Stability of Equivariant Vector Bundles over Toric Varieties

Jyoti Dasgupta, Arijit Dey, and Bivas Khan

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Abstract. We give a complete answer to the question of (semi)stability of tangent bundles on any nonsingular complex projective toric variety with Picard number 2 by using combinatorial criterion of (semi)stability of an equivariant sheaf. We also give a complete answer to the question of (semi)stability of tangent bundles on all toric Fano 4-folds with Picard number ≤ 3 which are classified by Batyrev [1]. We construct a collection of equivariant indecomposable rank 2 vector bundles on Bott towers and pseudo-symmetric toric Fano varieties. Furthermore, in case of Bott towers, we show the existence of an equivariant stable rank 2 vector bundle with certain Chern classes with respect to a suitable polarization.

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1 Introduction

Let X be a toric variety of dimension n, equipped with an action of the n-dimensional torus T with an associated fan Δ over an algebraically closed field k. A quasi-coherent sheaf $\mathcal{E}$ on X is said to be T-equivariant or simply an equivariant sheaf if it admits a lift of the T-action on X, which is linear on the stalks of $\mathcal{E}$. An equivariant structure on a sheaf $\mathcal{E}$ need not be unique. Any line bundle on a toric variety has an equivariant structure. It is well known that any locally free sheaf $\mathcal{E}$ on X is equivariant if and only if $t^*\mathcal{E} \cong \mathcal{E}$ for every $t \in T$ (see [26, Proposition 1.2.1]). Equivariant vector bundles over a nonsingular complete toric variety up to isomorphism were first classified by Kaneyama [20], [21] by involving both combinatorial and linear algebraic data modulo an equivalence relation. Later in a foundational paper [26], Klyachko classified equivariant vector bundles more systematically. In this paper, he gave a complete classification of equivariant bundles over an arbitrary toric variety in terms of a family of decreasing filtrations on a fixed finite dimensional vector space indexed by one dimensional cones satisfying certain compatibility condition [26, Theorem 2.2.1]. Most of the topological and algebraic invariants of equivariant vector bundles like Chern classes, global sections, cohomology spaces; he could decode from this filtration data. As a major application, later he used classification of equivariant vector bundles over $\mathbb{P}^2$ to prove Horn's conjecture on eigenvalues of sums of Hermitian matrices [27]. Recently this classification result has been generalized for equivariant principal G-bundles on any complex toric variety using two different approaches ([3], [22], [23]), where G is a complex linear algebraic group. In another work, the authors give a classification of equivariant principal G-bundles on any toric variety defined over an algebraically closed field when G is reductive, using a Tannakian approach [4].

In an unpublished preprint [25], Klyachko gave a generalization of the above classification theorem for equivariant torsion-free sheaves and gave a sketch without all details. Thereafter, Perling introduced the notion of Δ-families $\{E_m^\sigma\}_{\sigma\in\Delta,\ m\in M}$ for any quasi-coherent equivariant sheaf $\mathcal{E}$ which is constructed from the T-eigenspace decompositions of the modules of sections, together with the multiplication maps for regular T-eigenfunctions. He showed that the category of Δ-families is equivalent to the category of equivariant quasi-coherent sheaves [38]. When the sheaf $\mathcal{E}$ is torsion-free, corresponding Δ-family induces a family of multifiltrations of subspaces $\{E_m^\sigma\}_{\sigma\in\Delta,\ m\in M}$ on a fixed finite dimensional vector space $\mathbf{E}^0$ satisfying certain compatibility condition (see Theorem 2.2.8). Further, if we restrict ourselves to reflexive sheaves, then the entire Perling data becomes a family of increasing full finite dimensional filtered vector spaces $(\mathbf{E}^0, \{E^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}})$ without any compatibility condition, where $E_m^\rho = E^\rho(\langle m, v_\rho\rangle)$. Conversely, any such family of filtered vector spaces corresponds to an equivariant reflexive sheaf [38, Theorem 5.19]. This crucial observation of Perling is the starting point of the paper. Further, the first Chern class of an equivariant coherent sheaf can be computed from its associated Δ-family (see [30, Corollary 3.18]).

One of the most important and well-explored problems in moduli theory is the moduli problem of vector bundles of fixed topological type over a nonsingular projective variety. In general, we cannot expect that the set of all isomorphism classes of vector bundles over a nonsingular projective variety of a fixed topological type has an algebraic scheme structure. To overcome this, the notions of stable and semistable vector bundles were introduced by Mumford. When the underlying variety is a nonsingular projective curve, Mumford showed that the set of isomorphism classes of stable vector bundles of fixed rank and degree has the structure of a quasiprojective scheme [34]. Later this moduli space was compactified by Seshadri, using S-equivalence classes of semistable vector bundles [42]. Further, this work has been generalized for higher dimensional varieties by Gieseker [13], Maruyama [32, 33] and many others. It is natural to consider the equivariant versions of these notions for equivariant vector bundles over nonsingular projective toric varieties. Let X be a nonsingular complex projective toric variety and H be an equivariant very ample line bundle (equivalently, T-invariant very ample divisor) on X. An equivariant torsion-free sheaf $\mathcal{E}$ on X is said to be equivariantly stable (respectively, semistable) with respect to H if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$) for every proper equivariant subsheaf $\mathcal{F} \subset \mathcal{E}$ (see Section 2.3 for definition of μ). From the uniqueness of the Harder-Narasimhan filtration it follows easily that the notions of semistability and equivariant semistability of an equivariant torsion-free sheaf on a nonsingular projective toric variety are equivalent. Further, the notions of stability and equivariant stability also coincide for any equivariant torsion-free sheaf (see [2, Theorem 2.1]). When $\mathcal{E}$ is an equivariant reflexive sheaf, to determine its (semi)stability, it is enough to consider equivariant reflexive subsheaves of $\mathcal{E}$ (see Remark 2.3.2).

The purpose of this paper is twofold. First, we study (semi)stability of tangent bundle on a nonsingular projective toric variety with Picard number at most 3 (in Section 4 and 5). Secondly, we construct new examples of rank 2 equivariant vector bundles which are indecomposable or even stable over a large collection of nonsingular projective toric varieties of arbitrary dimension (in Section 6). Both these results rely on the key fact that one can combinatorially classify equivariant reflexive subsheaves of an equivariant reflexive sheaf (see Corollary 3.1.2). This turns out to be the central theme of the paper. In fact, with this technique, theoretically it is possible to check (semi)stability of any equivariant torsion-free sheaf on a nonsingular projective toric variety. Nevertheless, as the Picard number grows and the fan structure becomes more and more complicated, the task of computing degree of subsheaves becomes cumbersome. We hope one can write a computer program to check (semi)stability of any equivariant torsion-free sheaf from its given combinatorial data and the fan structure of the toric variety with respect to any polarization.

Tangent bundles $\mathcal{T}_X$ are natural examples of equivariant vector bundles on nonsingular toric varieties. The filtration data $\left(\mathscr{T}, \{\mathscr{T}^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ associated to $\mathcal{T}_X$ is relatively simple, it has a two step filtration of flag type $(1, n-1)$ for each $\rho \in \Delta(1)$ (see Corollary 2.2.17). The first main step of this paper is to show that equivariant reflexive subsheaves of $\mathcal{T}_X$ are in one-to-one correspondence with induced subfiltrations $\left(\mathbf{F}^0, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ of $\left(\mathscr{T}, \{\mathscr{T}^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$. In fact, this holds for all equivariant torsion-free (respectively, reflexive) subsheaves of any equivariant torsion-free (respectively, reflexive) sheaf (see Proposition 3.1.1, Corollary 3.1.2). This result is a natural generalization of [24, Proposition 4.1.1], where equivariant subbundles of an equivariant vector bundle were classified. We first apply this result to give a very simple proof of stability of tangent bundle on a projective space (see Proposition 4.1.1). Next, we study (semi)stability of tangent bundle on a nonsingular projective toric variety with Picard number 2. A theorem of Kleinschmidt [9, Theorem 7.3.7] tells us that any such variety X is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$, where $s, r \geq 1$, $s + r = \dim(X)$ and $0 \leq a_1 \leq \ldots \leq a_r$ are integers. In this case, tangent bundle will be always unstable with respect to any polarization whenever $(a_1, \ldots, a_r) \neq (0, 0, \ldots, 0, 1)$ (see Theorem 4.2.2). When $(a_1, \ldots, a_r) = (0, 0, \ldots, 0, 1)$, we give a necessary and sufficient condition for (semi)stability of tangent bundle with respect to any polarization (see Theorem 4.2.5). As a corollary we give a complete answer to (semi)stability of tangent bundle with respect to anticanonical divisor $-K_X$ for any Fano toric variety with Picard number 2 (see Corollary 4.2.7). This generalizes a very recent result of [2, Theorem 9.3].

By the result of Kobayashi [29] and Lübke [31], stability of tangent bundle with respect to $-K_X$ for a nonsingular Fano variety is considered to be an algebraic geometric analogue of existence of Kähler-Einstein metric on a smooth manifold. It is an open question if tangent bundle on a nonsingular Fano variety with Picard number 1 is stable with respect to $-K_X$. Though the conjecture is known in many cases (see [40],[39],[19],[44],[11] etc.), this question is wide open in general. If the Picard number is > 1, tangent bundle is not necessarily stable due to the geometry of contractions of extremal rays, for 3-folds this has been studied completely by Steffens [43]. By the result described in the previous paragraph, we have completely settled this question for any nonsingular toric Fano variety with Picard number 2. In Section 5, we study (semi)stability of tangent bundles on nonsingular Fano toric 4-folds with Picard number 3. In [1], [41], Batyrev and subsequently Sato have given a complete list of isomorphism classes of all nonsingular Fano toric 4-folds. There are in total 124 non-isomorphic toric Fano 4-folds. Among them there are 28 isomorphism classes with Picard number 3, out of which 8 are toric blow ups and 19 of them are projectivizations of split vector bundle over a toric variety and the last one is neither a blow up nor a projectivization of a splittable vector bundle. Among them 6 are stable, 3 are strictly semistable and rest have unstable tangent bundles with respect to the anticanonical polarization (see Table 1, Section 5).

In Section 6, first we consider a class of nonsingular projective toric varieties, known as Bott towers. A Bott tower of height n

$$M_n \to M_{n-1} \to \cdots \to M_2 \to M_1 \to M_0 = \{\text{point}\}$$

is defined inductively as an iterated projective bundle so that each stage M_k of the tower is of the form $\mathbb{P}(\mathcal{O}_{M_{k-1}} \oplus \mathcal{L})$ for an arbitrarily chosen line bundle $\mathcal{L}$ over the previous stage M_{k-1}. Bott towers were shown to be deformation equivalent to Bott-Samelson varieties by Grossberg and Karshon in [14]. In this section, we construct a (finite) collection of indecomposable rank 2 equivariant vector bundles over M_k $(k \geq 2)$ (Proposition 6.1.1). Further, we show that among these collections, there exists a stable rank 2 vector bundle over M_k $(k \geq 2)$, for certain Chern classes with respect to a suitable choice of polarization

(Proposition 6.1.5). The approach here is to construct a dimension 2 filtration corresponding to an equivariant vector bundle such that it does not have any induced subfiltration which violates the stability. In the next subsection, we consider pseudo-symmetric toric varieties which are very important examples of toric varieties and appear in classification of projective Fano toric varieties (see [41] for details). In [8], Cotignoli and Sterian have constructed a collection of indecomposable rank 2 vector bundles over pseudo-symmetric toric Fano varieties other than product of $\mathbb{P}^1$'s. It is not clear to us if they are equivariant or not. In this subsection, we construct a collection of rank 2 equivariant indecomposable vector bundles on any pseudo-symmetric toric Fano variety (Proposition 6.2.1).

We summarize our results as follows. Here all toric varieties are defined over complex numbers.

1. Classification of equivariant torsion-free (respectively, reflexive) subsheaves of a given equivariant torsion-free (respectively, reflexive) sheaf.
2. A simple proof for stability of tangent bundle on a projective space.
3. A necessary and sufficient condition for (semi)stability of tangent bundle on a nonsingular projective toric variety with Picard number 2 with respect to any polarization.
4. A complete answer to the question of (semi)stability of tangent bundles for Fano toric 4-folds with Picard number 3 from the classification due to Batyrev [1].
5. Construction of equivariant indecomposable as well as stable rank 2 vector bundles over a Bott tower.
6. Construction of equivariant indecomposable rank 2 vector bundles on pseudo-symmetric Fano toric variety.

After this work got complete, we came to know about the work of Hering, Nill and Süss, where they studied (semi)stability of tangent bundle on nonsingular toric variety for Picard number 2. Their result [17, Theorem 1.4] matches with our result in Section 4.

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2 Preliminaries and some basic facts

In this section, we briefly review some basic definitions and results on toric varieties and torus equivariant sheaves, which will be required later.

2.1 Toric Varieties

Let $T \cong (k^*)^n$ be an n-dimensional algebraic torus, where k is an algebraically closed field. A toric variety X of dimension n is a normal variety which contains T as an open dense subset such that the torus multiplication extends to an action of T on X. Toric varieties have a rich combinatorial structure which arises due to the action of the dense torus. We recall some basic facts about toric varieties which will be used in subsequent sections. For more details see [9], [12] and [35].

Let $M = \text{Hom}(T, k^*) \cong \mathbb{Z}^n$ be the character lattice of T and $N = \text{Hom}(M, \mathbb{Z})$ be the dual lattice. We denote by $\langle , \rangle : M \times N \to \mathbb{Z}$ the natural pairing between M and N. Let Δ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$, which defines a nonsingular projective toric variety $X = X(\Delta)$ of dimension n over k under the action of T. Let $S_\sigma := \sigma^\vee \cap M$ be the affine semigroup and $U_\sigma := \text{Spec } k[S_\sigma]$ be the affine toric variety corresponding to a cone $\sigma \in \Delta$. Let x_σ denote the distinguished point of U_σ (see [12, Section 2.1]). Then $T_\sigma := \text{Stab}(x_\sigma)$ is a subtorus of T with character lattice $M_\sigma := M/S_\sigma^\perp$, where $S_\sigma^\perp := \sigma^\perp \cap M$. The T-invariant closed subvariety corresponding to a cone σ is denoted by $V(\sigma)$, which is the closure of the T-orbit through x_σ and dim $V(\sigma) = n - \dim \sigma$. We denote the set of all cones of dimension d in Δ by $\Delta(d)$. Elements of $\Delta(1)$ are called rays. Each ray ρ has a unique minimal ray generator, which we denote by v_ρ. Sometime we will use the ray ρ and its minimal generator v_ρ interchangeably. Each ray ρ corresponds to a T-invariant prime divisor $D_\rho := V(\rho)$.

The following proposition on toric intersection theory will be extensively used in later sections, while computing slope of equivariant sheaves over a toric variety.

Proposition 2.1.1. *([9, Corollary 6.4.3, Lemma 6.4.4, Lemma 12.5.2]) Let $X(\Delta)$ be a nonsingular projective toric variety. To compute the intersection product $D_\rho \cdot V(\tau)$, where $\tau \in \Delta(n-1)$ is a wall, i.e. $\tau = \sigma \cap \sigma'$ for some $\sigma, \sigma' \in \Delta(n)$, write $\sigma = Cone(v_{\rho_1}, \dots, v_{\rho_n}), \sigma' = Cone(v_{\rho_2}, \dots, v_{\rho_{n+1}})$ and $\tau = Cone(v_{\rho_2}, \dots, v_{\rho_n})$. Then $v_{\rho_1}, \dots, v_{\rho_{n+1}}$ satisfy the linear relation $v_{\rho_1} + \sum_{i=2}^{n} b_i v_{\rho_i} + v_{\rho_{n+1}} = 0$, $b_i \in \mathbb{Z}$, called the wall relation. Then*

$$D_\rho \cdot V(\tau) = \begin{cases} 0 & \text{for all } \rho \notin \{\rho_1, \dots, \rho_{n+1}\} \\ 1 & \text{for } \rho \in \{\rho_1, \rho_{n+1}\} \\ b_i & \text{for } \rho = \rho_i,\ 2 \le i \le n. \end{cases}$$

More generally, for distinct rays $\rho_1, \dots, \rho_d \in \Delta(1)$ we have

$$D_{\rho_1} \cdot D_{\rho_2} \cdots D_{\rho_d} = \begin{cases} [V(\sigma)] \in A^\bullet(X) & \text{if } \sigma = Cone(\rho_1, \dots, \rho_d) \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Here $[V(\sigma)]$ denotes the rational equivalence class of $V(\sigma)$ in the Chow ring $A^\bullet(X)$.

We recall the fan structures of the following two classes of toric varieties which will be used in Section 5 while studying (semi)stability of tangent bundles on toric Fano 4-folds.

2.1.1 Projectivization of direct sum of line bundles on toric varieties

Let $D_0, D_1, \ldots, D_m$ be T-invariant Cartier divisors on a nonsingular toric variety $X = X(\Delta)$. Then the fan Δ' in $\mathbb{R}^n \oplus \mathbb{R}^m$ of $X' = \mathbb{P}(\mathcal{O}_X(D_0) \oplus \mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_m))$ is described as follows (see [35, Page 58] for details). Let $h_0, h_1, \ldots, h_m$ be the Δ-linear support functions corresponding to $D_0, D_1, \ldots, D_m$ respectively (see [35, Section 2.1]). Choose the standard $\mathbb{Z}$-basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$ and let $e_0 = -e_1 - \ldots - e_m$. Consider the $\mathbb{R}$-linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^m$, given by $y \mapsto (y, -\sum_{j=0}^{m} h_j(y)e_j)$. Now let $\tilde{\sigma}_i = \text{Cone}(e_0, \ldots, \widehat{e_i}, \ldots, e_m)$ for each $0 \leq i \leq m$ (henceforth by $\widehat{e_i}$ we mean that e_i is omitted from the relevant collection). Let $\tilde{\Delta}$ be the fan in $\mathbb{R}^m$ generated by $\tilde{\sigma}_i$ for $0 \leq i \leq m$. Then $\Delta' = \{\Phi(\sigma) + \tilde{\sigma} : \sigma \in \Delta, \tilde{\sigma} \in \tilde{\Delta}\}$.

2.1.2 Blowup of a toric variety along an invariant subvariety

Let $X = X(\Delta)$ be a nonsingular toric variety. Let τ be a cone in Δ and $\tilde{X} = Bl_{V(\tau)}(X)$ be the blowup of X along the T-invariant subvariety $V(\tau)$. Let $\tilde{\Delta}$ be the fan corresponding to $\tilde{X}$. Then $\tilde{\Delta} = \{\sigma \in \Delta : \tau \nsubseteq \sigma\} \cup \bigcup_{\tau \preceq \sigma} \Delta^*_\sigma(\tau)$, where $\Delta^*_\sigma(\tau) = \{\text{Cone}(A) : A \subseteq \{u_\tau\} \cup \sigma(1), \tau(1) \nsubseteq A\}$ for any cone σ containing τ, denoted as $\sigma \succeq \tau$ and $u_\tau = \sum_{\rho \in \tau(1)} v_\rho$ (see [9, Definition 3.3.17]).

2.2 Equivariant sheaves

We briefly recall the combinatorial description of equivariant sheaves on toric varieties introduced by Perling [38], which will be used later.

Let $X = X(\Delta)$ be a toric variety corresponding to a fan Δ. For each cone $\sigma \in \Delta$, define a relation $\leq_\sigma$ on M by setting $m \leq_\sigma m'$ if and only if $m' - m \in S_\sigma$. We write $m <_\sigma m'$ if $m \leq_\sigma m'$ holds but $m' \leq_\sigma m$ does not hold. A σ-family, denoted by $\widehat{E}^\sigma$, is a family of k-vector spaces $\{E^\sigma_m\}_{m \in M}$ together with linear maps $\chi^\sigma_{m,m'} : E^\sigma_m \to E^\sigma_{m'}$, whenever $m \leq_\sigma m'$ such that $\chi^\sigma_{m,m} = 1$ and $\chi^\sigma_{m,m''} = \chi^\sigma_{m',m''} \circ \chi^\sigma_{m,m'}$ for every triple $m \leq_\sigma m' \leq_\sigma m''$.

Let $\widehat{E}^\sigma$ and $\widehat{F}^\sigma$ be two σ-families with linear maps $\chi^\sigma_{m,m'}$ and $\psi^\sigma_{m,m'}$, respectively. Then a morphism $\widehat{\phi}^\sigma$ of σ-families from $\widehat{E}^\sigma$ to $\widehat{F}^\sigma$ is a set of linear maps $\{\phi^\sigma_m : E^\sigma_m \to F^\sigma_m\}_{m \in M}$ such that $\phi^\sigma_{m'} \circ \chi^\sigma_{m,m'} = \psi^\sigma_{m,m'} \circ \phi^\sigma_m$ for all $m, m' \in M$ with $m \leq_\sigma m'$.

Remark 2.2.1. Note that $\chi^\sigma_{m,m'}$ is an isomorphism whenever $m' - m \in S^\perp_\sigma$ (see [38, Lemma 5.3]). Hence we restrict our attention to σ-families having $\chi^\sigma_{m,m'} = 1$ (and hence $E^\sigma_m = E^\sigma_{m'}$) for all $m' - m \in S^\perp_\sigma$.

Let $\mathcal{E}$ be an equivariant quasi-coherent sheaf on the toric variety $X = X(\Delta)$ (see [38] for detailed definition of equivariant sheaves). The T-action on $\mathcal{E}$ gives rise to an isomorphism $\Phi_t : t^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$ for all $t \in T$. This induces an action of T on the space of global sections $E^\sigma := \Gamma(U_\sigma, \mathcal{E})$ given by $t \cdot f = \Phi_t(t^* f)$, where $f \in E^\sigma$ and $t^* f \in \Gamma(U_\sigma, t^*\mathcal{E})$ is its canonically lifted section. Hence we get the T-isotypical decomposition $E^\sigma = \bigoplus\limits_{m \in M} E^\sigma_m$, which makes E^σ naturally an M-graded $k[S_\sigma]$-module as follows. Recall that the action of the torus on the affine open variety U_σ induces the T-isotypical decomposition $k[S_\sigma] = \bigoplus\limits_{m \in S_\sigma} k\chi^m$. Then the M-graded $k[S_\sigma]$-module structure on E^σ is given by $\chi^\sigma_{m,m'} : E^\sigma_m \to E^\sigma_{m'}$, $e \mapsto \chi^{m'-m} \cdot e$, where $m, m' \in M$ and $m' - m \in S_\sigma$. Then the following three categories are equivalent (see [38, Proposition 5.5]):

(i) Equivariant quasi-coherent sheaves over U_σ,

(ii) M-graded $k[S_\sigma]$-modules with M-graded preserving homomorphisms,

(iii) σ-families.

For each pair $\tau \preceq \sigma$, we denote by $i_{\tau\sigma} : U_\tau \hookrightarrow U_\sigma$ the inclusion. Let $\widehat{E}^\sigma$ be a σ-family. We denote by $E^\sigma := \bigoplus_{m \in M} E^\sigma_m$ the corresponding M-graded $k[S_\sigma]$-module. The pull back $i^*_{\tau\sigma} E^\sigma = E^\sigma \otimes_{k[S_\sigma]} k[S_\tau]$ is naturally an M-graded $k[S_\tau]$-module (see [38, Section 5.2]) and hence corresponds to a τ-family (by the above equivalence of categories), which we denote by $i^*_{\tau\sigma}\widehat{E}^\sigma$.
The following notion of a Δ-family was introduced by Perling to go from affine toric varieties to general toric varieties.

Definition 2.2.2. *[38, Definition 5.8] A collection $\{\widehat{E}^\sigma\}_{\sigma \in \Delta}$ of σ-families is called a Δ-family, denoted by $\widehat{E}^\Delta$, if for each pair $\tau \preceq \sigma$ there exists an isomorphism of families $\eta_{\tau\sigma} : i^*_{\tau\sigma}(\widehat{E}^\sigma) \cong \widehat{E}^\tau$ such that for each triple $\rho \preceq \tau \preceq \sigma$ there is an equality $\eta_{\rho\sigma} = \eta_{\rho\tau} \circ i^*_{\rho\tau}\eta_{\tau\sigma}$.*
A morphism of Δ-families is a collection of morphisms $\{\widehat{\phi}^\sigma : \widehat{E}^\sigma \to \widehat{F}^\sigma\}_{\sigma \in \Delta}$ such that for all σ, τ and $\tau \preceq \sigma$, the following diagram commutes:

$$\begin{array}{ccc} i^*_{\tau\sigma}(\widehat{E}^\sigma) & \xrightarrow{\;i^*_{\tau\sigma}\widehat{\phi}^\sigma\;} & i^*_{\tau\sigma}(\widehat{F}^\sigma) \\ \downarrow{\scriptstyle \eta^E_{\tau\sigma}} & & \downarrow{\scriptstyle \eta^F_{\tau\sigma}} \\ \widehat{E}^\tau & \xrightarrow[\;\widehat{\phi}^\tau\;]{} & \widehat{F}^\tau. \end{array}$$

Theorem 2.2.3. *[38, Theorem 5.9] The category of Δ-families is equivalent to the category of equivariant quasi-coherent sheaves on X.*

In order to classify equivariant coherent sheaves, Perling introduced the following notion of finite Δ-family.

Definition 2.2.4. *[38, Definition 5.10] A σ-family $\widehat{E}^\sigma$ is said to be finite if:*

(i) E^σ_m's are finite dimensional for all $m \in M$,

(ii) for each chain $\ldots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \ldots$ of characters in M there exists an $i_0 \in \mathbb{Z}$ such that $E^\sigma_{m_i} = 0$ for all $i < i_0$,

(iii) there are only finitely many vector spaces E^σ_m such that the map $\bigoplus_{m' <_\sigma m} E^\sigma_{m'} \longrightarrow E^\sigma_m$, defined by the summation of the $\chi^\sigma_{m',m}$, is not surjective.

A Δ-family is said to be finite if all of its σ-families are finite.

Proposition 2.2.5. *[38, Proposition 5.11] A quasi-coherent equivariant sheaf is coherent if and only if its associated Δ-family is finite.*

Note that given a σ-family $\widehat{E}^\sigma$, the collection $\{E^\sigma_m, \chi^\sigma_{m,m'}\}$ forms a directed system of vector spaces. Let $\mathbf{E}^\sigma := \varinjlim_{m \in M} E^\sigma_m$. Any element of $\mathbf{E}^\sigma$ can be written as equivalence classes $[e, m]$, where $e \in E^\sigma_m$ and $[e, m] = [e', m']$ if and only if there exists $m'' \in M$ satisfying $m, m' \leq_\sigma m''$ such that $\chi^\sigma_{m,m''}(e) = \chi^\sigma_{m',m''}(e')$. Let $\mathcal{E}$ be an equivariant torsion-free sheaf of rank r on X. Then for all $\sigma \in \Delta$ and $m \leq_\sigma m'$, the maps in the following diagram are injective (see [38, Proposition 5.13]):

$$\begin{array}{ccc} E^\sigma_m & \xrightarrow{\chi^\sigma_{m,m'}} & E^\sigma_{m'} \\ & \searrow & \downarrow \\ & & \mathbf{E}^\sigma . \end{array} \tag{2.1}$$

Moreover, the restriction map $\Gamma(U_\sigma, \mathcal{E}) \to \Gamma(U_\tau, \mathcal{E})$ is injective whenever $\tau \preceq \sigma$. Now let m_τ be an integral element of the interior of $\sigma^\vee \cap \tau^\perp$ such that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_\tau)$. Note that $\Gamma(U_\tau, i^*_{\tau\sigma}(\mathcal{E}|_{U_\sigma})) = \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\tau] = \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[S_\sigma]} k[S_\sigma][\chi^{-m_\tau}]$. So there exists a natural inclusion of σ-families $\alpha_{\tau\sigma} : \widehat{E}^\sigma \to i^*_{\tau\sigma}\widehat{E}^\sigma$. Hence the following composition

$$\widehat{E}^\sigma \xhookrightarrow{\alpha_{\tau\sigma}} i^*_{\tau\sigma}\widehat{E}^\sigma \xrightarrow[\cong]{\eta_{\tau\sigma}} \widehat{E}^\tau \tag{2.2}$$

is injective (see Definition 2.2.2, [38, Proposition 5.14]), which further induces a natural injection

$$\tilde{\eta}_{\tau\sigma} : \ \mathbf{E}^\sigma \longhookrightarrow \mathbf{E}^\tau, \ [(e, m)] \mapsto [(\eta^m_{\tau\sigma} \circ \alpha^m_{\tau\sigma})(e), m)]. \tag{2.3}$$

The system of vector spaces $\{\mathbf{E}^\sigma\}_{\sigma \in \Delta}$ together with the homomorphisms $\tilde{\eta}_{\tau\sigma}$ for $\tau \preceq \sigma$, forms a directed partially ordered family whose direct limit can be identified with $\mathbf{E}^0$, here 0 denotes the zero cone. Note that we have the following isotypical decomposition

$$\Gamma(T,\mathcal{E}) = k[M]^r = \bigoplus_{m\in M} (\underbrace{k\chi^m \oplus \cdots \oplus k\chi^m}_{r \text{ times}})$$

and the homomorphisms $\chi^0_{m,m'}$ are isomorphisms. Hence we can identify $\mathbf{E}^0$ with $k\chi^m \oplus \cdots \oplus k\chi^m$ (r times) and thus it is a finite dimensional vector space of dimension r. Also, the natural inclusions $\mathbf{E}^\sigma \hookrightarrow \mathbf{E}^0$ obtained in (2.3) are isomorphisms $\mathbf{E}^\sigma \cong \mathbf{E}^0$ (see [38, Corollary 5.16]). Thus all the vector spaces in the Δ-family $\widehat{E}^\Delta$ can be realized as vector subspaces of $\mathbf{E}^0$.
The above technical reformulation of a finite Δ-family leads to the following definition of family of multifiltrations.

Definition 2.2.6. *[38, Definition 5.17] Let Δ be a fan, V be a finite-dimensional k-vector space, and let for each $\sigma \in \Delta$ a set of vector subspaces $\{E^\sigma_m\}_{m\in M}$ of V be given. We say that this system is a family of multifiltrations of V if:*

(i) For each $\sigma \in \Delta$ and $m \leq_\sigma m'$, E^σ_m is contained in $E^\sigma_{m'}$.

(ii) $V = \bigcup_{m\in M} E^\sigma_m$ for each $\sigma \in \Delta$.

(iii) For each chain $\ldots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \ldots$ of characters in M, there exists an $i_0 \in \mathbb{Z}$ such that $E^\sigma_{m_i} = 0$ for $i < i_0$.

(iv) For every $\sigma \in \Delta$, there exist only finitely many vector spaces E^σ_m such that $E^\sigma_m \nsubseteq \sum_{m'<_\sigma m} E^\sigma_{m'}$.

(v) (Compatibility condition.) For each $\tau \preceq \sigma$ with $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_\tau)$, we consider with respect to the preorder $\leq_\sigma$ the ascending chains $m + i \cdot m_\tau$ for $i \geq 0$. By condition (iv) and because V is finite dimensional, the sequence of subvector spaces $E^\sigma_{m+i\cdot m_\tau}$ necessarily becomes stationary for some $i^\tau_m \in \mathbb{Z}$. We require that $E^\tau_m = E^\sigma_{m+i^\tau_m \cdot m_\tau}$ for all $m \in M$.

A morphism between families of multifiltrations $\{E^\sigma_m\}_{m\in M,\sigma\in\Delta}$ and $\{F^\sigma_m\}_{m\in M,\sigma\in\Delta}$ is a homomorphism of the corresponding ambient vector spaces $\phi : \mathbf{E}^0 \to \mathbf{F}^0$ which is compatible with these multifiltratons, i.e. $\phi(E^\sigma_m) \subseteq F^\sigma_m$.

Remark 2.2.7. Note that in Definition 2.2.6, condition (iv) can be replaced with the following:
$(iv)'$ For every $\sigma \in \Delta$, there exist only finitely many vector spaces E^σ_m such that $E^\sigma_m \nsubseteq \bigcup_{m'<_\sigma m} E^\sigma_{m'}$ (see [37, Definition 4.19]).

Theorem 2.2.8. *[38, Theorem 5.18] The category of equivariant torsion-free sheaves is equivalent to the category of families of multifiltrations of finite-dimensional vector spaces.*

A coherent sheaf $\mathcal{E}$ on X is said to be reflexive if $\mathcal{E}$ is isomorphic to its double dual $\mathcal{E}^{**}$. Equivalently, a coherent sheaf $\mathcal{E}$ on X is reflexive if and only if $\mathcal{E}$ is torsion-free and for each open subset $U \subset X$ and each closed subset $Y \subset U$ of codimension ≥ 2, the restriction map $\Gamma(U, \mathcal{E}) \to \Gamma(U \setminus Y, \mathcal{E})$ is bijective (see [15, Proposition 1.6]). Dual of any coherent sheaf is an example of a reflexive sheaf (see [15, Corollary 1.2]). Note that any rank 1 reflexive sheaf is locally free. Now let $\mathcal{E}$ be an equivariant reflexive sheaf on $X = X(\Delta)$. Choose $Y = \bigcup_{\dim\, \tau \geq 2} V(\tau)$, a closed subset of X of codimension at least two. Then $\Gamma(X, \mathcal{E}) = \Gamma(X \setminus Y, \mathcal{E}) = \Gamma(X(\Delta_1), \mathcal{E})$, where $\Delta_1 = \Delta(0) \cup \Delta(1)$. In particular, for the affine toric variety U_σ, we have

$$\Gamma(U_\sigma, \mathcal{E}) = \Gamma\left(\bigcup_{\rho \in \sigma(1)} U_\rho, \mathcal{E}\right) = \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E}).$$

The above equality holds as vector subspaces of $\Gamma(T, \mathcal{E})$. Hence for each graded component of degree m, we have $\Gamma(U_\sigma, \mathcal{E})_m = \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E})_m$. Therefore, as vector subspaces of $\mathbf{E}^0$, we have $E^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho_m$. It follows that the compatibility condition (v) of Definition 2.2.6 is redundant. Thus the associated family of multifiltrations $\{E^\sigma_m\}_{m \in M, \sigma \in \Delta}$ of the equivariant reflexive sheaf $\mathcal{E}$ is completely determined by the family of multifiltrations $\{E^\rho_m\}_{m \in M, \rho \in \Delta(1)}$. Note that there is a canonical identification of $M/S^\perp_\rho$ with $\mathbb{Z}$ via the map $m \mapsto \langle m, v_\rho \rangle$. Hence identifying E^ρ_m with $E^\rho(\langle m, v_\rho \rangle)$, we get increasing full filtrations: $0 \subseteq \ldots \subseteq E^\rho(i) \subseteq E^\rho(i+1) \subseteq \ldots \subseteq \mathbf{E}^0$.
The following theorem shows that any equivariant reflexive sheaf arises from such filtrations.

Theorem 2.2.9. *[38, Theorem 5.19] The category of equivariant reflexive sheaves on a toric variety X is equivalent to the category of vector spaces with full filtrations associated to each ray in $\Delta(1)$. The morphisms in this category are linear maps which are compatible with the filtrations in the Δ-family sense.*

It is natural to ask for a combinatorial criterion for an equivariant reflexive sheaf to be locally free. This is given in the following proposition.

Proposition 2.2.10. *[37, Proposition 4.24] Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank r over X with corresponding filtrations $(\mathbf{E}^0, \{E^\rho(i)\}_{\rho \in \Delta(1)})$. Then $\mathcal{E}$ is locally free if and only if for each $\sigma \in \Delta$ there is an action of T_σ on $\mathbf{E}^0$ which gives a decomposition of $\mathbf{E}^0$ into T_σ-eigenspaces $\mathbf{E}^0 = \bigoplus_{m \in M/S^\perp_\sigma} \mathbf{E}^0_m$ such that*

$$E^\rho(i) = \bigoplus_{\substack{m \in M/S^\perp_\sigma \\ \langle m, v_\rho \rangle \leq i}} \mathbf{E}^0_m. \tag{2.4}$$

Remark 2.2.11. Recall that a family of linear subspaces $\{V_\lambda\}_{\lambda\in\Lambda}$ of a finite dimensional vector space V is said to form a distributive lattice if, there exists a basis B of V such that $B \cap V_\lambda$ is a basis of V_λ for every $\lambda \in \Lambda$. When X is nonsingular, the compatibility condition of locally free sheaves (2.4) in Proposition 2.2.10 is equivalent to the following: for each $\sigma \in \Delta$, the collection of subspaces $(\mathbf{E}^0, \{E^\rho(i)\}_{\rho\in\sigma(1)})$ forms a distributive lattice (cf. [26, Remark 2.2.2], arguments following Theorem 2.1.1 in [24]).

The following proposition tells us that the first Chern class of an equivariant coherent sheaf can be expressed using its associated Δ-family.

Proposition 2.2.12. *[30, Corollary 3.18] Let $X = X(\Delta)$ be a nonsingular projective toric variety. Let $\mathcal{E}$ be an equivariant coherent sheaf with associated Δ-family $\widehat{E}^\Delta$. Then we have*

$$c_1(\mathcal{E}) = -\sum_{\rho\in\Delta(1), i\in\mathbb{Z}} i \ \dim\, E^{[\rho]}(i) D_\rho,$$

where $E^{[\rho]}(i) = E^\rho(i)/E^\rho(i-1)$ and $E^\rho(i) = E^\rho_m$ such that $\langle m, v_\rho\rangle = i$.

Example 2.2.13 (Filtrations for line bundles). *[37, Section 4.7]* Let $\mathcal{L} = \mathcal{O}_X(D)$ be an equivariant line bundle on X for some T-invariant Cartier divisor $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$, $a_\rho \in \mathbb{Z}$. Then the associated filtrations $(L, \{L^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}})$ are given by:

$$L^\rho(i) = \begin{cases} 0 & i < -a_\rho \\ L(=k) & i \geq -a_\rho. \end{cases}$$

Next, we obtain filtrations for dual of an equivariant torsion-free sheaf following the proof of [37, Proposition 4.24], which will be useful to obtain filtrations of the tangent bundle from that of cotangent bundle.

Proposition 2.2.14. *Let $\mathcal{E}$ be an equivariant torsion-free sheaf with associated family of multifiltrations $\{E^\sigma_m\}_{\sigma\in\Delta,\ m\in M}$ of the vector space $\mathbf{E}^0$. Then filtrations associated to its dual reflexive sheaf $\mathcal{E}^*$ are given by $\big(F, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\big)$, where*

$$F = (\mathbf{E}^0)^* \text{ and } F^\rho(i) = \left(\frac{\mathbf{E}^0}{E^\rho(-i-1)}\right)^* \text{ for all } \rho\in\Delta(1),$$

where $E^\rho(i) = E^\rho_m$ for any $m \in M$ satisfying $\langle m, v_\rho\rangle = i$.

Proof. Since $\mathcal{E}$ is torsion-free, the singularity set $S(\mathcal{E}) = \{x \in X : \mathcal{E}_x$ is not free over $\mathcal{O}_{X,x}\}$ is of codimension at least two. Now for any ray $\rho \in \Delta(1)$, the affine toric variety $U_\rho = T \cup O(\rho)$, where $O(\rho) = T\cdot x_\rho$ is an orbit of dimension $n-1$. Since $\mathcal{E}$ is equivariant, it must be locally free over U_ρ.

Then $E^\rho := \Gamma(U_\rho, \mathcal{E})$ is an M-graded finitely generated free $k[S_\rho]$-module of rank r (see [38, Proposition 5.20]). We can write

$$E^\rho = \bigoplus_{j=1}^{r} k[S_\rho] e_j, \tag{2.5}$$

where $e_1, \ldots, e_r$ are homogeneous elements with deg $e_j = m_j$ for $j = 1, \ldots, r$. Equivalently, the T-action on E^ρ is given by $t \cdot e_j = \chi^{m_j}(t) e_j$ for $j = 1, \ldots, r$ (see [37, Proposition 2.31]). Set $L_j^\rho := k[S_\rho] e_j$ for $j = 1, \ldots, r$. Then for every j we have:

$$(L_j^\rho)_m = \begin{cases} 0 & m_j \nleq_\rho m \\ k\chi^{m-m_j} e_j & m_j \leq_\rho m. \end{cases}$$

We denote by $\mathbf{L}_j^\rho$ the direct limit of the directed family $\{(L_j^\rho)_m\}_{m \in M}$. Then we see that $(L_j^\rho)_m \cong \mathbf{L}_j^\rho$ for all $m \geq_\rho m_j$. In particular, we have the identification $\mathbf{L}_j^\rho = k e_j$. Thus for $i = \langle m, v_\rho \rangle$ we have

$$L_j^\rho(i) = \begin{cases} 0 & i < \langle m_j, v_\rho \rangle \\ \mathbf{L}_j^\rho & i \geq \langle m_j, v_\rho \rangle. \end{cases}$$

There is an action of T on the vector space $\mathbf{L}_j^\rho$ as follows:

$$T \times \mathbf{L}_j^\rho \longrightarrow \mathbf{L}_j^\rho, \ (t, l) \longmapsto \chi^{m_j}(t) l.$$

Since direct limit commutes with direct sum, we have $\mathbf{E}^\rho = \bigoplus_{j=1}^{r} \mathbf{L}_j^\rho$ and thus we get a diagonal action of T on $\mathbf{E}^\rho$ as follows:

$$T \times \mathbf{E}^\rho \longrightarrow \mathbf{E}^\rho, \ (t, e) \longmapsto \ \operatorname{diag}(\chi^{m_1}(t), \ldots, \chi^{m_r}(t)) e.$$

Furthermore, we have

$$E^\rho(i) = \bigoplus_{\langle m_j, v_\rho \rangle \leq i} \mathbf{L}_j^\rho. \tag{2.6}$$

Then using the following commutative diagram, we can transfer the T-action to $\mathbf{E}^0$ via the isomorphism (2.3)

$$\begin{array}{ccc} \mathbf{E}^\rho & \xrightarrow{\cong} & \bigoplus_{j=1}^{r} \mathbf{L}_j^\rho \\ \downarrow \cong & & \downarrow \cong \\ \mathbf{E}^0 & \xrightarrow{\cong} & \bigoplus_{j=1}^{r} \mathbf{L}_j^0. \end{array} \tag{2.7}$$

From (2.6) we have

$$E^\rho(i) = \bigoplus_{\langle m_j, v_\rho \rangle \leq i} \mathbf{L}_j^0. \tag{2.8}$$

Now for the dual sheaf $\mathcal{E}^*$ we have $F^\rho := \Gamma(U_\rho, \mathcal{E}^*) = \mathrm{Hom}_{k[S_\rho]}(E^\rho, k[S_\rho])$. Define $f_j \in F^\rho$ for $j = 1, \ldots, r$ by taking $f_j(e_i) = \delta_{ij}$ (see (2.5)). Since the dual action of T on F^ρ is compatible with the M-graded $k[S_\rho]$-module structure of F^ρ, it follows that the elements f_j are homogeneous of degree $-m_j$ for $j = 1, \ldots, r$. Thus we have $F^\rho = \bigoplus_{j=1}^{r} L'^\rho_j$ where $L'^\rho_j := k[S_\rho] f_j$. Then for every j we have:

$$(L'^\rho_j)_m = \begin{cases} 0 & -m_j \nleq_\rho m \\ k\chi^{m+m_j} f_j & -m_j \leq_\rho m. \end{cases}$$

As before, we see that $(L'^\rho_j)_m \cong \mathbf{L}'^\rho_j$ for all $m \geq_\rho -m_j$. In particular, we have the identification $\mathbf{L}'^\rho_j = k f_j = (\mathbf{L}^\rho_j)^*$. Now the action of T on $\mathbf{L}'^\rho_j$ is given by

$$T \times \mathbf{L}'^\rho_j \longrightarrow \mathbf{L}'^\rho_j,\ (t, l) \longmapsto \chi^{-m_j}(t) l.$$

Now $\mathbf{F}^\rho = \bigoplus_{j=1}^{r} \mathbf{L}'^\rho_j$ and hence $\mathbf{F}^\rho = (\mathbf{E}^\rho)^*$. As before, we get a diagonal action of T on $\mathbf{F}^\rho$ as follows:

$$T \times \mathbf{F}^\rho \longrightarrow \mathbf{F}^\rho,\ (t, e) \longmapsto \ \mathrm{diag}(\chi^{-m_1}(t), \ldots, \chi^{-m_r}(t)) e.$$

Using the dual diagram of (2.7), we transfer the T-action to $(\mathbf{E}^0)^*$. Thus we have

$$\begin{aligned} F^\rho(i) &= \bigoplus_{\langle -m_j, v_\rho \rangle \leq i} \mathbf{L}'^0_j \\ &= \left(\bigoplus_{\langle m_j, v_\rho \rangle \leq -i-1} \mathbf{L}^0_j \right)^\perp \\ &= (E^\rho(-i-1))^\perp \\ &= \left(\frac{\mathbf{E}^0}{E^\rho(-i-1)} \right)^*. \end{aligned}$$

Hence we get the desired filtrations for $\mathcal{E}^*$. □

Remark 2.2.15. Let $\mathcal{E}$ and $\mathcal{F}$ be equivariant reflexive sheaves with associated filtrations $\left(\mathbf{E}^0, \{E^\rho(i)\}_{\rho \in \Delta(1),\ i \in \mathbb{Z}}\right)$ and $\left(\mathbf{F}^0, \{F^\rho(i)\}_{\rho \in \Delta(1),\ i \in \mathbb{Z}}\right)$ respectively. Then arguing similarly as in the proof of Proposition 2.2.14, filtrations associated to their sum and tensor product are given as follows:

$$\left(\mathbf{E}^0 \oplus \mathbf{F}^0, \{(E \oplus F)^\rho(i)\}_{\rho \in \Delta(1),\ i \in \mathbb{Z}}\right),$$
$$\text{where } (E \oplus F)^\rho(i) = E^\rho(i) \oplus F^\rho(i),$$

$$\left(\mathbf{E}^0 \otimes \mathbf{F}^0, \{(E \otimes F)^\rho(i)\}_{\rho \in \Delta(1),\ i \in \mathbb{Z}}\right),$$
$$\text{where } (E \otimes F)^\rho(i) = \sum_{s+t=i} E^\rho(s) \otimes F^\rho(t).$$

Proposition 2.2.16. *Let $X = X(\Delta)$ be a nonsingular complete toric variety of dimension n. Then filtrations $\left(\Omega, \{\Omega^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ associated to cotangent bundle Ω_X are given by*

$$\Omega^\rho(i) = \begin{cases} 0 & i \leq -1 \\ Span(v_\rho)^\perp & i = 0 \\ M \otimes_{\mathbb{Z}} k & i \geq 1. \end{cases}$$

Proof. Let $\sigma \in \Delta(n)$ and write $\sigma = \text{Cone}(v_1, \ldots, v_n)$. Since X is nonsingular, $\{v_1, \ldots, v_n\}$ forms a $\mathbb{Z}$-basis of $N \cong \mathbb{Z}^n$. Let $u_1, \ldots, u_n$ be the corresponding dual basis of M. Then we have $U_\sigma = \text{Spec } k[\chi^{u_1}, \ldots, \chi^{u_n}]$. Set $z_j = \chi^{u_j}$ for $j = 1, \ldots, n$. Then $E^\sigma := \Gamma(U_\sigma, \Omega_X)$ is a free $k[S_\sigma]$-module generated by $dz_1, \ldots, dz_n$. The action of T on dz_j is given by $t \cdot dz_j = \chi^{u_j}(t) dz_j$. Thus as an M-graded $k[S_\sigma]$-module, E^σ is of the form $E^\sigma = \bigoplus_{j=1}^n k[S_\sigma] dz_j$. Set $L_j^\sigma = k[S_\sigma] dz_j$ for $j = 1, \ldots, n$. Then we have

$$(L_j^\sigma)_m = \begin{cases} 0 & u_j \not\leq_\sigma m \\ k\chi^{m-u_j} dz_j & u_j \leq_\sigma m. \end{cases}$$

Moreover, $\mathbf{L}_j^\sigma = (L_j^\sigma)_m$ for all $m \geq_\sigma u_j$. Hence taking $m = u_j$, we can identify $\mathbf{L}_j^\sigma = k\ dz_j$ for $j = 1, \ldots, n$. Note that from the proof of [37, Proposition 4.24], we have

$$E^\rho(i) = \bigoplus_{\langle u_j, v_\rho\rangle \leq i} \mathbf{L}_j^\sigma, \text{ for all } \rho \in \sigma(1). \tag{2.9}$$

Now $E^\rho(-1) = E_m^\rho$ be such that $\langle m, v_\rho\rangle = -1$. Hence from (2.9), we get $E^\rho(-1) = 0$, which implies $E^\rho(i) = 0$ for all $i \leq -1$. Similarly, $E^\rho(1) = E_m^\rho$ such that $\langle m, v_\rho\rangle = 1$. Thus from (2.9), we get $E^\rho(1) = k\ dz_1 \oplus \cdots \oplus k\ dz_n$ which can be identified with

$$M \otimes_{\mathbb{Z}} k = ku_1 \oplus \cdots \oplus ku_n \text{ via the identification } dz_j \mapsto u_j. \tag{2.10}$$

Thus we get $E^\rho(i) = M \otimes_{\mathbb{Z}} k$ for all $i \geq 1$.
Let $\rho = \text{Cone}\ (v_j)$. Now to compute $E^\rho(0)$, we consider an $m \in M$ satisfying $\langle m, v_\rho\rangle = 0$. Then from (2.9), we get $E^\rho(0) = \mathbf{L}_1^\sigma \bigoplus \cdots \bigoplus \widehat{\mathbf{L}}_j^\sigma \bigoplus \cdots \bigoplus \mathbf{L}_n^\sigma$. Hence, we can identify the space $E^\rho(0)$ with $\text{Span}(v_\rho)^\perp$ (see (2.10)). Thus we get the desired filtrations, which we denote by $\left(\Omega, \{\Omega^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$. ☐

The following corollary is immediate from Proposition 2.2.14 and Proposition 2.2.16.

Corollary 2.2.17. *Let $X = X(\Delta)$ be a nonsingular complete toric variety of dimension n. Then filtrations $\left(\mathscr{T}, \{\mathscr{T}^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ associated to tangent bundle $\mathcal{T}_X$ are given by*

$$\mathscr{T}^\rho(i) = \begin{cases} 0 & i \leq -2 \\ Span(v_\rho) & i = -1 \\ N \otimes_{\mathbb{Z}} k & i \geq 0. \end{cases}$$

2.3 Stability

Let X be a nonsingular projective variety of dimension n. Fix a polarization H, i.e., an ample divisor on X. For a torsion-free sheaf $\mathcal{E}$ over X, we have $\deg \mathcal{E} = c_1(\mathcal{E}) \cdot H^{n-1}$ and slope $\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank}(\mathcal{E})}$.
We consider stability in the sense of Mumford-Takemoto, which is also known as μ-stability. A subsheaf $\mathcal{F}$ of $\mathcal{E}$ is called proper if $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$. A torsion-free sheaf $\mathcal{E}$ over X is said to be (semi)stable with respect to H if, for any proper subsheaf $\mathcal{F}$ of $\mathcal{E}$, we have $\mu(\mathcal{F})(\leq) < \mu(\mathcal{E})$. We say $\mathcal{E}$ is strictly semistable if it is semistable but not stable. A torsion-free sheaf $\mathcal{E}$ is called unstable if it is not semistable.

Remark 2.3.1. *For a T-invariant divisor D_ρ on a nonsingular projective toric variety $X = X(\Delta)$, we have $\deg D_\rho > 0$ for all $\rho \in \Delta(1)$ by Nakai-Moishezon criterion [16, Theorem A.5.1].*

Remark 2.3.2. *Let X be a nonsingular complex toric variety. To check (semi)stability of a reflexive sheaf $\mathcal{E}$ on X, it suffices to consider only proper saturated subsheaves of $\mathcal{E}$ (see [18, Proposition 1.2.6]). Since saturated subsheaf of a reflexive sheaf is again reflexive (see [36, Lemma 1.1.16]), it is enough to consider only reflexive subsheaves of $\mathcal{E}$ for checking its (semi)stability. Furthermore, if $\mathcal{E}$ is equivariant, by [2, Theorem 2.1], it is enough to consider only equivariant reflexive subsheaves.*

3 Characterization of equivariant subsheaves of an equivariant sheaf

From now onwards, we take the underlying field k to be $\mathbb{C}$. In this section, we characterize all equivariant subsheaves of an equivariant torsion-free sheaf. Moreover, we give a combinatorial criterion of (semi)stability of equivariant torsion-free sheaves.

3.1 Equivariant subsheaves of equivariant torsion-free sheaves

In the following proposition, we give a combinatorial characterization of all equivariant subsheaves of an equivariant torsion-free sheaf.

Proposition 3.1.1. *Let $X = X(\Delta)$ be a complex toric variety and $\mathcal{E}$ be a torsion-free equivariant sheaf on X corresponding to a family of multifiltrations $\{E^\sigma_m\}_{\sigma \in \Delta,\ m \in M}$ of the vector space $\mathbf{E}^0$. There is a one-to-one correspondence between equivariant subsheaves of $\mathcal{E}$ and family of submultifiltrations $\{F^\sigma_m\}_{\sigma \in \Delta,\ m \in M}$ of the vector space $\mathbf{F}^0$, where $\mathbf{F}^0$ is a subspace of $\mathbf{E}^0$ and $F^\sigma_m = E^\sigma_m \cap \mathbf{F}^0$.*

Proof. Let $\mathcal{F}$ be an equivariant subsheaf of $\mathcal{E}$ and hence $F^\sigma := \Gamma(U_\sigma, \mathcal{F})$ is a T-stable subspace of $E^\sigma := \Gamma(U_\sigma, \mathcal{E})$ for any $\sigma \in \Delta$. The isotypical decomposition of both the spaces are given by $F^\sigma = \bigoplus\limits_{m \in M} F^\sigma_m$ and $E^\sigma = \bigoplus\limits_{m \in M} E^\sigma_m$, where

$F^\sigma_m = E^\sigma_m \cap F^\sigma$ since T is reductive (see [5, Theorem 1.23]). Thus we obtain σ-families $\widehat{F}^\sigma$ and $\widehat{E}^\sigma$ associated to $\mathcal{F}$ and $\mathcal{E}$ respectively, together with an inclusion of σ-families $\widehat{F}^\sigma \hookrightarrow \widehat{E}^\sigma$.
The Δ-family associated to $\mathcal{E}$ (respectively, $\mathcal{F}$) encodes the data for gluing the sheaves $\mathcal{E}_\sigma := \mathcal{E}|_{U_\sigma}$ (respectively, $\mathcal{F}_\sigma := \mathcal{F}|_{U_\sigma}$) on the affine open sets U_σ. Since the gluing data of $\mathcal{F}$ is the restriction of the gluing data of $\mathcal{E}$, we get the following commuting diagram as τ-families, where $\tau \preceq \sigma$:

$$\begin{array}{ccc} i^*_{\tau\sigma}\widehat{F}^\sigma & \xrightarrow[\cong]{\eta'_{\tau\sigma}} & \widehat{F}^\tau \\ \big\downarrow & & \big\downarrow \\ i^*_{\tau\sigma}\widehat{E}^\sigma & \xrightarrow[\cong]{\eta_{\tau\sigma}} & \widehat{E}^\tau . \end{array} \tag{3.1}$$

Here $\eta_{\tau\sigma}$ is as in Definition 2.2.2 of a Δ-family, similarly $\eta'_{\tau\sigma}$ denotes the corresponding isomorphism. Furthermore, the commutative diagram of $\mathbb{C}[S_\sigma]$-modules

$$\begin{array}{ccc} \Gamma(U_\sigma, \mathcal{F}) & \hookrightarrow & \Gamma(U_\tau, i^*_{\tau\sigma}\mathcal{F}) \\ \big\downarrow & & \big\downarrow \\ \Gamma(U_\sigma, \mathcal{E}) & \hookrightarrow & \Gamma(U_\tau, i^*_{\tau\sigma}\mathcal{E}) \end{array}$$

induces the following commutative diagram of σ-families (cf. (2.2))

$$\begin{array}{ccc} \widehat{F}^\sigma & \xhookrightarrow{\alpha'_{\tau\sigma}} & i^*_{\tau\sigma}\widehat{F}^\sigma \\ \big\downarrow & & \big\downarrow \\ \widehat{E}^\sigma & \xhookrightarrow[\alpha_{\tau\sigma}]{} & i^*_{\tau\sigma}\widehat{E}^\sigma . \end{array} \tag{3.2}$$

Combining (2.3) with the diagrams (3.1), (3.2) and [38, Proposition 5.15, Corollary 5.16], we get the following commutative diagram:

$$\begin{array}{ccc} \mathbf{F}^\sigma & \xrightarrow[\cong]{} & \mathbf{F}^0 \\ \big\downarrow & & \big\downarrow \\ \mathbf{E}^\sigma & \xrightarrow[\cong]{} & \mathbf{E}^0 . \end{array} \tag{3.3}$$

Hence we can realize all E^σ_m as subspaces of $\mathbf{E}^0$ and all F^σ_m as subspaces of $\mathbf{F}^0$ such that the collection of subspaces $\{E^\sigma_m\}_{\sigma\in\Delta,\, m\in M}$ (respectively,

$\{F_m^\sigma\}_{\sigma\in\Delta,\ m\in M})$ of $\mathbf{E}^0$ (respectively, $\mathbf{F}^0$) forms a family of multifiltrations. We have $F_m^\sigma \subseteq E_m^\sigma \cap \mathbf{F}^0$ for all $\sigma \in \Delta$ and $m \in M$. For the reverse inclusion, note that we have the following commutative diagram:

$$\begin{array}{ccc} F_m^\sigma & \hookrightarrow & \mathbf{F}^0 \\ \downarrow & & \downarrow \\ E_m^\sigma & \hookrightarrow & \mathbf{E}^0. \end{array} \tag{3.4}$$

By the diagrams (3.3) and (3.4), for $e \in E_m^\sigma \cap \mathbf{F}^0 = E_m^\sigma \cap \mathbf{F}^\sigma$, we have $[e, m] = [e', m'] \in \mathbf{F}^\sigma \subset \mathbf{E}^\sigma$, where $e' \in F_{m'}^\sigma$. Then there exists $m'' \in M$ such that $m, m' \leq_\sigma m''$ and $\chi_{m,m''}^\sigma(e) = \chi_{m',m''}^\sigma(e') \in F_{m''}^\sigma$. Since F^σ is a T-stable submodule of E^σ, it has a T-stable complement, say W^σ in E^σ. Thus $E_m^\sigma = F_m^\sigma \oplus W_m^\sigma$ for all $m \in M$. Let us write $e = e_1 + e_2$, where $e_1 \in F_m^\sigma$ and $e_2 \in W_m^\sigma$. Then we have that $\chi_{m,m''}^\sigma(e) = \chi_{m,m''}^\sigma(e_1) + \chi_{m,m''}^\sigma(e_2)$, where $\chi_{m,m''}^\sigma(e_1) \in F_{m''}^\sigma$ and $\chi_{m,m''}^\sigma(e_2) \in W_{m''}^\sigma$. It follows that $\chi_{m,m''}^\sigma(e_2) = 0$ and since $\chi_{m,m''}^\sigma$ is injective ($\mathcal{E}$ being torsion-free), we have $e_2 = 0$, i.e. $e \in F_m^\sigma$. This concludes the proof of the forward direction of the proposition.
Conversely, given a subspace $\mathbf{F}^0$ of $\mathbf{E}^0$, let us define $F_m^\sigma := E_m^\sigma \cap \mathbf{F}^0$ for $\sigma \in \Delta, m \in M$. Then by Definition 2.2.6 and Remark 2.2.7, $\{F_m^\sigma\}_{\sigma\in\Delta,\ m\in M}$ forms a family of multifiltrations of the vector space $\mathbf{F}^0$, and hence corresponds to a torsion-free equivariant sheaf $\mathcal{F}$ (see [38, Theorem 5.18]). It remains to show that $\mathcal{F}$ is an equivariant subsheaf of $\mathcal{E}$. This follows from $\Gamma(U_\sigma, \mathcal{F}) = \bigoplus_{m\in M} F_m^\sigma \subseteq \bigoplus_{m\in M} E_m^\sigma = \Gamma(U_\sigma, \mathcal{E})$. □

Recall that given a filtration $(V, \{F^pV\})$ on a vector space V and a subspace $W \subseteq V$, there is an induced subfiltration on W by setting $F^p(W) := W \cap F^p(V)$. As an immediate corollary of Proposition 3.1.1 we can characterize reflexive subsheaves in terms of induced subfiltrations.

Corollary 3.1.2. *Let $\mathcal{E}$ be an equivariant reflexive sheaf on X with associated filtrations $\big(\mathbf{E}^0, \{E^\rho(i)\}_{\rho\in\Delta(1)}\big)$. Then equivariant reflexive subsheaves of $\mathcal{E}$ are in one-to-one correspondence with induced subfiltrations $\big(\mathbf{F}^0, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\big)$ of $\big(\mathbf{E}^0, \{E^\rho(i)\}_{\rho\in\Delta(1),\ i\in Z}\big)$, where $\mathbf{F}^0$ is a subspace of $\mathbf{E}^0$.*

3.2 Stability of equivariant torsion-free sheaves

The following proposition provides a combinatorial criterion of (semi)stability of an equivariant torsion-free sheaf (cf. [28, Equation (19)], [30, Page 1730]).

Proposition 3.2.1. *Let $\mathcal{E}$ be an equivariant torsion-free sheaf on a nonsingular projective toric variety. Let $\{E_m^\sigma\}_{\sigma\in\Delta,\ m\in M}$ be the family of multifiltrations of the vector space $\mathbf{E}^0$ corresponding to $\mathcal{E}$. Then $\mathcal{E}$ is (semi)stable if and only if*

$$\frac{1}{\dim F}\left(-\sum_{\rho\in\Delta(1),\ i\in\mathbb{Z}} i \dim F^{[\rho]}(i) \deg D_\rho\right)$$
$$(\leq) < \frac{1}{\dim E}\left(-\sum_{\rho\in\Delta(1),\ i\in\mathbb{Z}} i \dim E^{[\rho]}(i) \deg D_\rho\right)$$

for every proper subspace F of E, where $F^\rho(i) = F \cap E^\rho(i)$ for any ray ρ.

Proof. By Proposition 2.2.12, we have

$$\mu(\mathcal{E}) = \frac{1}{\dim E}\left(-\sum_{\rho\in\Delta(1),\ i\in\mathbb{Z}} i \dim E^{[\rho]}(i) \deg D_\rho\right).$$

Since subsheaf of a torsion-free sheaf is again torsion-free, using Proposition 3.1.1 and Remark 2.3.2, the proposition follows. □

The following remark will help determine which subsheaves of tangent bundle have maximum possible slope.

REMARK 3.2.2. *Let $X = X(\Delta)$ be a nonsingular projective toric variety of dimension n. By Remark 2.3.2, to check (semi)stability of $\mathcal{T}_X$, it suffices to consider only proper equivariant reflexive subsheaves of $\mathcal{T}_X$. Let $\mathcal{F}$ be a proper equivariant reflexive subsheaf of $\mathcal{T}_X$ of rank $l \leq n-1$. Let $\left(F, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ be the filtrations associated to $\mathcal{F}$, where F is a vector subspace of $N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$ of dimension l and $F^\rho(i) = F \cap \mathscr{T}^\rho(i)$ (see Corollary 3.1.2). By Proposition 2.2.12, we have*

$$c_1(\mathcal{F}) = \begin{cases} \sum\limits_{v_\rho \in F\cap\Delta(1)} D_\rho & \text{if } F\cap\Delta(1) \neq \emptyset \\ 0 & \text{if } F\cap\Delta(1) = \emptyset. \end{cases} \tag{3.5}$$

Since we are interested in subsheaves of $\mathcal{T}_X$ with maximum possible slope and degree of D_ρ are positive (see Remark 2.3.1), it is enough to consider proper equivariant reflexive subsheaves with associated filtrations $\left(F, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\right)$ where $F = Span(F\cap\Delta(1))$.

4 STABILITY OF TANGENT BUNDLE ON A NONSINGULAR PROJECTIVE TORIC VARIETY WITH PICARD NUMBER ≤ 2

4.1 PICARD NUMBER 1

The only nonsingular projective toric variety with Picard group $\mathbb{Z}$ is the projective space (see [9, Exercise 7.3.10]). It is well known that the tangent bundle on projective space is stable (see [36, Theorem 1.3.2], [2, Theorem 7.1]). We give a simple proof of this fact using Proposition 3.2.1.

Proposition 4.1.1. *The tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ is stable for all $n > 0$.*

Proof. Let us fix some ample divisor H on $\mathbb{P}^n$. Let Δ denote the fan of $\mathbb{P}^n$ in the lattice $N = \mathbb{Z}^n$. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{Z}^n$ and set $e_0 = -e_1 - \cdots - e_n$. Then the fan consists of $n+1$ rays $e_0, e_1 \ldots, e_n$ and $n+1$ maximal cones $\text{Cone}(e_0, \ldots, \widehat{e}_i, \ldots, e_n)$, where $i = 0, \ldots, n$. We can assume $n \geq 2$ since the statement is trivial for $n = 1$. The divisors $D_0, \ldots, D_n$ corresponding to the rays $e_0, e_1 \ldots, e_n$ are all linearly equivalent and hence we have $\deg D_0 = \ldots = \deg D_n$. Note that $\mu(\mathcal{T}_{\mathbb{P}^n}) = (1 + \frac{1}{n}) \deg D_0$.
By Remark 3.2.2, let $\mathcal{F}$ be a proper equivariant reflexive subsheaf of $\mathcal{T}_{\mathbb{P}^n}$ of rank $l < n$ with associated filtrations $\big(F, \{F^\rho(i)\}_{\rho \in \Delta(1),\ i \in \mathbb{Z}}\big)$ where $F = \text{Span}(F \cap \Delta(1))$. Then we see that $\mu(\mathcal{F}) = \frac{p}{l} \deg D_0 \leq \deg D_0 < \mu(\mathcal{T}_{\mathbb{P}^n})$, where $|F \cap \Delta(1)| = p \leq l$. Hence by Proposition 3.2.1, $\mathcal{T}_{\mathbb{P}^n}$ is stable with respect to H. □

4.2 Picard number 2

Now we turn to nonsingular projective toric varieties with Picard group $\mathbb{Z}^2$, which were classified by Kleinschmidt (see [9, Theorem 7.3.7]). He showed that if X is any nonsingular projective toric variety with $\text{Pic}(X) \cong \mathbb{Z}^2$, then there are integers $s, r \geq 1$, $s + r = \dim(X)$ and $0 \leq a_1 \leq \ldots \leq a_r$ such that $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$. We recall the fan structure of X from [9, Example 7.3.5]. Let Δ be the fan of X in the lattice $N = \mathbb{Z}^s \times \mathbb{Z}^r$. Let $\{u_1, \ldots, u_s\}$ and $\{e'_1, \ldots, e'_r\}$ be standard bases of $\mathbb{Z}^s$ and $\mathbb{Z}^r$ respectively. Set

$$\begin{aligned} &v_i = (u_i, \mathbf{0}) \in N \text{ for } 1 \leq i \leq s \text{ ; } e_i = (\mathbf{0}, e'_i) \in N \text{ for } i = 1, \ldots, r, \\ &v_0 = -v_1 - \cdots - v_s + a_1 e_1 + \cdots + a_r e_r \text{ and } e_0 = -e_1 - \cdots - e_r. \end{aligned}$$

The rays of Δ are given by $v_0, v_1 \ldots, v_s, e_0, e_1, \ldots, e_r$ and the maximal cones are given by

$$\begin{aligned} \text{Cone}(v_0, \ldots, \widehat{v}_j, \ldots, v_s) + \text{Cone}(e_0, \ldots, \widehat{e}_i, \ldots, e_r), \\ \text{for all } j = 0, \ldots, s \text{ and } i = 0, \ldots, r. \end{aligned}$$

There are following relations among T-invariant prime divisors:

$$\begin{aligned} &\text{div}(\chi^{v_1^*}) = D_{v_1} - D_{v_0}, \ldots, \text{div}(\chi^{v_s^*}) = D_{v_s} - D_{v_0}, \\ &\text{div}(\chi^{e_i^*}) = D_{e_i} + a_i D_{v_0} - D_{e_0} \text{ for } i = 1, \ldots, r. \end{aligned} \tag{4.1}$$

Hence we have

$$D_{v_0} \sim_{\text{lin}} D_{v_i}, i = 1, \ldots, s \text{ and } D_{e_i} \sim_{\text{lin}} D_{e_0} - a_i D_{v_0}, i = 1, \ldots, r. \tag{4.2}$$

By (4.2), it follows that D_{v_0} and D_{e_0} generate $\text{Pic}(X)$. Now we show that D_{v_0} and D_{e_0} are not linearly equivalent. Consider the wall

$$\begin{aligned} \tau = \text{Cone}(v_0, \ldots, \widehat{v}_i, \ldots, v_s, e_0, \ldots, \widehat{e}_j, \ldots, \widehat{e}_k, \ldots, e_r), \\ \text{where } 0 \leq i \leq s,\ 0 \leq j < k \leq r. \end{aligned}$$

We can write $\tau = \text{Cone}(\tau, e_j) \cap \text{Cone}(\tau, e_k)$ and hence the wall relation is given by $e_0 + e_1 \cdots + e_r = 0$. Thus $D_{v_0} \cdot V(\tau) = 0$ and $D_{e_0} \cdot V(\tau) = 1$ (see Proposition 2.1.1). This implies that D_{v_0} and D_{e_0} are not numerically equivalent and hence not linearly equivalent. This also shows that D_{v_0} and D_{e_0} are $\mathbb{Z}$-linearly independent and hence we have $\text{Pic}(X) = \mathbb{Z}D_{v_0} \oplus \mathbb{Z}D_{e_0}$. In particular, anticanonical divisor is given by

$$-K_X = (s + 1 - a_1 - \cdots - a_r)D_{v_0} + (r + 1)D_{e_0}. \tag{4.3}$$

Proposition 4.2.1. *Let $D = aD_{v_0} + bD_{e_0}$, $a, b \in \mathbb{Z}$ be a T-invariant divisor on $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$. Then D is ample (respectively, nef) if and only if $a, b > 0$ (respectively, $a, b \geq 0$). In particular, X is Fano if and only if $a_1 + \cdots + a_r < s + 1$.*

Proof. Using toric Nakai criterion (see [35, Theorem 2.18]), we have D is ample if and only if $D \cdot V(\tau) > 0$ for all wall τ. Thus we need to compute $D \cdot V(\tau)$ for all walls $\tau \in \Delta(s + r - 1)$. Note that walls are of the following three types:
$\tau_{\{i,j\},0} = \text{Cone}(v_0, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_s, \widehat{e}_0, e_1, \ldots, e_r)$, $0 \leq i < j \leq s$,
$\tau_{\{i,j\},k} = \text{Cone}(v_0, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_s, e_0, \ldots, \widehat{e}_k, \ldots, e_r)$, $0 \leq i < j \leq s$, $0 < k \leq r$ and
$\tau_{i,\{j,k\}} = \text{Cone}(v_0, \ldots, \widehat{v}_i, \ldots, v_s, e_0, \ldots, \widehat{e}_j, \ldots, \widehat{e}_k, \ldots, e_r)$, $0 \leq i \leq s$, $0 \leq j < k \leq r$.
The wall relation corresponding to the wall $\tau_{\{i,j\},0} = \text{Cone}(\tau_{\{i,j\},0}, v_i) \cap \text{Cone}(\tau_{\{i,j\},0}, v_j)$ is given by

$$v_0 + \cdots + v_s - a_1e_1 - \cdots - a_re_r = 0,$$

which implies $D_{v_0} \cdot V(\tau_{\{i,j\},0}) = 1$ and $D_{e_0} \cdot V(\tau_{\{i,j\},0}) = 0$ (see Proposition 2.1.1). This gives

$$D \cdot V(\tau_{\{i,j\},0}) = a. \tag{4.4}$$

Similarly, the wall $\tau_{\{i,j\},k} = \text{Cone}(\tau_{\{i,j\},k}, v_i) \cap \text{Cone}(\tau_{\{i,j\},k}, v_j)$ gives the following relation

$$v_0 + \cdots + v_s + a_ke_0 + b_1e_1 + \cdots + \widehat{e}_k + \cdots + b_re_r = 0$$

for some integers $b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_r$. Thus $D_{v_0} \cdot V(\tau_{\{i,j\},k}) = 1$ and $D_{e_0} \cdot V(\tau_{\{i,j\},k}) = a_k$. Hence we have

$$D \cdot V(\tau_{\{i,j\},k}) = a + a_kb. \tag{4.5}$$

Finally the wall relation for $\tau_{i,\{j,k\}} = \text{Cone}(\tau_{i,\{j,k\}}, e_j) \cap \text{Cone}(\tau_{i,\{j,k\}}, e_k)$ is as follows

$$e_0 + e_1 \cdots + e_r = 0.$$

So we get $D_{v_0} \cdot V(\tau_{i,\{j,k\}}) = 0$ and $D_{e_0} \cdot V(\tau_{i,\{j,k\}}) = 1$. Hence we have

$$D \cdot V(\tau_{i,\{j,k\}}) = b. \tag{4.6}$$

Now considering the equations (4.4), (4.5) and (4.6), it follows that D is ample (respectively, nef) if and only if $a, b > 0$ (respectively, $a, b \geq 0$).
The second part of the proposition follows from (4.3). ∎

We fix a polarization $H = aD_{v_0} + bD_{e_0}$, $a, b > 0$. Then from (4.2), we have

$$\deg\, D_{v_0} = \deg\, D_{v_i} \text{ for } i = 1, \ldots, s \tag{4.7}$$

and

$$\deg\, D_{e_0} - \deg\, D_{e_i} = a_i \deg\, D_{v_0} \geq 0$$

for $i = 1, \ldots, r$ by Remark 2.3.1. So we have

$$\deg\, D_{e_0} \geq \deg\, D_{e_i} \text{ for } i = 1, \ldots, r. \tag{4.8}$$

Furthermore, we have $\deg\, D_{e_0} = \deg\, D_{e_r} + a_r \deg\, D_{v_0} > a_r \deg\, D_{v_0}$. Then if a_r is positive, we get

$$\deg\, D_{e_0} > \deg\, D_{v_0}. \tag{4.9}$$

From (4.3), we have

$$\mu(\mathcal{T}_X) = \left(\frac{s+1-a_1-\cdots-a_r}{s+r}\right) \deg\, D_{v_0} + \left(\frac{r+1}{s+r}\right) \deg\, D_{e_0}. \tag{4.10}$$

Denote by $\alpha = \dfrac{s+1-a_1-\cdots-a_r}{s+r}$ and $\beta = \dfrac{r+1}{s+r}$, then $\alpha < 1$ and $0 < \beta \leq 1$.

Theorem 4.2.2. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$, where $s, r \geq 1$, $0 \leq a_1 \leq \ldots \leq a_r$ and $a_r > 0$. Then tangent bundle $\mathcal{T}_X$ is unstable with respect to any polarization whenever $(a_1, \ldots, a_r) \neq (0, 0, \ldots, 0, 1)$.*

Proof. Note that $\mu(\mathcal{T}_X) < (\alpha+\beta)\deg\, D_{e_0}$ from (4.9) and (4.10). Observe that $\alpha+\beta > 1$ if and only if $a_1+\cdots+a_r \leq 1$, i.e. $(a_1, \ldots, a_r) = (0, 0, \ldots, 0, 1)$. When $\alpha+\beta \leq 1$, i.e. $(a_1, \ldots, a_r) \neq (0, 0, \ldots, 0, 1)$, then we see that $\mu(\mathcal{T}_X) < \deg\, D_{e_0}$. From Proposition 3.1.2, it follows that for $r = 1$ (respectively, $r \geq 2$), $\mathcal{O}_X(D_{e_0} + D_{e_1})$ (respectively, $\mathcal{O}_X(D_{e_0})$) is a rank 1 reflexive subsheaf of $\mathcal{T}_X$ corresponding to the vector subspace $\mathrm{Span}(e_0)$ of $N_{\mathbb{C}}$. Hence $\mathcal{T}_X$ is unstable. □

Next let us consider $X = \mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s, r \geq 1$. Then the relations in (4.2) simplify to the following form

$$\begin{aligned} &D_{v_0} \sim_{\text{lin}} D_{v_i}, i = 1, \ldots, s; \\ &D_{e_0} \sim_{\text{lin}} D_{e_i}, i = 1, \ldots, r-1 \text{ and } D_{e_r} \sim_{\text{lin}} D_{e_0} - D_{v_0}. \end{aligned} \tag{4.11}$$

Lemma 4.2.3. *$X = \mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s, r \geq 1$. Then,*

1. $\deg\, D_{v_0} = \sum_{i=r}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i$,

2. $\deg\, D_{e_0} = \sum_{i=r-1}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i$.

Proof. We can write H^{s+r-1} as

$$H^{s+r-1} = (aD_{v_0} + bD_{e_0})^{s+r-1} = \sum_{i=0}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i \, D_{v_0}^{r-1+s-i} D_{e_0}^{i}. \tag{4.12}$$

From (4.11), we have

$$\begin{aligned} &D_{v_0}^{r-1+s-i} = 0 \text{ for } i < r-1, \\ &D_{e_0}^{r+j} = D_{e_0} \cdots D_{e_{r-1}} (D_{e_r} + D_{v_0})^j = D_{e_0} \cdots D_{e_{r-1}} D_{v_0}^j \text{ for } j > 0. \end{aligned} \tag{4.13}$$

Put $i = r + j$, $j > 0$, then the i-th term of the binomial expression of H^{s+r-1} in (4.12) takes the form

$$D_{v_0}^{r-1+s-i} D_{e_0}^{i} = D_{v_0}^{s-1-j} D_{e_0}^{r+j} = D_{e_0} \cdots D_{e_{r-1}} D_{v_0}^{s-1}. \tag{4.14}$$

From (4.13) and (4.14), we see that

$$\begin{aligned} H^{s+r-1} = &\binom{r-1+s}{r-1} a^s b^{r-1} \, D_{e_0}^{r-1} D_{v_0}^{s} + \binom{r-1+s}{r} a^{s-1} b^r D_{e_0}^{r} D_{v_0}^{s-1} \\ &+ \sum_{i=r+1}^{r-1+s} \left(\binom{r-1+s}{i} \, a^{r-1+s-i} b^i \right) D_{e_0} \cdots D_{e_{r-1}} D_{v_0}^{s-1}. \end{aligned} \tag{4.15}$$

Now see that

$$D_{e_0}^{r-1} D_{v_0}^{s} D_{e_r} = 1, \; D_{e_0}^{r} D_{v_0}^{s-1} D_{e_r} = 0, \; D_{e_0} \cdots D_{e_{r-1}} D_{v_0}^{s-1} D_{e_r} = 0.$$

Hence, we have

$$\deg \, D_{e_r} = \binom{r-1+s}{r-1} a^s b^{r-1}. \tag{4.16}$$

Similarly, we can see that

$$\deg \, D_{v_0} = \sum_{i=r}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i. \tag{4.17}$$

Finally, from (4.11), (4.16) and (4.17), we get,

$$\deg \, D_{e_0} = \sum_{i=r-1}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i.$$

□

The following lemma is crucial in studying stability of tangent bundle on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^r \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s, r \geq 1$.

Lemma 4.2.4. *Let $X = \mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s, r \geq 1$. Then*

$$\begin{aligned} max\{\mu(\mathcal{F}) : \mathcal{F} \text{ is a proper subsheaf of } \mathcal{T}_X\} \\ = \deg D_{e_0} + \frac{1}{r}\left(\deg D_{e_0} - \deg D_{v_0}\right). \end{aligned}$$

Proof. Without loss of generality, we consider only proper equivariant reflexive subsheaves of $\mathcal{T}_X$ (see Remark 2.3.2). Let $\mathcal{F}$ be a proper equivariant reflexive subsheaf of $\mathcal{T}_X$ with associated filtrations $\left(F, \{F^\rho(i)\}_{\rho \in \Delta(1),\, i \in \mathbb{Z}}\right)$. In view of Remark 3.2.2, to find the maximum of $\{\mu(\mathcal{F}) : \mathcal{F}$ is a proper subsheaf of $\mathcal{T}_X\}$ it is enough to consider the following cases:

(i) $\text{rank}(\mathcal{F}) = r$ and $F \cap \Delta(1) = \{e_0, \ldots, e_r\}$. In this case $c_1(\mathcal{F}) = (r+1)D_{e_0} - D_{v_0}$ and hence $\mu(\mathcal{F}) = \deg D_{e_0} + \frac{1}{r}(\deg D_{e_0} - \deg D_{v_0})$.

(ii) $\text{rank}(\mathcal{F}) = j+1$ and $F \cap \Delta(1) = \{e_0, \ldots, e_j\}$, where $0 \leq j \leq r-2$. In this case $c_1(\mathcal{F}) = (j+1)D_{e_0}$ and hence $\mu(\mathcal{F}) = \deg D_{e_0}$.

(iii) $\text{rank}(\mathcal{F}) = j+k+2$ and $F \cap \Delta(1) = \{v_0, \ldots, v_j, e_0, \ldots, e_k\}$, where $0 \leq j < s$ and $-1 \leq k \leq r-2$ (here $k = -1$ should be interpreted as no e_k term belongs to $F \cap \Delta(1)$). In this case $c_1(\mathcal{F}) = (j+1)D_{v_0} + (k+1)D_{e_0}$ and hence $\mu(\mathcal{F}) = \frac{1}{j+k+2}((j+1)\deg D_{v_0} + (k+1)\deg D_{e_0}) < \deg D_{e_0}$.

(iv) $\text{rank}(\mathcal{F}) = s+j+2$ and $F \cap \Delta(1) = \{v_0, \ldots, v_s, e_r, e_0, \ldots, e_j\}$, where $-1 \leq j \leq r-3$ (here $j = -1$ should be interpreted as no e_j term belongs to $F \cap \Delta(1)$). In this case $c_1(\mathcal{F}) = (j+2)D_{e_0} + sD_{v_0}$ and hence $\mu(\mathcal{F}) = \frac{1}{s+j+2}((j+2)\deg D_{e_0} + s \deg D_{v_0}) < \deg D_{e_0}$.

(v) $\text{rank}(\mathcal{F}) = r+j+1$ and $F \cap \Delta(1) = \{v_0, \ldots, v_j, e_0, \ldots, e_r\}$, where $0 \leq j \leq s-2$. In this case $c_1(\mathcal{F}) = (r+1)D_{e_0} + jD_{v_0}$ and hence $\mu(\mathcal{F}) = \frac{1}{r+j+1}((r+1)\deg D_{e_0} + j \deg D_{v_0}) < \deg D_{e_0}$.

Thus the desired maximum is achieved from case (i). □

Theorem 4.2.5. *Let $X = \mathbb{P}(\mathcal{O}^r_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1))$, where $s \geq 1, r \geq 1$. Consider the polarization $H = aD_{v_0} + bD_{e_0}$, $a, b > 0$. Then tangent bundle $\mathcal{T}_X$ is H-(semi)stable if and only if*

$$\sum_{i=r-1}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i (\leq) < \frac{(sr+s+r)}{s(r+1)} \sum_{i=r}^{r-1+s} \binom{r-1+s}{i} a^{r-1+s-i} b^i.$$

Proof. Using Proposition 3.2.1 and Lemma 4.2.4, we obtain that

$$\begin{aligned} \mathcal{T}_X \text{ is (semi)stable } &\iff \deg D_{e_0} + \frac{1}{r}\left(\deg D_{e_0} - \deg D_{v_0}\right)(\leq) < \mu(\mathcal{T}_X) \\ &\iff \deg D_{e_0} (\leq) < \frac{(sr+s+r)}{s(r+1)} \deg D_{v_0} \text{ (see (4.10))}. \end{aligned}$$

Now the theorem follows by Lemma 4.2.3. □

The following remark is useful for studying (semi)stability of tangent bundles on product of nonsingular Fano varieties.

Remark 4.2.6. Let Y_1 and Y_2 be two nonsingular Fano varieties of dimension n_1 and n_2 respectively. Then $X = Y_1 \times Y_2$ is also a nonsingular Fano variety whose dimension is $n = n_1 + n_2$. Also one can see that $\mathcal{T}_X = \pi_1^*\mathcal{T}_{Y_1} \oplus \pi_2^*\mathcal{T}_{Y_2}$ and $\mu(\mathcal{T}_X) = \mu(\pi_1^*\mathcal{T}_{Y_1}) = \mu(\pi_2^*\mathcal{T}_{Y_2})$, where $\pi_i : X \to Y_i$, $i = 1, 2$ is the projection map. Now, if both $\mathcal{T}_{Y_1}$ and $\mathcal{T}_{Y_2}$ are semistable, then $\mathcal{T}_X$ is strictly semistable (see [43, Examples 3.2]).

Corollary 4.2.7. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^s}(a_r))$, $s, r \geq 1$, $0 \leq a_1 \leq \ldots \leq a_r$, with $a_1 + \cdots + a_r < s + 1$, i.e. X is Fano. Then with respect to the ample anticanonical divisor $-K_X$, we have the following:*

1. *$\mathcal{T}_X$ is unstable whenever $(a_1, \ldots, a_r) \neq (0, 0, \ldots, 0, 1)$ and $a_r > 0$.*
2. *If $r = 1$ and $a_1 = 1$, $\mathcal{T}_X$ is unstable for $s \geq 2$. It is strictly semistable for $s = 1$.*
3. *If $r > 1$ and $(a_1, \ldots, a_r) = (0, 0, \ldots, 0, 1)$, $\mathcal{T}_X$ is (semi)stable if and only if*

$$\sum_{i=r-1}^{r-1+s} \binom{r-1+s}{i} s^{r-1+s-i}(r+1)^i$$
$$(\leq) < \frac{(sr+s+r)}{s(r+1)} \sum_{i=r}^{r-1+s} \binom{r-1+s}{i} s^{r-1+s-i}(r+1)^i.$$

4. *If $a_r = 0$, $\mathcal{T}_X$ is strictly semistable.*

Proof. Clearly, (1) follows from Theorem 4.2.2.
For (2), using Theorem 4.2.5, we get that $\mathcal{T}_X$ is (semi)stable if and only if $(2s+1)s^s (\leq) < (s+2)^s$ (see also [2, Theorem 8.1]).
Note that for $s = 1$, the equality holds, hence in this case, $\mathcal{T}_X$ is strictly semistable.
For $s \geq 2$, using induction it can be shown that $(2s+1)s^s > (s+2)^s$ holds. Hence, $\mathcal{T}_X$ is unstable whenever $s \geq 2$.
Furthermore, (3) follows from Theorem 4.2.5, for the particular values $a = s$, $b = r + 1$. Finally, the assertion of (4) is immediate from Remark 4.2.6. ☐

5 Stability of tangent bundles on Fano 4-folds with Picard number 3

In this section, we are interested in (semi)stability of tangent bundles (with respect to the anticanonical divisor) on toric Fano 4-folds with Picard number 3 which were classified by Batyrev [1, Section 4].

5.1 Stability of tangent bundle on a $\mathbb{P}^2$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$

Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\alpha,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\beta,\gamma))$. Let Δ be the fan of X whose rays are given by

$$\mathbf{u}_0 = (-1,0,\alpha,\beta), \mathbf{u}_1 = (1,0,0,0), \mathbf{v}_0 = (0,-1,0,\gamma), \mathbf{v}_1 = (0,1,0,0) \text{ and}$$
$$\mathbf{e}_0 = (0,0,-1,-1), \mathbf{e}_1 = (0,0,1,0), \mathbf{e}_2 = (0,0,0,1),$$

and the maximal cones are given by

$$\text{Cone}(\mathbf{e}_0,\dots,\widehat{\mathbf{e}}_i,\dots,\mathbf{e}_2) + \text{Cone}(\mathbf{u}_j,\mathbf{v}_k), \text{ where } 0 \le i \le 2 \text{ and } 0 \le j,k \le 1.$$

We have the following relations

$$D_{\mathbf{u}_0} \sim_{\text{lin}} D_{\mathbf{u}_1}, D_{\mathbf{v}_0} \sim_{\text{lin}} D_{\mathbf{v}_1},$$
$$D_{\mathbf{e}_1} \sim_{\text{lin}} D_{\mathbf{e}_0} - \alpha D_{\mathbf{u}_0}, D_{\mathbf{e}_2} \sim_{\text{lin}} D_{\mathbf{e}_0} - \beta D_{\mathbf{u}_0} - \gamma D_{\mathbf{v}_0}.$$

Hence $\text{Pic}(X) = \mathbb{Z}D_{\mathbf{u}_0} \oplus \mathbb{Z}D_{\mathbf{v}_0} \oplus \mathbb{Z}D_{\mathbf{e}_0}$. Let $H = aD_{\mathbf{u}_0} + bD_{\mathbf{v}_0} + cD_{\mathbf{e}_0}$. Note that $D^2_{\mathbf{u}_0} = 0, D^2_{\mathbf{v}_0} = 0$ and hence we see that

$$H^3 = 3ac^2 D_{\mathbf{u}_0} D^2_{\mathbf{e}_0} + 3bc^2 D_{\mathbf{v}_0} D^2_{\mathbf{e}_0} + 6abc D_{\mathbf{u}_0} D_{\mathbf{v}_0} D_{\mathbf{e}_0} + c^3 D^3_{\mathbf{e}_0}.$$

Using the relations

$$D_{\mathbf{e}_0} D_{\mathbf{e}_1} D_{\mathbf{e}_2} = 0, D_{\mathbf{u}_0} D_{\mathbf{v}_0} D^2_{\mathbf{e}_0} = 1, D_{\mathbf{u}_0} D^3_{\mathbf{e}_0} = \gamma,$$
$$D_{\mathbf{v}_0} D^3_{\mathbf{e}_0} = \alpha + \beta, D^4_{\mathbf{e}_0} = \alpha\gamma + 2\beta\gamma,$$

we get

$$\begin{aligned} &\deg\, D_{\mathbf{u}_0} = 3bc^2 + c^3\gamma, \deg\, D_{\mathbf{v}_0} = 3ac^2 + c^3(\alpha+\beta), \\ &\deg\, D_{\mathbf{e}_0} = 3ac^2\gamma + 3bc^2(\alpha+\beta) + 6abc + c^3(\alpha\gamma + 2\beta\gamma). \end{aligned} \tag{5.1}$$

When $H = -K_X$, we have $a = 2-\alpha-\beta,\ b = 2-\gamma,\ c = 3$.
Following the notation of [1, Section 4], $X = D_7$ when $\alpha = 0, \beta = \gamma = 1$, and $X = D_{17}$ when $\alpha = 1, \beta = 0, \gamma = 1$.

Proposition 5.1.1. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\alpha,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\beta,\gamma))$. Then*

1. *$\mathcal{T}_X$ is unstable if $\alpha = 0, \beta = \gamma = 1$.*
2. *$\mathcal{T}_X$ is stable if $\alpha = 1, \beta = 0, \gamma = 1$.*

Proof. (1) $\alpha = 0, \beta = \gamma = 1$: Then $a = b = 1$ and $c = 3$.
We get $\deg\, D_{\mathbf{u}_0} = 54$, $\deg\, D_{\mathbf{v}_0} = 54$, $\deg\, D_{\mathbf{e}_0} = 126$ and $\mu(\mathcal{T}_X) = 121.5$. Note that $F = \text{Span}(e_0)$ corresponds to the destabilizing subsheaf $\mathcal{O}_X(D_{\mathbf{e}_0})$. Hence $\mathcal{T}_X$ is unstable.
(2) $\alpha = 1, \beta = 0, \gamma = 1$: Then $a = b = 1$ and $c = 3$.
We have $\deg\, D_{\mathbf{u}_0} = 54$, $\deg\, D_{\mathbf{v}_0} = 54$, $\deg\, D_{\mathbf{e}_0} = 99$, $\deg\, D_{\mathbf{e}_1} = 45$, $\deg\, D_{\mathbf{e}_2} =$

45 and $\mu(\mathcal{T}_X) = 101.25$. By Remark 3.2.2, to check (semi)stability, we only need to consider the following equivariant reflexive subsheaves $\mathcal{F}$ with associated filtrations $\big(F, \{F^\rho(i)\}_{\rho\in\Delta(1),\ i\in\mathbb{Z}}\big)$.

<u>$\mathrm{rank}(\mathcal{F}) = 1$</u>

$(i) F = \mathrm{Span}(\mathbf{u}_0),$ then $\mu(\mathcal{F}) = 54$. $(ii) F = \mathrm{Span}(\mathbf{v}_0),$ then $\mu(\mathcal{F}) = 54$.
$(iii) F = \mathrm{Span}(\mathbf{e}_0),$ then $\mu(\mathcal{F}) = 99$. $(iv) F = \mathrm{Span}(\mathbf{e}_1),$ then $\mu(\mathcal{F}) = 45$.
$(v) F = \mathrm{Span}(\mathbf{e}_2),$ then $\mu(\mathcal{F}) = 45$.

<u>$\mathrm{rank}(\mathcal{F}) = 2$</u>

(i) $F = \mathrm{Span}(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$, then $\mu(\mathcal{F}) = 94.5$.

(ii) $F = \mathrm{Span}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{e}_1)$, then $\mu(\mathcal{F}) = 76.5$.

(iii) $F = \mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{e}_2)$, then $\mu(\mathcal{F}) = 76.5$.

<u>$\mathrm{rank}(\mathcal{F}) = 3$</u>

(i) $F = \mathrm{Span}(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{u}_0, \mathbf{u}_1)$, then $\mu(\mathcal{F}) = 99$.

(ii) $F = \mathrm{Span}(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{v}_0, \mathbf{v}_1)$, then $\mu(\mathcal{F}) = 99$.

Hence we see that $\mathcal{T}_X$ is stable. □

5.2 Stability of tangent bundle on a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^2$

Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(\alpha, \beta))$ with associated fan Δ. The rays of Δ are given by

$$w_0 = (-1, 0, 0, \alpha), w_1 = (1, 0, 0, 0), z_0 = (0, -1, -1, \beta), z_1 = (0, 1, 0, 0),$$
$$z_2 = (0, 0, 1, 0), e_0 = (0, 0, 0, -1), e_1 = (0, 0, 0, 1)$$

and the maximal cones are given by

$$\mathrm{Cone}(w_i, z_0, \ldots, \widehat{z}_j, \ldots, z_2, e_k),\ \text{where } i = 0, 1,\ 0 \le j \le 2,\ k = 0, 1.$$

We have the following relations:

$$D_{w_0} \sim_{\mathrm{lin}} D_{w_1}, D_{z_0} \sim_{\mathrm{lin}} D_{z_j},\ \text{for } j = 1, 2; D_{e_1} \sim_{\mathrm{lin}} D_{e_0} - \alpha D_{w_0} - \beta D_{z_0}. \quad (5.2)$$

So we get $\mathrm{Pic}(X) = \mathbb{Z}D_{w_0} \oplus \mathbb{Z}D_{z_0} \oplus \mathbb{Z}D_{e_0}$. Let $H = aD_{w_0} + bD_{z_0} + cD_{e_0}$. Note that $D_{w_0}^2 = 0, D_{z_0}^3 = 0$ and hence

$$H^3 = 3ab^2 D_{w_0} D_{z_0}^2 + 3ac^2 D_{w_0} D_{e_0}^2 + 6abc D_{w_0} D_{z_0} D_{e_0} + 3bc^2 D_{z_0} D_{e_0}^2 + 3b^2 c D_{z_0}^2 D_{e_0} + c^3 D_{e_0}^3.$$

Also we have

$$D_{e_0} \cdot D_{e_1} = 0, D_{w_0} D_{z_0} D_{e_0}^2 = \beta,$$
$$D_{w_0} D_{e_0}^3 = \beta^2, D_{z_0}^2 D_{e_0}^2 = \alpha, D_{z_0} D_{e_0}^3 = 2\alpha\beta, D_{e_0}^4 = 3\alpha\beta^2.$$

Now we consider the polarization $H = -K_X$, i.e. $a = 2-\alpha$, $b = 3-\beta$, $c = 2$. Note that $X = D_1, D_6, D_{18}$ and D_{19} when $(\alpha,\beta) = (1,2), (1,1), (-1,2)$ and $(-1,1)$ respectively, following the notations of [1, Section 4].

Proposition 5.2.1. *Let* $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(\alpha,\beta))$. *Then*

1. $\mathcal{T}_X$ *is unstable for* $(\alpha,\beta) = (1,1), (1,2)$ *and* $(-1,2)$.
2. $\mathcal{T}_X$ *is stable for* $(\alpha,\beta) = (-1,1)$.

Proof. (1) The proof of the assertion follows from the following two tables

(α,β)	deg D_{w_0}	deg D_{z_0}	deg D_{e_0}	$\mu(\mathcal{T}_X)$
$(1,1)$	56	76	144	124
$(1,2)$	62	80	225	148
$(-1,2)$	62	64	75	100

(α,β)	F	$c_1(\mathcal{F})$	$\mu(\mathcal{F})$
$(1,1)$	Span(e_0)	$D_{e_0}+D_{e_1}$	156
$(1,2)$	Span(e_0)	$D_{e_0}+D_{e_1}$	228
$(-1,2)$	Span(w_0,e_0)	$D_{e_0}+D_{e_1}+D_{w_0}+D_{w_1}$	104

where $\mathcal{F}$ denotes the equivariant reflexive subsheaf of $\mathcal{T}_X$ corresponding to the subspace F of $\mathbb{C}^4$.
(2) $\underline{\alpha=-1,\beta=1}$: Then $a=3,\ b=2,\ c=2$.
We have deg $D_{w_0} = 56$, deg $D_{z_0} = 68$, deg $D_{e_0} = 48$ and $\mu(\mathcal{T}_X) = 100$. Also deg $(D_{e_0}+D_{e_1}) = 84$. Note that $\mathcal{O}_X(D_{w_0}),\ \mathcal{O}_X(D_{w_1}),\ \mathcal{O}_X(D_{z_0}),\ \mathcal{O}_X(D_{z_1}),\ \mathcal{O}_X(D_{z_2}),\ \mathcal{O}_X(D_{e_0}+D_{e_1})$ are the only rank 1 equivariant reflexive subsheaves of $\mathcal{T}_X$ and all of them has degree less than $\mu(\mathcal{T}_X)$.
Next, we consider higher rank equivariant reflexive subsheaves of $\mathcal{T}_X$. The maximum possible slope can occur only from the following situations.
$\underline{\text{rank}(\mathcal{F})=2}$

(i) $F = \text{Span}(e_0,e_1,z_j)$ for $j=0,1,2$, then $\mu(\mathcal{F}) = 76$.

(ii) $F = \text{Span}(w_0,w_1,e_0,e_1)$, then $\mu(\mathcal{F}) = 98$.

$\underline{\text{rank}(\mathcal{F})=3}$

(i) $F = \text{Span}(e_0,e_1,z_0,z_1,z_2)$, then $\mu(\mathcal{F}) = 96$.

(ii) $F = \text{Span}(w_0,w_1,e_0,e_1,z_j)$ for $j=0,1,2$, then $\mu(\mathcal{F}) = 88$.

Hence in this case $\mathcal{T}_X$ is stable. □

5.3 Stability of tangent bundle on a $\mathbb{P}^1$-bundle over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a_1))$, $a_1 = 1, 2$

Let $X = \mathbb{P}(\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(\alpha, \beta))$, where $X' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a_1))$. The rays of the fan Δ of X are

$$\mathbf{v}_0 = (-1, -1, a_1, \alpha), \mathbf{v}_1 = (1, 0, 0, 0), \mathbf{v}_2 = (0, 1, 0, 0),$$
$$\mathbf{e}'_0 = (0, 0, -1, \beta), \mathbf{e}'_1 = (0, 0, 1, 0), \mathbf{e}_1 = (0, 0, 0, 1), \mathbf{e}_0 = (0, 0, 0, -1)$$

and the maximal cones are

$$\text{Cone}(\mathbf{v}_0, \ldots, \widehat{\mathbf{v}}_j, \ldots, \mathbf{v}_2, \mathbf{e}'_p, \mathbf{e}_q), \text{ where } j = 0, 1, 2 \text{ and } 0 \leq p, q \leq 1.$$

Note that we have the following relations

$$D_{\mathbf{v}_0} \sim_{\text{lin}} D_{\mathbf{v}_1} \sim_{\text{lin}} D_{\mathbf{v}_2}, D_{\mathbf{e}'_1} \sim_{\text{lin}} D_{\mathbf{e}'_0} - a_1 D_{\mathbf{v}_0}, D_{\mathbf{e}_1} \sim_{\text{lin}} D_{\mathbf{e}_0} - \alpha D_{\mathbf{v}_0} - \beta D_{\mathbf{e}'_0}.$$

Hence $\text{Pic}(X) = \mathbb{Z}D_{\mathbf{v}_0} \oplus \mathbb{Z}D_{\mathbf{e}'_0} \oplus \mathbb{Z}D_{\mathbf{e}_0}$. Using toric Nakai criterion and the fact that $\mathbb{P}(\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(D)) \cong \mathbb{P}(\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(-D))$ for any divisor D on X', we only need to consider the following cases (comparing with the primitive relations listed in [1, Proposition 3.1.2]):

$$D_3 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(1, 1)), D_9 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(1, 0)),$$
$$D_8 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(0, 1)), D_{16} = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(-1, 1)),$$
$$D_2 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_1} \oplus \mathcal{O}_{\mathcal{B}_1}(0, 1)), D_5 = \mathbb{P}^1 \times \mathcal{B}_1, D_{12} = \mathbb{P}^1 \times \mathcal{B}_2,$$

where $X' = \mathcal{B}_1$ for $a_1 = 2$ and $X' = \mathcal{B}_2$ for $a_1 = 1$ from the notations of [1, Remark 2.5.10].
Let $H = aD_{\mathbf{v}_0} + bD_{\mathbf{e}'_0} + cD_{\mathbf{e}_0}$. We have the following relations

$$D^3_{\mathbf{v}_0} = D^3_{\mathbf{v}_1} = D^3_{\mathbf{v}_2} = 0, D_{\mathbf{e}'_0} D_{\mathbf{e}'_1} = 0, D_{\mathbf{e}_0} D_{\mathbf{e}_1} = 0.$$

So we have

$$H^3 = 3ab^2 D_{\mathbf{v}_0} D^2_{\mathbf{e}'_0} + 3ac^2 D_{\mathbf{v}_0} D^2_{\mathbf{e}_0} + 3a^2 b D^2_{\mathbf{v}_0} D_{\mathbf{e}'_0} + 6abc D_{\mathbf{v}_0} D_{\mathbf{e}'_0} D_{\mathbf{e}_0}$$
$$+ 3a^2 c D^2_{\mathbf{v}_0} D_{\mathbf{e}_0} + b^3 D^3_{\mathbf{e}'_0} + 3bc^2 D_{\mathbf{e}'_0} D^2_{\mathbf{e}_0} + 3b^2 c D^2_{\mathbf{e}'_0} D_{\mathbf{e}_0} + c^3 D^3_{\mathbf{e}_0}.$$

Furthermore, we have

$$D^2_{\mathbf{v}_0} D^2_{\mathbf{e}'_0} = 0, D^2_{\mathbf{v}_0} D^2_{\mathbf{e}_0} = \beta, D_{\mathbf{v}_0} D^3_{\mathbf{e}'_0} = 0, D_{\mathbf{v}_0} D_{\mathbf{e}'_0} D^2_{\mathbf{e}_0} = \alpha + a_1 \beta,$$
$$D_{\mathbf{v}_0} D^3_{\mathbf{e}_0} = 2\alpha\beta + a_1\beta^2, D_{\mathbf{v}_0} D^2_{\mathbf{e}'_0} D_{\mathbf{e}_0} = a_1, D^2_{\mathbf{e}'_0} D^2_{\mathbf{e}_0} = a_1\alpha + a_1^2\beta, D^3_{\mathbf{e}'_0} D_{\mathbf{e}_0} = a_1^2,$$
$$D_{\mathbf{e}'_0} D^3_{\mathbf{e}_0} = (\alpha + a_1\beta)^2, D^4_{\mathbf{e}'_0} = 0, D^4_{\mathbf{e}_0} = 3\alpha^2\beta + 3a_1\alpha\beta^2 + a_1^2\beta^3.$$

Now let us fix the polarization $H = -K_X$, i.e. $a = 3 - a_1 - \alpha$, $b = 2 - \beta$, $c = 2$.

Proposition 5.3.1. *The tangent bundles on the following toric Fano 4-folds are unstable*
(i)D_{12} *(ii)*D_9 *(iii)*D_8 *(iv)*D_3 *(v)*D_{16} *(vi)*D_5 *(vii)*D_2.

Proof. The proof of the proposition follows from the following two tables

a_1	(α,β)	deg $D_{\mathbf{v}_0}$	deg $D_{\mathbf{e}'_0}$	deg $D_{\mathbf{e}_0}$	$\mu(\mathcal{T}_X)$
1	$(0,0)$	72	96	56	112
1	$(1,0)$	72	98	98	116
1	$(0,1)$	74	98	117	120
1	$(1,1)$	78	104	189	140
1	$(-1,1)$	70	96	63	108
2	$(0,0)$	72	150	62	124
2	$(0,1)$	76	158	171	144

a_1	(α,β)	F	$c_1(\mathcal{F})$	$\mu(\mathcal{F})$
1	$(0,0)$	$\mathrm{Span}(\mathbf{e}_0)$	$D_{\mathbf{e}'_0}+D_{\mathbf{e}'_1}$	120
1	$(1,0)$	$\mathrm{Span}(\mathbf{e}'_0)$	$D_{\mathbf{e}'_0}+D_{\mathbf{e}'_1}$	124
1	$(0,1)$	$\mathrm{Span}(\mathbf{e}_0)$	$D_{\mathbf{e}_0}+D_{\mathbf{e}_1}$	136
1	$(1,1)$	$\mathrm{Span}(\mathbf{e}_0)$	$D_{\mathbf{e}_0}+D_{\mathbf{e}_1}$	196
1	$(-1,1)$	$\mathrm{Span}(\mathbf{e}'_0,\mathbf{e}_0)$	$D_{\mathbf{e}'_0}+D_{\mathbf{e}'_1}+D_{\mathbf{e}_0}+D_{\mathbf{e}_1}$	111
2	$(0,0)$	$\mathrm{Span}(\mathbf{e}'_0)$	$D_{\mathbf{e}'_0}+D_{\mathbf{e}'_1}$	156
2	$(0,1)$	$\mathrm{Span}(\mathbf{e}'_0)$	$D_{\mathbf{e}'_0}$	158

where $\mathcal{F}$ denotes the equivariant reflexive subsheaf of $\mathcal{T}_X$ corresponding to the subspace F of $\mathbb{C}^4$. □

5.4 Stability of tangent bundle on a $\mathbb{P}^2$-bundle over the Hirzebruch surface $\mathcal{H}_1$

Let $X=\mathbb{P}(\mathcal{O}_{\mathcal{H}_1}\oplus\mathcal{O}_{\mathcal{H}_1}\oplus\mathcal{O}_{\mathcal{H}_1}(\alpha,\beta))$. The rays of the fan Δ of X are given by:

$$\mathbf{v}_1=(1,0,0,0),\mathbf{v}_2=(0,1,0,0),\mathbf{v}_3=(-1,1,0,\alpha),\mathbf{v}_4=(0,-1,0,\beta),$$
$$\mathbf{e}_0=(0,0,-1,-1),\mathbf{e}_1=(0,0,1,0),\mathbf{e}_2=(0,0,0,1),$$

and the maximal cones are given by

$$\mathrm{Cone}(\mathbf{e}_0,\dots,\widehat{\mathbf{e}}_j,\dots,\mathbf{e}_2)+\mathrm{Cone}(\mathbf{v}_i,\mathbf{v}_{i+1}),\ \text{where } j=0,1,2 \text{ and } 1\le i\le 4.$$

Now we have the following relations

$$D_{\mathbf{v}_1}\sim_{\text{lin}} D_{\mathbf{v}_3}, D_{\mathbf{v}_2}\sim_{\text{lin}} D_{\mathbf{v}_4}-D_{\mathbf{v}_3},$$
$$D_{\mathbf{e}_1}\sim_{\text{lin}} D_{\mathbf{e}_0}, D_{\mathbf{e}_2}\sim_{\text{lin}} D_{\mathbf{e}_0}-\alpha D_{\mathbf{v}_3}-\beta D_{\mathbf{v}_4}.$$

Hence $\mathrm{Pic}(X)=\mathbb{Z}D_{\mathbf{v}_3}\oplus\mathbb{Z}D_{\mathbf{v}_4}\oplus\mathbb{Z}D_{\mathbf{e}_0}$. Note that, using toric Nakai criterion, $X=\mathbb{P}(\mathcal{O}_{X'}\oplus\mathcal{O}_{X'}\oplus\mathcal{O}_{X'}(\alpha,\beta))$ is Fano if and only if $\alpha=0,\beta=0,1$. We consider the case for $(\alpha,\beta)=(0,1)$, i.e. $X=D_{11}$ in the notation of [1, Section 4]).

Proposition 5.4.1. *Let* $X = \mathbb{P}(\mathcal{O}_{\mathcal{H}_1} \oplus \mathcal{O}_{\mathcal{H}_1} \oplus \mathcal{O}_{\mathcal{H}_1}(0,1))$. *Then* $\mathcal{T}_X$ *is unstable.*

Proof. Note that $-K_X = D_{\mathbf{v}_3} + D_{\mathbf{v}_4} + 3D_{\mathbf{e}_0}$. Since $D^2_{\mathbf{v}_3} = 0$, we have

$$(-K_X)^3 = 3D_{\mathbf{v}_3}D^2_{\mathbf{v}_4} + 27D_{\mathbf{v}_3}D^2_{\mathbf{e}_0} + 18D_{\mathbf{v}_3}D_{\mathbf{v}_4}D_{\mathbf{e}_0} + D^3_{\mathbf{v}_4} + 27D_{\mathbf{v}_4}D^2_{\mathbf{e}_0} + 9D^2_{\mathbf{v}_4}D_{\mathbf{e}_0} + 27D^3_{\mathbf{e}_0}.$$

Now we compute the following intersection products

$$D_{\mathbf{v}_3}D^3_{\mathbf{v}_4} = 0, D_{\mathbf{v}_3}D_{\mathbf{v}_4}D^2_{\mathbf{e}_0} = 1, D_{\mathbf{v}_3}D^2_{\mathbf{v}_4}D_{\mathbf{e}_0} = 0, D_{\mathbf{v}_3}D^3_{\mathbf{e}_0} = 1, D^4_{\mathbf{v}_4} = 0,$$
$$D^2_{\mathbf{v}_4}D^2_{\mathbf{e}_0} = 1, D^3_{\mathbf{v}_4}D_{\mathbf{e}_0} = 0, D_{\mathbf{v}_4}D^3_{\mathbf{e}_0} = 1, D^4_{\mathbf{e}_0} = 1.$$

Thus, we have $\deg D_{\mathbf{v}_3} = 54$, $\deg D_{\mathbf{v}_4} = 81$, $\deg D_{\mathbf{e}_0} = 108$ and $\mu(\mathcal{T}_X) = 114.75$.
Let $F = \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0)$. Then it corresponds to a rank 2 destabilizing reflexive subsheaf $\mathcal{F}$ of $\mathcal{T}_X$ with $\mu(\mathcal{F}) = 121.5$. Hence $\mathcal{T}_X$ is unstable. □

5.5 Stability of tangent bundle on a $\mathbb{P}^1$-bundle over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$

Let $X = \mathbb{P}(\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(\alpha, \beta))$, where $X' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. The rays of the fan Δ of X are given by

$$\mathbf{v}_0 = (-1,0,1,\alpha), \mathbf{v}_1 = (1,0,0,0), \mathbf{e}'_1 = (0,1,0,0), \mathbf{e}'_2 = (0,0,1,0),$$
$$\mathbf{e}'_0 = (0,-1,-1,\beta), \mathbf{e}_1 = (0,0,0,1), \mathbf{e}_0 = (0,0,0,-1),$$

and the maximal cones are given by

$$\mathrm{Cone}(\mathbf{v}_i, \mathbf{e}'_0, \ldots, \widehat{\mathbf{e}'_j}, \ldots, \mathbf{e}'_2, \mathbf{e}_k) \text{ for } i = 0,1, j = 0,1,2 \text{ and } k = 0,1.$$

We have the following relations

$$D_{\mathbf{v}_1} \sim_{\mathrm{lin}} D_{\mathbf{v}_0}, D_{\mathbf{e}'_1} \sim_{\mathrm{lin}} D_{\mathbf{e}'_0}, D_{\mathbf{e}'_2} \sim_{\mathrm{lin}} D_{\mathbf{e}'_0} - D_{\mathbf{v}_0},$$
$$D_{\mathbf{e}_1} \sim_{\mathrm{lin}} D_{\mathbf{e}_0} - \alpha_1 D_{\mathbf{v}_0} - \beta D_{\mathbf{e}'_0}.$$

Hence $\mathrm{Pic}(X) = \mathbb{Z}D_{\mathbf{v}_0} \oplus \mathbb{Z}D_{\mathbf{e}'_0} \oplus \mathbb{Z}D_{\mathbf{e}_0}$. Now using toric Nakai criterion, one can see that $X = \mathbb{P}(\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(\alpha,\beta))$ is Fano if and only if $\alpha = 0, -2 \leq \beta \leq 2$. It suffices to consider $\beta = 1,2$. Note that $X = D_{10}$ for $(\alpha,\beta) = (0,1)$ and $X = D_4$ for $(\alpha,\beta) = (0,2)$ from the notations of [1, Remark 2.5.10, Section 4].

Proposition 5.5.1. *Let* $X = \mathbb{P}(\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(0,\beta))$, *where* $X' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ *and* $\beta = 1,2$. *Then* $\mathcal{T}_X$ *is unstable.*

Proof. The anticanonical divisor is given by $-K_X = D_{\mathbf{v}_0} + (3-\beta)D_{\mathbf{e}'_0} + 2D_{\mathbf{e}_0}$. Since $D^2_{\mathbf{v}_0} = 0, D_{\mathbf{e}'_0}D_{\mathbf{e}'_1}D_{\mathbf{e}'_2} = 0, D_{\mathbf{e}_0}D_{\mathbf{e}_1} = 0$, we have

$$(-K_X)^3 = 3(3-\beta)^2 D_{\mathbf{v}_0}D^2_{\mathbf{e}'_0} + 12D_{\mathbf{v}_0}D^2_{\mathbf{e}_0} + 12(3-\beta)D_{\mathbf{v}_0}D_{\mathbf{e}'_0}D_{\mathbf{e}_0} + (3-\beta)^3 D^3_{\mathbf{e}'_0} + 12(3-\beta)D_{\mathbf{e}'_0}D^2_{\mathbf{e}_0} + 6(3-\beta)^2 D^2_{\mathbf{e}'_0}D_{\mathbf{e}_0} + 8D^3_{\mathbf{e}_0}.$$

Furthermore, we have the following relations

$$D_{\mathbf{v}_0}D^3_{\mathbf{e}'_0}=0, D_{\mathbf{v}_0}D_{\mathbf{e}'_0}D^2_{\mathbf{e}_0}=\beta, D_{\mathbf{v}_0}D^3_{\mathbf{e}_0}=\beta^2, D^2_{\mathbf{e}'_0}D^2_{\mathbf{e}_0}=\beta,$$
$$D^3_{\mathbf{e}'_0}D_{\mathbf{e}_0}=1, D_{\mathbf{e}'_0}D^3_{\mathbf{e}_0}=\beta^2, D^4_{\mathbf{e}_0}=\beta^3, D^4_{\mathbf{e}'_0}=0.$$

Hence, we have

$$\deg\ D_{\mathbf{v}_0}=\begin{cases}56 & \beta=1\\ 62 & \beta=2\end{cases};$$
$$\deg\ D_{\mathbf{e}'_0}=\begin{cases}92 & \beta=1\\ 98 & \beta=2\end{cases};$$
$$\deg\ D_{\mathbf{e}_0}=\begin{cases}112 & \beta=1\\ 200 & \beta=2.\end{cases}$$

Therefore, we obtain $\mu(\mathcal{T}_X)=\begin{cases}116 & \beta=1\\ 140 & \beta=2.\end{cases}$

Note that $\mathcal{O}_X(D_{\mathbf{e}_0}+D_{\mathbf{e}_1})$ is a rank 1 reflexive subsheaf of $\mathcal{T}_X$, whose degree is given by

$\deg\ (D_{\mathbf{e}_0}+D_{\mathbf{e}_1})=2\deg\ D_{\mathbf{e}_0}-\beta\deg\ D_{\mathbf{e}'_0}=\begin{cases}132 & \beta=1\\ 204 & \beta=2.\end{cases}$

Hence, $\mathcal{T}_X$ is unstable. □

5.6 Stability of tangent bundle on blow up of $\mathbb{P}^2$ on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(a_1)), a_1=0,1,2$

Let $X'=\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}\oplus\mathcal{O}_{\mathbb{P}^3}(a_1))$. The fan Δ' associated to X' is given as follows. Let u_1,u_2,u_3 be the standard basis of $\mathbb{Z}^3$ and e'_1 be that of $\mathbb{Z}$. Set $v_i=(u_i,0)\in\mathbb{Z}^4$ for $i=1,2,3$, $e_1=(0,0,0,e'_1)\in\mathbb{Z}^4$, $e_0=-e_1$ and $v_0=-v_1-v_2-v_3+a_1e_1$. Then $\Delta'(1)=\{v_0,v_1,v_2,v_3,e_0,e_1\}$ and maximal cones are of the form

$$\mathrm{Cone}(v_0,\ldots,\widehat{v}_j,\ldots,v_3,e_0)\text{ and }\mathrm{Cone}(v_0,\ldots,\widehat{v}_j,\ldots,v_3,e_1)\text{ for }j=0,1,2,3.$$

Note that $\mathrm{Pic}(X')=\mathbb{Z}D_{v_0}\oplus\mathbb{Z}D_{e_0}$. For $\tau=\mathrm{Cone}(v_0,e_1)\in\Delta'$, we have $V(\tau)=\mathbb{P}^2$. Let $X=Bl_{V(\tau)}(X)$ with associated fan Δ. Then the rays of Δ are

$$v_1=(1,0,0,0), v_2=(0,1,0,0), v_3=(0,0,1,0), v_0=(-1,-1,-1,a_1),$$
$$e_1=(0,0,0,1), e_0=(0,0,0,-1), u_\tau=(-1,-1,-1,a_1+1).$$

We have the following relations

$$D_{v_1}\sim_{\mathrm{lin}}D_{v_2}\sim_{\mathrm{lin}}D_{v_3}\sim_{\mathrm{lin}}D_{v_0}+D_{u_\tau}, D_{e_1}\sim_{\mathrm{lin}}D_{e_0}-a_1D_{v_0}-(a_1+1)D_{u_\tau}. \tag{5.3}$$

Hence, we have $\mathrm{Pic}(X)=\mathbb{Z}D_{v_0}\oplus\mathbb{Z}D_{e_0}\oplus\mathbb{Z}D_{u_\tau}$. The anticanonical divisor is given by $-K_X=(4-a_1)D_{v_0}+2D_{e_0}+(3-a_1)D_{u_\tau}$.

Note that $X=E_1,E_2,E_3$ for $a_1=2,1,0$ respectively in the notation of [1, Section 4].

Proposition 5.6.1. *Let* $X = Bl_{V(\tau)}(X')$*, where* $X' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(a_1))$ *and* $\tau = Cone(v_0, e_1) \in \Delta'$*. Then*

(1) $\mathcal{T}_X$ *is unstable for* $a_1 = 1, 2$.

(2) $\mathcal{T}_X$ *is stable for* $a_1 = 0$.

Proof. Let $-K_X = aD_{v_0} + 2D_{e_0} + bD_{u_\tau}$, where $a = 4 - a_1$ and $b = 3 - a_1$. Note that we have $D_{e_0}D_{u_\tau} = 0, D_{e_0}D_{e_1} = 0$ and $D_{v_0}D_{e_1} = 0$. So we have

$$(-K_X)^3 = a^3 D_{v_0}^3 + 12aD_{v_0}D_{e_0}^2 + 3ab^2 D_{v_0}D_{u_\tau}^2 + 6a^2 D_{v_0}^2 D_{e_0} + 8D_{e_0}^3 + 3a^2 bD_{v_0}^2 D_{u_\tau} + b^3 D_{u_\tau}^3.$$

Furthermore, we have

$$\begin{aligned} &D_{v_0}^4 = -a_1^2 - 3a_1 - 3, D_{v_0}^2 D_{e_0}^2 = a_1, D_{v_0}^2 D_{u_\tau}^2 = -a_1^2 - a_1, \\ &D_{v_0}^3 D_{e_0} = 1, D_{v_0} D_{e_0}^3 = a_1^2, D_{v_0} D_{u_\tau}^3 = a_1^2, \\ &D_{v_0}^3 D_{u_\tau} = (a_1 + 1)^2, D_{e_0}^4 = a_1^3, D_{u_\tau}^4 = -a_1^2 + a_1 - 1. \end{aligned}$$

(1)$\underline{a_1 = 2}$: Here $a = 2,\ b = 1$. We compute that deg $D_{v_0} = 76,$ deg $D_{e_0} = 216,$ deg $D_{u_\tau} = 21$ and $\mu(\mathcal{T}_X) = 151.25$.
Note that $\mathcal{O}_X(D_{e_0} + D_{e_1})$ is a rank 1 reflexive subsheaf of $\mathcal{T}_X$ with degree deg $(D_{e_0} + D_{e_1}) = 217$. Hence, $\mathcal{T}_X$ is unstable.
$\underline{a_1 = 1}$: Here $a = 3,\ b = 2$. We have deg $D_{v_0} = 61,$ deg $D_{e_0} = 125,$ deg $D_{u_\tau} = 28$ and $\mu(\mathcal{T}_X) = 122.25$.
Note that $\mathcal{O}_X(D_{e_0} + D_{e_1})$ is a rank 1 reflexive subsheaf of $\mathcal{T}_X$ with degree deg $(D_{e_0} + D_{e_1}) = 133$. Hence, $\mathcal{T}_X$ is unstable.
(2) $\underline{a_1 = 0}$: Here $a = 4,\ b = 3$. We have deg $D_{v_0} = 48,$ deg $D_{e_0} = 64,$ deg $D_{u_\tau} = 37$ and $\mu(\mathcal{T}_X) = 107.75$. Also deg $(D_{e_0} + D_{e_1}) = 91,$ deg $D_{v_1} = 85$. Note that rank 1 equivariant reflexive subsheaves are $\mathcal{O}_X(D_{v_0}),\ \mathcal{O}_X(D_{v_1}),\ \mathcal{O}_X(D_{v_2}),\ \mathcal{O}_X(D_{v_3}),\ \mathcal{O}_X(D_{e_0} + D_{e_1})$ and $\mathcal{O}_X(D_{u_\tau})$.
Next, we consider reflexive subsheaves of $\mathcal{T}_X$ of rank 2 and 3. The maximum possible slope can occur only from the following situations.
$\underline{\text{rank}(\mathcal{F}) = 2}$

(i) $F = \text{Span}(v_1, e_0, e_1)$, then $\mu(\mathcal{F}) = 88$.

(ii) $F = \text{Span}(v_0, e_0, e_1, u_\tau)$, then $\mu(\mathcal{F}) = 88$.

$\underline{\text{rank}(\mathcal{F}) = 3}$

(i) $F = \text{Span}(v_0, v_1, v_2, v_3)$, then $\mu(\mathcal{F}) = 101$.

(ii) $F = \text{Span}(v_0, v_1, e_0, e_1, u_\tau)$, then $\mu(\mathcal{F}) = 87$.

Hence, in this case, $\mathcal{T}_X$ is stable. ∎

5.7 Stability of tangent bundles on G_1-G_6 in the notation of [1, Section 4]

Let $X = G_1$. We write down the associated fan Δ using the primitive relations from [1, Proposition 3.1.2]). The rays of Δ are

$$v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (1,-1,-1,0), v_4 = (0,0,1,0),$$
$$v_5 = (0,0,0,1), v_6 = (2,0,-1,-1), v_7 = (-1,0,0,0),$$

and the maximal cones are given by the following condition

$$\sigma = \text{Cone}(v_i, v_j, v_k, v_l) \in \Delta \iff \text{Cone}(v_1, v_7), \text{Cone}(v_2, v_3, v_4), \text{Cone}(v_4, v_5, v_6), \text{Cone}(v_5, v_6, v_7), \text{Cone}(v_1, v_2, v_3) \nsubseteq \sigma.$$

We have the following relations

$$D_{v_2} \sim_{\text{lin}} D_{v_3}, D_{v_5} \sim_{\text{lin}} D_{v_6}, D_{v_4} \sim_{\text{lin}} D_{v_3} + D_{v_6}, D_{v_1} \sim_{\text{lin}} D_{v_7} - D_{v_3} - 2D_{v_6}.$$

Therefore, we have $\text{Pic}(X) = \mathbb{Z}D_{v_3} \oplus \mathbb{Z}D_{v_6} \oplus \mathbb{Z}D_{v_7}$. The anticanonical divisor is $-K_X = 2D_{v_3} + D_{v_6} + 2D_{v_7}$.

Proposition 5.7.1. *The tangent bundle on $X = G_1$ is unstable.*

Proof. We have

$$(-K_X)^3 = 8D_{v_3}^3 + 6D_{v_3}D_{v_6}^2 + 24D_{v_3}D_{v_7}^2 + 12D_{v_3}^2D_{v_6} + 24D_{v_3}D_{v_6}D_{v_7} + 24D_{v_3}^2D_{v_7} + D_{v_6}^3 + 12D_{v_6}D_{v_7}^2 + 6D_{v_6}^2D_{v_7} + 8D_{v_7}^3.$$

Using the following relations

$$D_{v_3}^4 = 1, D_{v_3}^2D_{v_6}^2 = 1, D_{v_3}^2D_{v_7}^2 = 1, D_{v_3}^3D_{v_6} = -1, D_{v_3}^2D_{v_6}D_{v_7} = 1,$$
$$D_{v_3}^3D_{v_7} = -1, D_{v_3}D_{v_6}D_{v_7}^2 = 1, D_{v_3}D_{v_6}^2D_{v_7} = 0, D_{v_3}D_{v_7}^3 = 3,$$
$$D_{v_3}D_{v_6}^3 = -1, D_{v_6}^2D_{v_7}^2 = 0, D_{v_6}^3D_{v_7} = 0, D_{v_6}D_{v_7}^3 = 1, D_{v_6}^4 = 1, D_{v_7}^4 = 5,$$

we have $\deg D_{v_3} = 61$, $\deg D_{v_6} = 55$, $\deg D_{v_7} = 176$ and $\mu(\mathcal{T}_X) = 132.25$. Note that $\mathcal{O}_X(D_{v_1} + D_{v_7})$ is a destabilizing subsheaf of $\mathcal{T}_X$ with degree 181, hence $\mathcal{T}_X$ is unstable. □

Let $X' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^2}(\beta))$. The fan Δ' associated to X' is given as follows. Let u_1, u_2 be the standard basis of $\mathbb{Z}^2$ and e_1', e_2' also denote the standard basis of $\mathbb{Z}^2$. Set $v_i = (u_i, 0, 0)$ for $i = 1, 2$ and $e_j = (0, 0, e_j')$ for $j = 1, 2$, $e_0 = -e_1 - e_2$ and $v_0 = -v_1 - v_2 + \alpha e_1 + \beta e_2$. Then $\Delta'(1) = \{v_0, v_1, v_2, e_0, e_1, e_2\}$ and the maximal cones are of the form

$$\text{Cone}(v_0, \dots, \widehat{v}_i, \dots, v_2, e_0, \dots, \widehat{e}_j, \dots, e_2) \text{ for } i, j = 0, 1, 2.$$

Note that $\text{Pic}(X') = \mathbb{Z}D_{v_0} \oplus \mathbb{Z}D_{e_0}$.

Proposition 5.7.2. *Let X be the blow up of $V(\tau)$ on X', where $\tau = Cone(v_0, e_2) \in \Delta'$ and $\alpha = 0$, $\beta = 1$ (note that $X = G_2$ in the notation of [1, Section 4]). Then $\mathcal{T}_X$ is unstable.*

Proof. Rays of the fan Δ associated to X are as follows

$$v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_0 = (-1,-1,0,1), e_1 = (0,0,1,0),$$
$$e_2 = (0,0,0,1), e_0 = (0,0,-1,-1), u_\tau = (-1,-1,0,2).$$

We have the following relations

$$D_{v_1} \sim_{\text{lin}} D_{v_2} \sim_{\text{lin}} D_{v_0} + D_{u_\tau}, D_{e_1} \sim_{\text{lin}} D_{e_0}, D_{e_2} \sim_{\text{lin}} D_{e_0} - D_{v_0} - 2D_{u_\tau}.$$

Hence, $\text{Pic}(X) = \mathbb{Z}D_{v_0} \oplus \mathbb{Z}D_{e_0} \oplus \mathbb{Z}D_{u_\tau}$. Then the anticanonical divisor is $-K_X = 2D_{v_0} + 3D_{e_0} + D_{u_\tau}$. We have

$$(-K_X)^3 = 8D_{v_0}^3 + 54D_{v_0}D_{e_0}^2 + 6D_{v_0}D_{u_\tau}^2 + 36D_{v_0}^2D_{e_0} + 36D_{v_0}D_{e_0}D_{u_\tau}$$
$$+ 12D_{v_0}^2D_{u_\tau} + 27D_{e_0}^3 + 9D_{e_0}D_{u_\tau}^2 + 27D_{e_0}^2D_{u_\tau} + D_{u_\tau}^3.$$

Using the following relations

$$D_{v_0}D_{e_2} = 0, D_{v_0}^4 = 5, D_{v_0}^3D_{u_\tau} = -4, D_{v_0}D_{e_0}D_{u_\tau}^2 = -1, D_{v_0}^2D_{e_0}D_{u_\tau} = 2,$$
$$D_{v_0}^3D_{e_0} = -3, D_{v_0}^2D_{u_\tau}^2 = 3, D_{v_0}D_{u_\tau}^3 = -2, D_{v_0}^2D_{e_0}^2 = 1, D_{v_0}D_{e_0}^2D_{u_\tau} = 0,$$
$$D_{v_0}D_{e_0}^3 = 1, D_{e_0}D_{u_\tau}^3 = 0, D_{e_0}^2D_{u_\tau}^2 = 0, D_{e_0}^3D_{u_\tau} = 0, D_{e_0}^4 = 1, D_{u_\tau}^4 = 1,$$

we have $\deg D_{v_0} = 44$, $\deg D_{e_0} = 111$, $\deg D_{u_\tau} = 29$ and $\mu(\mathcal{T}_X) = 112.5$. Note that $\deg D_{e_1} = 111$ and $\deg D_{e_2} = 9$. Now consider $F = \text{Span}(e_0, e_1, e_2)$, which corresponds to a rank 2 reflexive subsheaf of $\mathcal{T}_X$ with slope 115.5. Hence, $\mathcal{T}_X$ is unstable. ☐

Proposition 5.7.3. *Let X be the blow up of $V(\tau)$ on X', where $\tau = Cone(v_1, v_2, e_0) \in \Delta'$ and $\alpha = 1$, $\beta = 1$ (note that $X = G_3$ in the notation of [1, Section 4]). Then $\mathcal{T}_X$ is unstable.*

Proof. Rays of the fan Δ associated to X are as follows

$$v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_0 = (-1,-1,1,1), e_1 = (0,0,1,0),$$
$$e_2 = (0,0,0,1), e_0 = (0,0,-1,-1), u_\tau = (1,1,-1,-1).$$

We have the following relations

$$D_{v_1} \sim_{\text{lin}} D_{v_2} \sim_{\text{lin}} D_{v_0} - D_{u_\tau}, D_{e_1} \sim_{\text{lin}} D_{e_2} \sim_{\text{lin}} D_{e_0} - D_{v_0} + D_{u_\tau}.$$

Hence $\text{Pic}(X) = \mathbb{Z}D_{v_0} \oplus \mathbb{Z}D_{e_0} \oplus \mathbb{Z}D_{u_\tau}$. The anticanonical divisor is $-K_X = D_{v_0} + 3D_{e_0} + D_{u_\tau}$. Since $D_{v_0}D_{u_\tau} = 0$, we have

$$(-K_X)^3 = D_{v_0}^3 + 27D_{v_0}D_{e_0}^2 + 9D_{v_0}^2D_{e_0} + 27D_{e_0}^3$$
$$+ 9D_{e_0}D_{u_\tau}^2 + 27D_{e_0}^2D_{u_\tau} + D_{u_\tau}^3.$$

Using the following relations

$$D_{v_0}^4=0, D_{v_0}^2D_{e_0}^2=1, D_{v_0}^3D_{e_0}=0, D_{v_0}D_{e_0}^3=2, D_{e_0}^4=0,$$
$$D_{e_0}^2D_{u_\tau}^2=-1, D_{e_0}^3D_{u_\tau}=2, D_{e_0}D_{u_\tau}^3=0, D_{u_\tau}^4=1,$$

we have deg $D_{v_0}=81$, deg $D_{e_0}=108$, deg $D_{u_\tau}=28$ and $\mu(\mathcal{T}_X)=108.25$. Note that deg $D_{e_1}=$ deg $D_{e_2}=55$. Now consider $F=\text{Span}(e_0,e_1,e_2)$, which corresponds to a rank 2 reflexive subsheaf of $\mathcal{T}_X$ with slope 109. Hence, $\mathcal{T}_X$ is unstable. □

Proposition 5.7.4. *Let X be the blow up of $V(\tau)$ on X', where $\tau=$ Cone$(v_0,e_0)\in\Delta'$ and $(\alpha,\beta)=(0,0)$, $(0,1)$ and $(1,1)$ (note that $X=G_6,G_4,G_5$ respectively, in the notation of [1, Section 4]). Then $\mathcal{T}_X$ is stable.*

Proof. Rays of the fan Δ associated to X are as follows

$$v_1=(1,0,0,0), v_2=(0,1,0,0), v_0=(-1,-1,\alpha,\beta), e_1=(0,0,1,0),$$
$$e_2=(0,0,0,1), e_0=(0,0,-1,-1), u_\tau=(-1,-1,\alpha-1,\beta-1).$$

We have the following relations

$$D_{v_1}\sim_{\text{lin}} D_{v_2}\sim_{\text{lin}} D_{v_0}+D_{u_\tau}, D_{e_1}\sim_{\text{lin}} D_{e_0}-\alpha D_{v_0}-(\alpha-1)D_{u_\tau},$$
$$D_{e_2}\sim_{\text{lin}} D_{e_0}-\beta D_{v_0}-(\beta-1)D_{u_\tau}.$$

Hence $\text{Pic}(X)=\mathbb{Z}D_{v_0}\oplus\mathbb{Z}D_{e_0}\oplus\mathbb{Z}D_{u_\tau}$. The anticanonical divisor is $-K_X=(3-\alpha-\beta)D_{v_0}+3D_{e_0}+(5-\alpha-\beta)D_{u_\tau}$. Since $D_{v_0}D_{e_0}=0$, we have

$$(-K_X)^3=a^3D_{v_0}^3+3ab^2D_{v_0}D_{u_\tau}^2+3a^2bD_{v_0}^2D_{u_\tau}+27D_{e_0}^3 + 9b^2D_{e_0}D_{u_\tau}^2+27bD_{e_0}^2D_{u_\tau}+b^3D_{u_\tau}^3.$$

Now consider the following cases.
$\underline{(\alpha,\beta)=(0,0)}$: Then $a=3,\ b=5$. Using the following

$$D_{v_0}^4=3, D_{v_0}^3D_{u_\tau}=-2, D_{v_0}^2D_{u_\tau}^2=1, D_{v_0}D_{u_\tau}^3=0, D_{e_0}^4=3,$$
$$D_{e_0}^3D_{u_\tau}=-2, D_{e_0}^2D_{u_\tau}^2=1, D_{e_0}D_{u_\tau}^3=0, D_{u_\tau}^4=-1,$$

we have deg $D_{v_0}=$ deg $D_{e_0}=36$, deg $D_{u_\tau}=37$ and $\mu(\mathcal{T}_X)=100.25$. Note also that deg $D_{v_1}=$ deg $D_{v_2}=$ deg $D_{e_1}=$ deg $D_{e_2}=73$.
Next, we consider rank 2 equivariant reflexive subsheaves of $\mathcal{T}_X$. We list those having maximum possible slope below.

(i) $F=\text{Span}(v_0,e_0,u_\tau)$, then $\mu(\mathcal{F})=54.5$.

(ii) $F=\text{Span}(v_0,v_1,v_2)$ or $\text{Span}(e_0,e_1,e_2)$, then $\mu(\mathcal{F})=91$.

Finally, we list rank 3 equivariant reflexive subsheaves of $\mathcal{T}_X$ possibly having maximum slope.

(i) $F = \text{Span}(v_0, e_0, e_1, e_2, u_\tau)$ or $\text{Span}(v_0, v_1, v_2, e_0, u_\tau)$, then $\mu(\mathcal{F}) = 85$.

(ii) $F = \text{Span}(e_0, e_1, e_2, v_1)$, then $\mu(\mathcal{F}) = 85$.

Hence, $\mathcal{T}_X$ is stable.
$\underline{(\alpha, \beta) = (0, 1):}$ Then $a = 2,\ b = 4$. Using the following

$$D_{v_0}^4 = 2, D_{v_0}^3 D_{u_\tau} = -1, D_{v_0}^2 D_{u_\tau}^2 = 0, D_{v_0} D_{u_\tau}^3 = 1, D_{e_0}^4 = 1,$$
$$D_{e_0}^3 D_{u_\tau} = -1, D_{e_0}^2 D_{u_\tau}^2 = 1, D_{e_0} D_{u_\tau}^3 = 0, D_{u_\tau}^4 = -2,$$

we have deg $D_{v_0} = 32$, deg $D_{e_0} = 63$, deg $D_{u_\tau} = 41$ and $\mu(\mathcal{T}_X) = 104.25$. Note also that deg $D_{v_1} =$ deg $D_{v_2} = 73$, deg $D_{e_1} = 104$ and deg $D_{e_2} = 31$.
Next, we list down rank 2 equivariant reflexive subsheaves of $\mathcal{T}_X$ possibly giving maximum slope.

(i) $F = \text{Span}(e_0, e_1, e_2)$, then $\mu(\mathcal{F}) = 99$.

(ii) $F = \text{Span}(v_0, e_0, u_\tau)$, then $\mu(\mathcal{F}) = 68$.

Finally, consider the following rank 3 equivariant reflexive subsheaves of $\mathcal{T}_X$ contributing to maximum slope.

(i) $F = \text{Span}(v_0, e_0, e_1, e_2, u_\tau)$, then $\mu(\mathcal{F}) = 90.33$.

(ii) $F = \text{Span}(v_1, v_2, e_1, u_\tau)$, then $\mu(\mathcal{F}) = 97$.

(iii) $F = \text{Span}(v_0, v_1, v_2, e_2)$, then $\mu(\mathcal{F}) \sim 66.67$.

(iv) $F = \text{Span}(v_1, e_0, e_1, e_2)$, then $\mu(\mathcal{F}) = 90.33$.

Hence, $\mathcal{T}_X$ is stable.
$\underline{(\alpha, \beta) = (1, 1):}$ Then $a = 1,\ b = 3$. Using the following

$$D_{v_0}^4 = 1, D_{v_0}^3 D_{u_\tau} = 0, D_{v_0}^2 D_{u_\tau}^2 = -1, D_{v_0} D_{u_\tau}^3 = 2, D_{e_0}^4 = 0,$$
$$D_{e_0}^3 D_{u_\tau} = 0, D_{e_0}^2 D_{u_\tau}^2 = 1, D_{e_0} D_{u_\tau}^3 = 0, D_{u_\tau}^4 = -3,$$

we have deg $D_{v_0} = 28$, deg $D_{e_0} = 81$, deg $D_{u_\tau} = 45$ and $\mu(\mathcal{T}_X) = 101.5$. Note also that deg $D_{v_1} =$ deg $D_{v_2} = 73$, deg $D_{e_1} =$ deg $D_{e_2} = 53$.
Next, we consider rank 2 equivariant reflexive subsheaves of $\mathcal{T}_X$. We list those having maximum possible slope below.

(i) $F = \text{Span}(v_1, v_2, u_\tau)$, then $\mu(\mathcal{F}) = 95.5$.

(ii) $F = \text{Span}(v_0, e_0, u_\tau)$, then $\mu(\mathcal{F}) = 77$.

(iii) $F = \text{Span}(e_0, e_1, e_2)$, then $\mu(\mathcal{F}) = 93.5$.

Finally, we list rank 3 equivariant reflexive subsheaves of $\mathcal{T}_X$ having maximum possible slope.

(i) $F = \mathrm{Span}(v_0, e_0, e_1, e_2, u_\tau)$, then $\mu(\mathcal{F}) \sim 86.67$.

(ii) $F = \mathrm{Span}(v_0, v_1, v_2, e_0, u_\tau)$, then $\mu(\mathcal{F}) = 100$.

(iii) $F = \mathrm{Span}(v_1, e_0, e_1, e_2)$, then $\mu(\mathcal{F}) \sim 86.67$.

(iv) $F = \mathrm{Span}(v_0, v_1, e_0, u_\tau)$, then $\mu(\mathcal{F}) \sim 75.67$.

Hence, $\mathcal{T}_X$ is stable. □

In the following table, we summarize results regarding stability of tangent bundles on toric Fano 4-folds obtained in this paper, following the notations of Batyrev [1, Section 4].

Table 1: Stability of tangent bundles on toric Fano 4-folds

ρ	X	Stability of $\mathcal{T}_X$	Reference
1	$\mathbb{P}^4$	Stable	Prop 4.1.1
2	$B_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(3))$	Unstable	Cor 4.2.7, (1)
2	$B_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))$	Unstable	Cor 4.2.7, (1)
2	$B_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$	Unstable	Cor 4.2.7, (2)
2	$B_4 = \mathbb{P}^1 \times \mathbb{P}^3$	Strictly semistable	Remark 4.2.6
2	$B_5 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$	Strictly semistable	Cor 4.2.7, (3)
2	$C_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	Unstable	Cor 4.2.7, (1)
2	$C_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	Unstable	Cor 4.2.7, (3)
2	$C_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	Unstable	Cor 4.2.7, (1)
2	$C_4 = \mathbb{P}^2 \times \mathbb{P}^2$	Strictly semistable	Remark 4.2.6
3	$D_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,2))$	Unstable	Prop 5.2.1 (1)
3	$D_2 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_1} \oplus \mathcal{O}_{\mathcal{B}_1}(0,1))$	Unstable	Prop 5.3.1
3	$D_3 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(1,1))$	Unstable	Prop 5.3.1
3	$D_4 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_3} \oplus \mathcal{O}_{\mathcal{B}_3}(0,2))$	Unstable	Prop 5.5.1
3	$D_5 = \mathbb{P}^1 \times \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	Unstable	Prop 5.3.1
3	$D_6 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1))$	Unstable	Prop 5.2.1 (1)
3	$D_7 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1))$	Unstable	Prop 5.1.1 (1)
3	$D_8 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(0,1))$	Unstable	Prop 5.3.1
3	$D_9 = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(1,0))$	Unstable	Prop 5.3.1
3	$D_{10} = \mathbb{P}(\mathcal{O}_{\mathcal{B}_3} \oplus \mathcal{O}_{\mathcal{B}_3}(0,1))$	Unstable	Prop 5.5.1
3	$D_{11} = \mathbb{P}(\mathcal{O}_{\mathcal{H}_1} \oplus \mathcal{O}_{\mathcal{H}_1} \oplus \mathcal{O}_{\mathcal{H}_1}(0,1))$	Unstable	Prop 5.4.1
3	$D_{12} = \mathbb{P}^1 \times \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	Unstable	Prop 5.3.1
3	$D_{13} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	Strictly semistable	Remark 4.2.6

3	$D_{14} = \mathbb{P}^1 \times \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$	Strictly semistable	Remark 4.2.6, Cor 4.2.7, (3)
3	$D_{15} = \mathcal{H}_1 \times \mathbb{P}^2$	Strictly semistable	Remark 4.2.6
3	$D_{16} = \mathbb{P}(\mathcal{O}_{\mathcal{B}_2} \oplus \mathcal{O}_{\mathcal{B}_2}(-1, 1))$	Unstable	Prop 5.3.1
3	$D_{17} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1))$	Stable	Prop 5.1.1 (2)
3	$D_{18} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, 2))$	Unstable	Prop 5.2.1 (1)
3	$D_{19} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, 1))$	Stable	Prop 5.2.1 (2)
3	$E_1 = Bl_{\mathbb{P}^2}(B_2)$	Unstable	Prop 5.6.1 (1)
3	$E_2 = Bl_{\mathbb{P}^2}(B_3)$	Unstable	Prop 5.6.1 (1)
3	$E_3 = Bl_{\mathbb{P}^2}(B_4)$	Stable	Prop 5.6.1 (2)
3	G_1	Unstable	Prop 5.7.1
3	$G_2 = Bl_{\mathbb{P}^1 \times \mathbb{P}^1}(C_2)$	Unstable	Prop 5.7.2
3	$G_3 = Bl_{\mathbb{P}^1}(C_3)$	Unstable	Prop 5.7.3
3	$G_4 = Bl_{\mathcal{H}_1}(C_2)$	Stable	Prop 5.7.4
3	$G_5 = Bl_{\mathbb{P}^1 \times \mathbb{P}^1}(C_3)$	Stable	Prop 5.7.4
3	$G_6 = Bl_{\mathbb{P}^1 \times \mathbb{P}^1}(C_4)$	Stable	Prop 5.7.4

(Here ρ denotes the Picard number of X, Prop and Cor abbreviate Proposition and Corollary respectivey.)

6 Existence of equivariant indecomposable rank 2 vector bundles

In this section, we construct a collection of equivariant indecomposable rank 2 vector bundles over some special class of toric varieties of any dimension, namely Bott towers and pseudo-symmetric toric Fano varieties. Moreover, we show that in the case of Bott towers, among the constructed vector bundles, there is a vector bundle which is stable with respect to a suitable choice of polarization.

6.1 On Bott tower

A Bott tower is a tower $M_n \to M_{n-1} \to \cdots \to M_2 \to M_1 \to M_0 = \{\text{point}\}$, consisting of nonsingular projective toric varieties constructed as an iterated sequence of $\mathbb{P}^1$-bundles. We briefly recall the fan Δ_k of the k-th stage Bott tower M_k (see [7] for more details). Let $N = \mathbb{Z}^k$ with standard basis $e_1, \ldots, e_k$. Rays of Δ_k are given by

$$v_i = e_i \text{ for } i = 1, \ldots, k; \; v_{2k} = -e_k \text{ and}$$
$$v_{k+i} = -e_i + c_{i,i+1}e_{i+1} + \cdots + c_{i,k}e_k \text{ for } i = 1, \ldots, k-1,$$

where $c_{i,j}$'s are integers, called Bott numbers. There are 2^k maximal cones of dimension k generated by these rays such that no cone contains v_i and v_{k+i}

simultaneously for $i = 1, \ldots, k$. Let $D_i := D_{v_i}$ denote the invariant prime divisor corresponding to the edge v_i for $i = 1, \ldots, 2k$. We have the following relations among invariant prime divisors:

$$\begin{aligned} &D_{k+1} \sim_{\text{lin}} D_1, D_{k+2} \sim_{\text{lin}} D_2 + c_{1,2} D_{k+1}, \\ &D_{k+i} \sim_{\text{lin}} D_i + c_{1,i} D_{k+1} + \cdots + c_{i-1,i} D_{k+i-1} \text{ for } i = 3, \ldots, k, \end{aligned} \tag{6.1}$$

and the Picard group of the Bott tower is given by $\text{Pic}(M_k) = \mathbb{Z}D_{k+1} \oplus \cdots \oplus \mathbb{Z}D_{2k}$.

Proposition 6.1.1. *Let $X = M_k$ with $k \geq 2$ and $1 \leq p \leq k, 1 \leq q \leq 2k, q \neq p, k+p$. Then there exists a collection of rank 2 indecomposable equivariant vector bundles $\mathcal{E}_{p,q}$ on X with $c_1(\mathcal{E}_{p,q}) = D_p + D_q + D_{k+p}$.*

Proof. Consider the vector space $E = \mathbb{C}^2$ and three distinct one dimensional subspaces L_p, L_q and L_{k+p} in E. Now define the filtrations $\big(E, \{E^{v_j}_{p,q}(i)\}_{j=1,\ldots,2k}\big)$ as follows:
$E^{v_j}_{p,q}(i) = \begin{cases} 0 & i \leqslant -2 \\ L_j & i = -1 \\ E & i \geq 0 \end{cases}$ for $j = p, k+p, q$ and $E^{v_j}_{p,q}(i) = \begin{cases} 0 & i < 0 \\ E & i \geq 0 \end{cases}$ for all $j \neq p, q, k+p$.
Hence, the filtrations $\big(E, \{E^{v_j}_{p,q}(i)\}_{j=1,\ldots,2k}\big)$ correspond to a rank 2 equivariant reflexive sheaf on X, say $\mathcal{E}_{p,q}$ (see Proposition 2.2.9). Fix a maximal dimensional cone $\sigma \in \Delta_k$. To prove that $\mathcal{E}_{p,q}$ is also locally free, we need to show that the collection of subspaces $\mathfrak{E}^{\sigma}_{p,q} = \{\{E^{v_j}_{p,q}(i)\}_{v_j \in \sigma(1)}\}$ of E forms a distributive lattice (see Proposition 2.2.10, Remark 2.2.11). This follows because $\sigma(1)$ contains at most two of the ray generators v_p, v_q, v_{k+p}, since both v_p and v_{k+p} cannot belong to the same cone. Note that since L_p, L_q and L_{k+p} are distinct, the collection of subspaces $\{E^{v_j}_{p,q}(i)\}_{j=1,\ldots,2k}$ do not form a distributive lattice. Hence by [26, Corollary 2.2.3], $\mathcal{E}_{p,q}$ is in fact indecomposable.
Note that for $j = p, q, k+p$,

$$\dim\,(E^{[v_j]}(i)) = \begin{cases} 1 & i = -1, 0 \\ 0 & \text{otherwise} \end{cases}$$

and for $j \neq p, q, k+p$,

$$\dim\,(E^{[v_j]}(i)) = \begin{cases} 2 & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have $c_1(\mathcal{E}) = D_p + D_q + D_{k+p}$ using Proposition 2.2.12. □

Remark 6.1.2. *The above construction only depends on the choice of p, q. Any three distinct lines L_p, L_q and L_{k+p} will give rise to the same equivariant vector bundle $\mathcal{E}_{p,q}$, since any two sets of three distinct points in $\mathbb{P}^1$ are equivalent by an automorphism of $\mathbb{P}^1$. For $(p, q) \neq (p', q')$, the corresponding vector bundles $\mathcal{E}_{p,q}$ and $\mathcal{E}_{p',q'}$ are non-isomorphic by [26, Theorem 1.2.3, Corollary 1.2.4].*

We will show that the vector bundle $\mathcal{E}_{1,2}$ is stable with respect to a suitable choice of polarization. Henceforth we assume that the Bott numbers are non-negative. Let $H = \sum_{i=1}^{k} a_i D_{k+i}$ be a Cartier divisor on M_k. Then H is ample if and only if $a_i > 0$ for all $i = 1, \cdots, k$ (see [24, Theorem 3.1.1, Corollary 3.1.2]).

Lemma 6.1.3. *Let $H = D_{k+1} + bD_{k+2} + D_{k+3} + \cdots + D_{2k}$, where $b > 0$, be an ample divisor on M_k, $k \geq 2$. Then H^{k-1} is a non-negative integral combination of $V(\tau)$'s, where τ varies over all walls in Δ_k such that $\tau(1) \subseteq \{v_{k+1}, \ldots, v_{2k}\}$.*

Proof. Note that H^{k-1} is a positive integral combination of monomials of the form $D^{\underline{\alpha}} := D_{k+1}^{\alpha_1} \cdots D_{2k}^{\alpha_k}$ with non-negative integers $\alpha_1, \ldots, \alpha_k$ satisfying $\sum_{j=1}^{k} \alpha_j = k-1$. To prove the lemma, it suffices to write such a monomial $D^{\underline{\alpha}}$ as a non-negative integral combination of monomials of the form $D^{\underline{\beta}} = D_{k+1}^{\beta_1} \cdots D_{2k}^{\beta_k}$ with $\beta_j \in \{0, 1\}$ for $j = 1, \ldots, k$ (see [9, Lemma 12.5.2]).
Since $D_{k+1}^2 = 0$, without loss of generality we can assume $\alpha_1 \leq 1$. Now if $\alpha_2 > 1$, using relations in (6.1) and observing that v_2 and v_{k+2} do not form a cone, we can write

$$\begin{aligned} D^{\underline{\alpha}} &= D_{k+1}^{\alpha_1} D_{k+2}^{\alpha_2 - 1}(D_2 + c_{1,2} D_{k+1}) D_{k+3}^{\alpha_3} \cdots D_{2k}^{\alpha_k} \\ &= c_{1,2} D_{k+1}^{\alpha_1 + 1} D_{k+2}^{\alpha_2 - 1} D_{k+3}^{\alpha_3} \cdots D_{2k}^{\alpha_k} \\ &= c_{1,2} D_{k+1}^{\beta_1} D_{k+2}^{\alpha_2 - 1} D_{k+3}^{\alpha_3} \cdots D_{2k}^{\alpha_k} \text{ where } \beta_1 \leq 1 \text{ if the monomial is non-zero.} \end{aligned}$$

Hence we have reduced the exponent of D_{k+2} by one and repeating this process we can write $D^{\underline{\alpha}}$ as a non-negative integral combination of monomials of the form

$$D^{\underline{\beta}} = D_{k+1}^{\beta_1} \cdots D_{2k}^{\beta_k} \text{ with } \beta_1, \beta_2 \in \{0, 1\} \text{ and } \beta_3 = \alpha_3, \ldots, \beta_k = \alpha_k.$$

At the i-th stage, we arrive at monomials of the form $D^{\underline{\alpha}}$, where $\alpha_1, \ldots, \alpha_{i-1} \in \{0, 1\}$. Suppose $\alpha_i > 1$. Then again using relations in (6.1) and observing that v_i and v_{k+i} do not form a cone, we can write $D^{\underline{\alpha}}$ as a non-negative integral combination of monomials of the form $D^{\underline{\beta}}$'s with $\beta_i < \alpha_i$ and $\beta_{i+1} = \alpha_{i+1}, \ldots, \beta_k = \alpha_k$. If $\beta_j > 1$ for some $j = 1, \ldots, i-1$, appealing to Stage j, we will write this monomial as a non-negative integral combination of monomials of the form $D^{\underline{\beta'}}$'s with $\beta'_1, \ldots, \beta'_j \in \{0, 1\}$ and $\beta'_{j+1} = \beta_{j+1}, \ldots, \beta'_k = \beta_k$. Hence eventually we can write $D^{\underline{\alpha}}$ as a non-negative integral combination of monomials of the form $D^{\underline{\gamma}}$'s with $\gamma_1, \ldots, \gamma_i \in \{0, 1\}$ and $\gamma_{i+1} = \alpha_{i+1}, \ldots, \gamma_k = \alpha_k$. Continuing this process at the k-th stage we can express $D^{\underline{\alpha}}$ in the desired form. ☐

Remark 6.1.4. *Using Lemma 6.1.3, we can write $H^{k-1} = \sum_{\tau} a_\tau V(\tau)$, where τ varies over all such walls with $\tau(1) \subseteq \{v_{k+1}, \ldots, v_{2k}\}$ and $a_\tau \in \mathbb{Z}_{\geq 0}[b, c_{i,j} : 1 \leq i < j \leq k]$. Also observe that a_τ involves b only if $v_{k+2} \in \tau(1)$.*

Proposition 6.1.5. *Let $X = M_k$ ($k \geq 2$) be a Bott tower with non-negative Bott numbers. Consider the polarization $H = D_{k+1} + bD_{k+2} + D_{k+3} + \cdots + D_{2k}$ on X, where $b > 0$. Then there exists a rank 2 stable equivariant vector bundle $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 2D_1 + D_2$, which is H-stable for sufficiently large b.*

Proof. Consider the equivariant vector bundle vector $\mathcal{E}_{1,2}$ associated to the filtrations $\left(E, \{E_{1,2}^{v_j}(i)\}_{j=1,\ldots,2k}\right)$ from Proposition 6.1.1. Furthermore, $\deg \mathcal{E} = 2\deg D_1 + \deg D_2$. Hence $\mu(\mathcal{E}) = \deg D_1 + \frac{1}{2}\deg D_2$.
The only equivariant reflexive subsheaves of $\mathcal{E}$ are $\mathcal{O}_X(D_1)$, $\mathcal{O}_X(D_2)$, $\mathcal{O}_X(D_{k+1})$ and $\mathcal{O}_X$. Both $\deg D_1 (= \deg D_{k+1})$ and $\deg (\mathcal{O}_X)$ are less than $\mu(\mathcal{E})$. It remains to show that $\deg D_2 < \mu(\mathcal{E})$, i.e.,

$$\deg D_2 < 2 \deg D_1. \tag{6.2}$$

Now using Lemma 6.1.3 and Remark 6.1.4, we see that $\deg D_2 = P(c_{i,j} : 1 \leq i < j \leq k)$ and $\deg D_1 = b + Q(b, c_{i,j} : 1 \leq i < j \leq k)$, where $P(c_{i,j} : 1 \leq i < j \leq k) \in \mathbb{Z}_{\geq 0}[c_{i,j} : 1 \leq i < j \leq k]$ and $Q(c_{i,j} : 1 \leq i < j \leq k) \in \mathbb{Z}_{\geq 0}[b, c_{i,j} : 1 \leq i < j \leq k]$.
So (6.2) holds for sufficiently large b and hence we conclude that $\mathcal{E}$ is H-stable. □

Remark 6.1.6. *It can be shown that for the polarization $H = b_1 D_{k+1} + \ldots + b_k D_{2k}$ with $b_i > 0$ for all $i = 1, \ldots, k$, the vector bundle $\mathcal{E}$ constructed above is H-stable whenever $b_1 < b_2$ for the cases $k = 2, 3$.*

6.2 On pseudo-symmetric Fano toric varieties

A toric Fano variety is called pseudo-symmetric if its fan contains two centrally symmetric maximal cones, i.e., there exist maximal cones $\sigma, \sigma' \in \Delta$, such that $\sigma = -\sigma'$. For any pseudo-symmetric toric Fano variety X, there exists $s, p, q \in \mathbb{Z}_{\geq 0}$ and $k_1, \ldots, k_p, l_1, \ldots, l_q \in \mathbb{Z}_{\geq 0}$ such that

$$X \cong (\mathbb{P}^1)^s \times V^{2k_1} \times \ldots \times V^{2k_p} \times \widetilde{V}^{2l_1} \times \ldots \times \widetilde{V}^{2l_q}, \tag{6.3}$$

where V^n (respectively, $\widetilde{V}^n$) is an n-dimensional toric Fano variety called the n-dimensional Del Pezzo variety (respectively, pseudo Del Pezzo variety) (see [10]). We briefly recall the fan structures of V^n and $\widetilde{V}^n$ from [6, Section 3]. Let $v_1, \ldots, v_n$ be a basis of $N = \mathbb{Z}^n$, where n is even, say $n = 2r$. Set $v_0 = -v_1 - \cdots - v_n$ and $w_i = -v_i$ for $i = 0, \ldots, n$. Then $\Delta_{V^n}(1) = \{v_0, w_0, \ldots, v_n, w_n\}$ and $\Delta_{\widetilde{V}^n}(1) = \{v_0, v_1, w_1, \ldots, v_n, w_n\}$. Explicitly the fans are given as follows:

$$\Delta_{V^n} = \{\text{Cone}(v_i, w_j : i \in I^r, j \in J^r) \text{ and their faces} \mid I^r, J^r \subseteq \{0, \ldots, n\} \text{ disjoint}\};$$

$$\Delta_{\widetilde{V}^n} = \{\text{Cone}(v_0, v_i, w_j : i \in I^{r-1}, j \in J^r), \text{Cone}(v_i, w_j : i \in \widetilde{I}^{r+s}, j \in \widetilde{J}^{r-s}) \text{ and their faces} \mid I^{r-1}, J^r \subseteq \{1, \ldots, n\} \text{ disjoint}, s \in \{0, \ldots, r\} \text{ and } \widetilde{I}^{r+s}, \widetilde{J}^{r-s} \text{ a partition of } \{1, \ldots, n\}\}.$$

We construct a collection of equivariant indecomposable rank 2 vector bundles on X. When X is a product of $\mathbb{P}^1$'s, we are done by Proposition 6.1.1. Let us first prove the existence of a collection of equivariant indecomposable rank 2 vector bundles on Del Pezzo variety V^n.

Consider the vector space $E = \mathbb{C}^2$ and three distinct one dimensional subspaces L_a, L_b and L'_a in E where $0 \leq a, b \leq n$ and $a \neq b$. Define the filtrations $\Big(E, \{E^{\rho}_{\{a,b\},a}(i)\}\Big)$ as follows:

$$E^{v_j}_{\{a,b\},a}(i) = \begin{cases} 0 & i \leqslant -2 \\ L_j & i = -1 \\ E & i \geq 0, \end{cases} \text{ for } j = a, b; \qquad E^{w_a}_{\{a,b\},a}(i) = \begin{cases} 0 & i \leqslant -2 \\ L'_a & i = -1 \\ E & i \geq 0 \end{cases}$$

and

$$E^{\rho}_{\{a,b\},a}(i) = \begin{cases} 0 & i < 0 \\ E & i \geq 0, \end{cases} \text{ for any ray except } v_a, v_b, w_a.$$

By Proposition 2.2.9, the filtrations $\Big(E, \{E^{\rho}_{\{a,b\},a}(i)\}\Big)$ correspond to a rank 2 equivariant reflexive sheaf $\mathcal{E}_{\{a,b\},a}$ on V^n. Since the rays v_a, v_b, w_a do not form a cone in Δ_{V^n}, it follows that the filtrations satisfy the compatibility condition given in Remark 2.2.11, and hence $\mathcal{E}_{\{a,b\},a}$ is locally free by Proposition 2.2.10. As the one dimensional subspaces L_a, L_b, L'_a are distinct, the filtrations $\Big(E, \{E^{\rho}_{\{a,b\},a}(i)\}\Big)$ do not form a distributive lattice which implies that $\mathcal{E}_{\{a,b\},a}$ does not split and hence is indecomposable.

Consider three distinct one dimensional subspaces L_a, L'_a and L'_b in E where $0 \leq a, b \leq n, a \neq b$. By similar arguments, we have an equivariant indecomposable rank 2 locally free sheaf $\mathcal{E}_{a,\{a,b\}}$ on V^n associated to the filtrations $\Big(E, \{E^{\rho}_{a,\{a,b\}}(i)\}\Big)$ given as follows:

$$E^{v_a}_{a,\{a,b\}}(i) = \begin{cases} 0 & i \leqslant -2 \\ L_a & i = -1 \\ E & i \geq 0, \end{cases} ; \qquad E^{w_j}_{a,\{a,b\}}(i) = \begin{cases} 0 & i \leqslant -2 \\ L'_j & i = -1 \\ E & i \geq 0 \end{cases} \text{ for } j = a, b;$$

and

$$E^{\rho}_{a,\{a,b\}}(i) = \begin{cases} 0 & i < 0 \\ E & i \geq 0, \end{cases} \text{ for any ray except } v_a, w_a, w_b.$$

By similar arguments, we have a collection of equivariant indecomposable rank 2 vector bundles on pseudo Del Pezzo variety $\widetilde{V}^n$, given by $\mathcal{F}_{\{a,b\},a}$ associated to the filtrations $\Big(E, \{E^{\rho}_{\{a,b\},a}(i)\}\Big)$, $\mathcal{F}_{a,\{a,b\}}$ associated to the filtrations $\Big(E, \{E^{\rho}_{a,\{a,b\}}(i)\}\Big)$ for $1 \leq a, b \leq n, a \neq b$ and $\mathcal{F}_{\{0,a\},a}$ associated to the filtrations $\Big(E, \{E^{\rho}_{\{0,a\},a}(i)\}\Big)$ for $0 < a \leq n$.

When X is not a product of $\mathbb{P}^1$'s, from (6.3), at least one of k_p or l_q is positive. Without loss of generality, let us assume k_p is positive. We have an equivariant rank 2 indecomposable vector bundle on V^{2k_p}, say $\mathcal{E}^{V^{2k_p}}$. By pulling back $\mathcal{E}^{V^{2k_p}}$ to X via the projection map, we get an equivariant rank 2 vector bundle which is still indecomposable by [8, Remark 3.3].

From the above discussion, we get the following proposition.

Proposition 6.2.1. *Let X pseudo-symmetric toric Fano variety. There exists a collection of equivariant indecomposable rank 2 vector bundles on X.*

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Jyoti Dasgupta
Department of Mathematics
Indian Institute of
Technology-Madras
Chennai
India
jdasgupta.maths@gmail.com

Arijit Dey
Department of Mathematics
Indian Institute of
Technology-Madras
Chennai
India
arijitdey@gmail.com

Bivas Khan
Department of Mathematics
Indian Institute of
Technology-Madras
Chennai
India
bivaskhan10@gmail.com