The Bredon-Landweber Region in C_2-Equivariant Stable Homotopy Groups

Bertrand J. Guillou and Daniel C. Isaksen

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Abstract. We use the C_2-equivariant Adams spectral sequence to compute part of the C_2-equivariant stable homotopy groups $\pi^{C_2}_{n,n}$. This allows us to recover results of Bredon and Landweber on the image of the geometric fixed-points map $\pi^{C_2}_{n,n} \to \pi_0$. We also recover results of Mahowald and Ravenel on the Mahowald root invariants of the elements 2^k.

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1 Introduction

The goal of this article is to study some phenomena in the C_2-equivariant stable homotopy groups. Let $\mathbb{R}^{n,k}$ be the n-dimensional real representation of C_2 in which the nonidentity element of C_2 acts as -1 on the last k coordinates (and trivially on the first $n-k$), and let $S^{n,k}$ be its one-point compactification. Then $\pi^{C_2}_{n,k}$ is the set of 2-completed C_2-equivariant stable homotopy classes of maps $S^{n,k} \longrightarrow S^{0,0}$. In this article, we are primarily concerned with the groups $\pi^{C_2}_{k,k}$. We alert the reader to the fact that another notational convention is sometimes used for C_2-equivariant stable homotopy groups. Writing σ for the real sign representation of C_2, the representation $\mathbb{R}^{n,k}$ corresponds to $(n-k)+k\sigma$ in $RO(C_2)$. Thus the group which appears here as $\pi^{C_2}_{n,k}$ is also denoted $\pi^{C_2}_{n-k+k\sigma}$ in the literature. Our choice of notation works well in comparison to motivic homotopy theory, and furthermore it was the notation employed by Bredon [Br].

The classical Hopf map $\eta_{\mathrm{cl}} : S^3 \longrightarrow S^2$ can be modeled as the defining quotient map $\mathbb{C}^2 - \{0\} \longrightarrow \mathbb{CP}^1$ for complex projective space. When we remember the action of C_2 via complex conjugation, this represents a C_2-equivariant stable map η in $\pi_{1,1}^{C_2}$. Classically, η_{cl} is nilpotent in the stable homotopy ring, as is every element in positive stems [N]. However, the equivariant Hopf map η is not nilpotent because η induces the non-nilpotent element -2 on geometric fixed points. (The distinction between 2 and -2 depends on choices of orientations and is inconsequential to the argument.) We will concern ourselves with phenomena associated to the non-zero elements η^k in $\pi_{k,k}^{C_2}$.

Because the fixed points of the representation sphere $S^{k,k}$ consist of two points, the geometric fixed point homomorphism takes the form $\phi : \pi_{k,k}^{C_2} \longrightarrow \pi_0 \cong \mathbb{Z}$. Bredon [Br] and Landweber [L] proved that the image of ϕ is not in general generated by $\phi(\eta^k)$. For instance, $\phi(\eta^5) = (-2)^5 = -32$, but $\phi(\pi_{5,5}^{C_2}) = 16\mathbb{Z}$. In fact, the higher powers of η are increasingly divisible by 2 in the C_2-equivariant stable homotopy groups (Corollary 1.3).

Let $\rho : S^{0,0} \longrightarrow S^{1,1}$ be the inclusion of fixed points. This class is sometimes called a_σ in the literature. Our main result describes to what extent the powers of η are divisible by ρ, from which we will deduce several other results.

Theorem 1.1. *Let $k = 4j + \varepsilon \geq 1$, where $0 \leq \epsilon \leq 3$. If $\varepsilon = 0$, then the C_2-equivariant stable homotopy class η^k is divisible by ρ^{k-1} and no higher power of ρ. Otherwise, the C_2-equivariant stable homotopy class η^k is divisible by ρ^{4j} and no higher power of ρ.*

Our primary tool for studying C_2-equivariant stable homotopy groups is the equivariant Adams spectral sequence [G] [HK]. The Bredon-Landweber region refers to the subgroups of $\pi_{k,k}^{C_2}$ that are detected in Adams filtration greater than $\frac{1}{2}k - 1$. This region is displayed in the top part of Figure 2. We will show that the Bredon-Landweber region is additively generated by the elements $\rho^i \eta^k$, together with elements α such that $\rho^i \alpha = \eta^k$ for some i.

We recover the following theorem of Landweber [L, Theorem 2.2], which was originally conjectured by Bredon [Br].

Corollary 1.2. *Let $k = 8j + \varepsilon \geq 1$, with $0 \leq \varepsilon \leq 7$. The image of the geometric fixed points homomorphism $\pi_{k,k}^{C_2} \xrightarrow{\phi} \pi_0$ is generated by*

$$\begin{cases} 2^{4j+1} & \text{if} \quad \varepsilon = 0. \\ 2^{4j+\varepsilon} & \text{if} \quad 1 \leq \varepsilon \leq 4. \\ 2^{4j+4} & \text{if} \quad 5 \leq \varepsilon \leq 7. \end{cases}$$

Similarly, we have:

Corollary 1.3. *Let $k = 8j + \varepsilon \geq 5$, with $0 \leq \varepsilon \leq 7$. The C_2-equivariant* *stable homotopy class η^k is divisible by*

$$\begin{cases} 2^{4j-1} & \text{if} \quad \varepsilon = 0, \\ 2^{4j} & \text{if} \quad 1 \leq \varepsilon \leq 4, \\ 2^{4j+\varepsilon-4} & \text{if} \quad 5 \leq \varepsilon \leq 7, \end{cases}$$

and no higher power of 2*.*

The proofs of Theorem 1.1, Corollary 1.2, and Corollary 1.3 appear in Section 6. First, we must carry out some C_2-equivariant Adams spectral sequence calculations.

Our calculations can also be used to compute the classical Mahowald invariants of 2^k for all $k \geq 0$ (see Theorem 7.2). We directly apply the Bruner-Greenlees formulation [BG] of the Mahowald invariant that uses C_2-equivariant homotopy groups. These invariants were previously established by Mahowald and Ravenel [MR] using entirely different methods.

The charts were created using Hood Chatham's `spectralsequences` package.

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2 Notation

We continue with notation from [DI] and [GHIR] as follows.

1. $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau, \rho]$ is the motivic cohomology of $\mathbb{R}$ with $\mathbb{F}_2$ coefficients, where τ and ρ have bidegrees $(0,1)$ and $(1,1)$, respectively.

2. $\mathbb{M}_2^{C_2}$ is the bigraded C_2-equivariant Bredon cohomology of a point with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$.

3. $\mathcal{A}^{C_2}$ is the C_2-equivariant mod 2 Steenrod algebra, using coefficients $\underline{\mathbb{F}}_2$, and $\mathcal{A}^{C_2}(1)$ is the $\mathbb{M}_2^{C_2}$-subalgebra of $\mathcal{A}^{C_2}$ generated by Sq^1 and Sq^2.

4. $\mathrm{Ext}_{\mathrm{cl}}$, $\mathrm{Ext}_{\mathbb{C}}$, $\mathrm{Ext}_{\mathbb{R}}$, and Ext_{C_2} are the cohomologies of the classical, $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant mod 2 Steenrod algebras respectively. These objects are the E_2-pages of Adams spectral sequences.

5. $\pi_{*,*}^{C_2}$ and π_* are the stable homotopy rings of the 2-completed C_2-equivariant sphere spectrum and the 2-completed classical sphere spectrum respectively.

We will use some specific familiar elements of the Adams E_2-page. These elements lie near the "Adams edge" at the top of the Adams chart along a line of slope $1/2$. Our notation for these elements is standard. They include $P^k h_1$,

$P^k h_1^2$, $P^k h_1^3$, $P^k c_0$, $P^k h_1 c_0$, $P^k h_2$, and $P^k h_0 h_2$. In addition, we will consider the elements $P^k h_0 h_3$, $P^k h_0^2 h_3$, and $P^k h_0^3 h_3$. These slightly non-standard (but technically correct) names conveniently refer to a well-understood, regular family of elements in the Adams E_2-page. They are the top three elements in a tower of h_0-multiplications in stems congruent to 7 modulo 8. For more details, we refer the reader to any Adams chart, such as [I2] or [R, Figure A3.1].

We follow [I] in grading Ext groups according to (s, f, w), where:

1. f is the Adams filtration, i.e., the homological degree.
2. $s + f$ is the internal degree, i.e., corresponds to the first coordinate in the bidegree of the Steenrod algebra.
3. s is the stem, i.e., the internal degree minus the Adams filtration.
4. w is the weight.

Following this grading convention, the elements τ and ρ, as elements of $\mathrm{Ext}_{\mathbb{R}}$, have degrees $(0, 0, -1)$ and $(-1, 0, -1)$ respectively. Similarly,

$$h_0 \in \mathrm{Ext}_{\mathbb{R}}^{0,1,0}, \quad h_j \in \mathrm{Ext}_{\mathbb{R}}^{2^j-1,1,2^{j-1}} \text{ for } j > 0, \quad c_0 \in \mathrm{Ext}_{\mathbb{R}}^{8,3,5},$$

and the operator P increases degree by $(8, 4, 4)$.

We will also often refer to the COWEIGHT, which is defined to be $c = s - w$. Since both η and ρ have coweight 0, the Bredon-Landweber region consists entirely of elements of coweight 0. The coweight is also called the Milnor-Witt degree in the motivic context ([DI],[GI]).

3 The ρ-Bockstein spectral sequence

As an $\mathbb{M}_2^{\mathbb{R}}$-module, the equivariant coefficient ring $\mathbb{M}_2^{C_2}$ splits as $\mathbb{M}_2^{C_2} \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$, where NC is the "negative cone". The images of the $\mathbb{R}$-motivic classes ρ and τ in $\mathbb{M}_2^{C_2}$ are sometimes called a_σ and u_σ, respectively, in the equivariant literature. The negative cone has $\mathbb{F}_2$-basis $\{\frac{\gamma}{\rho^j \tau^{k+1}}\}$, where $j, k \geq 0$ and $\frac{\gamma}{\tau}$ lives in degree $(0, 0, 2)$. See [GHIR, Section 2.1] for more details. This splitting of $\mathbb{M}_2^{C_2}$ induces a splitting

$$\mathrm{Ext}_{C_2} \cong \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{NC},$$

where $\mathrm{Ext}_{NC} = \mathrm{Ext}_{\mathbb{R}}(NC, \mathbb{M}_2^{\mathbb{R}})$. The splitting of $\mathbb{M}_2^{C_2}$ also yields a splitting for the Bockstein spectral sequence, and we follow [GHIR, Proposition 3.1] in writing E_1^+ for the summand of the Bockstein E_1-term which converges to $\mathrm{Ext}_{\mathbb{R}}$ and E_1^- for the summand which converges to Ext_{NC}.

The $\mathbb{R}$-motivic ρ-Bockstein spectral sequence ([H, DI]) takes the form

$$E_1^+ = \mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}.$$

The groups $\mathrm{Ext}_{\mathbb{R}}$ are computed for low coweights in [DI]. In coweight 0, E_1^+ is $\mathbb{F}_2[h_0, h_1, \rho]/(h_0 h_1)$, and the only relevant Bockstein differential is $d_1(\tau) = \rho h_0$, giving

Lemma 3.1 ([DI]). *$\mathrm{Ext}_{\mathbb{R}}$ in coweight 0 is $\mathbb{F}_2[h_0, h_1, \rho]/(h_0 h_1, \rho h_0)$.*

The calculation of Ext_{NC} in coweight 0 is much more complicated. We have a short exact sequence [GHIR, Prop 3.1]

$$\bigoplus_{s\geq 0} \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \otimes_{\mathbb{F}_2[\tau]} \mathrm{Ext}_{\mathbb{C}} \to E_1^- \to \bigoplus_{s\geq 0} \mathrm{Tor}_{\mathbb{F}_2[\tau]}\left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}, \mathrm{Ext}_{\mathbb{C}}\right), \tag{3.2}$$

which we abbreviate as

$$\gamma E_1^- \longrightarrow E_1^- \longrightarrow QE_1^-.$$

As explained in [GHIR, Remark 3.5], for each class x in $\mathrm{Ext}_{\mathbb{C}}$ such that $\tau x = 0$, we get a class Qx in QE_1^-, and this element is infinitely divisible by ρ. The τ-torsion elements of $\mathrm{Ext}_{\mathbb{C}}$ in coweight 0 are h_1^k for $k \geq 4$, so these give rise to infinitely ρ-divisible classes Qh_1^k. This describes QE_1^- in coweight 0.
We now describe γE_1^- in coweight 0. First note that $\frac{\gamma}{\tau^i}$ has coweight $-i-1$. Now let x be a class in $\mathrm{Ext}_{\mathbb{C}}$ that is τ-free and not divisible by τ, and let $c \geq 0$ be the coweight of x. If $c \geq 2$, then $\frac{\gamma}{\tau^{c-1}}x$ is an element of γE_1^- in coweight 0 that is infinitely divisible by ρ. When $c \leq 1$, there is no corresponding element of γE_1^- in coweight 0.
This description of E_1^- is incomplete in the sense that it depends on the τ-free part of $\mathrm{Ext}_{\mathbb{C}}$, which is only known in a range. The τ-free part of $\mathrm{Ext}_{\mathbb{C}}$ corresponds precisely to classical $\mathrm{Ext}_{\mathrm{cl}}$ [I, Proposition 2.10]. In a range, information about $\mathrm{Ext}_{\mathrm{cl}}$ can be obtained from an Ext chart, such as [I2] or [R, Figure A.3.1].
In order to rule out certain Bockstein differentials later, we need some structural results for the Bockstein spectral sequence.

Proposition 3.3. *Let x be an element of E_r^- such that $d_r(x)$ is non-zero. Then x and $d_r(x)$ are both infinitely divisible by ρ in E_r^-.*

Proof. Let $E_r^-[k]$ be the part of E_r^- in Bockstein filtration k. Note that $E_1^-[k]$ is zero if $k > 0$, and that $\rho : E_1^-[k] \longrightarrow E_1^-[k+1]$ is an isomorphism if $k < 0$. The d_r differential takes the form $E_r^-[k] \longrightarrow E_r^-[k+r]$.
By induction, diagram chases show that $\rho : E_r^-[k] \longrightarrow E_r^-[k+1]$ is injective if $0 > k > -r$, and it is an isomorphism if $-r \geq k$. In particular, if $-r \geq k$, then every element of $E_r^-[k]$ is infinitely divisible by ρ.
Now let x be an element of $E_r^-[k]$ such that $d_r(x)$ is non-zero. This implies that $E_r^-[k+r]$ is non-zero, so $-r \geq k$, and x is infinitely divisible by ρ. Finally, the multiplicative structure implies that $d_r(x)$ must also be infinitely divisible by ρ. □

Remark 3.4. Proposition 3.3 is dual to [DI, Lemma 3.4], which shows that if $d_r(x)$ is non-zero in E_r^+, then $\rho^k x$ and $\rho^k d_r(x)$ are non-zero in E_r^+ for all $k \geq 0$. In fact, the proof of Proposition 3.3 dualizes line by line.

Proposition 3.5. *The C_2-equivariant Bockstein E_1-page is zero in degrees (s, f, w) such that the coweight $s - w$ is negative, the stem s is positive, and $f > \frac{1}{2}s + \frac{3}{2}$.*

Proof. The summand E_1^+ vanishes when $s - w < 0$, i.e., in negative coweights. Similarly, QE_1^- vanishes in negative coweights.

It only remains to consider γE_1^-. Consider a non-zero element of γE_1^- in degree (s, f, w) with $s > 0$. This element has the form $\frac{\gamma}{\rho^j \tau^k} x$, where x is τ-free in $\mathrm{Ext}_{\mathbb{C}}$. Moreover, the degree of x is $(s - j, f, w - j - k - 1)$.

Using a vanishing result for $\mathrm{Ext}_{\mathbb{C}}$ [GI2, Theorem 1.1], we know that

$$f \leq \frac{1}{2}(s - k) + \frac{3}{2}.$$

Since k is non-negative, it follows that $f \leq \frac{1}{2}s + \frac{3}{2}$. □

Lemma 3.6. *In coweight 1, the localization $E_1[h_1^{-1}]$ of the Bockstein E_1-page vanishes.*

Proof. We know that $\mathrm{Ext}_{\mathbb{C}}[h_1^{-1}]$ vanishes in coweight 1 [GI, Theorem 1.1]. Therefore $E_1^+[h_1^{-1}]$ and $QE_1^-[h_1^{-1}]$ both vanish. Finally, $\gamma E_1^-[h_1^{-1}]$ also vanishes because there are no τ-free classes in $\mathrm{Ext}_{\mathbb{C}}[h_1^{-1}]$. □

4 Some Bockstein differentials

The goal of this section is to compute some explicit Bockstein differentials that we will need for our analysis of the Bredon-Landweber region.

Lemma 4.1. *For $k \geq 0$,*

$$d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}} h_0.$$

$$d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) = \frac{\gamma}{\tau^{4k+3}} h_1.$$

$$d_3\left(\frac{\gamma}{\rho^3\tau^{4k+4}}\right) = 0.$$

Proof. These formulas follow from the Leibniz rule and the $\mathbb{R}$-motivic Bockstein differentials $d_1(\tau^{2k+1}) = \rho\tau^{2k} h_0$, $d_2(\tau^{4k+2}) = \rho^2\tau^{4k+1} h_1$, and $d_3(\tau^{4k+4}) = 0$ [DI, Proposition 3.2].

More specifically, start with the relation $\tau^{2k+1} \cdot \frac{\gamma}{\rho\tau^{2k+1}} = 0$. Apply the d_1 differential to obtain

$$0 = \rho\tau^{2k} h_0 \cdot \frac{\gamma}{\rho\tau^{2k+1}} + \tau^{2k+1} \cdot d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau} h_0 + \tau^{2k+1} \cdot d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right).$$

Therefore, $d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right)$ must equal $\frac{\gamma}{\tau^{2k+2}} h_0$.

The second and third formulas follow from a similar argument, starting with the relations $\tau^{4k+2} \cdot \frac{\gamma}{\rho^2\tau^{4k+2}} = 0$ and $\tau^{4k+4} \cdot \frac{\gamma}{\rho^2\tau^{4k+4}} = 0$. □

Lemma 4.2. *For $k \geq 0$, the elements $\tau P^k h_1$, $P^k h_2$, and $\tau P^k c_0$, are permanent cycles in the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence.*

Proof. We can express the classes $\tau P^k h_1$ recursively as matric Massey products [Q]:

$$\tau P^k h_1 = \left\langle [h_3\ c_0], \begin{bmatrix} h_0^4 \\ \rho^3 h_1^2 \end{bmatrix}, \tau P^{k-1} h_1 \right\rangle.$$

The May Convergence Theorem [M, Theorem 4.1] [I, Theorem 2.2.1], applied to the ρ-Bockstein spectral sequence, shows that $\tau P^k h_1$ is a permanent cycle. Similarly, we have recursive matric Massey products

$$P^k h_2 = \left\langle [h_3\ c_0], \begin{bmatrix} h_0^4 \\ \rho^3 h_1^2 \end{bmatrix}, P^{k-1} h_2 \right\rangle$$

$$\tau P^k c_0 = \left\langle [h_3\ c_0], \begin{bmatrix} h_0^4 \\ \rho^3 h_1^2 \end{bmatrix}, \tau P^{k-1} c_0 \right\rangle. \qquad \square$$

Lemma 4.3. *For $k \geq 0$,*

$$d_3(\tau^3 P^k h_0^3 h_3) = \rho^3 \tau P^{k+1} h_1.$$

$$d_3(\tau^3 P^k h_1 c_0) = \rho^3 P^{k+1} h_2.$$

Proof. By [DI, Theorem 4.1], the only classes in $\mathrm{Ext}_{\mathbb{R}}[\rho^{-1}]$ that survive the ρ-inverted Bockstein spectral sequence are those satifsying $s + f - 2w = 0$. Since $\tau P^k h_1$ does not satisfy this equation, either it supports a Bockstein differential, or $\rho^r \tau P^k h_1$ is hit by a Bockstein differential for some r. But Lemma 4.2 shows that $\tau P^k h_1$ does not support a differential. Therefore, $\rho^r \tau P^k h_1$ must be hit by some differential. By inspection, there is only one possibility. This establishes the first formula.

The same argument applies to establish the second formula. $\square$

Lemma 4.4. *For $k \geq 1$,*

$$d_{4k-1}\left(\frac{Q}{\rho^{4k-1}} h_1^{4k}\right) = \frac{\gamma}{\tau^{4k-1}} P^{k-1} h_0^3 h_3.$$

$$d_{4k}\left(\frac{Q}{\rho^{4k}} h_1^{4k+1}\right) = \frac{\gamma}{\tau^{4k}} P^k h_1.$$

Proof. The class $\frac{Q}{\rho^{4k}} h_1^{4k}$ restricts to a class of the same name in Ext over $\mathcal{A}(1)^{C_2}$. There, we have

$$d_{4k}\left(\frac{Q}{\rho^{4k}} h_1^{4k}\right) = \frac{\gamma}{\tau^{4k}} b^k$$

by [GHIR, Proposition 7.9]. The second formula follows immediately from this, since $P^k h_1$ restricts to $h_1 b^k$.

On the other hand, the Bockstein class $\frac{\gamma}{\tau^{4k}} b^k$ is not in the image of the restriction from the ρ-Bockstein spectral sequence over $\mathcal{A}^{C_2}$, so $\frac{Q}{\rho^{4k}} h_1^{4k}$ must support a shorter Bockstein differential over $\mathcal{A}^{C_2}$. The claimed differential is the only possibility. □

Table 1 summarizes the key differential calculations that we will need later.

Table 1: Key Bockstein differentials

c	(s, f, w)	element	r	d_r
$-2k-2$	$(1, 0, 2k+3)$	$\frac{\gamma}{\rho\tau^{2k+1}}$	1	$\frac{\gamma}{\tau^{2k+2}} h_0$
$-4k-3$	$(2, 0, 4k+5)$	$\frac{\gamma}{\rho^2\tau^{4k+2}}$	2	$\frac{\gamma}{\tau^{4k+3}} h_1$
$4k+6$	$(8k+7, 4k+4, 4k+1)$	$\tau^3 P^k h_0^3 h_3$	3	$\rho^3 \tau P^{k+1} h_1$
$4k+6$	$(8k+9, 4k+4, 4k+3)$	$\tau^3 P^k h_1 c_0$	3	$\rho^3 P^{k+1} h_2$
0	$(8k, 4k-1, 8k)$	$\frac{Q}{\rho^{4k-1}} h_1^{4k}$	$4k-1$	$\frac{\gamma}{\tau^{4k-1}} P^{k-1} h_0^3 h_3$
0	$(8k+2, 4k, 8k+2)$	$\frac{Q}{\rho^{4k+1}} h_1^{4k+1}$	$4k$	$\frac{\gamma}{\tau^{4k}} P^k h_1$

5 Ext_{C_2} in coweight 0

Proposition 5.1 explicitly describes a large part of Ext_{C_2} in coweight 0. This result is more easily understood in the Ext chart in Figure 1, where we are considering only elements above the shaded region.

Proposition 5.1. *In degrees (s, f, w) satisfying $s - w = 0$ and $f > \frac{1}{2}s - 1$, Ext_{C_2} consists of the following classes:*

1. *h_0^k for $k \geq 0$.*
2. *$\rho^j h_1^k$ for $j \geq 0$ and $k \geq 0$.*
3. *$\frac{Q}{\rho^j} h_1^{4k+\varepsilon}$, with $k \geq 1$, $0 \leq \varepsilon \leq 3$ and:*
   - *(a) $j \leq 4k-2$ when $\varepsilon = 0$.*
   - *(b) $j \leq 4k-1$ when $1 \leq \varepsilon \leq 3$.*

Proof. Lemma 3.1 explains how the classes h_0^k and $\rho^j h_1^k$ arise in $\mathrm{Ext}_{\mathbb{R}}$. It remains to study Ext_{NC}.

The desired elements of the form $\frac{Q}{\rho^j} h_1^{4k+\varepsilon}$ arise from the differentials in Lemma 4.4. Proposition 3.3 implies that these elements cannot be involved in any further differentials.

There are several additional Adams periodic families of elements in the Bockstein E_1-page that lie above the line $f = \frac{1}{2}s - 1$ in coweight 0. All of these elements are the targets of Bockstein differentials, as shown in Table 2, so they do not appear in Ext_{C_2}. Each differential in Table 2 follows from the Leibniz rule and the differentials in Table 1.

However, the last three calculations are not entirely obvious. In these cases, write

$$\frac{\gamma}{\rho^2\tau^{4k+1}}P^k c_0 = \frac{\gamma}{\rho^2\tau^{4k+2}} \cdot \tau P^k c_0$$

$$\frac{\gamma}{\rho^3\tau^{4k+1}}P^k h_0^3 h_3 = \frac{\gamma}{\rho^3\tau^{4k+4}} \cdot \tau^3 P^k h_0^3 h_3$$

$$\frac{\gamma}{\rho^3\tau^{4k+1}}P^k h_1 c_0 = \frac{\gamma}{\rho^3\tau^{4k+4}} \cdot \tau^3 P^k h_1 c_0,$$

and then apply the Leibniz rule to these products.

Table 2: Some Bockstein differentials in coweight 1

c	(s, f, w)	element	r	d_r
1	$(8k+3, 4k+1, 8k+2)$	$\frac{\gamma}{\rho^2\tau^{4k-2}}P^k h_1$	2	$\frac{\gamma}{\tau^{4k-1}}P^k h_1^2$
1	$(8k+4, 4k+1, 8k+3)$	$\frac{\gamma}{\rho\tau^{4k-1}}P^k h_2$	1	$\frac{\gamma}{\tau^{4k}}P^k h_0 h_2$
1	$(8k+4, 4k+2, 8k+3)$	$\frac{\gamma}{\rho\tau^{4k-1}}P^k h_0 h_2$	1	$\frac{\gamma}{\tau^{4k-1}}P^k h_1^3$
1	$(8k+8, 4k+2, 8k+7)$	$\frac{\gamma}{\rho\tau^{4k+1}}P^k h_0 h_3$	1	$\frac{\gamma}{\tau^{4k+2}}P^k h_0^2 h_3$
1	$(8k+8, 4k+3, 8k+7)$	$\frac{\gamma}{\rho\tau^{4k+1}}P^k h_0^2 h_3$	1	$\frac{\gamma}{\tau^{4k+2}}P^k h_0^3 h_3$
1	$(8k+10, 4k+3, 8k+9)$	$\frac{\gamma}{\rho^2\tau^{4k+1}}P^k c_0$	2	$\frac{\gamma}{\tau^{4k+2}}P^k h_1 c_0$
1	$(8k+10, 4k+4, 8k+9)$	$\frac{\gamma}{\rho^3\tau^{4k+1}}P^k h_0^3 h_3$	3	$\frac{\gamma}{\tau^{4k+3}}P^{k+1} h_1$
1	$(8k+12, 4k+4, 8k+11)$	$\frac{\gamma}{\rho^3\tau^{4k+1}}P^k h_1 c_0$	3	$\frac{\gamma}{\tau^{4k+4}}P^{k+1} h_2$

□

Figure 1 also shows some classes that are not part of the Bredon-Landweber region. These classes arise in the shaded part of the chart. The structure there is quite complicated, and it will be analyzed in a range in future work.

6 The Adams spectral sequence

We show in Proposition 6.1 that the entire Bredon-Landweber region described in Proposition 5.1 survives the C_2-equivariant Adams spectral sequence.

Proposition 6.1. *No element listed in Proposition 5.1 is either the target or the source of an Adams differential.*

Proof. Except for the elements h_0^k, all of the classes in the Bredon-Landweber region are h_1-periodic. Therefore, any class supporting an Adams differential into the Bredon-Landweber region would be h_1-periodic and of coweight 1. Lemma 3.6 shows that there are no such classes.

On the other hand, the elements h_0^k are h_0-periodic. By inspection in low dimensions, there are no h_0-periodic elements that could support differentials whose values are h_0^k.

Adams differentials on the classes in the Bredon-Landweber region lie in the vanishing region of Proposition 3.5. Therefore, these classes must be permanent cycles. □

Figure 2 shows the Bredon-Landweber region in the Adams E_∞-page. Similarly to the Adams E_2-page in Figure 1, there are additional classes in the shaded part of the chart that we will consider in future work.

We now analyze hidden ρ extensions in the Bredon-Landweber region. The key tool is Proposition 6.2.

Proposition 6.2. *The kernel of $U : \pi_{n,k}^{C_2} \longrightarrow \pi_n$, the underlying homomorphism, is the image of $\rho : \pi_{n-1,k-1}^{C_2} \longrightarrow \pi_{n,k}^{C_2}$.*

Proof. This follows immediately from the cofiber sequence

$$(C_2)_+ \longrightarrow S^{0,0} \xrightarrow{\rho} S^{1,1},$$

using the free-forgetful adjunction between equivariant homotopy classes $(C_2)_+ \longrightarrow X$ and classical homotopy classes from S^0 to the underlying spectrum of X. □

Lemma 6.3. *For $k \geq 4$, there is a hidden ρ extension from Qh_1^k to h_1^k.*

Proof. The classical Hopf map η_{cl} in π_1 is the underlying map of the equivariant Hopf map η in $\pi_{1,1}^{C_2}$. Let $k \geq 4$. Since $(\eta_{\mathrm{cl}})^k = 0$ in π_k, Proposition 6.2 implies that η^k must be a multiple of ρ. The only possibility is that there is a hidden ρ extension from Qh_1^k to h_1^k. □

The hidden extensions of Lemma 6.3 appear in Figure 2 as dashed lines of negative slope.

In principle, it would be possible for there to be additional hidden ρ extensions whose sources lie in the shaded part of Figure 2 and whose targets lie in the Bredon-Landweber region. Lemma 6.4 eliminates this possibility.

Lemma 6.4. *There are no hidden ρ-extensions whose targets lie in the Bredon-Landweber region.*

Proof. We use the unit map $S^{0,0} \longrightarrow ko_{C_2}$ for the C_2-equivariant connective real K-theory spectrum, as studied in [GHIR]. The entire Bredon-Landweber region is detected by the Adams E_∞-page for ko_{C_2}. Therefore, a hidden ρ extension into the Bredon-Landweber region would be detected by a ρ extension in the homotopy of ko_{C_2}.

There are in fact some elements in the homotopy of ko_{C_2} that support ρ extensions into the image of the Bredon-Landweber pattern. These elements are

detected by $\frac{Q}{\rho^{4k-4}} h_1^{4k-1}$ and $\frac{Q}{\rho^{4k-1}} h_1^{4k}$ in the Adams E_∞-page for ko_{C_2}. We must show that they do not lie in the image of the unit map.
These elements support infinite towers of h_0-multiplications in the Adams E_∞-page for ko_{C_2}. Therefore, they cannot lie in the image of the unit map, since the Adams E_∞-page in Figure 2 does not include elements that support infinite towers of h_0-multiplications in the relevant degrees. □

We have now collected enough results to prove Theorem 1.1, Corollary 1.2, and Corollary 1.3.

Proof of Theorem 1.1. The C_2-equivariant element η^k is detected by h_1^k in the Adams E_∞-page. Lemma 6.3 and Proposition 5.1 give a lower bound on the power of ρ that divides η^k, and Lemma 6.4 gives an upper bound on this power of ρ. □

Proof of Corollary 1.2. The geometric fixed points homomorphism ϕ takes the values $\phi(\rho) = 1$ and $\phi(\eta) = -2$. (The minus sign in $\phi(\eta)$ depends on choices of orientations and is inconsequential to the proof.) Consequently, $\phi(\alpha) = 0$ if α is not ρ-periodic, i.e., if α is detected below the Bredon-Landweber region. The corollary now follows from Theorem 1.1. □

Proof of Corollary 1.3. Recall [DI, Section 8] that h_0 detects $2+\rho\eta$. Therefore, for homotopy classes detected by h_0-torsion classes, multiplication by 2 coincides with multiplication by $-\rho\eta$. Thus η^k is divisible by 2^m if and only if η^{k-m} is divisible by ρ^m. This latter condition can be determined by Theorem 1.1. □

7 The Mahowald invariant of 2^k

The goal of this section is compute the Mahowald invariant of 2^k for all $k \geq 0$. We begin by determining the values of the underlying homomorphism $U : \pi_{s,s}^{C_2} \longrightarrow \pi_s$ on classes in the Bredon-Landweber region. Proposition 6.2 implies that $U(\alpha)$ is necessarily zero when α is divisible by ρ. On the other hand, $U(\alpha)$ is always non-zero when α is not divisible by ρ.

Theorem 7.1. *The underlying homomorphism $U : \pi_{s,s}^{C_2} \longrightarrow \pi_s$ takes values as described in Table 3.*

Proof. The underlying homomorphism U induces a map of Adams spectral sequences $\mathrm{Ext}_{C_2} \longrightarrow \mathrm{Ext}_{\mathrm{cl}}$. This map of spectral sequences detects the first four values in Table 3.
The summand Ext_{NC} lies in the kernel of this map of spectral sequences. Therefore, if α in $\pi_{n,k}^{C_2}$ is detected by an element of Ext_{NC}, then $U(\alpha)$ must be detected in strictly higher Adams filtration. For each of the last four entries in Table 3, there is only one possible element in higher Adams filtration. □

Table 3: Some values of the underlying homomorphism

s	α detected by	$U(\alpha)$ detected by
0	1	1
1	h_1	h_1
2	h_1^2	h_1^2
3	h_1^3	h_1^3
$8k-1$	$\frac{Q}{\rho^{4k-2}} h_1^{4k}$	$P^{k-1} h_0^3 h_3$
$8k+1$	$\frac{Q}{\rho^{4k-1}} h_1^{4k+1}$	$P^k h_1$
$8k+2$	$\frac{Q}{\rho^{4k-1}} h_1^{4k+2}$	$P^k h_1^2$
$8k+3$	$\frac{Q}{\rho^{4k-1}} h_1^{4k+3}$	$P^k h_1^3$

The underlying homomorphism U plays a central role in the Mahowald root invariant [MR]. The Bruner-Greenlees formulation of the Mahowald invariant of a homotopy class is given as follows [BG]. First, recall that the geometric fixed points homomorphism $\phi : \pi^{C_2}_{n,k} \longrightarrow \pi_{n-k}$ gives rise to an isomorphism [AI, Proposition 7.0]

$$\pi^{C_2}_{*,*}[\rho^{-1}] \cong \pi_*[\rho^{\pm 1}].$$

Then the Mahowald invariant is defined via the diagram

$$\begin{tikzcd}
\pi_* \arrow[r] \arrow[dr, dashed] \arrow[ddr, bend right, "M(-)"'] & \pi^{C_2}_{*,*}[\rho^{-1}] \\
& \pi^{C_2}_{*,*} \arrow[u] \arrow[d, "U"] \\
& \pi_*
\end{tikzcd}$$

Here the dashed arrow picks out an element from the largest possible stem. More precisely, given a classical stable homotopy class α in π_k, one first chooses an equivariant stable homotopy class β in $\pi^{C_2}_{n,n-k}$ such that $\phi(\beta) = \alpha$ and such that n is as large as possible. In particular, β is not divisible by ρ, for otherwise n would not be maximal. Then $M(\alpha)$ contains the element $U(\beta)$. Beware that there can be more than one choice for β, so the Mahowald invariant has indeterminacy in general.

Our C_2-equivariant calculations allow us to easily recover the Mahowald invariants of 2^k ([MR, Theorem 2.17]).

Theorem 7.2. *Let $k = 4j + \varepsilon \geq 4$ with $0 \leq \varepsilon \leq 3$. The Mahowald invariant of 2^k contains an element that is detected by*

$$\begin{array}{ll} P^{j-1}h_0^3h_3 & \textit{if } \varepsilon = 0. \\ P^j h_1 & \textit{if } \varepsilon = 1. \\ P^j h_1^2 & \textit{if } \varepsilon = 2. \\ P^j h_1^3 & \textit{if } \varepsilon = 3. \end{array}$$

Proof. Theorem 1.1 determines the value of β in the Bruner-Greenlees formulation of the Mahowald invariant. Then Theorem 7.1 gives the value of $U(\beta)$. $\square$

Remark 7.3. The indeterminacy in $M(2^k)$ is determined by the values of the underlying map on classes that are detected in the shaded region of Figure 2. It does not seem possible to predict the indeterminacy of $M(2^k)$ in general. However, inspection in low dimensions shows that the indeterminacy of $M(2^5)$ is generated by elements detected by $h_1^2h_3$ and h_1c_0, while $M(2^k)$ has no indeterminacy for all other values of $k \leq 8$. This indeterminacy calculation depends on a detailed analysis of the shaded region of Figure 2 and will be justified in future work.

8 Charts

Here is a key for reading the charts of Figure 1, Figure 2, and Figure 3:

1. Blue dots indicate copies of $\mathbb{F}_2$ from $\mathrm{Ext}_{\mathbb{R}}$ (or Ext over the $\mathbb{R}$-motivic version of $\mathcal{A}(1)$ in the case of Figure 3).

2. Gray dots indicate copies of $\mathbb{F}_2$ from Ext_{NC} (or from the negative cone part of Ext over $\mathcal{A}^{C_2}(1)$ in the case of Figure 3).

3. Horizontal lines indicate multiplications by ρ.

4. Dashed lines of negative slope indicate ρ extensions that are hidden in the Adams spectral sequence.

5. Vertical lines indicate multiplications by h_0.

6. Vertical arrows indicate infinite sequences of multiplications by h_0.

7. Lines of slope 1 indicate multiplications by h_1.

Figure 1

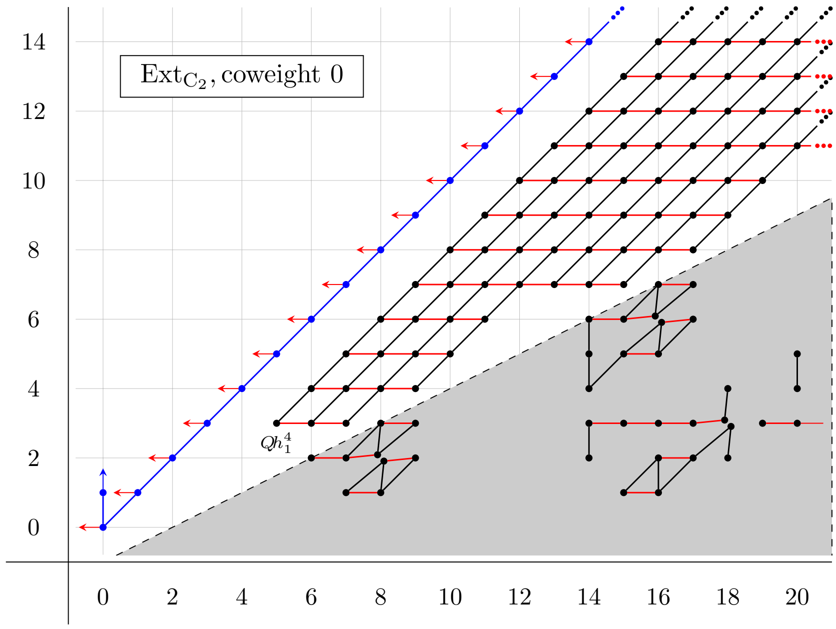


Figure 2

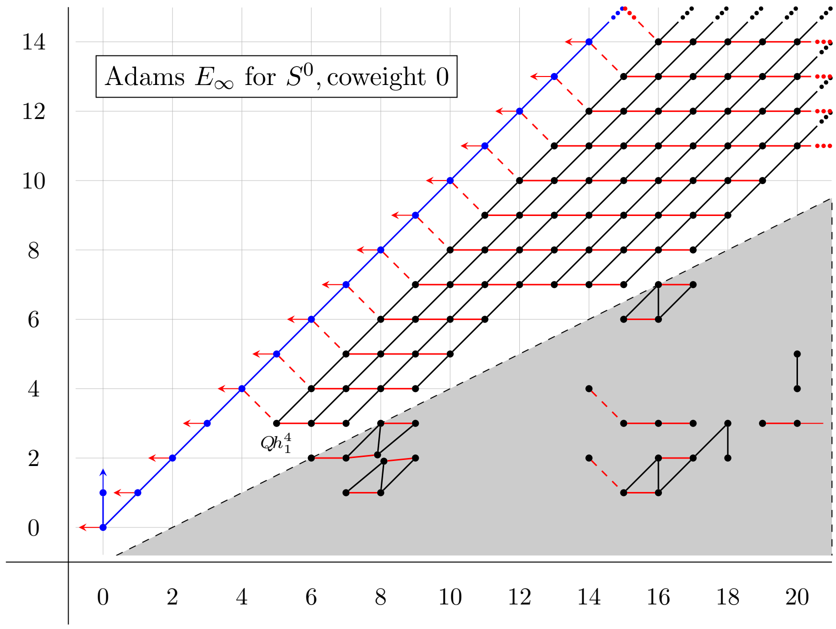
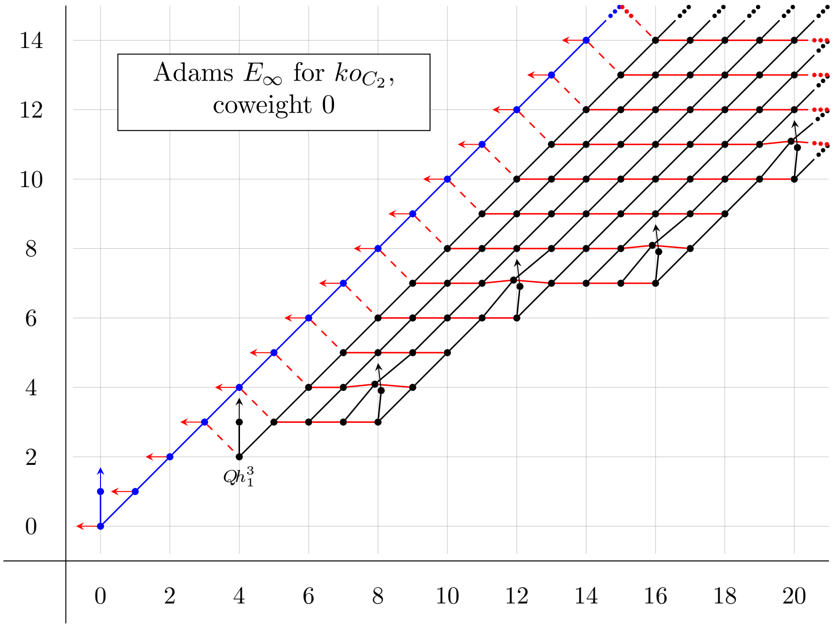


Figure 3



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Bertrand J. Guillou
Department of Mathematics
University of Kentucky
Lexington, KY 40506
USA
bertguillou@uky.edu

Daniel C. Isaksen
Department of Mathematics
Wayne State University
Detroit, MI 48202
USA
isaksen@wayne.edu