequence of bounded linear maps $X \to Y$. Assume that for each bounded submodule $M \subseteq X$ there is a bounded submodule $N \subseteq Y$ and a sequence of nonnegative integers (a_n) with $\lim a_n = \infty$ and $f_n(M) \subseteq \pi^{a_n} N$ for all $n \in \mathbb{N}$. Then the series $s(x) := \sum_n f_n(x)$ converges in $\overline{Y}$ for every $x \in X$, and the assignment $x \mapsto s(x)$ is bounded and linear. So it extends to a bounded linear map $s: \overline{X} \to \overline{Y}$.*

Proof. Straightforward. ☐

Definition 9.1.7. Let R be a torsion-free bornological V-algebra. A connection $\nabla: \overline{\Omega}^1(R) \to \overline{\Omega}^2(R)$ has *finite degree* on a bounded submodule $M \subseteq R$ if it has finite degree as a V-module map with respect to the filtrations on $\overline{\Omega}^1(M^{(\infty)})$ and $\overline{\Omega}^2(M^{(\infty)})$ from Example 9.1.2. A connection ∇ has *finite degree* on R if any bounded subset is contained in a bounded submodule of R on which ∇ has finite degree.

Remark 9.1.8. Lemma 9.1.5 implies that if ∇ has finite degree on M, then it also has finite degree on $M^{(b)}$ for all b. Then ∇ is a finite degree connection on $M^{(\infty)}$ with the bornology that is cofinally generated by $M^{(n)}$ for $n \in \mathbb{N}$.

The following theorem is an analytic version of the formal tubular neighbourhood theorem by Cuntz and Quillen in [10].

Theorem 9.1.9. *Let R be a complete, torsion-free bornological algebra. If $\overline{\Omega}^1(R)$ has a connection of finite degree, then $R^\dagger$ is analytically quasi-free.*

Proof. We introduce some notation on Hochschild cochains. If X is a complete, bornological R-bimodule and $\psi: R^{\overline{\otimes} n} \to X$ is an n-cochain, write $\delta(\psi)$ for its Hochschild coboundary. If $\xi: R^{\overline{\otimes} m} \to Y$ is another cochain, write $\psi \cup \xi: R^{\overline{\otimes} n+m} \to X \overline{\otimes}_R Y$ for the cup product. Let $\nabla: \overline{\Omega}^1 R \to \overline{\Omega}^2 R$ be a connection of finite degree, and let $M \subseteq R$ be a bounded submodule and $a \geq 0$ an integer such that ∇ has degree a on M. The connection ∇ is equivalent to a 1-cochain $\varphi_2: R \to \overline{\Omega}^2 R$ satisfying $\delta(\varphi_2) = \mathrm{d} \cup \mathrm{d}$, via $\nabla(x_0 \, \mathrm{d}x_1) = x_0 \varphi_2(x_1)$ for $x_0 \in R^+$, $x_1 \in R$. Then φ_2 raises the M-filtration degree by at most a. If X is a filtered R-bimodule and $\psi: R \overline{\otimes} R \to X$ is a 2-cocycle of degree at most b, then

$$\bar{\psi}: \overline{\Omega}^2 R \to X, \qquad \bar{\psi}(x_0 \, \mathrm{d}x_1 \, \mathrm{d}x_2) = x_0 \psi(x_1, x_2)$$

is a bimodule homomorphism. And the 1-cochain

$$\psi' = \bar{\psi} \circ \varphi_2$$

raises filtration degree by at most $a+b$ and satisfies $\delta(\psi') = \psi$. We inductively define 2-cocycles and 1-cochains with values in $\overline{\Omega}^{2(n+1)}R$ for $n \geq 1$ by

$$\psi_{2(n+1)} \coloneqq \sum_{j=0}^{n} \mathrm{d}\varphi_{2j} \cup \mathrm{d}\varphi_{2(n-j)} - \sum_{j=1}^{n} \varphi_{2j} \cup \varphi_{2(n+1-j)},$$
$$\varphi_{2(n+1)} \coloneqq \psi'_{2(n+1)}.$$

Put $\varphi_0 = \mathrm{id}\colon R \to R$. To see that the maps ψ_{2n} are cocycles, one proves first that

$$\delta(\mathrm{d}\varphi_{2n}) = -\sum_{j=0}^{n} \mathrm{d}(\varphi_{2j} \cup \varphi_{2(n-j)}).$$

Then a long but straightforward calculation using the Leibniz rule for both d and δ shows by induction that $\delta(\psi_{2n}) = 0$ (see [6, Theorem 2.1]). By construction, the bounded linear map $\varphi_{\leq 2n} \coloneqq \sum_{i=0}^{n} \varphi_{2i}$ is a section of the canonical projection $\mathsf{T}R \to R$, and its curvature vanishes modulo $\mathsf{J}R^{n+1}$. So it defines a bounded algebra homomorphism $R \to \mathsf{T}R/\mathsf{J}R^{n+1}$. Hence the infinite series $\sum_{i=0}^{\infty} \varphi_{2i}$ is an algebra homomorphism into the projective limit. It suffices to show that, for each m, the series $\sum_{i=0}^{\infty} \varphi_{2i}$ defines a bounded linear homomorphism $R_{\mathrm{lg}} \to (\mathcal{U}(\mathsf{T}R_{\mathrm{lg}}, \mathsf{J}R_{\mathrm{lg}}^m), \mathcal{U}(\mathsf{J}R_{\mathrm{lg}}, \mathsf{J}R_{\mathrm{lg}}^m))^\dagger$.
One checks by induction on n that $\varphi_{2n}(M^{(i)}) \subseteq \mathcal{F}^M_{i+(2n-1)a} \overline{\Omega}^{2n}R$. Hence

$$\varphi_{2n}(M^\diamond) \subseteq \sum_{i=0}^{\infty} \pi^i \mathcal{F}^M_{i+(2n-1)a+1} \overline{\Omega}^{2n}R. \tag{9.1.10}$$

Next let $m \geq 1$ and choose an integer $c > \max\{1, 2am\}$. Then

$$i + \left\lfloor \frac{n}{m} \right\rfloor - \left\lceil \frac{i+(2n-1)a+1}{c} \right\rceil \geq (1-1/c)i \geq 0 \tag{9.1.11}$$

for all $i \geq 0$ and sufficiently large n. Then $i \geq \left\lceil \frac{i+(2n-1)a+1}{c} \right\rceil - \left\lfloor \frac{n}{m} \right\rfloor$. Set $D(i,n,c) \coloneqq \left\lceil \frac{i+(2n-1)a+1}{c} \right\rceil$. Equations (9.1.10) and (9.1.11) and Lemmas 9.1.4 and 9.1.5 imply

$$\begin{aligned}\varphi_{2n}(M^\diamond) &\subseteq \sum_{i\geq 0} \pi^i \mathcal{F}^{M^{(c)}}_{D(i,n,c)+2n} \overline{\Omega}^{2n}(R) \\ &\subseteq \pi^{-\lfloor \frac{n}{m} \rfloor} \sum_{i\geq 0} \pi^{D(i,n,c)} \mathcal{F}^{M^{(c)}}_{D(i,n,c)+2n} \overline{\Omega}^{2n}(R) \subseteq \pi^{-\lfloor \frac{n}{m} \rfloor} \overline{\Omega}^{2n}((M^{(c)})^\diamond).\end{aligned}$$

By Proposition 4.4.16, the subset of infinite series $\sum_{n=0}^{\infty} \varphi_{2n}(M^\diamond)$ is bounded in $(\mathcal{U}(\mathsf{T}R_{\mathrm{lg}}, \mathsf{J}R_{\mathrm{lg}}^m), \mathcal{U}(\mathsf{J}R_{\mathrm{lg}}, \mathsf{J}R_{\mathrm{lg}}^m))^\dagger$. By Lemma 9.1.6, $\sum_{n=0}^{\infty} \varphi_{2n}$ defines a bounded homomorphism

$$R \to (\mathcal{U}(\mathsf{T}R_{\mathrm{lg}}, \mathsf{J}R_{\mathrm{lg}}^m), \mathcal{U}(\mathsf{J}R_{\mathrm{lg}}, \mathsf{J}R_{\mathrm{lg}}^m))^\dagger$$

for each $m \geq 1$; this completes the proof. □

Corollary 9.1.12. *Let R be as in Theorem 9.1.9. Then the natural map $\mathbb{HA}(R^\dagger) \to X(R^\dagger \otimes F)$ is a chain homotopy equivalence and $\mathrm{HA}_*(R)$ is isomorphic to the homology of $X(R^\dagger \otimes F)$.*

Proof. Immediate from Theorem 9.1.9 and Corollary 4.7.2. □

9.2 Filtered Noetherian rings and smooth algebras

We now show that some quasi-free algebras have a connection of finite degree. In particular, this includes smooth, commutative finitely generated V-algebras of relative dimension 1. For the remainder of this section, let R be a finitely generated V-algebra, equipped with the fine bornology. Let $S \subseteq R$ be a finite generating subset and let $S^{\le n}$ be the set of all products of elements of S of length at most n. As above, let $\mathcal{F}_n R \subseteq R$ be the V-submodule generated by $S^{\le n}$. By convention, $S^{\le 0} = \{1\}$ and $\mathcal{F}_0 R = V \cdot 1$. This is an increasing filtration on R. It induces filtrations on the bimodules $\Omega^l(R)$ as in Example 9.1.2. More concretely, $\mathcal{F}_n(\Omega^l(R))$ is the V-submodule of $\Omega^l(R)$ generated by $x_0\,\mathrm{d}x_1 \dots \mathrm{d}x_l$ with $x_0 \in \mathcal{F}_{n_0}(R)$ or $x_0 = 1$ and $n_0 = 0$, and $x_i \in \mathcal{F}_{n_i}(R)$ for $i = 1, \dots, l$, and $n_0 + \dots + n_l \le n$. By construction, the V-submodule $\mathcal{F}_n R \cdot \mathcal{F}_m R$ that is generated by products $x \cdot y$ with $x \in \mathcal{F}_n R$, $y \in \mathcal{F}_m R$ is equal to $\mathcal{F}_{n+m} R$ for all $n, m \in \mathbb{N}$. This is more than what is required for a filtered algebra, and the extra information is crucial for the filtration to generate the linear growth bornology.
Let M be an R-module with a finite generating set $S_M \subseteq M$. Then we define a filtration on M, called the *canonical filtration*, by letting $\mathcal{F}_n M$ be the V-submodule generated by $a \cdot x$ with $a \in \mathcal{F}_n R$ and $x \in S_M$. This satisfies $\mathcal{F}_m R \cdot \mathcal{F}_n M \subseteq \mathcal{F}_{n+m} M$ because $\mathcal{F}_m R \cdot \mathcal{F}_n R \subseteq \mathcal{F}_{n+m} R$. The following proposition characterises canonical filtrations by a universal property:

Proposition 9.2.1. *Let R be a filtered V-algebra and let M be a finitely generated R-module. Equip M with the filtration described above. Then any R-module map from M to a filtered R-module Y is of finite degree. The canonical filtrations for two different finite generating sets of M are shift equivalent.*

Proof. Let $\{m_1, \dots, m_n\}$ be a finite generating set for M as an R-module. Let $h\colon M \to Y$ be an R-module homomorphism into a filtered R-module Y. Since $Y = \bigcup \mathcal{F}_l Y$, there is an $l \in \mathbb{N}$ with $h(m_i) \in \mathcal{F}_l Y$ for all $i = 1, \dots, m$. Then $h(a \cdot m_i) \in \mathcal{F}_{n+l} R$ for $a \in \mathcal{F}_n R$. Hence $h(\mathcal{F}_n M) \subseteq \mathcal{F}_{n+l} Y$ for all $n \in \mathbb{N}$. That is, h has finite degree. In particular, if we equip M with another filtration $(\mathcal{F}'_n M)_{n \in \mathbb{N}}$, then the identity map has finite degree, that is, there is $l \in \mathbb{N}$ with $\mathcal{F}_n M \subseteq \mathcal{F}'_{n+l} M$ for all $n \in \mathbb{N}$. If the other filtration comes from another finite generating set, then we may reverse the roles and also get $l' \in \mathbb{N}$ with inclusions $\mathcal{F}'_n M \subseteq \mathcal{F}_{n+l'} M$ for all $n \in \mathbb{N}$. □

Definition 9.2.2. A filtered V-algebra R is called (left) *filtered Noetherian* if every left ideal I is finitely generated and the filtration $(\mathcal{F}_n R \cap I)_{n \in \mathbb{N}}$ is shift equivalent to the canonical filtration of Proposition 9.2.1 from a finite generating set. In other words, there are finitely many $x_1, \dots, x_n \in I$ and $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $y \in \mathcal{F}_m R \cap I$, there are $a_i \in \mathcal{F}_{m+l} R$ with $y = \sum_{i=1}^n a_i x_i$.

Lemma 9.2.3. *Let R be a finitely generated, quasi-free V-algebra. Assume that $R^+ \otimes (R^+)^{\mathrm{op}}$ is filtered Noetherian. Then $\Omega^1(R)$ has a connection of finite degree.*

Proof. Since R is quasi-free, the left multiplication map $R^+ \otimes \Omega^1(R) \twoheadrightarrow \Omega^1(R)$ splits by an R-bimodule homomorphism $s\colon \Omega^1(R) \to R^+ \otimes \Omega^1(R)$. By definition, $\Omega^1(R)$ is a left ideal in $R^+ \otimes (R^+)^{\mathrm{op}}$. By assumption, it is finitely generated as such, and the filtration on $R^+ \otimes (R^+)^{\mathrm{op}}$ restricted to $\Omega^1(R)$ is the canonical filtration on $\Omega^1(R)$ as a module over $R^+ \otimes (R^+)^{\mathrm{op}}$. Now Proposition 9.2.1 shows that the section s above has finite degree. The section s yields a connection $\nabla\colon \Omega^1(R) \to \Omega^2(R)$, which is defined by $\nabla(\omega) = 1 \otimes \omega - s(\omega)$. It follows that ∇ has finite degree. □

Our next goal is to show that a commutative, finitely generated V-algebra with the filtration coming from a finite generating set is filtered Noetherian. First consider the polynomial ring in n variables. The filtration defined by the obvious generating set is the total degree filtration, where $\mathcal{F}_m(V[x_1, \dots, x_n])$ is the V-submodule generated by the monomials of total degree at most m, that is, terms of the form $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $|\alpha| := \sum_{i=1}^n \alpha_i \le m$.

Theorem 9.2.4. *The polynomial ring $R = V[x_1, \dots, x_n]$ with the total degree filtration is filtered Noetherian.*

Proof. Let I be any ideal in R. Since R is Noetherian, I is finitely generated. Since V is a principal ideal domain, I has a finite, strong Gröbner basis with respect to any term order on the monomials x^α (see [2, Theorem 4.5.9]). We use the degree lexicographic order (see [2, Definition 1.4.3]); the only property we need is that $|\alpha| < |\beta|$ implies $x^\alpha \prec x^\beta$. The chosen order on monomials defines the *leading term* $\mathrm{lt}(f)$ of a polynomial f. Let $G = \{f_1, \dots, f_N\}$ be a strong Gröbner basis for I. By [2, Theorem 4.1.12], any $g \in I$ can be written as $g = \sum_{j=1}^M c_j t_j f_{i_j}$, where $M \in \mathbb{N}$, $c_j \in V$, t_j is a monomial in R, $i_j \in \{1, \dots, N\}$, and $\mathrm{lt}(t_j f_{i_j}) \prec \mathrm{lt}(g)$ for each j. So the total degree of $t_j f_{i_j}$ is at most the total degree of g for each $j = 1, \dots, M$, and this remains so for the total degree of t_j. Combining the monomials t_j with the same i_j, we write any element $g \in I$ of total degree at most m in the form $\sum_{i=1}^N p_j f_j$ with $p_j \in \mathcal{F}_m R$. □

Proposition 9.2.5. *A quotient of a filtered Noetherian V-algebra with the induced filtration is again filtered Noetherian.*

Proof. Let R be a filtered Noetherian V-algebra and let I be an ideal. Any ideal in the quotient ring R/I is of the form J/I for a unique ideal J in R containing I. Let $x_1, \dots, x_n \in J$ and $l \in \mathbb{N}$ be such that for all $m \in \mathbb{N}$ and $y \in \mathcal{F}_m R \cap I$, there are $a_i \in \mathcal{F}_{m+l} R$ with $y = \sum_{i=1}^n a_i x_i$. Then the images of $x_1, \dots, x_n$ in J/I and the same l will clearly work for the ideal J/I in the quotient R/I. □

Corollary 9.2.6. *Any finitely generated, commutative V-algebra is filtered Noetherian.*

Proof. Let A be a finitely generated, commutative V-algebra. Let S be any finite generating set. Turn it into a surjective homomorphism from the polynomial algebra $R = V[x_1, \dots, x_n]$ onto A. This identifies $A \cong R/I$ for an ideal I in R. The filtration on A defined by S is equal to the filtration on the quotient R/I defined by the degree filtration on R. Now the claim follows from Theorem 9.2.4 and Proposition 9.2.5. □

Proposition 9.2.7. *Let R be a smooth commutative V-algebra of relative dimension 1. Then R admits a connection of finite degree.*

Proof. The assumptions on R imply that $\Omega^1(R)$ a projective, finitely generated R-bimodule. Furthermore, by Corollary 9.2.6, R is filtered Noetherian. The result now follows from Lemma 9.2.3. □

Remark 9.2.8. In their seminal article [20], Paul Monsky and Gerard Washnitzer introduced the so-called *Monsky–Washnitzer cohomology* $H^*_{\mathrm{MW}}(A)$ for a smooth unital $\mathbb{F}$-algebra A that has a "very smooth" lift. This is a presentation $A = S/\pi S$ where S is dagger complete and very smooth ([20, Definition 2.5]); by definition, $H^*_{\mathrm{MW}}(A) = H_{\mathrm{dR}}(S \otimes F)$ is the de Rham cohomology of $S \otimes F$. As in the current article, Monsky and Washnitzer assumed that $\mathrm{char}(F) = 0$ but made no assumption about the characteristic of $\mathbb{F}$. The very smooth liftability assumption in [20] was crucial for their proof of the functoriality of H^*_{MW}. Later on, Marius van der Put [22] managed to remove that assumption; for any smooth commutative unital $\mathbb{F}$-algebra A of finite type, he defines $H^*_{\mathrm{MW}}(A)$ as the de Rham cohomology of the dagger completion of any smooth V-algebra R with $R/\pi R = A$. The existence of such a lift follows from a theorem of Renée Elkik [12]; van der Put proves functoriality of H^*_{MW} using Artin approximation. However, in his paper he assumes $\mathbb{F}$ to be finite. More recently, under very general assumptions (in particular, for $\mathbb{F}$ of arbitrary characteristic) Alberto Arabia [3] proved that every smooth $\mathbb{F}$-algebra admits a very smooth lift, and extended the original definition of Monsky and Washnitzer. In a parallel development, Pierre Berthelot introduced rigid cohomology $H^*_{\mathrm{rig}}(X)$ of general schemes X over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) > 0$, which for smooth affine $X = \mathrm{Spec}(A)$ agrees with $H^*_{\mathrm{MW}}(A)$. With no assumptions on $\mathrm{char}(\mathbb{F})$, Große-Klönne [13] introduced the de Rham cohomology of dagger spaces over V, and he related it to rigid cohomology in the case when $\mathrm{char}(\mathbb{F}) > 0$.

The following is one of the main applications of our theory:

Theorem 9.2.9. *Let X be a smooth affine variety over the residue field $\mathbb{F}$ of dimension 1 and let $A = \mathcal{O}(X)$ be its algebra of polynomial functions. Let R be a smooth, commutative algebra of relative dimension 1 with $R/\pi R \cong A$. Equip R with the fine bornology and let $R^\dagger$ be its dagger completion. If $* = 0, 1$, then $\mathrm{HA}_*(R^\dagger)$ is naturally isomorphic to the de Rham cohomology of $R^\dagger$. This is* *isomorphic to the Monsky–Washnitzer cohomology of A, which, if* $\operatorname{char}(\mathbb{F}) > 0$*, agrees with the rigid cohomology* $H^*_{\mathrm{rig}}(A, F)$ *of* X*.*

Proof. By our hypothesis and Proposition 9.2.7, R is quasi-free. Equipping R with the fine bornology, we are in the situation of Theorem 9.1.9. Then Corollary 9.1.12 and [7, Theorem 5.5] imply that $\mathrm{HA}_*(R^\dagger)$ is isomorphic to the de Rham cohomology of $R^\dagger$. Remark 9.2.8 discusses the generality in which the latter is known to be isomorphic to different cohomology theories over $\mathbb{F}$. □

Elkik [12] has shown that any smooth curve over $\mathbb{F}$ has a smooth lift over V. The following lemma shows that we may also arrange this lift to have relative dimension 1 as required in Theorem 9.2.9:

Lemma 9.2.10. *Let* R *be a smooth algebra,* $A = R/\pi R$*, and* $d = \dim A$*. Then* $R = R_1 \times R_2$*, where* R_1 *is smooth of relative dimension* d *and* $R_1/\pi R_1 = A$*.*

Proof. Since R is smooth, $\Omega^1_{R/V}$ is projective. So its rank r is a continuous function on $\operatorname{Spec}(R)$. Thus the set of primes $\mathfrak{P}$ where $r(\mathfrak{P}) = d$ is clopen. This clopen subset induces a product decomposition $R \cong R_1 \times R_2$. Since $\dim A = d$, the relative dimension is d at all primes containing π. Now the lemma follows. □

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