Schur-Finiteness (and Bass-Finiteness) Conjecture for Quadric Fibrations and Families of Sextic du Val del Pezzo Surfaces

Gonçalo Tabuada

Received: March 22, 2019
Revised: December 24, 2020

Communicated by Max Karoubi

Abstract. Let $Q \to B$ be a quadric fibration and $T \to B$ a family of sextic du Val del Pezzo surfaces. Making use of the theory of noncommutative mixed motives, we establish a precise relation between the Schur-finiteness conjecture for Q, resp. for T, and the Schur-finiteness conjecture for B. As an application, we prove the Schur-finiteness conjecture for Q, resp. for T, when B is low-dimensional. Along the way, we obtain a proof of the Schur-finiteness conjecture for smooth complete intersections of two or three quadric hypersurfaces. Finally, we prove similar results for the Bass-finiteness conjecture.

2020 Mathematics Subject Classification: 14A22, 14C15, 14D06
Keywords and Phrases: Schur-finiteness conjecture, Bass-finiteness conjecture, quadric fibrations, du Val del Pezzo surfaces, noncommutative algebraic geometry, noncommutative mixed motives

1 Introduction

Schur-finiteness conjecture

Let $\mathcal{C}$ be a $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal category. Given a partition λ of an integer $n \geq 1$, consider the corresponding $\mathbb{Q}$-linear representation V_λ of the symmetric group $\mathfrak{S}_n$ and the associated idempotent $e_\lambda \in \mathbb{Q}[\mathfrak{S}_n]$. Under these notations, the Schur-functor $S_\lambda \colon \mathcal{C} \to \mathcal{C}$ sends an object $a \in \mathcal{C}$ to the direct summand of $a^{\otimes n}$ determined by e_λ. Following Deligne [11, §1], an object $a \in \mathcal{C}$ is called *Schur-finite* if it is annihilated by some Schur-functor. Voevodsky introduced in [39] a triangulated category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (over a perfect base field k). By construction, this category is $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal, and comes equipped with a $\otimes$-functor $M(-)_{\mathbb{Q}}\colon \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ defined on smooth k-schemes of finite type. Given $X \in \mathrm{Sm}(k)$, an important conjecture in the theory of motives is the following:

Conjecture $\mathrm{S}(X)$: The geometric mixed motive $M(X)_{\mathbb{Q}}$ is Schur-finite.

Thanks to the (independent) work of Guletskii [12] and Mazza [28], the conjecture $\mathrm{S}(X)$ holds in the case where $\dim(X) \leq 1$. Thanks to the work of Kimura [21] and Shermenev [31], the conjecture $\mathrm{S}(X)$ also holds in the case where X is an abelian variety. Besides these cases (and some other cases scattered in the literature), the Schur-finiteness conjecture remains wide open. The main goal of this note is to prove the Schur-finiteness conjecture in the new cases of quadric fibrations and families of sextic du Val del Pezzo surfaces.

Quadric fibrations

Our first main result is the following:

Theorem 1. *Let $q\colon Q \to B$ a flat quadric fibration of relative dimension $d-2$. Assume that B and Q are k-smooth, that all the fibers of q have corank ≤ 1, and that the locus $D \subset B$ of the critical values of the fibration q is k-smooth. Under these assumptions, the following holds:*

(i) When d is even, we have $\mathrm{S}(Q) \Leftrightarrow \mathrm{S}(B) + \mathrm{S}(\widetilde{B})$, where $\widetilde{B}$ stands for the discriminant 2-fold cover of B (ramified over D).

(ii) When d is odd and $\mathrm{char}(k) \neq 2$, we have $\{\mathrm{S}(V_i)\} + \{\mathrm{S}(\widetilde{D}_i)\} \Rightarrow \mathrm{S}(Q)$, where V_i is any affine open of B and $\widetilde{D}_i$ is any Galois 2-fold cover of $D_i := D \cap V_i$.

To the best of the author's knowledge, Theorem 1 is new in the literature. Intuitively speaking, it relates the Schur-finiteness conjecture for the total space Q with the Schur-finiteness conjecture for certain coverings/subschemes of the base B. Among other ingredients, its proof makes use of Kontsevich's noncommutative mixed motives of twisted root stacks; consult §3-§4 below for details. Making use of Theorem 1, we are now able to prove the Schur-finiteness conjecture in new cases. Here are two low-dimensional examples:

Corollary 2 (Quadric fibrations over curves). *Let $q\colon Q \to B$ be a quadric fibration as in Theorem 1 with B a curve*[1]*. In this case, $\mathrm{S}(Q)$ holds.*

Corollary 3 (Quadric fibrations over surfaces). *Let $q\colon Q \to B$ be a quadric fibration as in Theorem 1 with B a surface and d odd. In this case, the implication $\mathrm{S}(B) \Rightarrow \mathrm{S}(Q)$ holds.*

Proof. Given a smooth k-surface X, we have $\mathrm{S}(X) \Leftrightarrow \mathrm{S}(U)$ for any open U of X. Therefore, thanks to Theorem 1(ii), the proof follows from the fact that when B is a surface, the conjectures $\{\mathrm{S}(V_i)\}$ can be replaced by the conjecture $\mathrm{S}(B)$. □

[1] Since B is a curve, the locus $D \subset B$ of the critical values of q is necessarily k-smooth.

Corollary 3 can be applied to the case where B is (an open subscheme of) an abelian surface or a smooth projective surface with $p_g = 0$ which satisfies Bloch's conjecture (see Guletskii-Pedrini [13, §4 Thm. 7]). Recall that Bloch's conjecture holds for surfaces not of general type (see Bloch-Kas-Leiberman [6]), for surfaces which are rationally dominated by a product of curves (see Kimura [21]), for Godeaux, Catanese and Barlow surfaces (see Voisin [40, 41]), etc.

Remark 4 (Related work). Let $q: Q \to B$ be a quadric fibration as in Theorem 1. In the particular case where Q and B are smooth *projective*, Bouali [9] and Vial [38, §4] "computed" the Chow motive $\mathfrak{h}(Q)_{\mathbb{Q}}$ of Q using smooth projective k-schemes of dimension $\leq \dim(B)$. Since the category of Chow motives (with $\mathbb{Q}$-coefficients) embeds fully-faithfully into $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (see [39, §4]), these computations lead to an alternative "geometric" proof of Corollaries 2-3. Note that in Theorem 1 and in Corollaries 2-3 we do *not* assume that Q and B are projective; we are (mainly) interested in geometric mixed motives and *not* in pure motives.

Intersections of quadrics

Let $Y \subset \mathbb{P}^{d-1}$ be a smooth complete intersection of m quadric hypersurfaces. The linear span of these quadric hypersurfaces gives rise to a flat quadric fibration $q: Q \to \mathbb{P}^{m-1}$ of relative dimension $d-2$, with Q k-smooth. Under these notations, our second main result is the following:

Theorem 5. *We have* $\mathrm{S}(Q) \Rightarrow \mathrm{S}(Y)$. *When* $2m \leq d$*, the converse also holds.*

By combining Theorem 5 with the above Corollaries 2-3, we hence obtain a proof of the Schur-finiteness conjecture in the following cases:

Corollary 6 (Intersections of two or three quadrics). *Assume that the quadric fibration* $q: Q \to \mathbb{P}^{m-1}$ *is as in Theorem 1. In this case, the conjecture* $\mathrm{S}(Y)$ *holds when* Y *is a smooth complete intersection of two, or of three odd-dimensional, quadric hypersurfaces.*

Families of sextic du Val del Pezzo surfaces

Recall that a *sextic du Val del Pezzo surface* X is a projective k-scheme with at worst du Val singularities and ample anticanonical class such that $K_X^2 = 6$. Consider a *family of sextic du Val del Pezzo surfaces* $f: T \to B$, *i.e.* a flat morphism f such that for every geometric point $x \in B$ the associated fiber T_x is a sextic du Val del Pezzo surface. Following Kuznetsov [26, §5], given $d \in \{2, 3\}$, let us write $\mathcal{M}_d$ for the relative moduli stack of semistable sheaves on fibers of T over B with Hilbert polynomial $h_d(t) := (3t + d)(t + 1)$, and Z_d for the coarse moduli space of $\mathcal{M}_d$. By construction, we have finite flat morphisms $Z_2 \to B$ and $Z_3 \to B$ of degrees 3 and 2, respectively. Under these notations, our third main result is the following:

Theorem 7. *Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces. Assume that* $\mathrm{char}(k) \notin \{2, 3\}$ *and that T is k-smooth. Under these assumptions, we have the equivalence of conjectures* $\mathrm{S}(T) \Leftrightarrow \mathrm{S}(B) + \mathrm{S}(Z_2) + \mathrm{S}(Z_3)$.

To the best of the author's knowledge, Theorem 7 is new in the literature. It leads to a proof of the Schur-finiteness conjecture in new cases. Here is an illustrative example:

Corollary 8 (Families of sextic du Val del Pezzo surfaces over curves). *Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7 with B a curve. In this case, the conjecture* $\mathrm{S}(T)$ *holds.*

Remark 9. Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7. To the best of the author's knowledge, the associated geometric mixed motive $M(T)_{\mathbb{Q}}$ has *not* been "computed" (in any non-trivial particular case). Nevertheless, consult Helmsauer [16] for the "computation" of the Chow motive $\mathfrak{h}(X)_{\mathbb{Q}}$ of certain *smooth* (projective) del Pezzo surfaces X.

Remark 10 (Conservativity conjecture). Given a field k equipped with a complex embedding $\sigma: k \to \mathbb{C}$, recall from Ayoub [3, Conj. 2.1] that the *conservativity conjecture* asserts that the Betti realization functor $\mathrm{B}_\sigma: \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \to \mathcal{D}(\mathbb{Q})$ is conservative. As explained in [3, Prop. 2.26], if the conservativity conjecture holds, then every object of the category $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is Schur-finite. In particular, the conjecture $\mathrm{S}(X)$ holds for every smooth k-scheme of finite type X (when k is equipped with a complex embedding). However, despite the (monumental) work of Ayoub [4], the conservativity conjecture remains wide open[2].

Bass-finiteness conjecture

Let k be a finite base field and X a smooth k-scheme of finite type. The Bass-finiteness conjecture $\mathrm{B}(X)$ (see [5, §9]) is one of the oldest and most important conjectures in algebraic K-theory. It asserts that the algebraic K-theory groups $K_n(X), n \geq 0$, are finitely generated. In the same vein, given an integer $r \geq 2$, we can consider the conjecture $\mathrm{B}(X)_{1/r}$, where $K_n(X)$ is replaced by $K_n(X)_{1/r} := K_n(X) \otimes \mathbb{Z}[1/r]$. Our fourth main result is the following:

Theorem 11. *The following holds:*

(i) *Theorem 1 and Corollaries 2-3 hold*[3] *similarly for the conjecture* $\mathrm{B}(-)_{1/2}$. *In Corollary 2, the groups $K_n(Q)_{1/2}, n \geq 2$, are moreover finite.*

(ii) *Theorem 5 holds similarly for the conjecture* $\mathrm{B}(-)$.

(iii) *Corollary 6 holds similarly for the conjecture* $\mathrm{B}(-)_{1/2}$. *In the case where Y is a smooth complete intersection of two quadric hypersurfaces, the groups $K_n(Y)_{1/2}, n \geq 2$, are moreover finite.*

[2] I hope that Ayoub manages to correct his work [4] in the (near) future.

[3] Corollary 3 (for the conjecture $\mathrm{B}(-)_{1/2}$) can also be applied to the case where B is (an open subscheme of) an abelian surface; see [19, Cor. 70 and Thm. 82].

(iv) *Theorem 7 and Corollary 8 hold similarly for the conjecture* $\mathrm{B}(-)_{1/6}$. *In Corollary 8, the groups* $K_n(T)_{1/6}, n \geq 2$, *are moreover finite.*

2 Preliminaries

In what follows, all schemes/stacks are of finite type over the perfect base field k.

Dg categories

For a survey on dg categories we invite the reader to consult [20]. In what follows, we will write $\mathrm{dgcat}(k)$ for the category of (essentially small) dg categories and dg functors. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks. Given a k-scheme X (or stack $\mathcal{X}$), the category of perfect complexes of $\mathcal{O}_X$-modules $\mathrm{perf}(X)$ admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; consult [20, §4.6] [27] for details. More generally, given a sheaf of $\mathcal{O}_X$-algebras $\mathcal{F}$, we can consider the dg category of perfect complexes of $\mathcal{F}$-modules $\mathrm{perf}_{\mathrm{dg}}(X;\mathcal{F})$.

Noncommutative mixed motives

For a book, resp. survey, on noncommutative motives we invite the reader to consult [33], resp. [32]. Recall from [33, §8.5.1] (see also [22, 23, 24]) the definition of Kontsevich's triangulated category of noncommutative mixed motives $\mathrm{NMot}(k)$. By construction, this category is idempotent complete, symmetric monoidal, and comes equipped with a $\otimes$-functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$. In what follows, given a k-scheme X (or stack $\mathcal{X}$) equipped with a sheaf of $\mathcal{O}_X$-algebras $\mathcal{F}$, we will write $U(X;\mathcal{F}) := U(\mathrm{perf}_{\mathrm{dg}}(X;\mathcal{F}))$.

3 Noncommutative mixed motives of twisted root stacks

Let X be a k-scheme, $\mathcal{L}$ a line bundle on X, $\sigma \in \Gamma(X,\mathcal{L})$ a global section, and $r > 0$ an integer. In what follows, we will write $D \subset X$ for the zero locus of σ. Recall from [10, Def. 2.2.1] (see also [1, Appendix B]) that the associated *root stack* $\mathcal{X}$ is defined as the following fiber-product of algebraic stacks

$$\begin{array}{ccc} \mathcal{X} := \sqrt[r]{(\mathcal{L},\sigma)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \theta_r} \\ X & \xrightarrow[(\mathcal{L},\sigma)]{} & [\mathbb{A}^1/\mathbb{G}_m]\,, \end{array}$$

where θ_r stands for the morphism induced by the r^{th} power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$. A *twisted root stack* $(\mathcal{X};\mathcal{F})$ consists of a root stack $\mathcal{X}$ equipped with a sheaf of Azumaya algebras $\mathcal{F}$. In what follows, we will write s for the product of the ranks of $\mathcal{F}$ (at each one of the connected components of $\mathcal{X}$). The following result, of independent interest, will play a key role in the proof of Theorem 1.

Theorem 12. *Assume that X and D are k-smooth.*

(i) We have an isomorphism $U(\mathcal{X}) \simeq U(X) \oplus U(D)^{\oplus(r-1)}$.

(ii) Assume moreover that $\mathrm{char}(k) \neq r$ and that k contains the r^{th} roots of unity. Under these extra assumptions, $U(\mathcal{X};\mathcal{F})_{1/rs}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/rs}$ containing the noncommutative mixed motives $\{U(V_i)_{1/rs}\}$ and $\{U(\widetilde{D}^l_i)_{1/rs}\}$, where V_i is any affine open subscheme of X and $\widetilde{D}^l_i$ is any Galois l-fold cover of $D_i := D \cap V_i$ with $l \nmid r$ and $l \neq 1$.

Proof. We start by proving item (i). Following [18, Thm. 1.6], the pull-back functor p^* is fully-faithful and we have the following semi-orthogonal decomposition[4] $\mathrm{perf}(X) = \langle \mathrm{perf}(D)_{r-1}, \ldots, \mathrm{perf}(D)_1, p^*(\mathrm{perf}(X)\rangle$. All the categories $\mathrm{perf}(D)_j$ are equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(D)$. Therefore, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the searched direct sum decomposition $U(\mathcal{X}) \simeq U(X) \oplus U(D)^{\oplus(r-1)}$.

Let us now prove item (ii). We consider first the particular case where $X = \mathrm{Spec}(A)$ is affine and the line bundle $\mathcal{L} = \mathcal{O}_X$ is trivial. Let μ_r be the group of r^{th} roots of unity and $\chi\colon \mu_r \to k^\times$ a (fixed) primitive character. Under these notations, consider the global quotient $[\mathrm{Spec}(B)/\mu_r]$, where $B := A[t]/\langle t^r - \sigma\rangle$ and the μ_r-action on B is given by $g \cdot t := \chi(g)^{-1}t$ for every $g \in \mu_r$ and by $g \cdot a := a$ for every $a \in A$. As explained in [10, Example 2.4.1], the root stack $\mathcal{X}$ agrees, in this particular case, with the global quotient $[\mathrm{Spec}(B)/\mu_r]$. By construction, the induced map $\mathrm{Spec}(B) \to X$ is a r-fold cover ramified over $D \subset X$. Moreover, for every l such that $l \mid r$ and $l \neq 1$, the associated closed subscheme $\mathrm{Spec}(B)^{\mu_l}$ agrees with the ramification divisor $D \subset \mathrm{Spec}(B)$. Therefore, since the functor $U(-)_{1/rs}\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)_{1/rs}$ is an additive invariant of dg categories in the sense of [33, Def. 2.1] (see [33, §8.4.5]), we conclude from [36, Cor. 1.28(ii)] that, in this particular case, $U(\mathcal{X};\mathcal{F})_{1/rs}$ belongs to the smallest thick additive subcategory of $\mathrm{NMot}(k)_{1/rs}$ containing the noncommutative mixed motives $U(\mathrm{Spec}(B))^{\mu_l}_{1/rs}$ and $\{U(\widetilde{D}^l)_{1/rs}\}$, where $\widetilde{D}^l$ is any Galois l-fold cover of D with $l \nmid r$ and $l \neq 1$. Furthermore, since the geometric quotient $\mathrm{Spec}(B)/\!\!/\mu_r$ agrees with X and the latter scheme is k-smooth, [36, Thm. 1.22] implies that $U(\mathrm{Spec}(B))^{\mu_l}_{1/rs}$ is isomorphic to $U(X)_{1/rs}$. This finishes the proof of item (ii) in the particular case where X is affine and the line bundle $\mathcal{L}$ is trivial.

Let us now prove item (ii) in the general case. As explained above, given any affine open subscheme V_i of X which trivializes the line bundle $\mathcal{L}$, the noncommutative mixed motive $U(\mathcal{V}_i;\mathcal{F}_i)_{1/rs}$, with $\mathcal{V}_i := p^{-1}(V_i)$ and $\mathcal{F}_i := \mathcal{F}_{|\mathcal{V}_i}$,

[4] Consult [7, 8] for the definition of semi-orthogonal decomposition.

belongs to the smallest thick additive subcategory of $\mathrm{NMot}(k)_{1/rs}$ containing $U(V_i)_{1/rs}$ and $\{U(\widetilde{D}_i^l)_{1/rs}\}$, where $\widetilde{D}_i^l$ is any Galois l-fold cover of $D_i := D \cap V_i$ with $l \mid r$ and $l \neq 1$. Let us then choose an affine open cover $\{W_i\}$ of X which trivializes the line bundle $\mathcal{L}$. Since X is quasi-compact (recall that X is of finite type over k), this affine open cover admits a *finite* subcover. Consequently, the proof follows by induction from the $\mathbb{Z}[1/rs]$-linearization of the distinguished triangles of Lemma 13 below. □

LEMMA 13. *Given an open cover $\{\mathcal{W}_1, \mathcal{W}_2\}$ of X, we have an induced Mayer-Vietoris distinguished triangle of noncommutative mixed motives*

$$U(\mathcal{X};\mathcal{F}) \longrightarrow U(\mathcal{W}_1;\mathcal{F}_1) \oplus U(\mathcal{W}_2;\mathcal{F}_2) \xrightarrow{\pm} U(\mathcal{W}_{12};\mathcal{F}_{12}) \xrightarrow{\partial} \Sigma U(\mathcal{X};\mathcal{F}), \quad (14)$$

where $\mathcal{W}_{12} := \mathcal{W}_1 \cap \mathcal{W}_2$ *and* $\mathcal{F}_{12} := \mathcal{F}_{|\mathcal{W}_{12}}$.

Proof. Consider the following commutative diagram of dg categories

$$\begin{array}{ccccc} \mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F}) & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_1;\mathcal{F}_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2) & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_{12};\mathcal{F}_{12}), \end{array}$$

where $\mathcal{Z}$ stands for the closed complement $\mathcal{X} - \mathcal{W}_1 = \mathcal{W}_2 - \mathcal{W}_{12}$ and $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}}$, resp. $\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}}$, stands for the full dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})$, resp. $\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)$, consisting of those perfect complexes of $\mathcal{F}$-modules, resp. $\mathcal{F}_2$-modules, that are supported on $\mathcal{Z}$. Both rows are short exact sequences of dg categories in the sense of Drinfeld/Keller (see [20, §4.6]) and the left vertical dg functor is a Morita equivalence. Therefore, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ is a localizing invariant of dg categories in the sense of [33, §8.1], we obtain the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}}) & \longrightarrow & U(\mathcal{X};\mathcal{F}) & \longrightarrow & U(\mathcal{W}_1;\mathcal{F}_1) & \xrightarrow{\partial} & \Sigma U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X};\mathcal{F})_{\mathcal{Z}}) \\ \downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\ U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}}) & \rightarrow & U(\mathcal{W}_2;\mathcal{F}_2) & \rightarrow & U(\mathcal{W}_{12};\mathcal{F}_{12}) & \xrightarrow{\partial} & \Sigma U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{W}_2;\mathcal{F}_2)_{\mathcal{Z}}). \end{array}$$

Finally, since the middle square is homotopy (co)cartesian, we hence obtain the claimed Mayer-Vietoris distinguished triangle (14). □

4 PROOF OF THEOREM 1

Following [25, §3] (see also [2, §1.2]), let E be a vector bundle of rank d on B, $q'\colon \mathbb{P}(E) \to B$ the projectivization of E on B, $\mathcal{O}_{\mathbb{P}(E)}(1)$ the Grothendieck line bundle on $\mathbb{P}(E)$, $\mathcal{L}$ a line bundle on B, and finally

$$\rho \in \Gamma(B, S^2(E^\vee) \otimes \mathcal{L}^\vee) = \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \mathcal{L}^\vee)$$

a global section. Given this data, recall that $Q \subset \mathbb{P}(E)$ is defined as the zero locus of ρ on $\mathbb{P}(E)$ and that $q\colon Q \to B$ is the restriction of q' to Q; note that the relative dimension of q is equal to $d-2$. Consider also the discriminant global section $\mathrm{disc}(q) \in \Gamma(B, \det(E^\vee)^{\otimes 2} \otimes (\mathcal{L}^\vee)^{\otimes d})$ and the associated zero locus $D \subset B$; note that D agrees with the locus of the critical values of q.

Recall from [25, §3.5] (see also [2, §1.6]) that when d is even, we can consider the *discriminant cover* $\widetilde{B} := \mathrm{Spec}_B(Z(\mathcal{C}l_0(q)))$ of B, where $Z(\mathcal{C}l_0(q))$ stands for the center of the sheaf $\mathcal{C}l_0(q)$ of even parts of the Clifford algebra associated to q; see [25, §3] (and also [2, §1.5]). By construction, $\widetilde{B}$ is a 2-fold cover ramified over D. Moreover, since D is k-smooth, $\widetilde{B}$ is also k-smooth.

Recall from [25, §3.6] (see also [2, §1.7]) that when d is odd and $\mathrm{char}(k) \neq 2$, we can consider the *discriminant stack* $\mathcal{X} := \sqrt[2]{(\det(E^\vee)^{\otimes 2} \otimes (\mathcal{L}^\vee)^{\otimes d}, \mathrm{disc}(q))/B}$. Since $\mathrm{char}(k) \neq 2$, $\mathcal{X}$ is a Deligne-Mumford stack with coarse moduli space B.

Proposition 15. *Under the above assumptions, the following holds:*

(i) When d is even, we have $U(Q)_{1/2} \simeq U(\widetilde{B})_{1/2} \oplus U(B)_{1/2}^{\oplus(d-2)}$.

(ii) When d is odd and $\mathrm{char}(k) \neq 2$, $U(Q)_{1/2}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/2}$ containing the noncommutative mixed motives $\{U(V_i)_{1/2}\}$ and $\{U(\widetilde{D}_i)_{1/2}\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of $D_i := D \cap V_i$.

Proof. As proved in [25, Thm. 4.2] (see also [2, Thm. 2.2.1]), we have the following semi-orthogonal decomposition

$$\mathrm{perf}(Q) = \langle \mathrm{perf}(B; \mathcal{C}l_0(q)), \mathrm{perf}(B)_1, \ldots, \mathrm{perf}(B)_{d-2} \rangle ,$$

where $\mathrm{perf}(B)_j := q^*(\mathrm{perf}(B)) \otimes \mathcal{O}_{Q/B}(j)$. All the categories $\mathrm{perf}(B)_j$ are equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(B)$. Therefore, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the decomposition $U(Q) \simeq U(B; \mathcal{C}l_0(q)) \oplus U(B)^{\oplus(d-2)}$.

We start by proving item (i). As explained in [25, §3.5] (see also [2, §1.6]), when d is even, the category $\mathrm{perf}(B; \mathcal{C}l_0(q))$ is equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(\widetilde{B}; \mathcal{F})$ where $\mathcal{F}$ is a certain sheaf of Azumaya algebras on $\widetilde{B}$ of rank $2^{\frac{d}{2}-1}$. This leads to an isomorphism $U(B; \mathcal{C}l_0(q)) \simeq U(\widetilde{B}; \mathcal{F})$. Making use of [37, Thm. 2.1], we hence conclude that $U(B; \mathcal{C}l_0(q))_{1/2}$ is isomorphic to $U(\widetilde{B}; \mathcal{F})_{1/2} \simeq U(\widetilde{B})_{1/2}$. Consequently, we obtain the isomorphism of item (i).

Let us now prove item (ii). As explained in [25, §3.6] (see also [2, §1.7]), when d is odd, the category $\mathrm{perf}(B; \mathcal{C}l_0(q))$ is equivalent (via a Fourier-Mukai type functor) to $\mathrm{perf}(\mathcal{X}; \mathcal{F})$ where $\mathcal{F}$ is a certain sheaf of Azumaya algebras on $\mathcal{X}$ of rank $2^{\frac{d-1}{2}}$. This leads to an isomorphism $U(B; \mathcal{C}l_0(q)) \simeq U(\mathcal{X}; \mathcal{F})$. By combining Theorem 12(ii) with the isomorphism $U(Q) \simeq U(\mathcal{X}; \mathcal{F}) \oplus U(B)^{\oplus(d-2)}$, we hence conclude that $U(Q)_{1/2}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/2}$ containing $U(B)_{1/2}$, $\{U(V_i)_{1/2}\}$, and $\{U(\widetilde{D}_i)_{1/2}\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of D_i. We now claim that $U(B)_{1/2}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{1/2}$ containing $\{U(V_i)_{1/2}\}$; note that this would conclude the proof. Choose an affine open cover $\{W_i\}$ of B. Since B is quasi-compact (recall that B is of finite type over k), this affine open cover admits a *finite* subcover. Therefore, similarly to the proof of Theorem 12, our claim follows from an inductive argument using the $\mathbb{Z}[1/2]$-linearization of the Mayer-Vietoris distinguished triangles $U(B) \to U(W_1) \oplus U(W_2) \xrightarrow{\pm} U(W_{12}) \xrightarrow{\partial} \Sigma U(B)$. □

As proved in [34, Thm. 2.8], there exists a $\mathbb{Q}$-linear, fully-faithful, $\otimes$-functor Φ making the following diagram commute

$$\begin{array}{ccc} \mathrm{Sm}(k) & \xrightarrow{X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)} & \mathrm{dgcat}(k) \\ \downarrow{\scriptstyle M(-)_{\mathbb{Q}}} & & \downarrow{\scriptstyle U(-)_{\mathbb{Q}}} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} & & \mathrm{NMot}(k)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})} \\ \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]} & \xrightarrow[\Phi]{} & \mathrm{NMot}(k)_{\mathbb{Q}}\,, \end{array} \tag{16}$$

where $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]}$ stands for the orbit category with respect to the Tate motive $\mathbb{Q}(1)[2]$ and $\underline{\mathrm{Hom}}(-,-)$ for the internal Hom of the monoidal structure; note that the functors $X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)$ and $\underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})$ are contravariant. By construction, π is a faithful $\otimes$-functor. Therefore, it follows from [28, Lem. 1.11] that we have the following equivalence:

$$\mathrm{S}(X) \Leftrightarrow \text{noncommutative mixed motive } (\Phi\circ\pi)(M(X)_{\mathbb{Q}}) \text{ is Schur-finite.} \tag{17}$$

We now have all the ingredients necessary to conclude the proof of Theorem 1.

ITEM (I)

The above functors π and $\underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})$ are $\mathbb{Q}$-linear. Therefore, by combining Proposition 15(i) with the commutative diagram (16), we conclude that

$$(\Phi\circ\pi)(M(Q)_{\mathbb{Q}}) \simeq (\Phi\circ\pi)(M(\widetilde{B})_{\mathbb{Q}}) \oplus (\Phi\circ\pi)(M(B)_{\mathbb{Q}})^{\oplus(d-2)}\,. \tag{18}$$

Since Schur-finiteness is stable under direct sums and direct summands, the proof of the equivalence $\mathrm{S}(Q) \Leftrightarrow \mathrm{S}(B) + \mathrm{S}(\widetilde{B})$ follows then from (17)-(18).

ITEM (II)

Recall from [33, §8.5.1-8.5.2] that, by construction, $\mathrm{NMot}(k)_{\mathbb{Q}}$ is a $\mathbb{Q}$-linear closed symmetric monoidal triangulated category in the sense of Hovey [17, §6-7]. As proved in [12, Thm. 1], this implies that Schur-finiteness has the 2-out-of-3 property with respect to distinguished triangles. The functor $\underline{\mathrm{Hom}}(-,U(k)_{\mathbb{Q}})$

is triangulated. Hence, by combining Proposition 15(ii) with the commutative diagram (16), we conclude that $(\Phi \circ \pi)(M(Q)_{\mathbb{Q}})$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_{\mathbb{Q}}$ containing the noncommutative mixed motives $\{(\Phi\circ\pi)(M(V_i)_{\mathbb{Q}})\}$ and $\{(\Phi\circ\pi)(M(\widetilde{D}_i)_{\mathbb{Q}})\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of D_i. Since by assumption the conjectures $\{\mathrm{S}(V_i)\}$ and $\{\mathrm{S}(\widetilde{D}_i)\}$ hold, (17) implies that the noncommutative mixed motives $\{(\Phi\circ\pi)(M(V_i)_{\mathbb{Q}})\}$ and $\{(\Phi\circ\pi)(M(\widetilde{D}_i)_{\mathbb{Q}})\}$ are Schur-finite. Therefore, making use of the 2-out-of-3 property of Schur-finiteness with respect to distinguished triangles (and of the stability of Schur-finiteness under direct summands), we conclude that $(\Phi \circ \pi)(M(Q)_{\mathbb{Q}})$ is also Schur-finite. The proof follows now from the above equivalence (17).

5 Proof of Theorem 5

Recall from the proof of Proposition 15 that we have the semi-orthogonal decomposition $\mathrm{perf}(Q) = \langle \mathrm{perf}(\mathbb{P}^{m-1}; \mathcal{C}l_0(q)), \mathrm{perf}(\mathbb{P}^{m-1})_1, \ldots, \mathrm{perf}(\mathbb{P}^{m-1})_{d-2}\rangle$, and consequently the following direct sum decomposition:

$$U(Q) \simeq U(\mathbb{P}^{m-1}; \mathcal{C}l_0(q)) \oplus U(\mathbb{P}^{m-1})^{\oplus(d-2)}\,. \tag{19}$$

As proved in [25, Thm. 5.5] (see also [2, Thm. 2.3.7]), the following also holds:

(a) When $2m < d$, we have the following semi-orthogonal decomposition $\mathrm{perf}(Y) = \langle \mathrm{perf}(\mathbb{P}^{m-1}; \mathcal{C}l_0(q)), \mathcal{O}(1), \ldots, \mathcal{O}(d-2m)\rangle$. Consequently, since the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the following direct sum decomposition $U(Y) \simeq U(\mathbb{P}^{m-1}; \mathcal{C}l_0(q)) \oplus U(k)^{\oplus(d-2m)}$.

(b) When $2m = d$, the category $\mathrm{perf}(Y)$ is equivalence (via a Fourier-Mukai type functor) to $\mathrm{perf}(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))$. Consequently, we obtain an isomorphism of noncommutative mixed motives $U(Y) \simeq U(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))$.

(c) When $2m > d$, $\mathrm{perf}(Y)$ is an admissible subcategory of $\mathrm{perf}(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))$. Hence, $U(Y)$ is a direct summand of $U(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))$.

Let us now prove the implication $\mathrm{S}(Q) \Rightarrow \mathrm{S}(Y)$. If the conjecture $\mathrm{S}(Q)$ holds, then it follows from the decomposition (19), from the commutative diagram (16), from the equivalence (17), and from the stability of Schur-finiteness under direct summands, that the noncommutative mixed motive $\underline{\mathrm{Hom}}(U(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))_{\mathbb{Q}}, U(k)_{\mathbb{Q}})$ is Schur-finite. Making use of the above descriptions (a)-(c) of $U(Y)$ and of the commutative diagram (16), we hence conclude that the noncommutative mixed motive $(\Phi\circ\pi)(M(Y)_{\mathbb{Q}})$ is also Schur-finite. Consequently, the conjecture $\mathrm{S}(Y)$ follows now from the above equivalence (17). Finally, note that when $2m \leq d$, a similar argument proves the converse implication $\mathrm{S}(Y) \Rightarrow \mathrm{S}(Q)$.

6 Proof of Theorem 7

Recall first from [26, Prop. 5.12] that since $\mathrm{char}(k) \notin \{2,3\}$ and T is k-smooth, the k-schemes B, Z_2 and Z_3 are also k-smooth.

Proposition 20. *We have* $U(T)_{1/6} \simeq U(B)_{1/6} \oplus U(Z_2)_{1/6} \oplus U(Z_3)_{1/6}$.

Proof. As proved in [26, Thm. 5.2 and Prop. 5.10], we have the semi-orthogonal decomposition $\mathrm{perf}(T) = \langle \mathrm{perf}(B), \mathrm{perf}(Z_2; \mathcal{F}_2), \mathrm{perf}(Z_3; \mathcal{F}_3)\rangle$, where $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) is a certain sheaf of Azumaya algebras over Z_2 (resp. Z_3) of order 2 (resp. 3). Recall that the functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ sends semi-orthogonal decompositions to direct sums. Hence, we obtain the direct sum decomposition:

$$U(T) \simeq U(B) \oplus U(Z_2; \mathcal{F}_2) \oplus U(Z_3; \mathcal{F}_3)\,. \tag{21}$$

Since $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) is of order 2 (resp. 3), the rank of $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) is necessarily a power of 2 (resp. 3). Making use of [37, Thm. 2.1], we hence conclude that the noncommutative mixed motive $U(Z_2; \mathcal{F}_2)_{1/2}$ (resp. $U(Z_3; \mathcal{F}_3)_{1/3}$) is isomorphic to $U(Z_2)_{1/2}$ (resp. $U(Z_3)_{1/3}$). Consequently, the proof follows now from the $\mathbb{Z}[1/6]$-linearization of (21). □

The functors π and $\underline{\mathrm{Hom}}(-, U(k)_{\mathbb{Q}})$ in (16) are $\mathbb{Q}$-linear. Therefore, similarly to the proof of item (i) of Theorem 1, by combining Proposition 20 with the commutative diagram (16), we conclude that

$$(\Phi\circ\pi)(M(T)_{\mathbb{Q}}) \simeq (\Phi\circ\pi)(M(\widetilde{B})_{\mathbb{Q}}) \oplus (\Phi\circ\pi)(M(Z_2)_{\mathbb{Q}}) \oplus (\Phi\circ\pi)(M(Z_3)_{\mathbb{Q}})\,. \tag{22}$$

Since Schur-finiteness is stable under direct sums and direct summands, the proof follows then from the combination of (22) with the equivalence (17).

7 Proof of Theorem 11

Item (i)

We start by proving the first claim. As explained in [33, §8.6] (see also [35, Thm. 15.10]), given $X \in \mathrm{Sm}(k)$, we have the isomorphisms of abelian groups:

$$\mathrm{Hom}_{\mathrm{NMot}(k)}(U(k), \Sigma^{-n} U(X)) \simeq K_n(X) \qquad n \in \mathbb{Z}\,. \tag{23}$$

Assume that d is even. By combining Proposition 15(i) with the $\mathbb{Z}[1/2]$-linearization of (23), we conclude that $K_n(Q)_{1/2} \simeq K_n(\widetilde{B})_{1/2} \oplus K_n(B)_{1/2}^{\oplus(d-2)}$. Therefore, since finite generation is stable under direct sums and direct summands, we obtain the equivalence $\mathrm{B}(Q)_{1/2} \Leftrightarrow \mathrm{B}(B)_{1/2} + \mathrm{B}(\widetilde{B})_{1/2}$. Assume now that d is odd and that $\mathrm{char}(k) \neq 2$. Finite generation has the 2-out-of-3 property with respect to (short or long) exact sequences and is stable under direct summands. Therefore, the proof of the following implication

$$\{\mathrm{B}(V_i)_{1/2}\} + \{\mathrm{B}(\widetilde{D}_i)_{1/2}\} \Rightarrow \mathrm{B}(Q)_{1/2}$$

follows from the combination of Proposition 15(ii) with the $\mathbb{Z}[1/2]$-linearization of (23). Finally, recall from [14, 29, 30] that the conjecture $\mathrm{B}(X)$ holds in the case where $\dim(X) \leq 1$. Therefore, the Corollaries 2-3 also hold similarly for the conjecture $\mathrm{B}(-)_{1/2}$.

We now prove the second claim. Let $q: Q \to B$ be a quadric fibration as in Theorem 1 with B a curve. Thanks to Corollary 2 (for the conjecture $\mathrm{B}(-)_{1/2}$), it suffices to show that the groups $K_n(Q), n \geq 2$, are torsion. Assume first that d is even. By combining Proposition 15(i) with the $\mathbb{Q}$-linearization of (23), we obtain an isomorphism $K_n(Q)_\mathbb{Q} \simeq K_n(\widetilde{B})_\mathbb{Q} \oplus K_n(B)_\mathbb{Q}^{\oplus(d-2)}$. Thanks to Proposition 24 below, we have $K_n(\widetilde{B})_\mathbb{Q} = K_n(B)_\mathbb{Q} = 0$ for every $n \geq 2$. Therefore, we conclude that the groups $K_n(Q), n \geq 2$, are torsion. Assume now that d is even and that $\mathrm{char}(k) \neq 2$. Thanks to Proposition 15(ii), $U(Q)_\mathbb{Q}$ belongs to the smallest thick triangulated subcategory of $\mathrm{NMot}(k)_\mathbb{Q}$ containing the noncommutative mixed motives $\{U(V_i)_\mathbb{Q}\}$ and $\{U(\widetilde{D}_i)_\mathbb{Q}\}$, where V_i is any affine open subscheme of B and $\widetilde{D}_i$ is any Galois 2-fold cover of D_i. Moreover, $U(Q)_\mathbb{Q}$ may be explicitly obtained from $\{U(V_i)_\mathbb{Q}\}$ and $\{U(\widetilde{D}_i)_\mathbb{Q}\}$ using solely the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles. Therefore, since $K_n(V_i)_\mathbb{Q} = 0$ for every $n \geq 2$ (see Proposition 24 below) and $K_n(\widetilde{D}_i)_\mathbb{Q} = 0$ for every $n \geq 1$ (see Quillen's computation [30] of the algebraic K-theory of a finite field), an inductive argument using the $\mathbb{Q}$-linearization of (23) and the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles implies that the groups $K_n(Q), n \geq 2$, are torsion.

Proposition 24. *We have $K_n(X)_\mathbb{Q} = 0, n \geq 2$, for every smooth k-curve X.*

Proof. In the particular case where X is affine, this result was proved in [15, Cor. 3.2.3] (see also [14, Thm. 0.5]). In the general case, choose an affine open cover $\{W_i\}$ of X. Since X is quasi-compact, this affine open cover admits a *finite* subcover. Therefore, the proof follows from an inductive argument (similar to the one in the proof of Theorem 12(ii)) using the $\mathbb{Q}$-linearization of (23) and the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles. □

Item (ii)

If the conjecture $\mathrm{B}(Q)$ holds, then it follows from the decomposition (19) and from the isomorphisms (23) that the algebraic K-theory groups $K_n(\mathrm{perf}_\mathrm{dg}(\mathbb{P}^{m-1}; \mathcal{C}l_0(q))), n \geq 0$, are finitely generated. Therefore, by combining the descriptions (a)-(c) of the noncommutative mixed motive $U(Y)$ (see the proof of Theorem 5) with (23), we conclude that the conjecture $\mathrm{B}(Y)$ also holds. Note that when $2m \leq d$, a similar argument proves the converse implication $\mathrm{B}(Y) \Rightarrow \mathrm{B}(Q)$.

Item (iii)

Items (i)-(ii) of Theorem 11 imply that Corollary 6 holds similarly for the conjecture $\mathrm{B}(-)_{1/2}$. We now address the second claim. Let $q: Q \to \mathbb{P}^1$ be the quadric fibration associated to the smooth complete intersection Y of two quadric hypersurfaces. Thanks to item (i), the groups $K_n(Q)_{1/2}, n \geq 2$, are finite. Hence, making use of the decomposition (19), of the $\mathbb{Z}[1/2]$-linearization of (23), and of the above descriptions (a)-(c) of $U(Y)$ (see the proof of Theorem 5), we conclude that the groups $K_n(Y)_{1/2}, n \geq 2$, are also finite.

ITEM (IV)

We start by proving the first claim. By combining Proposition 20 with the $\mathbb{Z}[1/6]$-linearization of (23), we conclude that

$$K_n(T)_{1/6} \simeq K_n(B)_{1/6} \oplus K_n(Z_2)_{1/6} \oplus K_n(Z_3)_{1/6}\,.$$

Therefore, since finite generation is stable under sums and direct summands, we obtain the equivalence $\mathrm{B}(T)_{1/6} \Leftrightarrow \mathrm{B}(B)_{1/6} + \mathrm{B}(Z_2)_{1/6} + \mathrm{B}(Z_3)_{1/6}$. As mentioned in the proof of item (i), the conjecture $\mathrm{B}(X)$ holds in the case where $\dim(X) \leq 1$. Hence, Corollary 8 also holds similarly for the conjecture $\mathrm{B}(-)_{1/6}$. We now prove the second claim. Let $f\colon T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7 with B a curve. Similarly to the proof of item (i) of Theorem 11, it suffices to show that the groups $K_n(T), n \geq 2$, are torsion. By combining Proposition 20 with the $\mathbb{Q}$-linearization of (23), we obtain an isomorphism $K_n(T)_\mathbb{Q} \simeq K_n(B)_\mathbb{Q} \oplus K_n(Z_2)_\mathbb{Q} \oplus K_n(Z_3)_\mathbb{Q}$. Thanks to Proposition 24, we have moreover $K_n(B)_\mathbb{Q} = K_n(Z_2)_\mathbb{Q} = K_n(Z_3)_\mathbb{Q} = 0$ for every $n \geq 2$. Therefore, we conclude that the groups $K_n(T), n \geq 2$, are torsion.

ACKNOWLEDGMENTS:

The author is grateful to Joseph Ayoub for useful e-mail exchanges concerning the Schur-finiteness conjecture, and to the anonymous referee for her/his comments and for suggesting Remark 10. The author also would like to thank the Hausdorff Institute for Mathematics (HIM) in Bonn for its hospitality. The author was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project `UIDB/00297/2020` (Centro de Matemática e Aplicações).

REFERENCES

[1] D. Abramovich, T. Graber and A. Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks.* Amer. J. Math. 130 (2008), no. 5, 1337–1398.

[2] A. Auel, M. Bernardara and M. Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems.* J. Math. Pures Appl. (9) 102 (2014), no. 1, 249–291.

[3] J. Ayoub, *Motives and algebraic cycles: a selection of conjectures and open questions.* Hodge theory and L^2-analysis, 87–125, Adv. Lect. Math. (ALM), vol. 39. Int. Press, Somerville, MA, 2017.

[4] J. Ayoub, *Topologie feuilletée et la conservativité des réalisations classiques en caractéristique nulle*. Available at http://user.math.uzh.ch/ayoub.

[5] H. Bass, *Some problems in classical algebraic K-theory*. Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 3–73. LNM 342, 1973.

[6] S. Bloch, A. Kas and D. Lieberman, *Zero cycles on surfaces with $p_g = 0$*. Compositio Math. 33 (1976), 135–145.

[7] A. Bondal and D. Orlov, *Derived categories of coherent sheaves*. Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 47–56.

[8] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*. arXiv:alg-geom/9506012.

[9] J. Bouali, *Motives of quadric bundles*. Manuscr. Math. 149 (2016), no. 3-4, 347–368.

[10] C. Cadman, *Using stacks to impose tangency conditions on curves*. Amer. J. Math. 129 (2007), no. 2, 405–427.

[11] P. Deligne, *Catégories tensorielles*. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. Mosc. Math. J. 2 (2002), no. 2, 227–248.

[12] V. Guletskii, *Finite-dimensional objects in distinguished triangles*. J. Number Theory 119 (2006), no. 1, 99–127.

[13] V. Guletskii and C. Pedrini, *Finite-dimensional motives and the conjectures of Beilinson and Murre*. Special issue in honor of Hyman Bass on his seventieth birthday. Part III. K-Theory 30 (2003), no. 3, 243–263.

[14] D. Grayson, *Finite generation of K-groups of a curve over a finite field (after Daniel Quillen)*. Algebraic K-theory, Part I (Oberwolfach, 1980), 69–90, LNM 966, 1982.

[15] G. Harder, *Die Kohomologie S-arithmetischer Gruppen über Funktionenkörpern*. Invent. Math. 42 (1977), 135–175.

[16] K. Helmsauer, *Chow Motives of del Pezzo surfaces of degree 5 and 6*. MSc. thesis (2013). Available at https://search.library.ualberta.ca/catalog/6504220.

[17] M. Hovey, *Model categories*. Mathematical Surveys and Monographs, vol. 63. American Mathematical Society, Providence, RI, 1999.

[18] A. Ishii and K. Ueda, *The special McKay correspondence and exceptional collections*. Tohoku Math. J. (2) 67 (2015), no. 4, 585–609.

[19] B. Kahn, *Algebraic K-theory, algebraic cycles and arithmetic geometry*. Handbook of Algebraic K-theory, 351–428, Berlin, New York. Springer-Verlag, 2005.

[20] B. Keller, *On differential graded categories*. International Congress of Mathematicians (Madrid), Vol. II, 151–190. Eur. Math. Soc., Zürich, 2006.

[21] S.-I. Kimura, *Chow groups are finite dimensional, in some sense*. Math. Ann. 331 (2005), no. 1, 173–201.

[22] M. Kontsevich, *Mixed noncommutative motives*. Talk at the Workshop on Homological Mirror Symmetry, Miami, 2010. Available at www-math.mit.edu/auroux/frg/miami10-notes.

[23] M. Kontsevich, *Notes on motives in finite characteristic*. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 213–247, Progr. Math., vol. 270, Birkhäuser Boston, Inc., Boston, MA, 2009.

[24] M. Kontsevich, *Noncommutative motives*. Talk at the IAS on the occasion of the 61. birthday of Pierre Deligne (2005). Available at http://video.ias.edu/Geometry-and-Arithmetic.

[25] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*. Adv. Math. 218 (2008), no. 5, 1340–1369.

[26] A. Kuznetsov, *Derived categories of families of sextic del Pezzo surfaces*. Available at arXiv:1708.00522. To appear in IMRN.

[27] V. Lunts and D. Orlov, *Uniqueness of enhancement for triangulated categories*. J. Amer. Math. Soc. 23 (2010), no. 3, 853–908.

[28] C. Mazza, *Schur functors and motives*. K-Theory 33 (2004), no. 2, 89–106.

[29] D. Quillen, *Finite generation of the groups K_i of rings of algebraic integers*. Cohomology of groups and algebraic K-theory, 479–488, Adv. Lect. Math. (ALM), 12 (2010).

[30] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*. Ann. of Math. (2) 96 (1972), 552–586.

[31] A. Shermenev, *The motive of an abelian variety*. Funct. Anal. 8 (1974), 47–53.

[32] G. Tabuada, *Recent developments on noncommutative motives*. New Directions in Homotopy Theory, Contemporary Mathematics 707 (2018), 143–173.

[33] G. Tabuada, *Noncommutative Motives*. With a preface by Yuri I. Manin. University Lecture Series, 63. American Mathematical Society, Providence, RI, 2015.

[34] G. Tabuada, *Voevodsky's mixed motives versus Kontsevich's noncommutative mixed motives*. Advances in Mathematics 264 (2014), 506–545.

[35] G. Tabuada, *Higher K-theory via universal invariants*. Duke Math. J. 145 (2008), no. 1, 121–206.

[36] G. Tabuada and M. Van den Bergh, *Additive invariants of orbifolds*. Geometry and Topology 22 (2018), 3003–3048.

[37] G. Tabuada and M. Van den Bergh, *Noncommutative motives of Azumaya algebras*. J. Inst. Math. Jussieu 14 (2015), no. 2, 379–403.

[38] C. Vial, *Algebraic cycles and fibrations*. Doc. Math. 18 (2013), 1521–1553.

[39] V. Voevodsky, *Triangulated categories of motives over a field*. Cycles, transfers, and motivic homology theories, 188–238, Ann. of Math. Stud., 143, Princeton, NJ, 2000.

[40] C. Voisin, *Bloch's conjecture for Catanese and Barlow surfaces*. J. Differential Geom. 97 (2014), no. 1, 149–175.

[41] C. Voisin, *Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme*. Ann. Scuola Norm. Sup. Pisa 19 (1992), 473–492.

Gonçalo Tabuada
Mathematics Institute
Zeeman Building
University of Warwick
Coventry CV4 7AL
UK
and
Departamento de Matemática and
Centro de Matemática e Aplicações (CMA)
FCT, UNL
Quinta da Torre
2829-516 Caparica
Portugal