Stationary Solutions to the Two-Dimensional Broadwell Model

Leif Arkeryd and Anne Nouri

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Abstract. Existence of renormalized solutions to the two-dimensional stationary Broadwell model in a square with given indata in L^1 is proven. Averaging techniques from the continuous velocity case being unavailable when the velocities are discrete, the approach is based on direct L^1-compactness arguments using the Kolmogorov-Riesz theorem.

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1 Introduction

The phase density f of a dilute gas evolves according to the Boltzmann equation, which writes

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f,f)(t,x,v), \quad t>0, \quad x \in \Omega, \quad v \in \mathbb{R}^3, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is the spatial domain. The left-hand side of (1.1) describes free transport, whereas the right-hand side describes the impact of collisions. In a discrete velocity model, the velocities are concentrated on a usually finite set of points $v_j \in \mathbb{R}^3$, $j \in J$, in the velocity space,

$$f(t,x,v) = \sum_{j \in J} F_j(t,x) \delta_{v=v_j}. \tag{1.2}$$

The Boltzmann equation is then changed into a nonlinear system of conservation laws

$$\frac{\partial F_j}{\partial t}(t,x) + v_j \cdot \nabla_x F_j(t,x) = \sum_{i,k,l} \Gamma_{ij}^{kl}(F_k F_l - F_i F_j)(t,x), \qquad t>0, \quad x \in \Omega, \quad j \in J, \tag{1.3}$$

where the constants Γ_{ij}^{kl} must be chosen so that (1.3) makes sense from a physical point of view, i.e. gives the right conservation laws and an entropy principle. Discrete velocity models are of conceptual interest in the kinetic theory of gases, and an interesting mathematical subject. Simple examples are important because they serve as study cases for general discrete velocity models and the full Boltzmann equation. The two simplest discrete velocity models are the Carleman and the Broadwell models. The model proposed by Carleman [11] describes a gas whose molecules move parallel to a given axis with constant, equal or opposite speeds. However, it is not a physical model because the conservation of momentum is not satisfied. The model proposed by Broadwell [10] describes a gas in which molecules travel with speed of constant magnitude in either direction along a coordinate axis. If particles traveling in opposite directions collide, they are equally likely to move after collision in each of the three coordinates directions, with velocities of opposite sign. In this paper, we consider the two-dimensional stationary Broadwell model in a square,

$$\begin{aligned} &\partial_x F_1 = F_3F_4 - F_1F_2, && F_1(0,\cdot) = f_{b1},\\ &-\partial_x F_2 = F_3F_4 - F_1F_2, && F_2(1,\cdot) = f_{b2},\\ &\partial_y F_3 = F_1F_2 - F_3F_4, && F_3(\cdot,0) = f_{b3},\\ &-\partial_y F_4 = F_1F_2 - F_3F_4, && F_4(\cdot,1) = f_{b4}, \end{aligned} \tag{1.4}$$

with unknown $(F_i)_{1\leq i\leq 4}$ defined on $[0,1]^2$, and given $(f_{bi})_{1\leq i\leq 4}$ defined on $[0,1]$. It is a four velocity model for the Boltzmann equation, with $F_i(x,y) = f(x,y,v_i)$,

$$v_1 = (1,0), \quad v_2 = (-1,0), \quad v_3 = (0,1), \quad v_4 = (0,-1).$$

In the two-dimensional setting of this paper, it describes a gas of particles with identical masses, moving along two perpendicular coordinate axis with the same modulus of velocity.

The boundary value problem (1.4) is considered in L^1 in one of the following equivalent forms,

the exponential multiplier form:

$$F_1(x,y) = f_{b1}(y)e^{-\int_0^x F_2(s,y)ds} + \int_0^x (F_3F_4)(s,y)e^{-\int_s^x F_2(\tau,y)d\tau}ds, \qquad \text{a.a. } (x,y) \in [0,1]^2, \tag{1.5}$$

and analogous equations for F_i, $2 \le i \le 4$,
the mild form:

$$F_1(x,y) = f_{b1}(y) + \int_0^x (F_3F_4 - F_1F_2)(s,y)ds, \quad \text{a.a. } (x,y) \in [0,1]^2, \qquad (1.6)$$

and analogous equations for F_i, $2 \le i \le 4$,
the renormalized form:

$$\partial_x \ln(1+F_1) = \frac{F_3F_4 - F_1F_2}{1+F_1}, \quad F_1(0,\cdot) = f_{b1}, \qquad (1.7)$$

in the sense of distributions, and analogous equations for F_i, $2 \le i \le 4$.
The entropy dissipation of a distribution function $F = (F_i)_{1\le i\le 4}$ is defined as

$$\int_{[0,1]^2} \big(F_1F_2 - F_3F_4\big) \ln \frac{F_1F_2}{F_3F_4}(x,y)dxdy.$$

The main result of the paper is the following.

Theorem 1.1.
Given a non-negative boundary value $f_b = (f_{bi})_{1\le i\le 4}$ with finite mass and entropy, i.e.

$$\sum_{i=1}^{2} \int_0^1 f_{bi}(1+\ln f_{bi})(y)dy + \sum_{i=3}^{4} \int_0^1 f_{bi}(1+\ln f_{bi})(x)dx < +\infty,$$

there exists a stationary non-negative renormalized solution in L^1 with finite entropy-dissipation to the Broadwell model (1.4).

Most mathematical results for discrete velocity models of the Boltzmann equation have been performed in one space dimension. An overview of early results is given in [14]. Half-space problems [5] and weak shock waves [6] for discrete velocity models have also been studied. In two dimensions, special classes of solutions are given in [7], [8], and [15]. [7] contains a detailed study of the stationary Broadwell equation in a rectangle with comparison to a Carleman-like system, and a discussion of (in)compressibility aspects. Discussion of normal discrete velocity models, i.e. conserving nothing but mass, momentum and energy, is done in [9].
The existence of continuous solutions to the two-dimensional stationary Broadwell model with continuous boundary data for a rectangle, is proven in [12]. That proof starts by solving the problem with a given gain term, and uses the compactness of the corresponding twice iterated solution operator to conclude by Schaeffer's fixed point theorem.
The present paper on the Broadwell model is set in a context of physically natural quantities. Mass and entropy flow at the boundary are given, and the solutions obtained, have finite mass and finite entropy dissipation. Averaging techniques from the continuous velocity case [13] being unavailable, a direct

compactness approach is used, based on the Kolmogorov-Riesz theorem.
The plan of the paper is the following. An approximation procedure for the construction of solutions to (1.1) is introduced in Section 2. The passage to the limit in the approximations is performed in Section 3. Here a compactness property of the approximated gain terms in mild form is carried over to the corresponding solutions themselves, using a particular sequence of successive alternating approximations and the Kolmogorov-Riesz theorem [16], [17].
A common approach to existence for stationary Boltzmann like equations is based on the regularizing properties of the gain term. In the continuous velocity case an averaging propery is available to keep this study of the gain term within a weak L^1 frame as in [3]. However, in the discrete velocity case, averaging is not available. Instead strong convergence of an approximating sequence is here directly proved from the regularizing properties for the gain term (cf Lemma 3.5 below). But the technique in that proof is restricted to two dimensional velocities, whereas the averaging technique in the continuous velocity case is dimension independent. Stationary solutions to discrete velocity models with arbitrarily many velocities have recently been obtained [1]. There the constancy of the sums F_1+F_2 and F_3+F_4 along characteristics, which in an essential way is used in the present paper, no longer holds.

2 Approximations

Denote by $L^1_+([0,1]^2)$ the set of non-negative integrable functions on $[0,1]^2$, and by $a\wedge b$ the minimum of two real numbers a and b. Let $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. Approximations to (1.4) to be used in the proof of Theorem 1.1, are introduced in the following lemma.

Lemma 2.1. *For any $k\in\mathbb{N}^*$, there exists a solution $F^k\in\big(L^1_+([0,1]^2)\big)^4$ to*

$$\partial_x F_1^k=\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}-\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}\,, \tag{2.1}$$

$$-\partial_x F_2^k=\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}-\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}\,, \tag{2.2}$$

$$\partial_y F_3^k=\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}-\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}\,, \tag{2.3}$$

$$-\partial_y F_4^k=\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}-\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}},\quad (x,y)\in[0,1]^2, \tag{2.4}$$

$$F_1^k(0,y)=f_{b1}(y)\wedge\frac{k}{2},\quad F_2^k(1,y)=f_{b2}(y)\wedge\frac{k}{2},\quad y\in[0,1], \tag{2.5}$$

$$F_3^k(x,0)=f_{b3}(x)\wedge\frac{k}{2},\quad F_4^k(x,1)=f_{b4}(x)\wedge\frac{k}{2},\quad x\in[0,1]. \tag{2.6}$$

Proof of Lemma 2.1.

The sequence of approximations $(F^k)_{k\in\mathbb{N}^*}$ is obtained in the limit of a further approximation with damping terms αF_j and convolutions in the collision operator.

Step I. Approximations with damping and convolutions.

Take $\alpha > 0$ and set

$$c_\alpha = \frac{1}{\alpha}\int_0^1 \sum_{i=1}^4 f_{bi}(u)du,$$
$$K_\alpha = \{f \in \big(L^1_+([0,1]^2)\big)^4; \sum_{i=1}^4 \int_{[0,1]^2} f_i(x,y)dxdy \leq c_\alpha\}. \tag{2.7}$$

Let μ_α be a smooth mollifier in (x,y) with support in the ball centered at the origin of radius α. Let $\mathcal{T}$ be the map defined on K_α by $\mathcal{T}(f) = F$, where $F = (F_i)_{1\leq i\leq 4}$ is the solution of

$$\alpha F_1 + \partial_x F_1 = \frac{F_3}{1+\frac{F_3}{k}}\frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}} - \frac{F_1}{1+\frac{F_1}{k}}\frac{f_2*\mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}}\,, \tag{2.8}$$
$$\alpha F_2 - \partial_x F_2 = \frac{f_3*\mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}}\frac{F_4}{1+\frac{F_4}{k}} - \frac{f_1*\mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}}\frac{F_2}{1+\frac{F_2}{k}}\,, \tag{2.9}$$
$$\alpha F_3 + \partial_y F_3 = \frac{F_1}{1+\frac{F_1}{k}}\frac{f_2*\mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}} - \frac{F_3}{1+\frac{F_3}{k}}\frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}}\,, \tag{2.10}$$
$$\alpha F_4 - \partial_y F_4 = \frac{f_1*\mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}}\frac{F_2}{1+\frac{F_2}{k}} - \frac{f_3*\mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}}\frac{F_4}{1+\frac{F_4}{k}}\,, \quad (x,y)\in[0,1]^2, \tag{2.11}$$
$$F_1(0,y) = f_{b1}(y)\wedge\frac{k}{2}, \quad F_2(1,y) = f_{b2}(y)\wedge\frac{k}{2}, \quad y\in[0,1], \tag{2.12}$$
$$F_3(x,0) = f_{b3}(x)\wedge\frac{k}{2}, \quad F_4(x,1) = f_{b4}(x)\wedge\frac{k}{2}, \quad x\in[0,1]. \tag{2.13}$$

$F = \mathcal{T}(f)$ is obtained as the limit in $L^1([0,1]^2)$ of the sequence $(F^n)_{n\in\mathbb{N}}$ defined by $F^0 = 0$ and

$$\begin{aligned}
&\alpha F_1^{n+1} + \partial_x F_1^{n+1} = \frac{F_3^n}{1+\frac{F_3^n}{k}} \frac{f_4 * \mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}} - \frac{F_1^{n+1}}{1+\frac{F_1^n}{k}} \frac{f_2 * \mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}},\\
&\alpha F_2^{n+1} - \partial_x F_2^{n+1} = \frac{f_3 * \mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}} \frac{F_4^n}{1+\frac{F_4^n}{k}} - \frac{f_1 * \mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}} \frac{F_2^{n+1}}{1+\frac{F_2^n}{k}},\\
&\alpha F_3^{n+1} + \partial_y F_3^{n+1} = \frac{F_1^n}{1+\frac{F_1^n}{k}} \frac{f_2 * \mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}} - \frac{F_3^{n+1}}{1+\frac{F_3^n}{k}} \frac{f_4 * \mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}},\\
&\alpha F_4^{n+1} - \partial_y F_4^{n+1} = \frac{f_1 * \mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}} \frac{F_2^n}{1+\frac{F_2^n}{k}} - \frac{f_3 * \mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}} \frac{F_4^{n+1}}{1+\frac{F_4^n}{k}},\\
&F_1^{n+1}(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad F_2^{n+1}(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1],\\
&F_3^{n+1}(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad F_4^{n+1}(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1], \quad n \in \mathbb{N}.
\end{aligned}$$

The sequence $(F^n)_{n\in\mathbb{N}}$ is monotone. Indeed, $F^0 \leq F^1$, by the exponential form of F^1. Moreover, assume $F^n \leq F^{n+1}$. It follows from the exponential form that $F^{n+1} - F^{n+2} \leq 0$. Moreover,

$$\begin{aligned}
&\alpha \sum_{i=1}^4 F_i^{n+1} + \partial_x (F_1^{n+1} - F_2^{n+1}) + \partial_y (F_3^{n+1} - F_4^{n+1})\\
&= \frac{f_1 * \mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}} \frac{F_2^n - F_2^{n+1}}{1+\frac{F_2^n}{k}} + \frac{f_2 * \mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}} \frac{F_1^n - F_1^{n+1}}{1+\frac{F_1^n}{k}}\\
&+ \frac{f_3 * \mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}} \frac{F_4^n - F_4^{n+1}}{1+\frac{F_4^n}{k}} + \frac{f_4 * \mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}} \frac{F_3^n - F_3^{n+1}}{1+\frac{F_3^n}{k}}\\
&\leq 0,
\end{aligned}$$

so that

$$\sum_{i=1}^4 \int_{[0,1]^2} F_i^{n+1}(x,y)\,dxdy \leq c_\alpha. \tag{2.14}$$

By the monotone convergence theorem, $(F^n)_{n\in\mathbb{N}}$ converges in $L^1([0,1]^2)$ to some solution F of (2.8)-(2.13). The solution of (2.8)-(2.13) is unique in the set of non-negative functions. Indeed, let $G = (G_i)_{1\leq i\leq 4}$ be a solution of (2.8)-(2.13) with $G_i \geq 0$, $1 \leq i \leq 4$. Let us prove by induction that

$$\forall n \in \mathbb{N}, \quad F_i^n \leq G_i, \quad 1 \leq i \leq 4. \tag{2.15}$$

(2.15) holds for $n = 0$, since $G_i \geq 0$, $1 \leq i \leq 4$. Assume (2.15) holds for n.

Using the exponential form of F_1^{n+1} implies

$$\begin{aligned}
F_1^{n+1}(x,y) &= (f_{b1}(y)\wedge \frac{k}{2})e^{-\alpha x-\int_0^x \frac{f_2*\mu_\alpha}{(1+\frac{F_1^n}{k})(1+\frac{f_2*\mu_\alpha}{k})}(X,y)dX}\\
&+\int_0^x \frac{F_3^n}{1+\frac{F_3^n}{k}}\frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}}(X,y)e^{-\alpha(x-X)-\int_X^x \frac{f_2*\mu_\alpha}{(1+\frac{F_1^n}{k})(1+\frac{f_2*\mu_\alpha}{k})}(r,y)dr}dX\\
&\le (f_{b1}(y)\wedge \frac{k}{2})e^{-\alpha x-\int_0^x \frac{f_2*\mu_\alpha}{(1+\frac{G_1}{k})(1+\frac{f_2*\mu_\alpha}{k})}(X,y)dX}\\
&+\int_0^x \frac{G_3}{1+\frac{G_3}{k}}\frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}}(X,y)e^{-\alpha(x-X)-\int_X^x \frac{f_2*\mu_\alpha}{(1+\frac{G_1}{k})(1+\frac{f_2*\mu_\alpha}{k})}(r,y)dr}dX\\
&= G_1(x,y),\quad (x,y)\in[0,1]^2.
\end{aligned}$$

The same argument can be applied to prove that $F_i^{n+1}\le G_i$, $2\le i\le 4$. Consequently,

$$F_i\le G_i,\quad 1\le i\le 4. \tag{2.16}$$

Moreover, substracting the sum of the partial differential equations satisfied by G_i from the sum of the partial differential equations satisfied by F_i, $1\le i\le 4$, and integrating the resulting equation on $[0,1]^2$, it results

$$\begin{aligned}
\alpha\sum_{i=1}^4\int_{[0,1]^2}(G_i-F_i)(x,y)dxdy+\int_0^1\big((G_1-F_1)(1,y)+(G_2-F_2)(0,y)\big)dy\\
+\int_0^1\big((G_3-F_3)(x,1)+(G_4-F_4)(x,0)\big)dx=0. \quad (2.17)
\end{aligned}$$

It results from (2.16)-(2.17) that $G=F$.
The map $\mathcal{T}$ is continuous in the L^1-norm topology (cf. [1] pages 124-5). Namely, let a sequence $(\varphi_l)_{l\in\mathbb{N}}$ in K_α converge in $L^1([0,1]^2)$ to $\varphi\in K_\alpha$. Set $\Phi_l=\mathcal{T}(\varphi_l)$. Because of the uniqueness of the solution to (2.8)-(2.13), it is enough to prove that there is a subsequence of (Φ_l) converging to $\Phi=\mathcal{T}(\varphi)$. Now there is a subsequence of (φ_l), still denoted (φ_l), such that decreasingly (resp. increasingly) $(G_l)=(\sup_{m\ge l}\varphi_m)$ (resp. $(g_l)=(\inf_{m\ge l}\varphi_m)$) converges to φ in L^1. Here $\sup_{m\ge l}\varphi_m$ (resp. $\inf_{m\ge l}\varphi_m$) means the vector equal to

$$\big(\sup_{m\ge l}\varphi_{m,i}\big)_{1\le i\le 4},\qquad (\text{resp.}\quad \big(\inf_{m\ge l}\varphi_{m,i}\big)_{1\le i\le 4}).$$

Let (S_l) (resp. (s_l)) be the sequence of solutions to

$$\alpha S_{l1}+\partial_x S_{l1}=\frac{S_{l3}}{1+\frac{S_{l3}}{k}}\frac{G_{l4}*\mu_\alpha}{1+\frac{G_{l4}*\mu_\alpha}{k}}-\frac{S_{l1}}{1+\frac{S_{l1}}{k}}\frac{g_{l2}*\mu_\alpha}{1+\frac{g_{l2}*\mu_\alpha}{k}}\,, \tag{2.18}$$

$$\alpha S_{l2}-\partial_x S_{l2}=\frac{G_{l3}*\mu_\alpha}{1+\frac{G_{l3}*\mu_\alpha}{k}}\frac{S_{l4}}{1+\frac{S_{l4}}{k}}-\frac{g_{l1}*\mu_\alpha}{1+\frac{g_{l1}*\mu_\alpha}{k}}\frac{S_{l2}}{1+\frac{S_{l2}}{k}}\,, \tag{2.19}$$

$$\alpha S_{l3}+\partial_y S_{l3}=\frac{S_{l1}}{1+\frac{S_{l1}}{k}}\frac{G_{l2}*\mu_\alpha}{1+\frac{G_{l2}*\mu_\alpha}{k}}-\frac{S_{l3}}{1+\frac{S_{l3}}{k}}\frac{g_{l4}*\mu_\alpha}{1+\frac{g_{l4}*\mu_\alpha}{k}}\,, \tag{2.20}$$

$$\alpha S_{l4}-\partial_y S_{l4}=\frac{G_{l1}*\mu_\alpha}{1+\frac{G_{l1}*\mu_\alpha}{k}}\frac{S_{l2}}{1+\frac{S_{l2}}{k}}-\frac{g_{l3}*\mu_\alpha}{1+\frac{g_{l3}*\mu_\alpha}{k}}\frac{S_{l4}}{1+\frac{S_{l4}}{k}}\,, \tag{2.21}$$

$$S_{l1}(0,y)=f_{b1}(y)\wedge\frac{k}{2},\quad S_{l2}(1,y)=f_{b2}(y)\wedge\frac{k}{2},\quad y\in[0,1], \tag{2.22}$$

$$S_{l3}(x,0)=f_{b3}(x)\wedge\frac{k}{2},\quad S_{l4}(x,1)=f_{b4}(x)\wedge\frac{k}{2},\quad x\in[0,1], \tag{2.23}$$

(resp.

$$\alpha s_{l1}+\partial_x s_{l1}=\frac{s_{l3}}{1+\frac{s_{l3}}{k}}\frac{g_{l4}*\mu_\alpha}{1+\frac{g_{l4}*\mu_\alpha}{k}}-\frac{s_{l1}}{1+\frac{s_{l1}}{k}}\frac{G_{l2}*\mu_\alpha}{1+\frac{G_{l2}*\mu_\alpha}{k}}\,, \tag{2.24}$$

$$\alpha s_{l2}-\partial_x s_{l2}=\frac{g_{l3}*\mu_\alpha}{1+\frac{g_{l3}*\mu_\alpha}{k}}\frac{s_{l4}}{1+\frac{s_{l4}}{k}}-\frac{G_{l1}*\mu_\alpha}{1+\frac{G_{l1}*\mu_\alpha}{k}}\frac{s_{l2}}{1+\frac{s_{l2}}{k}}\,, \tag{2.25}$$

$$\alpha s_{l3}+\partial_y s_{l3}=\frac{s_{l1}}{1+\frac{s_{l1}}{k}}\frac{g_{l2}*\mu_\alpha}{1+\frac{g_{l2}*\mu_\alpha}{k}}-\frac{s_{l3}}{1+\frac{s_{l3}}{k}}\frac{G_{l4}*\mu_\alpha}{1+\frac{G_{l4}*\mu_\alpha}{k}}\,, \tag{2.26}$$

$$\alpha s_{l4}-\partial_y s_{l4}=\frac{g_{l1}*\mu_\alpha}{1+\frac{g_{l1}*\mu_\alpha}{k}}\frac{s_{l2}}{1+\frac{s_{l2}}{k}}-\frac{G_{l3}*\mu_\alpha}{1+\frac{G_{l3}*\mu_\alpha}{k}}\frac{s_{l4}}{1+\frac{s_{l4}}{k}}\,, \tag{2.27}$$

$$s_{l1}(0,y)=f_{b1}(y)\wedge\frac{k}{2},\quad s_{l2}(1,y)=f_{b2}(y)\wedge\frac{k}{2},\quad y\in[0,1], \tag{2.28}$$

$$s_{l3}(x,0)=f_{b3}(x)\wedge\frac{k}{2},\quad s_{l4}(x,1)=f_{b4}(x)\wedge\frac{k}{2},\quad x\in[0,1]). \tag{2.29}$$

(S_l) is a non-increasing sequence, since that holds for the successive iterates defining the sequence. Then (S_l) decreasingly converges in L^1 to some S. Similarly (s_l) increasingly converges in L^1 to some s. The limits S and s satisfy (2.8)-(2.13) written for $(F,f)=(\Phi,\varphi)$. It follows by uniqueness that $s=\Phi=S$, hence that (Φ_l) converges in L^1 to Φ.
The map $\mathcal{T}$ is also compact in the L^1-norm topology. Indeed, let $(\varphi_l)_{l\in\mathbb{N}}$ be a sequence in K_α and $(\Phi_l)_{l\in\mathbb{N}}=(\mathcal{T}(\varphi_l))_{l\in\mathbb{N}}$. For any $|h|<1$, denote by $G_{l1}(x,y)=\Phi_{l1}(x,y+h)-\Phi_{l1}(x,y)$ and

$$\begin{aligned}H_{l1}(x,y)=&\frac{\Phi_{l3}}{1+\frac{\Phi_{l3}}{k}}\frac{\varphi_{l4}*\mu_\alpha}{1+\frac{\varphi_{l4}*\mu_\alpha}{k}}(x,y+h)-\frac{\Phi_{l3}}{1+\frac{\Phi_{l3}}{k}}\frac{\varphi_{l4}*\mu_\alpha}{1+\frac{\varphi_{l4}*\mu_\alpha}{k}}(x,y)\\&-\frac{\Phi_{l1}}{1+\frac{\Phi_{l1}}{k}}(x,y+h)\Big(\frac{\varphi_{l2}*\mu_\alpha}{1+\frac{\varphi_{l2}*\mu_\alpha}{k}}(x,y+h)-\frac{\varphi_{l2}*\mu_\alpha}{1+\frac{\varphi_{l2}*\mu_\alpha}{k}}(x,y)\Big).\end{aligned}$$

They satisfy

$$(\alpha+\frac{\varphi_{l2}*\mu_\alpha}{1+\frac{\varphi_{l2}*\mu_\alpha}{k}})G_{l1}+\partial_x G_{l1}=H_{l1},\quad G_{l1}(0,\cdot)=0,$$

so that

$$G_{l1}(x,y)=\int_0^x H_{l1}(X,y)e^{-\alpha(x-X)-\int_X^x\frac{\varphi_{l2}*\mu_\alpha}{1+\frac{\varphi_{l2}*\mu_\alpha}{k}}(u,y)du}dX,\quad (x,y)\in[0,1]^2.$$

The boundedness by k^2 of the integrands in the right-hand side of (2.8) and (2.10) induces uniform L^1-equicontinuity of $(\Phi_{l1})_{l\in\mathbb{N}}$ (resp. $(\Phi_{l3})_{l\in\mathbb{N}}$) with respect to the x (resp. y) variable. Together with the L^1-compactness of $(\varphi_l*\mu_\alpha)_{l\in\mathbb{N}}$, this implies uniform L^1-equicontinuity with respect to the y variable of $(H_{l1})_{l\in\mathbb{N}}$, then of $(\Phi_{l1})_{l\in\mathbb{N}}$. This proves the L^1 compactness of $(\Phi_{l1})_{l\in\mathbb{N}}$. The L^1 compactness of $(\Phi_{li})_{l\in\mathbb{N}}$, $2\le i\le 4$ can be proven similarly.
Hence by the Schauder fixed point theorem there is a fixed point $\mathcal{T}(F)=F$, i.e. a solution F to

$$\alpha F_1+\partial_x F_1=\frac{F_3}{1+\frac{F_3}{k}}\frac{F_4*\mu_\alpha}{1+\frac{F_4*\mu_\alpha}{k}}-\frac{F_1}{1+\frac{F_1}{k}}\frac{F_2*\mu_\alpha}{1+\frac{F_2*\mu_\alpha}{k}}\,, \tag{2.30}$$

$$\alpha F_2-\partial_x F_2=\frac{F_3*\mu_\alpha}{1+\frac{F_3*\mu_\alpha}{k}}\frac{F_4}{1+\frac{F_4}{k}}-\frac{F_1*\mu_\alpha}{1+\frac{F_1*\mu_\alpha}{k}}\frac{F_2}{1+\frac{F_2}{k}}\,, \tag{2.31}$$

$$\alpha F_3+\partial_y F_3=\frac{F_1}{1+\frac{F_1}{k}}\frac{F_2*\mu_\alpha}{1+\frac{F_2*\mu_\alpha}{k}}-\frac{F_3}{1+\frac{F_3}{k}}\frac{F_4*\mu_\alpha}{1+\frac{F_4*\mu_\alpha}{k}}\,, \tag{2.32}$$

$$\alpha F_4-\partial_y F_4=\frac{F_1*\mu_\alpha}{1+\frac{F_1*\mu_\alpha}{k}}\frac{F_2}{1+\frac{F_2}{k}}-\frac{F_3*\mu_\alpha}{1+\frac{F_3*\mu_\alpha}{k}}\frac{F_4}{1+\frac{F_4}{k}}\,,\quad (x,y)\in[0,1]^2 \tag{2.33}$$

$$F_1(0,y)=f_{b1}(y)\wedge\frac{k}{2},\quad F_2(1,y)=f_{b2}(y)\wedge\frac{k}{2},\quad y\in[0,1], \tag{2.34}$$

$$F_3(x,0)=f_{b3}(x)\wedge\frac{k}{2},\quad F_4(x,1)=f_{b4}(x)\wedge\frac{k}{2},\quad x\in[0,1]. \tag{2.35}$$

Step II. Removal of the damping and the convolutions in (2.30)-(2.35).

Let $k>1$ be fixed. Denote by F^α the solution to (2.30)-(2.35) defined in Step I. Each component of F^α being bounded by a multiple of k^2, $(F^\alpha)_{\alpha\in]0,1[}$ is weakly compact in $L^1([0,1]^2)$. Denote by F^k a limit of a subsequence for the weak topology of $L^1([0,1]^2)$. Let us prove that the convergence is strong in $L^1([0,1]^2)$. Consider the approximation scheme $(f_1^{\alpha,l}, f_2^{\alpha,l})_{l\in\mathbb{N}}$ of (F_1^α, F_2^α),

$$\begin{aligned}
&f_1^{\alpha,0} = f_2^{\alpha,0} = 0,\\
&\alpha f_1^{\alpha,l+1} + \partial_x f_1^{\alpha,l+1} = \frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}} \frac{F_4^\alpha * \mu_\alpha}{1+\frac{F_4^\alpha*\mu_\alpha}{k}} - \frac{f_1^{\alpha,l+1}}{1+\frac{f_1^{\alpha,l+1}}{k}} \frac{f_2^{\alpha,l} * \mu_\alpha}{1+\frac{f_2^{\alpha,l}*\mu_\alpha}{k}},\\
&f_1^{\alpha,l+1}(0,y) = f_{b1}(y)\wedge \frac{k}{2},\\
&\alpha f_2^{\alpha,l+1} - \partial_x f_2^{\alpha,l+1} = \frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}} \frac{F_4^\alpha * \mu_\alpha}{1+\frac{F_4^\alpha*\mu_\alpha}{k}} - \frac{f_1^{\alpha,l} * \mu_\alpha}{1+\frac{f_1^{\alpha,l}*\mu_\alpha}{k}} \frac{f_2^{\alpha,l+1}}{1+\frac{f_2^{\alpha,l+1}}{k}},\\
&f_2^{\alpha,l+1}(1,y) = f_{b2}(y)\wedge \frac{k}{2}, \quad l\in\mathbb{N}.
\end{aligned} \tag{2.36}$$

By induction on l it holds that

$$\begin{aligned}
&f_1^{\alpha,2l} \le f_1^{\alpha,2l+2} \le F_1^\alpha \le f_1^{\alpha,2l+3} \le f_1^{\alpha,2l+1},\\
&f_2^{\alpha,2l} \le f_2^{\alpha,2l+2} \le F_2^\alpha \le f_2^{\alpha,2l+3} \le f_2^{\alpha,2l+1}, \quad \alpha\in]0,1[, \quad l\in\mathbb{N}.
\end{aligned} \tag{2.37}$$

For every $l\in\mathbb{N}$, $(f_1^{\alpha,l})_{\alpha\in]0,1[}$ (resp. $(f_2^{\alpha,l})_{\alpha\in]0,1[}$) is translationally equicontinuous in the x-direction, since all integrands in its exponential form are bounded. It is translationally L^1-equicontinuous in the y-direction by induction on l. Indeed, it is so for (F_3^α) (resp. (F_4^α)) since $\partial_y(e^{\alpha y}F_3^\alpha)$ (resp. $\partial_y(e^{\alpha y}F_4^\alpha)$) is bounded by ek^2, and $(\frac{F_i^\alpha}{1+\frac{F_i^\alpha}{k}})_{\alpha\in]0,1[}$, $i\in\{3,4\}$, is bounded by k. Consequently, it is so for $(\frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}}\frac{F_4^\alpha*\mu_\alpha}{1+\frac{F_4^\alpha*\mu_\alpha}{k}})_{\alpha\in]0,1[}$. There is a limit sequence (g_1^l, g_2^l) in $(L^1([0,1]^2))^2$ such that up to subsequences $(f_1^{\alpha,l})$ (resp. $(f_2^{\alpha,l})$) converges to g_1^l (resp. g_2^l) in $L^1([0,1]^2)$ when $\alpha\to 0$. They satisfy

$$\begin{aligned}
&0\le g_1^{2l} \le g_1^{2l+2} \le F_1^k \le g_1^{2l+3} \le g_1^{2l+1},\\
&0\le g_2^{2l} \le g_2^{2l+2} \le F_2^k \le g_2^{2l+3} \le g_2^{2l+1}, \quad l\in\mathbb{N},\\
&\partial_x g_1^{2l+1} = G - \frac{g_1^{2l+1}}{1+\frac{g_1^{2l+1}}{k}} \frac{g_2^{2l}}{1+\frac{g_2^{2l}}{k}}, \quad \partial_x g_1^{2l} = G - \frac{g_1^{2l}}{1+\frac{g_1^{2l}}{k}} \frac{g_2^{2l-1}}{1+\frac{g_2^{2l-1}}{k}},\\
&-\partial_x g_2^{2l+1} = G - \frac{g_1^{2l}}{1+\frac{g_1^{2l}}{k}} \frac{g_2^{2l+1}}{1+\frac{g_2^{2l+1}}{k}}, \quad -\partial_x g_2^{2l} = G - \frac{g_1^{2l-1}}{1+\frac{g_1^{2l-1}}{k}} \frac{g_2^{2l}}{1+\frac{g_2^{2l}}{k}},\\
&g_1^l(0,y) = f_{b1}(y)\wedge\frac{k}{2}, \quad g_2^l(1,y) = f_{b2}(y)\wedge\frac{k}{2}, \quad y\in[0,1],
\end{aligned}$$

where G is the weak L^1 limit of $(\frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}}\frac{F_4^\alpha*\mu_\alpha}{1+\frac{F_4^\alpha*\mu_\alpha}{k}})_{\alpha\in]0,1[}$ when $\alpha\to 0$. In particular, $(g_1^{2l})_{l\in\mathbb{N}}$ and $(g_2^{2l})_{l\in\mathbb{N}}$ (resp $(g_1^{2l+1})_{l\in\mathbb{N}}$ and $(g_2^{2l+1})_{l\in\mathbb{N}}$) non-decreasingly (resp. non-increasingly) converge in L^1 to some g_1 and g_2 (resp. h_1 and h_2)

when $l \to +\infty$. The limits satisfy

$$
\begin{aligned}
&0 \le g_1 \le F_1^k \le h_1, \quad 0 \le g_2 \le F_2^k \le h_2,\\
&\partial_x h_1 = G - \frac{h_1}{1+\frac{h_1}{k}}\frac{g_2}{1+\frac{g_2}{k}}, \quad \partial_x g_1 = G - \frac{g_1}{1+\frac{g_1}{k}}\frac{h_2}{1+\frac{h_2}{k}},\\
&-\partial_x h_2 = G - \frac{g_1}{1+\frac{g_1}{k}}\frac{h_2}{1+\frac{h_2}{k}}, \quad -\partial_x g_2 = G - \frac{h_1}{1+\frac{h_1}{k}}\frac{g_2}{1+\frac{g_2}{k}},\\
&(h_1-g_1)(0,y)=0, \quad (h_2-g_2)(1,y)=0, \quad y\in[0,1].
\end{aligned}
$$

Hence,

$$
(h_2-g_2)(x,y) = (h_1-g_1)(x,y) - (h_1-g_1)(1,y), \quad (x,y)\in[0,1]^2,
$$

and

$$
\begin{aligned}
(h_1-g_1)(x,y) = &- (h_1-g_1)(1,y)\int_0^x \frac{h_1}{(1+\frac{h_1}{k})(1+\frac{g_2}{k})(1+\frac{h_2}{k})}(X,y)\\
&\exp\Big(-\int_X^x \frac{h_2(1+\frac{g_2}{k}) - h_1(1+\frac{g_1}{k})}{(1+\frac{g_1}{k})(1+\frac{h_1}{k})(1+\frac{g_2}{k})(1+\frac{h_2}{k})}(r,y)dr\Big)dX.
\end{aligned}
$$

The non-negativity of h_1-g_1, g_1, g_2, h_1 and h_2 implies that $h_1-g_1=0$. The same holds for h_2-g_2. Consequently

$$
g_1 = h_1 = F_1^k, \quad g_2 = h_2 = F_2^k.
$$

$(F_1^\alpha)_{\alpha\in]0,1[}$ converges to F_1^k in $L^1([0,1]^2)$ when $\alpha\to 0$. Indeed, given $\eta>0$, choose l_0 big enough so that $\parallel g_1^{2l_0+1} - g_1^{2l_0} \parallel_{L^1} < \eta$ and $\parallel g_1^{2l_0} - F_1^k \parallel_{L^1} < \eta$, then α_0 small enough so that

$$
\parallel f_1^{\alpha,2l_0+1} - g_1^{2l_0+1} \parallel_{L^1} \le \eta \quad \text{and} \quad \parallel f_1^{\alpha,2l_0} - g_1^{2l_0} \parallel_{L^1} \le \eta, \quad \alpha\in]0,\alpha_0[.
$$

Then split $\parallel F_1^\alpha - F_1^k \parallel_{L^1}$ as follows,

$$
\begin{aligned}
&\parallel F_1^\alpha - F_1^k \parallel_{L^1}\\
&\le \parallel F_1^\alpha - f_1^{\alpha,2l_0} \parallel_{L^1} + \parallel f_1^{\alpha,2l_0} - g_1^{2l_0} \parallel_{L^1} + \parallel g_1^{2l_0} - F_1^k \parallel_{L^1}\\
&\le \parallel f_1^{\alpha,2l_0+1} - f_1^{\alpha,2l_0} \parallel_{L^1} + 2\eta \quad \text{by} \quad (2.37)\\
&\le \parallel f_1^{\alpha,2l_0+1} - g_1^{2l_0+1} \parallel_{L^1} + \parallel g_1^{2l_0+1} - g_1^{2l_0} \parallel_{L^1} + \parallel g_1^{2l_0} - f_1^{\alpha,2l_0} \parallel_{L^1} + 2\eta\\
&\le 5\eta, \quad \alpha\in]0,\alpha_0[.
\end{aligned}
$$

The L^1 convergence of $(F_i^\alpha)_{k\in\mathbb{N}}$ to F_i^k, $2\le i\le 4$, can be proven similarly. Passing to the limit when $\alpha\to 0$ in (2.30)-(2.35) is straightforward. And so, F^k is a non-negative solution to (2.1)-(2.6). □

3 Passage to the limit when $k \to +\infty$

The study of the passage to the limit when $k \to +\infty$ in (2.1)-(2.6) is split into six lemmas. In Lemma 3.1, uniform bounds are obtained for mass, entropy and the entropy production term of the approximations. Lemma 3.2 splits $[0,1]^2$ into 'large' sets of type $0 \le x \le 1$ times a 'large' set in y for (F_1^k, F_2^k) (resp. a 'large' set in x times $0 \le y \le 1$ for (F_3^k, F_4^k)), where the approximations are uniformly bounded in L^∞, and their complements, where the mass of the approximations is small. Lemma 3.3 proves uniform equicontinuity with respect to the x (resp. y) variable of the two first (resp. last) components of the approximations. In Lemma 3.4, L^1-compactness of a truncated gain term of the approximations is proven. Lemma 3.5 proves that the approximations form a Cauchy sequence in $L^1([0,1]^2)$. Their limit is proven to be a renormalized solution to the Broadwell model in Lemma 3.6 .
In this section, c_b denotes constants that only depend on the given boundary value f_b.

Lemma 3.1.
There are constants c_b such that

$$\int_{[0,1]^2} F_i^k(x,y)dxdy \le c_b, \tag{3.1}$$

$$\int_{F_i^k(x,y)>k} F_i^k(x,y)dxdy \le \frac{c_b}{\ln k}, \quad i \in \{1, \cdots, 4\}, \tag{3.2}$$

$$\int \Big(\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}} - \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}\Big) \ln \frac{F_1^kF_2^k(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}{(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})F_3^kF_4^k}(x,y)dxdy$$
$$\le c_b, \quad k > 2. \tag{3.3}$$

Proof of Lemma 3.1.

Adding (2.1)-(2.4), integrating the resulting equation on $[0,1]^2$ and taking (2.5)-(2.6) into account, implies that total outflow equals total inflow. Also using $\partial_x(F_1^k + F_2^k) = \partial_y(F_3^k + F_4^k) = 0$ implies boundedness of the total mass $\sum_{i=1}^4 \int F_i^k(x,y)dxdy$. Multiply (2.1), (2.2), (2.3), (2.4) by $\ln \frac{F_1^k}{1+\frac{F_1^k}{k}}$, $\ln \frac{F_2^k}{1+\frac{F_2^k}{k}}$, $\ln \frac{F_3^k}{1+\frac{F_3^k}{k}}$, $\ln \frac{F_4^k}{1+\frac{F_1^k}{4}}$, respectively, add the corresponding equations, and integrate the resulting equation on $[0,1]^2$. Denoting by D^k the entropy production term for the approximation F^k,

$$D^k = \int \Big(\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}} - \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}\Big)$$
$$\ln \frac{F_1^kF_2^k(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}{(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})F_3^kF_4^k}(x,y)dxdy,$$

leads to

$$\begin{aligned}&\int_0^1 \Big(F_1^k \ln F_1^k - k(1+\frac{F_1^k}{k})\ln(1+\frac{F_1^k}{k})\Big)(1,y)dy\\&+\int_0^1 \Big(F_2^k \ln F_2^k - k(1+\frac{F_2^k}{k})\ln(1+\frac{F_2^k}{k})\Big)(0,y)dy\\&+\int_0^1 \Big(F_3^k \ln F_3^k - k(1+\frac{F_3^k}{k})\ln(1+\frac{F_3^k}{k})\Big)(x,1)dx\\&+\int_0^1 \Big(F_4^k \ln F_4^k - k(1+\frac{F_4^k}{k})\ln(1+\frac{F_4^k}{k})\Big)(x,0)dx + D^k \leq c_b.\end{aligned}$$

Moreover,

$$k\int \ln(1+\frac{F_i^k}{k}) \leq \int F_i^k \leq c_b, \quad 1\leq i\leq 4.$$

Hence

$$\begin{aligned}&\int_0^1 \Big(F_1^k \ln \frac{F_1^k}{1+\frac{F_1^k}{k}}(1,y) + F_2^k \ln \frac{F_2^k}{1+\frac{F_2^k}{k}}(0,y)\Big)dy\\&+\int_0^1 \Big(F_3^k \ln \frac{F_3^k}{1+\frac{F_3^k}{k}}(x,1) + F_4^k \ln \frac{F_4^k}{1+\frac{F^k 42}{k}}(x,0)\Big)dx + D^k \leq c_b.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_{F_1^k(1,y)>\frac{k}{k-1}} F_1^k \ln \frac{F_1^k}{1+\frac{F_1^k}{k}}(1,y)dy + \int_{F_2^k(0,y)>\frac{k}{k-1}} F_2^k \ln \frac{F_2^k}{1+\frac{F_2^k}{k}}(0,y)dy\\&+\int_{F_3^k(x,1)>\frac{k}{k-1}} F_3^k \ln \frac{F_3^k}{1+\frac{F_3^k}{k}}(x,1)dx + \int_{F_4^k(x,0)>\frac{k}{k-1}} F_4^k \ln \frac{F_4^k}{1+\frac{F_4^k}{k}}(x,0)dx\\&+D_k \leq c_b, \quad k>2.\end{aligned}$$

And so, (3.3) holds. Moreover, for any $\Lambda > 2$ and $k > 2$,

$$\begin{aligned}
&\ln \frac{\Lambda}{1+\frac{\Lambda}{k}}\Big(\int_{F_1^k(1,y)>k} F_1^k(1,y)dy + \int_{F_2^k(0,y)>k} F_2^k(0,y)dy \\
&\qquad + \int_{F_3^k(x,1)>k} F_3^k(x,1)dx + \int_{F_4^k(x,0)>k} F_4^k(x,0)dx\Big) \\
&\leq c_b + \int_{F_1^k(1,y)<\frac{k}{k-1}} F_1^k \mid \ln \frac{F_1^k}{1+\frac{F_1^k}{k}} \mid (1,y)dy \\
&+ \int_{F_2^k(0,y)<\frac{k}{k-1}} F_2^k \mid \ln \frac{F_2^k}{1+\frac{F_2^k}{k}} \mid (0,y)dy \\
&+ \int_{F_3^k(x,1)<\frac{k}{k-1}} F_3^k \mid \ln \frac{F_3^k}{1+\frac{F_3^k}{k}} \mid (x,1)dx \\
&+ \int_{F_4^k(x,0)<\frac{k}{k-1}} F_4^k \mid \ln \frac{F_4^k}{1+\frac{F_4^k}{k}} \mid (x,0)dx \\
&\leq c_b + 2, \quad k > 2.
\end{aligned} \tag{3.4}$$

In particular,

$$\begin{aligned}
&\int_{F_1^k(1,y)>k} F_1^k(1,y)dy + \int_{F_2^k(0,y)>k} F_2^k(0,y)dy \\
&+ \int_{F_3^k(x,1)>k} F_3^k(x,1)dx + \int_{F_4^k(x,0)>k} F_4^k(x,0)dx \leq \frac{c_b}{\ln k}, \quad k > 2.
\end{aligned} \tag{3.5}$$

Since

$$(F_1^k + F_2^k)(x,y) = F_1^k(1,y) + f_{b2}(y) \wedge \frac{k}{2}, \quad (x,y) \in [0,1]^2, \tag{3.6}$$

it holds that

$$F_1^k(x,y) > k \Rightarrow F^k(1,y) > \frac{k}{2}, \quad (x,y) \in [0,1]^2.$$

Consequently, for some subset ω_k of $[0,1]$ such that $|\omega_k| < \frac{c}{k}$,

$$\begin{aligned}
\int_{F_1^k(x,y)>k} F_1^k(x,y)dxdy &\leq \int_{F_1^k(1,y)>\frac{k}{2}} F_1^k(1,y)dy + \int_{\omega_k} f_{b2}(y)dy \\
&\leq \frac{c}{\ln k},
\end{aligned}$$

by (3.4) and the boundedness of the f_{b2} entropy. □

Lemma 3.2.
For $\epsilon > 0$, $\Lambda \geq \exp(\frac{2c_b}{\epsilon})$ and $k \geq \exp(\frac{3c_b}{\epsilon})$, there is a subset $\Omega_{k1}^{\epsilon\Lambda}$ of $[0,1]$ with

measure smaller than $\frac{c_b \epsilon}{\Lambda}$ *such that*

$$F_1^k(x,y) \leq \frac{\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}), \quad F_2^k(x,y) \leq \frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}),$$
$$x \in [0,1], \quad y \in [0,1] \setminus \Omega_{k1}^{\epsilon\Lambda}, \tag{3.7}$$

$$\int_0^1 \Big(\int_{\Omega_{k1}^{\epsilon\Lambda}} (F_1^k + F_2^k)(x,y)dy \Big) dx \leq c_b \epsilon. \tag{3.8}$$

Proof of Lemma 3.2.

Since $f_{b2} \in L^1([0,1])$ and

$$\int_0^1 (F_1^k(1,y) + F_2^k(0,y))dy + \int_0^1 (F_3^k(x,1) + F_4^k(x,0))dx \leq c_b,$$

the measure of the set

$$\Omega_{k1}^{\epsilon\Lambda} := \{y \in [0,1]; f_{b2}(y) \geq \frac{\Lambda}{\epsilon} \text{ or } F_1^k(1,y) \geq \frac{\Lambda}{\epsilon}\}, \tag{3.9}$$

is smaller than $\frac{c_b \epsilon}{\Lambda}$. (F_1^k, F_2^k) is uniformly bounded on $[0,1] \times ([0,1] \setminus \Omega_{k1}^{\epsilon\Lambda})$, since

$$\begin{aligned} F_1^k(x,y) &\leq F_1^k(1,y)\exp(\int_0^1 F_2^k(X,y)dX) \\ &\leq F_1^k(1,y)\exp(F_1^k(1,y) + f_{b2}(y)) \quad \text{by} \quad (3.6) \\ &\leq \frac{\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}), \end{aligned}$$

and

$$\begin{aligned} F_2^k(x,y) &\leq F_2^k(0,y)\exp(\int_0^1 F_1^k(X,y)dX) \\ &\leq (F_1^k(1,y) + f_{b2}(y))\exp(F_1^k(1,y) + f_{b2}(y)) \\ &\leq \frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}), \quad x \in [0,1], \quad y \in [0,1] \setminus \Omega_{k1}^{\epsilon\Lambda}. \end{aligned}$$

Moreover, for any $\Lambda \geq \exp(\frac{2c_b}{\epsilon})$ and $k \geq \exp(\frac{3c_b}{\epsilon})$,

$$
\begin{aligned}
&\int_0^1 \Big(\int_{\Omega_{k1}^{\epsilon\Lambda}} (F_1^k + F_2^k)(x,y)dy \Big) dx = \int_{\Omega_{k1}^{\epsilon\Lambda}} (F_1^k(1,y) + f_{b2}(y))dy \\
&\leq \int_{y\in\Omega_{k1}^{\epsilon\Lambda}; F_1^k(1,y)<\Lambda} F_1^k(1,y)dy + \int_{F_1^k(1,y)>\Lambda} F_1^k(1,y)dy \\
&+ \int_{y\in\Omega_{k1}^{\epsilon\Lambda}, f_{b2}(y)<\Lambda} f_{b2}(y)dy + \int_{f_{b2}(y)>\Lambda} f_{b2}(y)dy \\
&\leq 2\Lambda|\Omega_{k1}^{\epsilon\Lambda}| + \frac{c_b}{\ln\frac{\Lambda}{1+\frac{\Lambda}{k}}} + \frac{c_b}{\ln\Lambda} \\
&\qquad\qquad\qquad \text{by (3.4) and the boundedness of the entropy of } f_{b2} \\
&\leq c_b\epsilon.
\end{aligned}
$$

□

Lemma 3.3.
There is $c_b > 0$, and for $\epsilon > 0$ given there is $\delta > 0$ such that for $|h| < \delta$, uniformly in $k \in \mathbb{N}^*$,

$$
\begin{aligned}
&\int_{[0,1]^2} |F_i^k(x+h,y) - F_i^k(x,y)|dxdy \leq c_b\epsilon, \quad i \in \{1,2\}, \\
&\int_{[0,1]^2} |F_i^k(x,y+h) - F_i^k(x,y)|dxdy \leq c_b\epsilon, \quad i \in \{3,4\}. \qquad (3.10)
\end{aligned}
$$

Proof of Lemma 3.3.

The four cases F_1^k,..., F_4^k are analogous. The detailed estimates are carried out for F_1^k. The translational L^1 equicontinuity in the x-direction for $\ln(1+F_1^k)$ is obtained as follows from the ∂_x-term in the renormalized equation. Consider $h \in [0,1[$. Write the equation for F_1^k in renormalized form (1.7) integrated on $[x, x+h]$, where the integration from $x+h > 1$ tending to zero with h uniformly in k, is being omitted from the following computations;

$$
\begin{aligned}
&\ln(1+F_1^k(x+h,y)) - \ln(1+F_1^k(x,y)) \\
&= \int_x^{x+h} \frac{1}{1+F_1^k}\Big(\frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}} \Big)(X,y)dX. \qquad (3.11)
\end{aligned}
$$

Denote by sgn the sign function,

$$\mathrm{sgn(r)} = 1 \text{ if r} > 0, \quad \mathrm{sgn(r)} = -1 \text{ if r} < 0.$$

Multiply the previous equation by sgn $\big(\ln(1+F_1^k(x+h,y)) - \ln(1+F_1^k(x,y))\big)$

and integrate on $[0,1]^2$. Uniformly w.r.t. $k \in \mathbb{N}^*$,

$$\begin{aligned}
&\int_{[0,1]^2} |\ln(1+F_1^k(x+h,y)) - \ln(1+F_1^k(x,y))| dxdy \\
&\leq h \int_{[0,1]^2} \frac{1}{1+F_1^k} \Big| \frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}} \Big| (X,y) dX dy \\
&\leq h \Big(\int_{\frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} < \frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}}} \frac{F_1^k}{(1+F_1^k)(1+\frac{F_1^k}{k})} \frac{F_2^k}{1+\frac{F_2^k}{k}} (X,y) dX dy \\
&+ \int_{\frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}} < \frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} < 2 \frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}}} \frac{F_3^k}{(1+F_1^k)(1+\frac{F_3^k}{k})} \frac{F_4^k}{1+\frac{F_4^k}{k}} (X,y) dX dy \\
&\quad + \int_{\frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} > 2 \frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}}} \frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} (X,y) dX dy \Big) \\
&\leq h \Big(3 \int_{[0,1]^2} F_2^k(X,y) dX dy + \frac{2}{\ln 2} D^k \Big) \\
&\leq c_b h. \qquad (3.12)
\end{aligned}$$

Recall that for any non-negative real numbers $x_1 > x_2$, there is $\theta \in]0,1[$ such that

$$\begin{aligned}
x_1 - x_2 &= \exp(\ln(1+x_1)) - \exp(\ln(1+x_2)) \\
&= \exp\big(\theta \ln(1+x_1) + (1-\theta)\ln(1+x_2)\big)\big(\ln(1+x_1) - \ln(1+x_2)\big).
\end{aligned}$$

And so the L^1-norms of the translation differences of F_1^k and $\ln(1+F_1^k)$, are equivalent on $[0,1]\times([0,1]\setminus\Omega_{k1}^{\epsilon\Lambda})$ since F_1^k and $(x,y) \to F_1^k(x+h,y)$ are bounded in $L^\infty([0,1]\times([0,1]\setminus\Omega_{k1}^{\epsilon\Lambda}))$. There is also the small set $[0,1]\times\Omega_{k1}^{\epsilon\Lambda}$ with masses of F_1^k and $F_1^k(\cdot+h,\cdot)$ bounded by $c\epsilon$. Together with (3.12) this proves the translational equicontinuity in the x-direction for $k \geq \exp(\frac{3c_b}{\epsilon})$. The proof for $h \in]-1,0[$ is similar. □

Given $\epsilon > 0$, $\Lambda \geq \exp(\frac{2c_b}{\epsilon})$ and $k \geq \exp(\frac{3c_b}{\epsilon})$, let $\Omega_{k1}^{\epsilon\Lambda} \subset [0,1]$ as defined in Lemma 3.2, and take $\chi_{k1}^{\epsilon\Lambda}$ as the corresponding cutoff function,

$$\chi_{k1}^{\epsilon\Lambda}(y) = 1 \text{ if } \; y \notin \Omega_{k1}^{\epsilon\Lambda}, \qquad \chi_{k1}^{\epsilon\Lambda}(y) = 0 \text{ if } \; y \in \Omega_{k1}^{\epsilon\Lambda}.$$

Lemma 3.4.
Let $(\alpha^k)_{k\in\mathbb{N}}$ be a non-negative sequence bounded in L^∞ and compact in L^1 .

The sequences

$$\Big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^x \frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)e^{-\int_X^x \alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*}$$
$$\textit{and}\ \Big(\chi^{\epsilon\Lambda}_{k1}(y)\int_x^1 \frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)e^{-\int_x^X \alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*},$$
$$\Big(\textit{resp.}\ \big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^1 \frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}\Big),$$

are compact in $L^1([0,1]^2)$ *(resp. in* $L^1([0,1])$*).*

Proof of Lemma 3.4. For any $\gamma>1$, using Lemmas 3.1-3.2,

$$\begin{aligned}
&\int_{[0,1]^2}\chi^{\epsilon\Lambda}_{k1}(y)\Big|\int_0^{x+h}\frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)e^{-\int_X^{x+h}\alpha^k(u,y)du}dX\\
&\qquad -\int_0^{x}\frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)e^{-\int_X^{x}\alpha^k(u,y)du}dX\Big|dxdy\\
&\leq \int_{[0,1]^2}\chi^{\epsilon\Lambda}_{k1}(y)\Big|\int_x^{x+h}\frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)dX\Big|dxdy\\
&\quad +\int_{[0,1]^2}\chi^{\epsilon\Lambda}_{k1}(y)\int_0^{x}\frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)dX\Big|\int_x^{x+h}\alpha^k(u,y)du\Big|dxdy\\
&\leq \frac{c_b}{\ln\gamma}+\gamma h\int_{[0,1]^2}\chi^{\epsilon\Lambda}_{k1}(y)F^k_1F^k_2(x,y)dxdy\\
&\leq \frac{c_b}{\ln\gamma}+2\gamma h\big(\frac{\Lambda}{\epsilon}\big)^2e^{\frac{4\Lambda}{\epsilon}}.
\end{aligned}$$

Choosing γ big enough, then h small enough, proves the translational L^1 equicontinuity in the x direction of

$$\Big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^x \frac{F^k_3}{1+\frac{F^k_3}{k}}\frac{F^k_4}{1+\frac{F^k_4}{k}}(X,y)e^{-\int_X^x \alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*}.$$

Let us prove its translational L^1 equicontinuity in the y direction. Given $\tilde{\epsilon}>0$, let

$$\gamma>\exp\big(\frac{3c_b}{\tilde{\epsilon}}\big),\quad \epsilon_3<\frac{\tilde{\epsilon}}{6c_b\gamma}\big(\frac{\epsilon}{\Lambda}\big)^2e^{-\frac{4\Lambda}{\epsilon}},\quad \Lambda_3\geq\exp\big(\frac{2c_b}{\epsilon_3}\big). \tag{3.13}$$

Let $\Omega^{\epsilon_3\Lambda_3}_{k3}\subset[0,1]$ as defined in Lemma 3.2 for (F^k_3,F^k_4), and $\chi^{\epsilon_3\Lambda_3}_{k3}$ the corresponding cutoff function,

$$\chi^{\epsilon_3\Lambda_3}_{k3}(x)=1 \text{ if } x\notin\Omega^{\epsilon_3\Lambda_3}_{k3},\qquad \chi^{\epsilon\Lambda}_{k3}(x)=0 \text{ if } x\in\Omega^{\epsilon_3\Lambda_3}_{k3}.$$

First,

$$\int \Big(\int_{X\in[0,x];\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)>\gamma\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}(X,y)} \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\Big)dxdy$$
$$\leq \frac{c_b}{\ln\gamma} \leq \frac{\tilde{\epsilon}}{3}.$$

Moreover,

$$\int_{[0,1]^2} \chi_{k1}^{\epsilon\Lambda}(y) \int_{X\in[0,x];\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)<\gamma\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}(X,y)} (1-\chi_{k3}^{\epsilon_3\Lambda_3}(X))$$
$$\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dXdxdy \leq 2c_b\gamma\Big(\frac{\Lambda}{\epsilon}\Big)^2 e^{\frac{4\Lambda}{\epsilon}}\epsilon_3$$
$$\leq \frac{\tilde{\epsilon}}{3},$$

by definition of ϵ_3. Given the boundedness of $(F_3^k, F_4^k)_{k\geq \exp(\frac{3c_b}{\epsilon_3})}$ on $\big(\Omega_{k3}^{\epsilon_3\Lambda_3}\big)^c \times [0,1]$, and the statements of Lemmas 3.2-3.3 for (F_3^k, F_4^k), there is $h_3 > 0$ such that

$$\int\int_0^x \chi_{k3}^{\epsilon_3\Lambda_3}(X)\Big|\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y+h) - \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)\Big|dXdxdy$$
$$\leq \frac{\tilde{\epsilon}}{3},$$

for $h \in]0, h_3[$, uniformly with respect to $k \geq \exp(\frac{3c_b}{\epsilon_3})$.
The proofs of the $L^1([0,1]^2)$ (resp. $L^1([0,1])$) compactness of

$$\Big(\chi_{k1}^{\epsilon\Lambda}(y)\int_x^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_x^X \alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*},$$
$$\Big(\text{resp.}\big(\chi_{k1}^{\epsilon\Lambda}(y)\int_0^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}\Big)$$

are similar. □

Lemma 3.5.
$(F^k)_{k\in\mathbb{N}^*}$ is compact in $L^1([0,1]^2)$.

Proof of Lemma 3.5.

By (3.1)-(3.2), $(F^k)_{k\in\mathbb{N}^*}$ is weakly compact in $(L^1([0,1]^2))^4$. Denote by F the weak limit of a subsequence, still denoted by (F^k). Let us prove that $(F_1^k)_{k\in\mathbb{N}^*}$ is strongly compact in $L^1([0,1]^2)$. It is by (3.8) enough to prove that up to a subsequence, given $\epsilon>0$, for $\Lambda\geq e^{\frac{2c_b}{\epsilon}}$, $k\geq e^{\frac{3c_b}{\epsilon}}$ and $\Omega_{k1}^{\epsilon\Lambda}$ as defined in Lemma 3.2, $(\chi_{k1}^{\epsilon\Lambda}F_1^k)_{k\in\mathbb{N}^*}$ is strongly compact in $L^1([0,1]^2)$. For every F^k in the subsequence, consider the approximation scheme $(f_1^{k,l},f_2^{k,l})_{l\in\mathbb{N}}$ of (F_1^k,F_2^k), defined by

$$\begin{aligned}
&f_1^{k,-1}=f_2^{k,-1}=f_1^{k,0}=f_2^{k,0}=0,\\
&f_1^{k,l+1}(x,y)=f_{b1}(y)\\
&+\int_0^x\big(\chi_{k1}^{\epsilon\Lambda}(y)\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}-\frac{f_1^{k,l+1}}{1+\frac{f_1^{k,l-1}}{k}}\frac{f_2^{k,l}}{1+\frac{f_2^{k,l}}{k}}\big)(X,y)dX, \qquad (3.14)\\
&f_2^{k,l+1}(x,y)=f_{b2}(y)\\
&+\int_x^1\big(\chi_{k1}^{\epsilon\Lambda}(y)\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}-\frac{f_1^{k,l}}{1+\frac{f_1^{k,l}}{k}}\frac{f_2^{k,l+1}}{1+\frac{f_2^{k,l-1}}{k}}\big)(X,y)dX. \qquad (3.15)
\end{aligned}$$

By induction on l, and using an exponential form of $(f_1^{k,l+1},f_2^{k,l+1})$, it holds that

$$\begin{aligned}
&f_1^{k,2l}\leq f_1^{k,2l+2},\quad f_1^{k,2l+3}\leq f_1^{k,2l+1},\\
&f_2^{k,2l}\leq f_2^{k,2l+2},\quad f_2^{k,2l+3}\leq f_2^{k,2l+1},\quad (x,y)\in[0,1]^2,\quad k\in\mathbb{N}^*,\quad l\in\mathbb{N},
\end{aligned}\qquad (3.16)$$

and

$$\begin{aligned}
&f_1^{k,2l}\leq F_1^k\leq f_1^{k,2l+1},\quad f_2^{k,2l}\leq F_2^k\leq f_2^{k,2l+1},\\
&\qquad (x,y)\in[0,1]\times(\Omega_{k1}^{\epsilon\Lambda})^c,\quad k\in\mathbb{N}^*,\quad l\in\mathbb{N}. \qquad (3.17)
\end{aligned}$$

The sequence $(\chi_{k1}^{\epsilon\Lambda}f_1^{k,2l})_{k\geq e^{\frac{3c_b}{\epsilon}}}$ (resp. $(\chi_{k1}^{\epsilon\Lambda}f_2^{k,2l})_{k\geq e^{\frac{3c_b}{\epsilon}}}$) is bounded from above by $(\chi_{k1}^{\epsilon\Lambda}F_1^k)_{k\geq e^{\frac{3c_b}{\epsilon}}}$ (resp. $(\chi_{k1}^{\epsilon\Lambda}F_2^k)_{k\geq e^{\frac{3c_b}{\epsilon}}}$), hence by $\frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon})$. The sequence $(\chi_{k1}^{\epsilon\Lambda}f_1^{k,2l+1})_{k\geq e^{\frac{3c_b}{\epsilon}}}$ (resp. $(\chi_{k1}^{\epsilon\Lambda}f_2^{k,2l+1})_{k\geq e^{\frac{3c_b}{\epsilon}}}$) is bounded by $\frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon})(1+\frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}))$, since

$$\begin{aligned}
&\chi_{k1}^{\epsilon\Lambda}(y)f_1^{k,2l+1}(x,y)=\chi_{k1}^{\epsilon\Lambda}(y)F_1^k(x,y)+\chi_{k1}^{\epsilon\Lambda}(y)\int_0^x\frac{F_1^k}{1+\frac{F_1^k}{k}}\\
&\big(\frac{F_2^k}{1+\frac{F_2^k}{k}}-\frac{f_2^{k,l}}{1+\frac{f_2^{k,l}}{k}}\big)(X,y)e^{-\int_X^x\frac{f_2^{k,l}}{(1+\frac{f_2^{k,l}}{k})(1+\frac{f_1^{k,l-1}}{k})(1+\frac{F_1^k}{k})}(r,y)dr}dX\\
&\qquad\leq\chi_{k1}^{\epsilon\Lambda}F_1^k(x,y)+\chi_{k1}^{\epsilon\Lambda}(y)\int_0^xF_1^kF_2^k(X,y)dX.
\end{aligned}$$

By Lemma 3.4, there is a subsequence of $\big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}$, still denoted by $\big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}$, converging in $L^1([0,1])$ to some $\tilde{F}_1$. Given $\eta>0$, there is a subset ω_η of $[0,1]$ with measure smaller than η such that on ω_η^c the convergence of this sequence is uniform and $(\tilde{F}_1, f_{b1}, f_{b2})$ is bounded. It follows from (3.14)-(3.15) and the non-negativity of $(f_1^{k,2l}, f_2^{k,2l})_{(k,l)\in\mathbb{N}^2}$ that $(f_1^{k,2l}, f_2^{k,2l})_{(k,l)\in\mathbb{N}^2}$ is bounded on $[0,1]\times\omega_\eta^c$. Given these bounds, Lemma 3.4 and the expression of $(f_1^{k,l}, f_2^{k,l})$ in exponential form, it holds by induction that for each $l\in\mathbb{N}$, the sequence $(f_1^{k,2l}, f_2^{k,l})_{k\geq e^{\frac{3c_b}{\epsilon}}}$ is strongly compact in $L^1([0,1]\times\omega_\eta^c)$. Denote by (g_1^l, g_2^l) its limit up to a subsequence. By Lemma 3.4, let G (resp. H) with $\partial_x G=-\partial_x H$, be the limit in L^1 when $k\to+\infty$ of

$$
\begin{aligned}
&(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^x \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX)_{k\geq e^{\frac{3c_b}{\epsilon}}},\\
&\big(\text{resp. } (\chi^{\epsilon\Lambda}_{k1}(y)\int_x^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX)_{k\geq e^{\frac{3c_b}{\epsilon}}}\big).
\end{aligned}
$$

$(g_1^{2l}, g_2^{2l}, g_1^{2l+1}, g_2^{2l+1})$ satisfies

$$
\begin{aligned}
&g_1^0=g_2^0=0,\\
&g_1^{2l}(x,y)=f_{b1}(y)+G(x,y)-\int_0^x g_1^{2l}g_2^{2l-1}(X,y)dX, \quad l\in\mathbb{N}^*,\\
&g_1^{2l+1}(x,y)=f_{b1}(y)+G(x,y)-\int_0^x g_1^{2l+1}g_2^{2l}(X,y)dX, \quad l\in\mathbb{N},\\
&g_2^{2l}(x,y)=f_{b2}(y)+H(x,y)-\int_x^1 g_1^{2l-1}g_2^{2l}(X,y)dX, \quad l\in\mathbb{N}^*,\\
&g_2^{2l+1}(x,y)=f_{b2}(y)+H(x,y)-\int_x^1 g_1^{2l}g_2^{2l+1}(X,y)dX,\\
&\qquad\qquad l\in\mathbb{N}, \quad (x,y)\in[0,1]\times\omega_\eta^c.
\end{aligned}
\tag{3.18}
$$

By induction on l it holds that

$$
\begin{aligned}
&0\leq g_1^{2l}\leq g_1^{2l+2}\leq g_1^{2l+3}\leq g_1^{2l+1},\\
&0\leq g_2^{2l}\leq g_2^{2l+2}\leq g_2^{2l+3}\leq g_2^{2l+1}, \quad l\in\mathbb{N}.
\end{aligned}
\tag{3.19}
$$

Moreover,

$$
\begin{aligned}
\int_{[0,1]\times\omega_\eta^c} g_j^{2l}(x,y)dxdy \leq &\int_0^1 f_{bj}(y)dy\\
&+\int_{[0,1]\times\omega_\eta^c}(G+H)(x,y)dxdy, \quad j\in\{1,2\}, \quad l\in\mathbb{N}.
\end{aligned}
$$

By the monotone convergence theorem, $(g^{2l})_{l\in\mathbb{N}}$ (resp. $(g^{2l+1})_{l\in\mathbb{N}}$) increasingly (resp. decreasingly) converges in $L^1([0,1]\times\omega_\eta^c)$ and almost everywhere on $[0,1]\times\omega_\eta^c$ to some g (resp. h). By the dominated convergence theorem,

$$\lim_{l\to+\infty} g_1^{2l}g_2^{2l-1} = g_1h_2 \quad \text{and} \quad \lim_{l\to+\infty} g_1^{2l+1}g_2^{2l} = h_1g_2 \quad \text{in} \quad L^1([0,1]\times\omega_\eta^c).$$

Consequently,

$$\begin{aligned}
g_1(x,y) &= f_{b1}(y) + G(x,y) - \int_0^x g_1h_2(X,y)dX,\\
h_1(x,y) &= f_{b1}(y) + G(x,y) - \int_0^x h_1g_2(X,y)dX,\\
g_2(x,y) &= f_{b2}(y) + H(x,y) - \int_x^1 h_1g_2(X,y)dX\\
&\qquad\qquad = g_2(0,y) - G(x,y) + \int_0^x h_1g_2(X,y)dX,\\
h_2(x,y) &= f_{b2}(y) + H(x,y) - \int_x^1 g_1h_2(X,y)dX\\
&\qquad\qquad = h_2(0,y) - G(x,y) + \int_0^x g_1h_2(X,y)dX,\\
&\qquad\qquad\qquad (x,y)\in[0,1]\times\omega_\eta^c,
\end{aligned}$$

and

$$h_1 \geq g_1, \quad h_2 \geq g_2, \quad (x,y)\in[0,1]\times\omega_\eta^c. \tag{3.20}$$

Hence

$$(h_1-g_1)(1,y) = -(h_2-g_2)(0,y),$$

so that, by (3.20),

$$g_1(1,y) = h_1(1,y), \quad g_2(0,y) = h_2(0,y).$$

Consequently,

$$h_1 - g_1 = h_2 - g_2, \quad g_1h_2 - h_1g_2 = (g_1-g_2)(h_1-g_1),$$

and

$$(h_1-g_1)(x,y) = \int_0^x (g_1-g_2)(h_1-g_1)(X,y)dX. \tag{3.21}$$

It follows from $(h_1-g_1)(0,y)=0$ and the boundedness of (g_1,g_2) on $[0,1]\times\omega_\eta^c$ that $h_1-g_1=0$ and $(g_1,g_2)=(h_1,h_2)$ on $[0,1]\times\omega_\eta^c$. Hence the whole sequence $(g_1^l,g_2^l)_{l\in\mathbb{N}}$ converges to (g_1,g_2) in $L^1([0,1]\times\omega_\eta^c)$. Letting $\eta\to0$ and using (2.16), the convergence holds in $L^1([0,1]^2)$.

Given $\bar{\epsilon} > 0$, choose l_0 big enough so that $\| g_1^{2l_0} - g_1^{2l_0+1} \|_{L^1} < \bar{\epsilon}$, then k_0 big enough so that

$$\| f_1^{k,2l_0+1} - g_1^{2l_0+1} \|_{L^1} \leq \bar{\epsilon} \quad \text{and} \quad \| f_1^{k,2l_0} - g_1^{2l_0} \|_{L^1} \leq \bar{\epsilon}, \quad k \geq k_0.$$

Hence $\| f_1^{k,2l_0+1} - f_1^{k,2l_0} \|_{L^1} \leq 3\bar{\epsilon}$ for $k \geq k_0$. Then

$$\begin{aligned}
&\| F_1^k - F_1^{k'} \|_{L^1} \\
&\leq \| F_1^k - F_1^{k'} \|_{L^1((\Omega_{k1}^{\epsilon\Lambda})^c)} + 2c_b\epsilon \quad \text{by (3.8)} \\
&\leq \| F_1^k - f_1^{k,2l_0} \|_{L^1((\Omega_{k1}^{\epsilon\Lambda})^c)} + \| F_1^{k'} - f_1^{k',2l_0} \|_{L^1((\Omega_{k1}^{\epsilon\Lambda})^c)} \\
&+ \| f_1^{k,2l_0} - f_1^{k',2l_0} \|_{L^1} + 2c_b\epsilon \\
&\leq \| f_1^{k,2l_0+1} - f_1^{k,2l_0} \|_{L^1} + \| f_1^{k',2l_0+1} - f_1^{k',2l_0} \|_{L^1} \\
&+ \| f_1^{k,2l_0} - f_1^{k',2l_0} \|_{L^1} + 2c_b\epsilon \quad \text{by (3.17)} \\
&\leq 8\bar{\epsilon} + 2c_b\epsilon, \quad k \geq \max\{k_0, \exp(\frac{3c_b}{\epsilon})\}, \quad k' \geq \max\{k_0, \exp(\frac{3c_b}{\epsilon})\}.
\end{aligned}$$

And so (F_1^k) is a Cauchy sequence in $L^1([0,1]^2)$ with the limit equal to the weak limit F_1. Similarly, $(F_j^k)_{2\leq j\leq 4}$ is a Cauchy sequence in $(L^1([0,1]^2))^3$ with the limit equal to the weak limit $(F_j)_{2\leq j\leq 4}$. □

Lemma 3.6.
The limit F of $(F^k)_{k\in\mathbb{N}^*}$ in $L^1([0,1]^2)$ is a renormalized solution to the Broadwell model (1.4).

Proof of Lemma 3.6.

Start from a renormalized formulation of (2.1),

$$\begin{aligned}
&\int_0^1 \varphi_1(1,y) \ln\big(1 + F_1^k(1,y)\big)dy - \int_0^1 \varphi_1(0,y) \ln\big(1 + f_{b1}(y) \wedge \frac{k}{2}\big)dy \\
&- \int_{[0,1]^2} \ln\big(1 + F_1^k(x,y)\big)\partial_x\varphi_1(x,y)dxdy \\
&= \int_{[0,1]^2} \varphi_1(x,y)\frac{F_3^kF_4^k}{(1+F_1^k)(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}(x,y)dxdy \\
&- \int_{[0,1]^2} \varphi_1(x,y)\frac{F_1^kF_2^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}(x,y)dxdy,
\end{aligned} \tag{3.22}$$

for test functions $\varphi \in (C^1([0,1]^2))^4$. Using the strong L^1 convergence of the sequence (F^k) to pass to the limit when $k \to +\infty$ in the left hand side of (3.22), gives in the limit,

$$\int_0^1 \varphi_1(1,y)\ln\big(1+F_1(1,y)\big)dy - \int_0^1 \varphi_1(0,y)\ln\big(1+f_{b1}(y)\big)dy$$
$$-\int_{[0,1]^2} \ln\big(1+F_1(x,y)\big)\partial_x\varphi_1(x,y)dxdy.$$

For the passage to the limit when $k\to+\infty$ in the right hand side of (3.22), given $\eta>0$ there is a subset A_η of $[0,1]^2$ with $|A_\eta^c|<\eta$, such that up to a subsequence, $(F^k)_{k\in\mathbb{N}^*}$ uniformly converges to F on A_η and $F\in L^\infty(A_\eta)$. Passing to the limit when $k\to+\infty$ on A_η is straightforward. Moreover,

$$\lim_{\eta\to 0}\int_{A_\eta^c}\varphi\frac{F_1F_2}{1+F_1}(x,y)dxdy=0$$
$$\text{and}\quad \lim_{\eta\to 0}\int_{A_\eta^c}\varphi\frac{F_1^kF_2^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}(x,y)dxdy=0,$$

uniformly with respect to k, since

$$\frac{F_1}{1+F_1}\leq 1,\quad \frac{F_1^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}\leq 1,\quad\text{and}\quad \lim_{\eta\to 0}\int_{A_\eta^c}F_2^k=0,$$

uniformly with respect to k.
The gain term can be estimated as follows. The uniform boundedness of the entropy production term of (F^k) is given in Lemma 3.1. A convexity argument together with the L^1 convergence of (F^k) to F (see [13]), imply that

$$\int_{[0,1]^2}(F_1F_2-F_3F_4)\ln\frac{F_1F_2}{F_3F_4}(x,y)dxdy\leq c_b. \tag{3.23}$$

It follows that, for any $\gamma>1$,

$$\int_{A_\eta^c}|\varphi|\frac{F_3F_4}{1+F_1}(x,y)dxdy\leq\frac{c}{\ln\gamma}+c\gamma\int_{A_\eta^c}\frac{F_1F_2}{1+F_1}(x,y)dxdy$$
$$\leq\frac{c}{\ln\gamma}+c\gamma\int_{A_\eta^c}F_2(x,y)dxdy,$$

which tends to zero when $\eta \to 0$. Similarly, using (3.3),

$$\begin{aligned}
&\int_{A_\eta^c} |\varphi| \frac{F_3^k F_4^k}{(1+F_1^k)(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}(x,y)dxdy \\
&\le c \int_{A_\eta^c} \frac{F_3^k F_4^k}{(1+F_1^k)(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}(x,y)dxdy \\
&\le \frac{c}{\ln\gamma} + C\gamma \int_{A_\eta^c} \frac{F_1^k F_2^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}(x,y)dxdy \\
&\le \frac{C}{\ln\gamma} + C\gamma \int_{A_\eta^c} F_2^k(x,y)dxdy,
\end{aligned}$$

which tends to zero when $\eta \to 0$, uniformly in k. It follows that the right hand side of (3.22) converges to

$$\int_{[0,1]^2} \varphi(x,y)\frac{F_3F_4}{1+F_1}(x,y)dxdy - \int_{[0,1]^2} \varphi(x,y)\frac{F_1F_2}{1+F_1}(x,y)dxdy,$$

when $k \to +\infty$. Consequently, F satisfies the first equation of (1.4) in renormalized form. It can be similarly proven that F is solution to the last equations of (1.4). □

This completes the proof of Theorem 1.1.

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Leif Arkeryd
Mathematical Sciences
University of Göteborg and
Chalmers
Göteborg
Sweden
arkeryd@chalmers.se

Anne Nouri
Aix-Marseille University
CNRS, Centrale Marseille, I2M
Marseille
France
anne.nouri@univ-amu.fr