Intersection Space Constructible Complexes

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Abstract. We present an obstruction theoretic inductive construction of intersection space pairs, which generalizes Banagl's construction of intersection spaces for arbitraty depth stratifications. We construct intersection space pairs for pseudomanifolds with compatible trivial structures at the link fibrations; this includes the case of toric varieties. We define intersection space complexes in an axiomatic way, similar to Goresky-McPherson axioms for intersection cohomology. We prove that if the intersection space pair exists, then the pseudomanifold has an intersection space complex whose hypercohomology recovers the cohomology of the intersection space pair. We characterize existence and uniqueness of intersection space complexes in terms of the derived category of constructible complexes. In the case of algebraic varieties we show that a parallel obstruction theory in the derived category of Mixed Hodge Modules endowes intersection space cohomology with a Mixed Hodge Structure if the obstruction vanishes. We find classes of examples admitting intersection space complex, and counterexamples not admitting them which illustrate the use of the previously developed obstruction theory (counterexamples were known previously by various specialists). We prove that the (shifted) Verdier dual of an intersection space complex is an intersection space complex. We prove a generic Poincaré duality theorem for intersection space complexes.

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1   Introduction

Intersection spaces have been recently introduced by Banagl as a Poincaré duality homology theory for topological pseudomanifolds which is an alternative to Goresky and MacPherson intersection homology. When they are available they present the advantages of being spatial modifications of the given topological pseudomanifold, to which one can later apply algebraic topology functors in order to obtain invariants. In this sense, if one applies (reduced) singular cohomology one obtains a homology theory with internal cup products and a Poincaré duality is satisfied between the homology theories corresponding to complementary perversities. Moreover, one can apply many other functors leading to richer invariants. The idea of intersection spaces was sketched for the first time in [2], and was fully developed for spaces with isolated singularities in [3].

In [3] Banagl carefully analyzed the case of quintic 3-folds with ordinary double points appearing in the conifold transition and noticed that, in the same way that intersection cohomology gives the cohomology of a small resolution, cohomology of intersection spaces gives the cohomology of a smoothing in this case. This fitted with predictions motivated by string theory (see Banagl papers for full explanations). This motivated further work by Banagl, Maxim and Budur ([9], [10], [6], [19]) in which the relation between the cohomology of intersection spaces for the middle perversity and the Milnor fibre of a hypersurface X with isolated singularities is analyzed. The latest evolution of the results of these papers, contained in [6], is the construction of a perverse sheaf in X whose hypercohomology computes the cohomology of the intersection space of X in all degrees except for the top degree. Such a perverse sheaf is a modification of the nearby cycle complex and, in fact, when the monodromy is semi-simple in the eigenvalue 1, the middle perversity intersection space perverse sheaf is a direct summand of the nearby cycle complex.

The results described up to now are valid for only isolated singularities (sometimes even assuming that they are hypersurface singularities). In [3] Banagl generalizes the construction for the case of topological pseudomanifolds with two strata and trivial link fibration and sketches a method for more general class of non-isolated singularities. In [7] intersection spaces are constructed for the case of two strata assuming non-trivial conditions on the fibration by links. The first case in which intersection spaces are constructed for a topological pseudomanifold with more than two strata is in [4]. There, the depth 1 strata are circles or intervals, and the depth 2 strata are isolated singularities; in this case, it is the topology of the strata which is very restrictive. Finally, in [12], intersection spaces are constructed for a class of 3-strata spaces with flatness assumptions for the link bundles.

In [9] the following open questions are proposed: Is there a sheaf theoretic approach to intersection spaces, similar to the one of Goresky and MacPherson [15] for intersection cohomology? Up to which kind of singularities can the intersection space constructions be extended? Is intersection space cohomology

of algebraic varieties endowed with a Hodge structure?

The papers [10], [6] are contributions to the first and third question for the case of isolated singularities. The paper [8] is also a contribution towards the third question.

In a recent paper [13], Geske takes a new viewpoint: instead of only giving up the topological construction and focusing on producing a complex of sheaves at the original space, he constructs a complex of vector spaces which is related with the complex computing the (co)homology of the original space X, but that satisfies Poincaŕe duality. The construction is valid for any analytic variety (Poincaré duality is satisfied in the compact case). His construction is a bit further from Banagl's original ideas than ours, since his procedure is to make a modification which is global in a neighbourhood of the singular set, instead of stratifying it conveniently and making a fibrewise construction.

Finally, in order to finish our review of existing results, let us mention the rational Poincaré spaces approach developed in [16].

The present paper is a contribution to the three questions formulated above for the general singularity case. Before explaining our results in detail in the next Section, let us enumerate them in a very condensed way:

We realize that, for constructing Banagl's intersection spaces in a general pseudomanifold X, one needs to adopt the viewpoint of pairs of spaces and associate an *intersection space pair*, which is a spatial modification of the pair $(X, Sing(X))$. We find a procedure which runs inductively on the codimension of the strata and, if it is not obstructed, produces the intersection space pair. We show that, if the link fibrations of the pseudomanifold are trivial and the trivializations of these fibrations verify some compatibility conditions, the intersection space pair exists. This includes the case of toric varieties.

We prove that, if an intersection space pair exists for a topological pseudomanifold and a given perversity, then there exists a constructible complex of sheaves in our original space X that satisfies a set of properties of the same kind as those that characterize intersection cohomology complexes in [15]; we call this complex an *intersection space complex* for the given perversity. Its hypercohomology recovers the reduced cohomology of the intersection space in the case of isolated singularities. In the case of a depth 1 topological pseudomanifold, it recovers the cohomology of the intersection space relative to the singular stratum (like in [3]), which is the one that satisfies Poincaré duality for complementary perversities. For depth 2 and higher, if the dimension of the strata is sufficiently high, the intersection space construction is intrinsically a construction of pairs of spaces, as we will see below; the hypercohomology of our intersection space complex computes the rational cohomology of the pair of spaces.

Next, we leave the realm of topology and shift to a sheaf theoretic viewpoint, studying under which conditions intersection space complexes exist. We find obstructions for existence and uniqueness of intersection space complexes and give spaces parametrizing the possible intersection space complexes in case that the obstruction for existence vanish. Both of these obstructions vanish in the

case of isolated singularities and the obstruction for existence also vanish in the case considered in [4], as one should expect. If one assumes that the topological pseudomanifold is an algebraic variety, we notice that our constructions can be carried in the category of mixed Hodge modules, yielding a polarizable mixed Hodge structure in the hypercohomology of the intersection space complex, and hence in the cohomology of intersection spaces when the corresponding obstructions vanish.

We turn to analyze classes of topological pseudomanifolds in which we can prove the existence of intersection space complexes. We show that they exist for any perversity when the successive link fibrations are trivial (without the compatibility conditions needed to construct the intersection space pair). We also prove the existence if the homological dimension of the strata with respect to local systems is at most 1. This includes the case treated in [4]. On the other hand, building on the obstructions for existence, we produce examples of topological pseudomanifolds such that intersection space complexes do not exist for given perversities. As a consequence, Banagl intersection spaces can not exist either. We thought that this were the first examples, but the referee informed us that several specialists knew examples before, see for example Banagl-Chriestenson [7] (p. 180, Example 10.3, and p. 170, Example 6.13). One of the examples is a normal algebraic variety whose stratification has depth 1 and whose transversal singularity is an ordinary double point of dimension 3 (those appearing in the conifold transition examples); the perversity used is the middle one.

Finally, we turn to duality questions. We show that the Verdier dual of an intersection space complex with a given perversity is an intersection space complex with the complementary perversity. The proof resembles the one given in [15] for intersection cohomology complexes. However, since (unlike intersection cohomology complexes) intersection space complexes are not unique, this does not yield self dual sheaves for the middle perversity on algebraic varieties. In the case of depth 1 stratifications, we prove that generic choices of the intersection space complex yield the same Betti numbers in hypercohomology and we obtain Poincaré duality at the level of generic Betti numbers for complementary perversities.

Some open questions and further directions are hinted at the end of the next section.

We would like to thank the referee for his criticism and for many suggestions that have corrected some mistakes and improved the text a lot.

2 MAIN RESULTS

This section is the guide to the paper. Here we describe in detail the content of the paper section by section, with cross-references to the main results. The reader may jump to the corresponding sections for a full exposition.

2.1 TOPOLOGICAL CONSTRUCTIONS

The paper starts with a topological construction of pairs of intersection spaces in Section 3.

Banagl's construction of the intersection space [3] for a d-dimensional topological pseudomanifold X with isolated singularities for a given perversity $\bar{p}$ runs as follows. Let $\Sigma = \{p_1, ..., p_r\}$ be the singular set. Around p_i consider a conical neighbourhood B_i. Let L_i denote the link ∂B_i. Let $\bar{q}$ be the complementary perversity of $\bar{p}$. Consider a homological truncation

$$(L_i)_{\leq \bar{q}(d)} \to L_i.$$

This is a mapping of spaces inducing isomorphisms in homology in degrees up to $\bar{q}(d)$ and such that $H_j((L_i)_{\leq \bar{q}(m)})$ vanish for $j > \bar{q}(d)$. Assume for simplicity that the truncation map is an inclusion. Construct the intersection space replacing each of the B_i's by the cone over $(L_i)_{\leq \bar{q}(d)}$, call the resulting space Z. The vertices of the cones are called $\Sigma = \{p_1, ..., p_r\}$ as well. The intersection space is the result of attaching the cone over Σ to Z. If the truncation map is not an inclusion, one may force this using an appropriate homotopy model for it. The intersection space homology is the reduced homology of the intersection space.

The construction for the case of topological pseudomanifolds $X = X_d \supset X_{d-m}$ of dimension d with a single singular stratum of codimension m contained in [3] and [7] is the following generalization. Let T be a tubular neighbourhood of the singular set X_{d-m}. Consider the locally trivial fibration $T \to X_{d-m}$, and let $\partial T \to X_{d-m}$ be the associated fibration of links. Consider a fibrewise homology truncation

$$\partial T_{\leq \bar{q}(m)} \to \partial T.$$

This is a morphism of locally trivial fibrations which is a $\bar{q}(m)$-homology truncation at each fibre. Remove T from X and replace it by the fibrewise cone over $\partial T_{\leq \bar{q}(m)}$ (see Definition 3.3); call the resulting space Z. As before Σ is a subspace of Z. The intersection space is the result of attaching the cone over

Σ to Z and the intersection space homology is the reduced homology of the intersection space.

Notice that the intersection space homology coincides with the relative homology $H_*(Z, \Sigma)$. This observation is the starting point of our construction for more than two strata and of our constructible complex approach to intersection space homology. Consider a topological pseudomanifold with stratification

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

and a suitable system of tubular neighbourhoods of the strata (the conical structure of Definition 3.5). A great variety of topological pseudomanifolds can be endowed with this structure (see Remark 3.14).

Our topological construction of intersection spaces consists in modifying the space X inductively, going each step deeper in the codimension of the strata, by taking successive fibrewise homology truncations. In doing so, one necessarily modifies the singular set X_{d-2} and, as a result, the singular set is not going to be contained in the modified space Z. Instead, one obtains a modification Y of X_{d-2} contained in Z. One obtains a pair (Z, Y), and it is the homology of this pair that defines our intersection space homology. An important feature of the construction is that one needs to adopt the viewpoint of pairs of spaces right from the beginning if one wants to have a chance of proving duality results; this incarnates in the need of taking homology truncations of pairs of spaces. This new feature appears from the depth 2 strata and hence it did not appear in Banagl constructions explained above. It also may happen that, if the singular stratum, is of small dimension in comparison with the perversity, the situation does not appear at all.

As in the Banagl's construction, the homology truncations need not be inclusions. This forces us to work with an adequate homotopy model for X.

It is important to record for future reference that, at the k-th inductive step of the construction, one obtains a pair of spaces $(I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2}))$ which contains X_{d-k-1} and verifies

1. the pair $(I_k^{\bar{p}} X \setminus X_{d-k-1}, I_k^{\bar{p}}(X_{d-2}) \setminus X_{d-k-1})$ is an intersection space pair of $X \setminus X_{d-k-1}$.

2. there is a system of tubular neighbourhoods T_{k-1} of $X_{d-k-1} \setminus X_{d-k-2}$ in $I_k^{\bar{p}} X \setminus X_{d-k-2}$ such that we have locally trivial fibrations of pairs

$$T_{k-1} \cap (I_k^{\bar{p}} X \setminus X_{d-k-2}, I_k^{\bar{p}}(X_{d-2}) \setminus X_{d-k-2}) \to X_{d-k-1} \setminus X_{d-k-2},$$

$$\partial T_{k-1} \cap (I_k^{\bar{p}} X \setminus X_{d-k-2}, I_k^{\bar{p}}(X_{d-2}) \setminus X_{d-k-2}) \to X_{d-k-1} \setminus X_{d-k-2},$$

the first being the fibrewise cone over the second (see Definition 3.3).

The second locally trivial fibration is called *the fibration of link pairs at the k-th step of the construction*.

The construction follows the scheme of obstruction theory: it is inductive and, at each step, choices are made. The next step may be obstructed and this may

depend on the previous choices. The obstruction consists in the impossibility of constructing a fibrewise homology truncation of the fibration of link pairs at the k-th step of the construction.

When there is a set of choices so that the process terminates, we say that an intersection space pair exists. It needs not be unique.

In the case where the conical structure is trivial, that is, if the link fibrations are trivial and the trivializations are compatible with each other (see Definition 3.13), the intersection space pair exists (see Theorem 3.31). This includes the case of arbitrary toric varieties.

2.2 FROM TOPOLOGY TO CONSTRUCTIBLE COMPLEXES

In the rest of the paper, we investigate the existence and uniqueness of intersection space pairs and their duality properties by sheaf theoretic methods. For this, we associate to each intersection space pair an element in the derived category of constructible complexes whose hypercohomology computes the rational cohomology of the intersection space pair.

To get this, we need to construct a sequence of intersection space pairs which modify the pair (X_d, X_{d-2}) in increasingly smaller neighbourhoods of the strata of X. This is done in Section 4.

In Section 5, we exploit the sequence of intersection spaces to derive a constructible complex IS (see Definition 5.15) and prove, in Theorem 5.16, that the hypercohomology of IS recovers the cohomology of the intersection space pair. Finally, in Theorem 5.18, we prove that IS satisfies a set of properties in the same spirit as those that characterize intersection cohomology complexes in [15]. This is the basis for the axiomatic treatment of the next section.

2.3 A DERIVED CATEGORY APPROACH TO INTERSECTION SPACE (CO)HOMOLOGY

In Section 6, we take an axiomatic approach to intersection space complexes in the same way as Goresky-MacPherson approach to intersection cohomology in [15, section 3.3]. We define two sets of properties in the derived category of cohomologically constructible sheaves on X. The first set are the properties of the intersection cohomology sheaf composed with a shift. The second set of properties are inspired by Theorem 5.18.

We will call a complex of sheaves verifying the second set of properties intersection space complex of X (Definition 6.2). Theorem 5.18 implies that if there exists an intersection space pair of X (see Definition 3.27), then there exists an intersection space complex of X whose hypercohomology coincides with the cohomology of the intersection space pair. Moreover, we compare the support and cosupport properties of intersection cohomology complexes and intersection space complexes. From the comparison, one sees that intersection space complexes, except possibly in the case of isolated singularities, can not be perverse sheaves.

At this point, we investigate in a purely sheaf theoretical way under which conditions an intersection space complex may exist. For $k = 2, ..., d$, we define $U_k := X \setminus X_{d-k}$ and we denote the canonical inclusions by $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-(k+1)} \to U_{k+1}$.

In Theorem 7.3, we give necessary and sufficient conditions for the existence of intersection space complexes. As in the topological setting, the construction proceeds inductively on each time deeper strata. At the $(k + 1)$-th step, we have constructed an intersection space complex IS_k on $X \setminus X_{d-k-1}$ such that the complex $j_{k+1}^* i_{k+1*} IS_k$ on $X_{d-k-1} \setminus X_{d-k-2}$ is cohomologically locally constant. Comparing with the topological construction these local systems are the cohomology local systems of the fibration of link pairs at the $(k + 1)$-th step of the construction. One can consider the natural triangle in the derived category:

$$\tau_{\leq \bar{q}(k+1)} j_{k+1}^* i_{k+1*} IS_k \to j_{k+1}^* i_{k+1*} IS_k \to \tau_{>q(k+1)} j_{k+1}^* i_{k+1*} IS_k \xrightarrow{[1]} .$$

The obstruction to perform the next step in the construction is the obstruction to split the triangle in the derived category. This is the constructible sheaf counterpart of the possibility to construct fibrewise homology truncations in the topological world. In Theorem 7.3, we study the parameter spaces classifying the possible intersection space complexes in step $k + 1$ having fixed the step k (they are not unique in general). In the same theorem, for the sake of comparison we provide a proof of the existence and uniqueness of intersection cohomology complexes using the same kind of techniques.

There are extension groups controlling the obstructions for existence and uniqueness at each step. They are recorded in Corollary 7.6.

If X is an algebraic variety, we produce an analogue construction of intersection space complexes to the category of mixed Hodge modules over X. This is Theorem 8.1 and, as a corollary, one obtains a mixed Hodge structure in the cohomology of intersection spaces if the corresponding obstructions vanish. The obstructions for existence and uniqueness are the same kind of extension groups, but taken in the category of mixed Hodge modules (see Corollary 8.2). Using the same techniques, we show that, for an arbitrary perversity, the intersection cohomology complexes are mixed Hodge modules (Theorem 8.3). This puts a mixed Hodge structure on intersection cohomology with arbitrary perversity. We believe that this should be well known, but we provide a proof since it is a simple consequence of our ideas.

2.4 Classes of spaces admitting intersection space complexes and counterexamples

From the previous section, it is clear that the spaces admitting intersection space complexes need to be special. In this Section 9, we find two sufficient conditions for this, yielding an ample class of (yet special) examples.

By the results explained previously it is clear that pseudomanifolds admitting

a trivial conical structure (see Definition 3.13), the intersection space complex exists by the combination of Theorems 3.31 and 5.18.

The point is that if one only needs the existence of intersection space complexes one can relax the triviality properties. In Theorem 9.8 we prove that if the link fibrations are trivial in the sense of Definition 3.10, then the intersection space complex exists. This leaves out the hypothesis on the compatibility of trivializations.

The next class of examples admitting intersection space complexes are pseudomanifolds whose strata are of "cohomological dimension at most 1 for local systems" in the sense of Definition 9.10. This is proved in Theorem 9.11. This includes the case studied by Banagl in [4] and the case of complex analytic varieties with critical set of dimension 1 which are sufficiently singular, in the sense that there are no positive dimensional compact strata (Corollary 9.12).

From our previous results, it is clear that a necessary condition for the existence of an intersection space is the existence of an intersection space complex. We find a few examples (see Example 9.16, Example 9.17 and Example 9.18) not admitting an intersection space complex. The last example is an algebraic variety with two strata and the perversity is the middle one. So, one should not expect that algebraicity helps in the existence of intersection spaces. The idea to produce the examples is to observe that, if an space admits intersection space complexes, certain differentials in the Leray spectral sequence of the fibrations of links (which in our sheaf theoretic treatment is a local to global spectral sequence) have to vanish (Proposition 9.14 and Corollary 9.15). The referee informed us that the fact that nonvanishing differentials in the spectral sequence of the link bundle can obstruct fiberwise homology truncation has already been observed in [5, Theorem 5.2 and remarks on page 296], and that previous counterexamples to the existence of intersection spaces were found by several specialist (Banagl-Chriestenson [7] p. 180, Example 10.3, and p. 170, Example 6.13).

2.5 DUALITY

In Section 10, we prove that, if $\bar{p}$ and $\bar{q}$ are complementary perversities and $IS_{\bar{p}}$ is an intersection space complex for perversity $\bar{p}$, then its Verdier dual is an intersection space complex for perversity $\bar{q}$ (Theorem 10.1). So, the Verdier duality functor exchanges the sets of intersection space complexes for complementary perversities. The proof follows the axiomatic treatment of [15] for intersection cohomology complexes.

A surprising consequence is that the existence of intersection space complexes is equivalent for complementary perversities (Corollary 10.2). Then, we move to the case of depth 1 stratifications and prove, in Proposition 10.3, that, for generic choices of the intersection space complexes, the Betti numbers are always the same (they are minimal). Then, in Theorem 10.6, we show that the Betti numbers symmetry predicted by Poincaré duality for complementary perversities is satisfied for generic intersection space Betti numbers.

2.6   Open questions

Here is a list of natural questions for further study:

1. We conjecture that the intersection space complexes associated via Definition 5.15 to the intersection space pairs constructed for pseudomanifolds with trivial conical structure in Theorem 3.31 are self Verdier dual when the strata are of even codimension and the perversity is the middle one.

2. Assume that the intersection space complex exists. Does there exist an associated rational homotopy intersection space? Are there further restrictions than the existence of the intersection space complex?

3. If the intersection space complex exists, can one define on its hypercohomology a natural internal cup product? Can one find a product turning its space of sections into a differential graded algebra inducing a cup product? This would lead to a definition of "intersection space rational homotopy type".

4. Toric varieties have intersection space pairs. Compute their Betti numbers in terms of the combinatorics of the fan.

5. Generalize the generic Poincaré duality Theorem 10.6 to the case of arbitrary depth stratifications.

6. This is a suggestion of Banagl: relate our sheaf theoretical methods with the characteristic class obstructing Poincaré duality discussed in [7]. If the characteristic class vanishes, is the intersection space complex self-dual in the case of pseudomanifolds with trivial conical structure, even codimensional strata and middle perversity?

3   A topological construction of intersection spaces

3.1   Topological preliminaries

First, we give some basic definitions about t-uples of spaces in order to fix notation.

DEFINITION 3.1.   *1. A t-uple of spaces is an ordered set of topological spaces* $(Z_1, ..., Z_t)$.

2. A morphism from a t-uple of spaces into a space $(Z_1, ..., Z_t) \to Z$ *is a set of morphisms* $\varphi_i : Z_i \to Z$.

3. A morphism between t-uples of spaces $(Z_1, ..., Z_t) \to (Z'_1, ..., Z'_t)$ *is a set of morphisms* $\varphi_i : Z_i \to Z'_i$.

4. The mapping cylinder of a morphism $\varphi = (\varphi_1, ..., \varphi_t) : (Z_1, ..., Z_t) \to Z$, $cyl(\varphi)$, *is the union of the t-uple* $(Z_1, ..., Z_t) \times [0, 1]$ *with* $(Z, \operatorname{Im}(\varphi_2), ..., \operatorname{Im}(\varphi_t))$ *with the equivalence relation* $\sim$ *such that for* $i = 1, ..., t$ *and for every* $x \in Z_i$, *we have* $(x, 1) \sim \varphi_i(x)$.

Remark 3.2. *Remember that the mapping cylinder of a morphism of spaces* $f : X \to Y$ *is* $cyl(f) := (X \times [0,1] \sqcup Y)/\sim$ *where* $\sim$ *is the equivalence relation such that for every* $x \in X$, $(x,1) \sim f(x)$.

Definition 3.3. *Let* $\sigma : (Z_1, ..., Z_t) \to B$ *be a locally trivial fibration of t-uples of spaces. The* fibrewise cone *of* σ *over the base* B *is the locally trivial fibration*

$$\pi : cyl(\sigma) \to B,$$

where $cyl(\sigma)$ *is the mapping cylinder of* σ, $\pi(x,t) := \sigma(x)$ *for* $(x,t) \in (Z_1, ..., Z_t) \times [0,1]$ *and* $\pi(b) := b$ *for* $b \in B$ *(the definition of* π *is compatible with the identifications made to construct* $cyl(\sigma)$*). The fibrewise cone over a fibration has a canonical vertex section*

$$s : B \to cyl(\sigma)$$

sending any $b \in B$ *to the vertex of the cone* $(cyl(\sigma))_b$.

The following figure shows schematicatically $cyl(\sigma)$ when the fiber of σ is the pair $(\mathbb{T}, \Sigma)$ where $\mathbb{T}$ is a torus and Σ is isomorphic to S^1, and the base B is a circle. The fibre Σ is depicted into the torus $\mathbb{T}$ as a dotted circle.
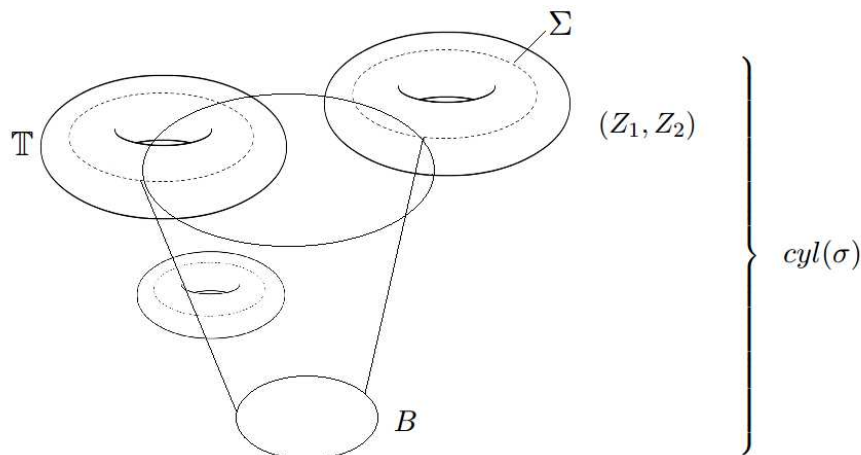
Figure 1

We use a definition of topological pseudomanifold similar to [1, Definition 4.1.1]. In the literature it is often assumed that the singular set has codimension at least 2. This is not necessary in our work, and some definitions fit better with each other not assuming it. So, we do not adopt this convention.

Definition 3.4. *A* topological pseudomanifold *is a paracompact Hausdorff topological space with a filtration by closed subspaces*

$$X = X_d \supset X_{d-1} \supset ... \supset X_0 \supset X_{-1} = \emptyset.$$

such that

- *Each pair (X_i, X_{i-1}) is a locally finite relative CW-complex.*

- *Every non-empty $X_{d-k} \setminus X_{d-k-1}$ is a topological manifold of dimension $d - k$ called pure stratum of X.*

- *$X \setminus X_{d-2}$ is dense in X.*

- Local normal triviality. *For each point $x \in X_{d-k} \setminus X_{d-k-1}$, there exists an open neighborhood U of x in X, a compact topological pseudomanifold L of dimension $k - 1$ with stratification*

$$L = L_{k-1} \supset L_{k-3} \supset ... \supset L_0 \supset L_{-1} = \emptyset$$

and a homeomorphism

$$\varphi : U \xrightarrow{\cong} \mathbb{R}^{d-k} \times c^\circ(L),$$

where $c^\circ(L)$ is the open cone of L, such that it preserves the strata, that is, $\varphi(U \cap X_{d-r}) = \mathbb{R}^{d-k} \times c^\circ(L_{k-r-1})$.

L is called the link of X over the point x.

The following new notion is important in our constructions.

Definition 3.5. *Let (X, Y) be a pair of topological spaces and let*

$$X_{d-k} \supset X_{d-k-1} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

be a topological pseudomanifold such that X_{d-k} is a subspace of Y. We say that the pair (X, Y) has a conical structure with respect to the stratified subspace if for every $r \geq k$ there exists an open neighbourhood TX_{d-r} of $X_{d-r} \setminus X_{d-r-1}$ in $X \setminus X_{d-r-1}$, with the following properties:

1. *Let $\overline{TX_{d-r}}$ be the closure of TX_{d-r} in X. There is a locally trivial fibration of $2(r - k + 1)$-uples of spaces*

$$
(\overline{TX_{d-r}} \setminus X_{d-r-1}) \cap (X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1})
$$
$$\downarrow{\sigma_{d-r}}$$
$$X_{d-r} \setminus X_{d-r-1}$$

such that its restriction to the boundary

$$
(\partial\overline{TX_{d-r}} \setminus X_{d-r-1}) \cap (X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1})
$$
$$\downarrow{\sigma^\partial_{d-r}}$$
$$X_{d-r} \setminus X_{d-r-1}$$

is a locally trivial fibration.

2. The fibration σ_{d-r} is the fibrewise cone of σ^{∂}_{d-r} over the base $X_{d-r} \setminus X_{d-r-1}$.

3. Let $k \leq r_1 < r_2 \leq d$ and consider the isomorphism induced by property (2)

$$\overline{TX_{d-r_1}} \cap (\overline{TX_{d-r_2}} \setminus X_{d-r_2-1}) \cong$$

$$\cong (\overline{TX_{d-r_1}} \cap (\partial\overline{TX_{d-r_2}} \setminus X_{d-r_2-1})) \times [0,1]/\sim$$

where $\sim$ is the equivalence relation of the mapping cylinder.

If we remove X_{d-r_1-1} in both parts of this isomorphism, we obtain an isomorphism

$$\phi_{r_1,r_2} : (\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \overline{TX_{d-r_2}} \cong$$

$$\cong ((\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \partial\overline{TX_{d-r_2}}) \times [0,1).$$

Note that since X_{d-r_2} is contained in X_{d-r_1-1}, the vertex section of the fibrewise cone is not included in the previous spaces.

With this notation, we have the equality

$$(\sigma_{d-r_1})_{|(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \overline{TX_{d-r_2}}} =$$

$$= \phi^{-1}_{r_1,r_2} \circ ((\sigma_{d-r_1})_{|(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \partial\overline{TX_{d-r_2}}}, Id_{[0,1)}) \circ \phi_{r_1,r_2},$$

that is, the fibration σ_{d-r_1} in the intersection $(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \overline{TX_{d-r_2}}$ is determined by its restriction to $(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \partial\overline{TX_{d-r_2}}$.

Figure 3 illustrates this property.

4. Let $k \leq r_1 < r_2 \leq d$. If $\partial\overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}) \neq \emptyset$, then we have the following equality of $2(r_1 - k + 1)$-uples

$$(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap (X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, \ldots$$

$$\overline{TX_{d-r_1+1}}, X_{d-r_1+1}) \cap \partial\overline{TX_{d-r_2}} = \sigma^{-1}_{d-r_1}(\partial\overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}))$$

and, in this space, we have

$$\sigma^{\partial}_{d-r_2} \circ \sigma_{d-r_1} = \sigma^{\partial}_{d-r_2}.$$

Figure 4 illustrates this property.

NOTATION 3.6. Consider a tuple $(Y_1, \ldots, Y_l)$ of subspaces whose components are a subset of the components of the tuple $(X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, \ldots, \overline{TX_{d-r+1}}, X_{d-r+1})$ considered in the previous definition. The fibre of the fibration σ^{∂}_{d-r} restricted to the tuple $(Y_1, \ldots, Y_l)$ is called the fibre of the link bundle of $(Y_1, \ldots, Y_l)$ over X_{d-r}.

REMARK 3.7. *The fibration σ_{d-r}^{∂} is the fibration of links of $X_{d-r} \setminus X_{d-r-1}$ and the fibration σ_{d-r} is the fibration associated to a tubular neighborhood.*

The fact that these fibrations of $2(r-k+1)$-uples are locally trivial yields that the intersection of the link L^x over the point $x \in X_{d-r} \setminus X_{d-r-1}$ with the open neighbourhoods $TX_{d-k}, TX_{d-k-1}, ..., TX_{d-r+1}$ only depends on the connected component of $X_{d-r} \setminus X_{d-r-1}$ containing x.

Let $k \leq r_1 < r_2 \leq d$. The following figure shows how the open neighbourhoods TX_{d-r} intersect each other.
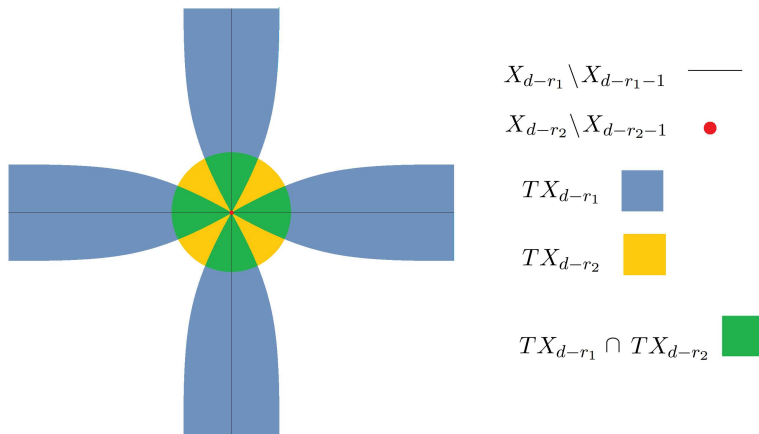

Figure 2

The following figures show how the morphisms σ_{d-r_1} and $\sigma_{d-r_2}^{\partial}$ behave in $(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \overline{TX_{d-r_2}}$ because of Properties (3) and (4) of Definition 3.5.
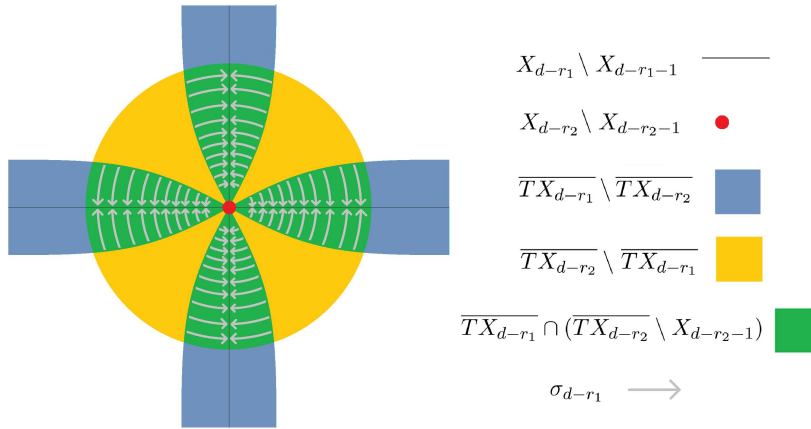
Figure 3: Property (3)
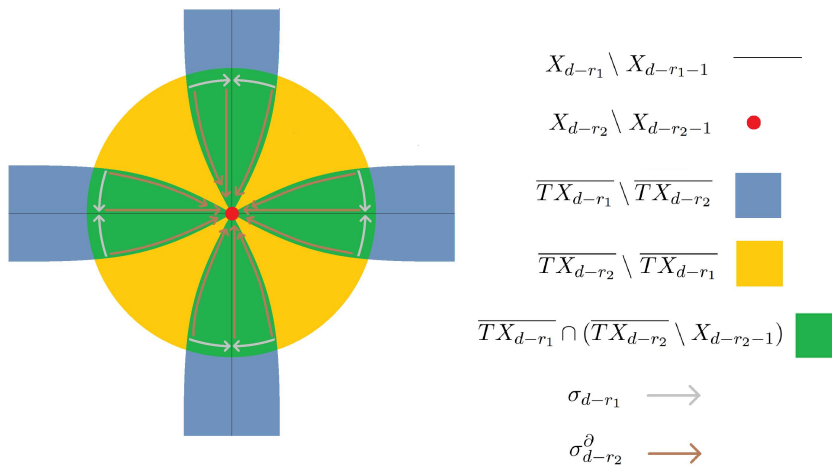
Figure 4: Property (4)

REMARK 3.8. *Using properties (3) and (4) of Definition 3.5, we can also deduce that*

$$(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap (X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ...$$

$$..., \overline{TX_{d-r_1+1}}, X_{d-r_1+1}) \cap \overline{TX_{d-r_2}} = \sigma_{d-r_1}^{-1}(\overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}))$$

and, in this space, we have

$$\sigma_{d-r_2} \circ \sigma_{d-r_1} = \sigma_{d-r_2}.$$

Notation 3.9. *Along this chapter, we will use the superindex ∂ to denote the fibrations of boundaries of suitable tubular neighborhoods.*

Definition 3.10. *We say that a conical structure verifies the r-th triviality property (T_r) if the locally trivial fibration σ^∂_{d-r} is trivial, that is, the following two properties hold for every connected component S_{d-r} of $X_{d-r} \setminus X_{d-r-1}$.*

1. *There exists an isomorphism*

$$(\sigma^\partial_{d-r})^{-1}(S_{d-r}) \cong L \times S_{d-r},$$

   where L denotes the $2(r-k+1)$-uple of the links of S_{d-r} in

   $$(X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1}).$$

2. *Under the identification given by property (1), σ^∂_{d-r} restricted to $L \times S_{d-r}$ is the canonical projection over S_{d-r}.*

Definition 3.11. *Let (X, Y) be a pair of spaces with a conical structure as in Definition 3.5 which verifies the r-th triviality property (T_r) for any r. Fix a trivialization*

$$(\sigma^\partial_{d-r})^{-1}(S_{d-r}) \cong L \times S_{d-r}$$

over each connected componet of each stratum. The set of all trivializations is called a system of trivializations *for the conical structure.*

Let (X, Y) be a pair of spaces with a conical structure as in Definition 3.5. Fix a system of trivializations for the conical structure, assuming that there is one. Let $k \leq r_1 < r_2 \leq d$ verifying that there exist connected components S_{d-r_1} and S_{d-r_2} of $X_{d-r_1} \setminus X_{d-r_1-1}$ and $X_{d-r_2} \setminus X_{d-r_2-1}$, respectively, such that $(\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1} \neq \emptyset$, or what is the same, that the closure $\overline{S_{d-r_1}}$ contains S_{d-r_2}.

By definition, $(\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(S_{d-r_1})$ and $\sigma^{-1}_{d-r_1}((\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$ are $2(k-r+1)$-uples. To simplify the notation, along the following reasoning we denote by $(\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(S_{d-r_1})$ and $\sigma^{-1}_{d-r_1}((\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$ the first components of these $2(k-r+1)$-uples. Consider also the space $(\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1}$.

Since we have fixed a system of trivializations, we have isomorphisms

$$L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2} \cong (\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1}$$

and

$$L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2} \times S_{d-r_2} \cong (\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(S_{d-r_1})$$

where $L^{S_{d-r_1}}_{d-r_2}$ denotes the fibre of the link bundle of S_{d-r_1} over S_{d-r_2} and $L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2}$ denotes the fibre of the link bundle of $\sigma^{-1}_{d-r_1}(S_{d-r_1})$ over S_{d-r_2}. Moreover, by the property (4) of Definition 3.5, we have an equality

$$\sigma^{-1}_{d-r_1}(S_{d-r_1}) \cap (\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) = \sigma^{-1}_{d-r_1}((\sigma^\partial_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$$

and, again using the fixed system of trivializations, we obtain an isomorphism

$$\sigma_{d-r_1}^{-1}((\sigma_{d-r_2}^{\partial})^{-1}(S_{d-r_2}) \cap S_{d-r_1}) \cong c(L_{d-r_1}^X) \times ((\sigma_{d-r_2}^{\partial})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$$

where $L_{d-r_1}^X$ is the fibre of the link bundle of X over S_{d-r_1}.
Combining the previous isomorphisms, we have

$$L_{d-r_2}^{\sigma_{d-r_1}^{-1}(S_{d-r_1})} \times S_{d-r_2} \cong c(L_{d-r_1}^X) \times ((\sigma_{d-r_2}^{\partial})^{-1}(S_{d-r_2}) \cap S_{d-r_1}) \cong$$

$$\cong c(L_{d-r_1}^X) \times L_{d-r_2}^{S_{d-r_1}} \times S_{d-r_2}$$

Let us denote by γ the isomorphism

$$\gamma : c(L_{d-r_1}^X) \times L_{d-r_2}^{S_{d-r_1}} \times S_{d-r_2} \to L_{d-r_2}^{\sigma_{d-r_1}^{-1}(S_{d-r_1})} \times S_{d-r_2}. \tag{1}$$

Since under the equivalences given by the trivializations the morphisms σ_{d-r_1} and $\sigma_{d-r_2}^{\partial}$ are the canonical projections, using the property (4) of Definition 3.5, the diagram

$$
\begin{array}{ccc}
c(L_{d-r_1}^X) \times L_{d-r_2}^{S_{d-r_1}} \times S_{d-r_2} & \longrightarrow & L_{d-r_2}^{S_{d-r_1}} \times S_{d-r_2} \\
\gamma \downarrow & & \downarrow \\
L_{d-r_2}^{\sigma_{d-r_1}^{-1}(S_{d-r_1})} \times S_{d-r_2} & \longrightarrow & S_{d-r_2}
\end{array}
$$

where all the morphisms except γ are the canonical projections, is commutative. So γ verifies the following condition:

$$\gamma : c(L_{d-r_1}^X) \times L_{d-r_2}^{S_{d-r_1}} \times S_{d-r_2} \longrightarrow L_{d-r_2}^{\sigma_{d-r_1}^{-1}(S_{d-r_1})} \times S_{d-r_2}$$

$$(x, y, z) \longrightarrow (\gamma_1(x, y, z), z)$$

DEFINITION 3.12. *We say that the system of trivializations is* compatible *if for any two connected components S_{d-r_1} and S_{d-r_2} as above, the map if γ_1 does not depend on z, that is, if there exists an isomorphism $\beta : c(L_{d-r_1}^X) \times L_{d-r_2}^{S_{d-r_1}} \to L_{d-r_2}^{\sigma_{d-r_1}^{-1}(S_{d-r_1})}$ such that $\gamma = (\beta, Id_{S_{d-r_2}})$.*

DEFINITION 3.13. *We say that the conical structure is* trivial *if it verifies the r-th triviality property (T_r) for any r and there exists a compatible system of trivializations.*

REMARK 3.14. *For a great variety of topological pseudomanifolds*

$$X = X_d \supset X_{d-1} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

the pair (X, X_{d-1}) has a conical structure with respect to the stratification

$$X_{d-2} \supset \dots \supset X_0 \supset X_{-1} = \emptyset.$$

Whitney stratifications, for instance, verify this property (see [14]). We assume that every topological pseudomanifold which appear in sections 3, 4 and 5 satisfies this. Moreover, we fix such a conical structure and denote the relevant neighbourhoods TX_{d-r} for r varying.
From Section 6, it is not necessary to adopt this assumption.

REMARK 3.15. *Toric varieties with their canonical stratification are topological pseudomanifolds which have a* trivial *conical structure with respect to the stratification. Details will appear in a forthcoming paper of T. Essig and the second author.*

3.2   An inductive construction of intersection spaces

Given a topological pseudomanifold we define an inductive procedure on the depth of the strata. The procedure depends on choices made at each inductive step, and may be obstructed for a given set of choices or carried until the deepest stratum. If for a given set of choices it can be carried until the end, it produces a pair of spaces which generalizes Banagl intersection spaces.

DEFINITION 3.16. *A perversity is a map $\bar{p} : \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 0}$ such that $\bar{p}(2) = 0$ and $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.*

Some special perversities are

- The zero perversity, $\bar{0}(k) = 0$.

- The total perversity, $\bar{t}(k) = k - 2$.

- The lower middle perversity, $\bar{m}(k) = \lfloor \frac{k}{2} \rfloor - 1$.

- The upper middle perversity, $\bar{n}(k) = \lceil \frac{k}{2} \rceil - 1$.

- Given a perversity $\bar{p}$, the complementary perversity is $\bar{t} - \bar{p}$. It is usually denoted by $\bar{q}$.

The lower and the upper middle perversities are complementary.
For our construction we need the notion of fibrewise homology truncation of fibrations of pairs of locally finite CW-complexes:

DEFINITION 3.17. *Let $\sigma : (X, Y) \to B$ be a locally trivial fibration. We say that σ admits a* fibrewise rational q-homology truncation *if there exists a morphism of pairs of spaces*

$$\phi : (X_{\leq q}, Y_{\leq q}) \to (X, Y)$$

such that $\sigma \circ \phi$ is a locally trivial fibration and, for any $b \in B$,

1. the homomorphism in homology of fibres

$$H_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q}) \to H_i(X_b, Y_b; \mathbb{Q})$$

is an isomorphism if $i \leq q$.

2. the homology group $H_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q})$ vanishes if $i > q$.

NOTATION 3.18. *Given a pair of spaces (X, Y) we denote by $(X, Y)_{\leq q}$ the pair $(X_{\leq q}, Y_{\leq q})$ appearing in the definition above. Note that the spaces $X_{\leq q}$ and $Y_{\leq q}$ are not homology truncations of X and Y.*

The following figure shows a fibrewise rational 1-homology truncation of the fibration σ in Figure 1. The fiber of the resulting fibration becomes a pointed circle.
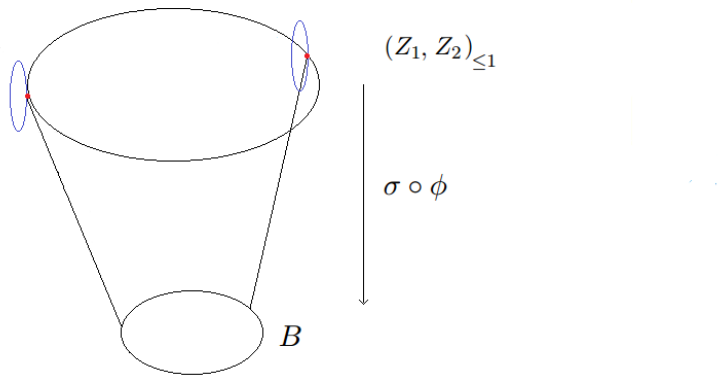
Figure 5

DEFINITION 3.19. *Given a pair of spaces (X, Y) with a conical structure as in Definition 3.5, a fibrewise rational q-homology truncation of σ_{d-r}^{∂}*

$$(\partial \overline{TX_{d-r}} \setminus X_{d-r-1} \cap (X, Y))_{\leq q}$$

$$\phi_{d-r}^{\partial} \quad \quad \overset{(\sigma_{d-r}^{\partial})_{\leq q}}{\searrow}$$

$$X_{d-r} \setminus X_{d-r-1}$$

$$\sigma_{d-r}^{\partial} \nearrow$$

$$\partial \overline{TX_{d-r}} \setminus X_{d-r-1} \cap (X, Y)$$

is compatible with the conical structure if, for every $r' > r$,

$$\sigma_{d-r'} \circ (\sigma_{d-r}^{\partial})_{\leq q} : (\sigma_{d-r}^{\partial})_{\leq q}^{-1}(\overline{TX_{d-r'}} \cap (X_{d-r} \setminus X_{d-r-1})) \to X_{d-r'} \setminus X_{d-r'-1}$$

is the fibrewise cone of

$$\sigma^{\partial}_{d-r'} \circ (\sigma^{\partial}_{d-r})_{\leq q} : (\sigma^{\partial}_{d-r})^{-1}_{\leq q}(\partial\overline{TX_{d-r'}} \cap (X_{d-r} \setminus X_{d-r-1})) \to X_{d-r'} \setminus X_{d-r'-1}$$

PROPOSITION 3.20. *Given a pair of spaces (X, Y) with a conical structure as in Definition 3.5, if there exists a fibrewise rational q-homology truncation of σ^{∂}_{d-r}, then there exists a fibrewise rational q-homology truncation of σ^{∂}_{d-r} compatible with the conical structure.*

Proof. Let us consider a fibrewise rational q-homology truncation of σ^{∂}_{d-r}, $(\sigma^{\partial}_{d-r})_{\leq q}$.
Then, the restriction of $(\sigma^{\partial}_{d-r})_{\leq q}$ to

$$(\sigma^{\partial}_{d-r})^{-1}_{\leq q}((X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r'>r} TX_{d-r'})$$

is a fibrewise rational q-homology truncation of the restriction of σ^{∂}_{d-r} to

$$(\sigma^{\partial}_{d-r})^{-1}((X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r'>r} TX_{d-r'})$$

Using the property (2) of Definition 3.5, we can extend the previous restriction to a fibrewise rational q-homology truncation over

$$(X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r''>r'} TX_{d-r''}$$

for any $r' > r$ inductively. Moreover, using the property (3) of Definition 3.5, we can check that these extensions are compatible with the conical structure. So, when $r' = d$, we obtain a fibrewise rational q-homology truncation of σ^{∂}_{d-r} compatible with the conical structure. $\square$

DEFINITION 3.21. *Given a pair of spaces (X, Y) with a conical structure verifying the triviality property (T_r) for any r (see Definition 3.10), choose a system of trivializations as in Definition 3.11. A fibrewise rational q-homology truncation of σ^{∂}_{d-r}*

is compatible with the trivialization *if the following conditions hold.*

1. *Given a connected component S_{d-r} of $X_{d-r} \setminus X_{d-r-1}$, if $L_{d-r} = (L_{d-r}^X, L_{d-r}^Y)$ denotes the fibre of the link bundle of (X,Y) over S_{d-r} and*

$$(\sigma_{d-r}^\partial)^{-1}(S_{d-r}) \cong L_{d-r} \times S_{d-r}$$

is the isomorphism induced by the system of trivializations, then there exists a pair of spaces $(L_{d-r})_{\leq q} := ((L_{d-r}^X)_{\leq q}, (L_{d-r}^Y)_{\leq q})$ such that the group $H_i((L_{d-r}^X)_{\leq q}, (L_{d-r}^Y)_{\leq q}; \mathbb{Q})$ vanishes if $i > q$, we have an isomorphism

$$(\sigma_{d-r}^\partial)_{\leq q}^{-1}(S_{d-r}) \cong (L_{d-r})_{\leq q} \times S_{d-r}$$

and, under these identifications, $((\sigma_{d-r}^\partial)_{\leq q})_{|(L_{d-r})_{\leq q} \times S_{d-r}}$ is the canonical projection and $(\phi_{d-r}^\partial)_{|(L_{d-r})_{\leq q} \times S_{d-r}} = (\phi_1, Id_{S_{d-r}})$ where $\phi_1 : (L_{d-r})_{\leq q} \to L_{d-r}$ is a morphism such that

$$H_i(\phi_1) : H_i((L_{d-r}^X)_{\leq q}, (L_{d-r}^Y)_{\leq q}; \mathbb{Q}) \to H_i(L_{d-r}^X, L_{d-r}^Y; \mathbb{Q})$$

is an isomorphism if $i \leq q$.

2. *Given $r' > r$ and a connected component $S_{d-r'}$ of $X_{d-r'} \setminus X_{d-r'-1}$ such that*

$$(\sigma_{d-r'}^\partial)^{-1}(S_{d-r'}) \cap S_{d-r} \neq \emptyset,$$

let $L_{d-r'}^{S_{d-r}}$ and $L_{d-r'}^{\sigma_{d-r}^{-1}(S_{d-r})}$ denote the fibres of the link bundles of S_{d-r} and $\sigma_{d-r}^{-1}(S_{d-r})$ over $S_{d-r'}$ respectively. Moreover, let

$$\gamma : c(L_{d-r}^X) \times L_{d-r'}^{S_{d-r}} \times S_{d-r'} \cong L_{d-r'}^{\sigma_{d-r}^{-1}(S_{d-r})} \times S_{d-r'}$$

be the isomorphism defined in Equation (1), in the discussion preceeding Definition 3.11. Then, the image of the composition

$$c((L_{d-r}^X)_{\leq q}) \times L_{d-r'}^{S_{d-r}} \times S_{d-r'} \xrightarrow{(c(\phi_1), Id)} c(L_{d-r}^X) \times L_{d-r'}^{S_{d-r}} \times S_{d-r'}$$

$$\downarrow \gamma$$

$$L_{d-r'}^{\sigma_{d-r}^{-1}(S_{d-r})} \times S_{d-r'}$$

is equal to $A \times S_{d-r'}$ for some subset $A \subset L_{d-r'}^{\sigma_{d-r}^{-1}(S_{d-r})}$.

REMARK 3.22. *If the conical structure is trivial (see Definition 3.13), the condition (1) of the previous definition implies the condition (2).*

REMARK 3.23. *If a fibrewise rational q-homology truncation of σ_{d-r}^∂ is compatible with the trivialization, then it is also compatible with the conical structure.*

The initial step of the induction.

Let X be a topological pseudomanifold such that the pair (X, X_{d-2}) has a conical structure with respect to the stratification, we consider the open neighbourhoods TX_{d-r} fixed in Remark 3.14.

Let $m \geq 2$ be the minimum such that $X_{d-m} \setminus X_{d-m-1} \neq \emptyset$. If the fibration

$$\sigma^{\partial}_{d-m} : \partial\overline{TX_{d-m}} \setminus X_{d-m-1} \to X_{d-m} \setminus X_{d-m-1}$$

predicted in Definition 3.5 does not admit a fibrewise rational $\bar{q}(m)$-homology truncation, then the intersection space does not exist. Otherwise we choose a fibrewise rational $\bar{q}(m)$-homology truncation compatible with the conical structure (see Proposition 3.20)

$$(\partial\overline{TX_{d-m}} \setminus X_{d-m-1})_{\leq \bar{q}(m)} \tag{2}$$

We construct a new space X'_m, a homotopy equivalence $\pi_m : X'_m \to X$ with contractible fibres and a subspace $I^{\bar{p}}_m X \hookrightarrow X'_m$ as follows.

Define the map

$$(\sigma_{d-m})_{\leq \bar{q}(m)} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \to X_{d-m} \setminus X_{d-m-1}$$

to be the fibrewise cone of the fibration $(\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}$ over $X_{d-m} \setminus X_{d-m-1}$. By property (2) of Definition 3.5 there exists a fibre bundle morphism

$$\phi_{d-m} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \to \overline{TX_{d-m}} \setminus X_{d-m-1}$$

over the base $X_{d-m} \setminus X_{d-m-1}$ which preserves the vertex sections. Let

$$\theta_{d-m} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \to X \setminus X_{d-m-1}$$

be the composition of the fibre bundle morphism ϕ_{d-m} with the natural inclusion of the closed subset $\overline{TX_{d-m}} \setminus X_{d-m-1}$ into $X \setminus X_{d-m-1}$. Let $cyl(\theta_{d-m})$ be the mapping cylinder of θ_{d-m}. It is by definition the union $cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \times [0,1] \coprod (X \setminus X_{d-m-1})$ under the usual equivalence relation. Denote by

$$s_{d-m} : X_{d-m} \setminus X_{d-m-1} \to cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)})$$

the vertex section. We define Z_m to be the result of quotienting $cyl(\theta_{d-m})$ by the equivalence relation which identifies, for any $x \in X_{d-m} \setminus X_{d-m-1}$, the subspace $s_{d-m}(x) \times [0,1]$ to a point.

In order to follow in an easier way our further constructions, observe at this point that the mapping cylinder $cyl(\phi^{\partial}_{d-m})$ of the vertical map ϕ^{∂}_{d-m} of diagram (2) above is a subspace both of $cyl(\theta_{d-m})$ and of Z_m.

The following figures show $cyl(\theta_{d-m})$ and Z_m.

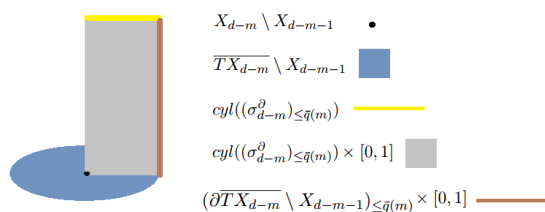|	
$X_{d-m} \setminus X_{d-m-1}$	•
$\overline{TX_{d-m}} \setminus X_{d-m-1}$	■
$cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)})$	—
$cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \times [0,1]$	▨
$(\partial \overline{TX_{d-m}} \setminus X_{d-m-1})_{\leq \bar{q}(m)} \times [0,1]$	—

Figure 6: $cyl(\theta_{d-m})$
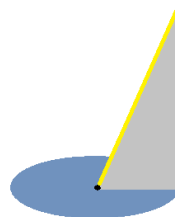
Figure 7: Z_m

The equivalence relation collapses the vertical line over the origin of the horizontal plane in Figure 6, therefore the yellow line in Figure 7 becomes diagonal.

We have a natural projection map $\pi_m : Z_m \to X \setminus X_{d-m-1}$ which is a homotopy equivalence whose fibres are contractible and has a natural section denoted by α_m. α_m is a closed inclusion of $X \setminus X_{d-m-1}$ into Z_m. In the previous figure, π_m is the projection onto the horizontal plane and α_m is the inclusion of the horizontal plane in the rest of the picture.

Define X'_m as the set $Z_m \cup X_{d-m-1}$. The projection map extends to a projection

$$\pi_m : X'_m \to X$$

Consider in X'_m the topology spanned by all the open subsets of Z_m and the collection of subsets of the form $\pi_m^{-1}(U)$ for any open subset U of X.

This projection is also a homotopy equivalence whose fibres are contractible and such that the natural section α_m extends to it giving a closed inclusion of X into X'_m.

Define the step m intersection space $I^{\bar{p}}_m X$ to be the subspace of X'_m given by

$$I^{\bar{p}}_m X := cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \times \{0\} \cup cyl(\phi^{\partial}_{d-m}) \cup (X \setminus TX_{d-m}),$$

with the restricted topology.

The following figure shows $I^{\bar{p}}_m X$.

Figure 8

REMARK 3.24. *Note that we have the equality $cyl(\phi^{\partial}_{d-m}) = \pi_m^{-1}(\partial\overline{TX_{d-m}} \setminus X_{d-m-1})$.*

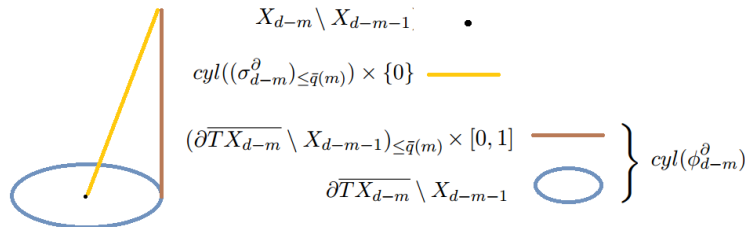With the above definitions we have the following chains of inclusions

$$X'_m \supset X \supset X_{d-m} \supset ... \supset X_0,$$

$$X'_m \supset I^{\bar{p}}_m X \supset X_{d-m} \supset ... \supset X_0,$$

where X is embedded in X'_m via the section α_m.
An immediate consequence of our construction is

LEMMA 3.25. *The pairs (X'_m, X_{d-m}) and $(I^{\bar{p}}_m X, X_{d-m})$ have a conical structure with respect to the stratified subspace $X_{d-m-1} \supset ... \supset X_0$, given by the following open neighbourhoods of $X_{d-r} \setminus X_{d-r-1}$: $\pi_m^{-1}(TX_{d-r})$ is a neighbourhood in $X'_m \setminus X_{d-r-1}$ and $\pi_m^{-1}(TX_{d-r}) \cap I^{\bar{p}}_m X$ is a neighbourhood in $I^{\bar{p}}_m X \setminus X_{d-r-1}$.*

THE INDUCTIVE STEP.
At this point we are ready to set up the inductive step of the construction of intersection spaces. The inductive step is different in nature to the initial step in the following sense. The necessary condition to be able to carry out the initial step is that a link fibration admits a fibrewise rational q-homology truncation. In the inductive step, this condition is replaced by the condition that a fibration of link pairs admits a fibrewise rational q-homology truncation. The smaller space in the pair is constructed by iterated modifications of $X_{d-2} = X_{d-m}$. Define

$$I^{\bar{p}}_m(X_{d-2}) := X_{d-2}.$$

We assume by induction that, for $k \geq m$, we have constructed

(i) a space X'_k and a projection

$$\pi_k : X'_k \to X$$

which is a homotopy equivalence with contractible fibres, together with a section α_k providing a closed inclusion of X into X'_k.

(ii) subspaces $I_k^{\bar{p}}(X_{d-2}) \subset I_k^{\bar{p}} X \subset X_k'$ such that, embedding X into X_k' via α_k, we have the topological pseudomanifold

$$X_0 \subset X_1 \subset ... \subset X_{d-k-1}$$

embedded into $I_k^{\bar{p}}(X_{d-2})$,

(iii) the pairs $(X_k', I_k^{\bar{p}} X)$, $(I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2}))$ have respective conical structures with respect to the stratified subspace described in the previous point. The open neighbourhoods of $X_{d-r} \setminus X_{d-r-1}$ appearing in these structures are $\pi_k^{-1}(TX_{d-r})$ in $X_k' \setminus X_{d-k-1}$ and $\pi_k^{-1}(TX_{d-r}) \cap I_k^{\bar{p}} X$ in $I_k^{\bar{p}} X \setminus X_{d-k-1}$ respectively.

If $X_{d-k-1} \setminus X_{d-k-2}$ is empty we define $X_{k+1}' := X_k'$, $\pi_{k+1} := \pi_k'$, $\alpha_{k+1} := \alpha_k$, $I_{k+1}^{\bar{p}} X := I_k^{\bar{p}} X$, and $I_{k+1}^{\bar{p}}(X_{d-2}) := I_k^{\bar{p}}(X_{d-2})$. It is clear that the required conditions are satisfied.

If $X_{d-k-1} \setminus X_{d-k-2}$ is not empty, since the pair $(I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2}))$ has a conical structure with respect to the stratified subspace

$$X_0 \subset X_1 \subset ... \subset X_{d-k-1},$$

we have a locally trivial fibration of pairs

$$\sigma_{d-k-1}^{\partial} : (\partial \overline{\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2})) \to X_{d-k-1} \setminus X_{d-k-2}.$$

If the fibration does not admit a fibrewise rational $\bar{q}(k+1)$-homology truncation, then the intersection space construction cannot be completed with the previous choices.

Otherwise we choose a fibrewise rational $\bar{q}(k+1)$-homology truncation compatible with the conical structure (see Proposition 3.20)

$$((\partial \overline{\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2}))_{\leq \bar{q}(k+1)}$$

We construct now a homotopy equivalence $\pi_{k+1} : X_{k+1}' \to X$ with contractible fibres and a pair of subspaces $(I_{k+1}^{\bar{p}} X, I_{k+1}^{\bar{p}}(X_{d-2})) \hookrightarrow X_{k+1}'$ as follows. Let

$$(\sigma_{d-k-2})_{\leq \bar{q}(k+1)} : cyl((\sigma_{d-k-1}^{\partial})_{\leq \bar{q}(k+1)}) \to X_{d-k-1} \setminus X_{d-k-2}$$

be the fibrewise cone of the fibration of pairs $(\sigma_{d-k-1}^{\partial})_{\leq \bar{q}(k+1)}$ over $X_{d-k-1} \setminus X_{d-k-2}$. Recall that, according with Definition 3.3, $cyl((\sigma_{d-k-1}^{\partial})_{\leq \bar{q}(k+1)})$ is a pair of spaces.

By property (2) of Definition 3.5 there exists a morphism of pairs of fibre bundles of pairs

$$\phi_{d-k-1} : cyl((\sigma^\partial_{d-k-1})_{\leq \bar{q}(k+1)}) \to \overline{(\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2}))$$

over the base $X_{d-k-1} \setminus X_{d-k-2}$ which preserves the vertex sections.
Let

$$\theta_{d-k-1} : cyl((\sigma^\partial_{d-k-1})_{\leq \bar{q}(k+1)}) \to X_k' \setminus X_{d-k-2}$$

be the composition of the fibre bundle morphism ϕ_{d-k-1} with the natural inclusion

$$\overline{(\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}} X, I_k^{\bar{p}}(X_{d-2})) \hookrightarrow X_k' \setminus X_{d-k-2}.$$

Let $cyl(\theta_{d-k-1})$ be the mapping cylinder of θ_{d-k-1}. It is by definition the union of the pair $cyl((\sigma^\partial_{d-k-1})_{\leq \bar{q}(k+1)}) \times [0,1]$ with the pair $(X_k' \setminus X_{d-k-2}, \mathrm{Im}(\phi_{d-k-1})_2)$ with the usual equivalence relation (where $(\phi_{d-k-1})_2$ denotes the second component of the fibre bundle of pairs ϕ_{d-k-1}).
Denote by

$$s_{d-k-1} : X_{d-k-1} \setminus X_{d-k-2} \to cyl((\sigma^\partial_{d-k-1})_{\leq \bar{q}(k+1)})$$

the vertex section. We define Z_{k+1} to be the pair of spaces which results of quotienting $cyl(\theta_{d-k-1})$ by the equivalence relation which identifies, for any $x \in X_{d-k-1} \setminus X_{d-k-2}$, the subspace $s_{d-k-1}(x) \times [0,1]$ to a point.
We denote the spaces forming the pair Z_{k+1} by $Z_{k+1} = (Z_{k+1}^1, Z_{k+1}^2)$. We have a natural projection map $\rho_{k+1} : Z_{k+1}^1 \to X_k' \setminus X_{d-k-2}$ which is a homotopy equivalence whose fibres are contractible, and has a natural section denoted by β_{k+1}. The composition

$$\pi_{k+1} := \pi_k|_{X_k' \setminus X_{d-k-2}} \circ \rho_{k+1} : Z_{k+1}^1 \to X \setminus X_{d-k-2}$$

is a homotopy equivalence with contractible fibres, and has a section $\alpha_{k+1} := \beta_{k+1} \circ \alpha_k|_{X \setminus X_{d-k-2}}$ providing a closed inclusion of $X \setminus X_{d-k-2}$ into Z_{k+1}^1.
Define X_{k+1}' as the set $Z_{k+1}^1 \cup X_{d-k-2}$. The projection maps ρ_{k+1} and π_{k+1} extends to projections

$$\rho_{k+1} : X_{k+1}' \to X_k' \tag{3}$$

$$\pi_{k+1} : X_{k+1}' \to X. \tag{4}$$

Consider the topology in X_{k+1}' spanned by the all the open subsets of Z_{k+1} and the collection of subsets of the form $\pi_{k+1}^{-1}(U)$ for any open subset U of X. With this topology the projections are also homotopy equivalences whose fibres are contractible, and such that the natural sections β_{k+1} and α_{k+1} extend to them as closed inclusions.
Define the step $k+1$ intersection space pair to be the pair of subspaces of X_{k+1}' given by

$$(I_{k+1}^{\bar{p}} X, I_{k+1}^{\bar{p}}(X_{d-2})) := cyl((\sigma^\partial_{d-k-1})_{\leq \bar{q}(k+1)}) \times \{0\} \cup$$

$$\cup cyl(\phi^{\partial}_{d-k-1}) \cup (I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2})) \setminus \pi_k^{-1}(TX_{d-k-1}),$$

with the restricted topology.

REMARK 3.26. *Note that we have the equality $cyl(\phi^{\partial}_{d-k-1}) = \pi_{k+1}^{-1}(\partial \overline{TX_{d-k-1}} \setminus X_{d-k-2})$.*

With the definitions above, and using that the homology truncation is compatible with the conical structure, it is easy to check that conditions (i)-(iii) are satisfied replacing k by $k+1$ and the induction step is complete.

DEFINITION 3.27. *Given a topological pseudomanifold $X_d \supset ... \supset X_0$ such that the pair (X_d, X_{d-2}) has a conical structure with respect to the stratification (see Definition 3.21 and Remark 3.14), we say that it has an intersection space pair if there exist successive choices of suitable fibrewise homology truncations so that the construction above can be carried up to $k = d$. In that case the pair*

$$(I^{\bar{p}}X, I^{\bar{p}}(X_{d-2})) = (I^{\bar{p}}_d X, I^{\bar{p}}_d(X_{d-2}))$$

is called an intersection space pair associated with the stratification.

REMARK 3.28. *Note that $I^{\bar{p}}X$ and $I^{\bar{p}}(X_{d-2})$ are not the intersection spaces of X and X_{d-2} respectively. Moreover, the absolute cohomology of these spaces is never to be considered, only the relative homology of the pair is relevant for the paper.*

DEFINITION 3.29. *We denote $X' := X'_d$. The* homotopy model *of X is the homotopy equivalence π_d which we denote $\pi : X' \to X$. The section α_d is denoted by $\alpha : X \to X'$ and provides a closed inclusion of X into X'.*

REMARK 3.30. *If the intersection space pair exists it does not have to be unique up to homotopy. The different choices of fibrewise homology truncations may yield different choices of intersection spaces. The construction of intersection spaces follows the scheme of obstruction theory in algebraic topology: previous choices of fibrewise homology truncation may affect the possibility of finishing the construction in the subsequent steps.*

3.3 Intersection space pairs for pseudomanifolds having trivial conical structures

Let X be a topological pseudomanifold with a trivial conical structure (see Definition 3.13). Fix a compatible system of trivializations (see Definitions 3.11 and 3.12). We carry out the inductive construction of the intersection space pair as above, but we add the following property to the properties (i)-(iii) which are checked along the induction:

(iv) the conical structures of the pairs $(X'_k, I^{\bar{p}}_k X)$, $(I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2}))$ with respect to

$$X_0 \subset X_1 \subset ... \subset X_{d-k-1}$$

are trivial, and a compatible system of trivializations is inherited from the inductive construction.

At the initial step of the construction we have a topological pseudomanifold

$$X \supset X_{d-m} \supset ... \supset X_0$$

with a trivial conical structure and a compatible set of trivializations (as before the codimension of the first non-open stratum is m).

The compatible system of trivializations gives us a fixed trivialization of the fibration

$$\sigma_{d-m}^{\partial} : \partial\overline{TX_{d-m}} \setminus X_{d-m-1} \to X_{d-m} \setminus X_{d-m-1}.$$

Choose a rational $\bar{q}(m)$-homology truncation of the fibre. This is always possible and elementary. Now, using the trivialization, the rational $\bar{q}(m)$-homology truncation of the fibre propagates to a fibrewise $\bar{q}(m)$-homology truncation of the fibration above. This is the truncation chosen at the initial step.

Now, using the compatibility of our system of trivializations, it is easy to show that the pairs $(X_k', I_m^{\bar{p}}X)$, $(I_m^{\bar{p}}X, I_m^{\bar{p}}(X_{d-2}))$ satisfy the required properties (i)-(iv). The compatible systems of trivializations required in property (iv) are inherited, by construction, by the compatible system of trivializations used at the beginning.

The inductive step of the construction is carried out in the same way: the fixed trivializations propagate to rational homology truncations of the corresponding fibrations of pairs of links.

We have proven:

THEOREM 3.31. *If X is a topological pseudomanifold with a trivial conical structure (see Definition 3.13,) then there exists an intersection space pair associated with it for every perversity.*

COROLLARY 3.32. *Let X be a toric variety, endowed with an stratification such that each stratum is an orbit by the torus action. Then, X has an intersection space pair for every perversity.*

Proof. The proof can be derived easily using the torus action to construct a compatible system of trivializations. A fully detailed proof is lengthy, and will appear in a forthcoming paper of T. Essig and the second author, where Poincaré Duality for intersection spaces associated with toric varieties is proved. $\square$

4   A sequence of Intersection Space pairs

Our aim is to associate with any choice of intersection space pair, a constructible complex on the original topological pseudomanifold X, whose hypercohomology coincides with the hypercohomology of the intersection space $I^{\bar{p}}X$. In order to do so, we define an increasing sequence of modified intersection space pairs, all of them included in the homotopy model X'. We provide precise definitions of the sequence, but leave much of the straightforward checking to the reader.

4.1 Systems of neighborhoods

Given a topological pseudomanifold

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

such that the pair (X, X_{d-2}) has a conical structure with respect to the stratification

$$X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

(see Remark 3.14), we denote the relevant neighbourhoods by TX_{d-r} for r varying.

Property (2) of Definition 3.5 states that the fibration σ_{d-r} is the fibrewise cone of the fibration σ_{d-r}^{∂} over the base $X_{d-r} \setminus X_{d-r-1}$. This means precisely that $\overline{TX_{d-r}} \setminus X_{d-r-1}$ is equal to the product

$$\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \times [0,1],$$

modulo the equivalence relation which identifies $(x, 1)$ and $(y, 1)$ if $\sigma_{d-r}^{\partial}(x)$ equals $\sigma_{d-r}^{\partial}(y)$.

For any $r \in 2, ..., d$ and any $n \in \mathbb{N}$ we define the open neighborhood $T^n X_{d-r}$ to be the quotient of

$$\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \times (1 - 1/(n+1), 1]$$

under the same equivalence relation.

The open subsets $T^n X_{d-r}$ for n varying, form a system of tubular neighborhoods of $X_{d-r} \setminus X_{d-r-1}$ in $X \setminus X_{d-r-1}$, whose intersection is the stratum $X_{d-r} \setminus X_{d-r-1}$. Moreover, for any fixed n the collection of neighborhoods $T^n X_{d-r}$, for r varying, give a conical structure to (X, X_{d-2}) with respect to the topological pseudomanifold $X_{d-2} \supset ... \supset X_0$.

4.2 The sequence of intersection space pairs

Suppose that there exists successive choices of suitable fibrewise homology truncations so that the construction of intersection space pairs described in the previous section can be carried up to $k = d$. Fix such a choice.

Fix $n \in \mathbb{N}$. Following the inductive construction of the previous section we produce a sequence of pairs

$$(I_k^{\bar{p},n} X, I_k^{\bar{p},n}(X_{d-2}))$$

for $k = 2, ..., r$ as follows.

Let m be the minimum such that $X_{d-m} \setminus X_{d-m-1} \neq \emptyset$. Define

$$K_m^n(X) := \pi_m^{-1}(\overline{TX_{d-m}} \setminus T^n X_{d-m}) \subset X_m',$$

$$I_m^{\bar{p},n}(X) := I_m^{\bar{p}}(X) \cup K_m^n(X),$$

$$C_m^n(X_{d-2}) := \emptyset,$$

$$I_m^{\bar{p},n}(X_{d-2}) := I_m^{\bar{p}}(X_{d-2}) = X_{d-2}.$$

REMARK 4.1. *Note that* $\pi_m^{-1}(\overline{(TX_{d-m}} \setminus T^n X_{d-m}) \setminus X_{d-m-1}) \cong cyl(\phi_{d-m}^\partial) \times$
$[0, 1 - 1/(n+1)]$ *(see Remark 3.24).*

The following figure shows the previous modification in Figure 8. $K_m^n(X)$ is
the union of blue set and brown set.

$X_{d-m} \setminus X_{d-m-1}$ •

$cyl((\sigma_{d-m}^\partial)_{\leq \bar{q}(m)}) \times \{0\}$ ——

$\overline{TX_{d-m}} \setminus T^n X_{d-m}$ ■

$\pi_m^{-1}(\overline{TX_{d-m}} \setminus T^n X_{d-m}) \setminus \overline{TX_{d-m}} \setminus T^n X_{d-m}$ ■
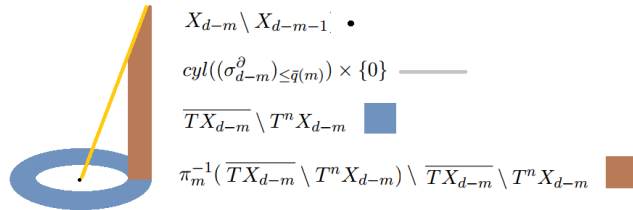
Figure 9

The following figure shows $\overline{TX_{d-m}} \setminus T^n X_{d-m}$ in more dimensions than in the
previous figure. $K_m^n(X)$ is the preimage of the blue set by the morphism π_m.

X_{d-m-1} •

$X_{d-m} \setminus X_{d-m-1}$ ——

$\overline{TX_{d-m}} \setminus T^n X_{d-m}$ ■
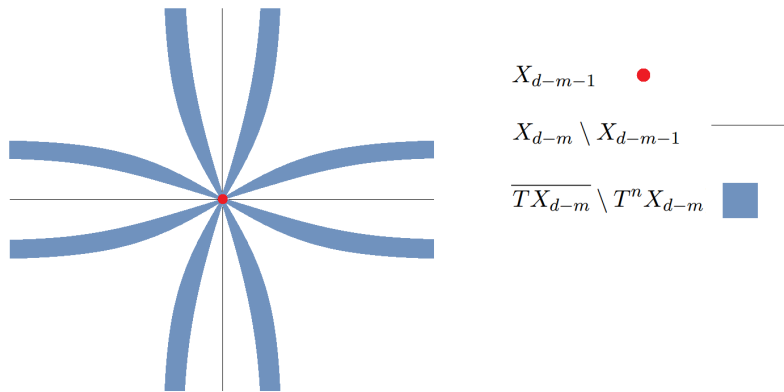
Figure 10

Assume that $K_k^n(X), C_k^n(X_{d-2}), I_k^{\bar{p},n}(X)$ and $I_k^{\bar{p},n}(X_{d-2})$ have been defined.
Recall that ρ_{k+1} is the projection defined in Equation (3). Define

$$K_{k+1}^n(X) := \rho_{k+1}^{-1}\big(((I_k^{\bar{p}} X \cap \pi_k^{-1}(\overline{TX_{d-(k+1)}})) \setminus \pi_k^{-1}(T^n X_{d-(k+1)})) \cup$$

$$\cup (K_k^n(X) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))\big),$$

$$I_{k+1}^{\bar{p},n} X := I_{k+1}^{\bar{p}} X \cup K_{k+1}^n(X),$$

$$C_{k+1}^n(X_{d-2}) := \rho_{k+1}^{-1}\big(((I_k^{\bar{p}}(X_{d-2}) \cap \pi_k^{-1}(\overline{TX_{d-(k+1)}})) \setminus \pi_k^{-1}(T^n X_{d-(k+1)})) \cup$$

$$\cup (C_k^n(X_{d-2}) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))),$$

$$I_{k+1}^{\bar{p},n}(X_{d-2}) := I_{k+1}^{\bar{p}}(X_{d-2}) \cup C_{k+1}^n(X_{d-2}).$$

The following figure illustrates the second induction step. Recall that the codimension of the biggest non-open stratum is m. The figure shows $(\overline{TX_{d-m}} \setminus T^n X_{d-m}) \setminus T^n X_{d-m-1}$ in blue and green and $(\overline{TX_{d-m-1}} \setminus T^n X_{d-m-1}) \setminus (\overline{TX_{d-m}} \setminus T^n X_{d-m})$ in yellow. $K_{m+1}^n(X)$ is the union of

- the preimage of the blue and green set by π_{m+1}

- the preimage of the yellow set by π_{m+1} intersected with the preimage of $I_m^{\bar{p}} X$ by ρ_{m+1}

$C_{m+1}^n(X_{d-2})$ is the union of

- the preimage of the blue and green set by π_{m+1}

- the preimage of the yellow set by π_{m+1} intersected with the preimage of $I_m^{\bar{p}}(X_{d-2})$ by ρ_{m+1}
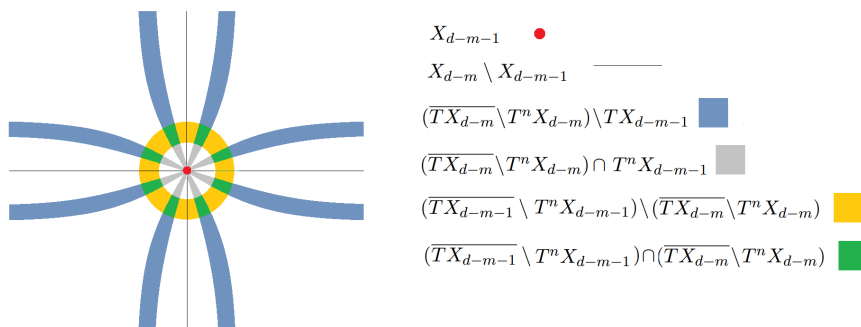
Figure 11

Iterate the construction until $k = d$ and define

$$(I^{\bar{p},n} X, I^{\bar{p},n}(X_{d-2})) := (I_d^{\bar{p},n} X, I_d^{\bar{p},n}(X_{d-2})),$$

which is a pair of subsets of X'.

Since the closed subsets $K_k^n(X), C_k^n(X_{d-2})$ are increasingly larger when n increases we have constructed a sequence of pairs of closed subsets

$$(I^{\bar{p}} X, I^{\bar{p}}(X_{d-2})) \subset ... \subset (I^{\bar{p},n} X, I^{\bar{p},n}(X_{d-2})) \subset (I^{\bar{p},n+1} X, I^{\bar{p},n+1}(X_{d-2})) \subset ...$$

An easy inspection on the construction shows:

PROPOSITION 4.2. *The previous construction has the following properties.*

1. *The inclusions $I^{\bar{p},n}X \subset I^{\bar{p},n+1}X$ and $I^{\bar{p},n}(X_{d-2}) \subset I^{\bar{p},n+1}(X_{d-2})$ are strong deformation retracts for any $n \in \mathbb{N}$. If we denote the inclusions by $\nu^n_{X_{d-2}} : I^{\bar{p},n}(X_{d-2}) \to I^{\bar{p},n}X$, $i^n_X : I^{\bar{p},n}X \to I^{\bar{p},n+1}X$ and $i^n_{X_{d-2}} : I^{\bar{p},n}(X_{d-2}) \to I^{\bar{p},n+1}(X_{d-2})$ and the retractions by $r^n_X : I^{\bar{p},n+1}X \to I^{\bar{p},n}X$ and $r^n_{X_{d-2}} : I^{\bar{p},n+1}(X_{d-2}) \to I^{\bar{p},n}(X_{d-2})$, we have commutative diagrams*

$$
\begin{array}{ccc}
I^{\bar{p},n}(X_{d-2}) & \xrightarrow{\ \nu^n\ } & I^{\bar{p},n}X \\
{\scriptstyle r^n_{X_{d-2}}}\big\uparrow\big\downarrow{\scriptstyle i^n_{X_{d-2}}} & & {\scriptstyle i^n_X}\big\downarrow\big\uparrow{\scriptstyle r^n_X} \\
I^{\bar{p},n+1}(X_{d-2}) & \xrightarrow{\ \nu^{n+1}\ } & I^{\bar{p},n+1}X.
\end{array}
\tag{5}
$$

2. *We have the equality $I^{\bar{p},n}(X_{d-2}) = I^{\bar{p},n+1}(X_{d-2}) \cap I^{\bar{p},n}(X)$.*

3. *For any $x \in X \setminus X_{d-2}$, there exists a small contractible neighbourhood U_x of x in X and a natural number n_0 such that, for every $n > n_0$, $\pi^{-1}(U_x)$ is contained in $I^{\bar{p},n}X$ and $\pi^{-1}(U_x) \cap I^{\bar{p},n}(X_{d-2}) = \emptyset$, where π is the homotopy equivalence from the homotopy model (see 3.29).*

4. *For any $x \in X_{d-r} \setminus X_{d-r-1}$, there exists a small contractible neighbourhood U_x of x in X and a natural number n_0 such that, for any $n > n_0$, the diagram (5) restricts to the diagram*

$$
\begin{array}{ccc}
I^{\bar{p},n}(X_{d-2}) \cap \pi^{-1}(U_x) & \xrightarrow{\ \nu^n\ } & I^{\bar{p},n}X \cap \pi^{-1}(U_x) \\
{\scriptstyle r^n_{X_{d-2}}}\big\uparrow\big\downarrow{\scriptstyle i^n_{X_{d-2}}} & & {\scriptstyle i^n_X}\big\downarrow\big\uparrow{\scriptstyle r^n_X} \\
I^{\bar{p},n+1}(X_{d-2}) \cap \pi^{-1}(U_x) & \xrightarrow{\ \nu^{n+1}\ } & I^{\bar{p},n+1}X \cap \pi^{-1}(U_x)
\end{array}
\tag{6}
$$

and we have the equalities $r^n_{X_{d-2}}(I^{\bar{p},n+1}(X_{d-2}) \cap \pi^{-1}(U_x)) = I^{\bar{p},n}(X_{d-2}) \cap \pi^{-1}(U_x)$ and $r^n_X(I^{\bar{p},n+1}(X) \cap \pi^{-1}(U_x)) = I^{\bar{p},n}(X) \cap \pi^{-1}(U_x)$.

Then, the inclusions

$$\pi^{-1}(U_x) \cap I^{\bar{p},n}X \hookrightarrow \pi^{-1}(U_x) \cap I^{\bar{p},n+1}X,$$

$$\pi^{-1}(U_x) \cap I^{\bar{p},n}(X_{n-2}) \hookrightarrow \pi^{-1}(U_x) \cap I^{\bar{p},n+1}(X_{n-2})$$

are strong deformation retracts.

Proof. (Sketch) For any r, we have an equality

$$(\overline{TX_{d-r}} \setminus T^n X_{d-r}) \setminus X_{d-r-1} = \partial \overline{TX_{d-r}} \setminus X_{d-r-1} \times [0, 1 - 1/(n+1)].$$

So, there are canonical retractions $\overline{TX_{d-r}} \setminus T^{n+1} X_{d-r} \to \overline{TX_{d-r}} \setminus T^n X_{d-r}$. These retractions induce retractions $K^{n+1}_r(X) \to K^n_r(X)$ and $C^{n+1}_r(X_{d-2}) \to C^n_r(X_{d-2})$ which produce the morphisms r^n_X and $r^n_{X_{d-2}}$ respectively.

Let $x \in X \setminus X_{d-2}$. A small ball U_x around of x verifies property (3) if there exists a natural number n_0 such that U_x does not intersect $T^{n_0} X_{d-r}$ for any r. Moreover, this number n_0 exists if and only if $U_x \cap X_{d-2}$ is empty.

Let $x \in X_{d-r} \setminus X_{d-r-1}$. A small neighbourhood of x, U_x, verifying property (4) can be constructed as follows: let V_x be a ball around x in the stratum $X_{d-r} \setminus X_{d-r-1}$. Take V_x small enough so that there exists a natural number n_0 such that V_x does not intersect $T^{n_0} X_{d-k}$ for any $k > r$. Consider the retraction σ_{d-r} appearing in Definition 3.5, (1). Define

$$U_x := \sigma_{d-r}^{-1}(V_x) \cap T^{n_0-1} X_{d-r}.$$

The following figure shows U_x where $x \in X_{d-m} \setminus X_{d-m-1}$ in Figure 11.

X_{d-m-1} •

$X_{d-m} \setminus X_{d-m-1}$ ———

$(\overline{TX_{d-m}} \setminus T^{n_0} X_{d-m}) \setminus TX_{d-m-1}$ ▮

$(\overline{TX_{d-m-1}} \setminus T^{n_0} X_{d-m-1}) \setminus (\overline{TX_{d-m}} \setminus T^{n_0} X_{d-m})$ ▮

$(\overline{TX_{d-m-1}} \setminus T^{n_0} X_{d-m-1}) \cap (\overline{TX_{d-m}} \setminus T^{n_0} X_{d-m})$ ▮

U_x
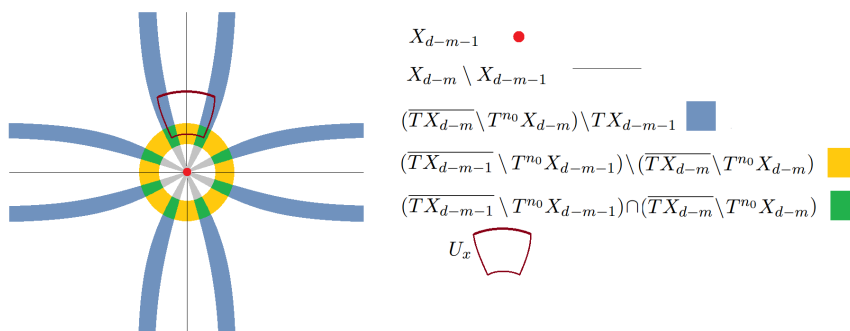
Figure 12

$\square$

DEFINITION 4.3. *Let x be any point of X. If $x \in X \setminus X_{d-2}$, a principal neighbourhood of x is a small neighbourhood which verifies Property (3) of Proposition 4.2. If $x \in X_{d-2}$, a principal neighbourhood of x is a small neighbourhood which verifies Property (4) of Proposition 4.2.*

DEFINITION 4.4. *Let $x \in X_{d-r} \setminus X_{d-r-1}$. A carved principal neighbourhood of x is an open subset U_x^* equal to $U_x \setminus X_{d-r}$ where U_x is a principal neighbourhood of x.*

Analogously to Property (4) of Proposition 4.2, we have

PROPOSITION 4.5. *If U_x^* is a carved principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$, there exists $n_0 \in \mathbb{N}$ such that, for every $n > n_0$, the diagram (5) restricts to the diagram*

$$
\begin{array}{ccc}
I^{\bar{p},n}(X_{d-2}) \cap \pi^{-1}(U_x^*) & \xrightarrow{\nu^n} & I^{\bar{p},n}X \cap \pi^{-1}(U_x^*) \\
{\scriptstyle r^n_{X_{d-2}}} \big\Updownarrow {\scriptstyle i^n_{X_{d-2}}} & & {\scriptstyle i^n_X} \big\Updownarrow {\scriptstyle r^n_X} \\
I^{\bar{p},n+1}(X_{d-2}) \cap \pi^{-1}(U_x^*) & \xrightarrow{\nu^{n+1}} & I^{\bar{p},n+1}X \cap \pi^{-1}(U_x^*)
\end{array}
\qquad (7)
$$

For the next propositions recall that σ_{d-r}^{∂} is the fibration of Definition 3.5, (2).

PROPOSITION 4.6. *If U_x is a principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$ for any $r > 0$ and $n \in \mathbb{N}$ big enough, the cohomology group*

$$H^i(I^{\bar{p},n}X \cap \pi^{-1}(U_x), I^{\bar{p},n}X_{d-2} \cap \pi^{-1}(U_x); \mathbb{Q})$$

is 0 if $i \leq \bar{q}(r)$ and isomorphic to the i-th cohomology group of the pair $(\sigma_{d-r}^{\partial})^{-1}(x) \subset (I_{r-1}^{\bar{p}}X, I_{r-1}^{\bar{p}}(X_{d-2}))$ if $i > \bar{q}(r)$.

Proof. Sketch of the proof. The pair $(I^{\bar{p},n}X \cap \pi^{-1}(U_x), I^{\bar{p},n}X_{d-2} \cap \pi^{-1}(U_x))$ is homotopy equivalent to the pair $(I^{\bar{p}}X \cap \sigma_{d-r}^{-1}(V_x), I^{\bar{p}}X_{d-2} \cap \sigma_{d-r}^{-1}(V_x))$ where V_x is a small ball around x in the stratum $X_{d-r} \setminus X_{d-r-1}$. With the notation of section 3, this space is equal to

$$cyl(((\sigma_{d-r}^{\partial})_{\leq \bar{q}(r)})_{|(\sigma_{d-r}^{\partial})^{-1}(V_x)}) \times \{0\} \cup cyl((\phi_{d-r}^{\partial})_{|(\sigma_{d-r}^{\partial})^{-1}(V_x)}) \cup$$

$$\cup (\pi^{-1}((\sigma_{d-r}^{\partial})^{-1}(V_x)) \cap (I_{r-1}^{\bar{p}}X, I_{r-1}^{\bar{p}}(X_{d-2}))),$$

which has the cohomology type of $\pi^{-1}((\sigma_{d-r}^{\partial})^{-1}(V_x)) \cap (I_{r-1}^{\bar{p}}X, I_{r-1}^{\bar{p}}(X_{d-2})) \cong \pi^{-1}((\sigma_{d-r}^{\partial})^{-1}(x)) \cap (I_{r-1}^{\bar{p}}X, I_{r-1}^{\bar{p}}(X_{d-2}))$ except for the first $q(r)$ cohomology groups which vanish by putting in $cyl(((\sigma_{d-r}^{\partial})_{\leq \bar{q}(r)})_{|(\sigma_{d-r}^{\partial})^{-1}(V_x)}) \times \{0\} \cup cyl((\phi_{d-r}^{\partial})_{|(\sigma_{d-r}^{\partial})^{-1}(V_x)})$. $\square$

PROPOSITION 4.7. *If U_x^* is a carved principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$ for any $r > 0$ and $n \in \mathbb{N}$ big enough, the cohomology group*

$$H^i(I^{\bar{p},n}X \cap \pi^{-1}(U_x^*), I^{\bar{p},n}X_{d-2} \cap \pi^{-1}(U_x^*); \mathbb{Q})$$

is isomorphic to the i-th cohomology group of the pair $(\sigma_{d-r}^{\partial})^{-1}(x) \subset (I_{r-1}^{\bar{p}}X, I_{r-1}^{\bar{p}}(X_{d-2}))$ for every $i \in \mathbb{Z}$.

5   Sheafification

5.1   Sheaf of cubical singular cochains

In this section, every topological space is hereditaly paracompact and locally contractible. In particular, the topological pseudomanifold and the intersection spaces of the previous section verify these properties.

In order to produce constructible complexes whose hypercohomology computes the cohomology of intersection space pairs we use sheaves of singular cohomology cochains. For technical reasons cubical cochains, as developed by Massey in [17, Chapters 7 and 12], adapt best to our construction. Here we sketch very briefly the main points we need; the reader should check [17] for complete definitions and proofs.

We denote by $(C_\bullet(X, \mathbb{Q}), \partial)$ the complex of cubical chains of a space X. The group $C_i(X, \mathbb{Q})$ is defined to be the quotient

$$C_i(X, \mathbb{Q}) := Q_i(X, \mathbb{Q})/D_i(X, \mathbb{Q}),$$

where $Q_i(X, \mathbb{Q})$ is the vector space spanned by maps from the i-cube to X and $D_i(X, \mathbb{Q})$ is the subspace of degenerate maps (maps which are constant in one direction of the cube). The differential ∂ is defined in the usual way. The functor given by the homology of the complex $(C_\bullet(\cdot, \mathbb{Q}), \partial)$ defines a homology theory with coefficients in $\mathbb{Q}$.

Let $(C^\bullet(X, \mathbb{Q}), \delta)$ be the complex of cubical cochains of X. It is by definition the dual of $(C_\bullet(X, \mathbb{Q}), \partial)$, and hence $C^i(X, \mathbb{Q})$ is the subspace of $Hom(Q_i(X, \mathbb{Q}), \mathbb{Q})$ formed by elements vanishing at $D_i(X, \mathbb{Q})$. The functor given by the cohomology of the complex $(C^\bullet(\cdot, \mathbb{Q}), \partial)$ defines a cohomology theory with coefficients in $\mathbb{Q}$.

Let $f : X \to Y$ be a continuous map. We denote by

$$f_{\# i} : C_i(X, \mathbb{Q}) \to C_i(Y, \mathbb{Q}),$$

$$f^{\# i} : C^i(Y, \mathbb{Q}) \to C^i(X, \mathbb{Q})$$

the associated transformations of complexes of cubical chains and cochains. They form morphisms of complexes

$$f_\# : (C_\bullet(X, \mathbb{Q}), \partial) \to (C_\bullet(Y, \mathbb{Q}), \partial),$$

$$f^\# : (C^\bullet(Y, \mathbb{Q}), \delta) \to (C^\bullet(X, \mathbb{Q}), \delta).$$

Let $f, g : X \to X$ two continuous maps. If f and g are homotopic, then $f_\#$ and $g_\#$ are homotopic morphisms of complexes, and the same happens for $f^\#$ and $g^\#$. We need for later use an explicit form of a homotopy of complexes between $f^\#$ and $g^\#$. Let

$$\rho : C_i(X, \mathbb{Q}) \to C_{i+1}(I \times X, \mathbb{Q})$$

be the morphism such that, if σ_i is a singular i-cube in X, $\rho(\sigma_i) = Id_I \times \sigma_i$ (the homomorphism ρ takes degenerate cubical chains to degenerate cubical chains). Let $H : I \times X \to Y$ be a homotopy between f and g, that is, $H_0 = f$ and $H_1 = g$. A homotopy between the morphism of complexes $f_\#$ and $g_\#$ is given by $H_\# \circ \rho$. The dual morphism of $H_\# \circ \rho$ is an homotopy between $f^\#$ and $g^\#$.

LEMMA 5.1. *Let $h : Z \to X$ be a continuous map. If $(H_t)_{|\mathrm{Im}(h)}$ is independent of $t \in I$, then for every $\sigma_i \in Q_i(Z, \mathbb{Q})$, $H_\# \circ \rho \circ h_\#(\sigma_i)$ is degenerate.*

Now we produce a sheafification of cubical cochains. This is an adaptation of the sheafification of singular chains appearing in [20].

DEFINITION 5.2. *For every $i \in \mathbb{Z}_{\geq 0}$, let C^i be the presheaf of vector spaces*

$$U \rightsquigarrow C^i(U, \mathbb{Q})$$

where the restriction morphisms are the obvious ones.
The sheaf of cubical singular i-cochains $\mathcal{C}_X^i$ is defined to be the sheafification of C^i.

For every $i \in \mathbb{Z}_{\geq 0}$, let $C_\circ^i(X)$ be the vector subspace of $C^i(X)$ given by the set of cochains $\xi^i \in C^i(X)$ such that there exists an open covering $\{U_j\}_{j \in J}$ of X such that $\xi_{|U_j}^i = 0$ for every $j \in J$. As in [20] one shows that the sheafification is defined by

$$\mathcal{C}_X^i(U) = C^i(U)/C_\circ^i(U).$$

At the level of sheaves we have functoriality as well. Let $f : X \to Y$ be a continuous map. Then, f induces a morphism of complexes of sheaves on Y

$$f^\# : \mathcal{C}_Y^\bullet \to f_* \mathcal{C}_X^\bullet.$$

As one expects, if X is contractible then

$$H^i(\mathcal{C}_X^\bullet(X)) \cong \left\{ \begin{array}{ll} \mathbb{Q} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0 \end{array} \right.$$

This implies that the complex of sheaves $\mathcal{C}_X^\bullet$ is a resolution of the constant sheaf $\mathbb{Q}_X$.

Moreover, for every $i \in \mathbb{Z}_{\geq 0}$, the sheaf $\mathcal{C}^i$ is flabby. Indeed, it is enough to prove the restriction morphisms of the presheaf, $C^i(X, \mathbb{Q}) \to C^i(U, \mathbb{Q})$, are surjective for every open subset $U \subset X$. Given $\xi \in C^i(U, \mathbb{Q})$, let $\xi_X \in C^i(X, \mathbb{Q})$ be the linear morphism $C_i(X, \mathbb{Q}) \to \mathbb{Q}$ such that, for every singular i-cube σ in X, we have

$$\xi_X(\sigma) = \left\{ \begin{array}{ll} \xi(\sigma) & \text{if} \quad \text{Im}(\sigma) \subset U \\ 0 & \text{if} \quad \text{Im}(\sigma) \not\subset U \end{array} \right.$$

Then, $(\xi_X)_{|U} = \xi$.

COROLLARY 5.3. *For every $i \in \mathbb{Z}_{\geq 0}$, the i-th cohomology group $H^i(X, \mathbb{Q})$ is isomorphic to i-th group of cohomology of the complex $\mathcal{C}_X^\bullet(X)$.*

5.2   The intersection space constructible complex

Let X be a topological pseudomanifold with stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

and a conical structure given by the stratification such that there exists a set of choices so that the inductive construction of the intersection space of X is not obstructed. Let X' be the homotopy model of X and $\pi : X' \to X$ the homotopy equivalence. Let $(I^{\bar{p}, n} X, I^{\bar{p}, n}(X_{d-2}))$ with $n \in \mathbb{N}$ be the associated sequence of intersection space pairs and

$$j^n : I^{\bar{p}, n} X \to X'$$

$$\mu^n : I^{\bar{p}, n}(X_{d-2}) \to X'$$

the canonical inclusions.

In order to lighten the formulas appearing in this section we denote by $\mathcal{C}_X^{n, \bullet}$ and $\mathcal{C}_{X_{d-2}}^{n, \bullet}$ the complex of sheaves of cubical singular cochains in $I^{\bar{p}, n} X$ and $I^{\bar{p}, n}(X_{d-2})$ respectively.

PROPOSITION 5.4. *For every $n \in \mathbb{N}$, there exists a commutative diagram*

$$
\begin{array}{ccc}
j_*^{n+1}\mathcal{C}_X^{n+1,\bullet} & \xrightarrow{\nu^{n+1\#}} & \mu_*^{n+1}\mathcal{C}_{X_{d-2}}^{n+1,\bullet} \\
\downarrow{\scriptstyle i_X^{n\#}} & & \downarrow{\scriptstyle i_{X_{d-2}}^{n\#}} \\
j_*^n\mathcal{C}_X^{n,\bullet} & \xrightarrow{\nu^{n\#}} & \mu_*^n\mathcal{C}_{X_{d-2}}^{n,\bullet}
\end{array}
\tag{8}
$$

where all the morphisms are surjective.

Proof. For every open subset $U \subset X'$, the inclusions of Diagram (5) restrict to a diagram

$$
\begin{array}{ccc}
I^{\bar{p},n}(X_{d-2}) \cap U & \xrightarrow{\nu^n} & I^{\bar{p},n}X \cap U \\
\downarrow{\scriptstyle i_{X_{d-2}}^n} & & \downarrow{\scriptstyle i_X^n} \\
I^{\bar{p},n+1}(X_{d-2}) \cap U & \xrightarrow{\nu^{n+1}} & I^{\bar{p},n+1}X \cap U
\end{array}
$$

So, we have the following diagram between the cubical cochain groups:

$$
\begin{array}{ccc}
C^i(I^{\bar{p},n+1}X \cap U, \mathbb{Q}) & \xrightarrow{\nu^{n+1\#}} & C^i(I^{\bar{p},n+1}(X_{d-2}) \cap U, \mathbb{Q}) \\
\downarrow{\scriptstyle i_X^{n\#}} & & \downarrow{\scriptstyle i_{X_{d-2}}^{n\#}} \\
C^i(I^{\bar{p},n}X \cap U, \mathbb{Q}) & \xrightarrow{\nu^{n\#}} & C^i(I^{\bar{p},n}(X_{d-2}) \cap U, \mathbb{Q})
\end{array}
$$

The morphisms of this diagram induce the morphisms of the proposition. Moreover, these morphisms are surjective since every inclusion of topological spaces induces a surjection between the corresponding cubical cochain groups. $\square$

Denote by $\mathcal{K}^{n,\bullet}$ the kernel of $\nu^{n\#}$. There is a canonical morphism

$$
i^{n\#} : \mathcal{K}^{n+1,\bullet} \to \mathcal{K}^{n,\bullet}.
$$

REMARK 5.5. *For every $i \in \mathbb{Z}_{\geq 0}$ and every $n \in \mathbb{N}$, the i-th rational cohomology group of the pair $(I^{\bar{p},n}X, I^{\bar{p},n}(X_{d-2}))$ is isomorphic to i-th cohomology group of the complex $\mathcal{K}^{n,\bullet}$.*

DEFINITION 5.6. *Given a pair of natural numbers $n_1 < n_2$, we will define*

$$
i^{n_1,n_2} := i^{n_1\#} \circ \ldots \circ i^{n_2-1\#} : \mathcal{K}^{n_2,\bullet} \to \mathcal{K}^{n_1,\bullet}
$$

Then, the complexes of sheaves $\{\mathcal{K}^{n,\bullet}\}_{n \in \mathbb{N}}$ and the morphisms i^{n_1,n_2} form an inverse system and we can consider the inverse limit

$$
\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet},
$$

which is a complex of sheaves.

Lemma 5.7. $(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_{X \setminus X_{d-2}}$ *is quasi-isomorphic to* $\mathbb{Q}_{X \setminus X_{d-2}}$.

Proof. Let $x \in X \setminus X_{d-2}$. We have the following equalities:

$$(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_x = \varinjlim_{x \in U \text{open}} \pi_*(\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})(U) = \varinjlim_{x \in U \text{open}} (\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})(\pi^{-1}(U)) =$$

$$= \varinjlim_{x \in U \text{open}} \varprojlim_{n \in \mathbb{N}} (\mathcal{K}^{n,\bullet}(\pi^{-1}(U))).$$

Let U_x be a principal neighbourhood of x (see definition 4.3). There exists $n_0 \in \mathbb{N}$ such that, for every open subset $U \subset U_x$ and for every $n > n_0$, we have

$$\pi^{-1}(U) \cap I^{\bar{p},n} X = \pi^{-1}(U)$$

and

$$\pi^{-1}(U) \cap I^{\bar{p},n}(X_{d-2}) = \emptyset$$

Consequently, $\mathcal{K}^{n,\bullet}(\pi^{-1}(U)) = j_*^n \mathcal{C}_X^{n,\bullet}(\pi^{-1}(U)) = \mathcal{C}_{X'}^{\bullet}(\pi^{-1}(U))$ where $\mathcal{C}_{X'}$ is the sheaf of singular i-cochains in X'.

Moreover, for every $n > n_0$, $i^{n\#} = i_X^{n\#} = Id_{\mathcal{C}_{X'}^{\bullet}(\pi^{-1}(U))}$. So, $\varprojlim_{n \in \mathbb{N}} (\mathcal{K}^{n,\bullet}(\pi^{-1}(U))) = \mathcal{C}_{X'}^{\bullet}(\pi^{-1}(U))$.

Thus, we have shown that $(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_{X \setminus X_{d-2}}$ is quasi-isomorphic to $\pi_* \mathbb{Q}_{\pi^{-1}(X \setminus X_{d-2})}$, and the later sheaf is quasi-isomorphic to $\mathbb{Q}_{X \setminus X_{d-2}}$ since $\pi|_{\pi^{-1}(X \setminus X_{d-2})}$ is a homotopy equivalence. □

We study now the cohomology of the complex $\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet}$ over each of the deeper strata of X and over the global sections. With this purpose, we study the cohomology of $\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet}$ in the principal and the carved principal neighbourhoods (see definitions 4.3 and 4.4) and in the total space.

Proposition 5.8. *Let* U *be equal to* X *or a principal neighbourhood or a carved principal neighbourhood of some* $x \in X_{d-k} \setminus X_{d-(k+1)}$. *Then,*

$$H^i(\varprojlim_{n \in \mathbb{N}} (\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))) \cong \varprojlim_{n \in \mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U))).$$

Now, we need some preliminary work in order to prove Proposition 5.8.

Lemma 5.9. *Let* U *be equal to* X *or a principal neighbourhood or a carved principal neighbourhood of some* $x \in X_{d-k} \setminus X_{d-(k+1)}$. *Then, there exists* $n_0 \in \mathbb{N}$ *such that if* $n > n_0$, *there exists a morphism*

$$r_U^{n\#} : \mathcal{K}^{n,\bullet}(\pi^{-1}(U)) \to \mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))$$

such that $i^{n\#}(\pi^{-1}(U)) \circ r_U^{n\#} = Id_{\mathcal{K}^{n,\bullet}(\pi^{-1}(U))}$ *and* $r_U^{n\#} \circ i^{n\#}(\pi^{-1}(U))$ *is homotopic to the identity.*

Moreover, there exists a homotopy h_U^n *between* $r_U^{n\#} \circ i^{n\#}(\pi^{-1}(U))$ *and* $Id_{\mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))}$ *such that* $i^{n\#}(\pi^{-1}(U)) \circ h_U^n = 0$.

Proof. Let U be a principal neighbourhood and let n_0 be the natural number of Proposition 4.2 (4). For every $n > n_0$, the diagram (6) induces a diagram

$$
\begin{array}{ccc}
j_*^{n+1}\mathcal{C}_X^{n+1,\bullet}(\pi^{-1}(U)) & \xrightarrow{\nu^{n+1\#}} & \mu_*^{n+1}\mathcal{C}_{X_{d-2}}^{n+1,\bullet}(\pi^{-1}(U)) \\
{\scriptstyle r_X^{n\#}}\Big\uparrow\Big\downarrow{\scriptstyle i_X^{n\#}} & & {\scriptstyle i_{X_{d-2}}^{n\#}}\Big\downarrow\Big\uparrow{\scriptstyle r_{X_{d-2}}^{n\#}} \\
j_*^{n}\mathcal{C}_X^{n,\bullet}(\pi^{-1}(U)) & \xrightarrow{\nu^{n\#}} & \mu_*^{n}\mathcal{C}_{X_{d-2}}^{n,\bullet}(\pi^{-1}(U))
\end{array}
$$

where $i_{X_{d-2}}^{n\#} \circ r_{X_{d-2}}^{n\#} = Id_{\mu_*^n\mathcal{C}_{X_{d-2}}^{n,\bullet}(\pi^{-1}(U))}$, $i_X^{n\#} \circ r_X^{n\#} = Id_{j_*^n\mathcal{C}_X^{n,\bullet}(\pi^{-1}(U))}$ and $r_{X_{d-2}}^{n\#} \circ i_{X_{d-2}}^{n\#}$ and $r_X^{n\#} \circ i_X^{n\#}$ are homotopic to the identity.
Then, we obtain a canonical morphism

$$
r_U^{n\#} : \mathcal{K}^{n,\bullet}(\pi^{-1}(U)) \to \mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))
$$

such that $i^{n\#}(\pi^{-1}(U)) \circ r_U^{n\#} = Id_{\mathcal{K}^{n,\bullet}(\pi^{-1}(U))}$ and $r_U^{n\#} \circ i^{n\#}(\pi^{-1}(U))$ is homotopic to the identity.
Consider the diagram (6) of Proposition 4.2 (4). There exist homotopies

$$
H_X^n : \pi^{-1}(U) \cap I^{\bar{p},n+1}X \times I \to \pi^{-1}(U) \cap I^{\bar{p},n+1}X
$$

between $i_X^n \circ r_X^n$ and $Id_{\pi^{-1}(U) \cap I^{\bar{p},n+1}X}$ and

$$
H_{X_{d-2}}^n : \pi^{-1}(U) \cap I^{\bar{p},n+1}(X_{d-2}) \times I \to \pi^{-1}(U) \cap I^{\bar{p},n+1}(X_{d-2})
$$

between $i_{X_{d-2}}^n \circ r_{X_{d-2}}^n$ and $Id_{\pi^{-1}(U) \cap I^{\bar{p},n+1}(X_{d-2})}$. Moreover, we can suppose that, for every $t \in I$, the restrictions of H_{Xt}^n and $H_{X_{d-2}t}^n$ to $\pi^{-1}(U) \cap I^{\bar{p},n}X$ and $\pi^{-1}(U) \cap I^{\bar{p},n}(X_{d-2})$ are the identity respectively.
Following the procedure explained in Section 5.1 the mapping H_X^n induces a homotopy between $i_X^{n\#} \circ r_X^{n\#}$ and $Id_{j_*^{n+1}\mathcal{C}_X^{n+1,\bullet}(\pi^{-1}(U))}$. Moreover, applying Lemma 5.1 we have that $i_X^{n\#} \circ h_X^n$ is equal to 0.
Similarly the mapping $H_{X_{d-2}}^n$ induce a homotopy $h_{X_{d-2}}^n$ between $i_{X_{d-2}}^{n\#} \circ r_{X_{d-2}}^{n\#}$ and $Id_{\mu_*^{n+1}\mathcal{C}_{X_{d-2}}^{n+1,\bullet}(\pi^{-1}(U))}$ such that $i_{X_{d-2}}^{n\#} \circ h_{X_{d-2}}^n$ is equal to 0.
So, there exists a homotopy h_U^n between $i^{n\#}(\pi^{-1}(U)) \circ r_U^{n\#}$ and $Id_{\mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))}$ such that $i^{n\#}(\pi^{-1}(U)) \circ h_U^n = 0$.
If U is equal to X or a carved principal neighbourhood we can apply the same method using the diagram (5) of Proposition 4.2 or the diagram of Proposition 4.5, respectively. $\square$

REMARK 5.10. *The diagram (5) of Proposition 4.2 is valid for every $n \in \mathbb{N}$. So, in the previous lemma, we can take $n_0 = 0$ if U is the total space.*

REMARK 5.11. *Note that the morphisms $r_U^{n\#}$ do not induce a morphism of complexes of sheaves since they are not defined for every open subset.*

DEFINITION 5.12. *Given a pair of natural numbers $n_1 < n_2$ such that $n_0 < n_1$, we define*

$$r_U^{n_1,n_2} := r_U^{n_2-1\#} \circ ... \circ r_U^{n_1\#} : \mathcal{K}^{n_1,\bullet}(\pi^{-1}(U)) \to \mathcal{K}^{n_2,\bullet}(\pi^{-1}(U))$$

REMARK 5.13. *For every $n > n_0$, the inclusions $\pi^{-1}(U) \cap I^{\bar{p},n}X \hookrightarrow \pi^{-1}(U) \cap I^{\bar{p},n+1}X$ and $\pi^{-1}(U) \cap I^{\bar{p},n}X_{d-2} \hookrightarrow \pi^{-1}(U) \cap I^{\bar{p},n+1}X_{d-2}$ are homotopy equivalences. So, $i_X^{n\#}(\pi^{-1}(U))$ and $i_{X_{d-2}}^{n\#}(\pi^{-1}(U))$ are quasi-isomorphisms. Then,*

$$i^{n\#}(\pi^{-1}(U)) : \mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))) \to \mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$$

is also a quasi-isomorphism and we have an isomorphism

$$\varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U))) \longrightarrow H^i(\mathcal{K}^{n_0+1,\bullet}((\pi^{-1}(U))))$$

$$\{[\xi_n^i]\}_{n\in\mathbb{N}} \longrightarrow [\xi_{n_0+1}^i]$$

NOTATION 5.14. *For every open subset $V \subset X'$, the elements of $H^i(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}(V))$ are equivalence classes of elements*

$$\{\xi_n^i\}_{n\in\mathbb{N}} \in \mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial_n^{i+1}(V)) \subset \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i}(V)$$

which we are going to denote with $[\{\xi_n^i\}_{n\in\mathbb{N}}]$.
In addition, given $n \in \mathbb{N}$ and an element $\xi_n^i \in \mathrm{Ker}(\partial_n^{i+1}(V)) \subset \mathcal{K}^{n,i}(V)$, we are going to denote its equivalence class in $H^i(\mathcal{K}^{n,\bullet}(V))$ with $[\xi_n^i]$.

Proof of Proposition 5.8. It is enough to prove that, if U is equal to X or a principal neighbourhood or a carved principal neighbourhood of x, then the morphism

$$\mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial_n^{i+1}(\pi^{-1}(U))) \xrightarrow{\alpha} \varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$$

$$\{\xi_n^i\}_{n\in\mathbb{N}} \longrightarrow \{[\xi_n^i]\}_{n\in\mathbb{N}}$$

factorizes into a morphism

$$H^i(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}(\pi^{-1}(U))) \xrightarrow{\beta} \varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$$

which is an isomorphism.
First, we prove α factorices. Let us consider an element $\{\xi_n^i\}_{n\in\mathbb{N}} \in \mathrm{Im}(\varprojlim_{n\in\mathbb{N}} \partial_n^i(\pi^{-1}(U)))$. Then, there exists an element $\{\delta_n^{i-1}\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i-1}(\pi^{-1}(U))$ such that

$$\varprojlim_{n\in\mathbb{N}} \partial_n^i(\pi^{-1}(U))(\{\delta_n^{i-1}\}_{n\in\mathbb{N}}) = \{\xi_n^i\}_{n\in\mathbb{N}}.$$

So, for every $n \in \mathbb{N}$, $\partial_n^i(\pi^{-1}(U))(\delta_n^{i-1}) = \xi_n^i$. Consequently, $\alpha(\{\xi_n^i\}_{n \in \mathbb{N}}) = \{[\xi_n^i]\}_{n \in \mathbb{N}} = 0$ and we conclude that the morphism factorizes.

Now, we prove that α and, consequently, β are surjective.

Because of Remark 5.13, it is enough to prove that, for every element $\xi \in \mathrm{Ker}(\partial_{n_0+1}^{i+1}((\pi^{-1}(U))))$, there exists an element $\{\xi_n^i\}_{n \in \mathbb{N}} \in \mathrm{Ker}(\varprojlim_{n \in \mathbb{N}} \partial_n^{i+1}(\pi^{-1}(U)))$ such that $\xi_{n_0+1}^i = \xi$.

Let us consider

$$\xi_n^i = \begin{cases} (i^{n,n_0+1}(\pi^{-1}(U)))(\xi) & \text{if} \quad n < n_0 + 1 \\ \xi & \text{if} \quad n = n_0 + 1 \\ r_U^{n_0+1,n}(\xi) & \text{if} \quad n > n_0 + 1 \end{cases}$$

Then, for every pair of natural numbers $n_1 < n_2$, we have the equality

$$((i^{n_1,n_2})(\pi^{-1}(U)))(\xi_{n_2}^i) = \xi_{n_1}^i.$$

Hence $\{\xi_n^i\}_{n \in \mathbb{N}}$ belongs to $\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,i}(\pi^{-1}(U))$.

Moreover

- we have the vanishing $(\partial_{n_0+1}^{i+1}(\pi^{-1}(U)))(\xi) = 0$,

- for every pair of natural numbers $n_1 < n_2$ we have the equality

$$\partial_{n_1}^i \circ i^{n_1,n_2} = i^{n_1,n_2} \circ \partial_{n_2}^i$$

and,

- if $n_1 > n_0$, we have the equality

$$\partial_{n_2}^i(\pi^{-1}(U)) \circ r_U^{n_1,n_2} = r_U^{n_1,n_2} \circ \partial_{n_1}^i(\pi^{-1}(U)).$$

Consequently $(\partial_n^{i+1}(\pi^{-1}(U)))(\xi_n^i) = 0$ for every $n \in \mathbb{N}$, $\{\xi_n^i\}_{n \in \mathbb{N}}$ belongs to $\mathrm{Ker}(\varprojlim_{n \in \mathbb{N}} \partial_n^{i+1}(\pi^{-1}(U)))$ and α and β are surjective.

Finally, we prove β is injective, that is, $\mathrm{Ker}(\alpha) = \mathrm{Im}(\varprojlim_{n \in \mathbb{N}} \partial_n^i(\pi^{-1}(U)))$.

Let $\{\xi_n^i\}_{n \in \mathbb{N}} \in \mathrm{Ker}(\varprojlim_{n \in \mathbb{N}} \partial_n^{i+1})(\pi^{-1}(U))$ such that $\alpha(\{\xi_n^i\}_{n \in \mathbb{N}}) = \{[\xi_n^i]\}_{n \in \mathbb{N}} = 0$. Then, $\xi_{n_0+1}^i \in \mathrm{Im}(\partial_{n_0+1}^i(\pi^{-1}(U)))$. So, there exists $\delta \in \mathcal{K}^{n_0+1,\,i-1}(\pi^{-1}(U))$ such that $(\partial_{n_0+1}^i(\pi^{-1}(U)))(\delta) = \xi_{n_0+1}^i$.

For every $n \in \mathbb{N}$, we define

$$\delta_n^{i-1} = \begin{cases} (i^{n,n_0+1}(\pi^{-1}(U)))(\delta) & \text{if} \quad n < n_0 + 1 \\ \delta & \text{if} \quad n = n_0 + 1 \\ r_U^{n_0+1,n}(\delta) & \text{if} \quad n > n_0 + 1 \end{cases}$$

Then, $\{\delta_n^{i-1}\}_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\,i-1}(\pi^{-1}(U))$.

Let us denote $\widetilde{\xi_n^i} := (\partial_n^i(\pi^{-1}(U)))(\delta_n^{i-1})$ for every $n \in \mathbb{N}$. Then we have the equality $(\varprojlim_{n \in \mathbb{N}} \partial_n^i(\pi^{-1}(U)))(\{\delta_n^{i-1}\}_{n \in \mathbb{N}}) = \{\widetilde{\xi_n^i}\}_{n \in \mathbb{N}}$ and $[\{\widetilde{\xi_n^i}\}_{n \in \mathbb{N}}] = 0$ in $H^i(\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$.

So, to prove β is injective, it is enough to prove the equality $[\{\widetilde{\xi_n^i}\}_{n\in\mathbb{N}}] = [\{\xi_n^i\}_{n\in\mathbb{N}}]$ in $H^i(\varprojlim_{n\in\mathbb{N}}\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$.

If $n < n_0 + 1$, we have

$$\widetilde{\xi_n^i} = \partial_n^i(\pi^{-1}(U))(\delta_n^{i-1}) = \partial_n^i(\pi^{-1}(U))(i^{n,n_0+1}(\pi^{-1}(U)))(\delta)) =$$

$$= i^{n,n_0+1}(\pi^{-1}(U))(\partial_{n_0+1}^i(\pi^{-1}(U))(\delta)) = i^{n,n_0+1}(\pi^{-1}(U))(\xi_{n_0+1}^i) = \xi_n^i.$$

If $n = n_0 + 1$, we have

$$\widetilde{\xi_{n_0+1}^i} = \partial_{n_0+1}^i(\pi^{-1}(U))(\delta) = \xi_{n_0+1}^i.$$

If $n > n_0 + 1$, we have

$$\widetilde{\xi_n^i} = \partial_n^i(\pi^{-1}(U))(\delta_n^{i-1}) = \partial_n^i(\pi^{-1}(U))(r_U^{n_0+1,n}(\delta)) =$$

$$= r_U^{n_0+1,n}(\partial_{n_0+1}^i(\pi^{-1}(U))(\delta)) = r_U^{n_0+1,n}(\xi_{n_0+1}^i) =$$

$$= r_U^{n_0+1,n}(i^{n_0+1,n}(\pi^{-1}(U))(\xi_n^i)).$$

For every $n > n_0 + 1$, let h_U^n be the homotopy defined in Lemma 5.9. A simple computation shows that, for every $n > n_0 + 1$, we have the equality:

$$\widetilde{\xi_n^i} - \xi_n^i = \partial_n^i\left(\sum_{k=n_0+1}^{n-1}(r_U^{k,n}\circ h_U^{k-1})(\xi_k^i) + h_U^{n-1}(\xi_n^i)\right).$$

Let

$$\epsilon_n^{i-1} = \begin{cases} 0 & \text{if } n \leq n_0 + 1 \\ \sum_{k=n_0+2}^{n-1}(r_U^{k,n}\circ h_U^{k-1})(\xi_k^i) + h_U^{n-1}(\xi_n^i) & \text{if } n > n_0 + 1 \end{cases}$$

Let us prove $\{\epsilon_n^{i-1}\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,i-1}(\pi^{-1}(U)))$.

Since, for every $n > n_0 + 1$, $i^{n\#}(\pi^{-1}(U)) \circ h_U^n = 0$ and $i^{n\#}(\pi^{-1}(U)) \circ r_U^{n\#} = Id_{\mathcal{K}^{n,\bullet}(\pi^{-1}(U))}$, if $n_1 > n_0 + 1$, we have the equality

$$i^{n_1,n_2}(\pi^{-1}(U))\left(\sum_{k=n_0+2}^{n_2-1}(r_U^{k,n_2}\circ h^{k-1})(\xi_k^i) + h_U^{n_2-1}(\xi_{n_2}^i)\right) =$$

$$= \sum_{k=n_0+2}^{n_1-1}(r_U^{k,n_1}\circ h_U^{k-1})(\xi_k^i) + h_U^{n_1-1}(\xi_{n_1}^i),$$

and if $n_1 \leq n_0 + 1$, we have

$$i^{n_1,n_2}(\pi^{-1}(U))\left(\sum_{k=n_0+2}^{n_2-1}(r_U^{k,n_2}\circ h_U^{k-1})(\xi_k^i) + h_U^{n_2-1}(\xi_{n_2}^i)\right) = 0.$$

Then, $\{\epsilon_n^{i-1}\}_{n \in \mathbb{N}}$ belongs to $\varprojlim_{n \in \mathbb{N}}(\mathcal{K}^{n,i-1}(\pi^{-1}(U)))$ and we have the equality

$$\varprojlim_{n \in \mathbb{N}} \partial^i(\pi^{-1}(U))(\{\epsilon_n^{i-1}\}_{n \in \mathbb{N}}) = \{\widetilde{\xi_n^i}\}_{n \in \mathbb{N}} - \{\xi_n^i\}_{n \in \mathbb{N}}.$$

Therefore, $[\{\widetilde{\xi_n^i}\}_{n \in \mathbb{N}}]$ equals $[\{\xi_n^i\}_{n \in \mathbb{N}}]$ in $H^i(\varprojlim_{n \in \mathbb{N}}(\mathcal{K}^{n,i}(\pi^{-1}(U))))$ and we conclude. $\square$

DEFINITION 5.15. *Let us define* $IS := \pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet}$.

THEOREM 5.16. *The hypercohomology of IS is isomorphic to the cohomology of the intersection space pair.*

Proof. The sheaves $\mathcal{K}^{n,i}$ are flabby. Let U be any open subset in X'. Every section in $\mathcal{K}^{n,i}(U)$, ξ, is also a section of $j_*^n \mathcal{C}^{n,i}(U)$. Then, ξ is the equivalence class of a singular cubical i-cochain $\xi' \in C^i(U \cap I^{\bar{p},n}X, \mathbb{Q})$. We can extend ξ' by 0 to get a singular cubical i-cochain $\xi'_{I^{\bar{p},n}X} \in C^i(I^{\bar{p},n}X, \mathbb{Q})$, that is, for every singular i-cube σ in $I^{\bar{p},n}X$, we have

$$\xi'_{I^{\bar{p},n}X}(\sigma) = \begin{cases} \xi'(\sigma) & \text{if} \quad \mathrm{Im}(\sigma) \subset U \\ 0 & \text{if} \quad \mathrm{Im}(\sigma) \not\subset U \end{cases}$$

The equivalence class of $\xi'_{I^{\bar{p},n}X}$ in the sheaf of singular cubical i-cochains is a section $\xi_{X'}$ in $j_*^n \mathcal{C}^{n,i}(X')$. It is easy to check that $\xi_{X'}$ is contained in $\mathcal{K}^{n,i}(X')$ since it is a extension by 0 of ξ.

Now, we prove $\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,i}$ is also flabby. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a section of $(\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,i})(U) = \varprojlim_{n \in \mathbb{N}}(\mathcal{K}^{n,i}(U))$. Since $\mathcal{K}^{n,i}(U)$ is flabby for every natural n, the sections ξ_n extend by 0 to sections $\xi_n^{X'}$ in $\mathcal{K}^{n,i}(X')$. It is easy to check that $i^{n_1,n_2}(X')(\xi_{n_2}^{X'})$ is equal to $\xi_{n_1}^{X'}$ for every pair of natural numbers $n_1 < n_2$. So, $\{\xi_n^{X'}\}_{n \in \mathbb{N}}$ is a global section of $\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,i}$.

Then, all the sheaves of the complex IS are flabby and, consequently, the hypercohomology of IS is equal to the cohomology of the global sections of IS. By Proposition 5.8, there is an isomorphism

$$H^i(\Gamma(X, IS)) \cong \varprojlim_{n \in \mathbb{N}} H^i(\Gamma(X', \mathcal{K}^n{}^\bullet)).$$

Since the cohomology $H^i(\Gamma(X', \mathcal{K}^n{}^\bullet))$ is the cohomology of the intersection space pair for every $n \in \mathbb{N}$, we conclude. $\square$

Now we prove a set of properties of the complex IS, in a similar vein as those satisfied by intersection cohomology sheaves.

DEFINITION 5.17. *For $k = 2, ..., d$, we define $U_k := X \setminus X_{d-k}$ and we denote the canonical inclusions with $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-(k+1)} \to U_{k+1}$.*

THEOREM 5.18. *The complex of sheaves IS satisfies the following properties.*

1. $IS_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2}$.

2. The cohomology sheaves $\mathcal{H}^i(IS)$ are 0 if $i \notin \{0, 1, ..., d\}$

3. For $k = 2, ..., d$, the cohomology sheaves $\mathcal{H}^i(j_k^* IS_{|U_{k+1}})$ are 0 if $i \leq \bar{q}(k)$.

4. For $k = 2, ..., d$, the usual morphisms between the cohomology sheaves $\mathcal{H}^i(j_k^* IS_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} IS_{|U_k})$ are isomorphisms if $i > \bar{q}(k)$.

Proof. (1) is shown in lemma 5.7

Let $x \in X_{d-k} \setminus X_{d-(k+1)}$ for some $k \in \{2, 3, ..., d\}$. Given a complex of sheaves we denote by H^i and $\mathcal{H}^i$ its i-th cohomology presheaf and sheaf respectively. We have the obvious chain of equalities:

$$\mathcal{H}^i(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_x = H^i(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_x = \varinjlim_{x \in U \text{open}} H^i((\varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})(\pi^{-1}(U))) =$$

$$= \varinjlim_{x \in U \text{open}} H^i(\varprojlim_{n \in \mathbb{N}}(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))).$$

Since the principal neighbourhoods form a system of neighborhoods for any point, we can suppose every open subset U appearing in the previous formula is a principal neighbourhood of x. Then, applying Proposition 5.8 we have

$$\mathcal{H}^i(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_x = \varinjlim_{\substack{U \text{ principal} \\ \text{neighbourhood of } x}} \varprojlim_{n \in \mathbb{N}} H^i((\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))).$$

So, applying proposition 4.6, $\mathcal{H}^i(\pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{K}^{n,\bullet})_x$ is 0 if $i \leq \bar{q}(k)$ and equal to

$$H^i((\sigma_{d-r}^\partial)^{-1}(x); \mathbb{Q})$$

if $i > \bar{q}(k)$.

Hence, we have proven (2) and (3) of the theorem.

Moreover, applying again Proposition 5.8

$$\mathcal{H}^i(j_k^* i_{k*} IS_{|U_k})_x = \varinjlim_{\substack{U \text{ principal} \\ \text{neighbourhood of } x}} H^i(\varprojlim_{n \in \mathbb{N}} \pi_* \mathcal{K}^{n,\bullet}(U \setminus X_{d-k})) =$$

$$= \varinjlim_{\substack{U \text{ principal} \\ \text{neighbourhood of } x}} \varprojlim_{n \in \mathbb{N}} H^i(\pi_* \mathcal{K}^{n,\bullet}(U \setminus X_{d-k}))$$

and, because of Proposition 4.7,

$$\mathcal{H}^i(j_k^* i_{k*} IS_{|U_k})_x = H^i((\sigma_{d-r}^\partial)^{-1}(x); \mathbb{Q})$$

for every $i \in \mathbb{Z}$, which concludes the proof. $\square$

6   AXIOMS OF INTERSECTION SPACE COMPLEXES

From now on, we do not need to assume that our topological pseudomanifold has a conical structure with respect to the stratification, like in Remark 3.14. Let X be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset \tag{9}$$

Let $U_k := X \setminus X_{d-k}$ and let $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ be the usual inclusions.

Let us denote by $D_{cc}^b(X)$ the bounded derived category of cohomologically constructible sheaves of rational vector spaces on X with the previous stratification. Fix a perversity $\bar{p}$ and let us consider the following sets of properties in this category:

1. We say that $B^\bullet \in D_{cc}^b(X)$ verifies $[AX1]_k$ for perversity $\bar{p}$ if:

   (a) $B^\bullet_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2}$,

   (b) the cohomology sheaf $\mathcal{H}^i(B^\bullet)$ is 0 if $i \notin \{0, 1, ..., d\}$,

   (c_k) $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}})$ is equal to 0 if $i > \bar{p}(k)$,

   (d_k) the natural morphism $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} B^\bullet_{|U_k})$ is an isomorphism if $i \leq \bar{p}(k)$.

2. Let $\bar{q}$ be the complementary perversity of $\bar{p}$. We say that $B^\bullet \in D_{cc}^b(X)$ verifies $[AXS1]_k$ for perversity $\bar{p}$ if:

   (a) $B^\bullet_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2}$,

   (b) the cohomology sheaf $\mathcal{H}^i(B^\bullet)$ is 0 if $i \notin \{0, 1, ..., d\}$,

   (c_k) $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}})$ is equal to 0 if $i \leq \bar{q}(k)$,

   (d_k) the natural morphism $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} B^\bullet_{|U_k})$ is an isomorphism if $i > \bar{q}(k)$.

REMARK 6.1. $B^\bullet$ *verifies* $[AX1]_k$ *for* $k = 2, ..., d$ *if and only if* $B^\bullet[d]$ *verifies the axioms [AX1] of [15, section 3.3], that is, if* $B^\bullet[d]$ *is the intersection cohomology sheaf of* X. *So, we will denote an object of* $D_{cc}^b(X)$ *verifying* $[AX1]_k$ *for* $k = 2, ..., d$ *by* $IC_{\bar{p}}[-d]$.

DEFINITION 6.2. *An* intersection space complex *of* X *with perversity* $\bar{p}$ *and stratification (9) is a complex of sheaves verifying* $[AXS1]_k$ *for* $k = 2, ..., d$. *We denote by* $IS_{\bar{p}}$ *a complex of sheaves in* X *with these properties.*

REMARK 6.3. *If the stratification of* X *induces a conical structure (see Definition 3.5) and there exists an intersection space pair of* X *with perversity* $\bar{p}$ *in the sense of Definition 3.27, then there exists an intersection space complex of* X *(see Theorem 5.18).*

In the sequel, we will need equivalent axioms to $[AXS1]_k$. In the following remark, we review the method of [15, section 3.4] to get equivalent axioms to $[AX1]_k$ and $[AXS1]_k$.

REMARK 6.4. *Using the long exact sequence of cohomology associated to the distinguished triangle*

$$j_k^! B^\bullet_{|U_{k+1}} \to j_k^* B^\bullet_{|U_{k+1}} \to j_k^* i_{k*} B^\bullet_{|U_k} \xrightarrow{[1]} \quad,$$

(c_k) and (d_k) of $[AX1]_k$ are equivalent to (c_k) and (d'_k) where (d'_k) is the following property:

$$\mathcal{H}^i(j_k^! B^\bullet_{|U_{k+1}}) = 0 \ \text{if } i \le \bar{p}(k) + 1.$$

Given $x \in X_{d-k} \setminus X_{d-k-1}$, let $u_x : \{x\} \to X_{d-k} \setminus X_{d-k-1}$ and $j_x : \{x\} \to X$ be the canonical inclusions. Then, following [15] 3.4, page 103, we have

$$j_x^! B^\bullet = u_x^! j_k^! B^\bullet_{|U_{k+1}} = u_x^* j_k^! B^\bullet_{|U_{k+1}}[k - d].$$

So, (d'_k) is equivalent to the following property (d''_k):
For every $x \in X_{d-k} \setminus X_{d-k-1}$, $\mathcal{H}^i(j_x^! B^\bullet_{|U_{k+1}}) = 0$ if $i \le \bar{p}(k) + 1 + d - k = d - \bar{q}(k) - 1$.
Now, we apply the same method to properties $[AXS1]_k$. Using again the long exact sequence of cohomology associated to

$$j_k^! B^\bullet_{|U_{k+1}} \to j_k^* B^\bullet_{|U_{k+1}} \to j_k^* i_{k*} B^\bullet_{|U_k} \xrightarrow{[1]},$$

we deduce (d_k) of $[AXS1]_k$ is equivalent to $(d1_k)$ and $(d2_k)$ where $(d1_k)$ and $(d2_k)$ are the following properties.

$(d1_k)$ $\mathcal{H}^i(j_k^! B^\bullet_{|U_{k+1}}) = 0$ *if* $i > \bar{q}(k) + 1$.

$(d2_k)$ *The canonical morphism $\mathcal{H}^{\bar{q}(k)+1}(j_k^! B^\bullet_{|U_{k+1}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_k^* B^\bullet_{|U_{k+1}})$ is the morphism 0.*

Moreover, using the property 1.13(15) of [15], these properties are equivalent to:

$(d1'_k)$ *For every $x \in X_{d-k} \setminus X_{d-k-1}$, $\mathcal{H}^i(j_x^! B^\bullet_{|U_{k+1}}) = 0$ if $i > \bar{q}(k) + 1 + d - k = d - \bar{p}(k) - 1$.*

$(d2'_k)$ *For every $x \in X_{d-k} \setminus X_{d-k-1}$, the canonical morphism*

$$\mathcal{H}^{d-\bar{p}(k)-1}(j_x^! B^\bullet_{|U_{k+1}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^* B^\bullet_{|U_{k+1}})$$

(given by property 1.13(15) of [15]) is the morphism 0.

Now, we recall useful definitions to compare the axioms $[AX1]_k$ with $[AXS1]_k$.

DEFINITION 6.5. *Let $B^\bullet$ be a complex of sheaves in a topological space X and, for every $x \in X$, let $j_x : \{x\} \to X$ be the canonical inclusion. Then,*

- *The* local support *of $B^\bullet$ in degree m is*

$$\{x \in X | \mathcal{H}^m(j_x^* B^\bullet) \neq 0\}$$

- *The* local cosupport *of $B^\bullet$ in degree m is*

$$\{x \in X | \mathcal{H}^m(j_x^! B^\bullet) \neq 0\}$$

The properties (c_k) of $[AX1]_k$, (c_k) of $[AXS1]_k$ and (d'_k), $(d1'_k)$, $(d2'_k)$ of Remark 6.4 can be defined in terms of support and cosupport.

Let us consider a complex stratified variety X. Then, the upper middle perversity and the lower middle perversity (see Definition 3.16) are equal over the codimension of the strata of X.

Let $\bar{m}$ be the middle perversity. The following table, taken from [11], illustrates the conditions of support and cosupport for a complex of sheaves $IC_{\bar{m}}[-d]$ verifying $[AX1]_k$ with perversity $\bar{m}$ for $k = 2, ..., d$.

degree		0	1	2	3	4
	8	c	c	c	c	c
	7			c	c	c
	6				c	c
	5					c
	4					
	3					$\times$
	2				$\times$	$\times$
	1			$\times$	$\times$	$\times$
	0	$\times$	$\times$	$\times$	$\times$	$\times$

complex codimension
of the strata

The symbol c means the complex can have local cosupport at that place, while the symbol $\times$ means the complex can have local support at that place.

The following tables illustrate the conditions of support and cosupport for an intersection space complex $IS_{\bar{m}}$ with perversity $\bar{m}$.

degree	0	1	2	3	4
8		×	×	×	×
7		×	×	×	×
6		×	×	×	×
5		×	×	×	×
4		×	×	×	×*
3		×	×	×*	
2		×	×*		
1		×*			
0	×				

complex codimension
of the strata

degree	0	1	2	3	4
8	c				
7		c^*			
6		c	c^*		
5		c	c	c^*	
4		c	c	c	c^*
3		c	c	c	c
2		c	c	c	c
1		c	c	c	c
0		c	c	c	c

complex codimension
of the strata

The symbol c means the complex can have local cosupport at that place, while the symbol × means the complex can have local support at that place. Moreover, the symbol ∗ means the support and the cosupport must verify a special condition given by $(d2'_k)$.

Note that in U_2, $(IS_{\bar{m}})_{|U_2} \cong (IC_{\bar{m}}[-d])_{|U_2} \cong \mathbb{Q}_{U_2}$. However, in X_{d-2}, the place at which $IS_{\bar{m}}$ can have support is exactly the place at which $IC_{\bar{m}}[-d]$ cannot have support and the place at which $IS_{\bar{m}}$ can have cosupport is exactly the place at which $IC_{\bar{m}}[-d]$ cannot have cosupport.

7　A derived category approach to intersection space complexes

In this section, we study necessary and sufficient conditions for the existence of an intersection space complex of X with a perversity $\bar{p}$. Unlike intersection cohomology sheaves, intersection space complexes are not unique. We study the space parametrizing the different choices of intersection space complexes for a fixed perversity.

7.1　Homological algebra review

We need the following lemma, that should be well known, but we give a proof for convenience of the reader.

Lemma 7.1. *Let*

$$A^\bullet \xrightarrow{\ f\ } B^\bullet \xrightarrow{\ g\ } C^\bullet$$
$$\xleftarrow[\ [1]\ ]{\phi}$$

be a distinguished triangle in the category $D^b_{cc}(X)$.
The following four conditions are equivalent:

  1. *f admits a retract*

  2. *g admits a section*

3. *There is an isomorphism in the derived category $\gamma : B^\bullet \cong A^\bullet \oplus C^\bullet$ such that $f = \gamma^{-1} \circ i_A$ and $g = p_C \circ \gamma$ where $i_A : A^\bullet \to A^\bullet \oplus C^\bullet$ is the natural inclusion and $p_C : A^\bullet \oplus C^\bullet \to C^\bullet$ is the natural projection.*

4. *ϕ is the morphism 0.*

Proof. First we prove that (1) implies (3). Let r be a retract of f. Then, $r \circ f = Id_{A^\bullet}$. So, the morphism induced by f between the cohomology sheaves $\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$ is injective for every $i \in \mathbb{Z}$. Hence, using the long exact sequence of cohomology

$$... \to \mathcal{H}^{i-1}(C^\bullet) \xrightarrow{\mathcal{H}^{i-1}(\phi)} \mathcal{H}^i(A^\bullet) \xrightarrow{\mathcal{H}^i(f)} \mathcal{H}^i(B^\bullet) \xrightarrow{\mathcal{H}^i(g)} \mathcal{H}^i(C^\bullet) \xrightarrow{\mathcal{H}^i(\phi)} \mathcal{H}^{i+1}(A^\bullet) \to ...,$$

we deduce $\mathcal{H}^i(\phi) = 0$ for every $i \in \mathbb{Z}$. Consequently, we have short exact sequences

$$0 \to \mathcal{H}^i(A^\bullet) \xrightarrow{\mathcal{H}^i(f)} \mathcal{H}^i(B^\bullet) \xrightarrow{\mathcal{H}^i(g)} \mathcal{H}^i(C^\bullet) \to 0 \qquad (10)$$

Now, let us consider the morphism

$$\gamma = \begin{pmatrix} r \\ g \end{pmatrix} : B^\bullet \to A^\bullet \oplus C^\bullet$$

The morphism induced by γ between the cohomology sheaves is

$$\mathcal{H}^i(\gamma) = \begin{pmatrix} \mathcal{H}^i(r) \\ \mathcal{H}^i(g) \end{pmatrix} : \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(A^\bullet) \oplus \mathcal{H}^i(C^\bullet),$$

which is an isomorphism, since $\mathcal{H}^i(r)$ is a retract of $\mathcal{H}^i(f)$ and (10) is exact. Moreover, since $f \circ g = 0$, we have $\gamma \circ f = i_A$ and it is clear that $p_C \circ \gamma = g$. So, we have proven (1) implies (3).

Now, we prove (2) implies (3). Let s be a section of g and let us consider the morphism

$$\gamma' = (f, s) : A^\bullet \oplus C^\bullet \to B^\bullet$$

In the same way as in the previous implication, we can show that γ' is a quasi-isomorphism, and that we have $f = \gamma' \circ i_A$ and $g \circ \gamma' = p_C$. So, $\gamma = (\gamma')^{-1}$ is the isomorphism which appears in condition (3).

Moreover, if condition (3) is true, $p_A : A^\bullet \oplus C^\bullet \to A^\bullet$ denotes the natural projection and $i_C : C^\bullet \to A^\bullet \oplus C^\bullet$ denotes the natural inclusion, then $p_A \circ \gamma$ is a retract of f and $\gamma^{-1} \circ i_C$ is a section of g. So, (3) implies (1) and (2). Now, it is enough to prove (3) $\Leftrightarrow$ (4). (3) implies that we have the following

isomorphism between distinguished triangles:

$$
\begin{array}{ccccc}
& & \xrightarrow{\ [1]\ } & & \\
& & \xrightarrow{\ \phi\ } & & \\
A^\bullet & \xrightarrow{\ f\ } & B^\bullet & \xrightarrow{\ g\ } & C^\bullet \\
\downarrow{\scriptstyle Id_{A^\bullet}} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Id_{C^\bullet}} \\
A^\bullet & \xrightarrow{\ i_A\ } & A^\bullet \oplus C^\bullet & \xrightarrow{\ p_C\ } & C^\bullet \\
& & \xrightarrow{\ 0\ } & & \\
& & \xrightarrow{\ [1]\ } & &
\end{array}
$$

So, ϕ is the morphism 0.

Now, suppose $\phi = 0$ and let us prove condition (3). By properties of the triangulated categories, we know

$$
C^\bullet[-1] \xrightarrow{\ -\phi[-1]\ } A^\bullet \xrightarrow{\ f\ } B^\bullet \xrightarrow{\ g\ }{[1]}
$$

is a distinguished triangle. So, if $\phi = 0$, there is an isomorphism $\gamma : B^\bullet \cong A^\bullet \oplus C^\bullet$ which completes the following isomorphism of triangles

$$
\begin{array}{ccccc}
& & \xrightarrow{\ [1]\ } & & \\
& & \xrightarrow{\ g\ } & & \\
C^\bullet[-1] & \xrightarrow{\ 0\ } & A^\bullet & \xrightarrow{\ f\ } & B^\bullet \\
\downarrow{\scriptstyle Id_{C^\bullet}[-1]} & & \downarrow{\scriptstyle Id_{A^\bullet}} & & \downarrow{\scriptstyle \gamma} \\
C^\bullet[-1] & \xrightarrow{\ 0\ } & A^\bullet & \xrightarrow{\ i_A\ } & A^\bullet \oplus C^\bullet \\
& & \xrightarrow{\ p_C\ } & & \\
& & \xrightarrow{\ [1]\ } & &
\end{array}
$$

γ is the isomorphism which appears in condition (3). $\qquad\square$

DEFINITION 7.2. *The triangle*

$$
A^\bullet \xrightarrow{\ f\ } B^\bullet \xrightarrow{\ g\ } C^\bullet \xrightarrow[\ \ [1]\ \ ]{\ \phi\ }
$$

is said to be split if it verifies the conditions of Lemma 7.1.

7.2 Characterization of existence and study of uniqueness

Now, we can state the main theorem in this section.

THEOREM 7.3. *The following holds:*

1. *[Goresky-MacPherson] There exists one object in $D^b_{cc}(X)$ verifying $[AX1]_k$ for $k = 2, ..., d$. This object is unique up to isomorphism.*

2. *Suppose there exists an intersection space complex in U_r, IS_{r-1}, that is, IS_{r-1} is an object in $D^b_{cc}(U_r)$ and it verifies $[AXS1]_k$ for $k = 2, ..., r-1$. Then, there exists an intersection space complex in U_{r+1}, IS_r, such that $(IS_r)_{|U_r} \cong IS_{r-1}$ if and only if the distinguished triangle:*

$$\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{f} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{[1]} \quad (11)$$

is split. Moreover, there is a bijection

$$\left\{ \begin{array}{c} \text{intersection space complexes} \\ IS_r \in D^b_{cc}(U_{r+1}) \text{ such} \\ \text{that } (IS_r)_{|U_r} \cong IS_{r-1} \end{array} \right\} / \{ \text{isomorphism} \} \longleftrightarrow \{ \text{retracts of } f \} / \sim$$

$$(12)$$

where $\sim$ is the equivalence relation such that $\lambda_1 \sim \lambda_2$ if and only if there exist isomorphisms $\alpha : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$ and $\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$ such that $\lambda_2 = \alpha \circ \lambda_1 \circ j_{r*} j_r^* \beta$.

Proof. 1. is proved in [15, section 3], but we give a proof adapted to our needs. There exists a unique object (up to isomorphism), $\mathbb{Q}_{U_2}$, in $D^b_{cc}(U_2)$ verifying (a) and (b) of $[AX1]_k$. Suppose there exists a unique object (up to isomorphism), $IC_{\bar{p} r-1}[-d]$, in $D^b_{cc}(U_r)$ verifying $[AX1]_k$ for $k = 2, ..., r-1$, and consider the following composition of natural morphisms:

$$i_{r*} IC_{\bar{p} r-1}[-d] \xrightarrow{\hspace{1.5cm} \phi_r \hspace{1.5cm}} j_{r*} j_r^* i_{r*} IC_{\bar{p} r-1}[-d] \longrightarrow \tau_{>\bar{p}(r)} j_{r*} j_r^* i_{r*} IC_{\bar{p} r-1}[-d]$$

Let us define $IC_{\bar{p} r}[-d] := cone(\phi_r)[-1]$. Then, there is a distinguished triangle:

$$IC_{\bar{p} r}[-d] \to i_{r*} IC_{\bar{p} r-1}[-d] \xrightarrow{\phi_r} \tau_{>\bar{p}(r)} j_{r*} j_r^* i_{r*} IC_{\bar{p} r-1}[-d] \xrightarrow{[1]} \quad (13)$$

Using the long exact sequence of cohomology associated to this triangle we can prove that $IC_r[-d]$ verifies $[AX1]_k$ for $k = 2, ..., r$.

Now, suppose there exists another object $B^\bullet$ in $D^b_{cc}(U_{r+1})$ verifying $[AX1]_k$ for $k = 2, ..., r$. Then, $B^\bullet_{|U_r}$ verifies $[AX1]_k$ for $k = 2, ..., r-1$. So, there exists an isomorphism $B^\bullet_{|U_r} \cong IC_{\bar{p} r-1}[-d]$.

Let $\varphi : B^\bullet \to i_{r*} IC_{\bar{p} r-1}[-d]$ be the composition of the canonical morphism $B^\bullet \to i_{r*} B^\bullet_{U_r}$ and an isomorphism $i_{r*} B^\bullet_{|U_r} \cong i_{r*} IC_{\bar{p} r-1}[-d]$ and let $C^\bullet := cone(\varphi)$. Since $i^* \varphi$ is an isomorphism, we have the isomorphism $i_r^* C^\bullet \cong 0$ in the derived category.

Then, the distinguished triangle

$$i_{r!} i_r^* C^\bullet \to C^\bullet \to j_{r*} j_r^* C^\bullet \xrightarrow{[1]}$$

implies there exists an isomorphism $C^\bullet \cong j_{r*}j_r^* C^\bullet$.
So, we have a distinguished triangle

$$B^\bullet \xrightarrow{\varphi} j_{r*}j_r^* i_{r*} IC_{\bar{p}\,r-1}[-n] \xrightarrow{\psi} C^\bullet \cong j_{r*}j_r^* C^\bullet$$

where ψ is the morphism following ϕ. Applying the functor $j_{r*}j_r^*$ we obtain the triangle

$$j_{r*}j_r^* B^\bullet \xrightarrow{j_{r*}j_r^*\varphi} j_{r*}j_r^* i_{r*} IC_{\bar{p}\,r-1}[-n] \xrightarrow{j_{r*}j_r^*\psi} j_{r*}j_r^* C^\bullet$$

Using the long exact sequence of cohomology associated to this triangle we prove $\mathcal{H}^i(j_{r*}j_r^* C^\bullet) = 0$ if $i \leq \bar{p}(r)$ and $\mathcal{H}^i(j_{r*}j_r^*\psi) : \mathcal{H}^i(j_{r*}j_r^* i_{r*} IC_{\bar{p}\,r-1}[-d]) \to \mathcal{H}^i(j_{r*}j_r^* C^\bullet)$ is an isomorphism if $i > \bar{p}(r)$. Then, we obtain isomorphisms

$$C^\bullet \cong \tau_{>\bar{p}(r)} j_{r*}j_r^* i_{r*} IC_{\bar{p}\,r-1}[-d] \qquad \text{and} \qquad B^\bullet \cong IC_{\bar{p}\,r}[-d].$$

Repeating this process finitely we obtain $IC_{\bar{p}}[-d] \in D_{cc}^b(X)$ verifying $[AX1]_k$ for $k = 2, ..., d$ and it is unique up to isomorphism.
Now, we prove 2. Let IS_{r-1} be an intersection space complex in U_r. We have to prove that there is a bijective map

$$\left\{ \begin{array}{c} \text{intersection space complexes} \\ IS_r \in D_{cc}^b(U_{r+1}) \text{ such} \\ \text{that } (IS_r)_{|U_r} \cong IS_{r-1} \end{array} \right\} / \{\text{isomorphism}\} \longleftrightarrow \{\text{retracts of } f\} / \sim$$

Let λ be a retract of f and consider the following composition of morphisms:

$$i_{r*} IS_{r-1} \xrightarrow{a} j_{r*}j_r^* i_{r*} IS_{r-1} \xrightarrow{\lambda} \tau_{\leq \bar{q}(r)} j_{r*}j_r^* i_{r*} IS_{r-1}$$

with φ_λ the arc over the composition.

where a is the canonical morphism.
Let us define $IS_r := cone(\varphi_\lambda)[-1]$. Then, there is a distinguished triangle:

$$IS_r \to i_{r*} IS_{r-1} \xrightarrow{\varphi_\lambda} \tau_{\leq \bar{q}(r)} j_{r*}j_r^* i_{r*} IS_{r-1} \xrightarrow{[1]} \qquad (14)$$

Since $(j_{r*}j_r^* i_{r*} IS_{r-1})_{|U_r}$ equals 0, $(IS_r)_{|U_r}$ is isomorphic to IS_r and, using the long exact sequence associated to the triangle, one proves that IS_r verifies $[AXS1]_k$ for $k = 2, ..., r$.
Let λ' be a different retract of f such that $\lambda \sim \lambda'$. We have to prove $cone(\varphi_\lambda) \cong cone(\varphi_{\lambda'})$.
Let $\alpha : \tau_{\leq \bar{q}(r)} j_{r*}j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*}j_r^* i_{r*} IS_{r-1}$ and $\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$ be isomorphisms such that $\lambda' = \alpha \circ \lambda \circ j_{r*}j_r^* \beta$.
Since we know the equalities $\varphi_\lambda = \lambda \circ a$ and $\varphi_{\lambda'} = \lambda' \circ a$, we have to prove the isomorphism $cone(\lambda \circ a) \cong cone(\alpha \circ \lambda \circ j_{r*}j_r^* \beta \circ a)$. Moreover, since α is an isomorphism, $cone(\alpha \circ \lambda \circ j_{r*}j_r^* \beta \circ a)$ is isomorphic to $cone(\lambda \circ j_{r*}j_r^* \beta \circ a)$.

Now, consider the following diagrams associated to the octahedral axiom of distinguished triangles.

$$i_{r*}IS_{r-1} \xrightarrow{\quad a \quad} j_{r*}j_r^*i_{r*}IS_{r-1} \xrightarrow{\quad \lambda \quad} \tau_{\le \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \qquad (15)$$

with upper arc labeled $\lambda \circ a$, and interior vertices $cone(a)$, $cone(\lambda)$, $cone(\lambda \circ a)$ connected by $[1]$-morphisms.

$$i_{r*}IS_{r-1} \xrightarrow{\quad a \quad} j_{r*}j_r^*i_{r*}IS_{r-1} \xrightarrow{\lambda \circ j_{r*}j_r^*\beta} \tau_{\le \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1}$$

with upper arc labeled $\lambda \circ j_{r*}j_r^*\beta \circ a$, and interior vertices $cone(a)$, $cone(\lambda \circ j_{r*}j_r^*\beta)$, $cone(\lambda \circ j_{r*}j_r^*\beta \circ a)$ connected by $[1]$-morphisms.

$$(16)$$

Let $\phi : cone(\lambda \circ j_{r*}j_r^*\beta) \to cone(\lambda)$ be an isomorphism completing the following isomorphism between triangles

$$
\begin{array}{ccccc}
j_{r*}j_r^*i_{r*}IS_{r-1} & \xrightarrow{\lambda \circ j_{r*}j_r^*\beta} & \tau_{\le \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} & \longrightarrow & cone(\lambda \circ j_{r*}j_r^*\beta) \\
\downarrow{\scriptstyle j_{r*}j_r^*\beta} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \phi} \\
j_{r*}j_r^*i_{r*}IS_{r-1} & \xrightarrow{\quad \lambda \quad} & \tau_{\le \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} & \longrightarrow & cone(\lambda)
\end{array}
$$

Then, if $\rho : cone(a) \to cone(a)$ is an isomorphism completing the triangles isomorphism

$$
\begin{array}{ccccc}
i_{r*}IS_{r-1} & \xrightarrow{\quad a \quad} & j_{r*}j_r^*i_{r*}IS_{r-1} & \longrightarrow & cone(a) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle j_{r*}j_r^*\beta} & & \downarrow{\scriptstyle \rho} \\
i_{r*}IS_{r-1} & \xrightarrow{\quad a \quad} & j_{r*}j_r^*i_{r*}IS_{r-1} & \longrightarrow & cone(a)
\end{array}
$$

the diagram

$$
\begin{array}{ccc}
cone(\lambda \circ j_{r*}j_r^*\beta) & \xrightarrow{\;[1]\;} & cone(a) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \rho} \\
cone(\lambda) & \xrightarrow{\;[1]\;} & cone(a)
\end{array}
$$

is commutative.

Therefore, there exists an isomorphism $cone(\lambda \circ j_{r*}j_r^*\beta \circ a) \cong cone(\lambda \circ a)$, completing the following morphism between triangles

$$
\begin{array}{ccccc}
cone(\lambda \circ j_{r*}j_r^*\beta) & \xrightarrow{\;[1]\;} & cone(a) & \longrightarrow & cone(\lambda \circ j_{r*}j_r^*\beta \circ a) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \rho} & & \downarrow \\
one(\lambda) & \xrightarrow[\;[1]\;]{} & cone(a) & \longrightarrow & cone(\lambda \circ a)
\end{array}
$$

Now, suppose there exists an intersection space complex in U_{r+1}, IS_r such that $(IS_r)_{|U_r} \cong IS_{r-1}$. We have to prove that the triangle (11) is split.

Let $h : IS_r \to i_{r*}IS_{r-1}$ be the composition of the canonical morphism $IS_r \to i_{r*}(IS_r)_{|U_r}$ and an isomorphism $i_{r*}(IS_r)_{|U_r} \cong i_{r*}IS_{r-1}$ and let $C^\bullet := cone(h)$. Then, there is a distinguished triangle:

$$
IS_r \xrightarrow{h} i_{r*}IS_{r-1} \xrightarrow{g} C^\bullet \xrightarrow{[1]}
$$

Note that $i_r^*h : i_r^*IS_r \to IS_{r-1}$ is an isomorphism. So, $i_r^*C^\bullet$ is isomorphic to 0 in the derived category. Then, the canonical triangle

$$
i_{r!}i_r^*C^\bullet \to C^\bullet \to j_{r*}j_r^*C^\bullet \xrightarrow{[1]}
$$

implies the isomorphism $C^\bullet \cong j_{r*}j_r^*C^\bullet$.

Moreover, the long exact sequence of cohomology associated to the triangle

$$
j_{r*}j_r^*IS_r \xrightarrow{j_{r*}j_r^*h} j_{r*}j_r^*i_{r*}IS_{r-1} \xrightarrow{j_{r*}j_r^*g} j_{r*}j_r^*C^\bullet \xrightarrow{[1]}
$$

implies that

$$
\mathcal{H}^i(j_{r*}j_r^*C^\bullet) \cong \begin{cases} 0 & \text{if } i > \bar{q}(r) \\ \mathcal{H}^i(j_{r*}j_r^*i_{r*}IS_{r-1}) & \text{if } i \leq \bar{q}(r) \end{cases}
$$

Applying the functor $\tau_{\leq \bar{q}(r)}$ to $j_{r*}j_r^*g$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} & \xrightarrow{\;a\;} & \tau_{\leq \bar{q}(r)}j_{r*}j_r^*C^\bullet \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle c} \\
j_{r*}j_r^*i_{r*}IS_{r-1} & \xrightarrow{\;b\;} & j_{r*}j_r^*C^\bullet
\end{array} \tag{17}
$$

where f and c are the canonical morphisms, $a = \tau_{\leq \bar{q}(r)}j_{r*}j_r^*g$ and $b = j_{r*}j_r^*g$. Moreover, a and c are isomorphisms and the composition $\lambda = a^{-1} \circ c^{-1} \circ b$ is a retract of f. So, the triangle (11) is split by Lemma 7.1.

Let $IS_r' \in D_{cc}^b(U_{r+1})$ be isomorphic to IS_r. Then, we have an isomorphism $(IS_r')_{|U_r} \cong IS_{r-1}$ and IS_r' is an intersection space complex in U_{r+1}.

Let $h' : IS'_r \to i_{r*}IS_{r-1}$ be the composition of the canonical restriction morphism and an isomorphism $i_{r*}(IS'_r)_{|U_r} \cong i_{r*}IS_{r-1}$, let $K^\bullet := cone(h')$ and consider the triangle

$$IS'_r \xrightarrow{h'} i_{r*}IS_r \xrightarrow{g'} K^\bullet \xrightarrow{[1]}$$

Applying the functor $\tau_{\leq \bar{q}(r)}$ to $j_{r*}j_r^* g'$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\tau_{\leq \bar{q}(r)}j_{r*}j_r^* i_{r*}IS_{r-1} & \xrightarrow{a'} & \tau_{\leq \bar{q}(r)}j_{r*}j_r^* K^\bullet \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle c'} \\
j_{r*}j_r^* i_{r*}IS_{r-1} & \xrightarrow{b'} & j_{r*}j_r^* K^\bullet
\end{array}
\tag{18}
$$

where h and c' are the canonical morphisms, $a' = \tau_{\leq \bar{q}(r)}j_{r*}j_r^* g'$ and $b' = j_{r*}j_r^* g'$. Moreover, a' and c' are isomorphisms and $\lambda' := a'^{-1} \circ c'^{-1} \circ b'$ is a retract of f. Let $\alpha : IS_r \to IS'_r$ be an isomorphism; let h be the composition of the canonical morphism $IS_r \to i_{r*}(IS_r)_{|U_r}$ and an isomorphism $\gamma : i_{r*}(IS_r)_{|U_r} \to i_{r*}IS_{r-1}$; let h' be the composition of the canonical morphism $IS'_r \to i_{r*}(IS'_r)_{|U_k}$ and an isomorphism $\gamma' : i_{r*}(IS'_r)_{|U_k} \to i_{r*}IS'_{r-1}$. Finally define $\beta := \gamma' \circ i_{r*}i_r^* \alpha \circ \gamma^{-1}$. Then, there is an isomorphism between triangles:

$$
\begin{array}{ccccc}
j_{r*}j_r^* IS_r & \xrightarrow{j_{r*}j_r^* h} & j_{r*}j_r^* i_{r*}IS_{r-1} & \xrightarrow{j_{r*}j_r^* g} & j_{r*}j_r^* C^\bullet \\
\downarrow{\scriptstyle j_{r*}j_r^* \alpha} & & \downarrow{\scriptstyle j_{r*}j_r^* \beta} & & \downarrow{\scriptstyle \delta} \\
j_{r*}j_r^* IS'_r & \xrightarrow{j_{r*}j_r^* h'} & j_{r*}j_r^* i_{r*}IS_{r-1} & \xrightarrow{j_{r*}j_r^* g'} & j_{r*}j_r^* K^\bullet
\end{array}
$$

where α, β and δ are isomorphisms.

Then, the morphisms of diagrams (17) and (18) have the following relations:

$$a' = \tau_{\leq \bar{q}(r)}\delta \circ a \circ (\tau_{\leq \bar{q}(r)}j_{r*}j_r^* \beta)^{-1}$$

$$b' = \delta \circ a \circ (j_{r*}j_r^* \beta)^{-1}$$

$$c' = \delta \circ c \circ (\tau_{\leq \bar{q}(r)}\delta)^{-1}$$

and, we obtain $\lambda = (\tau_{\leq \bar{q}(r)}j_{r*}j_r^* \beta)^{-1} \circ \lambda' \circ j_{r*}j_r^* \beta$. So, λ is equivalent to λ' by the equivalence relation $\sim$.

Summarizing, up to now we have defined an injective mappings α from left hand side to the right hand side of (12), and β in the opposite direction. In order to conclude it is enough to show that $\alpha \circ \beta$ equals the identity. This is a (lengthy) straightforward check which consists in running consecutively the constructions of both mappings given above. $\qquad \square$

REMARK 7.4. *There is a unique object in $D_{cc}^b(U_2)$ up to isomorphism verifying (a) and (b) from $[AXS1]_k$, $\mathbb{Q}_{U_2}$.*

7.3   The space of obstructions to existence and uniqueness

Now we are going to study the equivalence relation $\sim$, which appears in Theorem 7.3.

Remember that two retracts λ_1 and λ_2 of $\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{f} j_{r*} j_r^* i_{r*} IS_{r-1}$ are equivalent by $\sim$ if and only if there exist isomorphisms

$$\alpha : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$$

$$\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$$

such that $\lambda_2 = \alpha \circ \lambda_1 \circ j_{r*} j_r^* \beta$.

Let $\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$ be an isomorphism and let λ be a retract of f. Then

$$\tau_{\leq \bar{q}(r)} j_{r*} j_r^* \beta^{-1} : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$$

is an isomorphism and $\tau_{\leq \bar{q}(r)} j_{r*} j_r^* \beta^{-1} \circ \lambda \circ j_{r*} j_r^* \beta$ is a retract of f. Moreover, given another isomorphism $\alpha : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$, the composition $\alpha \circ \lambda \circ j_{r*} j_r^* \beta$ is a retract of f if and only if $\alpha \circ \lambda \circ j_{r*} j_r^* \beta \circ f = Id$. This happens if and only if

$$(\alpha - \tau_{\leq \bar{q}(r)} j_{r*} j_r^* \beta^{-1}) \circ \lambda \circ j_{r*} j_r^* \beta \circ f = 0.$$

Since $\lambda \circ j_{r*} j_r^* \beta \circ f$ is an isomorphism in the derived category we conclude that $\alpha = \tau_{\leq \bar{q}(r)} j_{r*} j_r^* \beta^{-1}$. So α is determined by β.

In particular, if r is the dimension of the largest non-trivial stratum, we have $U_r = U_2$. Then $Aut(i_{r*} IS_{r-1})$ is isomorphic to $Aut(i_{r*} \mathbb{Q}_{U_2})$. By adjunction $Hom(i_{r*} \mathbb{Q}_{U_2}, i_{r*} \mathbb{Q}_{U_2})$ is isomorphic to $Hom(i_r^* i_{r*} \mathbb{Q}_{U_2}, \mathbb{Q}_{U_2}) = Hom(\mathbb{Q}_{U_2}, \mathbb{Q}_{U_2}) \cong \mathbb{Q}$. Then the automorphism group $Aut(i_{r*} \mathbb{Q}_{U_2})$ can be identified with $\mathbb{Q}^*$ acting by multiplication, that is the group of homothetic transformations.

Moreover, if $\beta \in Aut(i_{r*} IS_{r-1})$ is a homothetic transformation then $\alpha = (\lambda \circ j_{r*} j_r^* \beta \circ f)^{-1}$ is the inverse homothetic transformation. So, if r is the dimension of the largest non-trivial stratum, the equivalence relation $\sim$ is trivial. Consequently, there is a bijective map

$$\left\{ \begin{array}{c} \text{intersection space complexes} \\ IS_r \in D_{cc}^b(U_{r+1}) \end{array} \right\} / \{\text{isomorphism}\} \longleftrightarrow \{\text{retracts of } f\}$$

REMARK 7.5. *For any r, the triangle (11) induces a exact sequence*

$$... \to [\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}[-1]] \to$$

$$\to [\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}] \xrightarrow{\tilde{g}} [j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}] \xrightarrow{\tilde{f}}$$

$$\xrightarrow{\tilde{f}} [\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}] \to ...$$

Moreover, the retracts of f are the elements λ contained in $[j_{r*}j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}]$ such that $\tilde{f}(\lambda)$ is the identity. So, the space $\{$retracts of $f\}$ is modulated by the vector space

$$[\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}]$$

The following is an immediate consequence of the previous construction.

COROLLARY 7.6. *Suppose there exists an intersection space complex IS_{r-1} in U_r, that is, IS_{r-1} is an object in $D_{cc}^b(U_r)$ and it verifies $[AXS1]_k$ for $k = 2, ..., r-1$.*

- *The obstruction to existence of intersection space in the next stratum lives in*

$$Ext^1(\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}) =$$
$$= [\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}[1]].$$

- *The space parametrizing isomorphism classes of intersection space complexes on U_{r+1} that extend a given IS_{r-1} on U_r is a quotient of the space*

$$Hom(\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}) =$$
$$= [\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}],$$

by the equivalence relation described above. The equivalence relation is trivial for the first stratum of positive codimension.

8 A MIXED HODGE MODULE STRUCTURE IN INTERSECTION SPACE COMPLEXES OF ALGEBRAIC VARIETIES

THEOREM 8.1. *Let X be a complex algebraic variety. Consider a stratification*

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

by algebraic subvarieties, which makes X a topological pseudomanifold. An intersection space complex on X associated with the stratification above admits a lifting to the derived category $D^b MHM(X)$ of mixed Hodge modules on X if and only if the choices of the retractions can be chosen as morphisms of mixed Hodge modules. In that case, its hypercohomology groups carry a rational polarizable mixed Hodge structure.

Proof. The proof follows the inductive construction of the intersection space complex considered in the proof of Theorem 7.3. To start with, we notice that there is a mixed Hodge module $\mathbb{Q}_{U_2}^H$ on U_2 such that $rat(\mathbb{Q}_{U_2}^H) = \mathbb{Q}_{U_2}$. In order to construct IS_r as an element in the derived category of mixed Hodge modules we proceed by induction. Assume, by induction, that there is an element IS_{r-1}^H in the derived category of mixed Hodge modules in U_r, which is transformed to an intersection space complex IS_{r-1} by rat.

In order to construct IS_r^H we proceed as follows. Consider the triangle

$$\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H \to j_{r*} j_r^* i_{r*} IS_{r-1}^H \to \tau_{> \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H \xrightarrow{[1]}$$

in the derived category of mixed Hodge modules in U_{r+1} with respect to Saito's anomalous t-structure (see [21] Remark 4.6 (2)). Applying the functor rat we obtain the distinguished triangle

$$\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{> \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{[1]} .$$

By Lemma 7.1, the triangle splits if and only if the connecting morphism in the first triangle equals. If this happens we consider a retraction $\lambda : j_{r*} j_r^* i_{r*} IS_{r-1}^H \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H$, which is a morphism in $D^b MHM(U_{r+1})$. Define IS_r^H to be the shifted cone $cone(\varphi_\lambda)[-1]$, where φ_λ is the composition of λ with the canonical morphism $a : i_{r*} IS_{r-1}^H \to j_{r*} j_r^* i_{r*} IS_{r-1}^H$ we have that IS_r^H belongs to $D^b MHM(U_{r+1})$, and its image under rat coincides with the intersection space complex IS_r obtained from the retraction $rat(\lambda)$.   □

The obstructions to existence and uniqueness can be lifted to mixed Hodge modules:

COROLLARY 8.2. *Let X be a complex algebraic variety. Consider an stratification*

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

by algebraic subvarieties, which makes X a topological pseudomanifold. Suppose there exists an intersection space complex IS_{r-1}^H in U_r which belongs to $D^b(MHM(U_{r+1}))$.

- *The obstruction to existence of intersection space in the category of Mixed Hodge modules in the next stratum is an element of*

$$Ext^1_{D^b MHM(U_{r+1})}(\tau_{> \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H).$$

- *If the obstruction vanishes, the different isomorphism classes of intersection space complexes in the category of mixed Hodge modules on U_{r+1} that extend a given IS_{r-1}^H on U_r are in bijection with the quotient of the space*

$$Hom_{D^b MHM(U_{r+1})}(\tau_{> \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}^H),$$

  modulo the equivalence relation similar to the one appearing in Corollary 7.6.

A simplification of the proof of Theorem 8.1 yields:

THEOREM 8.3. *Let X be a complex algebraic variety. Let $\bar{p}$ be any perversity. The intersection cohomology complex associated to it belongs to the derived category of mixed Hodge modules of X. Consequently the intersection homology complexes $IH_{\bar{p}}^k(X, \mathbb{Q})$ carry a canonical polarizable mixed Hodge structure.*

9   Classes of spaces admitting intersection space complexes and counterexamples

In this section we provide some examples and counterexamples to illustrate our theory. First, we introduce two classes of varieties which admit an intersection space complex for every perversity. The first class depends on the tubular neighbourhoods of the strata: if every stratum admits a trivial tubular neighbourhood, then there exists the intersection space complex for every perversity. The second class depends on the dimension of the strata: if every singular stratum has homological dimension for locally constant sheaves bounded above by 1, then there exists the intersection space complex for every perversity. Then, we find concrete examples of pseudomanifolds (including an algebraic variety) not admitting intersection space complexes and, hence, not admitting intersection space pairs.

9.1   Pseudomanifolds satisfying triviality properties

Let X be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset.$$

For $k = 2, ..., d+1$, let $U_k := X \setminus X_{d-k}$. We will also denote by $i_k : U_k \to U_{k+1}$, $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ and $i_{k_1,k_2} : U_{k_1} \to U_{k_2}$ the canonical inclusions. If X has a trivial conical structure then, by Theorem 3.31, there exists an intersection space pair. Hence, by Theorem 5.18, X has an intersection space complex. We have shown:

Corollary 9.1. *If X has a trivial conical structure, then it has an intersection space complex.*

Example 9.2. *Toric varieties admit an intersection space pair and intersection space complex for every perversity.*

However, as we prove below, in order to ensure the existence of the intersection space complex we can relax the triviality hypothesis on the stratification: one only needs that the triviality property (T_r) (see 3.10) is satisfied for any stratum. Having a trivial conical structure requires further compatibilities between the trivializations predicted by properties (T_r) (see Definition 3.12).

Definition 9.3. *A complex of sheaves is* formal and constant *if it is quasi-isomorphic to the direct sum of its cohomology sheaves with zero differentials and these cohomology sheaves are constant.*

A constructible complex of sheaves is formal and constant, for example, when the stratum is contractible (see [18] p. 410, Lemma 2.4.).

Definition 9.4. *If $B_{k-1}^\bullet$ is a complex of sheaves in U_k, we say $B_{k-1}^\bullet$ verifies the property (P_r), where $r \geq k$, if $j_r^* i_{k,r+1*} B_{k-1}^\bullet$ is formal and constant.*

Remark 9.5. *If $B^\bullet$ is a formal and constant complex of sheaves in U_k and (X, X_{d-2}) has a conical structure which verifies the property (T_r) of Definition 3.10 for some $r \geq k$, then $B^\bullet$ verifies the property (P_r).*

Proposition 9.6. *Given $k \in \{2, ..., d\}$, if there exists an intersection space complex IS_{k-1} with perversity $\bar{p}$ in U_k, which verifies (P_k), then there exists an intersection space complex IS_k with perversity $\bar{p}$ in U_{k+1}, such that $(IS_k)_{|U_k}$ is quasi-isomorphic to IS_{k-1}.*

Proof. $j_k^* i_{k*} IS_{k-1}$ is (up to isomorphism in the derived category) a complex of constant sheaves with zero differentials. So, the triangle

$$\tau_{\leq q} j_{k*} j_k^* i_{k*} IS_{k-1} \to j_{k*} j_k^* i_{k*} IS_{k-1} \to \tau_{>q} j_{k*} j_k^* i_{k*} IS_{k-1} \xrightarrow{[1]}$$

is split for every $q \in \mathbb{Z}$ and, applying Theorem 7.3, we conclude. $\square$

Lemma 9.7. *Let us suppose that there exists an intersection space complex IS_{k-1} with perversity $\bar{p}$ in U_k. If $(IS_{k-1})_{|U_{k-1}}$ verifies the properties (P_{k-1}) and (P_r) for a certain $r \geq k$ and (X, X_{d-2}) has a conical structure which verifies the property (T_r) of Definition 3.10, then IS_{k-1} verifies the property (P_r).*

Proof. By Theorem 7.3, $j_r^* i_{k,r+1*} IS_{k-1}[1]$ is quasi-isomorphic to the cone of a morphism

$$j_r^* i_{k-1,r+1*}(IS_{k-1})_{|U_{k-1}} \to j_r^* i_{k,r+1*} \tau_{\leq \bar{q}(k-1)} j_{k-1*} j_{k-1}^* i_{k-1*}(IS_{k-1})_{|U_{k-1}}$$

Since $(IS_{k-1})_{|U_{k-1}}$ verifies the properties (P_{k-1}) and (P_r), the complexes $j_r^* i_{k-1,r+1*}(IS_{k-1})_{|U_{k-1}}$ and $\tau_{\leq \bar{q}(k-1)} j_{k-1}^* i_{k-1*}(IS_{k-1})_{|U_{k-1}}$ are formal and constant. Then, using that (X, X_{d-2}) has a conical structure verifying the property (T_r), we deduce that the constructible complex $j_r^* i_{k,r+1*} \tau_{\leq \bar{q}(k-1)} j_{k-1*} j_{k-1}^* i_{k-1*}(IS_{k-1})_{|U_{k-1}}$ is also formal and constant. So, $j_r^* i_{k,r+1*} IS_{k-1}$ is formal and constant as well and we conclude. $\square$

Theorem 9.8. *If the pair (X, X_{d-2}) has a conical structure which verifies the property (T_r) of Definition 3.10 for any r, then there exists the intersection space complex of X for every perversity.*

Proof. The constant sheaf $\mathbb{Q}_{U_2}$ verifies (P_r) for every $r \geq 2$ such that the pair (X, X_{d-2}) has a conical structure with the property (T_r). So, if (X, X_{d-2}) has a conical structure which verifies the property (T_r) for any r, using Lemma 9.7 and Proposition 9.6, we can construct inductively for every k an intersection space complex with perversity $\bar{p}$ in U_k which verifies (P_r) for every $r \geq k$. $\square$

Corollary 9.9. *If every stratum of X is contractible, then there exists the intersection space complex of X for every perversity.*

9.2 Pseudomanifolds with strata of small homological dimension

Definition 9.10. *A space Y has homological dimension for locally constant sheaves bounded by m if any locally constant sheaf in Y has no cohomology in degree higher than m.*

Theorem 9.11. *Let X be a topological pseudomanifold with the following stratification:*
$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$
such that $X_{d-r} \backslash X_{d-r-1}$ has homological dimension for locally constant sheaves bounded above by 1 for any r.
Then, there exists the intersection space complex of X for every perversity $\bar{p}$.
Moreover, if $X_{d-r} \backslash X_{d-r-1}$ has homological dimension for locally constant sheaves bounded by 0 for any r, the intersection space complex is unique.

Proof. To prove the existence it is enough to prove that for any topological space Y which has homological dimension for locally constant sheaves bounded by 1, any cohomologically locally constant complex of sheaves $B^\bullet$ in $\mathcal{D}^b(Y)$ and any integer m, the triangle

$$\tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \xrightarrow{\phi} \tau_{\leq m}B^\bullet[1] \to ...$$

is split.
This triangle is split if and only if the morphism

$$\phi \in Ext^1(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$$

is 0. We prove the vanishing of $Ext^1(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$ by induction on the number N of integers k for which $\mathcal{H}^k(B^\bullet)$ is non-zero.
If m is greater or equal than the maximal k such that $\mathcal{H}^k(B^\bullet)$ is non-zero then $\tau_{>m}B^\bullet$ is equivalent to 0 in $\mathcal{D}^b(Y)$, and the assertion is obvious. If m is strictly smaller than the minimal k such that $\mathcal{H}^k(B^\bullet)$ is non-zero then $\tau_{\leq m}B^\bullet$ is equivalent to 0 in $\mathcal{D}^b(Y)$, and the assertion is also obvious.
If $N = 0, 1$ we are in one of the situations described above. Assume that $N > 1$ and that we are not in the situation described above. Then the number of different non-zero cohomology sheaves of $\tau_{\leq m}B^\bullet$ and $\tau_{>m}B^\bullet$ is strictly smaller than N. Applying the induction hypothesis several times we conclude that $\tau_{\leq m}B^\bullet$ and $\tau_{>m}B^\bullet$ are equivalent in the derived category to the direct sum of their cohomology sheaves. This reduces the statement to the case in which both $\tau_{\leq m}B^\bullet$ and $\tau_{>m}B^\bullet$ are equivalent in the derived category to $\mathcal{F}[-k_1]$ and and $\mathcal{G}[-k_2]$, where $\mathcal{F}[-k_1]$ is the locally constant sheaf $\mathcal{F}$ shifted to degree k_1, and $\mathcal{G}[-k_2]$ is analogous.
Assuming the reduction proved above we compute $Ext^1(\mathcal{G}[-k_2], \mathcal{F}[-k_1])$ using the local to global spectral sequence of $\mathcal{H}om^\bullet(\mathcal{G}[-k_2], \mathcal{F}[-k_1])$. Then,

$$E_2^{p,q} = \mathbb{H}^p(Y, \mathcal{E}xt^q(\mathcal{G}[-k_2], \mathcal{F}[-k_1])) = \mathbb{H}^{p+k_2-k_1+q}(Y, \mathcal{H}om(\mathcal{G}, \mathcal{F})).$$

If $p + q = 1$ then $p + k_2 - k_1 + q > 1$. So, since Y has homological dimension for locally constant sheaves bounded by 1, we have $E_2^{p,q} = 0$ if $p + q = 1$. We conclude that $Ext^1(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$ vanishes, as desired. $\square$

COROLLARY 9.12. *If the strata $X_{d-r} \setminus X_{d-r-1}$ have the homotopy type of a 1-dimensional CW-complex of dimension bounded by 1 for any r, then here exists the intersection space complex of X for every perversity $\bar{p}$.*

EXAMPLE 9.13. *Any complex algebraic variety with 1 dimensional critical set with a Whitney stratification satisfy the hypothesis of the previous corollary if no compact connected component of the critical set is equal a stratum. Hence it admits an intersection space complex.*

9.3   COUNTEREXAMPLES

Now, we illustrate the limits of our theory with a class of varieties which does not admit an intersection space complex for some perversities. With this purpose, the following proposition gives a necessary condition for the splitting of a triangle

$$\tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \xrightarrow{[1]} .$$

PROPOSITION 9.14. *Let X be a topological space, let $B^\bullet$ be a bounded complex of sheaves on X and let $E_r^{p,q}$ be the local to global spectral sequence of $B^\bullet$. Then, if the canonical triangle*

$$\tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \xrightarrow{[1]}$$

is split, the morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are 0 for every $r \geq 2$, $p \in \mathbb{Z}$ and $m < q \leq m + r - 1$.

Proof. The proof follows essentially Banagl's argument in [5, Theorem 5.2 and remarks on page 296]. Let us suppose the triangle is split and let $\lambda : B^\bullet \to \tau_{\leq m}B^\bullet$ be a retract of the canonical morphism.
Let $E_r^{p,q}$ be the local to global spectral sequence of $B^\bullet$, $E_r'^{p,q}$ the local to global spectral sequence of $\tau_{\leq m}B^\bullet$ and $\lambda_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ the morphism induced by λ.
For $r = 2$,

$$\lambda_2^{p,q} : \mathbb{H}^p(X, \mathcal{H}^q(B)) \to \mathbb{H}^p(X, \mathcal{H}^q(\tau_{\leq m}B))$$

is an isomorphism if $q \leq m$ and $\mathbb{H}^p(X, \mathcal{H}^q(\tau_{\leq m}B)) = 0$ if $q > m$.
Given $r \geq 2$, suppose $\lambda_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for every $q \leq m$ and $E_r'^{p,q} = 0$ for every $q > m$. Then, $E_{r+1}'^{p,q} = 0$ for every $q > m$.
Moreover, let us consider the commutative diagram:

$$
\begin{array}{ccc}
E_r^{p,q} & \xrightarrow{\lambda_r^{p,q}} & E_r'^{p,q} \\
\downarrow{\scriptstyle d_r^{p,q}} & & \downarrow{\scriptstyle d_r'^{p,q}} \\
E_r^{p+r,q-r+1} & \xrightarrow{\lambda_r^{p+r,q-r+1}} & E_r'^{p+r,q-r+1}
\end{array}
$$

If $q \leq m$, $\lambda_r^{p,q}$ induces an isomorphism between $\mathrm{Ker}(d_r^{p,q})$ and $\mathrm{Ker}(d_r'^{p,q})$ and $\lambda_r^{p+r,q-r+1}$ induces an isomorphism between $\mathrm{Im}(d_r^{p,q})$ and $\mathrm{Im}(d_r'^{p,q})$.

If $q > m$ and $q - r + 1 \leq m$, $E_r'^{p,q} = 0$ and, since the diagram is commutative, $d_r^{p,q} = d_r'^{p,q} = 0$. So, we deduce $d_r^{p,q}$ is 0 for every $m < q \leq m + r - 1$.

Moreover, $\mathrm{Im}(d_r^{p,q}) = \mathrm{Im}(d_r'^{p,q}) = 0$. Therefore, for every $q \leq m$, $\lambda_{r+1}^{p,q}$ is an isomorphism and we can finish the proof by induction. $\square$

COROLLARY 9.15. *Let X be a topological pseudomanifold with stratification*

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

and let k be the codimension of X_{d-2}, that is, $X_{d-2} = X_{d-k}$.

Let $\bar{p}$ a perversity and $\bar{q}$ its complementary perversity. If the local to global spectral sequence of $j_k^* i_{k*} \mathbb{Q}_{U_2}$ has any differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ different from 0 for some $r \geq 2$, $p \in \mathbb{Z}$ and $\bar{q}(k) < q \leq \bar{q}(k) + r - 1$, then there does not exist any intersection space complex of X with perversity $\bar{p}$.

The following example, based on Hopf's fibration, and also considered in Banagl-Chriestenson [7] (p. 180, Example 10.3, and p. 170, Example 6.13), verifies the conditions above.

EXAMPLE 9.16. *Let $\rho^\partial : S^3 \to S^2$ be the Hopf fibration and let $\rho : cyl(\rho^\partial) \to S^2$ be the fibrewise cone of the fibration (see definition 3.3). If $s : S^2 \to cyl(\rho^\partial)$ is the vertex section, we consider the space $X := cyl(\rho^\partial)$ with the stratification*

$$X \supset s(S^2)$$

Let $U := X \setminus s(S^2)$ and let $i : U \to X$ and $j : s(S^2) \to X$ be the canonical inclusions. Then, since the fiber of ρ^∂ is S^1

$$\mathcal{H}^i(j_* j^* i_* \mathbb{Q}_U) \begin{cases} \mathbb{Q}_{S^2} & if \, i = 0, 1 \\ 0 & otherwise \end{cases}$$

So, if $E_r^{p,q}$ is the hypercohomology spectral sequence of $j_* j^* i_* \mathbb{Q}_U$, $E_2^{p,q}$ is

$$q \uparrow \quad \begin{array}{c|c|c|c} 1 & \mathbb{Q} & 0 & \mathbb{Q} \\ \hline 0 & \mathbb{Q} & 0 & \mathbb{Q} \\ \hline & 0 & 1 & 2 \end{array}$$

$$\overrightarrow{p}$$

where the differential $d_2^{0,1}$ is different from 0.

Moreover, given any perversity $\bar{p}$, $\bar{p}(2) = 0$. So, applying Corollary 9.15, there does not exist an intersection space complex of X with stratification $X \supset s(S^2)$ with any perversity.

Hence, applying Remark 6.3, there does not exist any intersection space pair of X with the previous stratification.

The stratification of X given in the previous example is not natural. Since X is smooth, the natural stratification of X has no stratum different from X and $\emptyset$. The following example is more natural in the sense that the nontrivial stratum is the singular part of the variety.

EXAMPLE 9.17. *Let $\rho^\partial : S^3 \to S^2$ be the Hopf fibration and let us consider the locally trivial fibration*

$$\sigma^\partial : S^3 \times_{S^2} S^3 \to S^2$$

Moreover, let $\sigma : cyl(\sigma^\partial) \to S^2$ be the fibrewise cone of σ^∂ and $s : S^2 \to cyl(\sigma^\partial)$ the vertex section. Then, we define $X := cyl(\sigma^\partial)$ and we consider the stratification

$$X \supset s(S^2)$$

Let $U := X \setminus s(S^2)$ and let $i : U \to X$ and $j : s(S^2) \to X$ be the canonical inclusions. Then, since the fiber of σ^∂ is $S^1 \times S^1$

$$\mathcal{H}^i(j_* j^* i_* \mathbb{Q}_U) \begin{cases} \mathbb{Q}_{S^2} & if \quad i = 0, 2 \\ \mathbb{Q}^2_{S^2} & if \quad i = 1 \\ 0 & otherwise \end{cases}$$

So, if $E_r^{p,q}$ is the hypercohomology spectral sequence of $j_* j^* i_* \mathbb{Q}_U$, $E_2^{p,q}$ is

$$
q \uparrow
\begin{array}{c|c|c|c}
2 & \mathbb{Q} & 0 & \mathbb{Q} \\
\hline
1 & \mathbb{Q}^2 & 0 & \mathbb{Q}^2 \\
\hline
0 & \mathbb{Q} & 0 & \mathbb{Q} \\
\hline
& 0 & 1 & 2
\end{array}
$$

$$\overrightarrow{p}$$

where the differentials $d_2^{0,1}$ and $d_2^{0,2}$ are different from 0.
Moreover, given any perversity $\bar{p}$, either $\bar{p}(3) = 0$ or $\bar{p}(3) = 1$. So, applying Corollary 9.15, there does not exist an intersection space complex of X with stratification $X \supset s(S^2)$ with any perversity.
Hence, applying Remark 6.3, there does not exist any intersection space pair of X with the previous stratification.

A great number of examples can be constructed with this technique. For example, if one wishes to have simply connected link and strata one, can use instead of Hopf fibration the fibration $\phi : S^7 \to S^4$ with fibre S^3. In [7, Example 6.13 and Example 10.3] this technique is used to find spaces which does not admit an intersection space.
Now, we give an example of algebraic variety for which the intersection space does not exist for the middle perversity.

EXAMPLE 9.18. *Let $Fr(2,3)$ be the frame bundle over the Grassmannian $Gr(2,3)$, that is, $Fr(2,3) := \{M \in \mathrm{Mat}(2 \times 3, \mathbb{C}) | \mathrm{rk}(M) = 2\}$ and the canonical bundle*

$$\pi : Fr(2,3) \to Gr(2,3) \cong \mathbb{P}^2_\mathbb{C}$$

is a $GL(2,\mathbb{C})$-pincipal bundle with the action

$$\mathrm{GL}(2,\mathbb{C}) \times \mathrm{Fr}(2,3) \longrightarrow Fr(2,3)$$
$$(A,M) \longrightarrow A \cdot M$$

Let $R_1^2 := \{M \in \mathrm{Mat}(2 \times 2, \mathbb{C}) | \mathrm{rk(M)} \leq 1\}$ and let us consider the action

$$\mathrm{GL}(2,\mathbb{C}) \times \mathrm{R}_1^2 \longrightarrow R_1^2$$
$$(A,M) \longrightarrow A \cdot M$$

Let $X := Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2$. Since $Sing(R_1^2) = \{0\}$, we have the equality

$$Sing(X) = Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \{0\} \cong Gr(2,3) \cong \mathbb{P}_{\mathbb{C}}^2$$

and the induced fiber bundle

$$Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2 \setminus \{0\} \longrightarrow \mathbb{P}_{\mathbb{C}}^2$$
$$(M_1, M_2) \longrightarrow \pi(M_1)$$

is fiber homotopy equivalent to the fibration of links over the singularity. The fiber of this morphism is $R_1^2 \setminus \{0\}$.
Now, let us consider the action

$$\mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(A,(a,b)) \longrightarrow A \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

and let $Y := Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2$.
The morphism

$$\mathbb{C}^2 \xrightarrow{\ f\ } R_1^2$$
$$(a,b) \longrightarrow \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

is compatible with the actions. So, it induces a morphism $g : Y \to X$.
Moreover, $g^{-1}(Sing(X)) = Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \{0\} \cong Gr(2,3) \cong \mathbb{P}_{\mathbb{C}}^2$ and the fiber bundle

$$Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{P}_{\mathbb{C}}^2$$
$$(M_1, M_2) \longrightarrow \pi(M_1)$$

is fiber homotopy equivalent to the fibration of links. The fiber of this morphism is $\mathbb{C}^2 \setminus \{0\}$.

In addition,

$$Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\} \xrightarrow{\quad g \quad} Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2 \setminus \{0\}$$

$$\mathbb{P}_{\mathbb{C}}^2$$

is a morphism of fibrations which induces in the fiber the morphism $f : \mathbb{C}^2 \setminus \{0\} \to R_1^2 \setminus \{0\}$.

Let us denote $U_X := Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2 \setminus \{0\}$ *and* $U_Y := Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\}$. *Moreover, let* $j_X : \mathbb{P}_C^2 \to X$, $i_X : U_X \to X$, $j_Y : \mathbb{P}_C^2 \to Y$ *and* $i_Y : U_Y \to Y$ *be the canonical inclusions.*

The morphism between fibrations g *produces a morphism of complexes*

$$j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y} \xrightarrow{\gamma} j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X}.$$

Moreover, $\mathbb{C}^2 \setminus \{0\}$ *is homotopically equivalent to* S^3, $R_1^2 \setminus \{0\}$ *is homotopically equivalent to* $S^3 \times S^2$ *and* $f : \mathbb{C}^2 \setminus \{0\} \to R_1^2 \setminus \{0\}$ *induces an isomorphism between the 0-th and the third cohomology groups. Then* γ *induces an isomorphism between the cohomology sheaves*

$$\mathcal{H}^0(j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y}) \cong \mathcal{H}^0(j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X})$$

and

$$\mathcal{H}^3(j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y}) \cong \mathcal{H}^3(j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X}).$$

Let $E_r^{p,q}$ *be the local to global spectral sequence of* $j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X}$, *let* $E_r'^{p,q}$ *be the local to global spectral sequence of hypercohomology of* $j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y}$, *and* $\gamma_r^{p,q} : E_r'^{p,q} \to E_r^{p,q}$ *the morphism induced by* γ. *Then,*

$$\gamma_2^{p,q} : \mathbb{H}^p(\mathbb{P}_C^2, \mathcal{H}^q(j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y})) \to \mathbb{H}^p(\mathbb{P}_C^2, \mathcal{H}^q(j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X}))$$

is an isomorphism if $q = 0, 3$.
$E_2'^{p,q}$ *is*

$q \uparrow$					
3	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
	0	1	2	3	4

$$\overrightarrow{p}$$

and it does not degenerate in $r = 2$. *So,* $d_4'^{0,3}$ *is different from* 0 *since it is the unique differential different from* 0 *which can appear.*

Moreover,$E_2^{p,q}$ is

q ↑					
6	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
5	0	0	0	0	0
4	0	0	0	0	0
3	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
2	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
1	0	0	0	0	0
0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
	0	1	2	3	4

$$\overrightarrow{p}$$

We prove now $d_r^{0,3} : E_r^{0,3} \to E_r^{r,3-r+1}$ is different from 0 for some $r \geq 0$. If $d_2^{0,3} = 0$, we have the following isomorphisms

$$E_4^{0,3} \cong E_3^{0,3} \cong E_2^{0,3} \cong E_2'^{0,3} \cong E_4'^{0,3}.$$

Moreover,

$$E_4^{4,0} \cong E_2^{4,0} \cong E_2'^{4,0} \cong E_4'^{4,0}$$

and the diagram

$$
\begin{array}{ccc}
E_4^{0,3} & \xrightarrow{\gamma_4^{0,3}} & E_4'^{0,3} \\
\downarrow{\scriptstyle d_4^{0,3}} & & \downarrow{\scriptstyle d_4'^{0,3}} \\
E_4^{4,0} & \xrightarrow{\gamma_4^{4,0}} & E_4'^{4,0}
\end{array}
$$

is commutative.

Then, since $d_4'^{0,3}$ is not 0, $d_4^{0,3}$ is also different from 0.

If $\bar{p}$ is a perversity such that $\bar{p}(6) = 2$ and $\bar{q}$ is the complementary perversity, then we have $\bar{q}(6) = 2$. This happens for the middle perversity. Consequently, applying Corollary 9.15, there does not exist an intersection space complex of $X = Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2$ with perversity $\bar{p}$ and stratification $X \supset Sing(X) \cong \mathbb{P}_C^2$.

Hence, applying Remark 6.3, there does not exist any intersection space pair of X with perversity $\bar{p}$.

10 Duality

In this section, we establish the duality properties of the intersection space complexes. First, we study the Verdier dual of the intersection space complex. Next, we give a version of Poincaré duality for these complexes.

Let X be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

Let $U_k := X \setminus X_{d-k}$ and let $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ be the natural inclusions.

10.1 Verdier duality

Given a vector space E we we denote its by E^v its linear dual.

Theorem 10.1. *Let $IS_{\bar{p}}$ be an intersection space complex of X with perversity $\bar{p}$ and let $\bar{q}$ be the complementary perversity of $\bar{p}$. Then, $\mathcal{D}IS_{\bar{p}}[-d]$, where $\mathcal{D}$ denotes the Verdier dual, is an intersection space complex of X with perversity $\bar{q}$.*

Proof. We have to prove $\mathcal{D}IS_{\bar{p}}[-d]$ verifies $[AXS1]_k$ for perversity $\bar{q}$ for $k = 2, ..., d$.

(a) We have a chain of isomorphisms

$$(\mathcal{D}IS_{\bar{p}}[-d])_{|U_2} \cong \mathcal{D}\mathbb{Q}_{U_2}[-d] \cong \mathbb{Q}_{U_2}.$$

(b) Let $x \in X$, the group $\mathcal{H}^i(j_x^* \mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-i}(j_x^! IS_{\bar{p}})^v$, which is 0 if $i \notin \{0, 1, ..., d\}$.

(c_k) Let $x \in X_{d-k} \setminus X_{d-k-1}$, the group $\mathcal{H}^i(j_x^* \mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-i}(j_x^! IS_{\bar{p}})^v$, which (by property $(d1_k')$) is 0 if $d - i > d - \bar{p}(k) - 1$, that is, if $i \leq \bar{p}(k)$.

($d1_k'$) Let $x \in X_{d-k} \setminus X_{d-k-1}$, the group $\mathcal{H}^i(j_x^! \mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-i}(j_x^* IS_{\bar{p}})^v$, which (by property (c_k)) is 0 if $d - i \leq \bar{q}(k)$, that is, if $i > d - \bar{q}(k) - 1$.

($d2_k'$) Let $x \in X_{d-k} \setminus X_{d-k-1}$, the group $\mathcal{H}^{d-\bar{q}(k)-1}(j_x^! \mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{\bar{q}(k)+1}(j_x^* IS_{\bar{p}})^v$, the group $\mathcal{H}^{\bar{p}(k)+1}(j_x^* \mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-\bar{p}(k)-1}(j_x^! IS_{\bar{p}})^v$ and the canonical morphism

$$\mathcal{H}^{d-\bar{q}(k)-1}(j_x^! \mathcal{D}IS_{\bar{p}}[-d]) \to \mathcal{H}^{\bar{p}(k)+1}(j_x^* \mathcal{D}IS_{\bar{p}}[-d])$$

is the dual morphism of $\mathcal{H}^{d-\bar{p}(k)-1}(j_x^! IS_{\bar{p}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^* IS_{\bar{p}})$, which is the morphism 0 (by property $(d2_k')$).

$\square$

In Corollary 7.6 the space of obstructions for existence and uniqueness of intersection spaces are described. Verdier duality $\mathcal{D}$ interchanges intersection space complexes with complementary perversities. We deduce

Corollary 10.2. *Let X be a topological pseudomanifold as above. Let $\bar{p}$ and $\bar{q}$ complementary perversities. An intersection space complex for perversity $\bar{p}$ exists if and only if an intersection space complex for perversity $\bar{q}$ exists. Verdier duality induces a bijection between the set of intersection space complexes for perversity $\bar{p}$ and the set of intersection space complexes for perversity $\bar{q}$.*

10.2 Poincaré duality in the case of 2 strata

Now, suppose X has a unique non-trivial stratum. So, the stratification of X is

$$X \supset X_{d-k} \supset \emptyset$$

where k is the codimension of X_{d-k}.

According with Corollary 7.6, the obstruction for existence of intersection space for perversity $\bar{p}$ lives in $Ext^1(\tau_{>\bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}|_{U_k}, \tau_{\leq\bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}|_{U_k})$. Assume that the obstruction vanishes so that intersection space complexes exist. In this case the space of intersection space complexes for perversity $\bar{p}$ is parametrized by a quotient of the vector space

$$E_{\bar{p}} := Hom(\tau_{>\bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}|_{U_k}, \tau_{\leq\bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}|_{U_k}).$$

Corollary 10.2 implies that the obstruction for existence of intersection space for perversity $\bar{q}$ vanishes and that the space

$$E_{\bar{q}} := Hom(\tau_{>\bar{p}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}|_{U_k}, \tau_{\leq\bar{p}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}|_{U_k})$$

of intersection space complexes for perversity $\bar{q}$ is isomorphic to $E_{\bar{p}}$.

Proposition 10.3. *Let $E_{\bar{p}}$ be the space of all intersection space complexes of X with perversity $\bar{p}$ up to isomorphisms.*
The dimensions of the vector spaces $\mathbb{H}^i(X, IS_{\bar{p}})$ with $IS_{\bar{p}} \in E_{\bar{p}}$ have a minimum and the subset

$$\{IS_{\bar{p}} \in E_{\bar{p}} | \dim(\mathbb{H}^i(X, IS_{\bar{p}})) \text{ is minimum}\} \subset E_{\bar{p}}$$

is Zariski open for every i.

Proof. For every intersection space complex $IS_{\bar{p}} \in E_{\bar{p}}$, we have a triangle

$$IS_{\bar{p}} \to i_{k*}\mathbb{Q}_{U_2} \to \tau_{\leq\bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}$$

This triangle induces the long exact sequence of hypercohomology

$$... \to \mathbb{H}^i(X, IS_{\bar{p}}) \to \mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2}) \xrightarrow{\alpha^i(IS_{\bar{p}})} \mathbb{H}^i(X, \tau_{\leq\bar{q}(k)}j_{k*}j_k^*\mathbb{Q}_{U_2}) \to ...$$

So, for every $i \in \mathbb{Z}$, there is an isomorphism

$$\mathbb{H}^i(X, IS_{\bar{p}}) \cong \mathrm{Ker}(\alpha^i(IS_{\bar{p}})) \oplus \mathrm{CoKer}(\alpha^{i-1}(IS_{\bar{p}})).$$

Moreover,

$$\dim(\mathrm{CoKer}(\alpha^i(IS_{\bar{p}}))) = \dim(\mathbb{H}^i(X, \tau_{\leq\bar{q}(k)}j_{k*}j_k^*\mathbb{Q}_{U_2})) - \dim(\mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2})) +$$

$$+ \dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}})))$$

So, $\dim(\mathbb{H}^i(X, IS_{\bar{p}}))$ is minimum if and only if $\dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}})))$ and $\dim(\mathrm{Ker}(\alpha^{i-1}(IS_{\bar{p}})))$ are minimum.

The morphism $\alpha^i(IS_{\bar{p}})$ is the morphisms induced in hypercohomology by the composition

$$i_{k*}\mathbb{Q}_{U_2} \xrightarrow{a} j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2} \xrightarrow{\lambda(IS_{\bar{p}})} \tau_{\leq \bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}$$

where a is the canonical morphisms and $\lambda(IS_{\bar{p}})$ is a retraction of the natural truncation morphism $f : \tau_{\leq \bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2} \to j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}$.

Let us denote by

$$a^i : \mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}),$$

$$\lambda^i(IS_{\bar{p}}) : \mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}),$$

$$f^i : \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2})$$

the morphisms induced in hypercohomology. Then, we have

$$\dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}}))) = \dim((a^i)^{-1}(\mathrm{Ker}(\lambda^i(IS_{\bar{p}})))) =$$

$$= \dim(\mathrm{Ker}(a^i)) + \dim(\mathrm{Im}(a^i) \cap \mathrm{Ker}(\lambda^i(IS_{\bar{p}})))$$

Hence, $\dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}})))$ is minimum if and only if $\dim(\mathrm{Im}(a^i) \cap \mathrm{Ker}(\lambda^i(IS_{\bar{p}})))$ is minimum.

Since $\lambda^i(IS_{\bar{p}}) \circ f^i$ is the identity, the homomorphism $\lambda^i(IS_{\bar{p}})$ is surjective and we have the equality

$$\dim(\mathrm{Ker}(\lambda^i(IS_{\bar{p}}))) = \dim(\mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2})) - \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2})).$$

So, $\dim(\mathrm{Ker}(\lambda^i(IS_{\bar{p}})))$ is independent of $IS_{\bar{p}}$. Then, for every $IS_{\bar{p}} \in E_{\bar{p}}$, there is an isomorphism

$$\mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}))/\mathrm{Ker}(\lambda^i(IS_{\bar{p}})) \cong \mathbb{Q}^{d^i}$$

where $d^i := \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)}j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}))$.

Now, consider the composition of morphisms

$$
\begin{array}{ccc}
\mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2}) & \xrightarrow{\;\;a^i\;\;} & \mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}) \\
& \searrow{\scriptstyle \phi(IS_{\bar{p}})} & \downarrow{\scriptstyle \pi(IS_{\bar{p}})} \\
& & \mathbb{H}^i(X, j_{k*}j_k^* i_{k*}\mathbb{Q}_{U_2}))/\mathrm{Ker}(\lambda^i(IS_{\bar{p}})) \cong \mathbb{Q}^{d^i}
\end{array}
$$

where $\pi(IS_{\bar{p}})$ is the canonical projection.

Then, $\mathrm{Im}(a^i) \cap \mathrm{Ker}(\lambda^i(IS_{\bar{p}}))$ gets the minimum dimension when the morphism $\phi(IS_{\bar{p}})$ gets the maximum rank, which happens in an open subset. $\square$

Definition 10.4. *The* general *i*-th Betti number *of the intersection space complexes of X with perversity $\bar{p}$ is the minimum of the dimensions of the vector spaces* $\mathbb{H}^i(X, IS_{\bar{p}})$ *with* $IS_{\bar{p}} \in E_{\bar{p}}$.

Definition 10.5. *A* general intersection space complex *of X with perversity $\bar{p}$ is an intersection space complex $IS_{\bar{p}} \in E_{\bar{p}}$ such that $\dim(\mathbb{H}^i(X, IS_{\bar{p}}))$ is the general i-th Betti number for $i = 0, 1, ..., d$.*

Theorem 10.6. *Let $\bar{p}$ be a perversity and let $\bar{q}$ be its complementary perversity. If $IS_{\bar{p}}$ is a general intersection space complex of X with perversity $\bar{p}$ and $IS_{\bar{q}}$ is a general intersection space complex of X with perversity $\bar{q}$, then, for $i = 0, 1, ..., d$, there is an isomorphism of $\mathbb{Q}$-vector spaces*

$$\mathbb{H}^i(X, IS_{\bar{p}}) \cong \mathbb{H}^{d-i}(X, IS_{\bar{q}})^v$$

Proof. Given any intersection space complex $IS_{\bar{p}}$ of X with perversity $\bar{p}$, we have

$$\mathbb{H}^i(X, \mathcal{D}IS_{\bar{p}}[-d])^v \cong \mathbb{H}^{i-d}(X, \mathcal{D}IS_{\bar{p}})^v \cong \mathbb{H}^{d-i}(X, IS_{\bar{p}}).$$

Applying Theorem 10.1, the complex $\mathcal{D}IS_{\bar{p}}[-d]$ is an intersection space complex of X with perversity $\bar{q}$. We denote $IS_{\bar{q}} := \mathcal{D}IS_{\bar{p}}[-d]$.

Suppose $IS_{\bar{q}}$ is not a general intersection space complex of X. Then, there exists another intersection space complex of X with perversity $\bar{q}$, $IS'_{\bar{q}}$, such that we have the strict inequality

$$\sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS'_{\bar{q}})) < \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS_{\bar{q}})).$$

Consequently we have,

$$\sum_{i=0}^{d} \dim(\mathbb{H}^i(X, \mathcal{D}IS'_{\bar{q}}[-d])) = \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS'_{\bar{q}})^v) < \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS_{\bar{q}})^v) =$$

$$= \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, \mathcal{D}IS_{\bar{q}}[-d])) = \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS_{\bar{p}})).$$

So, $IS_{\bar{p}}$ is not a general intersection space complex of X.

We deduce that if $IS_{\bar{p}}$ is a general intersection space complex, then $IS_{\bar{q}}$ is also a general intersection space complex. So, there are isomorphisms

$$\mathbb{H}^i(X, IS_{\bar{p}}) \cong \mathbb{H}^{d-i}(X, IS_{\bar{q}})^v$$

for some general intersection space complexes $IS_{\bar{p}}$ and $IS_{\bar{q}}$.

Since the hypercohomology groups of general intersection space complexes with the same perversity are isomorphic, we conclude. $\square$

Remark 10.7. *We do not know of examples of intersection space complexes for the same perversity with different hypercohomology. It is an open question whether they exist.*

References

[1] M. Banagl. *Topological invariants of stratified spaces.* Springer, Berlin, 2007.

[2] M. Banagl. *Singular spaces and generalized Poincaré complexes*, Electron. Res. Announc. Math. Sci. 16 (2009), 63–73.

[3] M. Banagl. *Intersection spaces, spatial homology truncation and string theory.* Lecture Notes in Math., vol. 1997, Springer-Verlag, Berlin, Heidelberg, 2010.

[4] M. Banagl. *First cases of intersection spaces in stratification depth 2*, J. of Singularities 5 (2012), 57–84.

[5] M. Banagl. *Isometric group actions and the cohomology of flat fiber bundles*, Groups Geom. Dyn. 7, no. 2 (2013), 293–321

[6] M. Banagl, N. Budur, L. Maxim. *Intersection spaces, perverse sheaves and type IIB string theory*, Adv. Theor. Math. Phys. 18, no. 2 (2014), 363–399.

[7] M. Banagl, B. Chriestenson. *Intersection spaces, equivariant Moore approximation and the signature*, J. of Singularities 16 (2017), 141–179.

[8] M. Banagl, E. Hunsicker. *Hodge theory for intersection space cohomology*, Geometry and Topology, 23, no. 5 (2019), 2165–2225.

[9] M. Banagl, L. Maxim. *Intersection spaces and hypersurface singularities*, J. of Singularities 5 (2012), 48–56.

[10] M. Banagl, L. Maxim. *Deformation of singularities and the homology of intersection spaces*, J. Topol. Anal. 4, no. 4 (2012), 413–448.

[11] M. A. de Cataldo, L. Migliorini. *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. 46, no. 4 (2009), 535-633.

[12] J. T. Essig. Intersection space cohomology of three-strata pseudomanifolds, April 2018, to appear in J. Topol. Anal. arXiv e-prints, arXiv:1804.06690.

[13] C. Geske. *Algebraic intersection spaces*, J. Topol. Anal. 12, no. 4 (2020), 1157–1194.

[14] C. G. Gibson, K. Wirthmüller, A. A. du Plessis, E. J. N. Looijenga. *Topological stability of smooth mappings.* Lecture Notes in Math., vol. 552, Springer-Verlag, Berlin, New York, 1976.

[15] M. Goresky, R. MacPherson. *Intersection homology theory. II*, Invent. Math. 72, no. 1 (1983), 77–129.

[16] M. Klimczak. *Poincaré duality for spaces with isolated singularities*, Manuscripta Math. 153 (2017), 231–262.

[17] W. S. Massey. *A basic course in algebraic topology.* Graduate Texts in Mathematics, vol. 127, Springer-Verlag, New York, 1991.

[18] R. MacPherson, K. Vilonen, *Elementary construction of perverse sheaves*, Invent. Math. 84 (1986), 403–435.

[19] L. Maxim. *Intersection spaces, perverse sheaves and string theory*, J. of Singularities 15 (2016), 118–125

[20] S. Ramanan. *Global calculus.* Graduate Studies in Mathematics, vol. 65, American Mathematical Society, Providence, Rhode Island, 2004.

[21] M. Saito. *Mixed Hodge modules*, Publications of R.I.M.S. 26, no. 2 (1990), 221-333.

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