

# MONOIDAL STRUCTURES ON THE CATEGORIES OF QUADRATIC DATA

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Received: September 9, 2019

Revised: July 30, 2020

Communicated by Max Karoubi

**ABSTRACT.** The notion of 2-monoidal category used here was introduced by B. Vallette in 2007 for applications in the operadic context. The starting point for this article was a remark by Yu. Manin that in the category of quadratic algebras (that is, “quantum linear spaces”) one can also define 2-monoidal structure(s) with rather unusual properties. Here we give a detailed exposition of these constructions, together with their generalisations to the case of quadratic operads.

Their parallel exposition was motivated by the following remark. Several important operads/cooperads such as genus zero quantum cohomology operad, the operad classifying Gerstenhaber algebras, and more generally, (co)operads of homology/cohomology of some topological operads, start with collections of quadratic algebras/coalgebras rather than simply linear spaces.

Suggested here enrichments of the categories to which components of these operads belong, as well of the operadic structures themselves, might lead to a better understanding of these fundamental objects.

2020 Mathematics Subject Classification: Primary 18Mxx; Secondary 18M05, 18M50, 18M60, 18M70, 16S37

Keywords and Phrases: Monoidal categories, 2-monoidal categories, quadratic data, operads, black and white products, Koszul duality

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1	BRIEF SUMMARY AND PLAN OF EXPOSITION	

A *monoidal category*, as it was defined and studied in [MacL98, Chapter VII], is a category  $\mathcal{C}$  endowed with a bifunctor  $\boxtimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  satisfying the associativity axiom (“pentagon diagram”) and equipped with a (left and right) unit object. A *lax 2-monoidal category*, as it was introduced in [Val08], is a category  $\mathcal{C}$  endowed with *two* structures of monoidal category, whose respective bifunctors  $\boxtimes$  and  $\otimes$  are related by the natural transformation called *an interchange law*:

$$\varphi_{AA'BB'} : (A \otimes A') \boxtimes (B \otimes B') \rightarrow (A \boxtimes B) \otimes (A' \boxtimes B') \quad (1.1)$$

compatible with associativity of  $\boxtimes$  and  $\boxtimes$ -unit object in the sense made explicit in the Proposition 2 of [Va08]. Inverting all arrows (i.e. working in the opposite category), one gets the notion of *colax 2-monoidal category*. Finally, a 2-monoidal category equipped with a lax and a colax structure is simply called a *2-monoidal category*. A close but more restricted notion, which now often called *duoidal category* was coined by M. Aguiar and S. Mahajan in [AM10].

Notice that A. Joyal and R. Street, in the work [JS93] on braided tensor categories, came up with a notion of a category endowed with two monoidal products but related by a natural isomorphism, which forces the two monoidal structures to be isomorphic. C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt in [BFSV03] introduced a notion of iterated monoidal category in order to study iterated loop spaces. But in their framework, the units for the monoidal products should be equal. Neither of these two restrictions is imposed in our present examples.

Section 2 of our paper starts with a systematic formalization of the general notions: “algebra/operad defined by quadratic relations between (graded) generators” and their reduction to the notions of “quadratic data”. We then introduce various relevant categorical frameworks involving monoidal structures on the categories of such data, various canonical functors between them, and basic commutative diagrams relating these functors.

This is a development and generalization of constructions introduced in [Man88] as an approach to quantum algebra: quantum linear spaces, black and white products, bialgebras of their quantum endomorphisms, and quantum groups.

The central result of Section 3 is a new construction of 2-monoidal structures on the categories of quadratic data QD (defined in Section 2.2): we start with a simple construction of 2-monoidal structure on the category of graded vector spaces, and then show that it lifts to the category of quadratic data.

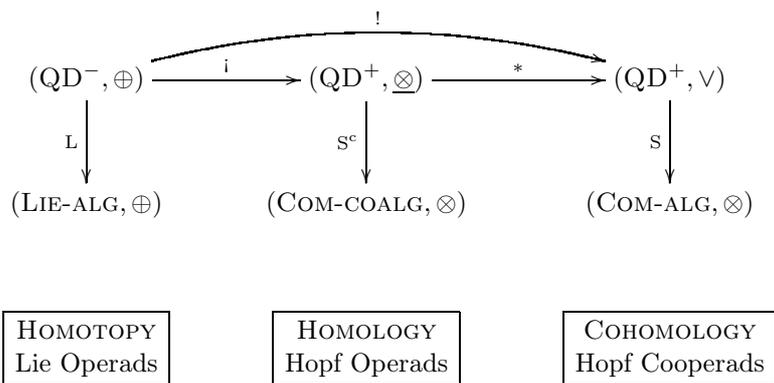
The central result of Section 4 is a generalization of this construction to the categories of binary operadic quadratic data BOQD defined in Section 4.2.

Finally, in Section 5, we return to the quantum picture of [Man88] and generalize it to our framework, as was done in [Man18] for the simplest case of the genus zero component of the quantum cohomology operad.

The most important new feature of our picture is the fact that there is an abundance of operads/cooperads with postulated properties arising naturally in various geometric contexts.

More precisely, any topological operad like the little discs operad (loop spaces) or the Deligne–Mumford operad of moduli spaces of stable genus 0 curves with marked points (quantum cohomology), induces a *homology* operad in the category of cocommutative coalgebras, a *cohomology* cooperad in the category of commutative algebras, and a “*homotopy*” operad in the category of Lie algebras. It is difficult to pass from one to another directly at the level of Lie algebras and (co)commutative (co)algebras.

It is however well-known that the Koszul duality of [GK94] between the two operads Lie – Com coincides with the duality Homotopy-(co)Homology in rational homotopy theory, see also [Ber14]. Our idea here is to lift these operadic structures on the level of simple categories of quadratic data without any loss of information (under Condition 1 of Proposition 5.3.4). In order to do so, we introduce the relevant notions of (symmetric, skew-symmetric) quadratic data together with suitable symmetric monoidal structures. On that level, we do have the Koszul duality and the linear duality functors. There are also “realisation” functors from these categories of quadratic data to categories of (co)algebras. Since all these functors are symmetric monoidal, they preserve (co)operad structures.



The simplest case is when one has to deal with operads in the category of skew-symmetric quadratic data  $QD^-$ , where the underlying monoidal structure is particularly easy: the direct sum. So this category is our favorite site to describe operadic structures. Then, we get for free (co)operad structures in all the other symmetric monoidal categories. It turns out that, this way, one can recover

many of the most important (co)operad structures present in the literature, like the graph (co)operads, the ones related to the little discs/configuration spaces of points  $\mathcal{D}_2(n) \sim \text{Conf}_n(\mathbb{R}^2)$ , the real locus of the moduli spaces of stable curves of genus 0 with marked points  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ , and their non-commutative versions. This point has two main interests: it makes particularly easy the passage between Lie operads and Hopf (co)operads and it allows us to organise the various operad structures in a common pattern. For instance, we construct a family of operads in skew-symmetric quadratic data whose first two cases are provided by the Drinfeld–Kohno quadratic data  $\text{Conf}_n(\mathbb{R}^2)$  and the Etingof–Henriques–Kamnitzer–Rains quadratic data  $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ ; we also give them a canonical operadic interpretation. This gives a new family of operads quite similar to the  $e_k$ -operads, except that instead of having a degree  $k - 1$  (binary) Lie bracket, we have a degree 1 “Lie bracket” of arity  $k$ .

With the same method, one can also study complex cases like the operad made up of the complex locus of the moduli spaces of stable curves of genus 0 with marked points  $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$  [KM94, Get95, KM96, Man99], whose cohomology rings admit a quadratic presentation by [Kee92]. There is also its non-commutative version  $\mathcal{B}(n)$  introduced in [DSV15] by means of toric varieties called brick manifolds and the dihedral topological operad  $\mathcal{M}_{0,n}^\delta(\mathbb{C})$  introduced by F. Brown in [Bro09] as a partial compactification with a view to understand multiple zeta values, see also [DV17, AP17]. The details are left to an interested reader.

#### ACKNOWLEDGEMENTS

We are grateful to Clemens Berger, Alexander Berglund, Ricardo Campos, Vladimir Dotsenko, Anton Khoroshkin, and Daniel Robert–Nicoud for interesting and useful discussions. The second author has been supported by the IUF and the grant ANR-14-CE25-0008-01 project SAT.

## 2 QUADRATIC DATA, MONOIDAL STRUCTURES, AND THEIR ALGEBRAIC REALISATIONS

### 2.1 NOTATIONS AND CONVENTIONS

We work over a ground field  $K$  of characteristic  $\neq 2$  and over the underlying category of finite dimensional  $\mathbb{Z}$ -graded  $K$ -vector spaces equipped with their morphisms of degree zero. The linear dual  $V^*$  is considered degree-wise:  $(V^*)_{-n} := \text{Hom}(V_n, K)$ . We equip this category with the usual tensor product  $(M \otimes N)_n := \bigoplus_{k+l=n} M_k \otimes N_l$  and with the natural isomorphisms  $\sigma(x \otimes y) := (-1)^{|x||y|} y \otimes x$  in order to make it into a symmetric monoidal category denoted simply by  $(\text{GRVECT}, \otimes)$ . We denote by  $s$  (respectively its linear dual  $s^{-1}$ ) the one-dimensional graded vector space concentrated in degree 1 (respectively  $-1$ ) and the degree shift operator by  $sV := s \otimes V$  (respectively  $s^{-1}V := s^{-1} \otimes V$ ).

2.2 CATEGORIES OF QUADRATIC DATA

For any graded vector space  $V$ , we consider the canonical decomposition  $V^{\otimes 2} \cong V^{\odot 2} \oplus V^{\wedge 2}$ , where

$$V^{\odot 2} := \langle x \odot y := x \otimes y + (-1)^{|x||y|} y \otimes x \rangle \quad \text{and}$$

$$V^{\wedge 2} := \langle x \wedge y := x \otimes y - (-1)^{|x||y|} y \otimes x \rangle .$$

DEFINITION 2.2.1 (Quadratic data). An object of the category QD of *quadratic data* is a pair  $(V, R)$  made up of a graded vector space  $V$  and a subspace  $R \subset V^{\otimes 2}$ . A morphism  $f : (V, R) \rightarrow (W, S)$  of quadratic data amounts to a morphism  $f : V \rightarrow W$  of graded vector spaces satisfying  $f^{\otimes 2}(R) \subset S$ .

The category of *symmetric quadratic data*  $\text{QD}^+$  (respectively *skew-symmetric quadratic data*  $\text{QD}^-$ ) is defined similarly with pairs  $(V, R)$  such that  $R \subset V^{\odot 2}$  (respectively  $R \subset V^{\wedge 2}$ ) this time.

2.3 FUNCTORS

There are first obvious “realisation” functors from the categories of quadratic data to the categories of unital associative algebras, unital commutative algebras, and Lie algebras respectively:

$$\begin{aligned} \text{A} : \text{QD} &\rightarrow \text{ASS-ALG} & \text{S} : \text{QD}^+ &\rightarrow \text{COM-ALG} \\ (V, R) &\mapsto \frac{T(V)}{(R)} & (V, R) &\mapsto \frac{S(V)}{(R)} , \\ \text{L} : \text{QD}^- &\rightarrow \text{LIE-ALG} \\ (V, R) &\mapsto \frac{\text{Lie}(V)}{(R)} . \end{aligned}$$

In order to lift the universal enveloping algebra functor

$$\begin{aligned} U : \text{LIE-ALG} &\rightarrow \text{ASS-ALG} \\ (\mathfrak{g}, [ , ]) &\mapsto U(\mathfrak{g}) := \frac{T(\mathfrak{g})}{(x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y])} \end{aligned}$$

to the quadratic data level, we consider the functor

$$\begin{aligned} \Lambda : \text{QD}^- &\rightarrow \text{QD} \\ (V, R) &\mapsto (V, \Lambda(R)) , \end{aligned}$$

where  $\Lambda(R) \in V^{\wedge 2} \subset V^{\otimes 2}$  is the natural inclusion.

Similarly, we lift the inclusion functor  $\text{COM-ALG} \hookrightarrow \text{ASS-ALG}$  to the quadratic data level by

$$\begin{aligned} \mathcal{S} : \text{QD}^+ &\rightarrow \text{QD} \\ (V, R) &\mapsto (V, \Sigma(R) \oplus V^{\wedge 2}) , \end{aligned}$$

where  $\Sigma(R) \in V^{\odot 2} \subset V^{\otimes 2}$  is the natural inclusion.

One can notice that the images of these algebraic realisation functors always produce a *weight graded* algebra, that is  $A \cong \bigoplus_{n \in \mathbb{N}} A^{(n)}$ , where each component  $A^{(n)}$  is finite dimensional. We denote the associated categories respectively by WG-ASS-ALG, WG-COM-ALG, and WG-LIE-ALG.

Dually, we consider the two categories of weight graded counital coassociative coalgebras WG-ASS-COALG and weight graded counital cocommutative coalgebras WG-COM-COALG, with finite dimensional components. There are also realisation functors from the categories of quadratic data to these two categories:

$$\begin{aligned} A^c : \text{QD} &\rightarrow \text{WG-ASS-COALG} && \text{and} \\ (V, R) &\mapsto T^c(V, R) \\ S^c : \text{QD}^+ &\rightarrow \text{WG-COM-COALG} \\ (V, R) &\mapsto S^c(V, R) \end{aligned}$$

where the quadratic coalgebra  $T^c(V, R)$  (and similarly the quadratic cocommutative coalgebra  $S^c(V, R)$ ) is initial object in the category of (conilpotent) counital coassociative coalgebras under  $T^c(V)$  such that the composite with the projection onto  $\frac{T^c(V)}{R}$  vanishes:

$$\begin{array}{ccccc} & & 0 & & \\ & & \curvearrowright & & \\ C & \xrightarrow{\quad} & T^c(V) & \twoheadrightarrow & \frac{T^c(V)}{R} \\ & \searrow & \nearrow & & \\ \exists & \downarrow & & & \\ T^c(V, R) & & & \xrightarrow{\quad} & \frac{T^c(V)}{R} \\ & & & \searrow & \\ & & & & 0 \end{array}$$

It is explicitly given by

$$T^c(V, R) \cong K \oplus V \oplus R \oplus \dots \oplus \left( \bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \dots ,$$

see [Val08, Section 2] or [LV12, Section 3.1.3] for more details. The category of cocommutative coalgebras naturally imbeds into the category of coassociative coalgebras: COM-COALG  $\hookrightarrow$  ASS-COALG and similarly WG-COM-COALG  $\hookrightarrow$  WG-ASS-COALG. These functors lift on the level of quadratic data by

$$\begin{aligned} \Sigma : \text{QD}^+ &\rightarrow \text{QD} \\ (V, R) &\mapsto (V, \Sigma(R)) . \end{aligned}$$

There are first *Koszul dual* functors

$$\begin{aligned} i : \text{QD} &\rightarrow \text{QD} && \text{and} && i : \text{QD}^\pm &\rightarrow \text{QD}^\mp \\ (V, R) &\mapsto (sV, s^2R) && && (V, R) &\mapsto (sV, s^2R) , \end{aligned}$$

where the double degree shift operator is defined by  $s(x \otimes y) := (-1)^{|x|} sx \otimes sy$  and which sends symmetric quadratic data to skew-symmetric quadratic data and vice versa. Notice that, all the above-mentioned functors are covariant.

Now, we consider the linear dual *contravariant* functors

$$* : \text{QD} \rightarrow \text{QD} \quad \text{and} \quad * : \text{QD}^\pm \rightarrow \text{QD}^\pm$$

$$(V, R) \mapsto (V^*, R^\perp) \quad (V, R) \mapsto (V^*, R^\perp),$$

In the former case, since  $R \subset V^{\otimes 2}$ , its orthogonal is understood in  $R^\perp \subset (V^*)^{\otimes 2} \cong (V^{\otimes 2})^*$ . In the latter case, since  $R \subset V^{\odot 2}$  (respectively  $R \subset V^{\wedge 2}$ ), its orthogonal is understood in  $R^\perp \subset (V^*)^{\odot 2}$  (respectively in  $R^\perp \subset (V^*)^{\wedge 2}$ ). One can iterate the above two types of functors to produce the *second Koszul dual* (contravariant) functors:

$$! :- *i : \text{QD} \rightarrow \text{QD} \quad \text{and}$$

$$(V, R) \mapsto (s^{-1}V^*, s^{-2}R^\perp)$$

$$! :- *i : \text{QD}^- \rightarrow \text{QD}^+$$

$$(V, R) \mapsto (s^{-1}V^*, s^{-2}R^\perp).$$

The weight-wise linear duality functor sends coalgebras to algebras (and vice-versa):

$$* : \text{WG-ASS-COALG} \rightarrow \text{WG-ASS-ALG} \quad \text{and}$$

$$\bigoplus_{n \in \mathbb{N}} C^{(n)} \mapsto \bigoplus_{n \in \mathbb{N}} (C^{(n)})^*$$

$$* : \text{WG-COM-COALG} \rightarrow \text{WG-COM-ALG}$$

$$\bigoplus_{n \in \mathbb{N}} C^{(n)} \mapsto \bigoplus_{n \in \mathbb{N}} (C^{(n)})^* .$$

PROPOSITION 2.3.1. *All these functors assemble into the commutative diagram of Figure 1.*

*Proof.* The commutativity of the extreme top face amounts to  $\Lambda(R)^\perp \cong \Sigma(s^{-2}R^\perp) \oplus (s^{-1}V^*)^{\wedge 2}$ , the commutativity of the left top face amounts to  $s^2\Lambda(R) \cong \Sigma(s^2R)$ , and the commutativity of the right top face amounts to  $\Sigma(R^\perp) \cong \Sigma(R)^\perp$ .

The commutativity of the front face is given by  $S^c(V, R)^* \cong S(V^*, R^\perp)$  and the commutativity of the back face is given by  $T^c(V, R)^* \cong T(V^*, R^\perp)$ . This comes from the fact that the universal property satisfied by quadratic algebras is categorically dual to the universal property defining quadratic coalgebras, see [LV12, Section 3.2.2].

The commutativity of the left-hand side vertical face comes from  $U\left(\frac{\text{Lie}(V)}{(R)}\right) \cong \frac{T(V)}{(\Lambda(R))}$ , and the commutativity of the right-hand side vertical face comes from  $\frac{S(V)}{(R)} \cong \frac{T(V)}{(\Sigma(R) \oplus V^{\wedge 2})}$ . The commutativity of the central horizontal face is obvious. It induces the commutativity of the central vertical face: the isomorphism  $S^c(V, R) \cong T^c(V, \Sigma(R))$  can be seen under the weight-wise linear dual from the above isomorphism and the dual characterisations of quadratic (co)algebras. (One can also prove that  $S^c(V, R)$  satisfies the universal property of coassociative quadratic coalgebra generated by  $(V, \Sigma(R))$ .)

The commutativity of the other faces involving only forgetful functors is straightforward. □



REMARK 2.3.2. We could also consider Koszul dual and linear dual inverse functors, going in the opposite direction, defined by formulas like  $(s^{-1}V, s^{-2}R)$  for  $i$  and  $(sV^*, s^2R^\perp)$  for  $*$ . We keep the exposition to the present degree of details for reasons that will be apparent in Section 5, when dealing with (co)operad structures. So far, we would like the vertex labeled by the category  $\text{QD}^-$  to be the unique top vertex of this diagram.

2.4 SYMMETRIC MONOIDAL STRUCTURES

We now enrich the above categories with symmetric monoidal structures. On the category  $\text{QD}$  of quadratic data, we consider the two symmetric monoidal products  $\otimes$  and  $\underline{\otimes}$ :

$$\begin{aligned} (V, S) \otimes (W, R) &:- (V \oplus W, R \oplus [V, W]_- \oplus S) \quad \text{and} \\ (V, S) \underline{\otimes} (W, R) &:- (V \oplus W, R \oplus [V, W]_+ \oplus S), \end{aligned}$$

where  $[V, W]_\pm := \langle v \otimes w \pm (-1)^{|v||w|} w \otimes v \mid v \otimes w \in V \otimes W \rangle$ .

The category  $\text{QD}^+$  of symmetric quadratic data is endowed with

$$\begin{aligned} (V, R) \vee (W, S) &:- (V \oplus W, R \oplus S) \quad \text{and} \\ (V, R) \underline{\otimes} (W, S) &:- (V \oplus W, R \oplus [V, W]_+ \oplus S) \end{aligned}$$

and the category  $\text{QD}^-$  of skew-symmetric quadratic data is endowed with

$$(V, R) \oplus (W, S) := (V \oplus W, R \oplus [V, W]_- \oplus S).$$

The bottom categories of algebras are equipped with the following monoidal products. We consider the direct sum  $\mathfrak{g} \oplus \mathfrak{h}$  of Lie algebras, where  $[x + y, x' + y'] := [x, x'] + [y, y']$ , for any  $x, x' \in \mathfrak{g}$  and  $y, y' \in \mathfrak{h}$ . This is the categorical product in the category  $\text{LIE-ALG}$ . The underlying tensor product  $\otimes$  of two associative algebras  $A, B \in \text{ASS-ALG}$  carries a natural associative product:  $\mu(a \otimes b, a' \otimes b') := (-1)^{|a'||b|} \mu_A(a, a') \otimes \mu_B(b, b')$ . If the two algebras happen to be commutative, so is their tensor product. The same holds true for the tensor product of coassociative or cocommutative coalgebras and in the weight graded case.

PROPOSITION 2.4.1. *The above-mentioned monoidal products endow their respective categories with a symmetric monoidal structure.*

*Proof.* Recall from [MacL98, Section XI.1] that to get a (strong) symmetric monoidal category besides monoidal products described above we have to define coherent objects (units) and coherent natural isomorphisms (associator, left and right unitors, and braiding). For the five categories  $(\text{QD}, \otimes)$ ,  $(\text{QD}, \underline{\otimes})$ ,  $(\text{QD}^+, \vee)$ ,  $(\text{QD}^+, \underline{\otimes})$ , and  $(\text{QD}^-, \oplus)$  of quadratic data, the unit is  $(0, 0)$ , the associator is  $(V \oplus W) \oplus Z \cong V \oplus (W \oplus Z)$ , the unitors are  $0 \oplus V \cong V \cong V \oplus 0$ , and the braiding is  $V \oplus W \cong W \oplus V$ . The symmetric monoidal structure on

(LIE-ALG,  $\oplus$ ) is given by a similar unit and by similar maps. For all monoidal categories of (possibly weight graded) algebras and coalgebras, the unit is  $K$ , the associator is  $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$ , the unitors are  $K \otimes V \cong V \cong V \otimes K$ , and the braiding is  $V \otimes W \cong W \otimes V$ . The various coherence diagrams are then straightforward to check.  $\square$

REMARK 2.4.2. Notice that the categories  $(\text{QD}^-, \oplus)$ ,  $(\text{QD}^+, \underline{\otimes})$ ,  $(\text{LIE-ALG}, \oplus)$  and the category of coaugmented (weight-graded) cocommutative coalgebras with  $\otimes$  are cartesian, that is their symmetric monoidal structure is given by their product and their terminal object. Dually the category  $(\text{QD}^+, \vee)$  and the category of augmented (weight-graded) commutative algebras with  $\otimes$  are cocartesian, that is their symmetric monoidal structure is given by their co-product and their initial object.

2.5 SYMMETRIC MONOIDAL FUNCTORS

We can now check the possible coherence between the various functors and symmetric monoidal structures introduced above.

THEOREM 2.5.1. *The commutative diagram described on Proposition 2.3.1 is made up of strong symmetric monoidal functors, see Figure 2.*

*Proof.* Recall from [MacL98, Section XI.2] that a strong symmetric monoidal functor  $F : (C, \otimes_C, 1_C) \rightarrow (D, \otimes_D, 1_D)$ , is a covariant functor between monoidal categories equipped with natural isomorphisms

$$\psi : 1_D \xrightarrow{\cong} F(1_C) \quad \text{and} \quad \varphi_{A,B} : F(A) \otimes_D F(B) \xrightarrow{\cong} F(A \otimes_C B) ,$$

subject to coherence diagrams with respect to the various associators, unitors, and braidings. Recall that the opposite of a symmetric monoidal category is again a symmetric monoidal category. A contravariant functor  $F : C \rightarrow D$  is called strong symmetric monoidal, when the induced covariant functor  $F^{\text{op}} : C^{\text{op}} \rightarrow D$  is strong symmetric monoidal.

Let us begin with the top faces functors. There, all the units are equal to  $(0, 0)$  and preserved by the various functors. The structural isomorphisms for the monoidal functor  $\Lambda$  are given by

$$\begin{aligned} \Lambda(V, R) \otimes \Lambda(W, S) &\cong (V \oplus W, \Lambda(R) \oplus [V, W]_- \oplus \Lambda(S)) \\ &\cong (V \oplus W, \Lambda(R \oplus [V, W]_- \oplus S)) \\ &\cong \Lambda((V, R) \oplus (W, S)) , \end{aligned}$$

the ones for the monoidal functor  $\Sigma$  are given by

$$\begin{aligned} \Sigma(V, R) \underline{\otimes} \Sigma(W, S) &\cong (V \oplus W, \Sigma(R) \oplus [V, W]_+ \oplus \Sigma(S)) \\ &\cong (V \oplus W, \Sigma(R \oplus [V, W]_+ \oplus S)) \\ &\cong \Sigma((V, R) \underline{\otimes} (W, S)) , \end{aligned}$$



and the ones for the monoidal functor  $\mathcal{S}$  are given by

$$\begin{aligned}\mathcal{S}(V, R) \otimes \mathcal{S}(W, S) &\cong (V \oplus W, \Sigma(R) \oplus V^{\wedge 2} \oplus [V, W]_- \oplus \Sigma(S) \oplus W^{\wedge 2}) \\ \hat{E} &\cong (V \oplus W, \Sigma(R \oplus S) \oplus (V \oplus W)^{\wedge 2}) \\ &\cong \mathcal{S}((V, R) \vee (W, S)) ,\end{aligned}$$

since

$$\begin{aligned}(V \oplus W)^{\otimes 2} &\cong (V \oplus W)^{\circ 2} \oplus (V \oplus W)^{\wedge 2} \\ &\cong (V^{\circ 2} \oplus [V, W]_+ \oplus W^{\circ 2}) \oplus (V^{\wedge 2} \oplus [V, W]_- \oplus W^{\wedge 2}) .\end{aligned}$$

The natural isomorphisms for the first Koszul duality functors  $i : (\text{QD}^-, \oplus) \rightarrow (\text{QD}^+, \underline{\otimes})$  are given by

$$\begin{aligned}(V, R)^i \underline{\otimes} (W, S)^i &\cong (sV \oplus sW, s^2R \oplus [sV, sW]_+ \oplus s^2S) \\ &\cong (s(V \oplus W), s^2(R \oplus [V, W]_- \oplus S)) \\ &\cong ((V, R) \oplus (W, S))^i ;\end{aligned}$$

the ones for  $i : (\text{QD}, \otimes) \rightarrow (\text{QD}, \underline{\otimes})$  are similar.

For the the linear dual functors, we consider the following isomorphisms

$$\begin{aligned}(V, R)^* \vee (W, S)^* &\cong (V^* \oplus W^*, R^\perp \oplus S^\perp) \cong (V \oplus W, R \oplus [V, W]_+ \oplus S)^* \\ &\cong ((V, R) \underline{\otimes} (W, S))^*\end{aligned}$$

and

$$\begin{aligned}(V, R)^* \otimes (W, S)^* &\cong (V^* \oplus W^*, R^\perp \oplus [V^*, W^*]_- \oplus S^\perp) \\ &\cong (V \oplus W, R \oplus [V, W]_+ \oplus S)^* \cong ((V, R) \underline{\otimes} (W, S))^* .\end{aligned}$$

Since the second Koszul duality functors  $!$  are the composites of two strong symmetric functors (see below), they are also strong symmetric monoidal.

Each of the functors of the the left-hand side face sends directly the unit to the unit, since  $L(0, 0) = 0$ ,  $A(0, 0) = K$ , and  $U(0) = K$ . The new structural isomorphisms are

$$\begin{aligned}L(V, R) \oplus L(W, S) &\cong L(V \oplus W, R \oplus [V, W]_- \oplus S) \cong L((V, R) \oplus (W, S)) , \\ A(V, R) \otimes A(W, S) &\cong A(V \oplus W, R \oplus [V, W]_- \oplus S) \cong A((V, R) \otimes (W, S)) ,\end{aligned}$$

and

$$U(\mathfrak{g} \oplus \mathfrak{h}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{h}) .$$

Regarding the right-hand side face, the respective units  $(0, 0)$  and  $K$ , are again directly sent to one another. In this case, the structural isomorphisms are

$$\mathcal{S}(V, R) \otimes \mathcal{S}(W, S) \cong \mathcal{S}(V \oplus W, R \oplus S) \cong \mathcal{S}((V, R) \vee (W, S))$$

and the identities for the forgetful functors.

In the middle horizontal face, we also consider the identities for the forgetful functors. For the two weight-wise linear dualisation functors  $* : \text{WG-COM-COALG} \hookrightarrow \text{WG-COM-ALG}$  and  $* : \text{WG-ASS-COALG} \hookrightarrow \text{WG-ASS-ALG}$  the natural maps  $C^* \otimes D^* \cong (C \otimes D)^*$  are isomorphisms since we are working with spaces with finite dimensional weight components.

Finally, the two vertical coalgebra realisations functors send the unit  $(0, 0)$  to the unit  $K$ . The structural isomorphisms are respectively

$$\begin{aligned} S^c(V, R) \otimes S^c(W, S) &\cong S^c(V \oplus W, R \oplus [V, W]_+ \oplus S) \cong S^c((V, R) \underline{\otimes} (W, S)) , \\ A^c(V, R) \otimes A^c(W, S) &\cong A^c(V \oplus W, R \oplus [V, W]_+ \oplus S) \cong A^c((V, R) \underline{\otimes} (W, S)) . \end{aligned}$$

They can be proved on two ways. One can first consider their respective weight-wise linear duals and apply the respective isomorphisms

$$\begin{aligned} \frac{S(V^*)}{(R^\perp)} \otimes \frac{S(W^*)}{(S^\perp)} &\cong \frac{S(V^* \oplus W^*)}{(R^\perp \oplus S^\perp)} \quad \text{and} \\ \frac{T(V^*)}{(R^\perp)} \otimes \frac{T(W^*)}{(S^\perp)} &\cong \frac{T(V^* \oplus W^*)}{(R^\perp \oplus [V^*, W^*]_- \oplus S^\perp)} . \end{aligned}$$

One can also show that the left-hand side coalgebra satisfies each time the universal property of quadratic coalgebras.

The commutativity of the various coherence diagrams for the symmetric monoidal functors on the top face come from the fact that their underlying functors on the category  $(\text{GRVECT}, \oplus)$  is either the identity, the degree shift, or the linear duality functor, which are symmetric monoidal. The bottom functors, namely the universal enveloping algebra functor, the inclusions, and the linear duality functor, are known to be symmetric monoidal. Finally, it is straightforward to check the various coherence diagrams satisfied by the symmetric monoidal functors  $A, S, L, S^c$ , and  $A^c$ .  $\square$

### 3 2-MONOIDAL STRUCTURES UPON QUADRATIC DATA

#### 3.1 NOTATION AND SETTING

As in [Man88, Man18], we will identify the category of quadratic algebras over  $K$  with the category QD of quadratic data, whose elements will be denoted by  $A = (A_1, R(A))$ .

In [Val08, Section 1], we introduced the notion of a *lax 2-monoidal category* as a category  $C$  endowed with two monoidal products  $\boxtimes$  and  $\otimes$  such that the functor  $\otimes : (C, \boxtimes)^2 \rightarrow (C, \boxtimes)$  is lax monoidal functor. Therefore, such a structure amounts to a natural transformation, called the *interchange law*,

$$\varphi_{AA'BB'} : (A \otimes A') \boxtimes (B \otimes B') \rightarrow (A \boxtimes B) \otimes (A' \boxtimes B')$$

satisfying the usual coherence diagrams. (Notice that the natural transformation might not be made up of isomorphisms, on the contrary to the strong

monoidal functors considered in Section 2.5.) Dually, the notion of a *colax 2-monoidal category* is obtained by a colax (or oplax) monoidal functor  $\otimes : (\mathcal{C}, \boxtimes)^2 \rightarrow (\mathcal{C}, \boxtimes)$ , that is by a natural transformation

$$\psi_{AA'BB'} : (A \boxtimes B) \otimes (A' \boxtimes B') \rightarrow (A \otimes A') \boxtimes (B \otimes B') .$$

satisfying the opposite coherent diagrams. A category equipped with two monoidal structures carrying two compatible lax and colax 2-monoidal structures is called a *2 monoidal-category*.

REMARK 3.1.1. In the [Val08, Proposition 2] (and its arXiv version as well), there are two misprints in the commutative triangle expressing compatibility with unit morphisms. First, the product  $(I \otimes A) \boxtimes (I \otimes A')$  should be replaced by  $(I \boxtimes A) \otimes (I \boxtimes A')$ . Second,  $F(A)$  should be replaced by  $F(A')$ .

### 3.2 THE INTERCHANGE LAWS IN $(\text{GRVECT}, \otimes, \oplus)$

The category  $\text{GRVECT}$  is endowed with two simple monoidal structures: *tensor product*  $\otimes$  over  $K$  and *direct sum*  $\oplus$ . They have unit objects and standard associativity morphisms. We may even assume these structures to be *strict* ones, and sometimes will do it for simplicity.

The interchange law in this context must be a natural monoidal transformation

$$\varphi_{AA'BB'} : (A \oplus A') \otimes (B \oplus B') \rightarrow (A \otimes B) \oplus (A' \otimes B') \quad (1)$$

with the following notation change:  $(\boxtimes, \otimes)$  are replaced here respectively by  $(\otimes, \oplus)$ .

Explicitly, we have natural identifications

$$(A \oplus A') \otimes (B \oplus B') \cong (A \otimes B) \oplus (A' \otimes B) \oplus (A \otimes B') \oplus (A' \otimes B') , \quad (2)$$

and we define the interchange law (1) as the projection  $pr_{14}$  of r.h.s. of (2) onto the sum of its first and fourth direct summand

$$pr_{14} : (A \oplus A') \otimes (B \oplus B') \rightarrow (A \otimes B) \oplus (A' \otimes B') . \quad (3)$$

In the other way round, the inclusion of the first and fourth direct summand defines the natural transformation

$$\psi_{AA'BB'} : (A \otimes B) \oplus (A' \otimes B') \rightarrow (A \oplus A') \otimes (B \oplus B') .$$

The compatibility of this projection and this inclusion with associativity and unit morphisms for  $\otimes$  defined in [Val08, Proposition 2] quoted above can be checked in a straightforward way. So the data  $(\text{GRVECT}, \otimes, \oplus, \varphi, \psi)$  form a 2-monoidal category.

Now we will state and prove the main result of this section.

3.3 FINAL NOTATION CHANGE: LIFTING (1) TO QUADRATIC DATA

Let us now reinterpret the players of the interchange law (1) as the first components of objects of QD, that is  $A = (A_1, R(A))$  etc. We lift the monoidal structures  $(\otimes, \oplus)$  on 1-components of quadratic data to monoidal structures in QD denoted in [Man88] as  $(\bullet, \underline{\otimes})$ , where

$$(A_1, R(A)) \bullet (B_1, R(B)) := (A_1 \otimes B_1, S_{(23)}(R(A) \otimes R(B))) .$$

Here  $R(A) \otimes R(B) \subset A_1^{\otimes 2} \otimes B_1^{\otimes 2}$ , and  $S_{(23)}$  interchanges the middle two components of the tensor product, so that the result lands in  $(A_1 \otimes B_1)^{\otimes 2}$  as it should be. For a description of  $\underline{\otimes}$  refer to the beginning of Section 2.4.

PROPOSITION 3.3.1. *The interchange morphism  $pr_{14}$  applied to 1-components of*

$$(A \underline{\otimes} A') \bullet (B \underline{\otimes} B') \rightarrow (A \bullet B) \underline{\otimes} (A' \bullet B') \tag{4}$$

*and then lifted to the tensor squares of these 1-components as  $pr_{14}^{\otimes 2}$ , sends the subspace of relations of the l.h.s. to the subspace of relations of the r.h.s.:*

$$R((A \underline{\otimes} A') \bullet (B \underline{\otimes} B')) \rightarrow R((A \bullet B) \underline{\otimes} (A' \bullet B')) \tag{5}$$

*and thus lifts to an interchange morphism in QD. The quadruple  $(\text{QD}, \bullet, \underline{\otimes}, \varphi)$  forms a lax 2-monoidal category.*

*Proof.* (i) *Preparation.* Since from now on the four graded vector spaces in (1)–(4) will be 1-components of quadratic data, we will add the subscript 1 in their notation. According to the definitions (5) and (7) on p. 19 of [Man88], the source of the arrow (5) can be explicitly written as

$$R((A \underline{\otimes} A') \bullet (B \underline{\otimes} B')) = S_{(23)}(\{R(A) \oplus [A_1, A'_1]_+ \oplus R(A')\} \otimes \hat{E} \{R(B) \oplus [B_1, B'_1]_+ \oplus R(B')\}) . \tag{6}$$

On the other hand, the target becomes

$$R((A \bullet B) \underline{\otimes} (A' \bullet B')) = S_{(23)}(R(A) \otimes R(B) \oplus [A_1 \otimes B_1, A'_1 \otimes B'_1]_+ \oplus S_{(23)}(R(A') \otimes R(B'))) . \tag{7}$$

On the respective 1-components of (4), the kernel of

$$pr_{14} : (A_1 \oplus A'_1) \otimes (B_1 \oplus B'_1) \rightarrow A_1 \otimes B_1 \oplus A'_1 \otimes B'_1$$

is

$$A_1 \otimes B'_1 \oplus A'_1 \otimes B_1 . \tag{8}$$

Therefore, whenever we apply  $pr_{14}^{\otimes 2}$  to a tensor monomial of degree four, whose first two or last two divisors (or both) look like  $a \otimes b'$  or  $a' \otimes b$ , then it is annihilated.

For brevity, we will call such monomials in  $[(A_1 \oplus A'_1) \otimes (B_1 \oplus B'_1)]^{\otimes 2}$  *vanishing ones*.

In the following sections of the proof we will apply this remark successively to various summands of (6):

$$S_{(23)}(\{R(A) \oplus [A_1, A'_1]_+ \oplus R(A')\} \otimes \{R(B) \oplus [B_1, B'_1]_+ \oplus R(B')\}) .$$

(ii) *Summands annihilated by  $pr_{14}^{\otimes 2}$ .*

(a) First, check that the whole subspace  $S_{(23)}(R(A) \otimes R(B'))$  is annihilated. In fact any element of it is a linear combination of vanishing monomials because after interchanging two middle elements in  $a_1 \otimes a_2 \otimes b'_1 \otimes b'_2$  where  $a_1, a_2 \in A_1$ ,  $b'_1, b'_2 \in B'_1$ , and both left half and right half binary products land in (8).

Essentially the same argument shows that  $S_{(23)}(R(A') \otimes R(B))$  is annihilated, and  $R(A') \otimes [B_1, B'_1]_+$  and  $[A_1, A'_1]_+ \otimes R(B')$  as well.

One can treat in the same way  $R(A) \otimes [B_1, B'_1]_+$  and  $[A_1, A'_1]_+ \otimes R(B)$ . The only difference is that after applying  $S_{(23)}$  to the respective monomials only *one half of the result*, either to the left, or to the right of the middle  $\otimes$  lands in the tensor square of (8).

(iii) *Summands upon which  $pr_{14}^{\otimes 2}$  is injective.* A direct observation shows that  $pr_{14}^{\otimes 2}$  restricted to  $S_{(23)}(R(A) \otimes R(B))$  is injective and in fact identifies it with the respective summand of (7).

Similarly,  $pr_{14}^{\otimes 2}$  identifies  $S_{(23)}(R(A') \otimes R(B'))$  identifies it with the respective summand of (7).

(iv) *Remaining terms.*

It remains to compare the terms

$$S_{(23)}([A_1, A'_1]_+ \otimes [B_1, B'_1]_+) \tag{9}$$

in the source of (5) with terms

$$[A_1 \otimes B_1, A'_1 \otimes B'_1]_+ \tag{10}$$

in its target.

The space (9) is spanned by linear combinations

$$\begin{aligned} & S_{(23)} \left( \left( a_i \otimes a'_j + (-1)^{|a_i||a'_j|} a'_j \otimes a_i \right) \otimes \left( b_k \otimes b'_l + (-1)^{|b_k||b'_l|} b'_l \otimes b_k \right) \right) \\ &= (-1)^{|a'_j||b_k|} a_i \otimes b_k \otimes a'_j \otimes b'_l + (-1)^{|b'_l|(|a'_j|+|b_k|)} a_i \otimes b'_l \otimes a'_j \otimes b_k \\ &+ (-1)^{|a_i|(|a'_j|+|b_k|)} a'_j \otimes b_k \otimes a_i \otimes b'_l + (-1)^{|b_k||b'_l|+|a_i|(|a'_j|+|b'_l|)} a'_j \otimes b'_l \otimes a_i \otimes b_k, \end{aligned} \tag{11}$$

where

$$a_i \in A_1, a'_j \in A'_1, b_k \in B_1, b'_l \in B'_1 .$$

Two middle terms in (11) are vanishing ones.

With the same notation, (10) is spanned by linear combinations

$$(-1)^{|a'_j||b_k|} a_i \otimes b_k \otimes a'_j \otimes b'_l + (-1)^{|b_k||b'_l|+|a_i|(|a'_j|+|b'_l|)} a'_j \otimes b'_l \otimes a_i \otimes b_k ,$$

which are exactly images of sums of the two remaining terms of (11) after application of  $pr_{14}^{\otimes 2}$ .

This completes the proof of the Proposition 3.3.1. □

### 3.4 THE DUAL PICTURE

COROLLARY 3.4.1. *The quadruple  $(\text{QD}, \otimes, \circ, \psi)$  is a lax 2-monoidal category as well.*

*Proof.* In the case when all involved quadratic data are finite-dimensional, the interchange law in  $(\text{QD}, \otimes, \circ, \psi)$  can be formally obtained by applying the linear duality functor  $*$  to the diagrams (4) and (5). Similarly, the commutativity of all relevant diagrams (compatibility with associativity of  $\otimes$  and with unity for  $\circ$ ) follows by duality from the respective facts for  $(\text{QD}, \bullet, \underline{\otimes}, \varphi)$ .

However, the statement itself of Corollary 3.4.1 remains true even without assumption of finite-dimensionality: to prove it one should develop detailed arguments parallel to those in given in the proof of Proposition 3.3.1.

Below we will only sketch the check that interchange laws in  $(\text{QD}, \otimes, \circ, \psi)$  and  $(\text{QD}, \bullet, \underline{\otimes}, \varphi)$  are  $*$ -dual.

Applying formally  $*$  to (4) and rewriting the left and right hand sides with the help of identifications, collected in [Man88, Section 3], especially in its subsection 5, we obtain a morphism

$$\{(A \bullet B) \underline{\otimes} (A' \bullet B')\}^* \rightarrow \{(A \underline{\otimes} A') \bullet (B \underline{\otimes} B')\}^* . \tag{12}$$

The l. h. s. of (12) can be rewritten as

$$(A \bullet B)^* \otimes (A' \bullet B')^* \cong (A^* \circ B^*) \otimes (A'^* \circ B'^*) .$$

Similarly, the r. h. s. of (12) is

$$(A \underline{\otimes} A')^* \circ (B \underline{\otimes} B')^* \cong (A^* \otimes A'^*) \circ (B^* \otimes B'^*) .$$

So finally (12) becomes

$$(A^* \circ B^*) \otimes (A'^* \circ B'^*) \rightarrow (A^* \otimes A'^*) \circ (B^* \otimes B'^*) . \tag{13}$$

Since  $*$  is a contravariant quasi-involution of QD that is,  $**$  is equivalent to  $\text{id}$ , (13) is the required interchange morphism, written for generic arguments. □

REMARK 3.4.2. Our two examples  $(\mathbf{QD}, \bullet, \underline{\otimes}, \varphi)$  and  $(\mathbf{QD}, \otimes, \circ, \psi)$  are lax 2-monoidal categories, but fail to be colax since the interchange laws  $\psi$  and  $\varphi$  from the 2-monoidal category  $\mathbf{GRV}\mathbf{ECT}$  do not lift to the appropriate level.

Several other pairs, consisting of  $\bullet$  and one of the monoidal structures from [Man88], are either simultaneously lax and colax, or neither lax/nor colax. These are less interesting cases. We will present in Section 4 their more interesting operadic versions.

### 3.5 APPLICATIONS

COROLLARY 3.5.1.

1. Let  $M, N$  be two monoids in  $\mathbf{QD}$  wrt the black product  $\bullet$ . Then  $M \underline{\otimes} N$  also has a natural structure of such a monoid.
2. Similarly, let  $M, N$  be two monoids in  $\mathbf{QD}$  wrt the tensor product  $\otimes$ . Then  $M \circ N$  also has a natural structure of such a monoid.

*Proof.* These statements are direct applications of the fact that lax monoidal functors preserve monoids. They are actually special cases of [Val08, Proposition 3], which was a motivation for the definition of the notion of lax 2-monoidal category.  $\square$

EXAMPLE 3.5.2. Let  $A := (A_1, R(A_1))$  be a quadratic data. The canonical map  $+$  :  $A_1 \oplus A_1 \rightarrow A_1$  induces a morphism of quadratic data  $A \otimes A \rightarrow A$  if and only if  $[A_1, A_1]_- \subset R(A_1)$ . Quadratic data satisfying this property actually form the image of the functor  $\mathcal{S}$  from symmetric quadratic data. Corollary 3.5.1 shows that their white product carries again a canonical  $\otimes$ -monoid structure.

This canonical  $\otimes$ -monoid structure on the quadratic data living in the image of the functor  $\mathcal{S}$  actually comes from the monoid structure on any symmetric quadratic data  $(V, R) \in (\mathbf{QD}^+, \vee)$  given by  $+$  :  $V \oplus V \rightarrow V$ . These two monoid structures induce respectively the concatenation product under the symmetric monoidal functor of algebraic realisations  $S(V, R)$  and  $A(V, R)$ .

## 4 2-MONOIDAL STRUCTURES UPON OPERADIC QUADRATIC DATA

### 4.1 OPERADS

Let us recall the *coordinate-free partial definition* of an operad. We denote by  $\mathbf{BIJ}$  the category of finite sets with bijections. Given any subset  $X \subset Y$ , we use the notation  $Y/X := (Y \setminus X) \sqcup \{*\}$ . Let  $(\mathbf{C}, \otimes, 1_{\mathbf{C}}, \alpha, \lambda, \rho, \tau)$  be a symmetric monoidal category.

DEFINITION 4.1.1 (Operad). An *operad* in  $\mathbf{C}$  is a presheaf  $\mathcal{P} : \mathbf{BIJ}^{\text{op}} \rightarrow \mathbf{C}$  endowed with partial operadic compositions  $\circ_{X \subset Y} : \mathcal{P}(Y/X) \otimes \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ , for any  $X \subset Y$ , and a unit  $\eta : 1_{\mathbf{C}} \rightarrow \mathcal{P}(\{*\})$  such that the following diagrams commute.

SEQUENTIAL AXIOM. For any  $X \subset Y \subset Z$ ,

$$\begin{array}{ccc}
 (\mathcal{P}(Z/Y) \otimes \mathcal{P}(Y/X)) \otimes \mathcal{P}(X) & \xrightarrow{\cong} & (\mathcal{P}((Z/X)/(Y/X)) \otimes \mathcal{P}(Y/X)) \otimes \mathcal{P}(X) \\
 \downarrow \alpha & & \downarrow \circ_{Y/X \subset Z/X} \otimes \text{id} \\
 \mathcal{P}(Z/Y) \otimes (\mathcal{P}(Y/X) \otimes \mathcal{P}(X)) & & \mathcal{P}(Z/X) \otimes \mathcal{P}(X) \\
 \downarrow \text{id} \otimes \circ_{X \subset Y} & & \downarrow \circ_{X \subset Z} \\
 \mathcal{P}(Z/Y) \otimes \mathcal{P}(Y) & \xrightarrow{\circ_{Y \subset Z}} & \mathcal{P}(Z) .
 \end{array}$$

PARALLEL AXIOM. For any  $X \sqcup Y \subset Z$ ,

$$\begin{array}{ccc}
 \mathcal{P}(((Z/X)/Y)) \otimes (\mathcal{P}(X) \otimes \mathcal{P}(Y)) & \xrightarrow{\text{id} \otimes \tau} & \mathcal{P}(((Z/X)/Y)) \otimes (\mathcal{P}(Y) \otimes \mathcal{P}(X)) \\
 \downarrow \alpha^{-1} & & \downarrow \alpha^{-1} \\
 (\mathcal{P}(((Z/X)/Y)) \otimes \mathcal{P}(X)) \otimes \mathcal{P}(Y) & & (\mathcal{P}(((Z/X)/Y)) \otimes \mathcal{P}(Y)) \otimes \mathcal{P}(X) \\
 \downarrow \cong & & \downarrow \circ_{Y \subset Z/X} \otimes \text{id} \\
 (\mathcal{P}(((Z/Y)/X)) \otimes \mathcal{P}(X)) \otimes \mathcal{P}(Y) & & \mathcal{P}(Z/X) \otimes \mathcal{P}(X) \\
 \downarrow \circ_{X \subset Z/Y} \otimes \text{id} & & \downarrow \circ_{X \subset Z} \\
 \mathcal{P}(Z/Y) \otimes \mathcal{P}(Y) & \xrightarrow{\circ_{Y \subset Z}} & \mathcal{P}(Z) .
 \end{array}$$

LEFT/RIGHT UNITAL AXIOMS. For any  $X \in \text{BIJ}$  and any  $x \in X$ ,

$$\begin{array}{ccc}
 1_{\mathbb{C}} \otimes \mathcal{P}(X) & \xrightarrow{\eta \otimes \text{id}} & \mathcal{P}(\{*\}) \otimes \mathcal{P}(X) \\
 \downarrow \lambda & & \downarrow \cong \\
 \mathcal{P}(X) & \xleftarrow{\circ_{X \subset X}} & \mathcal{P}(X/X) \otimes \mathcal{P}(X) \\
 \\ 
 \mathcal{P}(X) \otimes 1_{\mathbb{C}} & \xrightarrow{\text{id} \otimes \eta} & \mathcal{P}(X) \otimes \mathcal{P}(\{*\}) \\
 \downarrow \rho & & \downarrow \cong \\
 \mathcal{P}(X) & \xleftarrow{\circ_{\{x\} \subset X}} & \mathcal{P}(X/\{x\}) \otimes \mathcal{P}(\{x\}) .
 \end{array}$$

TOP EQUIVARIANCE. For any subset  $X \subset Y$  and any bijection  $f : X \rightarrow X$ , we consider the induced bijection  $\bar{f} : Y \rightarrow Y$ , which leaves the elements of  $Y \setminus X$  invariant,

$$\begin{array}{ccc}
 \mathcal{P}(Y/X) \otimes \mathcal{P}(X) & \xrightarrow{\circ_{X \subset Y}} & \mathcal{P}(Y) \\
 \downarrow \text{id} \otimes \mathcal{P}(f) & & \downarrow \mathcal{P}(\bar{f}) \\
 \mathcal{P}(Y/X) \otimes \mathcal{P}(X) & \xrightarrow{\circ_{X \subset Y}} & \mathcal{P}(Y) .
 \end{array}$$

BOTTOM EQUIVARIANCE. For any subset  $X \subset Y$  and any bijection  $f : Y/X \rightarrow Y/X$ , we consider  $\tilde{Y} := Y$  if  $f(*) = *$  and  $\tilde{Y} := Y \setminus \{f(*)\} \sqcup \{*\}$  otherwise. We also consider the bijection  $\tilde{Y} \rightarrow Y$  which sends  $*$  to  $f(*)$  and is equal to the identity otherwise. We denote by  $\mathring{f} : \tilde{Y}/X \rightarrow Y/X$  the induced bijection. Finally, we denote by  $\tilde{f} : Y \rightarrow \tilde{Y}$  the bijection which coincides with  $f$  except for the assignment  $f(*) \mapsto *$ .

$$\begin{array}{ccc}
 \mathcal{P}(Y/X) \otimes \mathcal{P}(X) & \xrightarrow{\mathcal{P}(\mathring{f}) \otimes \text{id}} & \mathcal{P}(\tilde{Y}/X) \otimes \mathcal{P}(X) & \xrightarrow{\circ_{X \subset \tilde{Y}}} & \mathcal{P}(\tilde{Y}) \\
 \downarrow \mathcal{P}(f) \otimes \text{id} & & & & \downarrow \mathcal{P}(\tilde{f}) \\
 \mathcal{P}(Y/X) \otimes \mathcal{P}(X) & \xrightarrow{\circ_{X \subset Y}} & & & \mathcal{P}(Y) .
 \end{array}$$

The skeletal category of BIJ is the groupoid  $\mathbb{S}$  whose objects are the sets  $\{1, \dots, n\}$ , for  $n \in \mathbb{N}$ , and whose morphisms are the elements of the symmetric groups  $\mathbb{S}_n$ . A presheaf on BIJ is thus equivalent to a collection  $\{\mathcal{P}(n)\}_{n \in \mathbb{N}}$  of right  $\mathbb{S}_n$ -modules, see [KM01, Section 1.1]. In these terms, the above structure of an operad is equivalent to partial composition products  $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \rightarrow \mathcal{P}(n+m-1)$ , for  $1 \leq i \leq n$ , and a unit map  $\eta : 1_{\mathbb{C}} \rightarrow \mathcal{P}(1)$  satisfying the analogous axioms, given in [LV12, Section 5.3.4] for example.

4.2 OPERADIC QUADRATIC DATA

The notions of black and white products were generalised to binary quadratic operads in [GK94, GK95] and then to quadratic operads (and cooperads) in [Val08].

In this section, we will work with the following analogous operadic notion of quadratic data.

DEFINITION 4.2.1 (Binary operadic quadratic data). A *binary operadic quadratic data* is a pair  $\mathcal{A} = (\mathcal{A}_1, R(\mathcal{A}))$  where  $\mathcal{A}_1$  is a graded  $K$ -linear representation of the symmetric group  $\mathbb{S}_2$  (that is, an  $\mathbb{S}_2$ -module) and  $R(\mathcal{A})$  is a  $\mathbb{S}_3$ -submodule of the part of arity 3 of the free operad  $\mathcal{T}(\mathcal{A}_1)$  generated by  $\mathcal{A}_1$ . A morphism  $f : (\mathcal{A}_1, R(\mathcal{A})) \rightarrow (\mathcal{B}_1, R(\mathcal{B}))$  is a map of  $\mathbb{S}_2$ -modules  $\mathcal{A}_1 \rightarrow \mathcal{B}_1$  whose extension  $\mathcal{T}(f)$  restricted to the arity 3 part of  $\mathcal{T}(\mathcal{A}_1)$  sends  $R(\mathcal{A})$  to  $R(\mathcal{B})$ . This category is denoted by BOQD.

If we assume additionally that our graded  $\mathbb{S}_2$ -modules are finite-dimensional, we can imitate the definition of the linear dualisation functor in our new context as the functor

$$\mathcal{A} = (\mathcal{A}_1, R(\mathcal{A})) \mapsto \mathcal{A}^* := (\mathcal{A}_1^*, R(\mathcal{A})^\perp) .$$

(Notice that this functor was denoted by  $!$  in [Val08, Section 2].) Otherwise, we can drop the finite-dimensionality restriction, and consider the Koszul dual functor  $i$  which produces quadratic cooperads, see [LV12, Section 7.1].

4.3 THE INTERCHANGE LAWS ON THE CATEGORY OF GRADED  $\mathbb{S}_2$ -MODULES

The category of graded  $\mathbb{S}_2$ -modules is endowed with two monoidal structures: the (Hadamard) *tensor product*  $\otimes$  and the *direct sum*  $\oplus$ , see for instance [KM01]. The unit of the former one is given by the trivial representation of  $\mathbb{S}_2$  and the unit of the latter one is given by the  $\mathbb{S}_2$ -module  $\{0\}$ . We refer the reader to Section 1.4 and Appendix A of [Val08] for more details.

We will consider the following two interchange maps:

$$\varphi_{\mathcal{A}_1 \mathcal{A}'_1 \mathcal{B}_1 \mathcal{B}'_1} : (\mathcal{A}_1 \oplus \mathcal{A}'_1) \otimes (\mathcal{B}_1 \oplus \mathcal{B}'_1) \rightarrow (\mathcal{A}_1 \otimes \mathcal{B}_1) \oplus (\mathcal{A}'_1 \otimes \mathcal{B}'_1) \quad (14)$$

and

$$\psi_{\mathcal{A}_1 \mathcal{A}'_1 \mathcal{B}_1 \mathcal{B}'_1} : (\mathcal{A}_1 \otimes \mathcal{B}_1) \oplus (\mathcal{A}'_1 \otimes \mathcal{B}'_1) \rightarrow (\mathcal{A}_1 \oplus \mathcal{A}'_1) \otimes (\mathcal{B}_1 \oplus \mathcal{B}'_1) . \quad (15)$$

As in Section 3.2 above, they can be naturally seen in the context of canonical identifications

$$(\mathcal{A}_1 \oplus \mathcal{A}'_1) \otimes (\mathcal{B}_1 \oplus \mathcal{B}'_1) \cong (\mathcal{A}_1 \otimes \mathcal{B}_1) \oplus (\mathcal{A}'_1 \otimes \mathcal{B}_1) \oplus (\mathcal{A}_1 \otimes \mathcal{B}'_1) \oplus (\mathcal{A}'_1 \otimes \mathcal{B}'_1) .$$

Namely, the law  $\varphi$  is the projection  $pr_{14}$  onto the first and the fourth summand of the right hand side, whereas the law  $\psi$  is the injection  $inj_{14}$  of the sum of the first and the fourth summands in the right hand side.

Together with the two interchange laws  $\varphi$  and  $\psi$ , we obtain a 2-monoidal structure on the category of graded  $\mathbb{S}_2$ -modules.

4.4 THE FIRST PRODUCT ON THE CATEGORY OF BINARY OPERADIC QUADRATIC DATA

We will now start preparing the construction of the black product on the category BOQD. The central piece of the construction is the analog of the map  $f^{\otimes 2}$  from Section 2.2, which was denoted  $\mathcal{T}(f)$  in the above definition of binary operadic quadratic data.

Let  $(\mathcal{A}_1, R(\mathcal{A}))$  be an object of BOQD and let  $a \in \mathcal{A}_1$ . Since the arity of  $a$  is 2, we will temporarily use the notation  $a(x, y)$  where  $x, y$  run over elements of an arbitrary algebra over the operad generated by  $(\mathcal{A}_1, R(\mathcal{A}))$ . Similarly, elements  $c$  of arity 3 of such an operad can be written as  $c(x, y, z)$  etc.

With this notation, there exists a basis of  $\mathcal{T}(\mathcal{A}_1)(3)$  consisting of bilinear expressions in  $a, a' \in \mathcal{A}_1$

$$\tau_1(a, a') , \quad \tau_2(a, a') , \quad \tau_3(a, a') ,$$

such that

$$\begin{aligned} \tau_1(a, a')(x, y, z) &:= a(a'(x, y), z) , \\ \tau_2(a, a')(x, y, z) &:= a(a'(y, z), x) , \\ \tau_3(a, a')(x, y, z) &:= a(a'(z, x), y) . \end{aligned}$$

This is a rewriting of the definition in [Val08, Section 4], where the language of planar rooted trees is used.

Now, for two binary operadic quadratic data  $(\mathcal{A}_1, R(\mathcal{A}))$  and  $(\mathcal{B}_1, R(\mathcal{B}))$ , we can calculate in terms of these bases the map of  $\mathbb{S}_3$ -modules

$$\Psi_{\mathcal{A}_1\mathcal{B}_1} : \mathcal{T}(\mathcal{A}_1(3)) \otimes \mathcal{T}(\mathcal{B}_1(3)) \otimes \text{sgn}_{\mathbb{S}_3} \longrightarrow \mathcal{T}(\mathcal{A}_1 \otimes \mathcal{B}_1 \otimes \text{sgn}_{\mathbb{S}_2})(3)$$

introduced in [GK94, GK95]; our presentation is due to [Val08].

Namely,

$$\Psi_{\mathcal{A}_1\mathcal{B}_1}(\tau_i(a, a') \otimes \tau_j(b, b')) := \delta_{ij} \tau_i(a \otimes b, a' \otimes b') .$$

The main statement of this subsection is the following one.

LEMMA 4.4.1. *There exists a well defined monoidal structure on BOQD, called the black product  $\bullet$ , given on objects by the formula*

$$\mathcal{A} \bullet \mathcal{B} := (\mathcal{A}_1 \otimes \mathcal{B}_1, \Psi_{\mathcal{A}_1\mathcal{B}_1}(R(\mathcal{A}) \otimes R(\mathcal{B}))) . \tag{16}$$

#### 4.5 THE SECOND PRODUCT ON THE CATEGORY OF BINARY QUADRATIC OPERADS

We will now define the product of binary operadic quadratic data  $\underline{\otimes}$  which will serve as an analog of the product  $\underline{\otimes}$  on the category of quadratic data.

First of all, for a graded  $\mathbb{S}_2$ -module  $\mathcal{A}_1$ , denote by  $\mathcal{A}_1^+$ , resp.  $\mathcal{A}_1^-$ , the submodule of  $\mathbb{S}_2$ -invariant elements, resp. (2,1)-antiinvariant elements of  $\mathcal{A}_1$ .

Furthermore, denote by  $\{\mathcal{A}_1, \mathcal{B}_1\}$  the sub- $\mathbb{S}_3$ -module of  $\mathcal{T}(\mathcal{A}_1 \oplus \mathcal{B}_1)(3)$  spanned by the elements  $\tau_1(a, b) + \tau_1(b, a)$ , where either  $(a, b) \in \mathcal{A}_1^+ \times \mathcal{B}_1^+$ , or  $(a, b) \in \mathcal{A}_1^- \times \mathcal{B}_1^-$ . Finally, put

$$\mathcal{A} \underline{\otimes} \mathcal{B} := (\mathcal{A}_1 \oplus \mathcal{B}_1, R(\mathcal{A}) \oplus \{\mathcal{A}_1, \mathcal{B}_1\} \oplus R(\mathcal{B})) .$$

We can now state and prove the analog of Proposition 3.3.1 in the operadic setting.

PROPOSITION 4.5.1. *The interchange law  $\varphi$  on the category of graded  $\mathbb{S}_2$ -modules induces morphisms in the category of binary operadic quadratic data*

$$\varphi_{\mathcal{A}\mathcal{A}'\mathcal{B}\mathcal{B}'} : (\mathcal{A} \underline{\otimes} \mathcal{A}') \bullet (\mathcal{B} \underline{\otimes} \mathcal{B}') \rightarrow (\mathcal{A} \bullet \mathcal{B}) \underline{\otimes} (\mathcal{A}' \bullet \mathcal{B}')$$

which define on  $(\text{BOQD}, \bullet, \underline{\otimes}, \varphi)$  the structure of a lax 2-monoidal category and on  $(\text{BOQD}, \underline{\otimes}, \bullet, \varphi)$  the structure of a colax 2-monoidal category.

*Proof.* As in the case of algebras (Proposition 3.3.1), we have to check that the morphism of graded  $\mathbb{S}_2$ -modules (14) induces a well defined morphism of graded  $\mathbb{S}_3$ -modules of relations

$$R((\mathcal{A} \underline{\otimes} \mathcal{A}') \bullet (\mathcal{B} \underline{\otimes} \mathcal{B}')) \rightarrow R((\mathcal{A} \bullet \mathcal{B}) \underline{\otimes} (\mathcal{A}' \bullet \mathcal{B}'))$$

The left hand side can be rewritten as

$$\Psi((R(\mathcal{A}) \oplus \{\mathcal{A}_1, \mathcal{A}'_1\} \oplus R(\mathcal{A}')) \otimes (R(\mathcal{B}) \oplus \{\mathcal{B}_1, \mathcal{B}'_1\} \oplus R(\mathcal{B}'))),$$

and the right hand side as

$$\Psi(R(\mathcal{A}) \otimes R(\mathcal{B})) \oplus \{\mathcal{A}_1 \otimes \mathcal{B}_1, \mathcal{A}'_1 \otimes \mathcal{B}'_1\} \oplus \Psi(R(\mathcal{A}') \otimes R(\mathcal{B}')) .$$

From (16), using the same arguments as in the case of quadratic data we conclude that the following summands of the left hand side get annihilated:

$$\begin{aligned} &\Psi(R(\mathcal{A}) \otimes \{\mathcal{B}_1, \mathcal{B}'_1\}), \quad \Psi(R(\mathcal{A}) \otimes R(\mathcal{B}')), \quad \Psi(\{\mathcal{A}_1, \mathcal{A}'_1\} \otimes R(\mathcal{B})), \\ &\Psi(\{\mathcal{A}_1, \mathcal{A}'_1\} \otimes R(\mathcal{B}')), \quad \Psi(R(\mathcal{A}') \otimes R(\mathcal{B})), \quad \Psi(R(\mathcal{A}') \otimes \{\mathcal{B}_1, \mathcal{B}'_1\}). \end{aligned}$$

The two summands in the left hand side

$$\Psi(R(\mathcal{A}) \otimes R(\mathcal{B})) \quad \text{and} \quad \Psi(R(\mathcal{A}') \otimes R(\mathcal{B}'))$$

map identically to the first and the last summands of the right hand side respectively.

It remains to show that the summand  $\Psi(\{\mathcal{A}_1, \mathcal{A}'_1\} \otimes \Psi(\{\mathcal{B}_1, \mathcal{B}'_1\}))$  lands in

$$\{\mathcal{A}_1, \mathcal{B}_1\} \otimes \{\mathcal{A}'_1, \mathcal{B}'_1\}.$$

From the definition of the brackets  $\{\mathcal{A}_1, \mathcal{B}_1\}$  given in Section 4.5, it follows that the graded  $\mathbb{S}_3$ -module  $\{\mathcal{A}_1, \mathcal{A}'_1\}$  is linearly spanned by the expressions  $\tau_i(a, a')$  where  $i = 1, 2, 3$ , and  $a \in \mathcal{A}_1, a' \in \mathcal{A}'_1$  are either simultaneously  $\mathbb{S}_2$ -even, or simultaneously  $\mathbb{S}_2$ -odd. This comes from the following facts

$$\begin{aligned} \tau_1(a, a')^{(12)} &= \pm \tau_1(a, a'), \quad \tau_2(a, a')^{(12)} = \pm \tau_3(a, a'), \\ \tau_1(a, a')^{(123)} &= \tau_2(a, a'), \quad \tau_2(a, a')^{(123)} = \tau_3(a, a'). \end{aligned}$$

Moreover,

$$\begin{aligned} &\Psi((\tau_i(a, a') + \tau_i(a', a)) \otimes (\tau_i(b, b') + \tau_i(b', b))) \\ &= (-1)^{|a'| |b|} \tau_i(a \otimes b, a' \otimes b') \\ &\quad + (-1)^{|a| |b'|} \tau_i(a' \otimes b', a \otimes b) \in \{\mathcal{A}_1 \otimes \mathcal{B}_1, \mathcal{A}'_1 \otimes \mathcal{B}'_1\}, \end{aligned}$$

whereas for  $i \neq j$ ,

$$\Psi((\tau_i(a, a') + \tau_i(a', a)) \otimes (\tau_j(a, a') + \tau_j(a', a))) = 0.$$

This concludes the proof. □

4.6 WHITE PRODUCT IN BOQD, YET ANOTHER PRODUCT, AND THE INTERCHANGE LAW  $\psi$

Similarly to what happens in the category of quadratic data, we can introduce the following white product in BOQD:

$$\mathcal{A} \circ \mathcal{B} := (\mathcal{A}_1 \otimes \mathcal{B}_1, \Phi^{-1}(R(\mathcal{A}) \otimes \mathcal{T}(\mathcal{B}_1)(3) + \mathcal{T}(\mathcal{A}_1)(3) \otimes R(\mathcal{B}))) ,$$

where  $\Phi$  is the natural map

$$\Phi = \Phi_{\mathcal{A}_1 \mathcal{B}_1} := \mathcal{T}(\mathcal{A}_1 \otimes \mathcal{B}_1)(3) \rightarrow \mathcal{T}(\mathcal{A}_1)(3) \otimes \mathcal{T}(\mathcal{B}_1)(3) ,$$

which duplicates the underlying tree. Black and white products are also related to each other by the operadic duality functor  $*$ .

Similarly, the product  $\underline{\circ}$  defined in Section 4.5 is sent to the following product  $\circ$  under the operadic Koszul duality functor  $*$ . We first consider the sub- $\mathbb{S}_3$ -module

$$[\mathcal{A}_1, \mathcal{B}_1] \subset \mathcal{T}(\mathcal{A}_1 \oplus \mathcal{B}_1)(3),$$

spanned by the elements  $\tau_1(a, b) - \tau_1(b, a)$  whenever  $a, b$  are simultaneously  $\mathbb{S}_2$ -even or odd, and in addition by the expressions  $\tau_1(a, b)$  when one of the arguments  $a, b$  is even and another is odd. Then, we define

$$\mathcal{A} \circledast \mathcal{B} := (\mathcal{A}_1 \oplus \mathcal{B}_1, R(\mathcal{A}) \oplus [\mathcal{A}_1, \mathcal{B}_1] \oplus R(\mathcal{B})) .$$

These two monoidal products are related by the interchange law  $\psi$  induced by (15).

PROPOSITION 4.6.1. *The interchange law  $\psi$  in the category of graded  $\mathbb{S}_2$ -modules lifts to morphisms in BOQD*

$$\psi_{\mathcal{A}\mathcal{A}'\mathcal{B}\mathcal{B}'} : (\mathcal{A} \circ \mathcal{A}') \circledast (\mathcal{B} \circ \mathcal{B}') \rightarrow (\mathcal{A} \circledast \mathcal{B}) \circ (\mathcal{A}' \circ \mathcal{B}')$$

which make  $(\text{BOQD}, \circledast, \circ, \psi)$  a lax 2-monoidal category and  $(\text{BOQD}, \circ, \circ, \psi)$  a colax 2-monoidal category.

*Proof.* As in the quadratic data case, this proposition is Koszul dual to Proposition 4.5.1 under finite dimensional assumptions. However, it holds in the general case by direct inspection. □

4.7 APPLICATIONS

COROLLARY 4.7.1.

1. Let  $M, N$  be two monoids in BOQD with respect to the black product  $\bullet$  (resp. the  $\circledast$  product). Then  $M \circledast N$  (resp.  $M \circ N$ ) also has a natural structure of a  $\bullet$ -monoid (resp. a  $\circledast$ -monoid).
2. Similarly, let  $M, N$  be two comonoids in BOQD with respect to the  $\circledast$  product (resp. the  $\circ$  product). Then  $M \bullet N$  (resp.  $M \circ N$ ) also has a natural structure of  $\circledast$ -comonoid (respectively  $\circ$ -comonoid).

*Proof.* Again the proof relies on the fact that lax monoidal functors preserve monoids and that colax monoidal functors preserve comonoids.  $\square$

COROLLARY 4.7.2.

1. Let  $\mathcal{P}, \mathcal{Q}$  be two operads in the symmetric monoidal category  $(\mathbf{QD}, \bullet)$ . Then their arity-wise  $\underline{\otimes}$ -product  $(\mathcal{P} \underline{\otimes} \mathcal{Q})(n) := \mathcal{P}(n) \underline{\otimes} \mathcal{Q}(n)$  is again an operad in  $(\mathbf{QD}, \bullet)$ .
2. Let  $\mathcal{P}, \mathcal{Q}$  be two operads in the symmetric monoidal category  $(\mathbf{QD}, \otimes)$ . Then their arity-wise white product  $(\mathcal{P} \circ \mathcal{Q})(n) := \mathcal{P}(n) \circ \mathcal{Q}(n)$  is again an operad in  $(\mathbf{QD}, \otimes)$ .

*Proof.* The statement of Proposition 3.3.1 actually says the functor  $\underline{\otimes}$  is a lax monoidal functor from  $(\mathbf{QD}, \bullet)^2$  to  $(\mathbf{QD}, \bullet)$ . It is straightforward to see that it is also symmetric. The first statement thus follows from Proposition 5.1.2. The second statement is proved in the way with the lax symmetric monoidal functor  $\circ$  from  $(\mathbf{QD}, \otimes)^2$  to  $(\mathbf{QD}, \otimes)$  of Corollary 3.4.1.  $\square$

This latter construction can be applied to the various examples of operads that we will give in Section 5.

REMARK 4.7.3. This result shows that one can refine the theory of 2-monoidal categories developed in [Val08]: one can define a notion of a *symmetric* 2-monoidal category by requiring that the structural interchange law be a *symmetric* monoidal functor. The present examples given in this paper will actually fall into this case; they provide us with symmetric 2-monoidal categories. We leave the details to the interested reader.

#### 4.8 SOME MORE MONOIDAL STRUCTURES AND KOSZUL DUALITIES

As in the case of quadratic data, mentioned in the last lines of Section 3, one can introduce several more pairs of monoidal structures in the context of binary operadic quadratic data. Here is a list of possibilities in BOQD, including the ones we have already considered.

We denote by  $\mathcal{A}_1 \circ_1 \mathcal{B}_1$  the sub- $\mathbb{S}_3$ -module of  $\mathcal{T}(\mathcal{A}_1 \oplus \mathcal{B}_1)(3)$  spanned by  $\tau_1(b, a)$ . We put

$$\begin{aligned} \mathcal{A} \vee \mathcal{B} &:= (\mathcal{A}_1 \oplus \mathcal{B}_1, R(\mathcal{A}) \oplus R(\mathcal{B})) , \\ \mathcal{A} \oplus \mathcal{B} &:= (\mathcal{A}_1 \otimes \mathcal{B}_1, R(\mathcal{A}) \oplus \mathcal{A}_1 \circ_1 \mathcal{B}_1 \oplus \mathcal{B}_1 \circ_1 \mathcal{A}_1 \oplus R(\mathcal{B})) , \\ \mathcal{A} \triangleleft \mathcal{B} &:= (\mathcal{A}_1 \oplus \mathcal{B}_1, R(\mathcal{A}) \oplus \mathcal{B}_1 \circ_1 \mathcal{A}_1 \oplus R(\mathcal{B})) , \\ \mathcal{A} \triangleright \mathcal{B} &:= (\mathcal{A}_1 \otimes \mathcal{B}_1, R(\mathcal{A}) \oplus \mathcal{A}_1 \circ_1 \mathcal{B}_1 \oplus R(\mathcal{B})) , \\ \mathcal{A} \underline{\otimes} \mathcal{B} &:= (\mathcal{A}_1 \oplus \mathcal{B}_1, R(\mathcal{A}) \oplus \{\mathcal{A}_1, \mathcal{B}_1\} \oplus R(\mathcal{B})) , \\ \mathcal{A} \odot \mathcal{B} &:= (\mathcal{A}_1 \oplus \mathcal{B}_1, R(\mathcal{A}) \oplus [\mathcal{A}_1, \mathcal{B}_1] \oplus R(\mathcal{B})) . \end{aligned}$$

PROPOSITION 4.8.1.

1. Let  $\mathcal{P}_{\mathcal{A}}$  denote the operad corresponding to the binary operadic quadratic data  $\mathcal{A}$ . Then  $\mathcal{P}_{\mathcal{A}\vee\mathcal{B}}$  is the coproduct of  $\mathcal{P}_{\mathcal{A}}$  and  $\mathcal{P}_{\mathcal{B}}$  in the category of operads, and furthermore

$$\mathcal{P}_{\mathcal{A}\oplus\mathcal{B}} \cong \mathcal{P}_{\mathcal{A}} \oplus \mathcal{P}_{\mathcal{B}}, \quad \mathcal{P}_{\mathcal{A}\triangleleft\mathcal{B}} \cong \mathcal{P}_{\mathcal{A}} \otimes \mathcal{P}_{\mathcal{B}}, \quad \mathcal{P}_{\mathcal{A}\triangleright\mathcal{B}} \cong \mathcal{P}_{\mathcal{B}} \otimes \mathcal{P}_{\mathcal{A}}.$$

2. These six monoidal structures are connected by the following Koszul duality involutions:

$$(\mathcal{A} \vee \mathcal{B})^* \cong \mathcal{A}^* \oplus \mathcal{B}^*, \quad (\mathcal{A} \triangleleft \mathcal{B})^* \cong \mathcal{A}^* \triangleright \mathcal{B}^*, \quad (\mathcal{A} \underline{\otimes} \mathcal{B})^* \cong \mathcal{A}^* \odot \mathcal{B}^*.$$

For more details, see [LV12, Section 8.6].

PROPOSITION 4.8.2. *The interchange laws  $\varphi$  and  $\psi$  in the category of graded  $\mathbb{S}_2$ -modules induce morphisms in the category BOQD which make the quintuples*

$$\begin{aligned} &(\text{BOQD}, \bullet, \vee, \varphi, \psi), \quad (\text{BOQD}, \bullet, \oplus, \varphi, \psi), \\ &(\text{BOQD}, \bullet, \triangleleft, \varphi, \psi), \quad (\text{BOQD}, \bullet, \triangleright, \varphi, \psi) \end{aligned}$$

and

$$\begin{aligned} &(\text{BOQD}, \circ, \vee, \varphi, \psi), \quad (\text{BOQD}, \circ, \oplus, \varphi, \psi), \\ &(\text{BOQD}, \circ, \triangleleft, \varphi, \psi), \quad (\text{BOQD}, \circ, \triangleright, \varphi, \psi) \end{aligned}$$

into 2-monoidal categories, i.e. simultaneously lax and colax.

*Proof.* The proof can be obtained by direct computations. □

## 5 LIE OPERADS AND HOPF (CO)OPERADS

The purpose of this section is to provide a simple categorical setting for the automatic construction of several (co)operads in categories of (co)algebras starting from just a single and simple operad structure. This framework applies to many operads which play a key role in the literature. In quantum groups, deformation quantization, algebraic topology and Grothendieck–Teichmüller groups, like in [Dri90, KM94, Tam03, SW11, LV14, Fre17], it is crucial to work with Lie operads or Hopf (co)operads, that is operads in the category of Lie algebras and (co)operads in the category of (co)algebras. These kind of (co)operad structures are produced here from topological operads; this way, we recover the ones present in the above-mentioned theories, as well as interesting new ones.

When dealing with symmetric monoidal categories which are obviously strong, we will drop this adjective for simplicity.

5.1 OPERADS, COOPERADS, AND SYMMETRIC MONOIDAL FUNCTORS

Since the opposite category of a symmetric monoidal category is again symmetric monoidal, we can consider the following notion dual to that of an operad.

DEFINITION 5.1.1 (Cooperad). A *cooperad*  $\mathcal{C}$  in  $\mathbf{C}$  is an operad in the opposite symmetric monoidal category  $\mathbf{C}^{\text{op}}$ .

This means that we are given a functor  $\mathcal{C} : \text{BIJ}^{\text{op}} \rightarrow \mathbf{C}^{\text{op}}$  (or equivalently  $\mathcal{C} : \text{BIJ} \rightarrow \mathbf{C}$ ) with partial decompositions maps in  $\mathbf{C}$ :

$$\delta_{X \subset Y} : \mathcal{C}(Y) \rightarrow \mathcal{C}(Y/X) \otimes \mathcal{C}(X) , \quad \text{for any } X \subset Y ,$$

and a counit  $\varepsilon : \mathcal{P}(\{*\}) \rightarrow 1_{\mathbf{C}}$  satisfying the dual commutative diagrams.

PROPOSITION 5.1.2.

1. Any covariant symmetric monoidal functor sends operads to operads and cooperads to cooperads.
2. Any contravariant symmetric monoidal functor sends operads to cooperads and cooperads to operads.

*Proof.* It is well-known that any covariant lax symmetric monoidal functor sends operads to operads. Thus any contravariant oplax symmetric monoidal functor, i.e. such that the associated covariant functor between the opposite categories, sends cooperads to cooperads. Let us just sketch the proof a little bit since we will use the transferred (co)operad structure later on.

Let  $(\mathbf{C}, \otimes_{\mathbf{C}}, 1_{\mathbf{C}}, \alpha_{\mathbf{C}}, \lambda_{\mathbf{C}}, \rho_{\mathbf{C}}, \tau_{\mathbf{C}})$ ,  $(\mathbf{D}, \otimes_{\mathbf{D}}, 1_{\mathbf{D}}, \alpha_{\mathbf{D}}, \lambda_{\mathbf{D}}, \rho_{\mathbf{D}}, \tau_{\mathbf{D}})$  be two symmetric monoidal categories and let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a covariant symmetric monoidal functor with structure maps

$$\psi : 1_{\mathbf{D}} \rightarrow F(1_{\mathbf{C}}) \quad \text{and} \quad \varphi_{A,B} : F(A) \otimes_{\mathbf{D}} F(B) \rightarrow F(A \otimes_{\mathbf{C}} B) .$$

For any operad  $\mathcal{P}$ , we consider the following structure maps of  $F\mathcal{P}$ :

$$F\mathcal{P}(Y/X) \otimes_{\mathbf{D}} F\mathcal{P}(X) \xrightarrow{\varphi_{\mathcal{P}(Y/X), \mathcal{P}(X)}} F(\mathcal{P}(Y/X) \otimes_{\mathbf{C}} \mathcal{P}(X)) \xrightarrow{F(\circ_{X \subset Y})} F\mathcal{P}(Y)$$

and  $1_{\mathbf{D}} \xrightarrow{\psi} F(1_{\mathbf{C}}) \xrightarrow{F(\eta)} F\mathcal{P}(\{*\}) .$

Dually, for any cooperad  $\mathcal{C}$ , we consider the following structure maps of  $F\mathcal{C}$ :

$$F\mathcal{C}(Y) \xrightarrow{F(\delta_{X \subset Y})} F(\mathcal{C}(Y/X) \otimes_{\mathbf{C}} \mathcal{C}(X)) \xrightarrow{\varphi_{\mathcal{C}(Y/X), \mathcal{C}(X)}^{\text{op}}} F\mathcal{C}(Y/X) \otimes_{\mathbf{D}} F\mathcal{C}(X)$$

and  $F\mathcal{C}(\{*\}) \xrightarrow{F(\varepsilon)} F(1_{\mathbf{C}}) \xrightarrow{\psi^{\text{op}}} 1_{\mathbf{D}} .$

It remains to check the various axioms of this new structure but this follows in a straightforward way from the defining axioms of the operad  $\mathcal{P}$  (or the cooperad  $\mathcal{C}$ ), of the two monoidal categories  $\mathbf{C}$  and  $\mathbf{D}$ , and the symmetric monoidal functor  $F$ .

The second assertion is less present in the literature. It is however a formal consequence of the first assertion. Let  $\mathcal{P} : \text{BIJ}^{\text{op}} \rightarrow \mathbf{C}$  be an operad in  $\mathbf{C}$ . By definition, this means that  $\mathcal{P}$  is a cooperad in the opposite category  $\mathbf{C}^{\text{op}}$ . It is thus sent to a cooperad in  $\mathbf{D}$  under the (covariant) symmetric monoidal functor  $F^{\text{op}} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ .  $\square$

We have already been applying this result in Corollary 4.7.2. Now Theorem 2.5.1 and Proposition 5.1.2 allow us to deduce seven operad structures and four cooperad structures out of the sole data of an operad structure in the category of skew-symmetric quadratic data. Since the monoidal product  $\oplus$  of this latter category is particularly simple, the data of an operad there is also not difficult to establish, as the following examples show.

REMARK 5.1.3. Notice that the left-to-right symmetric monoidal functors can all be inverted. So we could also induce transport (co)operad structures in the other way round. Moreover, one can often easily guess from a (co)operad structure in a category of (co)algebras the associated (co)operad structure in the above category of quadratic data. In the end, the global orientation of the diagram chosen here is not restrictive, but amounts rather to a choice of presentation.

DEFINITION 5.1.4 (Lie operad and (co)commutative Hopf (co)operad). An operad in the symmetric monoidal category  $(\text{LIE-ALG}, \oplus)$  of Lie algebras is called a *Lie operad*. An operad in the symmetric monoidal category  $(\text{COM-COALG}, \otimes)$  of cocommutative coalgebras is called a *cocommutative Hopf operad*. A cooperad in the symmetric monoidal category  $(\text{COM-ALG}, \otimes)$  of commutative algebras is called a *commutative Hopf cooperad*.

REMARK 5.1.5. The notion of a Lie operad should not be confused with the operad Lie encoding Lie algebras.

*From now on, we work over the field  $\mathbb{Q}$  of rational numbers.*

EXAMPLE 5.1.6. The homology group functor  $H_{\bullet}(-) := H_{\bullet}(-, \mathbb{Q})$  is a covariant symmetric monoidal functor and the cohomology group functor  $H^{\bullet}(-) := H^{\bullet}(-, \mathbb{Q})$  is a contravariant symmetric monoidal functor. The former sends a topological operad to a cocommutative Hopf operad and the latter sends it to a commutative Hopf cooperad.

## 5.2 LIE OPERADS FROM POINTED TOPOLOGICAL OPERADS

Following the same pattern, we aim at producing functorially Lie operads from topological operads using rational fundamental groups. Suppose now that every component  $\mathcal{O}(n)$  of the topological operad  $\mathcal{O}$  admits a base point  $*^n$  which is compatible with the operadic structure, i.e.  $*^n \circ_k *^m = *^{n+m-1}$ . In other words, this means that we consider an operad in the symmetric monoidal category of pointed topological spaces  $(\text{TOP}_{*}, \times)$ . In this case, one can consider the

fundamental groups  $\pi_1(\mathcal{O}(n))$  of each component and then their images under the Magnus construction [Mag37, Laz50]

$$\text{gr}(G) := \bigoplus_{k \geq 1} \Gamma_k G / \Gamma_{k+1} G ,$$

which associates a Lie algebra over  $\mathbb{Z}$  to any group  $G$  by means of its lower central series, defined inductively by  $\Gamma_1 G := G$  and  $\Gamma_{k+1} G := [\Gamma_k G, G]$ . Recall that the Lie bracket  $[x, y]$  is induced by the group commutator  $xyx^{-1}y^{-1}$ .

LEMMA 5.2.1.

1. *The fundamental group functor  $\pi_1 : (\text{TOP}_*, \times) \rightarrow (\text{GR}, \times)$  from the category of topological spaces to the category of groups is cartesian, i.e. strongly symmetric monoidal with respect to the products.*
2. *The Magnus functor  $\text{gr} : (\text{GR}, \times) \rightarrow (\text{LIE-ALG}_{\mathbb{Z}}, \oplus)$  from the category of groups to the category of Lie algebra over  $\mathbb{Z}$  is cartesian.*

*Proof.* The proof is straightforward. □

REMARK 5.2.2. As usual, in order to get a nice behaviour of topological spaces with respect to products, one needs to restrict to the category of compactly generated Hausdorff spaces with Kelly product, which we implicitly do here.

PROPOSITION 5.2.3. *Any pointed topological operad  $\mathcal{O}$  induces an operad in the category of Lie algebras over  $\mathbb{Z}$ :*

$$\text{gr}(\pi_1(\mathcal{O})) ,$$

*which is called the Magnus operad.*

*Proof.* This is a direct corollary of Proposition 5.1.2 and Lemma 5.2.1. □

### 5.3 OPERADIC QUADRATIC DATA FROM TOPOLOGICAL OPERADS

Now we study how the three aforementioned functors producing respectively “(co)homology” Hopf (co)operads and “homotopy” Lie operads from topological operads lift to the quadratic data level.

REMARK 5.3.1. In this paper, we need to put homology and cohomology on the same footing in order to treat them with the framework described in Section 2. Since we use the homological degree convention and since cohomology will always appear as linear dual of homology, the cohomology groups will be non-positively graded. In other words, we use the opposite of the usual convention.

Let  $\mathcal{O}$  be a topological operad. The restriction  $\cup : H^1(\mathcal{O}(n))^{\otimes 2} \subset H^1(\mathcal{O}(n))^{\otimes 2} \rightarrow H^2(\mathcal{O}(n))$  of the cup-product gives rise to the symmetric quadratic data

$$(H^1(\mathcal{O}(n)), \ker \cup) \in \text{QD}^+ ,$$

where  $H^1(\mathcal{O}(n))$  is concentrated in "homological degree"  $-1$ . When  $H^1(\mathcal{O}(n))$  is finite dimensional, for any  $n \geq 0$ , we consider the (degree-wise) linear dual symmetric quadratic data

$$(H^1(\mathcal{O}(n)), \ker \cup)^* \cong (H_1(\mathcal{O}(n)), \text{im } \Delta) \in \text{QD}^+ ,$$

where  $\Delta := {}^t\cup : H_2(\mathcal{O}(n)) \rightarrow H_1(\mathcal{O}(n))^{\otimes 2}$  is the restriction of the coproduct of the homology coalgebra. Finally, the Koszul duality functor gives rise to the following skew-symmetric quadratic data

$$(H_1(\mathcal{O}(n)), \text{im } \Delta)^i \cong (s^{-1}H_1(\mathcal{O}(n)), s^{-2}\text{im } \Delta) \in \text{QD}^- .$$

DEFINITION 5.3.2 (Holonomy Lie algebra, after Chen–Kohno [Che73, Koh85]). The holonomy Lie algebras of the topological spaces  $\mathcal{O}(n)$  are the quadratic Lie algebras induced by the above presentations:

$$\mathfrak{g}_{\mathcal{O}(n)} := \text{L}(s^{-1}H_1(\mathcal{O}(n)), s^{-2}\text{im } \Delta) .$$

When each component  $\mathcal{O}(n)$  is path connected, for  $n \geq 0$ , the (co)algebras  $H^\bullet(\mathcal{O}(n))$  (respectively  $H_\bullet(\mathcal{O}(n))$ ) are (co)augmented. In this case, the two (co)operad structures  $H^\bullet(\mathcal{O})$  and  $H_\bullet(\mathcal{O})$  induce respectively a cooperad structure on the collection of symmetric quadratic data  $\{(H^1(\mathcal{O}(n)), \ker \cup)\}$  in the symmetric monoidal category  $(\text{QD}^+, \vee)$  and an operad structure on the collection of quadratic data  $\{(H_1(\mathcal{O}(n)), \text{im } \Delta)\}$  in the symmetric monoidal category  $(\text{QD}^+, \otimes)$ . Since the Koszul duality functor  $i$  (in the opposite direction) is symmetric monoidal, it induces an operad structure on the collection of skew-symmetric quadratic data  $\{(s^{-1}H_1(\mathcal{O}(n)), s^{-2}\text{im } \Delta)\}$  in the symmetric monoidal category  $(\text{QD}^-, \oplus)$ . In the end, we get a canonical Lie operad structure  $\mathfrak{g}_{\mathcal{O}}$  on the level of the holonomy Lie algebras.

DEFINITION 5.3.3 (Holonomy operad). The *holonomy operad* is the operad  $\mathfrak{g}_{\mathcal{O}}$  made up of the holonomy Lie algebras associated to a path connected topological operad  $\mathcal{O}$ .

PROPOSITION 5.3.4. *Let  $\mathcal{O}$  be a topological operad satisfying the following condition.*

CONDITION 1. For any  $n \geq 0$ , the cohomology algebras  $H^\bullet(\mathcal{O}(n))$  admits a finitely generated homogenous quadratic presentation with generators in  $H^1(\mathcal{O}(n))$ .

*In this case, the canonical map  $H^1(\mathcal{O}(n)) \rightarrow H^\bullet(\mathcal{O}(n))$  induce the following isomorphism of commutative Hopf cooperads*

$$H^\bullet(\mathcal{O}) \cong \text{S}(H^1(\mathcal{O}), \ker \cup)$$

*and the following isomorphism of cocommutative Hopf operads*

$$H_\bullet(\mathcal{O}) \cong \text{S}^c(H_1(\mathcal{O}), \text{im } \Delta) .$$

*Proof.* The proof is straightforward. Condition 1 ensures first ensures that the underlying components of the topological operad are path connected and then provides us with the underlying isomorphisms. By definition, the (co)operad structure coincides on the level of the symmetric quadratic data. The universal property of the (co)free (co)commutative (co)algebra concludes the proof.  $\square$

REMARK 5.3.5. The above treatment holds true in the same way when the first (co)homology groups  $H^1$  and  $H_1$  are replaced by the first non-trivial (co)homology groups  $H^i$  and  $H_i$ , for  $i \geq 1$ .

So under Condition 1, the six above mentioned (co)operads contain the exact same amount of data; in other words, there is no loss of generality by considering the operadic structures on the level of quadratic data.

REMARK 5.3.6. Notice that any cocommutative Hopf operad induced by an operad in the category  $(\mathbf{QD}^+, \underline{\otimes})$  under the quadratic cocommutative coalgebra functor  $S^c$  contains canonically the operad  $\mathbf{uCom}$  encoding unital commutative algebras: this latter one is simply made up of the counits of each coalgebras.

Let us recall the following seminal result due to D. Sullivan.

THEOREM 5.3.7 ([Sul77], see also [Koh85]). *Let  $X$  be a pointed, path connected, and 1-finite topological space. When  $X$  is (rationally) 1-formal, its holonomy Lie algebra is isomorphic to its rational Magnus Lie algebra*

$$\mathfrak{g}_X \cong \mathrm{gr}(\pi_1(X)) \otimes \mathbb{Q}.$$

The proof of this statement falls into two parts. First, one shows that the (co)homological) degree 1 generators of the minimal model of the piece-wise linear forms  $A_{\mathrm{PL}}^\bullet(X)$  give the rational Magnus Lie algebra. Then, under the formality assumption, one just needs to coin the minimal model of the cohomology algebra  $H^\bullet(X)$ . The linear dual of the space of degree 1 generators is easily seen be the holonomy algebra, for instance by using the cobar-bar resolution and the homotopy transfer theorem.

In order to promote the above mentioned result to the operadic level (isomorphism between the holonomy operad  $\mathfrak{g}_{\mathcal{O}}$  and the rational Magnus operad  $\mathrm{gr}(\pi_1(\mathcal{O})) \otimes \mathbb{Q}$ ), one would need a *rational Hopf (1-)formality property* satisfied by the topological operad  $\mathcal{O}$  itself in order to control the operadic compatibility between the formality quasi-isomorphisms of dg commutative algebras

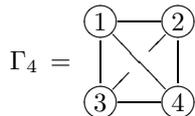
$$A_{\mathrm{PL}}^\bullet(\mathcal{O}(n)) \xleftarrow{\sim} \bullet \cdots \bullet \xrightarrow{\sim} H^\bullet(\mathcal{O}(n)).$$

This general question will be treated in the sequel of this paper, which will deal with the Hopf formality of topological operads.

We are now ready to give examples.

5.4 BERGER–KONTSEVICH–WILLWACHER, I.E. GRAPH OPERADS

For  $n \geq 2$ , we consider the complete graph  $\Gamma_n$  on  $n$  vertices labeled by  $\{1, \dots, n\}$ , that is with one and only one edge between every pair of distinct vertices. The edge between the vertices  $i$  and  $j$  is simply denoted by  $ij = ji$ .



Let us now introduce a topological version  $\text{Gra}_{S^1}$  of the complete graph operad due to C. Berger [Ber96], defined by the following (pointed) topological spaces

$$\text{Gra}_{S^1}(n) := \{*\}, \text{ for } n = 0 \text{ and } n = 1, \text{ and } \text{Gra}_{S^1}(n) := (S^1)^{\binom{n}{2}}, \text{ for } n \geq 2.$$

The elements  $\{\mu_{ij}\}$  of  $\text{Gra}_{S^1}(n)$  can be thought of as elements of the circle  $S^1$  labelling the edges  $ij$  of the complete graph  $\Gamma_n$ . The partial composition products  $\circ_p : \text{Gra}_{S^1}(n) \times \text{Gra}_{S^1}(m) \rightarrow \text{Gra}_{S^1}(n+m-1)$  of two collections  $\{\mu_{ij}\}$  and  $\{\mu'_{i'j'}\}$  are defined as follows. The idea is to insert the complete graph  $\Gamma_m$  at the  $p$ th vertex of the complete graph  $\Gamma_n$  and to relabel the vertices accordingly: the labels of the vertices  $1, \dots, p-1$  of  $\Gamma_n$  are stable, the labels of the vertices of  $\Gamma_m$  are shifted by  $p-1$ , and the labels of the remaining vertices  $p+1, \dots, n$  of  $\Gamma_n$  are shifted by  $m-1$ . If we denote the upshot of the partial composition product by  $\{\nu_{kl}\}$ , then  $\nu_{kl}$  is equal to the corresponding element  $\mu_{ij}$  when  $k, l \in \{1, \dots, p-1, p+m, \dots, n+m-1\}$ . It is equal to the corresponding element  $\mu'_{i'j'}$  when  $k, l \in \{p, \dots, p+m-1\}$ . When  $k \in \{1, \dots, p-1\}$  and  $l \in \{p, \dots, p+m-1\}$ , we set  $\nu_{kl} := \mu_{k,p}$  and when  $k \in \{p, \dots, p+m-1\}$  and  $l \in \{p+m, \dots, n+m-1\}$ , we set  $\nu_{kl} := \mu_{p,l-m+1}$ . With such a definition, the composite with  $\text{Gra}_{S^1}(1)$  is indeed the identity. The natural action of the symmetric group  $\mathbb{S}_n$  on the vertices of the graph  $\Gamma_n$  induces a right  $\mathbb{S}$ -module on  $\text{Gra}_{S^1}(n)$ . The composite on the right-hand side with  $\text{Gra}_{S^1}(0)$  amounts to forgetting some data which, with the symmetric group action, produces a FI-module structure [CEF15].

PROPOSITION 5.4.1. *The topological operad  $\text{Gra}_{S^1}$  is formal over  $\mathbb{Z}$ : there exists a quasi-isomorphism of dg operads over  $\mathbb{Z}$*

$$C_{\bullet}^{\text{sing}}(\text{Gra}_{S^1}, \mathbb{Z}) \xleftarrow{\sim} H_{\bullet}^{\text{sing}}(\text{Gra}_{S^1}, \mathbb{Z}).$$

*Proof.* The proof is straightforward and can be performed by the same arguments as in [DSV15, Section 8]. □

REMARK 5.4.2. Any topological space  $X$  can replace  $S^1$  in order to form a similar topological operad. For instance, any topological space homotopy equivalent to the circle, like  $\mathbb{C} \setminus \{0\}$  for instance, would produce a homotopy equivalent operad. In this case, the formality property can again be proved easily by hand; it can also be shown directly using [GSNPR05, CH17], which rely on mixed Hodge structures.

DEFINITION 5.4.3 (Berger–Kontsevich–Willwacher skew–symmetric quadratic data). The Berger–Kontsevich–Willwacher skew–symmetric quadratic data are spanned by

$$\text{BKW}(n) := (t_{ij}^n, t_{ij}^n \wedge t_{kl}^n) ,$$

where the set of generators  $t_{ij}^n$  of degree 0 runs over the set of edges  $ij$  of  $\Gamma_n$  and where the set of relations runs over all pairs  $(ij, kl)$  of edges in  $\Gamma_n$ . For  $n = 0$  and for  $n = 1$ , we set  $\text{BKW}(0) := (0, 0)$  and  $\text{BKW}(1) := (0, 0)$ .

We consider the following maps  $\circ_k : \text{BKW}(n) \oplus \text{BKW}(m) \rightarrow \text{BKW}(n + m - 1)$ :

$$\begin{aligned}
 t_{ij}^n &\mapsto \begin{cases} t_{i+m-1, j+m-1}^{n+m-1} & \text{for } k < i, j , \\ \sum_{l=i}^{i+m-1} t_{l, j+m-1}^{n+m-1} & \text{for } k = i , \\ t_{i, j+m-1}^{n+m-1} & \text{for } i < k < j , \\ \sum_{l=j}^{j+m-1} t_{i, l}^{n+m-1} & \text{for } k = j , \\ t_{ij}^{n+m-1} & \text{for } i, j < k , \end{cases} \\
 t_{ij}^m &\mapsto t_{i+k-1, j+k-1}^{n+m-1} .
 \end{aligned} \tag{17}$$

LEMMA 5.4.4. The above-mentioned data  $\text{BKW} := (\{\text{BKW}(n)\}, \{\circ_k\})$  forms an operad in the symmetric monoidal category  $(\text{QD}^-, \oplus)$ .

*Proof.* Since the spaces of relations are the full spaces  $R(n) = V(n)^{\wedge 2}$ , for any  $n \geq 0$ , the maps  $\circ_k$  are morphisms of quadratic data. It is straightforward to check the sequential and parallel axioms, the equivariance with respect to the symmetric groups action, as well as the axioms for the unit.  $\square$

The Lie operad  $L(\text{BKW})$  is thus made up of the graphs with one edge (of degree 0) and with similar partial composition maps.

PROPOSITION 5.4.5. The holonomy operad and the rational Magnus operad associated to  $\text{Gra}_{S^1}$  are isomorphic to the Lie operad associated to skew-symmetric quadratic data  $\text{BKW}$ :

$$\mathfrak{g}_{\text{Gra}_{S^1}} \cong L(\text{BKW}) \cong \text{gr}(\pi_1(\text{Gra}_{S^1})) \otimes \mathbb{Q} .$$

*Proof.* The first isomorphism is obtained directly from the definition of the holonomy operad. The cohomology algebra of the circle is the algebra of dual number  $H^*(S^1) \cong \mathbb{Q}1 \oplus \mathbb{Q}\varepsilon$ , that is the free commutative algebra on one degree one element  $\varepsilon$ . This shows that the cohomology symmetric quadratic data is trivial

$$(H^1(\text{Gra}_{S^1}(n)), \ker \cup) = (\varepsilon_{ij}^n, \emptyset) ,$$

and thus that the holonomy skew-symmetric data is the Berger–Kontsevich–Willwacher one

$$(s^{-1}H_1(\text{Gra}_{S^1}(n)), s^{-2}\text{im } \Delta) = (t_{ij}^n, t_{ij}^n \wedge t_{kl}^n) ,$$

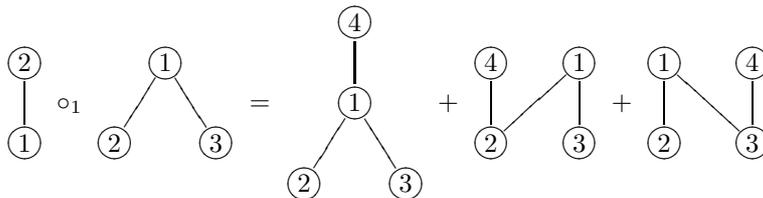
under the identification  $t_{ij}^n \cong s^{-1}(\varepsilon_{ij}^n)^*$ . In order to show that these isomorphisms commute with the respective operadic structures, one needs to describe the homology operad  $H_\bullet(\text{Gra}_{S^1})$ ; this computation is performed in the core of the proof of Proposition 5.4.6 below.

The second isomorphism is also straightforward from the definition of the rational Magnus operad. One has

$$\text{gr}(\pi_1(\text{Gra}_{S^1}(n))) \otimes \mathbb{Q} \cong \mathbb{Q}^{\binom{n}{2}}$$

and the partial composition maps agree. □

Let us recall from [Kon93, Kon97, Wil15] the definition of the *graph operad*  $\text{Gra}$  of natural operations of polyvector fields of  $\mathbb{R}^k$ . Its underlying  $\mathbb{S}$ -modules are spanned by subgraphs of  $\Gamma_n$ , that is graphs with  $n$  vertices labeled bijectively by  $\{1, \dots, n\}$  and possibly at most one edge of degree 1 between any pair of vertices. The partial composition product  $\gamma_1 \circ_k \gamma_2$  amounts to first inserting the graph  $\gamma_2$  at the  $k$ th vertex of  $\gamma_1$ , then relabelling accordingly the vertices, and finally consider the sum of all the possible ways to connect the edges in  $\gamma_1$  originally plugged to the vertex  $k$ , to any possible vertex of  $\gamma_2$ .



Every graded vector space  $\text{Gra}(n)$  forms a cocommutative coalgebra with the coproduct  $\Delta(\gamma)$  made up of the pairs of graphs  $\gamma' \otimes \gamma''$  with the same  $n$  vertices as  $\gamma$  but with edges from  $\gamma$  distributed on  $\gamma'$  and  $\gamma''$ . The partial composition products preserve these coproducts, thus  $\text{Gra}$  forms a cocommutative Hopf operad.

PROPOSITION 5.4.6. *The following three cocommutative Hopf operads are isomorphic*

$$H_\bullet(\text{Gra}_{S^1}) \cong S^c(\text{BKW}^i) \cong \text{Gra} .$$

*Proof.* The topological operad  $\text{Gra}_{S^1}$  satisfies Condition 1 and thus the first isomorphism is produced by Proposition 5.3.4 using Proposition 5.4.5. Using the fact that the homology coalgebra of the circle is the coalgebra of dual number on one degree one generator, one can directly prove the isomorphism of cocommutative Hopf operad  $H_\bullet(\text{Gra}_{S^1}) \cong \text{Gra}$ . The direct isomorphism  $S^c(\text{BKW}^i) \cong \text{Gra}$  can however be made explicit as follows using the previous sections. Let us denote by  $V(n) := \mathbb{Q}\{t_{ij}^n\}$  the space of generators of the quadratic data  $\text{BKW}(n)$ . The underlying  $\mathbb{S}$ -module of the cocommutative Hopf operad  $S^c(\text{BKW}^i) = (\{S^c(\text{BKW}^i(n))\}, \{\bar{\sigma}_k\})$ , obtained by applying the

symmetric monoidal functors  $i$  and then  $S^c$ , is made up of cofree cocommutative coalgebras  $S^c(sV(n)) = S^c(st_{ij}^n)$ , with  $|st_{ij}^n| = 1$ , which admits for basis the monomials  $st_{i_1 j_1}^n \odot \cdots \odot st_{i_k j_k}^n$ , where all the pairs  $ij$  are different. Such monomials are in one-to-one correspondence with the graphs of  $\text{Gra}(n)$ .

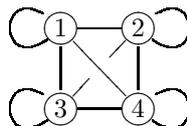
The partial composition products  $\bar{\circ}_k : S^c(sV(n)) \otimes S^c(sV(m)) \rightarrow S^c(sV(n + m - 1))$  of the operad  $S^c(\text{BKW}^1)$  are morphisms of cocommutative coalgebras; so they are characterised by their projections onto  $sV(n + m - 1)$ .

We denote by  $\emptyset^n$  the graph with  $n$  vertices and with no edge, that we identify with the counit  $\mathbf{1}^n$  of the cofree coalgebra  $S^c(sV(n))$ , i.e.  $\mathbf{1}^n \leftrightarrow \emptyset^n$ . We denote by  $\gamma_{ij}^n$  the graph with  $n$  vertices and with the only edge  $ij$ , that we identify with the generator  $st_{ij}^n$  of  $S^c(sV(n))$ , i.e.  $st_{ij}^n \leftrightarrow \gamma_{ij}^n$ . Under this correspondence, the isomorphism  $S^c(sV(n)) \otimes S^c(sV(m)) \cong S^c(s(V(n) \oplus V(m)))$  sends  $\emptyset^n \otimes \emptyset^m$  to  $\mathbf{1}^{n+m-1}$ ,  $\gamma_{ij}^n \otimes \emptyset^m$  to  $st_{ij}^n$ , and  $\emptyset^n \otimes \gamma_{ij}^m$  to  $st_{ij}^m$ . The images of these three latter elements under the partial composition products  $\circ_k$  of the operad  $\text{BKW}$  given in Equation (17) coincide, under the above identifications, to the images of the three former elements under the partial composition products of the operad  $\text{Gra}$ , which concludes the proof.  $\square$

The operad  $\text{UL}(\text{BKW}) = \text{AA}(\text{BKW})$  in associative algebras is similar to the operad  $\text{Gra}$  except that we consider graphs with possibly multiple edges (of degree 0) between vertices. The (algebra) product of two such graphs amounts to consider the union of their sets of edges.

REMARK 5.4.7. Using the recognition method of C. Berger [Ber96], one can see that  $\text{Gra}_{S^1}$  admits a (cellular) sub-operad which is an  $E_2$ -operad, that is a topological operad having the same homotopy type then the little disks operad  $\mathcal{D}_2(n) \sim \text{Conf}_n(\mathbb{R}^2)$ , see Section 5.6. The operad  $\text{Gra}$  admits a map from the operad encoding shifted Lie algebras, so it can be twisted à la Willwacher to produce a differential graded operad  $\text{TwGra}$ , see [Wil15] and [DSV18, Section 5] for more details. This latter operad plays a key role in the proof of the formality of the little disks operad in [Kon99, LV14, FW15]. Since  $\text{TwGra}$  forms a dg cocommutative Hopf operad, it is a good model for the rational homotopy type of the little disks operad; this point explains conceptually why the rational homotopy automorphism group of the little disks operad is isomorphic to the Grothendieck–Teichmüller group in [Wil15, Fre17].

One can perform the same arguments for the topological operad  $\text{Gra}_{S^1} \rtimes S^1$ , which is obtained by adding a copy of  $S^1$  at every input, see [SW03] for the semi-direct product of operads. This amounts to adding  $n$  generators  $t_i^n$ , for  $1 \leq i \leq n$ , to the skew-symmetric quadratic and again considering the full space of relations. The same results hold true *mutatis mutandis* by considering now graphs with possible tadpoles, that with possibly one loop attached to each vertex.



5.5 NONSYMMETRIC ANALOGUE OF THE LITTLE DISKS OPERAD, I.E.  $As_{S^1}$   
 AND  $As_{S^1} \rtimes S^1$ 

A noncommutative version of the notion of Gerstenhaber algebras was introduced in [DSV15, Section 3] in relation with noncommutative deformation theory. This notion is modelled by the nonsymmetric (pointed) topological operad which is defined in a way similar to the aforementioned topological operad  $Gra_{S^1}$  but starting from the complete linear graph  $\Theta_n$  instead of the complete graph  $\Gamma_n$ . Explicitely,  $\Theta_n$  is the graph on  $n$  vertices labeled by  $\{1, \dots, n\}$  from left to right with one and only one edge between every consecutive pair of vertices  $i$   $i + 1$ .

$$\Theta_n = \textcircled{1} \text{---} \textcircled{2} \text{---} \textcircled{3} \text{---} \dots \text{---} \textcircled{n}$$

The pointed topological ns operad  $As_{S^1}$  is defined by  $As_{S^1}(n) := \{*\}$ , for  $n = 0$  and  $n = 1$ , and by  $As_{S^1}(n) := (S^1)^{n-1}$ , for  $n \geq 2$ , with partial composition products are given by

$$(x_1, \dots, x_{n-1}) \circ_i (y_1, \dots, y_{m-1}) := (x_1, \dots, x_{i-1}, y_1, \dots, y_{m-1}, x_i, \dots, x_{n-1}) .$$

REMARK 5.5.1. This ns topological operad is formal [DSV15, Corollary 8.1.1]. Again, any topological space  $X$  can replace  $S^1$  in order to form a similar nonsymmetric topological operad. For instance, any topological space homotopy equivalent to the circle, like  $\mathbb{C} \setminus \{0\}$  for instance, would produce a homotopy equivalent operad, which is also formal.

DEFINITION 5.5.2 (skew-symmetric quadratic data LG). The *skew-symmetric quadratic data* LG, for Linear Graph, are spanned by

$$LG(n) := (e_{ii+1}^n, e_{ii+1}^n \wedge e_{jj+1}^n) ,$$

where the set of generators  $e_{ii+1}^n$  of degree 0 runs over the set of edges of  $\Theta_n$  and where the set of relations runs over all pairs of edges of  $\Theta_n$ . For  $n = 0$  and for  $n = 1$ , we set  $LG(0) := (0, 0)$  and  $LG(1) := (0, 0)$ .

The morphisms  $\circ_k : LG(n) \oplus LG(m) \rightarrow LG(n + m - 1)$  of skew-symmetric quadratic data defined by

$$\begin{aligned} e_{ii+1}^n &\mapsto \begin{cases} e_{i+m-1i+m}^{n+m-1} & \text{for } k \leq i , \\ e_{ii+1}^{n+m-1} & \text{for } k > i , \end{cases} \\ e_{ij}^m &\mapsto e_{j+k-1j+k}^{n+m-1} , \end{aligned}$$

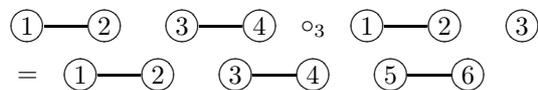
endow the collection  $\{LG(n)\}$  with a nonsymmetric operad structure in the symmetric monoidal category  $(\mathcal{QD}^-, \oplus)$ . The Lie operad  $L(LG)$  is made up of the graphs with one edge (of degree 0) and with similar partial composition maps.

PROPOSITION 5.5.3. *The holonomy operad and the rational Magnus operad associated to  $As_{S^1}$  are isomorphic to the Lie operad associated to the skew-symmetric data LG:*

$$\mathfrak{g}_{As_{S^1}} \cong L(LG) \cong \text{gr}(\pi_1(As_{S^1})) \otimes \mathbb{Q} .$$

*Proof.* This proof is similar to the proof of Proposition 5.4.5. □

Mimicking the above definition of the operad Gra, we introduce a nonsymmetric operad LGra made up of sub-graphs of the complete linear graph  $\Theta_n$  and with the insertion at vertex  $k$  for partial composition product:



This actually forms a cocommutative Hopf nonsymmetric operad with the co-product  $\Delta(\gamma) := \sum \gamma' \otimes \gamma''$  where the edges from  $\gamma$  are distributed on  $\gamma'$  and  $\gamma''$ .

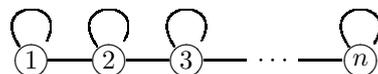
PROPOSITION 5.5.4. *The following three cocommutative Hopf operads are isomorphic*

$$H_{\bullet}(As_{S^1}) \cong S^c(LG^i) \cong LGra .$$

*Proof.* This proof is similar to the proof of Proposition 5.4.6. □

Again, the operad  $UL(LG) = A\Lambda(LG)$  in associative algebras is similar to the operad LGra but made up of linear graphs with possibly multiple edges (of degree 0) between consecutive vertices; the (algebra) product of two such graphs amounts to consider the union of their sets of edges.

A noncommutative version of the notion of Batalin–Vilkovisky algebras was introduced in [DSV15, Section 3]; it is modelled by the nonsymmetric topological operad  $As_{S^1} \rtimes S^1$ . The associated skew-symmetric quadratic data is similar but with  $n$  extra generators  $e_i^n$ , for  $1 \leq i \leq n$ . The same results hold true *mutatis mutandis* by considering now linear graphs with possible tadpoles.



REMARK 5.5.5. The homology nonsymmetric operads  $H_{\bullet}(As_{S^1})$  and  $H_{\bullet}(As_{S^1} \rtimes S^1)$  can also be twisted à la Willwacher to produce dg nonsymmetric operads. Their homology with respect to the twisted differential was computed in [DSV18, Section 6].

5.6 DRINFELD–KOHNO AND ARNOLD–ORLIK–SOLOMON, I.E.  $\mathcal{D}_2(n) \sim \text{Conf}_n(\mathbb{C})$

In this section, we refine the above mentioned Berger–Kontsevich–Willwacher skew–symmetric quadratic data following the works of Drinfeld [Dri90] and Kohno [Koh85]. We show that this refinement is canonical in a certain way. This theory corresponds to the topological operad  $\mathcal{D}_2$ , called the *little disks operad*, which is made up of configurations of disks inside the unit disk. It is the mother of operads (the father being the endomorphism operad), which arose from the recognition of double loop spaces in [BV73, May72]. Recall that the components of the little disks operad are homotopy equivalent to the configuration space of  $n$  points in the plane  $\mathcal{D}_2(n) \sim \text{Conf}_n(\mathbb{C})$ . Notice that the little disks operad fails to be well pointed.

DEFINITION 5.6.1 (Drinfeld–Kohno skew–symmetric quadratic data). The *Drinfeld–Kohno skew–symmetric quadratic data* are spanned by

$$\text{DK}(n) := (t_{ij}^n, t_{ij}^n \wedge t_{kl}^n \ \& \ t_{ij}^n \wedge (t_{ik}^n + t_{jk}^n)) \ ,$$

where the set of generators  $t_{ij}^n$  of degree 0 runs over the set of edges  $ij$  of  $\Gamma_n$ , and where the first set of relations runs over pairs  $(ij, kl)$  of disjoint edges and the second set of relations runs over triples of edges  $(ij, jk, kl)$  which form a triangle in  $\Gamma_n$ . For  $n = 0$  and for  $n = 1$ , we set  $\text{DK}(0) := (0, 0)$  and  $\text{DK}(1) := (0, 0)$ .

We consider the same partial composition products as the ones for the Berger–Kontsevich–Willwacher quadratic data given in Equation (17).

PROPOSITION 5.6.2. *The Drinfeld–Kohno skew–symmetric quadratic data  $\text{DK} := (\{\text{DK}(n)\}, \{\circ_k\})$  forms an operad in the symmetric monoidal category  $(\text{QD}^-, \oplus)$ .*

*Proof.* After the proof of Lemma 5.4.4, the only thing left to check is that the various maps  $\circ_k$  induce morphisms of quadratic data, that is

$$(\circ_k)^{\wedge 2}(R(n) \oplus [V(n), V(m)]_- \oplus R(m)) \subset R(n + m - 1) \ ,$$

where we use the notation  $\text{DK}(n) = (V(n), R(n))$ . This can be proved by straightforward but tedious computations. It becomes much easier with the previous interpretation in terms of graph operad: one can see that any first (respectively second) type relation in  $R(n)$  or  $R(m)$  is sent to any first (respectively second) type relation in  $R(n + m - 1)$ , that is pairs of disjoint edges (respectively graphs whose edges form a triangle). Regarding the relation  $[V(n), V(m)]_-$ , any of its elements  $t_{ij}^n \wedge t_{kl}^m$  is sent, under  $(\circ_k)^{\wedge 2}$ , to a sum of relations of first and second type.  $\square$

The canonical morphisms of quadratic data  $\text{DK}(n) \rightarrow \text{BKW}(n)$  induce a canonical morphism of operads  $\text{DK} \rightarrow \text{BKW}$  in  $\text{QD}^-$ . More generally, we call *sub-operad of BKW* any collection of skew–symmetric quadratic sub-data

$(V(n), R(n)) \subset \text{BKW}(n)$  stable under the partial composition products  $\circ_k$ , where  $V(n)$  is generated by the set of edges  $t_{ij}^n$ . As usual, the intersection of all such sub-operads, explicitly given by the intersection of all the spaces of relations  $R(n)$  for a fixed  $n$  each time, produces the smallest sub-operad of  $\text{BKW}$ . The following statement provides us with a universal operadic characterisation of the Drinfeld–Kohno quadratic data.

**THEOREM 5.6.3.** *The operad  $\text{DK}$  is the smallest sub-operad of  $\text{BKW}$ .*

*Proof.* Let us continue to use the notation  $\text{DK}(n) = (V(n), R(n))$  and let us consider a sub-operad  $\text{P}(n) := (V(n), S(n)) \subset \text{BKW}(n)$  of  $\text{BKW}$ . We have to show that  $R(n) \subset S(n)$  and this follows from the fact that the partial composition products  $\circ_p : \text{P}(n) \oplus \text{P}(m) \rightarrow \text{P}(n + m - 1)$  sends  $[V(n), V(m)]_-$  to  $S(n + m - 1)$  under  $(\circ_p)^{\wedge 2}$ . We begin with the relations of first type:  $t_{ij}^n \wedge t_{kl}^n$ . Using the action of the symmetric group, we can assume, without any loss of generality, that  $(i, j, k, l) = (1, 2, 3, 4)$  and we conclude with

$$(\circ_3)^{\wedge 2}(t_{12}^{n-1} \wedge t_{12}^2) = t_{12}^n \wedge t_{34}^n .$$

We treat now the relations of seconde type:  $t_{ij}^n \wedge (t_{ik}^n + t_{jk}^n)$ . Using again the action of the symmetric group, the proof reduces to the case  $(i, j, k) = (1, 2, 3)$ , which is given by

$$(\circ_1)^{\wedge 2}(t_{12}^{n-1} \wedge t_{12}^2) = (t_{13}^n + t_{23}^n) \wedge t_{12}^n .$$

□

In the lattice of operads made up of skew-symmetric data with generators  $t_{ij}^n$  and partial composition products  $\circ_k$ , the Berger–Kontsevich–Willwacher operad  $\text{BKW}$  is the maximal element and the Drinfeld–Kohno operad  $\text{DK}$  is the minimal element.

**PROPOSITION 5.6.4.** *The holonomy operad associated to  $\mathcal{D}_2$  is isomorphic to the Lie operad associated to skew-symmetric quadratic data  $\text{DK}$ :*

$$\mathfrak{g}_{\mathcal{D}_2} \cong \text{L}(\text{DK}) .$$

We will prove it below after the study of the (co)homology Hopf (co)operad. Since the little disks operad is not well pointed, we cannot consider directly a Lie operad of Magnus type here. Instead, one can consider the pointed topological operad  $\mathcal{K}_2$  introduced by M. Kontsevich in [Kon99], see also [Sin06], since this latter one is homotopy equivalent to the little disks operad. Notice that both operads,  $\mathcal{D}_2$  [LV14] and  $\mathcal{K}_2$  [ST18], are formal; they are even intrinsically Hopf formal by [FW15].

**REMARK 5.6.5.** Even if the little disks operad fails to be well pointed, its components  $\mathcal{D}_2(n) \sim \text{Conf}_n(\mathbb{C})$  are path connected with fundamental groups isomorphic to the pure braid groups  $\pi_1(\text{Conf}_n(\mathbb{C})) \cong \text{PB}_n$ . They are also

(rationally) formal by [Arn69]. So the Drinfeld–Kohno Lie algebras, as the holonomy Lie algebras of  $\text{Conf}_n(\mathbb{C})$  are the Lie algebras of infinitesimal braids

$$L(\text{DK}(n)) = \frac{\text{Lie}(t_{ij}^n)}{\left( [t_{ij}^n, t_{kl}^n], [t_{ij}^n, t_{ik}^n + t_{jk}^n] \right)} \cong \text{gr}(\text{PB}_n) \otimes \mathbb{Q} .$$

REMARK 5.6.6. The fact that the little disks operad fails to be well pointed should be seen as a richness. Instead of considering the fundamental group  $\pi_1$  of a pointed topological space, one can consider the fundamental groupoid  $\Pi_1$ . This latter functor  $\Pi_1 : (\text{TOP}, \times) \rightarrow (\text{GRP}, \times)$  is cartesian and thus sends topological operads to operads in groupoids. The operad in groupoids  $\Pi_1(\mathcal{D}_2)$  is equivalent to the operad in groupoids which encodes braided monoidal categories. Refining this operad with various "choices of base points" gives rise to various operads in groupoids and the morphisms between them define the notion of Drinfeld’s associators and Grothendieck–Teichmüller group(s), see [Fre17] for more details.

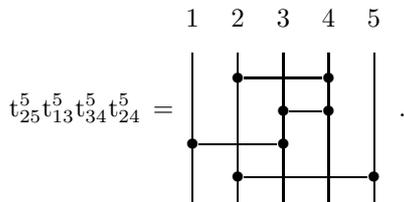
The *operad of chord diagrams* is the operad

$$\text{UL}(\text{DK}) = \text{A}\Lambda(\text{DK})$$

made up of the associative algebras of chord diagrams

$$\text{UL}(\text{DK}(n)) = \frac{T(t_{ij}^n)}{\left( [t_{ij}^n, t_{kl}^n], [t_{ij}^n, (t_{ik}^n + t_{jk}^n)] \right)} .$$

The name comes from the following pictorial way to represent its elements:



It plays a seminal role in the theory of Drinfeld’s associators [Dri90], Grothendieck–Teichmüller group(s) [Fre17], the formality of the little discs operad [Tam03, SW11, FW15] and Vassiliev knot invariants [BN95].

DEFINITION 5.6.7 (Arnold–Orlik–Solomon symmetric quadratic data). The *Arnold–Orlik–Solomon symmetric quadratic data* are spanned by

$$\text{AOS}(n) := \left( \omega_{ij}^n, \omega_{ij}^n \odot \omega_{jk}^n + \omega_{jk}^n \odot \omega_{ki}^n + \omega_{ki}^n \odot \omega_{ij}^n \right) ,$$

where the set of generators  $\omega_{ij}^n$  of degree  $-1$  runs over the set of edges  $ij$  of  $\Gamma_n$ , and where the set of relations runs over increasing triples  $i < j < k$ . For  $n = 0$  and for  $n = 1$ , we set  $\text{AOS}(0) := (0, 0)$  and  $\text{AOS}(1) := (0, 0)$ .

Arnold proved in [Arn69] that the Orlik-Solomon algebras

$$S(\text{AOS}(n)) = \frac{S(\omega_{ij}^n)}{\left(\omega_{ij}^n \odot \omega_{jk}^n + \omega_{jk}^n \odot \omega_{ki}^n + \omega_{ki}^n \odot \omega_{ij}^n\right)} \cong H^\bullet(\mathcal{D}_2(n))$$

compute the cohomology algebras of the configuration spaces of points in the plane. One can see by a direct computation that

$$\text{DK}^! \cong \text{AOS} .$$

Equivalently, this means that  $\text{DK}^i \cong \text{AOS}^*$ , which provides us with the following presentation of the cocommutative coalgebras underlying the homology operad

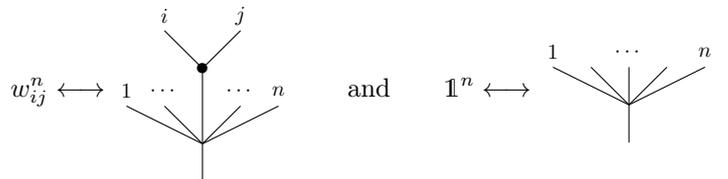
$$S^c(\text{AOS}^*(n)) = S^c(w_{ij}^n, w_{ij}^n \odot w_{kl}^n \ \& \ w_{ij}^n \odot (w_{ik}^n + w_{jk}^n)) \cong H_\bullet(\mathcal{D}_2(n)) ,$$

where  $w_{ij}^n = (\omega_{ij}^n)^* = \text{st}_{ij}^n$  has degree 1. The presentation of the homology operad was given by F.R. Cohen in [Coh76]: it is shown to be isomorphic to the operad encoding Gerstenhaber algebras  $H_\bullet(\mathcal{D}_2) \cong \text{Gerst}$ , see [LV12, Section 13.3].

*Proof of Proposition 5.6.4.* We go back to the definition and we follow the same kind of arguments as in the proof of Proposition 5.4.6. If we denote the operadic structure maps of DK by  $\circ_k$ , the ones of the homology operad  $H_\bullet(\mathcal{D}_2)$  by  $\tilde{\circ}_k$ , and the counits of the homology coalgebras by  $\mathbb{1}^n \in H_\bullet(\mathcal{D}_2(n))$ , we have the following commutative diagram

$$\begin{array}{ccc} H_1(\mathcal{D}_2(n)) \oplus H_1(\mathcal{D}_2(m)) & \xrightarrow{\circ_k} & H_1(\mathcal{D}_2(n+m-1)) \\ \downarrow \cong & & \parallel \\ H_1(\mathcal{D}_2(n)) \otimes \mathbb{Q}\mathbb{1}^m \oplus \mathbb{Q}\mathbb{1}^n \otimes H_1(\mathcal{D}_2(m)) & \xrightarrow{\tilde{\circ}_k} & H_1(\mathcal{D}_2(n+m-1)) . \end{array}$$

The isomorphism of operads  $H_\bullet(\mathcal{D}_2) \cong \text{Gerst}$  of [Coh76], see also the survey [Sin13], identifies the following elements



where  $\bullet$  denotes the shifted Lie bracket and where the bottom corollas denote the iterations of the commutative product. Under this correspondence, the operad structure on Gerst produces the formulæ given in Equation (17). Let

us illustrate this on the less trivial case: the partial composite  $w_{ij}^n \circ_i \mathbb{1}^m$  amounts to graft the above right-hand side corolla with  $m$  leaves at the input  $i$  of the left-hand side corolla. Using iteratively the Leibniz relation, one rewrites this 3-vertices trees into a sum of 2-vertices trees, which correspond to

$$w_{ij}^n \circ_i \mathbb{1}^m = w_{i\ j+m-1}^{n+m-1} + w_{i+1\ j+m-1}^{n+m-1} + \cdots + w_{i+m-2\ j+m-1}^{n+m-1} + w_{i+m-1\ j+m-1}^{n+m-1} .$$

□

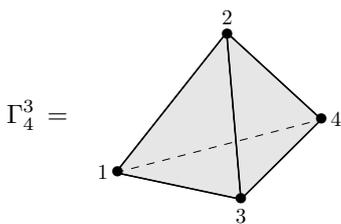
The canonical morphism of operads  $DK \rightarrow BKW$  in  $QD^-$  induces a canonical morphism of operads in associative algebras between the operad of chord diagrams and the operad of graphs with multiple edges mentioned at the end of Section 5.4. It also induces the canonical morphism  $Gerst \rightarrow Gra$  of cocommutative Hopf operads, whose deformation complex gives the Grothendieck–Teichmüller Lie algebra  $grt$  in [Wil15].

### 5.7 HYPERGRAPHS

The purpose of this subsection is to extend Section 5.4 from graphs to hypergraphs. This latter notion amounts to “graphs” where “edges” can now join an arbitrary number of vertices.

DEFINITION 5.7.1 (Hypergraph). An *hypergraph* is a pair  $(V, E)$  where  $V$  is a set of vertices and where  $E$  is a set of subsets of  $V$ , called *hyperedges*.

In the sequel, we will mainly consider the sets  $V = \underline{n} = \{1, \dots, n\}$ , for  $n \geq 2$ . We will only consider hypergraphs where the elements of  $E$  have all cardinal equal to  $k$ , for  $k \geq 2$ ; they will be called *k-hypergraphs*. For example, the *complete k-hypergraph*  $\Gamma_n^k$  on  $n$  vertices is  $(V, E)$ , where  $E$  is the set of all subset of  $\{1, \dots, n\}$  with  $k$ -elements. In the case  $k = 2$ , we recover the complete graph  $\Gamma_n = \Gamma_n^2$  of Section 5.4.



We define the topological operad of complete  $k$ -hypergraph by

$$\text{Gra}_{S^1}^k(n) := \{*\}, \text{ for } n < k, \quad \text{and} \quad \text{Gra}_{S^1}^k(n) := (S^1)^{\binom{n}{k}}, \text{ for } n \geq k .$$

The elements  $\{\mu_I \hat{E}; I \subset \underline{n}, |I| = k\}$  of  $\text{Gra}_{S^1}^k(n)$  are thought of as collections of labels, living in the circle  $S^1$ , for every hyperedges  $I$  of the complete  $k$ -hypergraph  $\Gamma_n^k$ . The partial composition products  $\circ_p : \text{Gra}_{S^1}^k(n) \times \text{Gra}_{S^1}^k(m) \rightarrow$

$\text{Gra}_{S^1}^k(n+m-1)$  of two collections  $\{\mu_I\}$  and  $\{\mu'_J\}$  are defined in a way similar to that of the operad  $\text{Gra}_{S^1}$ . We first insert the complete  $k$ -hypergraph  $\Gamma_m^k$  at the  $p$ th vertex of the complete  $k$ -hypergraph  $\Gamma_n^k$  and then we relabel the vertices accordingly. The hyperedges coming from  $\Gamma_n^k$  (respectively  $\Gamma_m^k$ ) are labeled by the according  $\mu_I$  (respectively  $\mu'_J$ ). The hyperedges made up of  $k-1$  vertices  $i_1, \dots, i_{k-1}$  from  $\Gamma_n^k$  and one vertex from  $\Gamma_m^k$  are labelled by  $\mu_{i_1, \dots, i_{k-1}, p}$ . All the other hyperedges are labelled by the base point  $*$ .

PROPOSITION 5.7.2. *The data  $\text{Gra}_{S^1}^k := (\{\text{Gra}_{S^1}^k(n)\}, \{\circ_p\})$  forms a pointed topological operad, which is formal over  $\mathbb{Z}$ .*

*Proof.* It is straightforward to check the sequential and parallel axioms, the equivariance with respect to the symmetric groups action, as well as the axioms for the unit. The formality property is proved by the same arguments and computations as in [DSV15, Section 8]. □

The special case  $k = 2$  gives back the operad  $\text{Gra}_{S^1} = \text{Gra}_{S^1}^2$  of Section 5.4.

DEFINITION 5.7.3 ( $k$ -Hypergraph skew-symmetric quadratic data). The  $k$ -Hypergraph skew-symmetric quadratic data are spanned by

$$k\text{-HG}(n) := (t_I^n, t_I^n \wedge t_J^n) ,$$

where the set of generators  $t_I^n$  of degree 0 runs over the set of hyperedges  $I$  of  $\Gamma_n^k$  and where the set of relations runs over all pairs  $(I, J)$  of hyperedges of  $\Gamma_n^k$ . For  $n < k$ , we set  $k\text{-HG}(n) := (0, 0)$ .

We consider the following maps  $\circ_p : k\text{-HG}(n) \oplus k\text{-HG}(m) \rightarrow k\text{-HG}(n+m-1)$ . Let us denote  $I = \{i_1, \dots, i_k\}$  and use the notation  $I+a := \{i_1+a, \dots, i_k+a\}$ .

$$\begin{aligned}
 t_I^n &\mapsto \begin{cases} t_{I+m-1}^{n+m-1} & \text{for } p < i_1 , \\ \sum_{j=0}^{m-1} t_{i_1, \dots, i_{l-1}, i_l+j, i_{l+1}+m-1, \dots, i_k+m-1}^{n+m-1} & \text{for } p = i_l , \\ t_{i_1, \dots, i_l, i_{l+1}+m-1, \dots, i_k+m-1}^{n+m-1} & \text{for } i_l < p < i_{l+1}, \\ t_I^{n+m-1} & \text{for } i_k < p , \end{cases} \\
 t_I^m &\mapsto t_{I+p-1}^{n+m-1} .
 \end{aligned}
 \tag{18}$$

LEMMA 5.7.4. *The aforementioned data  $k\text{-HG} := (\{k\text{-HG}(n)\}, \{\circ_p\})$  forms an operad in the symmetric monoidal category  $(\text{QD}^-, \oplus)$ .*

*Proof.* Since the spaces of relations are the full spaces, the maps  $\circ_p$  are morphisms of quadratic data. One can check directly that they form an operad structure. This can be done easily by viewing the elements  $t_I^n$  as the  $k$ -hypergraph with one hyperedge  $I$  and respectively by inserting the empty  $k$ -hypergraph at its  $p$ th vertex or by inserting it into the empty  $k$ -hypergraph. □

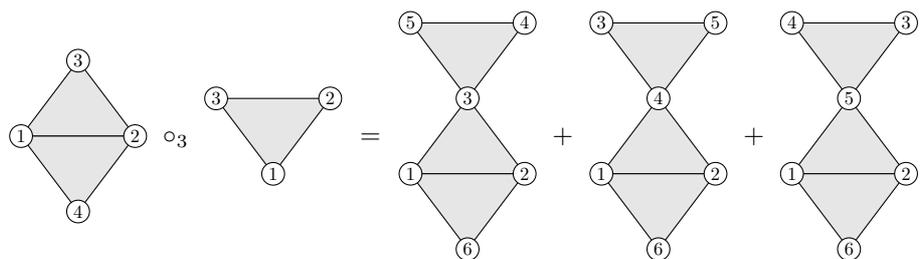
In the special case  $k = 2$ , we recover the Berger–Kontsevich–Willwacher operad  $\text{BKW} = 2\text{-HG}$ .

PROPOSITION 5.7.5. *The holonomy operad and the rational Magnus operad associated to  $\text{Gra}_{S^1}^k$  are isomorphic to the Lie operad associated to skew-symmetric quadratic data  $k\text{-HG}$ :*

$$\mathfrak{g}_{\text{Gra}_{S^1}^k} \cong L(k\text{-HG}) \cong \text{gr} \left( \pi_1(\text{Gra}_{S^1}^k) \right) \otimes \mathbb{Q} .$$

*Proof.* This proof is the same *mutatis mutandis* as the one of Proposition 5.4.5 □

Let us introduce the *k-hypergraph graded operad*  $\text{Gra}^k$ . Its underlying  $\mathbb{S}$ -modules are spanned by sub-hypergraphs of  $\Gamma_n^k$ , where each hyperedge receives degree 1. The partial composition product  $\gamma_1 \circ_p \gamma_2$  amounts to first inserting the  $k$ -hypergraph  $\gamma_2$  at the  $p$ th vertex of  $\gamma_1$ , then relabelling accordingly the vertices, and finally considering the sum of all the possible ways to connect the hyperedges in  $\gamma_1$  containing the vertex  $p$ , to any possible vertex of  $\gamma_2$ .



Every graded vector space  $\text{Gra}^k(n)$  forms a cocommutative coalgebra with the coproduct  $\Delta(\gamma)$  made up of the pairs of graphs  $\gamma' \otimes \gamma''$  with the same  $n$  vertices as  $\gamma$  but with hyperedges from  $\gamma$  distributed on  $\gamma'$  and  $\gamma''$ . The partial composition products preserve these coproducts, thus  $\text{Gra}^k$  forms a cocommutative Hopf operad.

PROPOSITION 5.7.6. *The following three cocommutative Hopf operads are isomorphic*

$$H_\bullet(\text{Gra}_{S^1}^k) \cong S^c(k\text{-HG}^1) \cong \text{Gra}^k .$$

*Proof.* This proof is similar to that of Proposition 5.4.6. □

REMARK 5.7.7. For any  $k \geq 2$ , there is a canonical morphism  $s\text{Lie}^k \rightarrow \text{Gra}^k$  from the operad of shifted  $k$ -Lie algebras which sends its generator to  $t_{1\dots k}^k$ . This latter notion is made up of a “Lie bracket” of degree 1 with  $k$ -inputs satisfying a generalised Jacobi relation, see [LV12, Section 13.11.3]. Since this operad (unshifted) is the unit for the black product of  $k$ -ary quadratic operads, one can develop a similar twisting procedure as that of [Wil15] according to [DSV18, Remark 5.8]. The study of the resulting dg operad  $\text{TwGra}^k$  is a very interesting subject.

5.8 ETINGOF–HENRIQUES–KAMNITZER–RAINS, I.E.  $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$

In the very same way as the Drinfeld–Kohno quadratic data refines the Berger–Kontsevich–Willwacher quadratic data in canonical way, the Etingof–Henriques–Kamnitzer–Rains quadratic data refines the 3-Hypergraph quadratic data in a canonical way. This new one actually comes from the topological operad made up of the *real locus* of the moduli spaces of stable curves of genus 0 with marked points  $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ , studied in depth in [EHKR10].

DEFINITION 5.8.1 (Etingof–Henriques–Kamnitzer–Rains skew-symmetric quadratic data). The *Etingof–Henriques–Kamnitzer–Rains skew-symmetric quadratic data* are spanned by

$$\text{EHKR}(n) := \left( t_{ijk}^n, t_{ijk}^n \wedge t_{lmn}^n \ \&\mathcal{L} \ t_{ijk}^n \wedge (t_{lmi}^n + t_{lmj}^n + t_{lmk}^n) \right) ,$$

where the set of generators  $t_{ijk}^n$  of degree 0 runs over the set of hyperedges  $ijk$  of  $\Gamma_n^3$ , and where the first set of relations runs over pairs  $(ijk, lmn)$  of disjoint hyperedges and the second set of relations runs over pairs  $(ijk, \{l, m\})$  formed by an hyperedge and two separate vertices of  $\Gamma_n^3$ . For  $n < k$ , we set  $\text{EHKR}(n) := (0, 0)$ .

We consider the same partial composition products as the ones for the 3-Hypergraph quadratic data given in Equation (18).

PROPOSITION 5.8.2. *The Etingof–Henriques–Kamnitzer–Rains skew-symmetric quadratic data  $\text{EHKR} := (\{\text{EHKR}(n)\}, \{\circ_p\})$  forms an operad in the symmetric monoidal category  $(\text{QD}^-, \oplus)$ .*

*Proof.* As in the proof of Proposition 5.6.2, we only need to check that the various maps  $\circ_p$  induce morphisms of quadratic data. Again, this can be achieved easily with the 3-hypergraph description:

the first (respectively second) type relation in  $R(n)$  or  $R(m)$  is sent to any first (respectively second) type relation in  $R(n + m - 1)$ , that is pairs of disjoint hyperedges (respectively a sum of hypergraphs based on pentagons with a distinguished triangle). Any element  $t_{ijk}^n \wedge t_{lmn}^m$  of the relation  $[V(n), V(m)]_-$  is sent to a sum of relations of first and second type under  $(\circ_p)^{\wedge 2}$ .  $\square$

The canonical morphisms of quadratic data  $\text{EHKR}(n) \rightarrow 3\text{-HG}(n)$  induces a canonical morphism of operads  $\text{EHKR} \rightarrow 3\text{-HG}$  in  $\text{QD}^-$ . The following statement is a universal operadic characterisation of the Etingof–Henriques–Kamnitzer–Rains skew-symmetric quadratic data.

THEOREM 5.8.3. *The operad  $\text{EHKR}$  is the smallest sub-operad of 3-HG.*

*Proof.* This proof is similar to that of Theorem 5.6.3. It is also the particular case  $n = 3$  of Theorem 5.10.3.  $\square$

In the lattice of operads made up of skew-symmetric data with generators  $t_{ijk}^n$  and partial composition products  $\circ_p$ , the 3-hypergraphs operad 3-HG is the maximal element and the Etingof–Henriques–Kamnitzer–Rains operad EHKR is the minimal element.

PROPOSITION 5.8.4. *The holonomy operad associated to  $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$  is isomorphic to the Lie operad associated to skew-symmetric quadratic data EKHR:*

$$\mathfrak{g}\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \cong L(\text{EHKR}) .$$

REMARK 5.8.5. The topological operad  $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$  fails to be well pointed; its components are connected with fundamental groups  $\pi_1(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})) \cong \text{PC}_n$  called the pure cactus group in [EHKR10]. It important to notice that these topological spaces however fail to be formal for  $n \geq 6$ . It is however conjectured in [EHKR10] that the Etingof–Henriques–Kamnitzer–Rains holonomy Lie algebras are isomorphic to the Magnus construction

$$L(\text{EHKR}(n)) = \frac{\text{Lie}(t_{ijk}^n)}{\left( [t_{ijk}^n, t_{lmn}^n], [t_{ijk}^n, t_{lmi}^n + t_{lmj}^n + t_{lmk}^n] \right)} \cong \text{gr}(\text{PC}_n) \otimes \mathbb{Q} .$$

The operad in groupoids  $\Pi_1(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}))$  is equivalent to the operad in groupoids which encodes coboundary monoidal categories, see [HK06].

It is proved in [EHKR10, Proposition 3.1] that the Koszul dual symmetric data  $\text{EHKR}^1$  is equal to

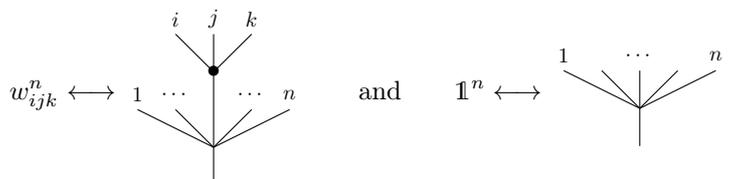
$$\text{EHKR}^1(n) = \left( \omega_{ijk}^n, \omega_{ijk}^n \odot \omega_{klm}^n + \omega_{jkl}^n \odot \omega_{lmi}^n + \omega_{klm}^n \odot \omega_{mij}^n + \omega_{lmi}^n \odot \omega_{ijk}^n + \omega_{mij}^n \odot \omega_{jkl}^n \right) ,$$

where the generator have (homological) degree  $-1$ . The main theorem of [EHKR10] asserts that this quadratic data provides us with a presentation of the cohomology algebra:

$$\begin{aligned} S(\text{EHKR}^1(n)) &= \frac{S(\omega_{ijk}^n)}{\left( \omega_{ijk}^n \odot \omega_{klm}^n + \omega_{jkl}^n \odot \omega_{lmi}^n + \omega_{klm}^n \odot \omega_{mij}^n + \omega_{lmi}^n \odot \omega_{ijk}^n + \omega_{mij}^n \odot \omega_{jkl}^n \right)} \\ &\cong H^\bullet(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})) . \end{aligned}$$

A presentation for the homology operad is also given in *loc. cit.*: it is shown to be isomorphic to the the operad encoding unital 2-Gerstenhaber algebras  $H_\bullet(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})) \cong 2\text{-uGerst}$ . This kind of algebraic structure is made up of a degree 0 unital commutative product and a degree 1 skew-symmetric “2-Lie bracket” of arity 3 which satisfy generalised Leibniz and Jacobi relations.

*Proof of Proposition 5.8.4.* This proof is similar to the one of Proposition 5.6.4. The isomorphism of operads  $H_\bullet(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})) \cong 2\text{-uGerst}$  of [EHKR10] identifies the following elements



where the degree 1 element  $w_{ijk}^n$  stands for  $(\omega_{ijk}^n)^* = st_{ijk}^n$  and where  $\bullet$  denotes the shifted 2-Lie bracket. Under this correspondence, the operad structure on 2-uGerst produces the formulæ given in Equation (18). For instance, the partial composite  $w_{ijk}^n \circ_i \mathbb{1}^m$  gives

$$w_{ijk}^n \circ_i \mathbb{1}^m = \sum_{l=0}^{m-1} w_{i+l,j+m-1,k+m-1}^{n+m-1} ,$$

by the generalised Leibniz relation. □

The canonical morphism of operads  $\text{EHKR} \rightarrow 3\text{-HG}$  in  $\text{QD}^-$  induces a morphism of cocommutative Hopf operads

$$\begin{aligned} 2\text{-uGerst} &\cong H_\bullet(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})) \cong S^c(\text{EHKR}^i) \\ &\rightarrow \text{Gra}^3 \cong H_\bullet(\text{Gra}_{S^1}^3) \cong S^c(3\text{-HG}^i) . \end{aligned}$$

In the light of [Wil15], the study of the deformation complex of this morphism of operads is a very interesting question. What is the analogue of the Grothendieck–Teichmüller Lie algebra  $\text{grt}$  (case  $n = 2$ ) here (case  $n = 3$ )?

### 5.9 LINEAR HYPERGRAPHS AND REAL BRICK MANIFOLDS $\mathcal{B}_{\mathbb{R}}(n)$

Following the same pattern, one can give a  $k$ -hypergraph generalisation of Section 5.5 too. The starting point amounts to considering only *linear  $k$ -hypergraphs*, i.e. the ones made up of intervals of length  $k$ . We denote the complete linear  $k$ -hypergraph by  $\Theta_n^k$ . We define the pointed topological ns operad  $\text{As}_{S^1}^k$  by  $\text{As}_{S^1}^k(n) := \{*\}$ , for  $n < k$ , and by  $\text{As}_{S^1}(n) := (S^1)^{n-k+1}$ , for  $n \geq k$ . Its elements  $(x_{1k}, \dots, x_{n-k+1n})$  are seen as labels, living in  $S^1$ , of the intervals of lengths  $k$  of  $\{1, n\}$ . The partial composition products are given by  $(x_{1k}, \dots, x_{n-k+1n}) \circ_i (y_{1k}, \dots, y_{m-k+1m}) :=$

$$(x_{1k}, \dots, x_{i-k+1i}, \underbrace{*, \dots, *}_{k-2}, y_{1k}, \dots, y_{m-k+1m}, \underbrace{*, \dots, *}_{k-2}, x_{ii+k-1}, \dots, x_{n-k+1n}) .$$

The special case  $k = 2$  gives back the operad  $\text{As}_{S^1} = \text{As}_{S^1}^2$  of Section 5.5.

DEFINITION 5.9.1 (Linear  $k$ -Hypergraph skew-symmetric quadratic data). The *Linear  $k$ -Hypergraph skew-symmetric quadratic data* are spanned by

$$k\text{-LHG}(n) := (t_{ii+k-1}^n, t_{ii+k-1}^n \wedge t_{jj+k-1}^n) ,$$

where the set of generators  $t_{ii+k-1}^n$  of degree 0 runs over the set of linear hyperedges of  $\Theta_n^k$  and where the set of relations runs over all pairs of hyperedges of  $\Theta_n^k$ . For  $n < k$ , we set  $k\text{-LHG}(n) := (0, 0)$ .

We consider the following maps  $\circ_p : k\text{-LHG}(n) \oplus k\text{-LHG}(m) \rightarrow k\text{-LHG}(n+m-1)$ .

$$\begin{aligned} t_{ii+k-1}^n &\mapsto \begin{cases} t_{i+m-1i+k+m-2}^{n+m-1} & \text{for } p \leq i , \\ 0 & \text{for } i < p < i+k-1 , \\ t_{ii+k-1}^{n+m-1} & \text{for } i+k-1 \leq p , \end{cases} \\ t_{jj+k-1}^m &\mapsto t_{j+p-1j+k+p-2}^{n+m-1} . \end{aligned} \tag{19}$$

LEMMA 5.9.2. *The aforementioned data  $k\text{-LHG} := (\{k\text{-LHG}(n)\}, \{\circ_p\})$  forms a nonsymmetric operad in the symmetric monoidal category  $(\mathbb{QD}^-, \oplus)$ .*

*Proof.* The proof is straightforward. □

In the special case  $k = 2$ , we recover the Linear Graph nonsymmetric operad  $\text{LG} = 2\text{-LHG}$ .

PROPOSITION 5.9.3. *The holonomy operad and the rational Magnus operad associated to  $\text{As}_{S^1}^k$  are isomorphic to the Lie operad associated to skew-symmetric quadratic data  $k\text{-LHG}$ :*

$$\mathfrak{g}_{\text{As}_{S^1}^k} \cong \text{L}(k\text{-LHG}) \cong \text{gr}(\pi_1(\text{As}_{S^1}^k)) \otimes \mathbb{Q} .$$

*Proof.* This proof is the same *mutatis mutandis* as the one of Proposition 5.4.5 □

One can define a *linear  $k$ -hypergraph graded operad*  $\text{LGr}^k$ . Its underlying  $\mathbb{N}$ -modules are spanned by sub-hypergraphs of  $\Theta_n^k$ , where each hyperedge receives degree 1. The partial composition product  $\gamma_1 \circ_p \gamma_2$  amounts to first inserting the linear  $k$ -hypergraph  $\gamma_2$  at the  $p$ th vertex of  $\gamma_1$ , then relabelling accordingly the vertices, and finally keeping only the hyperedges of  $\gamma_1$  which do not contain  $p$ , or for which  $p$  is a minimum or a maximum element.

PROPOSITION 5.9.4. *The following three cocommutative Hopf operads are isomorphic*

$$H_\bullet(\text{As}_{S^1}^k) \cong \text{S}^c(k\text{-LHG}^i) \cong \text{LGr}^k .$$

*Proof.* This proof is similar to that of Proposition 5.4.6. □

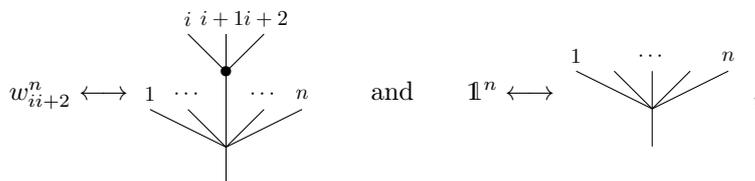
REMARK 5.9.5. For any  $k \geq 2$ , there is a canonical morphism  $\text{spAs}^k \rightarrow \text{LGr}^k$  from the nonsymmetric operad of (shifted) partially associative  $k$ -algebras [LV12, Section 13.11.1] which sends its generator to  $t_{1k}^k$ . Since this operad (unshifted) is the unit for the black product of  $k$ -ary quadratic nonsymmetric operads, one can develop a similar twisting procedure as that of [Wil15] according to [DSV18, Remark 5.8]. The study of the resulting dg nonsymmetric operad  $\text{TwLGr}^k$  is again an interesting subject.

A non-commutative version for the moduli spaces  $\overline{\mathcal{M}}_{0,n+1}$  of stable curves with marked points was given in [DSV15] by means of toric varieties called *brick manifolds* and denoted by  $\mathcal{B}(n)$ . This family was endowed with a topological nonsymmetric operad structure. The linear 3-hypergraph quadratic data is related to this ns operad  $\mathcal{B}_{\mathbb{R}}$  in the real case.

PROPOSITION 5.9.6. *The holonomy operad associated to  $\mathcal{B}_{\mathbb{R}}$  is isomorphic to the Lie operad associated to skew-symmetric quadratic data 3-LHG:*

$$\mathfrak{g}_{\mathcal{B}_{\mathbb{R}}} \cong \text{L}(3\text{-LHG}) .$$

*Proof.* The proof is similar to the proof of Proposition 5.8.4. It relies on the isomorphism of ns operads  $H_{\bullet}(\mathcal{B}_{\mathbb{R}}) \cong 2\text{-ncGerst}$  from [DSV15, Theorem 9.3.1] which identifies the following elements



where the degree 1 element  $w_{ii+2}^n$  stands for  $st_{ii+2}^n$  and where  $\bullet$  denotes the shifted 2-partially associative product. Under this correspondence, the ns operad structure on 2-ncGerst produces exactly the formulæ given in Equation (19).  $\square$

### 5.10 GENERALISATION

Even if the following definition is not prompted by a family of already known topological operads, it is still possible to produce these skew-symmetric quadratic data  $k\text{-}\overline{\text{HG}}$  refining  $k\text{-HG}$ , for any  $k \geq 2$ , following a general canonical procedure which coincides to the aforementioned examples in the cases  $k = 2$  (Section 5.6) and  $k = 3$  (Section 5.8).

DEFINITION 5.10.1 (Refined  $k$ -Hypergraph skew-symmetric quadratic data). The *refined  $k$ -Hypergraph skew-symmetric quadratic data* are spanned by

$$k\text{-}\overline{\text{HG}}(n) := (t_I^n, t_I^n \wedge t_J^n \ \& \ t_{i_1 \dots i_k}^n \wedge (t_{J,i_1}^n + \dots + t_{J,i_k}^n)) ,$$

where the set of generators  $t_I^n$  of degree 0 runs over the set of hyperedges  $I$  of  $\Gamma_n^k$ , and where the first set of relations runs over pairs  $(I, J)$  of disjoint hyperedges of  $\Gamma_n^k$ , i.e.  $I \cap J = \emptyset$ , and the second set of relations runs over pairs  $(I = \{i_1, \dots, i_k\}, J)$  formed by an hyperedge  $I$  and a disjoint set  $J$  of  $k - 1$  vertices of  $\Gamma_n^k$ . For  $n < k$ , we set  $k\text{-}\overline{\text{HG}}(n) := (0, 0)$ .

We consider the same partial composition products as the ones for the  $k$ -Hypergraph quadratic data given in Equation (18).

**PROPOSITION 5.10.2.** *The refined  $k$ -Hypergraph skew-symmetric quadratic data  $k\text{-}\overline{\text{HG}} := (\{k\text{-}\overline{\text{HG}}(n)\}, \{\circ_p\})$  forms an operad in the symmetric monoidal category  $(\text{QD}^-, \oplus)$ .*

*Proof.* The proof is similar to that of Proposition 5.6.2 and Proposition 5.8.2. Relations of first type (respectively second type) are sent to relations of first type (respectively second type) under the partial composition maps  $(\circ_p)^{\wedge 2}$ . Any element of the relation  $[V(n), V(m)]_-$  is sent to a sum of relations of first and second type under  $(\circ_p)^{\wedge 2}$ : for instance, the image of  $t_{J,p}^n \wedge t_I^m$  is equal to

$$\begin{aligned} & \left( \sum_{i \in I} t_{\tilde{J}, i+p-1}^{n+m-1} + \sum_{i \notin I} t_{\tilde{J}, i+p-1}^{n+m-1} \right) \wedge t_{I+p-1}^{n+m-1} \\ &= \left( \sum_{i \in I} t_{\tilde{J}, i+p-1}^{n+m-1} \right) \wedge t_{I+p-1}^{n+m-1} + \sum_{i \notin I} \left( t_{\tilde{J}, i+p-1}^{n+m-1} \wedge t_{I+p-1}^{n+m-1} \right), \end{aligned}$$

where  $\tilde{J}$  is the "image" of  $J$  in the complete  $k$ -hypergraph  $\Gamma_{n+m-1}^k$ , which is produced after relabelling. The first term on the right-hand side is a relation of second type and the second term on the right-hand side is a sum of relations of first type. □

The canonical morphisms of quadratic data  $k\text{-}\overline{\text{HG}}(n) \rightarrow k\text{-HG}(n)$  induces a canonical morphism of operads  $k\text{-}\overline{\text{HG}} \rightarrow k\text{-HG}$  in  $\text{QD}^-$ . The refined  $k$ -Hypergraph skew-symmetric quadratic data is characterized by the following universal operadic property.

**THEOREM 5.10.3.** *The operad  $k\text{-}\overline{\text{HG}}$  is the smallest sub-operad of  $k\text{-HG}$ .*

*Proof.* We proceed in the same way in the proof of Theorem 5.6.3. Let us use the notation  $k\text{-}\overline{\text{HG}}(n) = (V(n), R(n))$  and let us consider a sub-operad  $P(n) := (V(n), S(n)) \subset k\text{-HG}(n)$  of  $k\text{-HG}$ . We show that  $R(n) \subset S(n)$ . We begin with the relations of first type:  $t_I^n \wedge t_J^n$ . Using the action of the symmetric group, we can assume, without any loss of generality, that  $I = \{1, \dots, k\}$  and  $J = \{k + 1, \dots, 2k - 1\}$ . We conclude with

$$(\circ_{k+1})^{\wedge 2}(t_I^{n-k+1} \wedge t_J^k) = t_I^n \wedge t_J^n .$$

We treat now the relations of second type:  $t_{i_1, \dots, i_k}^n \wedge (t_{J, i_1}^n + \dots + t_{J, i_k}^n)$ . Using again the action of the symmetric group, the proof reduces to the case  $(i, j, k, l) = (1, 2, 3, 4)$ , which is given by

$$(\circ_1)^{\wedge 2} (t_I^{n-k+1} \wedge t_J^k) = (t_{J,1}^n + \dots + t_{J,k}^n) \wedge t_I^n .$$

□

In the lattice of operads made up of skew-symmetric data with generators  $t_I^n$  and partial composition products  $\circ_p$ , the  $k$ -hypergraphs operad  $k\text{-HG}$  is the maximal element and the refined  $k$ -hypergraphs operad  $k\text{-}\overline{\text{HG}}$  is the minimal element.

DEFINITION 5.10.4 (Unital  $(k - 1)$ -Gerstenhaber algebra). A *unital  $(k - 1)$ -Gerstenhaber algebra*, for  $k \geq 2$ , is a chain complex  $A$  equipped with an element  $u \in A_0$  and two operations  $\mu : A^{\odot 2} \rightarrow A$  of degree 0 and  $\beta : A^{\odot k} \rightarrow A$  of degree 1 satisfying the following relations.

UNIT RELATIONS  $\mu(u, -) = \text{id}$  and  $\beta(u, -, \dots, -) = 0$  .

ASSOCIATIVITY RELATION  $\mu \circ_1 \mu = \mu \circ_2 \mu$  .

LEIBNIZ RELATION  $\beta \circ_1 \mu = (\mu \circ_1 \beta)^{(2 \ 3 \ \dots \ k \ k+1)} + \mu \circ_2 \beta$  .

JACOBI RELATION  $\sum_{\sigma \in \text{Sh}_{k, k-1}^{-1}} (\beta \circ_1 \beta)^\sigma = 0$  ,

where  $\text{Sh}_{k, k-1}^{-1}$  denotes the set of inverse of  $(k, k - 1)$ -shuffles, also known as  $(k, k - 1)$ -unshuffles.

We denote the associated operad by  $\text{uGerst}_{(k-1)}$ , which is generated by three generators, that we still denote respectively by  $u, \mu$ , and  $\beta$ . We endow it with a cocommutative Hopf operad structure by the following assignment:

$$\begin{aligned} \Delta(u) &:= u \otimes u \\ \Delta(\mu) &:= \mu \otimes \mu \\ \Delta(\beta) &:= \mu^{k-1} \otimes \beta + \beta \otimes \mu^{k-1} \hat{E} , \end{aligned}$$

where  $\mu^{k-1} := \underbrace{\mu \circ_1 \mu \circ_1 \dots \mu \circ_1 \mu}_{k-1 \text{ times } \mu}$  .

LEMMA 5.10.5. *The above assignment defines a cocommutative Hopf operad structure on  $\text{uGerst}_{(k-1)}$ .*

*Proof.* We first need to show that the coproduct  $\Delta$  is well-defined on the quotient of the free operad on  $u, \mu$ , and  $\beta$  by the above relations. One can treat in a straightforward way the unit and the Leibniz relations. The case of the

associativity relation is given by the following computation performed in the free operad

$$\begin{aligned} \Delta(\mu \circ_1 \mu - \mu \circ_2 \mu) &= \mu \circ_1 \mu \otimes \mu \circ_1 \mu - \mu \circ_2 \mu \otimes \mu \circ_2 \mu \\ &= (\mu \circ_1 \mu - \mu \circ_2 \mu) \otimes \mu \circ_1 \mu + \mu \circ_2 \mu \otimes (\mu \circ_1 \mu - \mu \circ_2 \mu) . \end{aligned}$$

The case of the Jacobi relation is treated as follows. Notice first that iterating the Leibniz relation, one gets the relation

$$\beta \circ_1 \mu^{k-1} = \sum_{i=1}^k (\mu^{k-1} \circ_1 \beta)^{\sigma_i} ,$$

where

$$\sigma_i := \begin{bmatrix} 1 & 2 & \dots & k & k+1 & \dots & k+i-1 & k+i & \dots & 2k-1 \\ i & k+1 & \dots & 2k-1 & 1 & \dots & i-1 & i+1 & \dots & k \end{bmatrix}$$

We denote the induced element in the free operad by  $\mathbb{L} := \beta \circ_1 \mu^{k-1} - \sum_{i=1}^k (\mu^{k-1} \circ_1 \beta)^{\sigma_i}$ . Similarly, the element representing the Jacobi relation in the free operad is denoted by  $\mathbb{J}$ . We conclude with the following computation:

$$\begin{aligned} \Delta(\mathbb{J}) &= \sum_{\sigma \in \text{Sh}_{k,k-1}^{-1}} \left( (\beta \circ_1 \beta)^\sigma \otimes \mu^{2(k-1)} + \mu^{2(k-1)} \otimes (\beta \circ_1 \beta)^\sigma \right. \\ &\quad \left. + (\beta \circ_1 \mu^{k-1})^\sigma \otimes (\mu^{k-1} \circ_1 \beta)^\sigma - (\mu^{k-1} \circ_1 \beta)^\sigma \otimes (\beta \circ_1 \mu^{k-1})^\sigma \right) \\ &= \mathbb{J} \otimes \mu^{2(k-1)} + \mu^{2(k-1)} \otimes \mathbb{J} + \sum_{\sigma \in \text{Sh}_{k,k-1}^{-1}} (\mathbb{L} \otimes \mu^{k-1} \circ_1 \beta - \mu^{k-1} \circ_1 \beta \otimes \mathbb{L})^\sigma . \end{aligned}$$

In the end, it is enough to check the cocommutativity and the coassociativity of the coproduct  $\Delta$  on the generators  $u$ ,  $\mu$ , and  $\beta$ . □

PROPOSITION 5.10.6 ([Kho19, Theorem 3.39]). *The cocommutative Hopf operad  $S^c(k\text{-}\overline{\text{HG}}^i)$  is isomorphic to the cocommutative Hopf operad encoding unitary  $(k-1)$ -Gerstenhaber algebras, i.e.*

$$S^c(k\text{-}\overline{\text{HG}}^i) \cong \text{uGerst}_{(k-1)} .$$

*Proof.* A full proof based on the Koszul duality theory can be find in [Kho19]. Let us however sketch the strategy which extends the method of [MR96, BDK07] to the general case  $k \geq 2$ .

Let us denote by  $\text{uCom}$  the operad encoding unital commutative algebras and by  $\text{sLie}_{(k-1)}$  the operad encoding shifted Lie  $(k-1)$ -algebras [HW95]. The

defining relations of the operad  $\text{uGerst}_{(k-1)}$  can be interpreted as rewriting rules which induces a distributive law [Mar96] and [LV12, Section 8.6]. As a consequence, the underlying  $\mathbb{S}$ -module of the operad  $\text{uGerst}_{(k-1)}$  is isomorphic to the operadic composite product

$$\text{uGerst}_{(k-1)} \cong \text{uCom} \circ \text{sLie}_{(k-1)} .$$

This latter  $\mathbb{S}$ -module admits a basis made up of (commutative) forests of  $(k-1)$ -trees, that is rooted trees with all vertices of valence equal to  $k$ , modulo the Jacobi relation.

On the other hand, one can see that the quadratic algebras  $\text{S}(k\text{-}\overline{\text{HG}}^1(n))$ , for  $n \geq k$ , admit the following Koszul dual presentation

$$k\text{-}\overline{\text{HG}}^1(n) = (\omega_I^n, \omega_{i_1 \dots i_k}^n \odot \omega_{i_k \dots i_{2k-1}}^n + \omega_{i_2 \dots i_{k+1}}^n \odot \omega_{i_{k+1} \dots i_{2k-1} i_1}^n + \dots + \omega_{i_{2k-1} i_1 \dots i_{k-1}}^n \odot \omega_{i_{k-1} \dots i_{2k-2}}^n) ,$$

where the set of generators  $\omega_I^n = s^{-1} (t_{1 \dots k}^k)^*$  of degree  $-1$  runs over the set of hyperedges  $I$  of  $\Gamma_n^k$ , and where the set of relations runs over increasing  $(2k-1)$ -tuples  $i_1 < \dots < i_{2k-1}$ .

We consider the pairing  $\langle , \rangle : (\text{uCom} \circ \mathcal{T}(\beta))(n) \otimes \text{S}(\omega_I^n)$  defined by

$$\langle \mu^{n-1}, 1^n \rangle = 1 \quad \text{and} \quad \langle (\mu^{n-k} \circ_1 \beta)^\sigma, \omega_I^n \rangle = 1 \quad \text{and} \quad \langle \chi, \omega_I^n \rangle = 0 ,$$

where  $\sigma$  is the  $(k, n-k)$  shuffle which sends  $\{1, \dots, k\}$  to  $I$  and  $\{k+1, \dots, n\}$  to  $\underline{n} \setminus I$  and where  $\chi$  is an basis element of  $\text{uCom} \circ \mathcal{T}(\beta)$  different from  $(\mu^{n-k} \circ_1 \beta)^\sigma$ . It induces a well-defined and non-degenerate pairing  $\langle , \rangle : (\text{uCom} \circ \text{sLie}_{(k-1)})(n) \otimes \text{S}(k\text{-}\overline{\text{HG}}^1(n))$ , which proves the isomorphism on the level of the underlying  $\mathbb{S}$ -modules

$$\text{S}^c(k\text{-}\overline{\text{HG}}^1) \cong \text{uGerst}_{(k-1)} .$$

By the definition of the pairing, this isomorphism respects to the arity-wise coalgebra structures. It remains to show that it also respects the partial composition products: this can be done in a straightforward way by a computation similar to the ones performed in the proofs of Proposition 5.6.4 and Proposition 5.8.4.

The most difficult part of the proof, not covered here, amounts to proving that the pairing  $\langle , \rangle$  is non-degenerate. This can be done by cumbersome combinatorics or using the Koszul duality results [Kho19, Theorem 3.19 & Theorem 3.21]. □

REMARK 5.10.7. Considering the partitions of  $\{1, \dots, n(k-1) + 1\}$  of size  $l(k-1) + 1$ , for  $0 \leq l \leq n$ , together with their refinement, one gets a poset denoted by  $\Pi_{n(k-1)+1}^{(k)}$ , see [HW95]. This poset is actually the operadic partition poset associated to the set-theoretical operad encoding algebras made up of a commutative operation of arity  $k$  satisfying a totally associative relation, see [Val07]. These posets are Cohen–Macaulay and their top Whitney homology groups produce the cooperad  $(k-1)\text{-uGerst}^*$ .

REMARK 5.10.8. We refer the reader to the forthcoming paper [Kho19] for the homological properties of the quadratic data  $k\text{-}\overline{\text{HG}}(n)$  and their associated operads.

“In the other way round”, one can define a family of pointed topological operads from the aforementioned skew-symmetric quadratic data as follows. We first consider the operad  $\widehat{\text{L}}(k\text{-}\overline{\text{HG}})$  in the category of *complete* Lie algebras, see [DSV18, Section 2]. Then we consider, from [BFMT15, Rob17], the following pair of adjoint functors

$$\text{L} : \text{SIMPLICIAL SETS} \xrightleftharpoons[\perp]{} \text{COMPLETE DG LIE ALGEBRAS} : \text{R} .$$

The right adjoint functor R is the *representation functor* defined by

$$\text{R}(\mathfrak{g}) := \text{Hom}_{\text{COMPLETE DG-LIE-ALG}}(\mathfrak{mc}^\bullet, \mathfrak{g}) ,$$

where the cosimplicial complete dg Lie algebra  $\mathfrak{mc}^\bullet$  is given by quasi-free complete dg Lie algebras

$$\mathfrak{mc}^n := \left( \widehat{\text{Lie}}(s^{-1}\Delta^n), d\widehat{E} \right)$$

on the desuspension of the standard  $n$ -simplices. The left adjoint functor is given by the left Kan extension

$$\text{L} := \text{Lan}_Y \mathfrak{mc}^\bullet$$

along the Yoneda embedding  $Y : \Delta \rightarrow \text{SIMPLICIAL SETS}$ . As a right adjoint, the representation functor R from complete dg Lie algebras to pointed simplicial sets is cartesian, it thus sends operads to operads. This produces a pointed simplicial operad  $\text{R}(\widehat{\text{L}}(k\text{-}\overline{\text{HG}}))$ . Finally, the geometric realisation functor, again cartesian, provides us with the pointed topological operads

$$\mathcal{O}_k := \left| \text{R}(\widehat{\text{L}}(k\text{-}\overline{\text{HG}})) \right| .$$

More details on the above mentioned cartesian functor R, intimately related to the rational homotopy theory of operads, will be given in the sequel of this paper. Meanwhile, it is worth noticing that the representation functor R is equal to the restriction from complete  $L_\infty$ -algebras to complete dg Lie algebras of Getzler’s functor  $\gamma$ , see [Get09].

PROPOSITION 5.10.9. *The holonomy operad and the rational Magnus operad associated to  $\mathcal{O}_k$  are isomorphic to the Lie operad associated to skew-symmetric quadratic data  $k\text{-}\overline{\text{HG}}$ :*

$$\mathfrak{g}_{\mathcal{O}_k} \cong \text{L}(k\text{-}\overline{\text{HG}}) \cong \text{gr}(\pi_1(\mathcal{O}_k)) \otimes \mathbb{Q} .$$

*Proof.* We first claim that, for any rational Lie algebra  $\mathfrak{g}$ , the rational Magnus Lie algebra of  $|\text{R}(\mathfrak{g})|$  is isomorphic to  $\mathfrak{g}$ . We notice first that, for any Lie algebra  $\mathfrak{g}$ , the above definition of the functor R implies  $\text{R}(\mathfrak{g})_0 \cong \{0\}$  and

$$\text{R}(\mathfrak{g})_n \cong \left\{ (x_{ij})_{0 \leq i < j \leq n} \in \mathfrak{g}^{\binom{n+1}{2}} \mid \text{BCH}(x_{ij}, x_{jk}) = x_{ik}, \text{ for } 1 \leq i < j < k \leq n \right\}$$

for  $n \geq 1$ , where BCH stands for the Baker–Campbell–Hausdorff formula. Its simplicial maps are given by

$$s_k((x_{ij})) = \begin{cases} x_{ij} & \text{for } i < j \leq k, \\ x_{ij-1} & \text{for } i < k < j, \\ x_{i-1j-1} & \text{for } k < i < j, \\ 0 & \text{for } i = k, j = k + 1, \end{cases} \quad \text{and}$$

$$d_k((x_{ij})) = \begin{cases} x_{ij} & \text{for } i < j \leq k, \\ x_{ij+1} & \text{for } i < k \leq j, \\ x_{i+1j+1} & \text{for } k \leq i < j. \end{cases}$$

It is a Kan complex canonically pointed by 0, see for instance [RV20] for details. It is straightforward to compute its first simplicial homotopy group:  $(\pi_1(\mathbf{R}(\mathfrak{g})), \cdot) \cong (\mathfrak{g}, \text{BCH})$ . This produces the first isomorphism of Lie algebras

$$\text{gr}(\pi_1(|\mathbf{R}(\mathfrak{g})|)) \otimes \mathbb{Q} \cong \mathfrak{g},$$

by [Laz50].

Finally, we claim that, for any skew-symmetric quadratic data  $(V, R)$ , the holonomy Lie algebra of  $|\mathbf{R}(\widehat{\mathbf{L}}(V, R))|$  is isomorphic to the quadratic Lie algebra  $\mathbf{L}(V, R)$ . With the above description, it is straightforward to compute the rational simplicial groups of  $\mathbf{R}(\mathfrak{g})$ , which gives  $H_1(\mathbf{R}(\mathfrak{g})) \cong sV$  and  $\text{im } \Delta \cong s^2R$ . This implies the isomorphism of Lie algebras:

$$\mathfrak{g}_{|\mathbf{R}(\widehat{\mathbf{L}}(V, R))|} \cong \mathbf{L}(V, R).$$

All these isomorphisms are natural and respect the operad structures. □

Let us sum up the results of the previous sections into the following table.

	$k = 2$	$k = 3$	$k \geq 4$
MAX	BKW $\leftrightarrow$ $\text{Gra}_{S^1}$	3-HG $\leftrightarrow$ $\text{Gra}_{S^1}^3$	$k$ -HG $\leftrightarrow$ $\text{Gra}_{S^1}^k$
MIN	DK $\leftrightarrow$ $\mathcal{D}_2$	EHKR $\leftrightarrow$ $\overline{\mathcal{M}}_{0, n+1}(\mathbb{R})$	$k$ - $\overline{\text{HG}}$ $\leftrightarrow$ $\mathbf{R}(\widehat{\mathbf{L}}(k\text{-}\overline{\text{HG}}))$

REMARK 5.10.10. In the case of linear hypergraphs, one can see that the lattice of nonsymmetric operads made up of skew-symmetric data with generators  $t_{ii+k-1}^n$  and partial composition products  $\circ_p$  contains only one element: the nonsymmetric operad  $k$ -LHG. Therefore, there is no way to refine it following the above pattern.

The canonical morphism of operads  $k\text{-}\overline{\text{HG}} \rightarrow k\text{-HG}$  in  $\text{QD}^-$  induces a morphism of cocommutative Hopf operads

$$(k-1)\text{-uGerst} \cong H_\bullet \left( \left| \mathbf{R}(\widehat{\mathbf{L}}(k\text{-}\overline{\text{HG}})) \right| \right) \cong S^c(k\text{-}\overline{\text{HG}}^i) \\ \rightarrow \text{Gra}^k \cong H_\bullet(\text{Gra}_{S^1}^k) \cong S^c(k\text{-HG}^i).$$

Studying the associated deformation complex would solve the following question: what is the  $k^{\text{th}}$  analogue (case  $n = k$ ) of the Grothendieck–Teichmüller Lie algebra  $\text{grt}$  (case  $n = 2$ )?

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