# Deformation of Multiple Zeta Values and Their Logarithmic Interpretation in Positive Characteristic

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Abstract. Pellarin introduced the deformation of multiple zeta values of Thakur as elements over Tate algebras. In this paper, we relate these values to a certain coordinate of the logarithm of a higher dimensional Drinfeld module over the Tate algebra which we will introduce. Moreover, we define multiple polylogarithms in our setting and represent deformation of multiple zeta values as a linear combination of multiple polylogarithms. As an application of our results, we also write Dirichlet-Goss multiple $L$-values as a linear combination of twisted multiple polylogarithms at algebraic points.

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## 1 Introduction

### 1.1 Background

Multiple zeta values were introduced by Euler as the infinite sum

$$\zeta(s_1,\dots,s_r) := \sum_{\substack{n_1>n_2>\cdots>n_r>0 \\ n_1,\dots,n_r\in\mathbb{Z}_{\geq 1}}} \frac{1}{n_1^{s_1}\dots n_r^{s_r}} \in \mathbb{R}$$

for positive integers $s_1,\dots,s_r$ such that $s_1 > 1$. These values can be seen as a generalization of special values $\zeta(n)$ of Riemann zeta function for a positive integer $n > 1$. Their motivic interpretation is given by Terasoma [Ter02] and

Goncharov independently. Moreover for the tuple $\mathfrak{s} = (s_1, \dots, s_r)$, the multiple polylogarithm $\mathrm{Li}_{\mathfrak{s}}(z_1, \dots, z_r)$ is defined by

$$\mathrm{Li}_{\mathfrak{s}}(z_1, \dots, z_r) := \sum_{\substack{n_1 > n_2 > \cdots > n_r > 0 \\ n_1, \dots, n_r \in \mathbb{Z}_{\geq 1}}} \frac{z_1^{n_1} \dots z_r^{n_r}}{n_1^{s_1} \dots n_r^{s_r}} \in \mathbb{Q}[[z_1, \dots, z_r]],$$

and its specialization at $z_1 = \cdots = z_r = 1$ gives the value of $\zeta(s_1, \dots, s_r)$. We refer the reader to [Wal00] and [Zha16] for interesting properties of those objects.

In this paper, we are interested in the function field analogue of multiple zeta values and their deformation in positive characteristic. Let $q$ be a power of a prime $p$. We let $\mathbb{F}_q$ be the finite field with $q$ elements. We set $A := \mathbb{F}_q[\theta]$ as the polynomial ring in the variable $\theta$ with coefficients from $\mathbb{F}_q$, and $A_+$ as the set of monic polynomials of $A$. We let $K$ be the function field $\mathbb{F}_q(\theta)$ and ord be the valuation corresponding to the infinite place normalized so that $\mathrm{ord}(\theta) = -1$. Moreover, we define the norm $|\cdot|_\infty$ corresponding to ord so that $|\theta|_\infty = q$. We also let $K^{\mathrm{perf}}$ be the perfect closure of $K$ and $\overline{K}$ be the algebraic closure of $K$. The completion of $K$ with respect to $|\cdot|_\infty$ is denoted by $\mathbb{K}_\infty$ and the completion of an algebraic closure of $\mathbb{K}_\infty$ is denoted by $\mathbb{C}_\infty$.

We define the Carlitz-Goss zeta value $\zeta_A(n)$ at a positive integer $n$ by the infinite series

$$\zeta_A(n) = \sum_{a \in A_+} \frac{1}{a^n} \in \mathbb{K}_\infty^\times,$$

which can be seen as a function field analogue of $\zeta(n)$. The arithmetic of these special values were studied by Carlitz [Car35], Gekeler [Gek88], Goss [Gos96] and Thakur [Tha90]. Also their transcendental behavior over $K$ was discovered by Chang and Yu [CY07] and Yu [Yu91].

Let $\mathfrak{s} = (s_1, \dots, s_r)$ be a tuple in $\mathbb{Z}_{\geq 1}^r$ for some positive integer $r$ and set $w := \sum s_i$. Then the multiple zeta value $\zeta_A(\mathfrak{s})$ of weight $w$ and depth $r$ is defined by Thakur in [Tha04, Sec. 5.10] as the infinite sum

$$\zeta_A(\mathfrak{s}) := \sum_{\substack{|a_1|_\infty > |a_2|_\infty > \cdots > |a_r|_\infty \geq 0 \\ a_1, \dots, a_r \in A_+}} \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in \mathbb{K}_\infty.$$

In 2009, Thakur [Tha09, Thm. 4] proved that $\zeta_A(\mathfrak{s})$ is non-zero. Furthermore, Anderson and Thakur [AT09] give the realization of multiple zeta values as periods of a certain $t$-motive (see [BP20, §4] for more details on $t$-motives).

In 2014, Chang [Cha14] defined the multiple polylogarithm $\mathrm{Li}_{\mathfrak{s}}(z_1, \dots, z_r)$ by

$$\mathrm{Li}_{\mathfrak{s}}(z_1, \dots, z_r) := \sum_{i_1 > i_2 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \dots z_r^{q^{i_r}}}{\ell_{i_1}^{s_1} \dots \ell_{i_r}^{s_r}} \in K[[z_1, \dots, z_r]], \tag{1.1}$$

where $\ell_i := (\theta - \theta^q)\dots(\theta - \theta^{q^i})$ for $i \geq 0$ and $\ell_0 := 1$. When $r = 1$, it becomes the Carlitz $n$-th polylogarithm

$$\log_n(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{\ell_i^n} \in K[[z]]$$

defined by Anderson and Thakur [AT90]. Moreover, Anderson and Thakur [AT90] represents $\zeta_A(n)$ as a $K$-linear combination of $\log_n(\theta^j)$ where $j < nq/(q-1)$.

Unlike classical case, relating multiple zeta values to multiple polylogarithms is not trivial in function field setting. Using $t$-motivic interpretation of multiple zeta values in [AT09], Chang [Cha14] clarified this phenomenon for higher depths stating that there exist tuples $(a_j, (u_{j1}, \dots, u_{jr})) \in A \times A^r$ where $j$ is in a finite index set $J$, and $\Gamma_{\mathfrak{s}} \in A$, which all can be explicitly defined, such that

$$\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s}) = \sum_{j \in J} a_j \operatorname{Li}_{\mathfrak{s}}(u_{j1}, \dots, u_{jr}). \tag{1.2}$$

Later Chang and Mishiba [CM20, Thm. 1.4.1] related $\zeta_A(\mathfrak{s})$ to a certain coordinate of the logarithm of a $t$-module (see [And86] for details on $t$-modules) by proving that there exist a uniformizable $t$-module $G_{\mathfrak{s}}$ of dimension $k_{\mathfrak{s}}$ defined over $K$, a special point $v_{\mathfrak{s}} \in G_{\mathfrak{s}}(K)$ and an element $Z_{\mathfrak{s}} \in G_{\mathfrak{s}}(\mathbb{K}_\infty)$ such that $\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})$ occurs as the $w$-th coordinate of $Z_{\mathfrak{s}}$ and $\exp_{G_{\mathfrak{s}}}(Z_{\mathfrak{s}}) = v_{\mathfrak{s}}$. Thus, the logarithmic interpretation of multiple zeta values allow them to verify the function field analogue of Furusho's conjecture (see [Fur06] and [Fur07]).

## 1.2 Tate algebras

Let $U \subset \mathbb{Z}_{\geq 1}$ be any finite set and let $\mathbb{T}_U$ be the Tate algebra on the closed unit polydisc over $\mathbb{C}_\infty$ with independent variables $t_i$ for $i \in U$. Let $\Sigma \subset \mathbb{Z}_{\geq 1}$ be a finite union of finite sets $U_i \subset \mathbb{Z}_{\geq 1}$. The Frobenius automorphism $\tau : \mathbb{T}_\Sigma \to \mathbb{T}_\Sigma$ is given by raising the coefficients of the given infinite series to their $q$-th power and fixing the independent variables $t_i$ (see §2.1 for details). Furthermore, for any $\mathbb{F}_q$-algebra $R$, set $R[\underline{t}_U] := R[t_i : i \in U]$ and $R(\underline{t}_U)$ to be the fraction field of the polynomial ring $R[\underline{t}_U]$.

For a moment let us concentrate on $\mathbb{T}_\Sigma$ where $\Sigma = \{1, \dots, n\}$. In 2012, Pellarin [Pel12] defined the following $L$-series

$$\zeta_C\binom{\Sigma}{s} := \sum_{a \in A_+} \frac{a(t_1)\dots a(t_n)}{a^s} \in \mathbb{T}_\Sigma^\times \tag{1.3}$$

for some positive integer $s$ as a deformation of Carlitz-Goss zeta value $\zeta_A(s)$. Similar notion of deformation has been also carried to multiple zeta values of Thakur by Pellarin in [Pel16] and [Pel17] as follows: For any $a \in A_+$ and an independent variable $t$, we set $a(t) := a_{|\theta=t}$. We now define the map $\sigma_U : A_+ \to$

$\mathbb{F}_q[\underline{t}_\Sigma]$ by $\sigma_U(a) := 1$ if $U = \emptyset$ and $\sigma_U(a) := \prod_{i\in U} a(t_i) \in \mathbb{F}_q[\underline{t}_\Sigma]$ otherwise. For some $r \in \mathbb{Z}_{\geq 1}$, we call

$$\mathcal{C} = \begin{pmatrix} U_1, \dots, U_r \\ s_1, \dots, s_r \end{pmatrix} \tag{1.4}$$

a composition array of weight $w =: \mathrm{wght}(\mathcal{C})$ and depth $r := \mathrm{dep}(\mathcal{C})$. Now we define

$$\zeta_C(\mathcal{C}) := \sum_{\substack{|a_1|_\infty > |a_2|_\infty > \cdots > |a_r|_\infty \geq 0 \\ a_1,\dots,a_r \in A_+}} \frac{\sigma_{U_1}(a_1)\sigma_{U_2}(a_2)\dots\sigma_{U_r}(a_r)}{a_1^{s_1}a_2^{s_2}\dots a_r^{s_r}} \in \mathbb{T}_\Sigma, \tag{1.5}$$

to be the multiple zeta value corresponding to $\mathcal{C}$. Observe that when

$$\mathcal{C} = \begin{pmatrix} \emptyset, \dots, \emptyset \\ s_1, \dots, s_r \end{pmatrix}, \tag{1.6}$$

we see that $\zeta_C(\mathcal{C}) = \zeta_A(\mathfrak{s})$. Using the non-vanishing of $\zeta_A(\mathfrak{s})$ and a specialization argument, Pellarin [Pel17, Prop. 3] proved that the multiple zeta values $\zeta_C(\mathcal{C})$ are non-zero as elements of $\mathbb{T}_\Sigma$.

For any $i, j \geq 1$, we define the element $b_i(t_j) := \prod_{k=0}^{i-1}(t_j - \theta^{q^k}) \in A[\underline{t}_\Sigma]$ and $b_0(t_j) := 1$. We also let $b_i(U) := 1$ if $U = \emptyset$ and $b_i(U) := \prod_{j\in U} b_i(t_j)$ otherwise. For some tuple $(u_1, \dots, u_r) \in \mathbb{T}_\Sigma^r$ living in a certain subset of $\mathbb{T}_\Sigma$ (see §2.3 for details), we define the multiple polylogarithm $\mathrm{Li}_\mathcal{C}(u_1, \dots, u_r)$ by the infinite series

$$\mathrm{Li}_\mathcal{C}(u_1, \dots, u_r) := \sum_{i_1 > i_2 > \cdots > i_r \geq 0} \frac{b_{i_1}(U_1)\dots b_{i_r}(U_r)\tau^{i_1}(u_1)\dots\tau^{i_r}(u_r)}{\ell_{i_1}^{s_1}\dots\ell_{i_r}^{s_r}} \in \mathbb{T}_\Sigma.$$

Our first result (stated as Theorem 2.24 later) is as follows.

THEOREM 1.7. *For any composition array $\mathcal{C}$ of depth $r$ defined as in (1.4), there exist tuples $(a_j, (u_{j1}, \dots, u_{jr})) \in A \times K^{perf}(\underline{t}_\Sigma)^r$ where $j$ is in a finite index set $\mathfrak{I}$, and $\Gamma_\mathcal{C} \in A[\underline{t}_\Sigma]$, which all can be explicitly defined, such that*

$$\Gamma_\mathcal{C}\zeta_C(\mathcal{C}) = \sum_{j\in\mathfrak{I}} a_j \,\mathrm{Li}_\mathcal{C}(u_{j1}, \dots, u_{jr}).$$

Note that Theorem 1.7 can be seen as a generalization of Chang's identity (1.2) and that identity follows from our result by choosing the composition array $\mathcal{C}$ as in (1.6) (see §2.3 for details).

### 1.3 DIRICHLET-GOSS MULTIPLE $L$-VALUES

We denote the algebraic closure of $\mathbb{F}_q$ by $\overline{\mathbb{F}}_q$. For any $1 \leq i \leq r$, let $\chi_i : A \to \overline{\mathbb{F}}_q$ be a Dirichlet character. Furthermore let $(s_1, \dots, s_r)$ be a tuple of positive

integers. We define the Dirichlet-Goss multiple $L$-value $L(\chi_1,\dots,\chi_r;s_1,\dots,s_r)$ of weight $w$ and depth $r$ by the infinite series

$$L(\chi_1,\dots,\chi_r;s_1,\dots,s_r):=\sum_{\substack{|a_1|_\infty>|a_2|_\infty>\cdots>|a_r|_\infty\geq 0\\ a_1,\dots,a_r\in A_+}}\frac{\chi_1(a_1)\dots\chi_r(a_r)}{a_1^{s_1}\dots a_r^{s_r}}\in\mathbb{C}_\infty.$$

One of the advantages of studying $\zeta_C(\mathcal{C})$ is to be able to deduce some properties of Dirichlet-Goss multiple $L$-values. Consider the composition array $\mathcal{C}$ defined as in (1.4) with pairwise disjoint sets $U_i$ such that $\Sigma=\sqcup_{i=1}^r U_i$. For any $1\leq i\leq r$ and $j\in U_i$, let $\mathfrak{p}_{ij}\in A_+$ be the minimal polynomial of $\xi_{i,j}\in\overline{\mathbb{F}}_q$. Since $\zeta_C(\mathcal{C})$ converges in $\mathbb{T}_\Sigma$, evaluating (1.5) at $t_j=\xi_{ij}$ produces the Dirichlet-Goss multiple $L$-value $L(\chi_1,\dots,\chi_r;s_1,\dots,s_r)$ such that $\chi_i:A\to\overline{\mathbb{F}}_q$ is the map sending $a\in A$ to $\prod_{j\in U_i}a(\xi_{ij})$ which is actually the Dirichlet character modulo the ideal generated by $\prod_{j\in U_i}\mathfrak{p}_{ij}$ in $A$.

For any $j\in\mathbb{Z}_{\geq 0}$, we set elements $B_{\chi_i,j}\in\mathbb{C}_\infty^\times$ corresponding to the character $\chi_i$ for all $1\leq i\leq r$ (see §2.4 for details). We define

$$\begin{aligned}&\mathrm{Li}_{\binom{\chi_1,\dots,\chi_r}{s_1,\dots,s_r}}(z_1,\dots,z_r)\\ &\qquad:=\sum_{i_1>i_2>\cdots>i_r\geq 0}\frac{B_{\chi_1,i_1}\dots B_{\chi_r,i_r}}{\ell_{i_1}^{s_1}\dots\ell_{i_r}^{s_r}}z_1^{q^{i_1}}\dots z_r^{q^{i_r}}\in\mathbb{C}_\infty[[z_1,\dots,z_r]].\end{aligned}$$

Let $\mathfrak{p}$ be irreducible in $A_+$ of degree $d$ and $\lambda_\mathfrak{p}\in\mathbb{C}_\infty^\times$ be a $\mathfrak{p}$-torsion point. Let $K_\mathfrak{p}:=K(\lambda_\mathfrak{p})$ be the $\mathfrak{p}$-th cyclotomic field extension of $K$ and $\Delta_\mathfrak{p}$ be its Galois group (see [Ros02, Chap. 12] for the details of cyclotomic field extensions over function fields). Consider the unique group isomorphism $v_\mathfrak{p}:\Delta_\mathfrak{p}\to\mathbb{F}_{q^d}^\times$ induced by the Teichmüller character corresponding to a fixed choice of a root $\xi_\mathfrak{p}$ of $\mathfrak{p}$ and let $g(v_\mathfrak{p})$ be the Gauss-Thakur sum (see §2.4 and [AP15] for the details). We obtain the following corollary of Theorem 1.7.

COROLLARY 1.8. *Fix a positive integer $r$. For any $1\leq i\leq r$ and $j_1,\dots,j_r\in\mathbb{Z}_{\geq 1}$, let $\xi_{i1},\dots,\xi_{ij_i}$ be elements in $\overline{\mathbb{F}}_q$ whose minimal polynomials are $\mathfrak{p}_{i1},\dots,\mathfrak{p}_{ij_i}$ respectively and define the Dirichlet character $\chi_i:A\to\overline{\mathbb{F}}_q$ given by $\chi_i(a)=a(\xi_{i1})\dots a(\xi_{ij_i})$. Then there exist elements $b_j\in\overline{K}$ and $\eta_{ji}\in K^{perf}$ for $1\leq i\leq r$ where $j$ is in a finite index set $\mathfrak{I}$ such that*

$$\begin{aligned}&\prod_{1\leq k\leq j_1}g(v_{\mathfrak{p}_{1k}})\cdots\prod_{1\leq k\leq j_r}g(v_{\mathfrak{p}_{rk}})L(\chi_1,\dots,\chi_r;s_1,\dots,s_r)\\ &\qquad=\sum_{j\in\mathfrak{I}}b_j\,\mathrm{Li}_{\binom{\chi_1,\dots,\chi_r}{s_1,\dots,s_r}}(\eta_{j1},\dots,\eta_{jr}).\end{aligned}$$

For a special class of Dirichlet characters, we can deduce more about Dirichlet-Goss multiple $L$-values and their transcendence. The next theorem will be proved in §2.4.

THEOREM 1.9. *For a finite set $U \subset \mathbb{Z}_{\geq 1}$ and a tuple $\xi = (\xi_i| \quad i \in U) \in \mathbb{F}_q^{|U|}$ where $|U|$ is the cardinality of $U$, let $\chi_{U,\xi} : A \to \mathbb{F}_q$ be the Dirichlet character given by*

$$\chi_{U,\xi}(a) = \prod_{i \in U} a(\xi_i). \tag{1.10}$$

*(i) Let $U_1, \dots, U_r$ be finite subsets of $\mathbb{Z}_{\geq 1}$, $(s_1, \dots, s_r)$ a tuple of positive integers and $\xi_j = (\xi_{ji}| \quad i \in U_j) \in \mathbb{F}_q^{|U_j|}$ for $1 \leq j \leq r$. If $\underline{L}(\chi_{U_1,\xi_1}, \dots, \chi_{U_r,\xi_r}; s_1, \dots, s_r)$ is non-zero, then it is transcendental over $\overline{K}$.*

*(ii) Fix positive integers $m, j_1, \dots, j_m$. For any $1 \leq i \leq m$ and $1 \leq k \leq j_i$, let $U_{ik}$ be a finite subset of $\mathbb{Z}_{\geq 1}$, $\xi_{ik} = (\xi_{ikj}| \quad j \in U_{ik}) \in \mathbb{F}_q^{|U_{ik}|}$ and $(s_{i1}, \dots, s_{ij_i})$ be a tuple of positive integers. Furthermore set $w_i = \sum_{b=1}^{j_i} s_{ib}$ and assume that $w_i \neq w_j$ if $i \neq j$ for $1 \leq i, j \leq m$. If $L(\chi_{U_{i1},\xi_{i1}}, \dots, \chi_{U_{ij_i},\xi_{ij_i}}; s_{i1}, \dots, s_{ij_i})$ is non-zero for each $1 \leq i \leq m$, then the set $\{L(\chi_{U_{i1},\xi_{i1}}, \dots, \chi_{U_{ij_i},\xi_{ij_i}}; s_{i1}, \dots, s_{ij_i})| \quad 1 \leq i \leq m\}$ is $\overline{K}$-linearly independent.*

## 1.4 ANDERSON $A[\underline{t}_\Sigma]$-MODULES

Let $\mathrm{Mat}_n(\mathbb{T}_\Sigma)[\tau]$ be the twisted polynomial ring in $\tau$ with coefficients in $\mathrm{Mat}_n(\mathbb{T}_\Sigma)$ (see §2.1 for details). Inspired by [Dem14, §2], we call an Anderson $A[\underline{t}_\Sigma]$-module $\phi : A[\underline{t}_\Sigma] \to \mathrm{Mat}_n(\mathbb{T}_\Sigma)[\tau]$ of dimension $n$ defined over $\mathbb{T}_\Sigma$ as an $\mathbb{F}_q[\underline{t}_\Sigma]$-linear homomorphism given by

$$\phi(\theta) = A_0 + A_1\tau + \cdots + A_s\tau^s$$

for some $s$ and $(\theta\,\mathrm{Id}_n - A_0)^n = 0$. We should highlight the fact that Anglès, Pellarin and Tavares Ribeiro [APTR16] and Anglès and Tavares Ribeiro [ATR17] have already studied $n = 1$ case, called Drinfeld $A[\underline{t}_\Sigma]$-modules, due to their relation with log-algebraic identities, Taelman's class modules and Pellarin $L$-series.

In this paper, we focus on a special class of Anderson $A[\underline{t}_\Sigma]$-modules, given by

$$\phi(\theta) = \theta\,\mathrm{Id}_n + N + E\tau \tag{1.11}$$

where $N \in \mathrm{Mat}_n(\mathbb{F}_q)$ is a nilpotent matrix and $E \in \mathrm{Mat}_n(\mathbb{T}_\Sigma)$. For such $\phi$, similar to Anderson $t$-modules, we can assign an exponential function, which is a vector valued function denoted by

$$\exp_\phi : \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma),$$

and we show that it has an infinite radius of convergence (see §3.1). We also call $\phi$ uniformizable if $\exp_\phi$ is a surjective function. Our next result (stated as Theorem 4.46 later) is as follows.

Theorem 1.12. *For any composition array $\mathcal{C}$ of weight $w$, there exist a uniformizable Anderson $A[\underline{t}_\Sigma]$-module $G_\mathcal{C}$ of dimension $k_\mathcal{C}$ defined over $\mathbb{T}_\Sigma$, a special point $v_\mathcal{C} \in K^{perf}(\underline{t}_\Sigma)^{k_\mathcal{C}}$ and an element $Z_\mathcal{C} \in \mathbb{T}_\Sigma^{k_\mathcal{C}}$, which all can be explicitly defined, such that*

*(i)* $\Gamma_\mathcal{C}\zeta_C(\mathcal{C})$ *occurs as the $w$-th coordinate of $Z_\mathcal{C}$.*

*(ii)* $\exp_{G_\mathcal{C}}(Z_\mathcal{C}) = v_\mathcal{C}$.

*Remark* 1.13. Let $\omega_i$ be the Anderson-Thakur element corresponding to $t_i - \theta$ and $\tilde{\pi}$ be the Carlitz period (see §2.1 for details). Assume that $\Sigma = \{1, \dots, n\}$ and $s$ is a positive integer. Finally we set $\alpha := \prod_{i=1}^{n}(t_i - \theta)$. Using Anderson's ideas (see [GP19, §4.5] for Drinfeld modules over Tate algebras), we can show that the generator of the kernel of $\exp_{C_\alpha^{\otimes s}}$ (see §3.2 for the definition of $C_\alpha^{\otimes s}$) is a vector whose last coordinate is given by $\frac{\tilde{\pi}^s}{\omega_1 \dots \omega_n}$. Together with the use of Theorem 1.12, we are able to prove that if the point $Z_\mathcal{C}$ is an $A[\underline{t}_\Sigma]$-torsion point for $C_\alpha^{\otimes s}$, then $L(\chi_{t_1} \dots \chi_{t_n}, s)$ is Eulerian in the sense that

$$\omega_1 \dots \omega_n \frac{L(\chi_{t_1} \dots \chi_{t_n}, s)}{\tilde{\pi}^s} \in \mathbb{F}_q(\underline{t}_\Sigma, \theta)$$

where $\mathbb{F}_q(\underline{t}_\Sigma, \theta)$ is the fraction field of the polynomial ring $\mathbb{F}_q[\underline{t}_\Sigma][\theta]$. The opposite direction is expected to hold but due to lack of an analogue of Yu's transcendence theory [Yu91] in our setting, it is still an open problem. We can also ask about the Eulerian criterion for $\zeta_C(\mathcal{C})$ in higher depth case similar to the criterion given in [CPY19]. In this case, calculations show that not all the generators of the kernel of $\exp_{G_\mathcal{C}}$ take $\tilde{\pi}^w$ times the inverse of Anderson-Thakur elements in their $w$-th coordinate and that causes difficulties even proving the direction we show in depth 1 case. One should also understand the generalized version [CGM20, Lem. 4.1.7] of Yu's theorem [Yu91, Thm. 2.3] in this setting. The author hopes to tackle this problem in the near future.

*Remark* 1.14. We remark that although Chang and Mishiba use dual $t$-motives and their fiber coproducts to prove uniformizability of $G_\mathfrak{s}$ in [CM20, Thm. 1.4.1], we use a different method as it is still not clear how we should define the dual $t$-motives for Anderson $A[\underline{t}_\Sigma]$-modules. In our method, we first prove the uniformizability of Anderson $A[\underline{t}_\Sigma]$-modules given of the form (4.4) by following ideas modified from [GP19, §4], and we construct $G_\mathcal{C}$ in (4.38) from those modules. Then using the map $\lambda$ defined in (4.39), we prove that $G_\mathcal{C}$ is also uniformizable (Proposition 4.45).

### 1.5 Outline of the Paper

The outline of the paper can be given as follows: In §2 we cover some necessary notation and background for the rest of the paper and recall recent developments on power sums. We prove Corollary 1.8 and continue to §2 by proving Theorem 1.7. Basically our method is to modify Chang's ideas in the proof of [Cha14, Thm. 5.5.2]. The main difficulty in our case is to determine the

radius of convergence of an infinite series defined in (2.31). This was overcome in [Cha14] by using a property [ABP04, Prop. 3.1.1] of matrices satisfying a certain functional equation. In our setting, we are able to prove the same property (Theorem 2.40) after some analysis on the norm of a solution of a functional equation (Lemma 2.37) and determining the solution in terms of Anderson-Thakur elements (Proposition 2.38). We finish §2 by introducing multiple star polylogarithms and expressing multiple polylogarithms in terms of multiple star polylogarithms (Theorem 2.44).
In §3, we discuss Anderson $A[\underline{t}_\Sigma]$-modules and introduce some properties of a special class of such modules defined as in (1.11). Furthermore, we define the notion of uniformizability and give an example. Finally, we finish the section by introducing Frobenius modules corresponding to Anderson $A[\underline{t}_\Sigma]$-modules.
In §4, we give the definition of $A[\underline{t}_\Sigma]$-module $G$ and make some analysis on the coefficients of the logarithm function of $G$ (see §3.1 for the details on logarithm function) using Chang and Mishiba's methods in [CM19] and [CM20]. We introduce the Anderson $A[\underline{t}_\Sigma]$-module $G_{\mathcal{C}}$ and show that $G_{\mathcal{C}}$ is uniformizable (Proposition 4.45). Moreover we give the proof of Theorem 1.12 and discuss Example 4.48.
We conclude our paper with an Appendix to give the proof of Theorem 4.17 which relates rigid analytic triviality to uniformizability. We note that similar result was proved by the author and Papanikolas in [GP19, Thm. 4.5.5] for Anderson $A[\underline{t}_\Sigma]$-modules of dimension 1 over Tate algebras using Anderson's ideas in his unpublished work. Here we modify those techniques for Anderson $A[\underline{t}_\Sigma]$-module $G$ of higher dimension.

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## 2 Multiple Polylogarithms

### 2.1 Preliminaries

Let $U \subset \mathbb{Z}_{\geq 1}$ be a finite set. We denote the cardinality of $U$ by $|U|$. Assume that $\mu = (\mu_1, \dots, \mu_{|U|}) \in \mathbb{Z}_{\geq 0}^{|U|}$ and set $\underline{t}_U^\mu := \prod_{i \in U} t_i^{\mu_i}$. Recall from §1 that $\Sigma \subset \mathbb{Z}_{\geq 0}$ is a finite union of finite sets. We can write any element $f \in \mathbb{T}_\Sigma$ as $f = \sum_{\mu \in \mathbb{Z}_{\geq 0}^{|\Sigma|}} f_\mu \underline{t}_\Sigma^\mu$ where $f_\mu \in \mathbb{C}_\infty$ such that $|f_\mu|_\infty \to 0$ as $\sum_{i \in \Sigma} \mu_i \to \infty$. Furthermore we let $\mathbb{T}_\Sigma(\mathbb{K}_\infty) \subset \mathbb{T}_\Sigma$ to be the set of elements $f = \sum_{\mu \in \mathbb{Z}_{\geq 0}^{|\Sigma|}} f_\mu \underline{t}_\Sigma^\mu \in \mathbb{T}_\Sigma$ such that $f_\mu \in \mathbb{K}_\infty$. We define the Gauss norm $\|\cdot\|_\infty$ in $\mathbb{T}_\Sigma$ by setting

$$\|f\|_\infty := \sup\{|f_\mu|_\infty \mid \mu \in \mathbb{Z}_{\geq 0}^{|\Sigma|}\},$$

and its corresponding valuation $\mathrm{ord}_\infty$ given by

$$\mathrm{ord}_\infty(f) := \inf\{\mathrm{ord}(f_\mu) \ \mid \ \mu \in \mathbb{Z}_{\geq 0}^{|\Sigma|}\}.$$

Note that $\mathbb{T}_\Sigma$ is complete with respect to $\|\cdot\|_\infty$. Moreover the Tate algebras $\mathbb{T}_t$ and $\mathbb{T}_{\Sigma,t}$ in variables $t$ and $t$ and $t_i$ for $i \in \Sigma$ respectively can be defined similarly. For more details on Tate algebras, we refer the reader to [BGR84] and [FVP04].

We consider the Frobenius automorphism $\tau : \mathbb{T}_\Sigma \to \mathbb{T}_\Sigma$ by $\tau(f) = \sum_{\mu \in \mathbb{Z}_{\geq 0}^{|\Sigma|}} f_\mu^q \underline{t}_\Sigma^\mu$ and we set $f^{(n)} := \tau^n(f)$ for any integer $n \in \mathbb{Z}$. The extension of the homomorphism $\tau$ to $\mathbb{T}_{\Sigma,t}$ can be defined similarly.

For $k, d \in \mathbb{Z}_{\geq 1}$ and any matrix $M = (M_{ij}) \in \mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)$, we define $M^{(n)}$ by applying the automorphism $\tau^n$ to each entry of $M$. We set

$$\|M\|_\infty := \sup_{i,j}\{\|M_{ij}\|_\infty\},$$

and consider $\mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)[[\tau]]$ the non commutative ring of power series in $\tau$ with coefficients in $\mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)$ such that for any element $M \in \mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)[[\tau]]$ we have

$$\tau M = M^{(1)}\tau.$$

Finally, we set $\mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)[\tau] \subset \mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)[[\tau]]$ for the subring of polynomials in $\tau$ with coefficients in $\mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)$.

Now we start to define some special elements which will be in use throughout the paper. For any $j \in \Sigma$ and $i \in \mathbb{Z}$ we define $b_i(t_j) \in K^{\mathrm{perf}}(\underline{t}_\Sigma)^\times$ by

$$b_i(t_j) := \left\{ \begin{array}{ll} \prod_{k=0}^{i-1}(t_j - \theta^{q^k}), & \text{if } i \geq 1 \\ 1, & \text{if } i = 0 \\ \prod_{k=0}^{-i-1}(t_j - \theta^{q^{-k-1}})^{-1}, & \text{if } i \leq -1 \end{array} \right\}.$$

Note that one can also define $b_i(t)$ for any $i \in \mathbb{Z}$ similarly.

Lemma 2.1. *[Dem15, Lem. 3.3.2]*

*(i) For any integer $i, d \in \mathbb{Z}$, we have*

$$\tau^d(b_i(t_j)) = \frac{b_{d+i}(t_j)}{b_d(t_j)} = \frac{b_i(t_j)}{b_d(t_j)}\tau^i(b_d(t_j)).$$

*(ii)*

$$\frac{1}{\tau(b_i(t_j))}\bigg|_{t_j=\theta} = \left\{ \begin{array}{ll} \ell_i^{-1}, & \text{if } i \geq 0 \\ 0, & \text{if } i \leq -1 \end{array} \right\}.$$

After fixing a $(q-1)$-st root of $-\theta$, we define the function $\Omega(t)$ as the following infinite product

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty}\left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}_t^\times.$$

One can observe that $\Omega(t)$ has infinite radius of convergence as a function of $t$ and satisfies

$$\Omega^{(-1)}(t) = (t-\theta)\Omega(t). \tag{2.2}$$

Moreover for any $n \in \mathbb{Z}_{\geq 1}$, we have

$$\Omega^{(n)}(t) = \frac{\Omega(t)}{(t-\theta^q)(t-\theta^{q^2})\dots(t-\theta^{q^n})}. \tag{2.3}$$

Note also that

$$\Omega(\theta) = \tilde{\pi}^{-1} \tag{2.4}$$

where we define

$$\tilde{\pi} := \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1-\theta^{1-q^i}\right)^{-1} \in \mathbb{C}_\infty^\times,$$

the Carlitz period. Let $\beta$ be a unit in $\mathbb{T}_\Sigma$. Then one can find an element $y \in \mathbb{C}_\infty$ such that $\|\beta - y\|_\infty < \|\beta\|_\infty$. Choose an element $\gamma \in \mathbb{C}_\infty^\times$ such that $\gamma^{q-1} = y$. We now define the infinite product

$$\omega_\beta := \gamma \prod_{i\geq 0} \frac{y^{q^i}}{\tau^i(\beta)},$$

which converges in $\mathbb{T}_\Sigma^\times$ by the choice of the element $y$ (see [APTR16, Sec. 6] for more details). The element $\omega_\beta \in \mathbb{T}_\Sigma^\times$ is called the Anderson-Thakur element corresponding to $\beta$ and defines up to the multiplication by an element in $\mathbb{F}_q^\times$. One also notes that

$$\tau(\omega_\beta) = \beta\omega_\beta. \tag{2.5}$$

Furthermore, by [APTR16, §6.1], we have $\|\omega_\beta\|_\infty = q^{\frac{-\operatorname{ord}(\beta)}{q-1}}$ and if $\beta_1, \beta_2 \in \mathbb{T}_\Sigma^\times$, then we obtain $\omega_{\beta_1\beta_2} = c\omega_{\beta_1}\omega_{\beta_2}$ for some $c \in \mathbb{F}_q^\times$. We set $\omega_i := \omega_{t_i-\theta} \in \mathbb{T}_\Sigma^\times$ where $i \in \Sigma$. For any integer $n$, it satisfies that

$$\omega_i^{(n)} = b_n(t_i)\omega_i. \tag{2.6}$$

Now for any $U \subset \mathbb{Z}_{\geq 1}$, let us set $\omega_U := \prod_{i\in U} \omega_i$. Recall from §1 that $\Sigma = \cup_{i=1}^r U_i$. For any $1 \leq i \leq r$, we define $\alpha_i := \prod_{j\in U_i}(t_j - \theta)$. Thus we see that $\|\alpha_i\|_\infty = q^{|U_i|}$. We also set $\alpha_k := 1$ and $\omega_{U_k} := 1$ if $U_k = \emptyset$. Using (2.6) one can obtain that

$$\omega_{U_i}^{(-1)} = \frac{\omega_{U_i}}{\alpha_i^{(-1)}}. \tag{2.7}$$

## 2.2 Power Sums

In most of this section, we summarize the work of Anglès, Pellarin and Tavares Riberio [APTR18, §6] and Demeslay [Dem15, §3.3.1] on power sums.

We denote the set of degree $d$ polynomials in $A_+$ by $A_{+,d}$. Let $z$ be an indeterminate over $\mathbb{C}_\infty$. Following the notation in [APTR18], for any $N \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 1}$, we define $L(N,s,z)$ by

$$L(N,s,z) := \sum_{d\geq 0} z^d \sum_{a\in A_{+,d}} \frac{a(t_1)\dots a(t_s)}{a^N} \in K[t_1,\dots,t_s][[z]].$$

Let us set $\exp_z := \sum_{i\geq 0} \frac{b_i(t_1)\dots b_i(t_s)}{D_i} z^i\tau^i \in K[t_1,\dots,t_s,z][[\tau]]$, where $D_0 := 1$ and $D_i := (\theta^{q^i}-\theta)D_{i-1}^q$. By [APTR18, Thm. 4.6], we know that $\exp_z(L(1,s,z)) \in A[t_1,\dots,t_s][z]$. Therefore for some $m \in \mathbb{Z}_{\geq 0}$ we can let $\exp_z(L(1,s,z)) = \sum_{i=0}^m \sigma_{s,i}(\mathbf{t})z^i$ so that $\sigma_{s,i}(\mathbf{t}) \in A[t_1,\dots,t_s]$ for any $i \in \{0,\dots,m\}$.

PROPOSITION 2.8. *[APTR18, Prop. 5.6] We have* $\deg_z(\exp_z(L(1,s,z))) \leq \frac{s-1}{q-1}$.

We further define $\log_{N,z}$ by

$$\log_{N,z} = \sum_{i\geq 0} \frac{b_i(t_1)\dots b_i(t_s)}{\ell_i^N} z^i\tau^i.$$

Let us fix $N \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 1}$ and set $r \in \mathbb{Z}_{\geq 1}$ as a positive integer so that $N \leq q^r$. Moreover we set $s := q^r - N + n$. Thus by Proposition 2.8 we see that the $z$-degree of $\exp_z(L(1,s,z))$ only depends on integers $N$ and $n$. For any $0 \leq i \leq m$ and $0 \leq j \leq B := (q^r-N)(r-1+m)$, we have the elements $g_{i,j} := \sum_{i_{n+1}+\dots+i_s=j} f_{i_{n+1},\dots,i_s} \in A[t_1,\dots,t_n]$ where $f_{i_{n+1},\dots,i_s} \in A[t_1,\dots,t_n]$ so that

$$b_r(t_1)\dots b_r(t_n)\tau(b_{r-1}(t_{n+1}))\dots\tau(b_{r-1}(t_s))\tau^r(\sigma_{s,i}(\mathbf{t})) = \sum_{i_{n+1},\dots,i_s} f_{i_{n+1},\dots,i_s} t_{n+1}^{i_{n+1}}\dots t_s^{i_s}.$$

By [APTR18, Thm. 6.2] we obtain

$$\sum_{d\geq 0} z^d \sum_{a\in A_{+,d}} \frac{a(t_1)\dots a(t_n)}{a^N} = \frac{1}{\ell_{r-1}^{q^r-N} b_r(t_1)\dots b_r(t_n)} \sum_{j\geq 0} \theta^j \log_{N,z}\left(\sum_{i=0}^m z^i g_{i,j}\right). \tag{2.9}$$

If we analyze the coefficients of $z^d$ on both sides of (2.9) we get for any $d \geq 0$,

$$\sum_{a\in A_{+,d}} \frac{a(t_1)\dots a(t_n)}{a^N} = \frac{1}{\ell_{r-1}^{q^r-N} b_r(t_1)\dots b_r(t_n)} \sum_{i=0}^{\min\{m,d\}} \frac{b_{d-i}(t_1)\dots b_{d-i}(t_n)}{\ell_{d-i}^N} \sum_{k=0}^B g_{i,k}^{(d-i)}\theta^k. \tag{2.10}$$

Let $K^{\mathrm{perf}}(t_1,\dots,t_n)$ be the fraction field of the polynomial ring $K^{\mathrm{perf}}[t_1,\dots,t_n]$. We now define the polynomial $Q_{n,N}(t) \in K^{\mathrm{perf}}(t_1,\dots,t_n)[t]$ by

$$Q_{n,N}(t) := \sum_{k=0}^{B}\sum_{i=0}^{m} \frac{b_{-i}(t_1)\dots b_{-i}(t_n)}{\tau(b_{-i}(t))^N}\tau^{-i}(g_{i,k})t^k. \tag{2.11}$$

*Remark* 2.12. It is important to notice that it follows from the definition (2.11) of $Q_{n,N}(t)$ that the $t$-coefficients of $Q_{n,N}(t)$ lie also in the Tate algebra $\mathbb{T}_\Sigma$ where $\Sigma = \{1,\dots,n\}$.

Note that

$$\tau^d(Q_{n,N}(t)) = \sum_{k=0}^{B}\sum_{i=0}^{m} \frac{\tau^d(b_{-i}(t_1)\dots b_{-i}(t_n))}{\tau^d(\tau(b_{-i}(t))^N)}\tau^{d-i}(g_{i,k})t^k. \tag{2.13}$$

By Lemma 2.1(i), we have that $\tau^d(b_{-i}(t_1)\dots b_{-i}(t_n)) = \frac{b_{d-i}(t_1)\dots b_{d-i}(t_n)}{b_d(t_1)\dots b_d(t_n)}$ and

$$\tau^d(\tau(b_{-i}(t))^N) = \tau(\tau^d(b_{-i}(t))^N) = \frac{\tau(b_{d-i}(t)^N)}{\tau(b_d(t))^N)}.$$

Thus by (2.13) we have

$$\tau^d(Q_{n,N}(t)) = \frac{\tau(b_d(t))^N)}{b_d(t_1)\dots b_d(t_n)}\sum_{k=0}^{B}\sum_{i=0}^{m} \frac{b_{d-i}(t_1)\dots b_{d-i}(t_n)}{\tau(b_{d-i}(t)^N)}\tau^{d-i}(g_{i,k})t^k. \tag{2.14}$$

Using Lemma 2.1(ii) and (2.14) we see that

$$\tau^d(Q_{n,N}(t))_{|t=\theta} = \frac{\ell_d^N}{b_d(t_1)\dots b_d(t_n)}\sum_{k=0}^{B}\sum_{i=0}^{\min\{m,d\}} \frac{b_{d-i}(t_1)\dots b_{d-i}(t_n)}{\ell_{d-i}^N}\tau^{d-i}(g_{i,k})\theta^k. \tag{2.15}$$

Combining (2.10) with (2.15), we see that

$$\sum_{a\in A_{+,d}} \frac{a(t_1)\dots a(t_n)}{a^N} = \frac{1}{\ell_{r-1}^{q^r-N} b_r(t_1)\dots b_r(t_n)}\frac{b_d(t_1)\dots b_d(t_n)}{\ell_d^N}\tau^d(Q_{n,N}(t))_{|t=\theta}. \tag{2.16}$$

Observe that $L(N,s,z)$ converges for $z=1$. Thus we have by (2.16) that

$$\frac{b_d(t_1)\dots b_d(t_n)\tau^d(Q_{n,N}(t))_{|t=\theta}}{\ell_d^N} \to 0 \tag{2.17}$$

as $d\to\infty$. Note that as $d$ gets arbitrarily large, we have $\|Q_{n,N}^{(d)}(t)_{|t=\theta}\|_\infty = \|Q_{n,N}(t)\|_\infty^{q^d} + \epsilon_d$ where $\epsilon_d \in \mathbb{R}$ is a constant depending on $d$ such that

$\lim_{d\to\infty} \epsilon_d/q^d = 0$. Thus, after calculating the norm of the terms in the left hand side of (2.17), for some constant $C \in \mathbb{R}^\times$, we obtain

$$Cq^{q^d\left(\frac{n-Nq}{q-1}+\log_q(\|Q_{n,N}(t)\|_\infty)+\frac{\epsilon_d}{q^d}\right)} \to 0$$

as $d$ goes to infinity. This can only happen if $\|Q_{n,N}(t)\|_\infty < q^{\frac{Nq-n}{q-1}}$.
Now for any $d, N \in \mathbb{Z}_{\geq 0}$ and $U \subset \mathbb{Z}_{\geq 1}$, we denote the power sum $S_d(U,N)$ by

$$S_d(U,N) := \sum_{a\in A_{+,d}} \frac{\sigma_U(a)}{a^N} \in K[\underline{t}_U],$$

and state the following theorem.

THEOREM 2.18. *[Dem15, Thm. 3.3.6, Lem. 3.3.9] Let $N \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 1}$ be such that $N \leq q^r$. Let $U \subset \mathbb{Z}_{\geq 1}$ be a non-empty finite set. Then there exists a unique polynomial $Q_{U,N}(t) \in K^{perf}(\underline{t}_U)[t]$ of the form*

$$Q_{U,N}(t) = \sum_{k=0}^{B_N}\sum_{i=0}^{m_N} \frac{\prod_{j\in U} b_{-i}(t_j)}{\tau(b_{-i}(t))^N}\tau^{-i}(g_{N,i,k})t^k \tag{2.19}$$

*for some $B_N, m_N \in \mathbb{Z}_{\geq 0}$ and $g_{N,i,k} \in A[\underline{t}_U]$ such that*

$$S_d(U,N) = \sum_{a\in A_{+,d}} \frac{\sigma_U(a)}{a^N} = \frac{b_d(U)}{\ell_d^N \ell_{r-1}^{q^r-N} b_r(U)}(\tau^d(Q_{U,N}(t)))_{|t=\theta} \tag{2.20}$$

*for all $d \geq 0$. In particular,*

$$\frac{\ell_{r-1}^{q^r-N} b_r(U)\omega_U}{\tilde{\pi}^N} S_d(U,N) = (\omega_U Q_{U,N}(t)\Omega^N(t))^{(d)}_{|t=\theta}.$$

*Moreover, $\|Q_{U,N}(t)\|_\infty < q^{\frac{Nq-|U|}{q-1}}$.*

*Proof.* Due to above discussion, only remaining part is to prove uniqueness. Suppose there exist two polynomials $Q_{U,N}(t)$ and $Q'_{U,N}(t)$ in $K^{\mathrm{perf}}(\underline{t}_U)[t]$ satisfying (2.20). Then we have $(\tau^d(Q_{U,N}(t) - Q'_{U,N}(t)))_{|t=\theta} = 0$. Since $t - \theta^{q^{-d}}$ is monic in $K^{\mathrm{perf}}(\underline{t}_U)[t]$, it divides $Q_{U,N}(t) - Q'_{U,N}(t)$ for any non-negative integer $d$ in $K^{\mathrm{perf}}(\underline{t}_U)[t]$. But since $Q_{U,N}(t)$ and $Q'_{U,N}(t)$ are polynomials, it is possible only if $Q_{U,N}(t) = Q'_{U,N}(t)$. □

For the completeness of this section, we also state Anderson and Thakur's result on power sums using our notation as follows.

THEOREM 2.21. *[AT90, Eq. 3.7.3-3.7.4] Let $N \in \mathbb{Z}_{\geq 0}$ be such that $N = \sum n_i q^i$ where $0 \leq n_i \leq q-1$ and set $\Gamma_N := \prod D_i^{n_i}$. Then there exists a unique polynomial $Q_{\emptyset,N}(t) \in A[t]$ such that*

$$S_d(\emptyset,N) = \sum_{a\in A_{+,d}} \frac{1}{a^N} = \frac{1}{\ell_d^N \Gamma_N}(\tau^d(Q_{\emptyset,N}(t)))_{|t=\theta}$$

*for all $d \geq 0$. In particular,*

$$\frac{\Gamma_N}{\widetilde{\pi^N}} S_d(\emptyset, N) = (\Omega^N(t) Q_{\emptyset,N}(t))^{(d)}_{|t=\theta}.$$

*Moreover, $\|Q_{\emptyset,N}(t)\|_\infty < q^{\frac{Nq}{q-1}}$.*

### 2.3 Multiple Zeta Values over Tate algebras

Throughout this section we assume that $(s_1, s_2, \ldots, s_r) \in \mathbb{Z}^r_{\geq 1}$ and $\Sigma = \cup_{i=1}^r U_i$ unless otherwise stated.

Definition 2.22. Let $\mathcal{C}$ be a composition array defined as in (1.4). We define the multiple zeta values $\zeta_C(\mathcal{C})$ over Tate algebras in the sense of the definition of Pellarin [Pel16], [Pel17] as the following object which converges in $\mathbb{T}_\Sigma$:

$$\begin{aligned}\zeta_C(\mathcal{C}) &= \sum_{\substack{|a_1|_\infty > \cdots > |a_r|_\infty \geq 0 \\ a_1,\ldots,a_r \in A_+}} \frac{\sigma_{U_1}(a_1)\sigma_{U_2}(a_2)\ldots\sigma_{U_r}(a_r)}{a_1^{s_1} a_2^{s_2} \ldots a_r^{s_r}} \\ &= \sum_{i_1 > i_2 > \cdots > i_r \geq 0} S_{i_1}(U_1, s_1)\ldots S_{i_r}(U_r, s_r).\end{aligned}$$

We define $n_l := |U_l|$ and consider the set $D'_{\mathcal{C}} \subset \mathbb{T}^r_\Sigma$ given by

$$D'_{\mathcal{C}} := \{(f_1, \ldots, f_r) \in \mathbb{T}^r_\Sigma \;\mid\; \|f_i\|_\infty < q^{\frac{s_i q - n_i}{q-1}} \text{ for } i = 1, \ldots, r\}.$$

For an $r$-tuple $(f_1, \ldots, f_r) \in D'_{\mathcal{C}}$, we set

$$\mathrm{Li}_{\mathcal{C}}(f_1, \ldots, f_r) := \sum_{i_1 > i_2 > \cdots > i_r \geq 0} \frac{b_{i_1}(U_1)\ldots b_{i_r}(U_r)\tau^{i_1}(f_1)\ldots\tau^{i_r}(f_r)}{\ell_{i_1}^{s_1}\ldots\ell_{i_r}^{s_r}}.$$

Using the definition of elements $b_i(U)$ and $\ell_i$, we note that $\mathrm{Li}_{\mathcal{C}}(f_1, \ldots, f_r)$ converges in $\mathbb{T}_\Sigma$ if $(f_1, \ldots, f_r) \in D'_{\mathcal{C}}$.
Let us fix a composition array $\mathcal{C}$ of depth $r$ as in (1.4) and let $C_i$ be the $t$-degree of the polynomial $Q_{U_i,s_i}(t) = \sum_{j\geq 0} u_{ij} t^j \in K^{\mathrm{perf}}(\underline{t}_{U_i})[t]$ for $1 \leq i \leq r$. Then consider the index set

$$\mathfrak{I} := \{0, \ldots, C_1\} \times \cdots \times \{0, \ldots, C_r\}.$$

For each $i = (j_1, \ldots, j_r) \in \mathfrak{I}$, we set

$$u_i := (u_{1j_1}, \ldots, u_{rj_r}) \in K^{\mathrm{perf}}(\underline{t}_{U_1}) \times \cdots \times K^{\mathrm{perf}}(\underline{t}_{U_r}). \tag{2.23}$$

Furthermore, we set $a_i := \theta^{j_1 + \cdots + j_r}$.

Theorem 2.24. *Let $\mathfrak{I}_1$ be the set of indices $i$ such that $U_i \neq \emptyset$ and $\mathfrak{I}_2$ be the set of $i$ such that $U_i = \emptyset$. Then we have*

$$\zeta_C(\mathcal{C}) = \frac{1}{\prod_{i\in\mathfrak{I}_1} \ell_{r_{s_i}}^{q^{r_{s_i}} - s_i} b_{r_{s_i}}(U_i) \prod_{i\in\mathfrak{I}_2} \Gamma_{s_i}} \sum_{i\in\mathfrak{I}} a_i \,\mathrm{Li}_{\mathcal{C}}(u_i) \tag{2.25}$$

*where $r_{s_i} \geq 1$ is an integer such that $s_i \leq q^{r_{s_i}}$ for $i \in \mathfrak{I}_1$.*

In §2.5, we give a proof for Theorem 2.24. The main idea is to modify the idea of the proof of [ABP04, Prop. 3.1.1] and combine it with the idea of the proof of [Cha14, Thm. 5.5.2]. Before stating the proof, we discuss some applications of our result.

## 2.4 Applications of Theorem 2.24 to Dirichlet-Goss multiple $L$-values

First we briefly discuss Gauss-Thakur sums. We recall the notation from §1.3 and for any $j = 0, \dots, d-1$ define the Gauss-Thakur sum

$$g(v_{\mathfrak{p}}^{q^j}) = -\sum_{\delta \in \Delta_{\mathfrak{p}}} v_{\mathfrak{p}}(\delta^{-1})^{q^j} \delta(\lambda_{\mathfrak{p}}) \in \mathbb{F}_{q^d}[\theta][\lambda_{\mathfrak{p}}].$$

By [Tha88, Thm. I], we know that $g(v_{\mathfrak{p}})$ is non-zero. Moreover, by [AP15, Thm. 2.9], we have

$$g(v_{\mathfrak{p}}) = \mathfrak{p}'(\xi_p)^{-1}\omega(\xi_p) \tag{2.26}$$

where $\mathfrak{p}'$ is the first derivative of $\mathfrak{p}$ with respect to $\theta$, $\omega \in \mathbb{T}_t$ is the Anderson-Thakur element corresponding to $t - \theta$ defined by

$$\omega := (-\theta)^{\frac{1}{q-1}} \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \in \mathbb{T}_t^{\times},$$

and $\omega(\xi_p) = \omega_{|t=\xi_p}$. For further details about Gauss-Thakur sums, we refer the reader to [AP14], [AP15], [Gos96] and [Tha88].

*Proof of Corollary 1.8.* Consider the composition array $\mathcal{C}$ as in (1.4) with pairwise disjoint subsets $U_i$ such that $|U_i| = j_i$ for all $1 \le i \le r$. By the definition of the polynomials $Q_{U_i,s_i}(t)$, there exists $m_i \in \mathbb{Z}_{\ge 0}$ such that

$$Q_{U_i,s_i}(t) = \prod_{k \in U_i} b_{-m_i}(t_k) \sum_{l \ge 0} c_{i,l} t^l \in K^{\mathrm{perf}}(\underline{t}_{U_i})[t]$$

where $c_{i,l} = \sum_{\mu \in \mathbb{Z}_{\ge 0}^{j_i}} c_{\mu,i,l} \underline{t}_{U_i}^{\mu} \in K^{\mathrm{perf}}[\underline{t}_{U_i}]$ with $c_{\mu,i,l} \in K^{\mathrm{perf}}$ and $c_{\mu,i,l} = 0$ for $l \gg 0$ and all but finitely many tuple $\mu \in \mathbb{Z}_{\ge 0}^{j_i}$. Up to permutation of elements $\xi_{ik} \in \overline{\mathbb{F}}_q$ for $1 \le i \le r$ and $1 \le k \le j_i$, assume that the Dirichlet character $\chi_i : A \to \overline{\mathbb{F}}_q$ given by $\chi_i(a) = a(\xi_{i1}) \dots a(\xi_{ij_i})$ is also given by $\chi_i(a) = \prod_{k \in U_i} a(t_k)_{|t_k = \xi_{ik}}$. For any $l \in \mathbb{Z}_{\ge 0}$, set $B_{\chi_i,l} := \prod_{k \in U_i} \omega_k^{(l)}(\xi_{ik}) b_{-m_i}(t_k)^{(l)}_{|t_k=\xi_{ik}} \in \mathbb{C}_{\infty}^{\times}$. Finally for $1 \le i \le r$, set $\xi_i := (\xi_{i1}, \dots, \xi_{ij_i})$ and for any tuple $\mu = (\mu_1, \dots, \mu_{j_i}) \in \mathbb{Z}_{\ge 0}^{j_i}$, define $\xi_i^{\mu} := \prod_{1 \le k \le j_i} \xi_k^{\mu_k}$. Thus by using Lemma 2.1 and (2.6), for any tuple

$(u_{1j_1},\dots,u_{rj_r})\in D'_{\mathcal{C}}$ as in (2.23), we have

$$\begin{aligned}
&\mathrm{Li}_{\mathcal{C}}(u_{1j_1},\dots,u_{rj_r})\big|_{\substack{t_k=\xi_{ik}\\ k\in U_i}}\\
&=\frac{1}{\prod_{k\in U_1}\omega_k(\xi_{1k})\cdots\prod_{k\in U_r}\omega_k(\xi_{rk})}\times\\
&\qquad\sum_{\substack{\mu_i\in\mathbb{Z}_{\geq 0}^{j_i}\\ l_1,\dots,l_r\geq 0}}\xi_1^{\mu_1}\dots\xi_r^{\mu_r}\sum_{i_1>i_2>\cdots>i_r\geq 0}\frac{B_{\chi_1,i_1}\dots B_{\chi_r,i_r}c_{\mu_1,1,l_1}^{q^{i_1}}\cdots c_{\mu_r,r,l_r}^{q^{i_r}}}{\ell_{i_1}^{s_1}\dots\ell_{i_r}^{s_r}}\\
&=\frac{1}{\prod_{k\in U_1}\omega_k(\xi_{1k})\cdots\prod_{k\in U_r}\omega_k(\xi_{rk})}\times\\
&\qquad\sum_{\substack{\mu_i\in\mathbb{Z}_{\geq 0}^{j_i}\\ l_1,\dots,l_r\geq 0}}\xi_1^{\mu_1}\dots\xi_r^{\mu_r}\,\mathrm{Li}_{\binom{\chi_1,\dots,\chi_r}{s_1,\dots,s_r}}(c_{\mu_1,1,l_1},\dots,c_{\mu_r,r,l_r}).
\end{aligned}$$

Since the coefficients $c_{\mu,i,j}=0$ for sufficiently large $j$ and all but finitely many $\mu$ when $1\leq i\leq r$, the sum in right hand side of the second equality above has finitely many terms. Thus evaluating both sides of (2.25) at $t_k=\xi_{ik}$ for $1\leq i\leq r$, $k\in U_i$ and using (2.26) finish the proof. □

Next we prove Theorem 1.9 by using Theorem 2.24 and the transcendence theory developed in [Cha14].

*Proof of Theorem 1.9.* For some $1\leq i\leq m$ and $j_i\leq i$, consider the Dirichlet-Goss multiple $L$-value $L(\chi_{U_{i1}},\xi_{i1},\dots,\chi_{U_{ij_i}},\xi_{ij_i};s_{i1},\dots,s_{ij_i})$. We continue with the notation of the proof of Corollary 1.8. For $1\leq k\leq j_i$, set $P_{ik,l}(t_{U_{ik}}):=\sum_{\mu\in\mathbb{Z}_{\geq 0}^{j_{ik}}}c_{\mu,(ik),l}\underline{t}_{U_{ik}}^{\mu}\in K^{\mathrm{perf}}[\underline{t}_{U_i}]$ so that $Q_{U_{ik},s_{ik}}(t)=\prod_{v\in U_{ik}}b_{-m_{ik}}(t_v)\sum_{l\geq 0}P_{ik,l}(t_{U_{ik}})t^l$. Since the field $\mathbb{F}_q$ is invariant under the automorphism $\tau$, by using (2.5), we see that $\omega_v(\xi_{ikv})$ is algebraic over $K$ for $1\leq k\leq j_i$ and $v\in U_{ik}$. Moreover, for any $n\in\mathbb{Z}_{\geq 0}$, we also see that

$$B_{\chi_{U_{ik}},n}=\prod_{v\in U_{ik}}\omega_v^{(n)}(\xi_{kv})b_{-m_{ik}}(t_v)^{(n)}_{|t_v=\xi_{ikv}}=\Big(\prod_{v\in U_{ik}}\omega_v(\xi_{ikv})b_{-m_{ik}}(\xi_{ikv})\Big)^{q^n} \tag{2.27}$$

where $b_{-m_{ik}}(\xi_{ikv})=b_{-m_i}(t_v)_{|t_v=\xi_{ikv}}$. Furthermore one can also obtain that

$$P_{ik,l}(t_{U_{ik}})^{(n)}_{\substack{t_v=\xi_{ikv}\\ v\in U_{ik}}}=\Big(P_{ik,l}(t_{U_{ik}})_{\big|_{\substack{t_v=\xi_{ikv}\\ v\in U_{ik}}}}\Big)^{q^n}. \tag{2.28}$$

Now for any tuple $l=(l_1,\dots,l_{j_i})\in\mathfrak{I}':=\{0,\dots,C_{i1}\}\times\{0,\dots,C_{ij_i}\}$ where $C_{ih}$ is the degree of $Q_{U_{ih},s_{ih}}(t)$ as a polynomial of $t$ for $1\leq h\leq j_i$, set $\mu_l$ to be

an element of $\overline{K}^{j_i}$ given by

$$\Big( \prod_{v\in U_{i1}} \omega_v(\xi_{i1v}) b_{-m_{i1}}(\xi_{i1v}) P_{i1,l_1}(t_{U_{i1}})\big|_{\substack{t_v=\xi_{i1v},\dots,\\ v\in U_{i1}}} \\ \prod_{v\in U_{ij_i}} \omega_v(\xi_{ij_iv}) b_{-m_{ij_i}}(\xi_{ij_iv}) P_{ij_i,l_{j_i}}(t_{U_{ij_i}})\big|_{\substack{t_v=\xi_{ij_iv}\\ v\in U_{ij_i}}} \Big).$$

Consider the composition array $\mathcal{C}_i = \begin{pmatrix} U_{i1} & \dots & U_{ij_i} \\ s_{i1} & \dots & s_{ij_i} \end{pmatrix}$. Then, for any $(u_{i1l_1},\dots,u_{ij_il_{j_i}}) \in D'_{\mathcal{C}_i}$ corresponding to the tuple $l$ as in (2.23), using (2.27) and (2.28), we immediately see that

$$\mathrm{Li}_{\mathcal{C}_i}(u_{i1l_1},\dots,u_{ij_il_{j_i}})\Big|_{\substack{t_v=\xi_{ikv}\\ v\in U_{ik}}} = \frac{1}{\prod_{v\in U_{ik}}\omega_v(\xi_{i1v})\cdots\prod_{v\in U_{ij_i}}\omega_v(\xi_{ij_iv})}\,\mathrm{Li}_{\mathfrak{s}_i}(\mu_l) \tag{2.29}$$

where $\mathfrak{s}_i = (s_{i1},\dots,s_{ij_i})$ and $\mathrm{Li}_{\mathfrak{s}_i}$ is the Carlitz multiple polylogarithm defined as in (1.1). Thus we obtain that the Dirichlet-Goss $L$-value $L(\chi_{U_{11},\xi_{11}},\dots,\chi_{U_{i1},\xi_{ij_i}};s_{i1},\dots,s_{ij_i})$ can be written as a $\overline{K}$-linear combination of multiple polylogarithms $\mathrm{Li}_{\mathfrak{s}}$ at algebraic points by Theorem 2.24. Finally the result now follows from [Cha14, Thm. 3.4.5] and [Cha14, Prop. 5.4.1]. □

*Remark* 2.30. In [CM20, Thm. 5.2.5], Chang and Mishiba introduced the multiple zeta value $\zeta_A(\mathfrak{s})$ as a $K$-linear combination of multiple star polylogarithms at some algebraic points $v_1,\dots,v_m$ for some $m\in\mathbb{Z}_{\geq 1}$. Furthermore, they showed that those points are related to dual $t$-motives of certain Anderson $t$-modules via the isomorphism between $\mathrm{Ext}^1$-modules and Anderson $t$-modules (see [CPY19, Thm. 5.2.3] and [CM19, Rem. 3.3.5]). In our case, when the Dirichlet characters are of the form as in Theorem 1.9, one can write a Dirichlet-Goss multiple $L$-value as a $\overline{K}$-linear combination of multiple star polylogartihms at some algebraic points $\tilde{v}_1,\dots,\tilde{v}'_m$ for some $m'\in\mathbb{Z}_{\geq 1}$ by using Theorem 2.44 which will be proved in §2.6 and form a similar relation between those points and dual $t$-motives. However, since the elements of $\overline{\mathbb{F}}_q\setminus\mathbb{F}_q$ are not invariant under the automorphism $\tau$, we do not know the corresponding Anderson $t$-modules to these points when arbitrary Dricihlet characters are used to construct Dirichlet-Goss multiple $L$-values. This is due to the fact that these points can be only expressed after specializing variables by using our methods. It would be interesting to construct these $t$-modules directly to make the relation between dual $t$-motives and the points $\tilde{v}_1,\dots,\tilde{v}_m$ more transparent.

## 2.5 Proof of Theorem 2.24

For any $1 \leq l < j \leq r+1$, we define the following objects:

$$\begin{aligned} L_{j,l}(t) &:= \\ &\sum_{i_l > \cdots > i_{j-1} \geq 0} (\omega_{U_l} \Omega^{s_l}(t) Q_{U_l,s_l}(t))^{(i_l)} \dots (\omega_{U_{j-1}} \Omega^{s_{j-1}}(t) Q_{U_{j-1},s_{j-1}}(t))^{(i_{j-1})} \\ &= \Omega^{s_l+\cdots+s_{j-1}}(t) \prod_{i=l}^{j-1} \omega_{U_i} \times \\ &\qquad \sum_{i_l > \cdots > i_{j-1} \geq 0} \frac{Q^{(i_l)}_{U_l,s_l}(t) \dots Q^{(i_{j-1})}_{U_{j-1},s_{j-1}}(t) b_{i_l}(U_l) \dots b_{i_{j-1}}(U_{j-1})}{((t-\theta)\dots(t-\theta^{q^{i_l}}))^{s_l} \dots ((t-\theta)\dots(t-\theta^{q^{i_{j-1}}}))^{s_{j-1}}}. \end{aligned} \tag{2.31}$$

Moreover we set $L_{j,l}(t) := 1$ if $j = l$. Observe that for some $\epsilon_l, \dots, \epsilon_{j-1} \in \mathbb{R}_{>0}$, we have by Theorem 2.18 and Theorem 2.21 that

$$\begin{aligned} &\frac{\|Q^{(i_l)}_{U_l,s_l}(t) \dots Q^{(i_{j-1})}_{U_{j-1},s_{j-1}}(t)\|_\infty \|b_{i_l}(U_l) \dots b_{i_{j-1}}(U_{j-1})\|_\infty}{\|((t-\theta)\dots(t-\theta^{q^{i_l}}))^{s_l} \dots ((t-\theta)\dots(t-\theta^{q^{i_{j-1}}}))^{s_{j-1}}\|_\infty} \\ &= q^{q^{i_l}\left(\frac{s_l q - |U_l|}{q-l} - \epsilon_l\right)} \dots q^{q^{i_{j-1}}\left(\frac{s_{j-1} q - |U_{j-1}|}{q-1} - \epsilon_{j-1}\right)} \times \\ &\qquad q^{\frac{q^{i_l}|U_l| - |U_l|}{q-1}} \dots q^{\frac{q^{i_{j-1}}|U_{j-1}| - |U_{j-1}|}{q-1}} q^{-\left(s_l q\left(\frac{q^{i_l}-1}{q-1}\right) + \cdots + s_{j-1} q\left(\frac{q^{i_{j-1}}-1}{q-1}\right)\right)} \\ &= q^{q^{i_l}\left(\frac{s_l q - |U_l|}{q-1} + \frac{|U_l|}{q-1} - \frac{s_l q}{q-1} - \epsilon_l\right)} \dots q^{q^{i_{j-1}}\left(\frac{s_{j-1} q - |U_{j-1}|}{q-1} + \frac{|U_{j-1}|}{q-1} - \frac{s_{j-1} q}{q-1} - \epsilon_{j-1}\right)} \times \\ &\qquad q^{\frac{-|U_l| - \cdots - |U_{j-1}| + q(s_l + \cdots + s_{j-1})}{q-1}} \\ &= q^{-q^{i_l}\epsilon_l} \dots q^{-q^{i_{j-1}}\epsilon_{j-1}} q^{\frac{-|U_l| - \cdots - |U_{j-1}| + q(s_l + \cdots + s_{j-1})}{q-1}} \to 0 \end{aligned}$$

whenever $0 \leq i_{j-1} < \cdots < i_l \to \infty$. Thus we conclude that $L_{j,l}(t) \in \mathbb{T}_{\Sigma,t}$.

Proposition 2.32. *For any $1 \leq l < j \leq r+1$, we have*

$$L^{(-1)}_{j,l}(t) = L_{j,l}(t) + Q^{(-1)}_{U_{j-1},s_{j-1}}(t) \frac{(t-\theta)^{s_{j-1}}}{\alpha^{(-1)}_{j-1}} \Omega^{s_{j-1}}(t) \omega_{U_{j-1}} L_{j-1,l}(t). \tag{2.33}$$

*Proof.* Observe that

$$L_{j,l}(t) = \Omega^{s_l+\cdots+s_{j-1}}(t)\prod_{i=l}^{j-1}\omega_{U_i}\times$$
$$\Bigg[\sum_{i_l>\cdots>i_{j-1}\geq 1}\frac{Q^{(i_l)}_{U_l,s_l}(t)\dots Q^{(i_{j-1})}_{U_{j-1},s_{j-1}}(t)b_{i_l}(U_l)\dots b_{i_{j-1}}(U_{j-1})}{((t-\theta^q)\dots(t-\theta^{q^{i_l}}))^{s_l}\dots((t-\theta^q)\dots(t-\theta^{q^{i_{j-1}}}))^{s_{j-1}}}$$
$$+\,Q_{U_{j-1},s_{j-1}}(t)\times$$
$$\sum_{i_l>\cdots>i_{j-2}\geq 1}\frac{Q^{(i_l)}_{U_l,s_l}(t)\dots Q^{(i_{j-2})}_{U_{j-2},s_{j-2}}(t)b_{i_l}(U_l)\dots b_{i_{j-2}}(U_{j-2})}{((t-\theta^q)\dots(t-\theta^{q^{i_l}}))^{s_l}\dots((t-\theta^q)\dots(t-\theta^{q^{i_{j-2}}}))^{s_{j-2}}}\Bigg].$$

Therefore using (2.2) and (2.7) together with the above equation, we see that the equality in (2.33) holds. □

We recall the polynomials $Q_{U_i,s_i}(t) \in K^{\mathrm{perf}}(\underline{t}_{U_i})[t]$ from Theorem 2.18 and Theorem 2.21, and consider the matrix $\Phi_l \in \mathrm{Mat}_{r+2-l}(\mathbb{T}_\Sigma[t])$ defined by

$$\Phi_l := \begin{bmatrix} \frac{(t-\theta)^{s_l+\cdots+s_r}}{\prod_{i=l}^r \alpha_i^{(-1)}} & 0 & \cdots & \cdots & 0 \\ \frac{Q^{(-1)}_{U_l,s_l}(t)(t-\theta)^{s_l+\cdots+s_r}}{\prod_{i=l}^r \alpha_i^{(-1)}} & & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{(t-\theta)^{s_r}}{\alpha_r^{(-1)}} & 0 \\ 0 & \cdots & 0 & \frac{Q^{(-1)}_{U_r,s_r}(t)(t-\theta)^{s_r}}{\alpha_r^{(-1)}} & 1 \end{bmatrix}. \tag{2.34}$$

and the matrix $\Psi_l$ defined by

$$\Psi_l := \begin{bmatrix} \Omega^{s_l+\cdots+s_r}(t)\prod_{i=l}^r \omega_{U_i} \\ L_{(l+1),l}(t)\Omega^{s_{l+1}+\cdots+s_r}(t)\prod_{i=l+1}^r \omega_{U_i} \\ \vdots \\ L_{r,l}(t)\Omega^{s_r}(t)\omega_{U_r} \\ L_{r+1,l}(t) \end{bmatrix} \in \mathrm{Mat}_{(r+2-l)\times 1}(\mathbb{T}_{\Sigma,t}). \tag{2.35}$$

Using (2.2), (2.7) and Proposition 2.32 one can prove the following lemma.

LEMMA 2.36. *We have*

$$\Psi_l^{(-1)} = \Phi_l\Psi_l.$$

In order to prove that the function $L_{j,l}(t)$ has infinite radius of convergence, we need to state a technical lemma.

LEMMA 2.37. *Let* $\{s_1, \dots, s_k\} \in \mathbb{Z}^k_{\geq 0}$. *Let* $G, Q \in \mathbb{T}_\Sigma$ *be such that* $\|G\|_\infty \leq q^{\frac{q(s_i+\cdots+s_j)-(|U_i|+\cdots+|U_j|)}{q-1}}$ *and* $\|Q\|_\infty < q^{\frac{qs_{i-1}-|U_{i-1}|}{q-1}}$ *for some* $2 \leq i \leq j \leq k$. *Let also* $H \in \mathbb{T}^\times_\Sigma$ *such that* $\|H\|_\infty = q^{\left(s_{i-1}+\cdots+s_j-\frac{|U_{i-1}|+\cdots+|U_j|}{q}\right)}$. *Then there exists an element* $F \in \mathbb{T}_\Sigma$ *such that*

$$H(F^{(-1)} - Q^{(-1)}G^{(-1)}) = F.$$

*Moreover,* $\|F\|_\infty \leq q^{\frac{q(s_{i-1}+\cdots+s_j)-(|U_{i-1}|+\cdots+|U_j|)}{q-1}}$.

*Proof.* We define the potential solution $F$ as the following infinite sum

$$F := QG + Q^{(1)}G^{(1)}(H^{-1})^{(1)} + \sum_{r=2}^{\infty} Q^{(r)}G^{(r)}(H^{-1})^{(r)}(H^{-1})^{(r-1)} \dots (H^{-1})^{(1)}.$$

One can see that $F$ satisfies the desired equality in the lemma. We need to show that $F$ is a well-defined element in $\mathbb{T}_\Sigma$. By the assumptions on elements $G, H$ and $Q$, for some $\epsilon > 0$ and $r \to \infty$, we have the following estimate.

$$\begin{aligned}
&\|Q^{(r)}G^{(r)}(H^{-1})^{(r)} \dots (H^{-1})^{(1)}\|_\infty \\
&\leq q^{\frac{q^r}{q-1}\left(qs_{i-1}-|U_{i-1}|-\epsilon+q(s_i+\cdots+s_j)-(|U_i|+\cdots+|U_j|)\right)} \times \\
&\qquad\qquad q^{\frac{q^r-1}{q-1}(-q(s_{i-1}+\cdots+s_j)+|U_{i-1}|+\cdots+|U_j|)} \\
&= q^{\frac{-q^r\epsilon}{q-1}} q^{\frac{-1}{q-1}(-q(s_{i-1}+\cdots+s_j)+|U_{i-1}|+\cdots+|U_j|)} \to 0.
\end{aligned}$$

Thus we can conclude that as $r \to \infty$, the norm of the general term of the sum approaches to 0. Therefore the sum is well defined and since $\mathbb{T}_\Sigma$ is a complete normed space, $F$ is in $\mathbb{T}_\Sigma$. On the other hand, by the assumptions and the properties of the non-archimedean norm $\|\cdot\|_\infty$, we have $\|F\|_\infty \leq \max\{\|HQ^{(-1)}G^{(-1)}\|_\infty, \|F^{(-1)}H\|_\infty\}$.
*Case 1:* $\|F\|_\infty = \|HQ^{(-1)}G^{(-1)}\|_\infty$.
By the assumption, we have the following estimate.

$$\begin{aligned}
\|F\|_\infty &< q^{(s_{i-1}+\cdots+s_j-(|U_{i-1}|+\cdots+|U_j|)/q)} q^{\frac{qs_{i-1}-|U_{i-1}|}{q(q-1)}} q^{\frac{q(s_i+\cdots+s_j)-(|U_i|+\cdots+|U_j|)}{q(q-1)}} \\
&= q^{\frac{(q-1)(s_{i-1}+\cdots+s_j)}{q-1}-(|U_{i-1}|+\cdots+|U_j|)/q} \times \\
&\qquad q^{\frac{1}{q(q-1)}\left(q(s_{i-1}+s_i+\cdots+s_j)-(|U_{i-1}|+\cdots+|U_j|)\right)} \\
&= q^{\frac{q(s_{i-1}+\cdots+s_j)}{q-1}-\frac{(|U_{i-1}|+\cdots+|U_j|)}{q}\left(1+\frac{1}{q-1}\right)} = q^{\frac{q(s_{i-1}+\cdots+s_j)-(|U_{i-1}|+\cdots+|U_j|)}{q-1}}.
\end{aligned}$$

*Case 2:* $\|F\|_\infty = \|HF^{(-1)}\|_\infty$.
We note that $\|F^{(-1)}\|_\infty = \|F\|_\infty^{1/q}$. Thus, in this case we have that

$$\begin{aligned}
\|F\|_\infty &= \|H\|_\infty^{\frac{q}{q-1}} \\
&= q^{\frac{q}{q-1}\left((s_{i-1}+\cdots+s_j)-\frac{(|U_{i-1}|+\cdots+|U_j|)}{q}\right)} \\
&= q^{\frac{q(s_{i-1}+\cdots+s_j)-(|U_{i-1}|+\cdots+|U_j|)}{q-1}}.
\end{aligned}$$

By the analysis of these two cases, we deduce the last statement of the lemma. □

Now recall the matrix $\Phi_l$ from (2.34) and let us define elements $b_i \in \mathrm{Mat}_{r+2-l}(\mathbb{T}_\Sigma)$ such that $\Phi_l = \sum_{i=0}^{m} b_i t^i$ for some $m \in \mathbb{Z}_{\geq 0}$.

PROPOSITION 2.38. *There exists a matrix $U \in \mathrm{GL}_{r+2-l}(\mathbb{T}_\Sigma)$ such that $U^{(-1)}b_0 = U$.*

*Proof.* Without loss of generality let us take $l = 1$. To avoid heavy notation let $Q_{U_i,s_i} \in \mathbb{T}_\Sigma$ be the constant term of the polynomial $Q_{U_i,s_i}(t)$ for any $1 \leq i \leq r$. Let $\{s_1, \ldots, s_r\} \in \mathbb{Z}^r_{\geq 1}$. Observe that

$$b_0 = \begin{bmatrix} \frac{(-\theta)^{s_1+\cdots+s_r}}{\prod_{i=1}^r \alpha_i^{(-1)}} & 0 & 0 & \cdots & 0 \\ \frac{Q^{(-1)}_{U_1,s_1}(-\theta)^{s_1+\cdots+s_r}}{\prod_{i=1}^r \alpha_i^{(-1)}} & \frac{(-\theta)^{s_2+\cdots+s_r}}{\prod_{i=2}^r \alpha_i^{(-1)}} & 0 & \cdots & 0 \\ 0 & \frac{Q^{(-1)}_{U_2,s_2}(-\theta)^{s_2+\cdots+s_r}}{\prod_{i=2}^r \alpha_i^{(-1)}} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{(-\theta)^{s_r}}{\alpha_r^{(-1)}} & 0 \\ 0 & \cdots & 0 & \frac{Q^{(-1)}_{U_r,s_r}(-\theta)^{s_r}}{\alpha_r^{(-1)}} & 1 \end{bmatrix}.$$

Let the matrix $U = (a_{ij})$ be defined as a potential solution of the equation in the proposition. Therefore the $i$-th row $R_i$ of the matrix $U^{(-1)}b_0$ appears as

$$R_i := \Bigg[\frac{(-\theta)^{s_1+\cdots+s_r}}{\prod_{i=1}^r \alpha_i^{(-1)}}(a^{(-1)}_{i,1} + Q^{(-1)}_{U_1,s_1}a^{(-1)}_{i,2}), \frac{(-\theta)^{s_2+\cdots+s_r}}{\prod_{i=2}^r \alpha_i^{(-1)}}(a^{(-1)}_{i,2} + Q^{(-1)}_{U_2,s_2}a^{(-1)}_{i,3}) \\ , \ldots, \frac{(-\theta)^{s_r}}{\alpha_r^{(-1)}}(a^{(-1)}_{i,r} + Q^{(-1)}_{U_r,s_r}a^{(-1)}_{i,r+1}), a^{(-1)}_{i,r+1}\Bigg]. \quad (2.39)$$

Now our aim is to pick elements $a_{i,j} \in \mathbb{T}_\Sigma$ in a way that the desired equality would be satisfied. First for $1 \leq i \leq r$, we set $a_{i,j} = 0$ when $j > i$. In order to see how we can pick the other elements let us analyze the $k$-th row of $U$ where $1 \leq k \leq r$. By the above setting, we know $a_{k,k+1}$=0. Then by (2.39), in order to give the desired equality we want to have

$$\frac{(-\theta)^{s_k+\cdots+s_r}}{\prod_{i=k}^r \alpha_i^{(-1)}} a^{(-1)}_{k,k} = a_{k,k}.$$

Set $\beta_k := \frac{(-\theta)^{q(s_k+\cdots+s_r)}}{\prod_{i=k}^r \alpha_i}$. Since $\beta_k \in \mathbb{T}^\times_\Sigma$, by (2.7) we can pick $a_{k,k} = \omega_{\beta_k} \in \mathbb{T}^\times_\Sigma$. Note also that $\|\omega_{\beta_k}\|_\infty = q^{\frac{q(s_k+\cdots+s_r)-(|U_k|+\cdots+|U_r|)}{q-1}}$.
Now we need to find $a_{k,k-1}$ such that

$$\frac{(-\theta)^{s_{k-1}+\cdots+s_r}}{\prod_{i=k-1}^r \alpha_i^{(-1)}}(a^{(-1)}_{k,k-1} - Q^{(-1)}_{U_{k-1,s_{k-1}}} a^{(-1)}_{k,k}) = a_{k,k-1}.$$

Since $\|Q_{U_{k-1},s_{k-1}}\|_\infty < q^{\frac{qs_{k-1}-|U_{k-1}|}{q-1}}$, we have that the element $a_{k,k-1}$ is in $\mathbb{T}_\Sigma$ and $\|a_{k,k-1}\|_\infty \leq q^{\frac{q(s_{k-1}+\cdots+s_r)-(|U_{k-1}|+\cdots+|U_r|)}{q-1}}$ by Lemma 2.37. The other elements of the $k$-th row of $U$ can be found by using the same idea together with Lemma 2.37. Thus, we determine the $k$-th row of $U$ recursively when $1 \leq k \leq r$ and conclude that all elements in the $k$-th row is in $\mathbb{T}_\Sigma$.
To determine the last row, we let $s_{r+1} = 0$ and $U_{r+1} = \emptyset$. Then if we apply the same idea above we see that we let $\beta_{r+1} = \frac{(-\theta)^{qs_{r+1}}}{\alpha_{r+1}^{(-1)}} = 1$ and therefore we can pick $a_{r+1,r+1} = 1$. We can now pick the other elements of the last row from $\mathbb{T}_\Sigma$ by again using Lemma 2.37.
According to our selection for the elements $a_{i,j}$ we now see that

$$U = \begin{bmatrix} \omega_{\beta_1} & & & & \\ a_{2,1} & \omega_{\beta_2} & & & \\ \vdots & & \ddots & & \\ \vdots & & & \omega_{\beta_r} & \\ a_{r+1,1} & \dots & \dots & a_{r+1,r} & 1 \end{bmatrix}.$$

Since $U$ is a lower triangular matrix, one can obtain $\det(U) = \prod_{i=1}^{r} \omega_{\beta_i} \in \mathbb{T}_\Sigma^\times$. Thus, we conclude that $U \in \mathrm{GL}_{r+1}(\mathbb{T}_\Sigma)$. □

THEOREM 2.40. *The function $\Psi_l(t) := \Psi_l$ has infinite radius of convergence. In particular the function $L_{j,l}(t)$ is well defined for any values of $t \in \mathbb{T}_\Sigma$.*

*Proof.* We modify the ideas of the proof of Proposition 3.1.3 of [ABP04]. Recall that $\Phi_l = \sum_{i=0}^{m} b_i t^i$. By Proposition 2.38 there exists a matrix $U \in \mathrm{GL}_{r+2-l}(\mathbb{T}_\Sigma)$ such that $U^{(-1)} b_0 U^{-1} = \mathrm{Id}_{r+2-l}$. Now set $\Psi_l' := U\Psi_l$ and $\Phi_l' := U^{(-1)}\Phi_l U^{-1}$ and let $\Phi_l' = \sum_{i=0}^{m} b_i' t^i$ so that $b_0' = \mathrm{Id}_{r+2-l}$ and $\Psi_l' = \sum_{i\geq 0} g_i' t^i$. By Lemma 2.36, one can see that $\Psi_l'^{(-1)} = \Phi_l'\Psi_l'$. Therefore for all $n \geq 1$, we have that

$$g_n'^{(-1)} - g_n' = \sum_{i=1}^{\min\{n,m\}} b_i' g'_{n-i}.$$

Since $\Psi \in \mathrm{Mat}_{(r+2-l)\times 1}(\mathbb{T}_{\Sigma,t})$ and $U \in \mathrm{GL}_{(r+2-l)}(\mathbb{T}_\Sigma)$, $\lim_{n\to\infty}\|g_n'\|_\infty = 0$. Let us set

$$\tilde{g}_n := \sum_{\nu=1}^{\infty} \Big(\sum_{i=1}^{m} b_i' g_{n-i}'\Big)^{(\nu)}.$$

Note that $\tilde{g}_n$ also converges for all sufficiently large $n$ and we have that $\lim_{n\to\infty}\|\tilde{g}_n\|_\infty = 0$. We also have that

$$\tilde{g}_n^{(-1)} = \sum_{i=1}^{m} b_i' g_{n-i}' + \tilde{g}_n = g_n'^{(-1)} - g_n' + \tilde{g}_n.$$

Thus $(\tilde{g}_n - g'_n)^{(-1)} = \tilde{g}_n - g'_n$ for $n \gg 0$. When $n \gg 0$, we see that $\tilde{g}_n - g'_n$ is invariant under twisting, thus by [GP19, Lem. 2.5.1], we have that $\tilde{g}_n - g_n \in \mathbb{F}_q[\underline{t}_\Sigma]$. But for sufficiently large $n$ the norm of $\tilde{g}_n - g'_n$ is arbitrarily small. Therefore we conclude that $\tilde{g}_n - g'_n = 0$. Therefore for $n \gg 0$ and a fixed real number $C > 1$ we have that

$$\begin{aligned} C^n\|g'_n\|_\infty &\leq \max_{i=1}^{m} C^n\|b'_i\|_\infty^q\|g'_{n-i}\|_\infty^q \\ &\leq (\max_{i=1}^{m} C^i\|b'_i\|_\infty^q)(\max_{i=1}^{m}\|g'_{n-i}\|_\infty)^{q-1}\max_{i=1}^{m} C^{n-i}\|g'_{n-i}\|_\infty \\ &\leq \max_{i=n-m}^{n-1} C^i\|g'_i\|_\infty \end{aligned}$$

where the last inequality comes from the fact that

$$(\max_{i=1}^{m} C^i\|b'_i\|_\infty^q)(\max_{i=1}^{m}\|g'_{n-i}\|_\infty)^{q-1} \leq 1$$

when $n$ is sufficiently large. Therefore $\sup_{n=0}^{\infty} C^n\|g'_n\|_\infty < \infty$ and it implies that all entries of $\Psi'_l = U\Psi_l$ has infinite radius of convergence. Multiplying the column matrix $\Psi'_l$ containing functions with infinite radius of convergence with $U^{-1}$ from the left then implies that functions in the entries of $\Psi_l$ have also infinite radius of convergence. □

*Proof of Theorem 2.24.* The proof uses the ideas from the proof of Theorem 5.5.2 of [Cha14]. Let $\mathcal{C}$ be the composition array as in (1.4) so that $\mathfrak{I}_1$ is the set of indices $i$ such that $U_i \neq \emptyset$ and $\mathfrak{I}_2$ is the set of $i$'s such that $U_i = \emptyset$. We denote $L_{r+1}(t) := L_{r+1,1}(t)$. By Theorem 2.40 we have that the function $L_{r+1}(t)$ has infinite radius of convergence. Recall that

$$L_{r+1}(t) = \Omega^{s_1+\cdots+s_r}(t)\prod_{i=1}^{r}\omega_{U_i}\times \sum_{i_1>i_2>\cdots>i_r\geq 0}\frac{Q^{(i_1)}_{U_1,s_1}(t)\dots Q^{(i_r)}_{U_r,s_r}(t)b_{i_1}(U_1)\dots b_{i_r}(U_r)}{((t-\theta^q)\dots(t-\theta^{q^{i_1}}))^{s_1}\dots((t-\theta^q)\dots(t-\theta^{q^{i_r}}))^{s_r}}.$$

Since $L_{r+1}(t)$ is well-defined at $t=\theta$, by Theorem 2.18, Theorem 2.21 and the equalities (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned} L_{r+1}(\theta) &= \sum_{i_1>i_2>\cdots>i_r\geq 0}(\omega_{U_1}\Omega^{s_1}Q_{U_1,s_1})^{(i_1)}(\theta)\dots(\omega_{U_r}\Omega^{s_r}Q_{U_r,s_r})^{(i_r)}(\theta) \\ &= \frac{\prod_{i=1}^{r}\omega_{U_i}\prod_{i\in\mathfrak{I}_1}\ell_{r_{s_i}-1}^{q^{r_{s_i}}-s_i}b_{r_{s_i}}(U_i)\prod_{i\in\mathfrak{I}_2}\Gamma_{s_i}}{\tilde{\pi}^{s_1+\cdots+s_r}}\times \\ &\qquad \sum_{i_1>i_2>\cdots>i_r\geq 0}S_{i_1}(U_1,s_1)\dots S_{i_r}(U_r,s_r) \\ &= \frac{\prod_{i=1}^{r}\omega_{U_i}\prod_{i\in\mathfrak{I}_1}\ell_{r_{s_i}-1}^{q^{r_{s_i}}-s_i}b_{r_{s_i}}(U_i)\prod_{i\in\mathfrak{I}_2}\Gamma_{s_i}}{\tilde{\pi}^{s_1+\cdots+s_r}}\zeta_C(\mathcal{C}). \end{aligned} \tag{2.41}$$

Observe that

$$\frac{L_{r+1}(t)}{\prod_{i=1}^{r}\omega_{U_i}\Omega^{s_1+\cdots+s_r}(t)} = \sum_{i_1>i_2>\cdots>i_r\geq 0}\frac{Q^{(i_1)}_{U_1,s_1}(t)\dots Q^{(i_r)}_{U_r,s_r}(t)b_{i_1}(U_1)\dots b_{i_r}(U_r)}{((t-\theta^q)\dots(t-\theta^{q^{i_1}}))^{s_1}\dots((t-\theta^q)\dots(t-\theta^{q^{i_r}}))^{s_r}}. \quad (2.42)$$

Applying $t=\theta$ to both sides of (2.42) and combining it with (2.41) by using (2.4), we get that

$$\zeta_C(\mathcal{C})\prod_{i\in\mathfrak{I}_1}\ell_{r_{s_i}-1}^{q^{r_{s_i}}-s_i}b_{r_{s_i}}(U_i)\prod_{i\in\mathfrak{I}_2}\Gamma_{s_i}=\sum_{i\in\mathfrak{I}}a_i\,\mathrm{Li}_{\mathcal{C}}(u_i)$$

where the right hand side is justified by the fact that $\|Q_{U_j,s_j}(t)\|_\infty < q^{\frac{s_jq-|U_j|}{q-1}}$ for any $j\in\{1,\dots,r\}$ and therefore $\mathrm{Li}_{\mathcal{C}}(u)$ converges for any $u\in S$. □

## 2.6 Multiple Star Polylogarithms

Let $\mathcal{C}$ be a composition array as in (1.4) such that $r>1$. Recall that $n_l=|U_l|$ for $1\leq l\leq r$. We define the subset $D''_{\mathcal{C}}\subset\mathbb{T}^r_\Sigma$ by

$$\{f=(f_1,\dots,f_r)\in\mathbb{T}^r_\Sigma \mid \|f_1\|_\infty < q^{\frac{s_1q-n_1}{q-1}}, \|f_i\|_\infty\leq q^{\frac{s_iq-n_i}{q-1}} \text{ for } i=2,\dots,r\}$$

Inspired by the work of Chang and Mishiba in [CM20, Sec. 2.2], for $u=(u_1,\dots,u_r)\in D''_{\mathcal{C}}$, we define the multiple star polylogarithm $\mathrm{Li}^*_{\mathcal{C}}(u)$ corresponding to the composition array $\mathcal{C}$ by the infinite series

$$\mathrm{Li}^*_{\mathcal{C}}(u):=\sum_{i_1\geq i_2\geq\cdots\geq i_r\geq 0}\frac{b_{i_1}(U_1)\dots b_{i_r}(U_r)\tau^{i_1}(u_1)\dots\tau^{i_r}(u_r)}{\ell_{i_1}^{s_1}\dots\ell_{i_r}^{s_r}}.$$

Observe that $\mathrm{Li}^*_{\mathcal{C}}(u)$ converges in $\mathbb{T}_\Sigma$ if $u\in D''_{\mathcal{C}}$.
Let $\mathcal{C}_i=\binom{U_i}{s_i}$ be a composition array for all $1\leq i\leq r$. We define the addition '+' between composition array $\mathcal{C}_i$ and $\mathcal{C}_j$ by

$$\mathcal{C}_i+\mathcal{C}_j=\binom{U_i\cup U_j}{s_i+s_j}$$

and the operation ',' by

$$(\mathcal{C}_i,\mathcal{C}_j)=\binom{U_i,U_j}{s_i,s_j}$$

Observe that $(\mathcal{C}_1,\dots,\mathcal{C}_r)=\mathcal{C}$.
Now similar to [CM20, Sec. 5.2], we define the set $S$ whose elements are symbols ',' and '+' and the set $S^\times$ containing symbols ',' and '×'. We define the map

$$f: S^{r-1}\to S^{\times r-1}$$

in a way that $f$ fixes the symbol ',' and sending '+' to '×'. As an example, if $v = (v_1, v_2) \in S^2$ so that $v_1 =$ ',' and $v_2 =$ '+' then $f(v) = v' = (v_1', v_2')$ where $v_1' =$ ',' and $v_2' =$ '×'.

We continue with further definition. For any $v = (v_1, \dots, v_{r-1}) \in S^{r-1}$ and any composition array $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$, we define $v(\mathcal{C}) = (\mathcal{C}_1 v_1 \mathcal{C}_2 v_2 \dots v_{r-1} \mathcal{C}_r)$. For any $u = (u_1, \dots, u_r) \in \mathbb{T}^r_\Sigma$, we set $v^\times(u) := (u_1 f(v_1) u_2 f(v_2) \dots u_r f(v_r))$. We also set $\gamma(v)$ to be the number of '+' in $v$. As an example, let $v = (v_1, v_2, v_3)$ be such that $v_1 =$',', $v_2 =$'+' and $v_3 =$ ','. Let

$$\mathcal{C} = \begin{pmatrix} U_1, U_2, U_3, U_4 \\ s_1, s_2, s_3, s_4 \end{pmatrix}$$

and $\mathcal{C}_i = \binom{U_i}{s_i}$ for $i = 1, 2, 3, 4$. Then

$$v(\mathcal{C}) = \begin{pmatrix} U_1, U_2 \cup U_3, U_4 \\ s_1, s_2 + s_3, s_4 \end{pmatrix}$$

and $v^\times(u) = (u_1, u_2 u_3, u_4)$. Note also that $\gamma(v) = 1$.

Observe that by the properties of non-archimedean geometry if $u \in D'_{\mathcal{C}}$ ($D''_{\mathcal{C}}$ resp.) then $v^\times(u)$ is also in $D'_{\mathcal{C}}$ ($D''_{\mathcal{C}}$ resp.) for any $v = (v_1, \dots, v_{r-1}) \in S^{r-1}$. Finally for $r = 1$ we define $S^{r-1} = S^0$ and for any $v \in S^0$ we have $v(\mathcal{C}) := \mathcal{C}$, $v^\times(u) := u$ and $\gamma(v) := 0$.

Now using the inclusion-exclusion principle on the set $\{i_1 \geq i_2 \geq \dots i_r \geq 0\}$ as in [CM20, Prop. 5.2.3] we have that

$$\mathrm{Li}_{\mathcal{C}}(u) = \sum_{v \in S^{r-1}} (-1)^{\gamma(v)} \mathrm{Li}^*_{v(\mathcal{C})}(v^\times(u)). \tag{2.43}$$

THEOREM 2.44 (cf. [CM20, Thm. 5.2.5]). *Let $\mathcal{C}$ be a composition array of depth $r$ as in (1.4). Let $\mathfrak{I}_1$ be the set of indices $i$ such that $U_i \neq \emptyset$ and $\mathfrak{I}_2$ be the set of $i$ such that $U_i = \emptyset$. Then there exist composition arrays $\mathcal{C}_l$ with $\mathrm{wght}(\mathcal{C}) = \mathrm{wght}(\mathcal{C}_l)$ and $\mathrm{dep}(\mathcal{C}_l) \leq r$, elements $a_l \in A$ and $u_l \in K^{perf}(\underline{t}_\Sigma)^{\mathrm{dep}(\mathcal{C}_l)}$ such that*

$$\prod_{i \in \mathfrak{J}_2} \Gamma_{s_i} \prod_{i \in \mathfrak{J}_1} \ell_{r_{s_i}-1}^{q^{r_{s_i}} - s_i} b_{r_{s_i}}(U_i) \zeta_C(\mathcal{C}) = \sum_l a_l (-1)^{\mathrm{dep}(\mathcal{C}_l)-1} \mathrm{Li}^*_{\mathcal{C}_l}(u_l)$$

*where $r_{s_i} \geq 1$ is an integer such that $s_i \leq q^{r_{s_i}}$ for $i \in \mathfrak{I}_1$.*

*Proof.* Using Theorem 2.24 and (2.43) we observe that

$$\begin{aligned} &\prod_{i \in \mathfrak{J}_2} \Gamma_{s_i} \prod_{i \in \mathfrak{J}_1} \ell_{r_{s_i}-1}^{q^{r_{s_i}} - s_i} b_{r_{s_i}}(U_i) \zeta_C(\mathcal{C}) \\ &\quad = \sum_{i \in \mathfrak{J}} a_i \mathrm{Li}_{\mathcal{C}}(u_i) \\ &\quad = \sum_{i \in \mathfrak{J}} a_i \sum_{V \in S^{r-1}} (-1)^{\gamma(V)} \mathrm{Li}^*_{V(\mathcal{C})}(V^\times(u_i)) \\ &\quad = \sum_{i \in \mathfrak{J}} \sum_{V \in S^{r-1}} (-1)^{r-1} a_i (-1)^{\mathrm{dep}(V(\mathcal{C}))-1} \mathrm{Li}^*_{V(\mathcal{C})}(V^\times(u_i)) \end{aligned}$$

where the last line follows from the fact that $\gamma(V) + \deg(V(\mathcal{C})) = r$ for $V \in S^{r-1}$. Thus we can define the tuple $(a_l, \mathcal{C}_l, u_l)$ by

$$(a_l, \mathcal{C}_l, u_l) = ((-1)^{r-1}a_i, V(\mathcal{C}), V^{\times}(\mathcal{C}))$$

for $i \in \mathfrak{I}$ and $V \in S^{r-1}$. □

## 3 Higher Dimensional Drinfeld modules over Tate algebras

### 3.1 Anderson $A[\underline{t}_\Sigma]$-modules

The idea of Drinfeld modules over Tate algebras was first introduced by Anglès, Pellarin and Tavares Ribeiro [APTR16]. In this section, we introduce the concept of Anderson $A[\underline{t}_\Sigma]$-modules which can be seen as higher dimensional Drinfeld modules over Tate algebras.

Definition 3.1. An Anderson $A[\underline{t}_\Sigma]$-module $\phi : A[\underline{t}_\Sigma] \to \mathrm{Mat}_n(\mathbb{T}_\Sigma)[\tau]$ of dimension $n$ defined over $\mathbb{T}_\Sigma$ is an $\mathbb{F}_q[\underline{t}_\Sigma]$-linear homomorphism such that

$$\phi(\theta) = A_0 + A_1\tau + \cdots + A_s\tau^s$$

for some $s$ and $(\theta\,\mathrm{Id}_n - A_0)^n = 0$.
Any Anderson $A[\underline{t}_\Sigma]$-module $\phi$ defines an $A[\underline{t}_\Sigma]$-module action on $\mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ given by

$$\phi(\theta) \cdot f = A_0 f + A_1\tau(f) + \cdots + A_s\tau^s(f), \quad f \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma).$$

We also define the $\mathbb{F}_q[\underline{t}_\Sigma]$-linear homomorphism $\partial_\phi : A[\underline{t}_\Sigma] \to \mathrm{Mat}_n(\mathbb{T}_\Sigma)$ by $\partial_\phi(\theta) = A_0$ and its action on $\mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ by $\partial_\phi(\theta) \cdot f = A_0 f$ for any $f \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$.

*Remark* 3.2. We note that any $t$-module in the sense of Anderson [And86] can be also seen as an Anderson $A[\underline{t}_\Sigma]$-module over $\mathbb{C}_\infty \subset \mathbb{T}_\Sigma$.

Let $\phi_1$ and $\phi_2$ be Anderson $A[\underline{t}_\Sigma]$-modules of dimension $n_1$ and $n_2$ respectively. An Anderson $A[\underline{t}_\Sigma]$-module homomorphism $\varphi : \phi_1 \to \phi_2$ is defined as an element $\varphi \in \mathrm{Mat}_{n_2\times n_1}(\mathbb{T}_\Sigma)[\tau]$ such that

$$\phi_2(\theta)\varphi = \varphi\phi_1(\theta).$$

We now discuss the exponential and logarithm function of some class of Anderson $A[\underline{t}_\Sigma]$-modules. It is important to point out that one can define an exponential and logarithm function corresponding to any Anderson $A[\underline{t}_\Sigma]$-module using methods of Anderson [And86] but concerning the purpose of the present paper, we only analyze special cases.
Let $\phi$ be an Anderson $A[\underline{t}_\Sigma]$-module defined by

$$\phi(\theta) = \theta\,\mathrm{Id}_n + N + E\tau \tag{3.3}$$

for some $N \in \mathrm{Mat}_n(\mathbb{F}_q)$ such that $N^n = 0$ and $E \in \mathrm{Mat}_n(\mathbb{T}_\Sigma)$.

For any square matrices $X_1$ and $X_2$, we first define $[X_1, X_2] := X_1X_2 - X_2X_1$. Then we set $\mathrm{ad}(X_1)^0(X_2) := X_2$ and for $j \geq 1$, $\mathrm{ad}(X_1)^j(X_2) := [X_1, \mathrm{ad}(X_1)^{j-1}(X_2)]$.

LEMMA 3.4. *[Pap, Lem. 3.2.9] Let $Y, N \in \mathrm{Mat}_n(\mathbb{T}_\Sigma)$. Then we have*

$$\mathrm{ad}(N)^j(Y) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} N^m Y N^{j-m}.$$

*Moreover, if $N$ is a nilpotent matrix so that $N^n = 0$, then $\mathrm{ad}(N)^j(Y) = 0$ for $j > 2n-2$.*

*Proof.* Using the definition of $\mathrm{ad}(N)^j(Y)$ and an induction argument imply the above formula. Now assume that $N$ is a nilpotent matrix such that $N^n = 0$. Thus for $j > 2n-2$ and $0 \leq m \leq j$, we have $N^m Y N^{j-m}$ is 0 as either $m \geq n$ or $j - m \geq n$ and therefore we have either $N^m = 0$ or $N^{j-m} = 0$. □

PROPOSITION 3.5. *Let*

$$\exp_\phi = \sum_{i \geq 0} \beta_i \tau^i \in \mathrm{Mat}_n(\mathbb{T}_\Sigma)[[\tau]]$$

*be the infinite series such that $\beta_0 = \mathrm{Id}_n$ and*

$$\exp_\phi \partial_\phi(\theta) = \phi(\theta) \exp_\phi \tag{3.6}$$

*holds in $\mathrm{Mat}_n(\mathbb{T}_\Sigma)[[\tau]]$. Then we have*

$$\beta_{i+1} = \sum_{j=0}^{2n-2} \frac{\mathrm{ad}(N)^j(E\beta_i^{(1)})}{[i+1]^{j+1}}. \tag{3.7}$$

*Moreover $\|\beta_i\|_\infty \leq \|\theta\|_\infty^{-iq^i} \|E\|_\infty^i$ for $i \geq 0$.*

*Proof.* By comparing the coefficients of $\tau$ in (3.6) we see that

$$\beta_{i+1}(\theta^{q^{i+1}} \mathrm{Id}_n + N) = \theta \mathrm{Id}_n \beta_{i+1} + N\beta_{i+1} + E\beta_i^{(1)} \tag{3.8}$$

After some arrangement we see that (3.8) can be rewritten as

$$\beta_{i+1} = \frac{[N, \beta_{i+1}]}{\theta^{q^{i+1}} - \theta} + \frac{E\beta_i^{(1)}}{\theta^{q^{i+1}} - \theta}.$$

Thus a similar calculation as in [AT90, Eq. 2.2.3] implies that the formula for $\beta_{i+1}$ in (3.7) holds. Now we claim that $\|\beta_i\|_\infty \leq \|\theta\|_\infty^{-iq^i} \|E\|_\infty^i$ for $i \geq 0$. We

do induction on $i$. If $i = 0$, then $\beta_0 = \mathrm{Id}_n$ and the claim holds. Assume it also holds for $i$. Then by the induction hypothesis and (3.7) we have

$$\begin{aligned}\|\beta_{i+1}\|_\infty &\leq \max_{0\leq j\leq 2n-2} \frac{\|\beta_i^{(1)}\|_\infty \|E\|_\infty}{[i+1]^{j+1}} \\ &= \|\theta\|_\infty^{-iq^{i+1}} \|E\|_\infty^{i+1} \|\theta\|_\infty^{-q^{i+1}} \\ &= \|\theta\|_\infty^{-(i+1)q^{i+1}} \|E\|_\infty^{i+1}\end{aligned}$$

as desired. □

We call the infinite series $\exp_\phi = \sum \beta_i \tau^i \in \mathrm{Mat}_n(\mathbb{T}_\Sigma)[[\tau]]$ in Proposition 3.5 the exponential series of $\phi$. The exponential series $\exp_\phi$ induces an $\mathbb{F}_q[\underline{t}_\Sigma]$-linear homomorphism $\exp_\phi : \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ defined by

$$\exp_\phi(f) = \sum_{i\geq 0} \beta_i \tau^i(f), \quad f \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma).$$

Moreover by Proposition 3.5 we see that the function $\exp_\phi$ converges everywhere on $\mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$.
Using a similar argument as in the proof of [GP19, Thm. 3.3.2] together with Proposition 3.5, we deduce the following lemma.

LEMMA 3.9 (cf. [GP19, Lem. 3.3.2]). *Let $\phi$ be the Anderson $A[\underline{t}_\Sigma]$-module defined as in (3.3). Then there exists $\varepsilon_\phi > 0$ such that the open ball $\{f \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \mid \|f\|_\infty < \varepsilon_\phi\} \subset \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ can be mapped $\|\cdot\|_\infty$-isometrically by $\exp_\phi$ to itself.*

Let $P_0 = \mathrm{Id}_n$. We define the logarithm series

$$\log_\phi = \sum_{j\geq 0} P_j \tau^j \in \mathrm{Mat}_n(\mathbb{T}_\Sigma)[[\tau]]$$

as the formal inverse of $\exp_\phi$ in $\mathrm{Mat}_n(\mathbb{T}_\Sigma)[[\tau]]$ such that

$$\exp_\phi \log_\phi = \log_\phi \exp_\phi = \mathrm{Id}_n \tag{3.10}$$

and it also satisfies

$$\partial_\phi(\theta) \log_\phi = \log_\phi \phi(\theta). \tag{3.11}$$

The logarithm series $\log_\phi$ induces an $\mathbb{F}_q[\underline{t}_\Sigma]$-linear homomorphism $\log_\phi : \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ defined by

$$\log_\phi(f) = \sum_{i\geq 0} P_i \tau^i(f), \quad f \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$$

which has a finite radius of convergence by Lemma 3.9. It also implies that $\exp_\phi$ is an automorphism with its inverse $\log_\phi$ on $\{f \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \mid \|f\|_\infty < \varepsilon_\phi\} \subset \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$.

Using a similar calculation to [CM19, Eq. (3.2.4)] we can obtain

$$P_{i+1} = -\sum_{j=0}^{2d'-2} \frac{\mathrm{ad}(N)^j(P_i E^{(i)})}{(\theta^{q^{i+1}} - \theta)^{j+1}} \tag{3.12}$$

where $d'$ is a positive integer such that $N^{d'} = 0$ and $N^k \neq 0$ for $k < d'$.

## 3.2 Uniformizability

We now discuss the uniformizability of Anderson $A[\underline{t}_\Sigma]$-modules. We can refer the reader to [APTR16, Sec. 3,6] and [GP19] for more details about uniformizability of Anderson $A[\underline{t}_\Sigma]$-modules of dimension 1.

Definition 3.13. We call an Anderson $A[\underline{t}_\Sigma]$-module $\phi$ of dimension $n$ uniformizable if $\exp_\phi : \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ is a surjective function.

Example 3.14. Assume that $\Sigma = \{1, \dots, s\}$ for some $s \in \mathbb{Z}_{\geq 1}$ and let $f = \sum_{\mu \in \mathbb{Z}_{\geq 1}^{|\Sigma|}} f_\mu \underline{t}_\Sigma^\mu$ be in $\mathbb{T}_\Sigma$. For any positive integer $n$, we define the Anderson $A[\underline{t}_\Sigma]$-module $C^{\otimes n} : A[\underline{t}_\Sigma] \to \mathrm{Mat}_n(\mathbb{T}_\Sigma)[\tau]$, the $n$-th tensor power of the Carlitz module by

$$C^{\otimes n}(\theta) = \theta\,\mathrm{Id}_n + \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 1 & \dots & \dots & 0 \end{bmatrix} \tau.$$

When $n = 1$, we call $C : A[\underline{t}_\Sigma] \to \mathbb{T}_\Sigma[\tau]$ given by $C(\theta) = \theta + \tau$ the Carlitz module (see [Gos96, §3] for details). By [BP20, Sec. 4.3] we know that $\exp_{C^{\otimes n}} : \mathrm{Mat}_{n\times 1}(\mathbb{C}_\infty) \to \mathrm{Mat}_{n\times 1}(\mathbb{C}_\infty)$ is surjective. Indeed we can show that $\exp_{C^{\otimes n}} : \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$ is also surjective as follows. Let

$$y = \big[\sum a_{1,\mu}\underline{t}_\Sigma^\mu, \dots, \sum a_{n,\mu}\underline{t}_\Sigma^\mu\big]^\intercal \in \mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$$

for some $a_{j,\mu} \in \mathbb{C}_\infty$ where $1 \leq j \leq n$ and $\mu \in \mathbb{Z}_{\geq 0}^{|\Sigma|}$. Then for any $\mu$, there exists $x_\mu = [x_{1,\mu}, \dots, x_{n,\mu}]^\intercal \in \mathbb{C}_\infty$ such that $\exp_{C^{\otimes n}}(x_\mu) = [a_{1,\mu}, \dots, a_{n,\mu}]^\intercal$. Note that the entries of $y$ are elements in $\mathbb{T}_\Sigma$. Thus by Lemma 3.9, for any $j$, there exists $N_j \in \mathbb{Z}$ such that $a_{j,\mu}$ is in the radius of convergence of $\log_{C^{\otimes n}}$ for any $s$-tuple $\mu$ whose sum of the entries is bigger than $N_j$. Thus for such tuple $(a_{1,\mu}, \dots, a_{n,\mu})^\intercal$, we can choose $x_\mu = \log_{C^{\otimes n}}((a_{1,\mu}, \dots, a_{n,\mu})^\intercal)$ such that $\|(a_{1,\mu}, \dots, a_{n,\mu})^\intercal\|_\infty = \|x_\mu\|_\infty$ and $\exp_{C^{\otimes n}}(x_\mu) = [a_{1,\mu}, \dots, a_{n,\mu}]^\intercal$. Therefore we guarantee that the element

$$x := \begin{pmatrix} \sum x_{1,\mu}\underline{t}_\Sigma^\mu \\ \vdots \\ \sum x_{n,\mu}\underline{t}_\Sigma^\mu \end{pmatrix}$$

lives in $\mathrm{Mat}_{n\times 1}(\mathbb{T}_\Sigma)$. Furthermore, by the $\mathbb{F}_q[\underline{t}_\Sigma]$-linearity of $\exp_{C^{\otimes n}}$, we have $\exp_{C^{\otimes n}}(x) = y$.

Recall that $\Sigma = \cup_{i=1}^r U_i \subset \mathbb{Z}_{\geq 1}$. We define $\alpha := \prod_{i=1}^r \prod_{j\in U_i}(t_j - \theta)$. Following Demeslay [Dem14, Sec. 4.1], we define the Anderson $A[\underline{t}_\Sigma]$-module $C_\alpha^{\otimes n} : A[\underline{t}_\Sigma] \to \mathrm{Mat}_n(\mathbb{T}_\Sigma)[\tau]$ by

$$C_\alpha^{\otimes n}(\theta) = \theta\,\mathrm{Id}_n + \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \alpha & \dots & \dots & 0 \end{bmatrix} \tau. \tag{3.15}$$

Note that if $\Sigma = \emptyset$, then $C_\alpha^{\otimes n} = C^{\otimes n}$ for any positive integer $n$. As an example, when $r = 1$, $n = 1$ and $\Sigma = U_1 = \{1, \dots, s\}$ for some $s \in \mathbb{Z}_{\geq 1}$, we have that

$$C_\alpha(\theta) = \theta + (t_1 - \theta)\dots(t_s - \theta)\tau, \tag{3.16}$$

and when $r = 1$, $n = 2$ and $U_1 = \{1, 2\}$ we have

$$C_\alpha^{\otimes 2}(\theta) = \begin{bmatrix} \theta & 1 \\ 0 & \theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ (t_1 - \theta)(t_2 - \theta) & 0 \end{bmatrix} \tau.$$

### 3.3 Frobenius Modules

We now investigate the idea of Frobenius modules in our setting. Alert reader might notice that the terminology was also used in [CPY19, Sec. 2.2] and [GP19, Sec. 4].

For any $d, n \in \mathbb{Z}_{\geq 1}$, we define the non-commutative polynomial ring $\mathrm{Mat}_{d\times n}(\mathbb{T}_\Sigma)[\sigma]$ subject to the relation

$$\sigma f = f^{(-1)}\sigma, \quad f \in \mathrm{Mat}_{d\times n}(\mathbb{T}_\Sigma)[\sigma].$$

Moreover for any $M = (M_{ij}) \in \mathrm{Mat}_{d\times n}(\mathbb{T}_\Sigma)$, we define the non-archimedean norm $\|\cdot\|_\sigma$ by

$$\|M\|_\sigma := \sup\{\|M_{ij}\|_\infty\}.$$

Let $\varphi = A_0 + A_1\tau + \dots + A_k\tau^k \in \mathrm{Mat}_{d\times n}(\mathbb{T}_\Sigma)[\tau]$. Then we define the map $* : \mathrm{Mat}_{d\times n}(\mathbb{T}_\Sigma)[\tau] \to \mathrm{Mat}_{n\times d}(\mathbb{T}_\Sigma)[\sigma]$ by

$$\varphi^* = A_0^\intercal + A_1^{\intercal(-1)}\sigma + \dots + A_k^{\intercal(-k)}\sigma^k.$$

We also define the ring $\mathbb{T}_\Sigma[t, \sigma] := \mathbb{T}_\Sigma[\sigma][t]$ with respect to the condition

$$ct = tc, \quad c^{(-1)}\sigma = \sigma c, \quad t\sigma = \sigma t, \quad c \in \mathbb{T}_\Sigma.$$

Definition 3.17. Let $\phi$ be an Anderson $A[\underline{t}_\Sigma]$-module of dimension $n$ defined as in (3.3). We call a $\mathbb{T}_\Sigma[t, \sigma]$-module $H(\phi)$ the Frobenius module corresponding to $\phi$ if it is free of rank $n$ over $\mathbb{T}_\Sigma[\sigma]$ and its $\mathbb{T}_\Sigma[t]$-action is defined by

$$ct \cdot h = ch\phi(\theta)^* = ch(\theta\,\mathrm{Id}_n + N^\intercal + E^{\intercal(-1)}\sigma)$$

for any $h \in H(\phi)$ and $c \in \mathbb{T}_\Sigma$.

Example 3.18. Let $\alpha = \prod_{i\in\Sigma}(t_i - \theta)$. Consider the left $\mathbb{T}_\Sigma[t,\sigma]$-module $H := \mathbb{T}_\Sigma[t]$ whose $\mathbb{T}_\Sigma[\sigma]$-action is given by $c\sigma \cdot f = cf^{(-1)}\alpha^{(-1)^{-1}}(t-\theta)^n$ for any $c \in \mathbb{T}_\Sigma$ and $f \in H$. One can see that $H$ is free of rank $n$ over $\mathbb{T}_\Sigma[\sigma]$ with the basis $\{v_n, \dots, v_1\}$ where $v_i = (t-\theta)^{i-1}$ for $1 \le i \le n$. Let $H(C_\alpha^{\otimes n}) := \mathrm{Mat}_{1\times n}(\mathbb{T}_\Sigma[\sigma])$ be the Frobenius module corresponding to $C_\alpha^{\otimes n}$ whose $\mathbb{T}_\Sigma[t]$-module action is defined by $ct \cdot h = ch(C_\alpha^{\otimes n})^*$ for any $c \in \mathbb{T}_\Sigma$ and $h \in H(C_\alpha^{\otimes n})$. We define $e_i \in \mathrm{Mat}_{1\times n}(\mathbb{T}_\Sigma[\sigma])$ to be the row matrix whose $i$-th coordinate is 1 and the rest is zero. One notes that $\{e_n\}$ forms a $\mathbb{T}_\Sigma[t]$-basis for $H(C_\alpha^{\otimes n})$. There exists a $\mathbb{T}_\Sigma[t]$-module isomorphism $g : H \to H(C_\alpha^{\otimes n})$ given by $g(1) = e_n$. Furthermore $g$ also respects the $\mathbb{T}_\Sigma$-linear action of $\sigma$ and therefore $g$ is a $\mathbb{T}_\Sigma[t,\sigma]$-module isomorphism.

## 4 Anderson $A[\underline{t}_\Sigma]$-Module $G_{\mathcal{C}}$

### 4.1 The Construction of Anderson $A[\underline{t}_\Sigma]$-Module $G$

For the rest of the paper, for any matrix $M \in \mathrm{Mat}_k(\mathbb{T}_\Sigma)$ of the form

$$M = \begin{bmatrix} M[11] & \cdots & M[1r] \\ \vdots & & \vdots \\ M[r1] & \cdots & M[rr] \end{bmatrix} \tag{4.1}$$

such that $M[ij] \in \mathrm{Mat}_{d_i\times d_j}(\mathbb{T}_\Sigma)$, we call $M[ij]$ the $(i,j)$-th block matrix of $M$. We fix a composition array $\mathcal{C}$ defined as in (1.4) and consider $(u_1, \dots, u_r) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$. For any $1 \le j \le r$ we set $d_j := s_j + \cdots + s_r$ and $k := d_1 + \cdots + d_r$. For each $j$, we define the matrices $N_j \in \mathrm{Mat}_{d_j}(\mathbb{F}_q)$ and $N \in \mathrm{Mat}_k(\mathbb{F}_q)$ by:

$$N_j = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad N := \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_r \end{bmatrix}.$$

Recall from §2.1 that $\Sigma \subset \mathbb{Z}_{\ge 1}$ is a union of finite sets given by $\Sigma = \cup_{i=1}^r U_i$ and $\alpha_i = \prod_{j\in U_i}(t_j - \theta)$. Set

$$E[jm] := \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ \prod_{n=m}^r \alpha_n & 0 & \dots & 0 \end{bmatrix} \in \mathrm{Mat}_{d_j}(\mathbb{T}_\Sigma) \text{ if } j = m \tag{4.2}$$

and if $j < m$ we define

$$E[jm] := \begin{bmatrix} & 0 & & \dots & \dots & 0 \\ & \vdots & & \ddots & & \vdots \\ & 0 & & & \ddots & \vdots \\ (-1)^{m-j}\prod_{i=j}^{m-1} u_i \prod_{n=m}^{r} \alpha_n & & 0 & \dots & 0 \end{bmatrix} \in \mathrm{Mat}_{d_j \times d_m}(\mathbb{T}_\Sigma). \tag{4.3}$$

Using (4.2) and (4.3) we define the block matrix $E$ by

$$E := \begin{bmatrix} E[11] & E[12] & \dots & E[1r] \\ & E[22] & \ddots & \vdots \\ & & \ddots & E[(r-1)r] \\ & & & E[rr] \end{bmatrix} \in \mathrm{Mat}_k(\mathbb{T}_\Sigma).$$

Finally we define the Anderson $A[\underline{t}_\Sigma]$-module $G : A[\underline{t}_\Sigma] \to \mathrm{Mat}_k(\mathbb{T}_\Sigma)[\tau]$ by

$$G(\theta) = \theta\,\mathrm{Id}_k + N + E\tau.$$

Set $\mathfrak{a}_j := \prod_{i=j}^{r} \alpha_i$. Then we can also write the Anderson $A[\underline{t}_\Sigma]$-module $G$ of dimension $k$ corresponding to $\mathcal{C}$ and $(u_1, \dots, u_r) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$ as

$$G(\theta) = \begin{bmatrix} C_{\mathfrak{a}_1}^{\otimes d_1}(\theta) & E[12] & \dots & E[1r] \\ & C_{\mathfrak{a}_2}^{\otimes d_2}(\theta) & & E[2r] \\ & & \ddots & \vdots \\ & & & C_{\mathfrak{a}_r}^{\otimes d_r}(\theta) \end{bmatrix}. \tag{4.4}$$

By definition, the Frobenius module $H(G)$ of $G$ carries a $\mathbb{T}_\Sigma[t]$-module structure such that for any $x \in H(G)$ the $t$-action is given by

$$t \cdot x = xG(\theta)^* \tag{4.5}$$

and is free of rank $k$ over $\mathbb{T}_\Sigma[\sigma]$. We now claim that $H(G)$ can be given by the direct sum of $\mathbb{T}_\Sigma[t, \sigma]$-modules

$$H(G) = H(C_{\mathfrak{a}_1}^{\otimes d_1}) \oplus \cdots \oplus H(C_{\mathfrak{a}_r}^{\otimes d_r}) \tag{4.6}$$

and therefore is free of rank $r$ over $\mathbb{T}_\Sigma[t]$. If $r = 1$, then we see that $G = C_{\mathfrak{a}_1}^{\otimes d_1}$ and the Frobenius module $H(G)$ is free of rank 1 over $\mathbb{T}_\Sigma[t]$ by Example 3.18. Suppose that $r = 2$. We consider the short exact sequence

$$0 \longrightarrow H(C_{\mathfrak{a}_1}^{\otimes d_1}) \xrightarrow{\rho_1} H(G) \xrightarrow{\rho_2} H(C_{\mathfrak{a}_2}^{\otimes d_2}) \longrightarrow 0 \tag{4.7}$$

of free $\mathbb{T}_\Sigma[\sigma]$-modules such that for any $x = (x_1, \dots, x_{d_1}) \in H(C^{\otimes d_1})$ and $y = (y_1, \dots, y_{d_2}) \in H(C^{\otimes d_2})$, we have $\rho_1(x) = (x, 0, \dots, 0)$ and $\rho_2(x, y) = y$.

Note that

$$\begin{aligned}\rho_1(t\cdot x)=\rho_1(xC_{\mathfrak{a}_1}^{\otimes d_1}(\theta)^*)&=\rho_1((\theta x_1+x_2,\dots,x_1\mathfrak{a}_1^{(-1)}\sigma+\theta x_{d_1}))\\&=(\theta x_1+x_2,\dots,x_1\mathfrak{a}_1^{(-1)}\sigma+\theta x_{d_1},0,\dots,0)\\&=(x,0,\dots,0)G(\theta)^*\\&=t\cdot\rho_1(x).\end{aligned}$$

Similar calculation can be applied to see that $\rho_2(t\cdot(x,y))=t\cdot\rho_2((x,y))$. Thus the maps $\rho_1$ and $\rho_2$ are compatible with the $t$-action of $\mathbb{T}_\Sigma[t]$-modules $H(C_{\mathfrak{a}_1}^{\otimes d_1})$, $H(C_{\mathfrak{a}_2}^{\otimes d_2})$ and $H(G)$. Therefore the short exact sequence in (4.7) is also a short exact sequence of $\mathbb{T}_\Sigma[t]$-modules. Since $H(C_{\mathfrak{a}_1}^{\otimes d_1})$ and $H(C_{\mathfrak{a}_2}^{\otimes d_2})$ are free over $\mathbb{T}_\Sigma[t]$ of rank 1, $H(G)$ is also free of rank 2 over $\mathbb{T}_\Sigma[t]$ with the basis $\{m_1,m_2\}$ such that under the projection map $\mathrm{proj}_i : H(G)\to H(C_{\mathfrak{a}_i}^{\otimes d_i})$, $\mathrm{proj}_i(m_i)$ is a $\mathbb{T}_\Sigma[t]$-basis for $H(C_{\mathfrak{a}_i}^{\otimes d_i})$ when $i=1,2$.

To show the claim for any $r$, we just replace the short exact sequence in (4.7) with

$$0\longrightarrow H(C_{\mathfrak{a}_1}^{\otimes d_1})\xrightarrow{\rho_1}H(G)\xrightarrow{\rho_2}H(G_{r-1})\longrightarrow 0$$

such that

$$G_{r-1}(\theta)=\begin{bmatrix}C_{\mathfrak{a}_2}^{\otimes d_2}(\theta) & E[23] & \dots & E[2r]\\ & C_{\mathfrak{a}_3}^{\otimes d_3}(\theta) & & E[3r]\\ & & \ddots & \vdots\\ & & & C_{\mathfrak{a}_r}^{\otimes d_r}(\theta)\end{bmatrix}$$

and apply the same argument above.

*Remark* 4.8. We observe from the above discussion and Example 3.18 that if $\{m_1,\dots,m_r\}$ is a $\mathbb{T}_\Sigma[t]$-basis for $H(G)$, then the set $\{(t-\theta)^{d_1-1}\cdot m_1,\dots,m_1,\dots,(t-\theta)^{d_r-1}\cdot m_r,\dots,m_r\}$ is a $\mathbb{T}_\Sigma[\sigma]$-basis for $H(G)$. Let us choose $p_{ij}=(t-\theta)^{d_i-j}\cdot m_i$ for $1\le i\le r$ and $1\le j\le d_i$. Since $H(G)$ is isomorphic to $\mathrm{Mat}_{1\times k}(\mathbb{T}_\Sigma)[\sigma]$ as a $\mathbb{T}_\Sigma[\sigma]$-module, by the change of basis, we can identify each $p_{ij}$ with $e_{ij}=(*,\dots,0,1,0,\dots,*)$ where 1 appears in the $(d_1+\cdots+d_{i-1}+j)$-th place and the other entries are zero.

We let $\mathbf{m}:=[m_1,\dots,m_r]^\intercal\in\mathrm{Mat}_{r\times 1}(H(G))$ be the column vector containing $\mathbb{T}_\Sigma[t]$-basis elements of $H(G)$ and consider the matrix $\Phi\in\mathrm{GL}_r(\mathbb{T}_\Sigma[t])$ defined by

$$\Phi:=\begin{bmatrix}\frac{(t-\theta)^{d_1}}{\mathfrak{a}_1^{(-1)}} & & & \\ \frac{u_1^{(-1)}(t-\theta)^{d_1}}{\mathfrak{a}_1^{(-1)}} & \frac{(t-\theta)^{d_2}}{\mathfrak{a}_2^{(-1)}} & & \\ & \frac{u_2^{(-1)}(t-\theta)^{d_2}}{\mathfrak{a}_2^{(-1)}} & \ddots & \\ & & \ddots & \\ & & \frac{u_{r-1}^{(-1)}(t-\theta)^{d_{r-1}}}{\mathfrak{a}_{r-1}^{(-1)}} & \frac{(t-\theta)^{d_r}}{\mathfrak{a}_r^{(-1)}}\end{bmatrix}. \tag{4.9}$$

PROPOSITION 4.10. *We have* $\sigma\mathbf{m} = \Phi\mathbf{m}$.

*Proof.* Let $p_{ij}$ and $e_{ij}$ be as in Remark 4.8. Set $c \cdot m_j := 0 \in H(G)$ for any $c \in \mathbb{T}_\Sigma[t]$ and $j \leq 0$. Claim that for any $1 \leq i \leq r$ we have

$$\frac{u_{i-1}^{(-1)}(t-\theta)^{d_{i-1}}}{\mathfrak{a}_{i-1}^{(-1)}} \cdot m_{i-1} + \frac{(t-\theta)^{d_i}}{\mathfrak{a}_i^{(-1)}} \cdot m_i = \sigma m_i. \tag{4.11}$$

We do induction on $i$. First we see that

$$\begin{aligned} t \cdot p_{11} &= t \cdot e_{11} \\ &= (1, 0, \ldots, 0)G(\theta)^* \\ &= (\theta, \ldots, \mathfrak{a}_1^{(-1)}\sigma, 0, \ldots, 0) \\ &= \theta p_{11} + \mathfrak{a}_1^{(-1)}\sigma p_{1d_1} \\ &= \theta p_{11} + \mathfrak{a}_1^{(-1)}\sigma m_1 \end{aligned} \tag{4.12}$$

where $\mathfrak{a}_1^{(-1)}\sigma$ appears in the $d_1$-th place. Since $p_{11} = (t-\theta)^{d_1-1} \cdot m_1$, it follows from (4.12) that $\sigma m_1 = \frac{(t-\theta)^{d_1}}{\mathfrak{a}_1^{(-1)}} \cdot m_1$. Assume that the equality in (4.11) holds for $i-1$. By using the induction hypothesis and the $t$-action defined in (4.5) we obtain

$$\begin{aligned} &t \cdot p_{i1} \\ &= t \cdot e_{i1} \\ &= \theta p_{i1} + \mathfrak{a}_i^{(-1)}\Big((-1)^{i-1}u_1^{(-1)}\ldots u_{i-1}^{(-1)}\sigma \cdot p_{1d_1} + (-1)^{i-2}u_2^{(-1)}\ldots u_{i-1}^{(-1)}\sigma \cdot p_{2d_2} \\ &\quad + \cdots + (-1)u_{i-1}^{(-1)}\sigma \cdot p_{(i-1)d_{i-1}}\Big) + \mathfrak{a}_i^{(-1)}\sigma p_{id_i} \\ &= \theta p_{i1} + \mathfrak{a}_i^{(-1)}\Big((-1)^{i-1}u_1^{(-1)}\ldots u_{i-1}^{(-1)}\frac{(t-\theta)^{d_1}}{\mathfrak{a}_1^{(-1)}} \cdot m_1 \\ &\quad + (-1)^{i-2}u_1^{(-1)}\ldots u_{i-1}^{(-1)}\frac{(t-\theta)^{d_1}}{\mathfrak{a}_1^{(-1)}} \cdot m_1 + (-1)^{i-3}u_2^{(-1)}\ldots u_{i-1}^{(-1)}\frac{(t-\theta)^{d_2}}{\mathfrak{a}_2^{(-1)}} \cdot m_2 \\ &\quad + (-1)^{i-2}u_2^{(-1)}\ldots u_{i-1}^{(-1)}\frac{(t-\theta)^{d_2}}{\mathfrak{a}_2^{(-1)}} \cdot m_2 \\ &\quad + \cdots + \frac{(-1)^{i-(i-2)}u_{i-2}^{(-1)}u_{i-1}^{(-1)}(t-\theta)^{d_{i-2}}}{\mathfrak{a}_{i-2}^{(-1)}} \cdot m_{i-2} \\ &\quad + \frac{(-1)u_{i-1}^{(-1)}(t-\theta)^{d_{i-1}}}{\mathfrak{a}_{i-1}^{(-1)}} \cdot m_{i-1}\Big) + \mathfrak{a}_i^{(-1)}\sigma p_{id_i} \\ &= \theta p_{i1} + \mathfrak{a}_i^{(-1)}\sigma m_i - \frac{u_{i-1}^{(-1)}(t-\theta)^{d_{i-1}}}{\alpha_{i-1}^{(-1)}} \cdot m_{i-1}. \end{aligned} \tag{4.13}$$

Since $p_{i1} = (t-\theta)^{d_i-1} \cdot p_{id_i} = (t-\theta)^{d_i-1} \cdot m_i$, the claim follows from the calculation in (4.13). Thus the definition of the matrix $\Phi$ and $\mathbf{m}$ together with (4.11) imply the proposition. □

## 4.2 Rigid Analytic Trivialization of $G$

In this section, we introduce the idea of rigid analytic triviality for Anderson $A[\underline{t}_\Sigma]$-module $G$ of dimension $k$ defined as in (4.4). We start with explaining necessary background and at the end, we relate them to the uniformizability of Anderson $A[\underline{t}_\Sigma]$-modules.
Let $H(G)$ be the Frobenius module corresponding to $G$ which is free of rank $r$ over $\mathbb{T}_\Sigma[t]$ and $\mathbf{m} = [m_1, \dots, m_r]^\intercal$ be the column vector consisting of $\mathbb{T}_\Sigma[t]$-basis elements of $H(G)$. Then for any $h = [h_1, \dots, h_r] \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$, we define the map

$$\iota : \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t]) \to \mathrm{Mat}_{1\times k}(\mathbb{T}_\Sigma[\sigma])$$

by $\iota(h) = h \cdot \mathbf{m} = h_1 \cdot m_1 + \cdots + h_r \cdot m_r$ where the action $\cdot$ is given by the $\mathbb{T}_\Sigma[t]$-action on $H(G)$.

Lemma 4.14. *We have*

*(i) For any $h = [h_1, \dots, h_r] \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$, we have $\iota(h^{(-1)}\Phi) = \sigma\iota(h)$.*

*(ii) For all $h = [h_1, \dots, h_r] \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$, we have $\iota(th) = \iota(h)G(\theta)^*$.*

*Proof.* To prove the first part we observe by Proposition 4.10 that

$$\sigma\iota(h) = \sigma[h_1, \dots, h_r] \cdot \mathbf{m} = [h_1^{(-1)}, \dots, h_r^{(-1)}]\sigma \cdot \mathbf{m} = h^{(-1)}\Phi \cdot \mathbf{m} = \iota(h^{(-1)}\Phi).$$

On the other hand, we have by (4.5) that

$$\iota(th) = t[h_1, \dots, h_r] \cdot \mathbf{m} = t \cdot \iota(h) = \iota(h)G(\theta)^*$$

which proves the second part. □

We call the tuple $(\iota, \Phi)$ a $t$-frame of $G$.
Before we state the definition of rigid analytic triviality, for any $f \in \mathbb{T}_\Sigma$, we define the ring $\mathbb{T}_\Sigma\{t/f\}$ by

$$\mathbb{T}_\Sigma\{t/f\} := \left\{ \sum_{i\geq 0} a_i t^i \in \mathbb{T}_\Sigma[[t]] \;\middle|\; \|f\|_\infty^i \|a_i\|_\infty \to 0 \right\}.$$

Moreover, we define the norm $\|g\|_f = \|\sum a_i t^i\|_f = \sup\{\|f\|_\infty^i \|a_i\|_\infty\}$ so that the ring $\mathbb{T}_\Sigma\{t/f\}$ is complete with respect to the norm $\|\cdot\|_f$. Furthermore, for any $M = (M_{ij}) \in \mathrm{Mat}_{n\times l}(\mathbb{T}_\Sigma\{t/f\})$ we set $\|M\|_f := \max_{i,j}\|M_{ij}\|_f$.

DEFINITION 4.15. Let $(\iota, \Phi)$ be a $t$-frame of $G$ defined as in (4.4) and let $\Psi \in \mathrm{GL}_r(\mathbb{T}_\Sigma\{t/\theta\})$ be a matrix such that

$$\Psi^{(-1)} = \Phi\Psi.$$

Then we call $(\iota, \Phi, \Psi)$ a rigid analytic trivialization of $G$ and say $G$ is rigid analytically trivial.

*Remark* 4.16. We recall Example 3.18. Note that the $\mathbb{T}_\Sigma[\sigma]$-action on the $\mathbb{T}_\Sigma[t]$-basis $v_1$ of $H(C_\alpha^{\otimes n})$ is represented by $\Phi = \frac{(t-\theta)^n}{\alpha^{(-1)}}$. By (2.2) and (2.6) we see that $\Psi = \omega_\alpha \Omega(t)^n$ satisfies the equality $\Psi^{(-1)} = \Phi\Psi$. Moreover since $\Omega(t) \in \mathbb{T}_\Sigma\{t/\theta\}$ by [GP19, Cor. 6.2.10] so is $\Psi$. Thus $C_\alpha^{\otimes n}$ is rigid analytically trivial.

Now we finish this section with a fundamental theorem which will be useful to prove Theorem 1.12.

THEOREM 4.17 (cf. [GP19, Thm. 4.5.5]). *Let $\phi$ be an Anderson $A[\underline{t}_\Sigma]$-module defined in (3.3). If $\phi$ has a rigid analytic trivialization $(\iota, \Phi, \Psi)$, then $\phi$ is uniformizable.*

*Proof.* See Appendix A. □

As an immediate corollary of Remark 4.16 and Theorem 4.17, we deduce the following.

COROLLARY 4.18. *The Anderson $A[\underline{t}_\Sigma]$-module $C_\alpha^{\otimes n}$ defined in (3.15) is uniformizable.*

We now discuss the rigid analytic triviality of the Anderson $A[\underline{t}_\Sigma]$-module $G$ whose corresponding Frobenius module $H(G)$ given as in (4.6). We prove the following proposition.

PROPOSITION 4.19. *Let $G$ be the Anderson $A[\underline{t}_\Sigma]$-module defined as in (4.4) corresponding $\mathcal{C}$ and the tuple $u = (u_1, \dots, u_r) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$ such that $\|u_i\|_\infty < q^{\frac{s_i q - |U_i|}{q-1}}$ for $1 \le i \le r$. Then $G$ is rigid analytically trivial. In particular, the exponential function $\exp_G$ is surjective.*

*Proof.* Let $\Phi$ be given as in (4.9). For any $1 \le l < j \le r+1$ and $u_{l,j} = (u_l, \dots, u_{j-1}) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$ such that $\|u_i\|_\infty < q^{\frac{s_i q - |U_i|}{q-1}}$ for $l \le i \le j-1$, set

$$\begin{aligned} L_{u_{l,j}}(t) :&= \sum_{i_l > \cdots > i_{j-1} \ge 0} (\omega_{U_l} \Omega^{s_l}(t) u_l)^{(i_l)} \dots (\omega_{U_{j-1}} \Omega^{s_{j-1}}(t) u_{j-1})^{(i_{j-1})} \\ &= \Omega^{s_l + \cdots + s_{j-1}}(t) \prod_{i=l}^{j-1} \omega_{U_i} \times \\ &\quad \sum_{i_l > \cdots > i_{j-1} \ge 0} \frac{u_l^{(i_l)} \dots u_{j-1}^{(i_{j-1})} b_{i_l}(U_l) \dots b_{i_{j-1}}(U_{j-1})}{((t-\theta^q)\dots(t-\theta^{q^{i_l}}))^{s_l} \dots ((t-\theta^q)\dots(t-\theta^{q^{i_{j-1}}}))^{s_{j-1}}}. \end{aligned} \tag{4.20}$$

Consider the matrix $\Psi \in \mathrm{Mat}_r(\mathbb{T}_{\Sigma,t})$ by

$$\Psi = \begin{bmatrix} \Omega(t)^{d_1}\prod_{i=1}^r \omega_{U_i} & & & & \\ L_{u_{1,2}}(t)\Omega(t)^{d_2}\prod_{i=2}^r \omega_{U_i} & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & & \\ L_{u_{1,r}}(t)\Omega(t)^{d_r}\omega_{U_r} & & \dots & L_{u_{r-1,r}}(t)\Omega(t)^{d_r}\omega_{U_r} & \Omega(t)^{d_r}\omega_{U_r} \end{bmatrix}.$$

Since $\Omega(t)$ and $\omega_{U_i}$ are invertible in $\mathbb{T}_{\Sigma,t}$, by Theorem 2.40, we see that $\Psi \in \mathrm{GL}_r(\mathbb{T}_\Sigma\{t/\theta\})$. Thus the proposition follows from the same argument of the proof of Proposition 2.32. □

### 4.3 Analysis on the Coefficients of The Logarithm Function

We continue with the notation from §4.1 and furthermore for any $1 \le i \le r$, we define $E_i \in \mathrm{Mat}_k(\mathbb{T}_\Sigma)$ so that its $(i,i)$-th block matrix is $E[ii]$ and the rest is zero matrix.
We denote the logarithm function $\log_G$ by $\log_G = \sum_{i\ge 0} P_i\tau^i$ where

$$P_0 = \mathrm{Id}_k, \quad P_i = \begin{bmatrix} P_i[11] & \cdots & P_i[1r] \\ \vdots & & \vdots \\ P_i[r1] & \cdots & P_i[rr] \end{bmatrix} \in \mathrm{Mat}_k(\mathbb{T}_\Sigma), \quad P_i[jk] \in \mathrm{Mat}_{d_j\times d_k}(\mathbb{T}_\Sigma).$$

Proposition 4.21 (cf. [CM19, Prop. 3.2.1]). *We have $P_i[lm] = 0$ for $l > m$. For $l \le m$, we denote the lower most right corner of $P_i[lm]$ by $y_i[lm]$. Then $y_i[lm] = \frac{\prod_{j=m}^r b_i(U_j)}{L_i^{d_m}}$ if $l = m$ and when $l < m$, we have*

$$y_i[lm] = (-1)^{m-l}\times \sum_{0\le i_l\le\cdots\le i_{m-1}<i} \frac{u_l^{(i_l)}\dots u_{m-1}^{(i_{m-1})} b_{i_l}(U_l)\dots b_{i_{m-1}}(U_{m-1})\prod_{j=m}^r b_i(U_j)}{\ell_{i_l}^{s_l}\dots\ell_{i_{m-1}}^{s_{m-1}}\ell_i^{d_m}}. \tag{4.22}$$

*Proof.* We follow the ideas of Chang and Mishiba in [CM19, Prop. 3.2.1]. By (3.12) we have

$$P_{i+1} = -\sum_{j=0}^{2d_1-2} \frac{\mathrm{ad}(N)^j(P_iE^{(i)})}{(\theta^{q^{i+1}}-\theta)^{j+1}} \tag{4.23}$$

similar to the identity (2.1.3) in [AT90]. Note that the upper bound of $j$ in (4.23) is determined by the fact that $N^{d_1} = 0$. Moreover, we have by (4.23) and the definition of the matrix $E$ and $N$ that

$$P_i[lm] = 0 \quad \text{for } l > m. \tag{4.24}$$

Observe that for a block matrix $Y \in \mathrm{Mat}_k(\mathbb{T}_\Sigma)$ of the form (4.1), if $y$ is the element in the lower most right corner of the $(l,m)$-th block matrix, then $E_l^\intercal Y E_m$ has all entries zero except the upper most left corner of the $(l,m)$-th block matrix which is $y\prod_{j=l}^r \alpha_j \prod_{j=m}^r \alpha_j$.

Note that $E_l^\intercal N = 0$. On the other hand, for any $j$, we can write $\mathrm{ad}(N)^j(P_iE^{(i)}) = NM + (-1)^J P_iE^{(i)}N^j$ for some $M \in \mathrm{Mat}_k(\mathbb{T}_\Sigma)$. Thus, using [CM19, Eq. (3.2.6)] we see that

$$E_l^\intercal P_{i+1}E_m = \frac{E_l^\intercal P_iE^{(i)}N^{d_m-1}E_m}{(\theta-\theta^{q^{i+1}})^{d_m}}. \tag{4.25}$$

Observe that $E^{(i)}N^{j-1}E_m = 0$ if $j \neq d_m - 1$ and $E^{(i)}N^{d_m-1}E_m$ has all zero columns except the $(d_1+\cdots+d_{m-1}+1)$-st to $(d_1+\cdots+d_m)$-th columns which are of the form

$$\prod_{j=m}^r \alpha_j[E[1m]^{(i)},\dots,E[mm]^{(i)},\dots,0]^\intercal.$$

Moreover $E_l^\intercal P_i$ has all zero rows except the $(d_1+\cdots+d_{l-1}+1)$-th row which is of the form $\prod_{j=l}^r \alpha_j[*,\dots,y_i[l1],\dots,y_i[l2],\dots,y_i[lr]]$ where $y_i[lw]$ appears in the $(d_1+\cdots+d_w)$-th place for $1 \leq w \leq r$.

Now comparing the upper most left corner of the $(l,m)$-th block matrix of both sides of (4.25) by using (4.24) we observe that if $l = m$ we have

$$\ell_{i+1}y_{i+1}[lm] = \prod_{j=l}^r \alpha_j^{(i)} y_i[lm]\ell_i^{d_m} \tag{4.26}$$

and if $m > l$, then we obtain

$$\ell_{i+1}^{d_m}y_{i+1}[lm] = \prod_{j=m}^r \alpha_j^{(i)}\left(\ell_i^{d_m}\sum_{n=l}^{m-1} y_i[ln](-1)^{m-n}\prod_{e=n}^{m-1}u_e^{(i)} + \ell_i^{d_m}y_i[lm]\right). \tag{4.27}$$

We apply induction on $i$. Note that if $m = l$ then the first part of the proposition can be easily shown by using (4.26) as $y_0[lm] = 1$ in this case.

When $m > l$ we assume the proposition holds for $y_i[ln]$ where $l \leq n < m$ and

$i \geq 0$. Then using (4.27) we have that

$$
\begin{aligned}
&\ell_{i+1}^{d_m} y_{i+1}[lm] \\
&= \ell_i^{d_m} \sum_{n=l+1}^{m-1} (-1)^{n-l} \times \\
&\qquad \sum_{0 \leq i_l \leq \cdots \leq i_{n-1} < i} \frac{\prod_{j=l}^{n-1} u_j^{(i_j)} b_{i_j}(U_j) \prod_{k=n}^{r} b_i(U_k) \alpha_k^{(i)}}{\ell_{i_l}^{s_l} \dots \ell_{i_{n-1}}^{s_{n-1}} \ell_i^{d_n}} (-1)^{m-n} \prod_{e=n}^{m-1} u_e^{(i)} \\
&\quad + \ell_i^{d_m} \frac{1}{\ell_i^{d_l}} (-1)^{m-l} u_l^{(i)} \dots u_{m-1}^{(i)} b_i(U_m) \dots b_i(U_r) \prod_{k=l}^{m-1} b_i(U_l) \prod_{j=m}^{r} \alpha_j^{(i)} \\
&\qquad + \ell_i^{d_m} y_i[lm] \prod_{j=m}^{r} \alpha_j^{(i)} \\
&= (-1)^{m-l} \times \\
&\qquad \sum_{n=l+1}^{m-1} \sum_{0 \leq i_l \leq \cdots \leq i_{n-1} < i} \frac{\prod_{j=l}^{n-1} u_j^{(i_j)} b_{i_j}(U_j) u_n^{(i)} \dots u_{m-1}^{(i)} b_{i+1}(U_n) \dots b_{i+1}(U_r)}{\ell_{i_l}^{s_l} \dots \ell_{i_{n-1}}^{s_{n-1}} \ell_i^{s_n} \dots \ell_i^{s_{m-1}}} \\
&\quad + (-1)^{m-l} \frac{b_{i+1}(U_m) \dots b_{i+1}(U_r) \prod_{j=l}^{m-1} u_j^{(i)} b_i(U_j)}{\ell_i^{s_l} \dots \ell_i^{s_{m-1}}} + \ell_i^{d_m} y_i[lm] \prod_{j=m}^{r} \alpha_j^{(i)} \\
&= (-1)^{m-l} \sum_{\substack{0 \leq i_l \leq \cdots \leq i_{m-1} \\ i_{m-1} = i}} \frac{b_{i+1}(U_m) \dots b_{i+1}(U_r) \prod_{j=l}^{m-1} u_j^{(i_j)} b_{i_j}(U_j)}{\ell_{i_l}^{s_l} \dots \ell_{i_{m-1}}^{s_{m-1}}} \\
&\qquad + \ell_i^{d_m} y_i[lm] \prod_{j=m}^{r} \alpha_j^{(i)}.
\end{aligned}
\tag{4.28}
$$

Since $y_0[lm] = 0$ we obtain

$$
\begin{aligned}
&\prod_{j=m}^{r} \alpha_j^{(i)} \ell_i^{d_m} y_i[lm] \\
&= (-1)^{m-l} \sum_{h=0}^{i-1} \sum_{\substack{0 \leq i_l \leq \cdots \leq i_{m-1} \\ i_{m-1} = h}} \frac{b_i(U_m) \dots b_i(U_r) \prod_{j=m}^{r} \alpha_j^{(i)} \prod_{j=l}^{m-1} u_j^{(i_j)} b_{i_j}(U_j)}{\ell_{i_l}^{s_l} \dots \ell_{i_{m-1}}^{s_{m-1}}}
\end{aligned}
$$

This also implies that

$$\prod_{j=m}^{r}\alpha_j^{(i)}\ell_i^{d_m}y_i[lm]=(-1)^{m-l}\times \sum_{0\le i_l\le\cdots\le i_{m-1}<i}\frac{u_l^{(i_l)}\dots u_{m-1}^{(i_{m-1})}b_{i_l}(U_l)\dots b_{i_{m-1}}(U_{m-1})b_{i+1}(U_m)\dots b_{i+1}(U_r)}{\ell_{i_l}^{s_l}\dots\ell_{i_{m-1}}^{s_{m-1}}}. \tag{4.29}$$

Thus the proposition follows from combining (4.28) and (4.29). □

LEMMA 4.30 (cf. [CM20, Lem. 4.2.1]). *Let $\mathcal{C}$ be a composition array as in (1.4) and $u=(u_1,\dots,u_r)\in(\mathbb{T}_\Sigma\setminus\{0\})^r$. Let also $n_l=|U_l|$ be the nonnegative integer for $1\le l\le r$. If $\|u_l\|_\infty\le q^{\frac{s_lq-n_l}{q-1}}$ for each $1\le l<r$, then*

$$\|P_iN^{d_l-j}E_l\|_\infty\le q^{(d_l-j)q^i-(d_lq^i-d_1)\frac{q}{q-1}+(n_l+\cdots+n_r)\frac{q^i}{q-1}} \tag{4.31}$$

*for each $i,j,k$ where $i\ge0$, $1\le l\le r$, and $1\le j\le d_l$.*

*Proof.* We note that the matrix $P_iN^{d_l-j}E_l$ has all zero columns except the $(d_1+\cdots+d_{l-1}+1)$-th column which is $\prod_{j=l}^r\alpha_j$ multiple of the $(d_1+\cdots+d_{l-1}+j)$-th column of $P_i$. If $i=0$ then the lemma holds. Assume by induction that the inequality (4.31) holds for $i$ and we show that it also holds for $i+1$. By (4.23) we have that

$$P_{i+1}N^{d_l-j}E_l=-\sum_{m=0}^{2d_1-2}\frac{1}{(\theta^{q^{i+1}}-\theta)^{m+1}}\sum_{n=0}^{m}(-1)^n\binom{m}{n}N^{m-n}P_iE^{(i)}N^{n+d_l-j}E_l. \tag{4.32}$$

We observe that $E^{(i)}N^{n+d_l-j}E_l=0$ for $n\ne j-1$. Moreover by the definition of $N$ we have that $N^{m-n}=0$ for $m-n\ge d_1$. Therefore using the definition of the matrices $E_l$ and $E$ we have

$$\begin{aligned}P_{i+1}N^{d_l-j}E_l&=\sum_{m=j-1}^{d_1+j-2}\frac{(-1)^j}{(\theta^{q^{i+1}}-\theta)^{m+1}}\binom{m}{j-1}N^{m-j+1}P_iE^{(i)}N^{d_l-1}E_l\\&=\sum_{m=j-1}^{d_1+j-2}\frac{(-1)^j}{(\theta^{q^{i+1}}-\theta)^{m+1}}\binom{m}{j-1}N^{m-j+1}\times\\&\qquad\sum_{n=1}^{l}(-1)^{l-n}P'_{i,l,n}\prod_{n\le e\le l-1}u_e^{(i)}\prod_{h=l}^{r}\alpha_h\alpha_h^{(i)}\end{aligned}$$

where we define the matrix $P'_{i,l,n}$ as the matrix whose $(d_1+\cdots+d_{l-1}+1)$-th column is the $(d_1+\cdots+d_{n-1}+d_n)$-th column of $P_i$ and all the other columns

are zero. Thus taking $l = n$ and $j = d_n$ in the inequality (4.31) and using induction hypothesis we see that

$$\begin{aligned}
&\|P'_{i,l,n} \prod_{n\le e\le l-1} u_e^{(i)} \prod_{h=l}^{r} \alpha_h \alpha_h^{(i)}\|_\infty \\
&\le q^{-(d_n q^i - d_1)\frac{q}{q-1} + \frac{q^i(n_n+\cdots+n_r)}{q-1}} \prod_{n\le e\le l-1} q^{\frac{q^i(s_e q - n_e)}{q-1}} \prod_{h=l}^{r} \|\alpha_h^{(i)}\|_\infty \\
&\le q^{-(d_n q^i - d_1)\frac{q}{q-1} + \frac{q^i(n_n+\cdots+n_r)}{q-1}} q^{(d_n - d_l)\frac{q^{i+1}}{q-1}} q^{-\frac{q^i(n_n+\cdots+n_{l-1})}{q-1}} q^{(n_l+\cdots+n_r)q^i} \\
&\le q^{-(d_l q^i - d_1)\frac{q}{q-1} + \frac{q^i(n_l+\cdots+n_r)}{q-1}} q^{(n_l+\cdots+n_r)q^i} \\
&\le q^{-(d_l q^i - d_1)\frac{q}{q-1} + \frac{q^{i+1}(n_l+\cdots+n_r)}{q-1}}.
\end{aligned} \tag{4.33}$$

Thus using (4.32) and (4.33) we see that

$$\begin{aligned}
\|P_{i+1} N^{d_l - j} E_l\|_\infty &\le \max_{j-1\le k\le d_1+j-2} q^{-(k+1)q^{i+1} - (d_l q^i - d_1)\frac{q}{q-1}} q^{\frac{(n_l+\cdots+n_r)q^{i+1}}{q-1}} \\
&= q^{-jq^{i+1} - (d_l q^i - d_1)\frac{q}{q-1}} q^{\frac{(n_l+\cdots+n_r)q^{i+1}}{q-1}} \\
&= q^{(d_l - j)q^{i+1} - (d_l q^{i+1} - d_1)\frac{q}{q-1} + (n_l+\cdots+n_r)\frac{q^{i+1}}{q-1}}
\end{aligned}$$

which concludes the proof. □

We continue with the notation in the statement of Lemma 4.30.

PROPOSITION 4.34 (cf. [CM20, Prop. 4.2.2]). *Let* $\|u_l\|_\infty \le q^{\frac{s_l q - n_l}{q-1}}$ *for* $1 \le l < r$ *and* $x = (x_i) \in \mathbb{T}_\Sigma^k$ *be a point such that* $\|x_{d_1+\cdots+d_{l-1}+j}\|_\infty < q^{-(d_l - j) + \frac{d_l q}{q-1} - \frac{(n_l+\cdots+n_r)}{q-1}}$. *Then* $\log_G$ *converges at* $x$ *in* $\mathbb{T}_\Sigma^k$.

*Proof.* The proof follows from the standard estimation in non-archimedean analysis. In particular since $\|E_l\|_\infty = q^{(n_l+\cdots+n_r)}$, by Lemma 4.30 we have

$$\begin{aligned}
&\|P_i x^{(i)}\|_\infty \\
&\le \max_{j,l}\{q^{-(n_l+\cdots+n_r) + (d_l - j)q^i - (d_l q^i - d_1)\frac{q}{q-1} + (n_l+\cdots+n_r)\frac{q^i}{q-1}} \|x_{d_1+\cdots+d_{l-1}+j}\|_\infty^{q^i}\} \\
&= \max_{j,l}\left\{q^{-(n_l+\cdots+n_r) + \frac{d_1 q}{q-1}} \left(\|x_{d_1+\cdots+d_{l-1}+j}\|_\infty / q^{-(d_l - j) + \frac{d_l q}{q-1} - \frac{(n_l+\cdots+n_r)}{q-1}}\right)^{q^i}\right\}.
\end{aligned} \tag{4.35}$$

But by the assumption on the element $x$ when $i$ goes to infinity, the last term in (4.35) approaches to 0 and thus this proves the statement in the proposition. □

The special point $v_{\mathcal{C},u} \in \mathbb{T}_\Sigma^k$ corresponding to a composition array $\mathcal{C}$ and $u = (u_1, \dots, u_r) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$ is defined by

$$v = v_{\mathcal{C},u} := [0, \dots, 0, (-1)^{r-1}u_1 \dots u_r, 0, \dots, 0, (-1)^{r-2}u_2 \dots u_r, 0, \dots, 0, u_r]^\intercal \tag{4.36}$$

where the entry $(-1)^{r-j}u_j \dots u_r$ for $1 \le j \le r$ appears in $(d_1 + \dots + d_j)$-th place.

We continue with some notation. For any composition array

$$\mathcal{C} = \begin{pmatrix} U_j, \dots, U_i \\ s_j, \dots, s_i \end{pmatrix}$$

where $1 \le j \le i$, we define the composition array $\tilde{\mathcal{C}}$ by

$$\tilde{\mathcal{C}} := \begin{pmatrix} U_i, U_{i-1}, \dots, U_j \\ s_i, s_{i-1}, \dots, s_j \end{pmatrix}.$$

Furthermore, for any $u = (u_j, \dots, u_i) \in (\mathbb{T}_\Sigma \setminus \{0\})^{j-i+1}$, we define $\tilde{u} := (u_i, u_{i-1}, \dots, u_j) \in (\mathbb{T}_\Sigma \setminus \{0\})^{j-i+1}$.

THEOREM 4.37 (cf. [CM19, Thm. 3.3.3]). *Let $\mathcal{C}$ be a composition array of depth $r$ defined as in (1.4) and $u = (u_1, \dots, u_r) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$ be such that $\tilde{u} \in D''_{\tilde{\mathcal{C}}}$. Let $G$ and $v$ be defined as in (4.4) and (4.36) respectively corresponding to $\mathcal{C}$ and $u$. Then $\log_G$ converges at $v$. Moreover for any $1 \le l \le r$, the element in $(d_1 + \dots + d_l)$-th place of $\log_G(v)$ is equal to $(-1)^{r-l}\operatorname{Li}^*_{\tilde{\mathcal{C}}_l}(\tilde{u}_l)$ where $\mathcal{C}_l = \binom{U_l, \dots, U_r}{s_l, \dots, s_r}$ and $u_l = (u_l, \dots, u_r)$.*

*Proof.* We follow the technique in [CM19, Thm. 3.3.3]. By the assumption on the element $u$, the norm of the $(d_1 + \dots + d_{l-1} + d_l)$-th coordinate of $v$ is

$$\|(-1)^{r-l}u_l \dots u_r\|_\infty < q^{\frac{s_l q - n_l}{q-1}} \dots q^{\frac{s_r q - n_r}{q-1}} = q^{\frac{d_l q}{q-1} - \frac{(n_l + \dots + n_r)}{q-1}}.$$

Then by Proposition 4.34 we see that $\log_G$ converges at $v$. Using Proposition 4.21 and the definition of $v$ we see that the $(d_1 + \dots + d_l)$-th coordinate of

$\log_G(v)$ is

$$\begin{aligned}
&\sum_{i\geq 0}\sum_{m=l}^{r} y_i[lm](-1)^{r-m}u_m^{(i)}\dots u_r^{(i)}\\
&=\sum_{i\geq 0}\frac{(-1)^{r-l}\prod_{j=l}^{r} b_i(U_j)u_l^{(i)}\dots u_r^{(i)}}{\ell_i^{d_l}}\\
&\quad+\sum_{i\geq 0}\sum_{m=l+1}^{r}(-1)^{m-l}\times\\
&\qquad\sum_{0\leq i_l\leq\dots\leq i_{m-1}<i}\frac{\prod_{j=l}^{m-1}u_j^{(i_j)}b_{i_j}(U_j)}{\ell_{i_l}^{s_l}\dots\ell_{i_{m-1}}^{s_{m-1}}\ell_i^{d_m}}(-1)^{r-m}u_m^{(i)}\dots u_r^{(i)}\prod_{j=m}^{r}b_i(U_j)\\
&=(-1)^{r-l}\Bigg(\sum_{i\geq 0}\frac{\prod_{j=l}^{r}u_j^{(i)}b_i(U_j)}{\ell_i^{s_l}\dots\ell_i^{s_r}}\\
&\qquad+\sum_{i\geq 0}\sum_{m=l+1}^{r}\sum_{0\leq i_l\leq\dots\leq i_{m-1}<i}\frac{\prod_{j=l}^{m-1}u_j^{(i_j)}b_i(U_j)u_m^{(i)}\dots u_r^{(i)}\prod_{j=m}^{r}b_i(U_j)}{\ell_{i_l}^{s_l}\dots\ell_{i_{m-1}}^{s_{m-1}}\ell_i^{s_m}\dots\ell_i^{s_r}}\Bigg)
\end{aligned}$$

Thus, by the above calculation, we obtain

$$\begin{aligned}
&\sum_{i\geq 0}\sum_{m=l}^{r} y_i[lm](-1)^{r-m}u_m^{(i)}\dots u_r^{(i)}\\
&\qquad=(-1)^{r-l}\sum_{0\leq i_l\leq\dots\leq i_r}\frac{u_l^{(i_l)}\dots u_r^{(i_r)}b_{i_l}(U_l)\dots b_{i_r}(U_r)}{\ell_{i_l}^{s_l}\dots\ell_{i_r}^{s_r}}\\
&\qquad=(-1)^{r-l}\operatorname{Li}^{*}_{\tilde{\mathcal{C}}_l}(\tilde{u}_l).
\end{aligned}$$

□

### 4.4 The Construction of the Anderson $A[\underline{t}_\Sigma]$-module $G_{\mathcal{C}}$

Let $\mathcal{C}$ be a composition array as in (1.4) such that $\operatorname{wght}(\mathcal{C}) = w$ and $\operatorname{dep}(\mathcal{C}) = r$, and let $u = (u_1,\dots,u_r) \in (\mathbb{T}_\Sigma \setminus \{0\})^r$. We recall the set of tuples $\{(a_l,\mathcal{C}_l,u_l)|\ \ 1\leq l\leq n\}$ for some $n\in\mathbb{Z}_{\geq 1}$ from Theorem 2.44 and without loss of generality, for $1\leq l\leq s$, let $\mathcal{C}_l$ be a composition array such that $\operatorname{dep}(\mathcal{C}_l)=1$ and for $s+1\leq l\leq n$, let $\mathcal{C}_l$ be a composition array whose depth is bigger than 1. Assume that $\operatorname{dep}(\mathcal{C}_l)=m_l$. Let $G_l$ be the Anderson $A[\underline{t}_\Sigma]$-module corresponding to the tuple $(\tilde{\mathcal{C}}_l,\tilde{u_l})$ defined as in (4.4). We also recall the notation from §4.1 and set $k_l'=\sum_{j=2}^{m_l}d_{lj}$ for any $1\leq l\leq n$. It is

easy to see from the definition that $w = d_{l1}$ for $1 \leq l \leq n$. Now define

$$G'_l := \begin{bmatrix} C^{\otimes d_{l2}} & E_l[23] & \dots & \dots & E_l[2m_l] \\ & C^{\otimes d_{l3}} & E_l[34] & \dots & E_l[3m_l] \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & & & & C^{\otimes d_{lm_l}} \end{bmatrix} \in \mathrm{Mat}_{k'_l}(\mathbb{T}_\Sigma)[\tau]$$

and $G''_l := \begin{bmatrix} E_l[12] & E_l[13] & \dots & E_l[1m_l] \end{bmatrix} \in \mathrm{Mat}_{w \times k'_l}(\mathbb{T}_\Sigma)$. Observe that

$$G_l(\theta) = \begin{bmatrix} C^{\otimes w}_{\mathfrak{a}_1}(\theta) & G''_l \\ & G'_l \end{bmatrix}.$$

Let us set $k_{\mathcal{C}} := w + \sum_{l=1}^{n} k'_l$. Then we define the Anderson $A[\underline{t}_\Sigma]$-module $G_{\mathcal{C}} : A[\underline{t}_\Sigma] \to \mathrm{Mat}_{k_{\mathcal{C}}}(\mathbb{T}_\Sigma)[\tau]$ by

$$G_{\mathcal{C}}(\theta) = \begin{bmatrix} C^{\otimes d_1}_{\mathfrak{a}_1} & G''_{s+1} & G''_{s+2} & \dots & G''_n \\ & G'_{s+1} & & & \\ & & G'_{s+2} & & \\ & & & \ddots & \\ & & & & G'_n \end{bmatrix} \in \mathrm{Mat}_{k_{\mathcal{C}}}(\mathbb{T}_\Sigma)[\tau]. \tag{4.38}$$

Using the definition of matrices $G'_l$ and $G''_l$ we see that $G_{\mathcal{C}}$ can be rewritten as in (3.3) and therefore it has an exponential function $\exp_{G_{\mathcal{C}}} : \mathrm{Mat}_{k_{\mathcal{C}} \times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{k_{\mathcal{C}} \times 1}(\mathbb{T}_\Sigma)$ which is everywhere convergent by Proposition 3.5.

For the rest of this section we aim to prove that $\exp_{G_{\mathcal{C}}}$ is a surjective function. Now let $k_l := \sum_{j=1}^{m_l} d_{lj} = w + k'_l$ for $1 \leq l \leq n$. First we give the definition of the following map $\lambda : \mathbb{T}_\Sigma^{\sum_{l=1}^{n} k_l} \to \mathbb{T}_\Sigma^{k_{\mathcal{C}}}$ by

$$\lambda : \begin{pmatrix} z_{11} \\ \vdots \\ z_{1k_1} \\ \vdots \\ z_{n1} \\ \vdots \\ z_{nk_n} \end{pmatrix} \to \Lambda \begin{pmatrix} z_{11} \\ \vdots \\ z_{1k_1} \\ \vdots \\ z_{n1} \\ \vdots \\ z_{nk_n} \end{pmatrix} = \begin{bmatrix} z_{11} + \dots + z_{s1} + z_{(s+1)1} + \dots + z_{n1} \\ \vdots \\ z_{1w} + \dots + z_{sw} + z_{(s+1)w} + \dots + z_{nw} \\ z_{(s+1)(w+1)} \\ \vdots \\ z_{(s+1)k_{s+1}} \\ \vdots \\ z_{n(w+1)} \\ \vdots \\ z_{nk_n} \end{bmatrix}, \tag{4.39}$$

where the matrix $\Lambda \in \mathrm{Mat}_{k_{\mathcal{C}} \times \sum_{l=1}^{n} k_l}(\mathbb{T}_\Sigma)$ defined by the following block matrix

$$\begin{bmatrix} I_{w\times sw} & \mathrm{Id}_w & O_{w\times k'_{s+1}} & \mathrm{Id}_w & O_{w\times k'_{s+2}} & \dots & \mathrm{Id}_w & O_{w\times k'_n} \\ & O_{k'_{s+1}\times w} & \mathrm{Id}_{k'_{s+1}} & & & & & \\ & & & O_{k'_{s+2}\times w} & \mathrm{Id}_{k'_{s+2}} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & O_{k'_n\times w} & \mathrm{Id}_{k'_n} \end{bmatrix} \tag{4.40}$$

so that $I_{w\times sw}$ is the block matrix $[\mathrm{Id}_w, \dots, \mathrm{Id}_w] \in \mathrm{Mat}_{w\times sw}(\mathbb{T}_\Sigma)$ and $O_{i\times j} \in \mathrm{Mat}_{i\times j}(\mathbb{T}_\Sigma)$ is the $i \times j$ zero matrix. Before we prove our next lemma, it should be noted that we define the Anderson $A[\underline{t}_\Sigma]$-module $\oplus_{l=1}^{n} G_l$ of dimension $\sum_{l=1}^{n} k_l$ by

$$\begin{aligned} \oplus_{l=1}^{n} G_l(\theta) &:= \begin{bmatrix} G_1(\theta) & & & \\ & G_2(\theta) & & \\ & & \ddots & \\ & & & G_n(\theta) \end{bmatrix} \\ &= \begin{bmatrix} G(\theta) & & & & & \\ & C_{\mathfrak{a}_1}^{\otimes w}(\theta) & G''_{s+1} & & & \\ & & G'_{s+1} & & & \\ & & & \ddots & & \\ & & & & C_{\mathfrak{a}_1}^{\otimes w}(\theta) & G''_n \\ & & & & & G'_n \end{bmatrix} \end{aligned} \tag{4.41}$$

where $G(\theta) \in \mathrm{Mat}_{sw}(\mathbb{T}_\Sigma)[\tau]$ defined as

$$G(\theta) = \begin{bmatrix} C_{\mathfrak{a}_1}^{\otimes w}(\theta) & & \\ & \ddots & \\ & & C_{\mathfrak{a}_1}^{\otimes w}(\theta) \end{bmatrix}.$$

Moreover its exponential function $\exp_{\oplus_{l=1}^{n} G_l} : \mathrm{Mat}_{(sw+\sum_{l=s+1}^{n} k_l)\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{(sw+\sum_{l=s+1}^{n} k_l)\times 1}(\mathbb{T}_\Sigma)$ is given by

$$\exp_{\oplus_{l=1}^{n} G_l} : \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \to \begin{pmatrix} \exp_{G_1}(f_1) \\ \vdots \\ \exp_{G_n}(f_n) \end{pmatrix}$$

where $f_j \in \mathrm{Mat}_{k_l\times 1}(\mathbb{T}_\Sigma)$ for $1 \le j \le n$. Using the matrices given in (4.38), (4.40) and (4.41), we immediately prove the following lemma.

LEMMA 4.42. *We have*

$$G_{\mathcal{C}}(\theta)\Lambda = \Lambda G_{\oplus_{l=1}^{n} G_l}(\theta).$$

*In other words, $\Lambda$ is an Anderson $A[\underline{t}_\Sigma]$-module homomorphism.*

Our next lemma introduces the relation between the matrix $\Lambda$, the infinite series $\exp_{G_\mathcal{C}}$ and $\exp_{\oplus_{l=1}^n G_l}$.

LEMMA 4.43. *We have the following equality over* $\mathrm{Mat}_{\sum_{l=1}^n k_l \times k_\mathcal{C}}(\mathbb{T}_\Sigma)[[\tau]]$*:*

$$\exp_{G_\mathcal{C}} \Lambda = \Lambda \exp_{\oplus_{l=1}^n G_l}.$$

*In particular, for any $f \in \mathrm{Mat}_{\sum_{l=1}^n k_l \times 1}(\mathbb{T}_\Sigma)$, we have $\lambda(\exp_{\oplus_{l=1}^n G_l}(f)) = \exp_{G_\mathcal{C}}(\lambda(f))$.*

*Proof.* Let us set $G_\mathcal{C}(\theta) = \theta \,\mathrm{Id}_{k_\mathcal{C}} + N_1 + E_1\tau$ for the nilpotent matrix $N_1$ such that $N_1^w = 0$ and $E_1 \in \mathrm{Mat}_{k_\mathcal{C}}(\mathbb{T}_\Sigma)$. Similarly, let $\oplus_{l=1}^n G_l(\theta) = \theta \,\mathrm{Id}_{\sum_{l=1}^n k_l} + N_2 + E_2\tau$ such that $E_2 \in \mathrm{Mat}_{\sum_{l=1}^n k_l}(\mathbb{T}_\Sigma)$. By the definition of $\oplus_{l=1}^n G_l$ we know that $N_2^w = 0$. Since $\Lambda$ is invariant under the automorphism $\tau$, by Lemma 4.42 we have that

$$(\theta \,\mathrm{Id}_{k_\mathcal{C}} + N_1 + E_1\tau)\Lambda = \theta \,\mathrm{Id}_{k_\mathcal{C}} \Lambda + N_1\Lambda + E_1\Lambda\tau = \Lambda\theta \,\mathrm{Id}_{\sum k_l} + \Lambda N_2 + \Lambda E_2\tau.$$

Since $\Lambda\theta \,\mathrm{Id}_{\sum k_l} = \theta \,\mathrm{Id}_{k_\mathcal{C}} \Lambda$, comparing coefficients of $\tau^0$ and $\tau$ above, we see that $N_1\Lambda = \Lambda N_2$ and $E_1\Lambda = \Lambda E_2$. Now let $\exp_{G_\mathcal{C}} = \sum_{i\geq 0} \beta_{1,i}\tau^i$ and $\exp_{\oplus_{l=1}^n G_l} = \sum_{i\geq 0} \beta_{2,i}\tau^i$. We claim that $\beta_{1,i}\Lambda = \Lambda\beta_{2,i}$ for all $i \geq 0$. We do induction on $i$. For $i = 0$, the claim holds. Assume that it is true for $i$. By (3.7) we have that

$$\beta_{m,i+1} = \sum_{j=0}^{2w-2} \frac{\mathrm{ad}(N_m)^j(E_m\beta_{m,i}^{(1)})}{[i+1]^{j+1}}, \quad m = 1, 2. \tag{4.44}$$

Moreover by the induction argument, Lemma 3.4 and commuting of $N_i$ and $E_i$ with $\Lambda$ for $i = 1, 2$ and for any $0 \leq j \leq 2w-2$ we have

$$\mathrm{ad}(N_1)^j(E_1\beta_{1,i}^{(1)})\Lambda = \Lambda \,\mathrm{ad}(N_2)^j(E_2\beta_{2,i}^{(1)}).$$

Thus the claim follows from (4.44). □

PROPOSITION 4.45. *The exponential function* $\exp_{G_\mathcal{C}} : \mathrm{Mat}_{k_\mathcal{C}\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{k_\mathcal{C}\times 1}(\mathbb{T}_\Sigma)$ *is surjective. In other words, the Anderson $A[\underline{t}_\Sigma]$-module $G_\mathcal{C}$ is uniformizable.*

*Proof.* Let $G_1, \ldots, G_s, G_{s+1}, \ldots, G_n$ be the Anderson $A[\underline{t}_\Sigma]$-modules that are used to construct $G_\mathcal{C}$ such that for $1 \leq j \leq s$, $G_j = C^{\otimes w}$ and $G_j \neq C^{\otimes w}$ when $j \geq s+1$. Let $y = [y_1, \ldots, y_w, y_{s+1,w+1}, \ldots, y_{s+1,k_{s+1}}, \ldots, y_{n,w+1}, \ldots, y_{n,k_n}]^\intercal$ be in $\mathrm{Mat}_{k_\mathcal{C}\times 1}(\mathbb{T}_\Sigma)$. Let $E_{ij} \in \mathrm{Mat}_{k_i\times 1}(\mathbb{F}_q)$ be the column matrix such that $j$-th entry is 1 and the other entries are zero. We now define elements $Y_j \in \mathrm{Mat}_{k_j\times 1}(\mathbb{T}_\Sigma)$ for different cases. If $n \geq w \geq s$, then we set

$$Y_j := \left\{ \begin{array}{ll} y_j E_{jj}, & \text{if } 1 \leq j \leq s \\ y_j E_{jj} + \sum_{l=w+1}^{k_j} y_{jl} E_{jl}, & \text{if } s < j \leq w \\ \sum_{l=w+1}^{k_j} y_{jl} E_{jl}, & \text{if } w < j \leq n \end{array} \right\}.$$

If $n \geq s \geq w$, we set

$$Y_j := \left\{ \begin{array}{ll} y_j E_{jj}, & \text{if } 1 \leq j \leq w \\ O_{jj}, & \text{if } w < j \leq s \\ \sum_{l=w+1}^{k_j} y_{jl} E_{jl}, & \text{if } s < j \leq n \end{array} \right\}.$$

where $O_{jj}$ is the $k_j \times 1$-zero matrix. Finally, if $w \geq n \geq s$, then we define

$$Y_j := \left\{ \begin{array}{ll} y_j E_{jj}, & \text{if } 1 \leq j \leq s \\ y_j E_{jj} + \sum_{l=w+1}^{k_j} y_{jl} E_{jl}, & \text{if } s < j \leq n-1 \\ \sum_{l=n}^{w} y_l E_{jl} + \sum_{l=w+1}^{k_j} y_{jl} E_{jl}, & \text{if } j = n \end{array} \right\}.$$

By Corollary 4.18 and Proposition 4.19, $\exp_{G_j}$ is surjective for all $j$. So there exist elements $X_j \in \mathrm{Mat}_{k_j \times 1}(\mathbb{T}_\Sigma)$ such that $\exp_{G_j}(X_j) = Y_j$. Now let $x := [X_1, \ldots, X_n]^\intercal \in \mathrm{Mat}_{\sum k_l \times 1}(\mathbb{T}_\Sigma)$ and $Y := [Y_1, \ldots, Y_n]^\intercal \in \mathrm{Mat}_{\sum k_l \times 1}(\mathbb{T}_\Sigma)$. Thus by the definition of the map $\lambda$ and Lemma 4.43 we see that

$$\lambda(\exp_{\oplus_{l=1}^n G_l}(x)) = \lambda(Y) = y = \exp_{G_\mathcal{C}}(\lambda(x))$$

which gives the surjectivity of $\exp_{G_\mathcal{C}}$. □

## 4.5 Proof of Theorem 1.12

In this subsection we give the proof of our following result and introduce an example.

Theorem 4.46. *Let $\mathcal{C}$ be a composition array as in (1.4) of weight $w$. Let also $\mathfrak{I}_1$ be the set of indices $i$ such that $U_i \neq \emptyset$ and $\mathfrak{I}_2$ be the set of $i$'s such that $U_i = \emptyset$. Then there exist a uniformizable Anderson $A[\underline{t}_\Sigma]$-module $G_\mathcal{C}$ of dimension $k_\mathcal{C}$ defined over $\mathbb{T}_\Sigma$, a special point $v_\mathcal{C} \in \mathrm{Mat}_{k_\mathcal{C} \times 1}(K^{perf}(\underline{t}_\Sigma))$ and an element $Z_\mathcal{C} \in \mathrm{Mat}_{k_\mathcal{C} \times 1}(\mathbb{T}_\Sigma)$ such that*

*(i)* $\prod_{i \in \mathfrak{I}_1} \ell_{r_{s_i}-1}^{q^{r_{s_i}}-s_i} b_{r_{s_i}}(U_i) \prod_{i \in \mathfrak{I}_2} \Gamma_{s_i} \zeta_C(\mathcal{C})$ *occurs as the $w$-th coordinate of $Z_\mathcal{C}$ where $r_{s_i} \geq 1$ is an integer such that $s_i \leq q^{r_{s_i}}$ for $i \in \mathfrak{I}_1$.*

*(ii)* $\exp_{G_\mathcal{C}}(Z_\mathcal{C}) = v_\mathcal{C}$.

*Proof.* We recall the construction of the Anderson $A[\underline{t}_\Sigma]$-module $G_\mathcal{C}$ from §4.4 and elements $a_l \in A$ coming from the tuples $(a_l, \mathcal{C}_l, u_l)$ in Theorem 2.44. By Proposition 4.45, we know that $G_\mathcal{C}$ is uniformizable. We set $Z_l := \log_{G_l}(v_{\tilde{\mathcal{C}}_l, \tilde{u}_l})$ and $v_l := \exp_{G_l}(Z_l) = v_{\tilde{\mathcal{C}}_l, \tilde{u}_l} \in \mathrm{Mat}_{k_l \times 1}(K^{\mathrm{perf}}(\underline{t}_\Sigma))$ where the last equality comes from the functional equation (3.10) and the definition of $Z_l$ makes sense by Theorem 4.37. We define

$$Z_\mathcal{C} := \lambda((\partial_{G_1}(a_1) \cdot Z_1, \ldots, \partial_{G_n}(a_n) \cdot Z_n)^\intercal) \in \mathrm{Mat}_{k_\mathcal{C} \times 1}(\mathbb{T}_\Sigma)$$

and

$$v_\mathcal{C} := \lambda((G_1(a_1) \cdot v_1, \ldots, G_n(a_n) \cdot v_n)^\intercal) \in \mathrm{Mat}_{k_\mathcal{C} \times 1}(K^{\mathrm{perf}}(\underline{t}_\Sigma)).$$

Note that by Theorem 4.37, the $w$-th coordinate of $Z_l$ is equal to $(-1)^{\mathrm{dep}(\tilde{\mathcal{C}}_l)-1}\mathrm{Li}^*_{\tilde{\mathcal{C}}_l}(\tilde{\tilde{u}_l}) = (-1)^{\mathrm{dep}(\mathcal{C}_l)-1}\mathrm{Li}^*_{\mathcal{C}_l}(u_l)$. We observe that by the definition of Anderson $A[\underline{t}_\Sigma]$-modules $G_l$, for any $a \in A[\underline{t}_\Sigma]$ and $f = (f_1, \dots, f_{k_l})^\intercal \in \mathrm{Mat}_{k_l\times 1}(\mathbb{T}_\Sigma)$, the $w$-th coordinate of $\partial_{G_l}(a)\cdot f$ is equal to $af_w$. Thus, using the definition of the map $\lambda$, we see that the $w$-th coordinate of $Z_{\mathcal{C}}$ is equal to $\sum_l a_l(-1)^{\mathrm{dep}(\mathcal{C}_l)-1}\mathrm{Li}^*_{\mathcal{C}_l}(u_l)$. But by Theorem 2.44, we see that the sum is equal to $\prod_{i\in\mathfrak{I}_1}\ell_{r_{s_i}-1}^{q^{r_{s_i}}-s_i}b_{r_{s_i}}(U_i)\prod_{i\in\mathfrak{I}_2}\Gamma_{s_i}\zeta_C(\mathcal{C})$ which proves the first part.
To prove part (ii), we use the equality (3.6) and Lemma 4.43 to see that

$$\begin{aligned}\exp_{G_{\mathcal{C}}}(Z_{\mathcal{C}}) &= \exp_{G_{\mathcal{C}}}(\lambda((\partial_{G_1}(a_1)\cdot Z_1,\dots,\partial_{G_n}(a_n)\cdot Z_n)^\intercal))\\ &= \lambda\exp_{\oplus_{l=1}^n G_l}((\partial_{G_1}(a_1)\cdot Z_1,\dots,\partial_{G_n}(a_n)\cdot Z_n)^\intercal))\\ &= \lambda((G_1(a_1)\cdot\exp_{G_1}(Z_1),\dots,G_n(a_n)\cdot\exp_{G_n}(Z_n))^\intercal)\\ &= \lambda((G_1(a_1)\cdot v_1,\dots,G_n(a_n)\cdot v_n)^\intercal)\\ &= v_{\mathcal{C}}.\end{aligned}$$

. □

*Remark* 4.47. One can observe that we can capture Chang and Mishiba's result [CM20, Thm. 1.4.1] by defining the composition array $\mathcal{C}$ as in (1.6).

EXAMPLE 4.48. Let $\Sigma = \{1,\dots,n\}$ and let $L\binom{\Sigma}{s}$ be the Pellarin $L$-series defined as in (1.3). By Theorem 2.18, for any $d \geq 0$, there exists a polynomial

$$Q_{\Sigma,s}(t) = \sum_{l\geq 0} u_l t^l \in K^{\mathrm{perf}}(\underline{t}_\Sigma)[t]$$

such that

$$\sum_{a\in A_{+,d}}\frac{a(t_1)\dots a(t_n)}{a^s} = \frac{b_d(\Sigma)}{\ell_{r-1}^{q^r-s}\ell_d^s b_r(\Sigma)}\tau^d(Q_{\Sigma,s}(t))_{|t=\theta}$$

where $r \geq 1$ is an integer satisfying $q^r \geq s$. Choose $\beta = (t_1-\theta)\dots(t_n-\theta)$ and set $G := C_\beta^{\otimes s}$. For any $l$, we define $v_l := (0,\dots,0,u_l)$ and $Z_l := \log_G(v_l)$ which is a well-defined element in $\mathbb{T}^s_\Sigma$ by Theorem 2.18 and Proposition 4.34. Finally we set $v_{\mathcal{C}} := \sum_{l\geq 0} G(\theta^l)\cdot v_l \in K^{\mathrm{perf}}(\underline{t}_\Sigma)^s$ and $Z_{\mathcal{C}} := \sum_{l\geq 0}\partial_G(\theta^l)\cdot Z_l \in \mathbb{T}^s_\Sigma$. Thus by the proof of Theorem 4.46, we see that

$$\exp_G(Z_{\mathcal{C}}) = \exp_G((*,\dots,*,\ell_{r-1}^{q^r-s}b_r(\Sigma)L(\chi_{t_1}\dots\chi_{t_n},s))^\intercal) = v_{\mathcal{C}}.$$

For the case $n = s = 1$, using [Per14, Thm. 4.16], we immediately see that $Q_{\Sigma,1}(t) = t_1 - t$. Thus, $v_{\mathcal{C}} = t_1 - \theta - t_1 + \theta = 0$ and by [APTR16, Rem. 5.13, Lem. 6.8, Lem. 7.1] (see also [Pel12, Thm. 1]) we have

$$Z_{\mathcal{C}} = (t_1-\theta)L\binom{\Sigma}{s} = (t_1-\theta)\log_G(1) = -\frac{\tilde{\pi}}{\omega_1} \in \mathbb{T}_\Sigma(\mathbb{K}_\infty).$$

*Remark* 4.49. We continue with the notation of Example 4.48 and recall the definition of $\mathbb{T}_\Sigma(\mathbb{K}_\infty)$ from §2.1. We set

$$U_{G_\mathcal{C}} := \{x \in \mathrm{Mat}_{k_\mathcal{C}\times 1}(\mathbb{T}_\Sigma(\mathbb{K}_\infty)) \mid \exp_{G_\mathcal{C}}(x) \in \mathrm{Mat}_{k_\mathcal{C}\times 1}(A[\underline{t}_\Sigma])\}.$$

Using the action of $A[\underline{t}_\Sigma]$ to $\mathrm{Mat}_{k_\mathcal{C}\times 1}(\mathbb{T}_\Sigma(\mathbb{K}_\infty))$ by left multiplication, one can see that $U_{G_\mathcal{C}}$ is an $A[\underline{t}_\Sigma]$-module. We call $U_{G_\mathcal{C}}$ the unit module (see [ATR17] and [ANDTR18] for more details). By Example 4.48, we see that $Z_\mathcal{C} \in U_{G_\mathcal{C}}$ when $n = s = 1$. The situation is more interesting when $n$ is larger. Set $n = q$ and $s = 1$. By [Dem15, Ex. 3.3.7], we have

$$Q_{\Sigma,1}(t) = (t_1 - t)\dots(t_q - t)\left(1 - \frac{(t-\theta)}{(t_1 - \theta^{1/q})\dots(t_q - \theta^{1/q})}\right) \in K^{\mathrm{perf}}(\underline{t}_\Sigma)[t].$$

A small calculation shows that $v_\mathcal{C} \in A[\underline{t}_\Sigma]$ and therefore $Z_\mathcal{C} \in U_{G_\mathcal{C}}$ for $n = q$ and $s = 1$. In other words, although the elements $v_l$ in the proof of Theorem 4.46 constructing the special point $v_\mathcal{C}$ for this case are not in $A[\underline{t}_\Sigma]$, $v_\mathcal{C}$ is itself in $A[\underline{t}_\Sigma]$. It would be interesting to analyze under what conditions $Z_\mathcal{C}$ lies in $U_{G_\mathcal{C}}$.

## A The Proof of Theorem 4.17

Throughout this section we let $G$ be the Anderson $A[\underline{t}_\Sigma]$-module of dimension $k$ defined as in (4.4). We should also mention that unlike the rest of the paper we use the notation $G_\theta$ for the matrix $G(\theta)$ in (4.4) and $G_\theta(f)$ for $G(\theta)\cdot f$ for any $f \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$ in this section.

### A.1 Operators

Let $\delta_0, \delta_1 : \mathrm{Mat}_{1\times d}(\mathbb{T}_\Sigma[\sigma]) \to \mathrm{Mat}_{d\times 1}(\mathbb{T}_\Sigma)$ be the maps given by $\delta_0\Big(\sum_{i=0}^n a_i\sigma^i\Big) = a_0^\intercal$ and $\delta_1\Big(\sum_{i=0}^n a_i\sigma^i\Big) = \sum_{i=0}^n a_i^{\intercal(i)}$. Furthermore for any $f = \sum a_i\tau^i \in \mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma[\tau])$ we recall the definition of $f^*$ in §3.3 and define the map $f^* : \mathrm{Mat}_{1\times d}(\mathbb{T}_\Sigma[\sigma]) \to \mathrm{Mat}_{1\times k}(\mathbb{T}_\Sigma[\sigma])$ by

$$f^*(g) = gf^*.$$

Then we state the following lemma whose proof can be given similar to the proof of [GP19, Lem. 4.2.2] and [Jus10, Lem. 1.1.21- 1.1.22].

Lemma A.1. *Let $f = \sum_{j=0}^n f_j\tau^j \in \mathrm{Mat}_{k\times d}(\mathbb{T}_\Sigma)[\tau]$.*

*(a) Let us define $\partial_0 f : \mathrm{Mat}_{d\times 1}(\mathbb{T}_\Sigma) \to \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$ by $\partial_0 f(g) = f_0 g$. The following diagram commutes with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Mat}_{1\times d}(\mathbb{T}_s[\sigma]) & \xrightarrow{\sigma(\cdot)} & \mathrm{Mat}_{1\times d}(\mathbb{T}_s[\sigma]) & \xrightarrow{\delta_0} & \mathrm{Mat}_{d\times 1}(\mathbb{T}_s[\sigma]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \partial_0 f} & & \\
0 & \longrightarrow & \mathrm{Mat}_{1\times k}(\mathbb{T}_s[\sigma]) & \xrightarrow{\sigma(\cdot)} & \mathrm{Mat}_{1\times k}(\mathbb{T}_s[\sigma]) & \xrightarrow{\delta_0} & \mathrm{Mat}_{k\times 1}(\mathbb{T}_s) & \longrightarrow & 0
\end{array}$$

*(b)*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Mat}_{1\times d}(\mathbb{T}_s[\sigma]) & \xrightarrow{(\sigma-1)(\cdot)} & \mathrm{Mat}_{1\times d}(\mathbb{T}_s[\sigma]) & \xrightarrow{\delta_1} & \mathrm{Mat}_{d\times 1}(\mathbb{T}_s) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \mathrm{Mat}_{1\times k}(\mathbb{T}_s[\sigma]) & \xrightarrow{(\sigma-1)(\cdot)} & \mathrm{Mat}_{1\times k}(\mathbb{T}_s[\sigma]) & \xrightarrow{\delta_1} & \mathrm{Mat}_{k\times 1}(\mathbb{T}_s) & \longrightarrow & 0 \end{array}$$

*In particular, we have* $G_\theta\delta_1 = \delta_1 G_\theta^*$.

## A.2 Division Towers

We start with a definition.

DEFINITION A.2. For any $x \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$, we call a sequence $\{f_n\}_{n=0}^\infty$ in $\mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$ a convergent $\theta$-division tower above $x$ if

- $\lim_{n\to\infty}\|f_n\|_\infty = 0$.
- $G_\theta(f_{n+1}) = f_n$ for all $n \geq 0$.
- $G_\theta(f_0) = x$.

We now give the following theorem whose proof uses similar ideas as in the proof of [GP19, Thm. 4.3.2].

THEOREM A.3 (cf. [GP19, Thm. 4.3.2]). *Let* $x \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$ *Then there exists a canonical bijection*

$$F : \{\zeta \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma) \,|\, \exp_G(\zeta) = x\} \to \{\textit{convergent } \theta\textit{-division towers above } x\}$$

*defined by* $F(\zeta) = \{\exp_G(\partial_G(\theta)^{-(n+1)}\zeta)\}_{n=0}^\infty$. *Moreover, if* $\{f_n\}_{n=0}^\infty$ *is a convergent* $\theta$*-division tower above* $x$*, then with respect to* $\|\cdot\|_\infty$*, we have* $\lim_{n\to\infty}\partial_G(\theta)^{n+1}f_n = \zeta$.

*Proof.* Note that by the functional equation (3.6) we have

$$G_\theta(\exp_G(\partial_G(\theta)^{-(n+1)}\zeta)) = \exp_G(\partial_G(\theta)^{-n}\zeta)$$

and $G_\theta(\exp_G(\partial_G(\theta)^{-1}\zeta)) = \exp_G(\zeta) = x$. We also see by Lemma 3.9 that for arbitrarily large $n$, we have $\|\exp_G(\partial_G(\theta)^{-n}\zeta)\|_\infty = \|\partial_G(\theta)^{-n}\zeta\|_\infty$. So the sequence given as $F(\zeta)$ converges to 0 and is actually a $\theta$-division sequence above $x$. Thus the map $F$ is well-defined. For the injectivitiy, let us assume that $\exp_G(\partial_G(\theta)^{-(n+1)}\zeta_1) = \exp_G(\partial_G(\theta)^{-(n+1)}\zeta_2)$ for some $\zeta_1, \zeta_2 \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$ and any $n \in \mathbb{Z}_{\geq 0}$. Then we have $\exp_G(\partial_G(\theta)^{-(n+1)}(\zeta_1 - \zeta_2)) = 0$. But by Lemma 3.9 we see that $\partial_G(\theta)^{-(n+1)}(\zeta_1 - \zeta_2)$ should be equal to zero matrix for sufficiently large $n$. Thus, one can deduce by the invertibility of $\partial_G(\theta)$ that $\zeta_1 = \zeta_2$.
We prove the surjectivity as follows. Let $\{f_n\}_{n=0}^\infty$ be a convergent $\theta$-division tower above $x$. By the convergence of the sequence, there exists $N \in \mathbb{Z}_{\geq 0}$ such

that for any $n \geq N$, $f_n$ is in the radius of convergence of $\log_G$. Now we set $\zeta = \partial_G(\theta^{n+1}) \log_G(f_n)$ for any $n \geq N$. Then by (3.11) we have

$$\partial_G(\theta^{n+2}) \log_G(f_{n+1}) = \partial_G(\theta^{n+1}) \log_G(G_\theta(f_{n+1})) = \partial_G(\theta^{n+1}) \log_G(f_n).$$

Thus, our choice for $\zeta$ is independent of $n$. Therefore we have $f_n = \exp_G(\partial_G(\theta^{n+1})^{-1}\zeta)$. Then for any $n < N$, we obtain by using (3.7) that

$$f_n = G_{\theta^{N-n}}(f_N) = G_{\theta^{N-n}}(\exp_G(\partial_G(\theta^{N+1})^{-1}\zeta)) = \exp_G(\partial_G(\theta^{n+1})^{-1}\zeta).$$

Thus we see that $F(\zeta) = \{f_n\}_{n=0}^{\infty}$. For the last assertion, we observe that for any $n \geq 0$, $\zeta - \partial_G(\theta^{n+1}) \exp_G(\partial_G(\theta^{n+1})^{-1}\zeta) = \partial_G(\theta)^{n+1} \sum_{j \geqslant 1} \beta_j \partial_G(\theta)^{-q^j(n+1)} \zeta^{(j)}$ where $\exp_G = \sum_{j \geq 0} \beta_j \tau^j$. Notice that for each $j$, $\|\partial_G(\theta^{n+1})\beta_j \partial_G(\theta)^{-q^j(n+1)}\zeta^{(j)}\|_\infty \to 0$ as $n \to \infty$ because by Proposition 3.5 we see that $\lim_{j\to\infty} \|\beta_j\|_\infty R^{q^j} = 0$ for any $R \in \mathbb{R}_{>0}$. Thus $\lim_{n\to\infty} \|\zeta - \partial_G(\theta^{n+1}) f_n\|_\infty = 0$. □

*Remark* A.4. We recall the definition of the row matrix $\mathbf{m}$ from §4.1. By (4.13), for any $1 \leq j \leq r$, we have

$$\frac{(t-\theta)^{d_j}}{\prod_{i=j}^{r} \alpha_i^{(-1)}} \cdot m_j = \sigma m_j + \sum_{w=1}^{j-1} (-1)^w \prod_{n=1}^{w} u_{j-n}^{(-1)} \sigma m_{j-w}. \tag{A.5}$$

We now consider the map $\delta_0 \circ \iota : (\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t]), \|\cdot\|_\theta) \to (\mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma), \|\cdot\|_\infty)$. Let $h = [h_1, \ldots, h_r] \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$ be such that $h_i = \sum_{j=0}^{\infty} h_{ij}(t-\theta)^j$ and $h_{ij} = 0$ for $j \gg 0$. By the identification of $\mathbb{T}_\Sigma[\sigma]$-basis of $H(G)$ as vectors $e_{ij}$ as in Remark 4.8 and (A.5) we see that

$$\begin{aligned} \delta_0 \circ \iota(h) &= \delta_0(h_1 \cdot m_1 + \cdots + h_r \cdot m_r) \\ &= (h_{1(d_1-1)}, \ldots, h_{10}, \ldots, h_{r(d_r-1)}, \ldots, h_{r0}). \end{aligned} \tag{A.6}$$

Lemma A.7. *Let $h \in \mathbb{T}_\Sigma[t]$ be such that $h = \sum_{j=0}^{l} h_j(t-\theta)^j$ for some $l \in \mathbb{Z}_{\geq 0}$. Then*

$$\|h\|_\theta = \sup\{\|h_i\|_\infty \|\theta\|_\infty^i \mid i \in \mathbb{Z}_{\geq 0}\}.$$

*Proof.* Assume that $h = \sum_{j=0}^{l} h_j(t-\theta)^j = \sum_{j=0}^{l} g_j t^j$ for some $g_j \in \mathbb{T}_\Sigma$. By the assumption on $g_j$ and $h_j$, we see that $g_j = h_j - \binom{j+1}{j} h_{j+1}\theta^{j+1-j} + \cdots - \binom{l}{j} h_l \theta^{l-j}$. Thus we obtain

$$\begin{aligned} \|h\|_\theta &= \sup_j \{\|g_j\|_\infty \|\theta\|_\infty^j\} \\ &\leq \sup_j \Big( \sup_{j \leq i \leq l} \{\|h_i\|_\infty |\theta|_\infty^{i-j} |\theta|_\infty^j\} \Big) \\ &\leq \sup_i \{\|h_i\|_\infty \|\theta\|_\infty^i\}. \end{aligned}$$

On the other hand, again by the assumption, we have that $h_j = \sum_{i=j}^{l} \binom{i}{j} g_i \theta^{i-j}$. Thus similarly we have

$$\begin{aligned}\sup\{\|h_j\|_\infty \|\theta\|_\infty^j\} &\leq \sup_j \Big( \sup_{j\leq i\leq l} \{\|g_i\|_\infty \|\theta\|_\infty^i\}\Big) \\ &= \sup_j \Big( \sup_{j\leq i\leq l} \{\|g_i\|_\infty |\theta|_\infty^{i-j} |\theta|_\infty^j\}\Big) \\ &= \|h\|_\theta\end{aligned}$$

which completes the proof. □

Thus, using (A.6) and Lemma A.7, we obtain $\|\delta_0 \circ \iota(h)\|_\infty \leq \|h\|_\theta$. Therefore the map $\delta_0 \circ \iota$ is bounded. By the fact that $\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$ is $\|\cdot\|_\theta$-dense in $\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma\{t/\theta\})$, we extend the map $\delta_0 \circ \iota$ to a map $D : (\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma\{t/\theta\}), \|\cdot\|_\theta) \to (\mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma), \|\cdot\|_\infty)$ of complete normed modules.

THEOREM A.8 (cf. [GP19, Thm. 4.4.6]). *Let $(\iota, \Phi)$ be a t-frame for $G$. Moreover let $h \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma)[t]$ and assume that there exists a matrix $g \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma\{t/\theta\})$ such that $g^{(-1)}\Phi - g = h$. If $v = \delta_1(\iota(h)) \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$ and $\zeta = D(g + h)$, then we have $\exp_G(\zeta) = v$.*

*Proof.* The proof follows the ideas of the proof of Theorem 4.4.6 of [GP19]. We first let $g = \sum_{i=0}^{\infty}$ where $g_i \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma)$ and define $g_{\leq n} = \sum_{i\leq n} g_i t^i$ and $g_{>n} = \sum_{i>n} g_i t^i$ for $n \geq 0$. Furthermore we set

$$h_n := \frac{h + g_{\leq n} - g_{\leq n}^{(-1)}\Phi}{t^{n+1}} = \frac{g_{>n}^{(-1)}\Phi - g_{>n}}{t^{n+1}} \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[[t]]). \tag{A.9}$$

Observe that since $g^{(-1)}\Phi - g = h$, the second expression in (A.9) is a power series in $t$ and divisible by $t^{n+1}$. But since $\deg_t(g_{\leq n}) \leq n$, $h_n$ should be a polynomial in $t$ and therefore $h_n \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$. Moreover, $\deg_t(h_n) \leq \max\{\deg_t(h) - n - 1, 0\}$. Thus the degree of $h_n$ in $t$ does not depend on $n$ and therefore $h_n$ can be seen as an element of a free and finitely generated $\mathbb{T}_\Sigma$-module $M$ of $\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$. We now prove several claims.
*Claim 1*: The sequence $\{\delta_1(\iota(h_n))\}_{n=0}^{\infty}$ is a convergent $\theta$-division tower above $\delta_1(\iota(h))$.

*Proof of Claim 1:* Using Lemma 4.14 and Lemma A.1, we see that

$$\begin{aligned}\delta_1(\iota(h_n)) - G_\theta(\delta_1(\iota(h_{n+1}))) &= \delta_1(\iota(h_n)) - \delta_1(G_\theta^*(\iota(h_{n+1}))) \\ &= \delta_1(\iota(h_n)) - \delta_1(\iota(h_{n+1})G_\theta^*) \\ &= \delta_1(\iota(h_n) - t h_{n+1}).\end{aligned} \tag{A.10}$$

From the definition of $h_n$ and using Lemma 4.14 we obtain

$$\delta_1(\iota(h_n - t h_{n+1})) = \delta_1 \iota\left(\left(\frac{g_{n+1}}{t^{n+1}}\right)^{(-1)} - \frac{g_{n+1}}{t^{n+1}}\right) = \delta_1\left((\sigma - 1)\iota\left(\frac{g_{n+1}}{t^{n+1}}\right)\right) = 0 \tag{A.11}$$

where the last equality follows from Lemma A.1(b). Thus (A.10) and (A.11) imply that $\delta_1(\iota(h_n)) = G_\theta(\delta_1(\iota(h_{n+1})))$ for $n \geq 0$. Similar calculation as above also shows that $G_\theta(\iota(h_0)) = v = \delta_1(\iota(h))$.
Recall the definition of the norm $\|\cdot\|_\sigma$ from §3.3 and the norm $\|\cdot\|_1$ from §4.2 to observe that since $\|g_n\|_\infty \to 0$ as $n \to \infty$, we obtain $\|g_{>n}\|_1 \to 0$ as $n \to \infty$. Moreover we have

$$\|h_n\|_1 \leq \max\{\|g_{>n}^{(-1)}\Phi\|_1, \|g_{>n}\|_1\} = \max\{\|g_{>n}\|_1^{1/q}\|\Phi\|_1, \|g_{>n}\|_1\}.$$

Thus $\|h_n\|_1 \to 0$ as $n \to \infty$. By [GP19, Lem. 2.2.2], the norms $\|\cdot\|_1$ and $\|\iota(\cdot)\|_\sigma$ are equivalent on $M$ and therefore $\|\iota(h_n)\|_\sigma \to 0$ when $n \to \infty$. Since the $t$-degree of $h_n$ is bounded indepedent of $n$, we can also see that the $\sigma$-degree of $\iota(h_n)$ is also bounded and say for arbitrarily large $n$, $\iota(h_n) = \sum_{j=0}^{N} a_j \sigma^j$ such that $\|a_j\|_\sigma < 1$. Thus we have

$$\|\delta_1(\iota(h_n))\|_\infty = \|\sum_{j=0}^{N} a_j^{\intercal(j)}\|_\infty \leq \sup\{\|a_j\|_\infty\} = \|\iota(h_n)\|_\sigma. \tag{A.12}$$

Thus (A.12) implies that $\|\delta_1(\iota(h_n))\|_\infty \to 0$ as $n \to \infty$ and therefore the sequence $\{\delta_1(\iota(h_n))\}_{n=0}^{\infty}$ is a convergent $\theta$-division tower above $v = \delta_1(\iota(h))$. □

*Claim 2*: We have $\lim_{n\to\infty}\|\partial_G(\theta^{n+1})\|_\infty\|h_n\|_\theta = 0$.

*Proof of Claim 2:* Let us set $h_n = \sum_{i=0}^{N_0} c_i t^i$ where $c_i \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma)$ and by the discussion in the beginning of the proof we know the existence of some positive integer $N_0$ which is independent of $n$. We have from the definition of $h_n$ that

$$\begin{aligned}
\|\partial_G(\theta^{n+1})\|_\infty\|h_n\|_\theta &= \|\partial_G(\theta)\|_\infty^{n+1}\|\frac{g_{>n}^{(-1)}\Phi - g_{>n}}{t^{n+1}}\|_\theta \\
&= \|\partial_G(\theta)\|_\infty^{n+1}\sup\{\|\theta\|_\infty^i\|c_i\|_1\} \\
&= \sup\{\|\theta\|_\infty^{n+1+i}\|c_i\|_1\} \\
&= \|c_0 t^{n+1} + \cdots + c_{N_0} t^{n+N_0+1}\|_\theta \\
&= \|g_{>n}^{(-1)}\Phi - g_{>n}\|_\theta.
\end{aligned} \tag{A.13}$$

But observe that $g \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma\{t/\theta\}$ so $\|g_{>n}\|_\theta \to 0$ as $n \to \infty$ which implies the claim together with (A.13). □

*Claim 3:* We have $\lim_{n\to\infty} \partial_G(\theta^{n+1})\delta_1(\iota(h_n)) = \zeta$.

*Proof of Claim 3:* Using the definition of $h_n$, Lemma 4.14 and Lemma A.1, observe that

$$\begin{aligned}
\lim_{n\to\infty} \delta_0(\iota(t^{n+1}h_n + g_{\leq n}^{(-1)}\Phi)) &= \lim_{n\to\infty} \delta_0(\iota(h_n)G^*_{\theta^{n+1}}) \\
&= \lim_{n\to\infty} \partial_G(\theta^{n+1})\delta_0(\iota(h_n)).
\end{aligned}$$

Thus using the definition of $\zeta$, we see that $\zeta = \lim_{n\to\infty} \partial_G(\theta^{n+1})\delta_0(\iota(h_n))$. Therefore we need to show that $\lim_{n\to\infty} \partial_G(\theta^{n+1})(\delta_1(\iota(h_n)) - \delta_0(\iota(h_n))) = 0$. Observe that

$$\|\delta_1(\iota(h_n)) - \delta_0(\iota(h_n))\|_\infty \leq \|\sum_{j=1}^{N} a_j^{\intercal(j)}\|_\infty \leq \sup\{\|a_j\|_\infty^{q^j}\} \leq \|\iota(h_n)\|_\sigma^q. \quad \text{(A.14)}$$

Thus (A.14) implies that the claim is equivalent to showing that $\lim_{n\to\infty} \partial_G(\theta^{n+1})\|\iota(h_n)\|_\sigma^q = 0$. Since for sufficiently large $n$, $\|\iota(h_n)\|_\sigma \leq 1$, we have to show that $\lim_{n\to\infty} \partial_G(\theta^{n+1})\|\iota(h_n)\|_\sigma$. By the equivalence of the norms $\|\cdot\|_\theta$ and $\|\iota(\cdot)\|_\sigma$ on $M$, the claim follows from Claim 2. □

Thus the proof of the theorem follows easily as by Claim 1, Claim 3 and Theorem A.3 we see that $\exp_G(\zeta) = \delta_1(\iota(h)) = v$. □

*Proof of Theorem 4.17.* Let us choose an arbitrary element

$$\tilde{h} = [h_{1(d_1-1)}, \dots, h_{10}, \dots, h_{r(d_r-1)}, \dots, h_{r0}]^\intercal \in \mathrm{Mat}_{k\times 1}(\mathbb{T}_\Sigma)$$

and let $h = [\sum_{j=0}^{d_1-1} h_{1j}(t-\theta)^j, \dots, \sum_{j=0}^{d_r-1} h_{rj}(t-\theta)^j] \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$. Since $\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$ is $\|\cdot\|_\theta$-dense in $\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma\{t/\theta\})$, we can write $h\Psi = u + h$ so that $u \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$ and $\|h\|_\theta < 1$. We also have that $\|h^{(n)}\|_\theta \leq \|h\|_\theta^{q^n}$ holds for all $n \geq 0$. Then the series $H := \sum_{n=1}^{\infty} h^{(n)}$ converges to an element of $\mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma\{t/\theta\})$ because $\|h\|_\theta < 1$. Moreover, $H^{(-1)} - H = h$. By Proposition [GP19, Prop. 4.5.2], there exists $U \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\Sigma[t])$ such that $U^{(-1)} - U = u$. Set $x := (U + H)\Psi^{-1}$. Then one can see that

$$x^{(-1)}\Phi - x = (u + h)\Psi^{-1} = h.$$

Using Remark 4.8 and Remark A.4 one can observe that $\delta_1(\iota(h)) = \tilde{h}$. Therefore by Theorem A.8 we have that $\exp_G(D(x + h)) = \delta_1(\iota(h)) = \tilde{h}$. So $\exp_G$ is surjective. □

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