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A Class of Polynomials from Banach Spaces into Banach Algebras

By

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Abstract

Let E be a complex Banach space and F be a complex Banach algebra. We will be interested in the subspace $\mathbb{P}_f({}^nE, F)$ of $P({}^nE, F)$ generated by the collection of functions $\varphi^n (n \in \mathbb{N}, \varphi \in L(E, F))$ where $\varphi^n(x) = (\varphi(x))^n$ for each $x \in E$.

§1. Introduction

Let E be a complex Banach space. When F is a complex Banach algebra it is natural to consider the space generated by $\{\varphi^n : \varphi \in L(E, F)\}$ where $n \in \mathbb{N}$ is fixed and $\varphi^n(x) := (\varphi(x))^n$. Our purpose in this paper is to study this space. In fact, we are going to compare it with the space of n-homogenous polynomials of finite type and to get its dual in a convenient norm. This paper provides a number of illustrative examples and counterexamples that lead to a better understanding of the space defined by us.

The Banach space of all continuous n-linear mappings A from E^n into F endowed with the norm $\|A\| = \sup\{\|A(x_1, x_2, \ldots, x_n)\| : \|x_j\| \leq 1, j = 1, \ldots, n\}$ will be denoted by $L({}^nE, F)$. As usual we will write E' for $L(E, \mathbb{C})$. We denote by $\mathcal{K}(E, F)$ the space of all compact linear operators.

We denote by $P({}^nE, F)$ the Banach space of all continuous n-homogeneous polynomials P from E into F with the norm $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$, and by $P_f({}^nE, F)$ the space of all finite type n-homogeneous polynomials

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from E into F, i.e, the space generated by the mappings $x \longmapsto (\varphi(x))^n \cdot b$ where $\varphi \in E'$ and $b \in F$. We also denote by $P_{wu}(^nE, F)$ the space of all continuous n-homogeneous polynomials from E into F that are uniformly weakly continuous when restricted to the bounded subsets of E. More generally if $\varphi_i \in E'(i = 1, \ldots, n)$ and $b \in F$ we denote by $L_f(^nE, F)$ the space generated by the mappings $(x_1, x_2, \ldots, x_n) \in E^n \longmapsto \varphi_1(x_1) \cdot \varphi_2(x_2) \ldots \varphi_n(x_n) \cdot b \in F$ and by $L_{wu}(^nE, F)$ the space of the elements of $L(^nE, F)$ that are uniformly weakly continuous when restricted to any bounded subset of E^n.

For each mapping $f : E \longrightarrow \mathbb{C}$ and $b \in F$ we set $f \otimes b(x) = f(x) \cdot b$ for all $x \in E$.

As usual we will always omit F in the notation in case $F = \mathbb{C}$.

For background on Banach algebras and on continuous n-homogeneous polynomials on Banach spaces we refer to [2].

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§2. Polynomials

Let E be a complex Banach space and F be a complex Banach algebra. We define $I\!P_f(^nE, F)$ as the space of all $P = \sum_{i=1}^{k} \varphi_i^n$ where $\varphi_i \in L(E, F)(i = 1, \cdots, k)$. If $F = \mathbb{C}$, it is clear that $I\!P_f(^nE, \mathbb{C}) = P_f(^nE)$ for all E and $n \in \mathbb{N}$. On the other hand, if $E = \mathbb{C}$, we have $I\!P_f(^n\mathbb{C}, F) \subset P_f(^n\mathbb{C}, F)$ for every F and $n \in \mathbb{N}$. Indeed, since $\lambda = \lambda \cdot 1$ for all $\lambda \in \mathbb{C}$, it is clear that $\varphi^n(\lambda) = \lambda^n(\varphi(1))^n$ for every $\varphi \in L(\mathbb{C}, F)$ and, consequently, if $P = \sum_{i=1}^{k} \varphi_i^n$ $(\varphi_i \in L(\mathbb{C}, F))$ we have $P(\lambda) = \lambda^n \sum_{i=1}^{k} (\varphi_i(1))^n = id_C^n \otimes b(\lambda)$ where $b = \sum_{i=1}^{k} (\varphi_i(1))^n \in F$.

In this section we are going to establish necessary and sufficient conditions in order to have $P_f(^nE, F) \subset I\!P_f(^nE, F)$.

Definition 2.1. (1) For each $n \in \mathbb{N}$, we say that F has the r_n-property if given any $b \in F$ there exists $\{a_1, a_2, \ldots, a_p\} \subset F$ such that $b = \sum_{i=1}^{p} \lambda_i \cdot a_i^n$ where $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$.

(2) We say that an algebra F has the r-property if F has the r_n-property $\forall n \in \mathbb{N}$.

Proposition 2.1. *Every complex algebra with identity has the r-property.*

Proof. Let F be a complex algebra with identity 1_F. Given an arbitrary $n \in \mathbb{N}$ and $b \in F$ we have $b = \sum_{l=1}^{n} \lambda_l \cdot a_l^n$ where $\lambda_l = (1/n^2)e^{2\pi i l/n}$ and $a_l = b + e^{2\pi i l/n} \cdot 1_F$ for each $l = 1, 2, \ldots, n$. Indeed

$$\begin{aligned}\sum_{l=1}^{n} e^{\frac{2\pi i l}{n}} \cdot (b + e^{\frac{2\pi i l}{n}} \cdot 1_F)^n &= \sum_{l=1}^{n} e^{\frac{2\pi i l}{n}} \cdot \sum_{r=0}^{n} \binom{n}{r} b^r \cdot e^{\frac{2\pi i l(n-r)}{n}} \\ &= \sum_{r=0}^{n} \binom{n}{r} b^r \cdot \sum_{l=1}^{n} e^{\frac{2\pi i (n-r+1) l}{n}} \\ &= \sum_{l=1}^{n} e^{\frac{2\pi i (n+1) l}{n}} + nb \cdot \left(\sum_{l=1}^{n} e^{2\pi i l} \right) \\ &\quad + \sum_{r=2}^{n} \binom{n}{r} b^r \cdot \sum_{l=1}^{n} e^{\frac{2\pi i (n-r+1) l}{n}}.\end{aligned}$$

As $\sum_{l=1}^{n} e^{2\pi i l} = n$ and $\sum_{l=1}^{n} e^{2\pi i (n-r+1)l/n} = 0$ $\forall r = 0, 2, 3, \cdots, n$ the statement follows. □

We remark that given an arbitrary Banach space $(F, || \ ||)$ we can always define a product $\odot$ on F in order that $(F, +, \odot)$ is an algebra with identity. We can also define in F a norm $||| \ |||$ that is equivalent to the original norm $|| \ ||$ and such that $(F, ||| \ |||)$ with above operations is a Banach algebra (with identity). Indeed, let G be a closed hyperplane of F and $e \in F \setminus G$ be such that $||e|| = 1$. Given $u = ae + x$ and $v = be + y$ (where $a, b \in \mathbb{C}$ and $x, y \in G$) we define $u \odot v := abe + (bx + ay)$. It is easy to verify that $(F, +, \odot)$ is an algebra with identity e. Now, if we define $|||u||| = |a| + ||x||$ for $u = ae + x$ ($a \in \mathbb{C}$ and $x \in G$) we get an equivalent norm on F and $(F, ||| \ |||)$ endowed with $+$ and $\odot$ is a Banach algebra with identity e.

Proposition 2.2. *Let E be a Banach space and F be a Banach algebra. Then $P_f(^nE, F) \subset I\!P_f(^nE, F)$ if and only if F has the r_n-property.*

Proof. If $P_f(^nE, F) \subset I\!P_f(^nE, F)$, then $\varphi^n \otimes b \in I\!P_f(^nE, F)$, for every $\varphi \in E'$ and $b \in F$. So, there exist $T_1, T_2, \cdots, T_p \in L(E, F)$ such that $\varphi^n \otimes b = \sum_{i=1}^{p} T_i^n$. Now, if $b \neq 0$ it is enough to take $\varphi \not\equiv 0$ and $x_0 \in E$ such that $\varphi(x_0) = 1$ in order to get $b = \varphi^n \otimes b(x_0) = \sum_{i=1}^{p} (T_i(x_0))^n$. The case $b = 0$ is trivial.

Reciprocally, let $P \in P_f(^nE, F)$. By definition there exist $\varphi_1, \varphi_2, \ldots, \varphi_p \in E'$ and $b_1, \ldots, b_p \in F$ such that $P = \sum_{i=1}^{p} \varphi_i^n \otimes b_i$. So, it is enough to show that $\varphi^n \otimes b \in I\!P_f(^nE, F)$ for every $\varphi \in E'$ and $b \in F$. As F has the r_n-property, there exist $b_1, \ldots, b_p \in F$ such that $b = \lambda_1 b_1^n + \cdots + \lambda_p b_p^n$ and consequently $\varphi^n \otimes b = \sum_{i=1}^{p} (\beta_i \varphi \otimes b_i)^n$ where $\beta_i^n = \lambda_i (i = 1, \ldots, p)$. □

As a consequence of Proposition 2.2 we get

Proposition 2.3. *Let E be a Banach space. The following are equivalent*:

(a) *E is a finite dimensional space*

(b) *$P_f(^nE, F) = I\!P_f(^nE, F)$ for every Banach algebra F with the r_n-property $\forall n \in \mathbb{N}$.*

(c) *$P_f(^nE, F) = I\!P_f(^nE, F)$ for every Banach algebra F with identity $\forall n \in \mathbb{N}$.*

Proof. By Proposition 2.2 we have $P_f(^nE, F) \subset I\!P_f(^nE, F)$ $\forall n \in \mathbb{N}$ whenever F is a Banach algebra with the r_n-property. Since $P_f(^nE, F) = P(^nE, F)$ if E is a finite dimensional space, we get $(a) \Longrightarrow (b)$. From Proposition 2.1 we have $(b) \Longrightarrow (c)$.

Now suppose that $P_f(^nE, F) = I\!P_f(^nE, F)$ for all $n \in \mathbb{N}$ and for every Banach algebra F with identity. Since we can define in E a product $\odot$ such that $(E, +, \odot)$ is an algebra with identity, we have that $P_f(^1E, E) = I\!P_f(^1E, E)$.

Now, $\overline{id_E(B_E)} = \overline{B_E}$ is compact since $P_f(^1E, E) = E' \otimes E \subset \mathcal{K}(E, E)$ and $id_E \in L(E, E) = I\!P_f(^1E, E)$. Consequently, $\dim E < \infty$. □

Next we are going to show by examples that when F has the r_n-property we may have $\overline{P_f(^nE, F)} \subsetneqq \overline{I\!P_f(^nE, F)}$ and $I\!P_f(^nE, F) \subsetneqq P(^nE, F)$.

Example 2.1. Let $E = c_0$ endowed with the pointwise product and let F be a Banach algebra with the r_n-property and such that $c_0 \subset F$ as a subalgebra. An example of such an algebra is given by l_∞ endowed with the pointwise product. We define $P : c_0 \longrightarrow F$ by $P(x) = x^n + \sum_{k=1}^{\infty} \lambda_k x_k^n \cdot e_k$ for all $x = (x_k)_{k\in\mathbb{N}}$ where $(\lambda_k)_{k\in\mathbb{N}} \in l_1$ with $\lambda_k > 0$ $\forall k \in \mathbb{N}$ and $\{e_k : k \in \mathbb{N}\}$ is the canonical basis of c_0. In other words, $P = id_{c_0}^n + \sum_{k=1}^{\infty} \lambda_k \pi_k^n \otimes e_k$, where, for each k, $\pi_k((x_i)_{i\in\mathbb{N}}) = x_k$.

Suppose that $||e_k||_F \leq 1, (k = 1, 2, \cdots)$. For every $x \in c_0$ we have $||\sum_{k=k_0}^{\infty} \lambda_k x_k^n \cdot e_k|| \leq \sum_{k=k_0}^{\infty} |\lambda_k| \cdot |x_k|^n \leq (\sum_{k=k_0}^{\infty} |\lambda_k|) \cdot ||x||^n$. Since $(\lambda_k)_{k\in\mathbb{N}} \in l_1$, given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0}^{\infty} |\lambda_k| < \epsilon$ and, consequently, $||\sum_{k=1}^{\infty} \lambda_k x_k^n \cdot e_k - \sum_{k=1}^{k_0-1} \lambda_k x_k^n \cdot e_k|| \leq \epsilon$ for all $x \in c_0$ such that $||x|| \leq 1$. So $P_m = id_{c_0}^n + \sum_{k=1}^{m} \lambda_k \pi_k^n \otimes e_k \in I\!P_f(^nc_0, F)$ is such that $P_m \overset{||\,||}{\longrightarrow} P$ and, consequently, $P \in \overline{I\!P_f(^nc_0, F)}$.

If $P \in \overline{P_f({}^n c_0, F)}$ we would have $P - \sum_{k=1}^{\infty} \lambda_k \pi_k^n \otimes e_k \in \overline{P_f({}^n c_0, F)}$. But this is a contradiction since $P - \sum_{k=1}^{\infty} \lambda_k \pi_k^n \otimes e_k = id_{c_0}^n$ and every element of $\overline{P_f({}^n c_0, F)}$ is compact. So, $P \in \overline{I\!P_f({}^n c_0, F)} \setminus \overline{P_f({}^n c_0, F)}$.

Example 2.2. Let $E = l_2$ and $F = l_1$ endowed with the pointwise product. Since $L(l_2, l_1) = \mathcal{K}(l_2, l_1)$, we have that every element of $I\!P_f({}^n l_2, l_1)$ is compact for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}, n \geq 2$, let $Q : l_2 \longrightarrow l_1$ be defined by $Q((x_k)_k) := (x_k^n)_k$. Since $(Q(e_k)) := (e_k) \subset Q(B_{l_2})$ doesn't admit convergent subsequence, we have that $Q \in P({}^n l_2, l_1) \setminus I\!P_f({}^n l_2, l_1)$.

Next we are going to give an example of E and F such that $\overline{P_f({}^n E, F)} \not\subset I\!P_f({}^n E, F)$ although $P_f({}^n E, F) \subsetneqq I\!P_f({}^n E, F)$.

For the construction of the first example we need the following result:

Proposition 2.4. *Let $E = c_0$ and $F = l_\infty$ endowed both with the pointwise product. If $P \in I\!P_f({}^n c_0, l_\infty)$, then $\pi_j \circ P \in P_f({}^n c_0), \forall j \in \mathbb{N}$.*

Proof. Let $P \in I\!P_f({}^n c_0, l_\infty)$. For each $j \in \mathbb{N}$, we define $P_j := \pi_j \circ P$ where $\pi_j((x_k)_k) = x_j$. We are going to show that $P_j \in P_f({}^n c_0)$ for all $j \in \mathbb{N}$. By definition, $P \in I\!P_f({}^n c_0, l_\infty)$ if and only if there exist $T_1, T_2, \ldots, T_p \in L(c_0, l_\infty)$ such that $P(x) = \sum_{k=1}^{p} T_k^n(x)$ for all $x \in c_0$. Consequently, $P_j(x) = \sum_{k=1}^{p} (\pi_j \circ T_k)^n(x)$ for every $x \in c_0$ and it is clear that $P_j \in P_f({}^n c_0)$ $\forall j \in \mathbb{N}$. $\square$

Example 2.3. Let $P : c_0 \longrightarrow l_\infty$ be defined by $P((x_k)_k) := a \cdot (\sum_{k=0}^{\infty} (1/2^k) x_k^n)$ where $a = (a_j) \in c_0, a \neq 0$. Since $a \neq 0$, there exists j such that $a_j \neq 0$ and, consequently, $\pi_j \circ P(x) = a_j \cdot (\sum_{k=0}^{\infty} (1/2^k) x_k^n) \notin P_f({}^n c_0)$. By Proposition 2.4 we get $P \notin I\!P_f({}^n c_0, l_\infty)$. On the other hand, for every $x \in c_0$ such that $\|x\| \leq 1$ we have

$$\left\| P(x) - a \cdot \sum_{k=0}^{p} \frac{1}{2^k} x_k^n \right\| = \left\| a \cdot \sum_{k=p+1}^{\infty} \frac{1}{2^k} x_k^n \right\|$$
$$= \sup_j \left| a_j \cdot \sum_{k=p+1}^{\infty} \frac{1}{2^k} x_k^n \right| \leq \|a\| \cdot \sum_{k=p+1}^{\infty} \frac{1}{2^k} \longrightarrow 0.$$

Consequently, $a \cdot (\sum_{k=0}^{p} (1/2^k) x_k^n) \longrightarrow P(x)$ in the usual norm in $P({}^n c_0, l_\infty)$ and, since $a \cdot (\sum_{k=0}^{p} (1/2^k) x_k^n) = \sum_{k=0}^{p} (1/2^k) \pi_k^n \otimes a(x)$, we have $P \in \overline{P_f({}^n c_0, l_\infty)}$. So, $P \in \overline{P_f({}^n c_0, l_\infty)} \setminus I\!P_f({}^n c_0, l_\infty)$ although $P_f({}^n c_0, l_\infty) \subsetneqq I\!P_f({}^n c_0, l_\infty)$

Even if E is not a subspace of F we may have $\overline{P_f({}^nE,F)} \setminus I\!P_f({}^nE,F) \neq \emptyset$ as the following example shows:

Example 2.4. Let $P : l_2 \longrightarrow l_2'$ be defined by $P((x_k)_k) := ((1/k^2)\cdot x_k^{n-1})_k$, i.e, $P = \sum_{k=1}^{\infty}(1/k^2)\pi_k^{n-1}\otimes e_k$. It is known that there exists a symmetric $(n-1)-$linear mapping $A : l_2^{n-1} \longrightarrow l_2'$ such that $P(x) = A(x,\ldots,x)$ for every $x \in l_2$. Since $\{A(\sum_{i=1}^p e_i,\ldots,\sum_{i=1}^p e_i) : p \in \mathbb{N}\}$ is linearly independent, we have that $\dim A(l_2\times\ldots\times l_2) = \infty$ and, consequently, $A \notin L_f({}^{n-1}l_2,l_2')$. Using the canonical isomorphism between $L({}^{n-1}l_2,l_2')$ and $L({}^nl_2)$, we associate to A an element $B \in L({}^nl_2)$ such that $A((x_{1k}),\ldots,(x_{n-1k}))(x_{nk}) = B((x_{1k}),\ldots,(x_{nk}))$. As this isomorphism identifies $L_f({}^{n-1}l_2,l_2')$ with $L_f({}^nl_2)$, we have $B \notin L_f({}^nl_2)$. Consequently, if $Q(x) = B(x,\ldots,x)$ we have $Q \notin P_f({}^nl_2)$.

On the other hand, $P \in \overline{P_f({}^{n-1}l_2,l_2')} \subset P_{wu}({}^{n-1}l_2,l_2')$ and, consequently, $A \in L_{wu}({}^{n-1}l_2,l_2')$. Since the above mentioned canonical isomorphism identifies $L_{wu}({}^{n-1}l_2,l_2')$ with $L_{wu}({}^nl_2)$, we have $B \in L_{wu}({}^nl_2)$ and, consequently, $Q \in P_{wu}({}^nl_2)$. Since l_2 has the a.p., $Q \in \overline{P_f({}^nl_2)}$. So, there exists $Q \in \overline{P_f({}^nl_2)} \setminus P_f({}^nl_2)$. Now, let $R : l_2 \longrightarrow l_1$ be defined by $R(x) = (Q\otimes e_1)(x) = Q(x)\cdot e_1$. It is clear that $R \in P({}^nl_2,l_1)$.

If we consider l_1 endowed with the pointwise product, we can prove as in Proposition 2.4, that $R \in I\!P_f({}^nl_2,l_1)$ implies $\pi_j \circ R \in P_f({}^nl_2)\ \forall j$. But, $\pi_1 \circ R = Q \notin P_f({}^nl_2)$ and so $R \notin I\!P_f({}^nl_2,l_1)$. Finally, we have $R \in \overline{P_f({}^nl_2,l_1)}$ and so $\overline{P_f({}^nl_2,l_1)} \setminus I\!P_f({}^nl_2,l_1) \neq \emptyset$.

We remark that we don't know if l_1 has the r_n-property when we consider l_1 endowed with the pointwise product. So, in this case, we don't know if $P_f({}^nl_2,l_1) \subset I\!P_f({}^nl_2,l_1)$.

By considering l_1 endowed with a convenient product, we will show in Example 2.5 that in fact, we can have E and F such that $P_f({}^nE,F) \subsetneqq I\!P_f({}^nE,F) \subsetneqq \overline{P_f({}^nE,F)}$.

Example 2.5. Let $E = l_2$ and $F = l_1$ endowed with the following product: $(x_k)\odot(y_k)_k := (z_k)_k$ where $z_1 = x_1\cdot y_1$ and $z_k = x_1\cdot y_k + y_1\cdot x_k\ \forall k \geq 2$. We remark that, since $(x_k)_k = x_1\cdot e_1 + \sum_{k=2}^{\infty} x_k\cdot e_k$ and $(y_k) = y_1\cdot e_1 + \sum_{k=2}^{\infty} y_k\cdot e_k$, this is the product defined after Proposition 2.1, and so $l_1 = (l_1,+,\odot)$ is a comutative algebra with identity.

From Propositions 2.1 and 2.2 we have $P_f({}^nl_2,l_1) \subset I\!P_f({}^nl_2,l_1)$. If $T : l_2 \longrightarrow l_1$ is defined by $T((x_k)) := ((1/k)x_k)_k$ for every $(x_k) \in l_2$, we have clearly $T^n \in I\!P_f({}^nl_2,l_1)$. On the other hand, it is easy to check that, $T^n(\sum_{i=1}^p e_i) =$

$(1, n/2, \cdots, n/p, 0, 0, \cdots)$ for each $p \in \mathbb{N}$ and so $\{T^n(\sum_{i=1}^p e_i) : p \in \mathbb{N}\}$ is linearly independent. Consequently, $\dim T^n(l_2) = \infty$ and $T^n \notin P_f(^n l_2, l_1)$.

Since $L(l_2, l_1) = \mathcal{K}(l_2, l_1) = L_{wu}(l_2, l_1)$, it is clear that $I\!P_f(^n l_2, l_1) \subset P_{wu}(^n l_2, l_1) = \overline{P_f(^n l_2, l_1)}$.

Finally, we are going to show that $\overline{P_f(^n l_2, l_1)} \setminus I\!P_f(^n l_2, l_1) \neq \emptyset$.

If $P \in I\!P_f(^n l_2, l_1)$, there exist $T_1, \ldots, T_p \in L(l_2, l_1)$ such that $P = \sum_{i=1}^p T_i^n$. For each $x \in l_2$, let $T_{1i}(x) = \pi_1 \circ T_i(x) \in \mathbb{C}$. From the definition of $\odot$ we have that $\pi_1 \circ T_i^n(x) = \pi_1[T_i(x) \odot \ldots \odot T_i(x)] = (T_{1i}(x))^n = (\pi_1 \circ T_i)^n(x)$ for every $x \in l_2$. So, $\pi_1 \circ P = \sum_{i=1}^p (\pi_i \circ T_i)^n$ and, since $\pi_1 \circ T_i \in l_2'$ for all $i = 1, \ldots, p$ we get $\pi_1 \circ P \in P_f(^n l_2)$.

Now, take $Q \in \overline{P_f(^n l_2)} \setminus P_f(^n l_2)$ and define $R : l_2 \longrightarrow l_1$ by $R = Q \otimes e_1$. From $\pi_1 \circ R = Q \notin P_f(^n l_2)$ we infer $R \notin I\!P_f(^n l_2, l_1)$. Since $Q \in \overline{P_f(^n l_2)}$, it follows that for arbitrary $\epsilon > 0$, there exist $\varphi_1, \cdots, \varphi_p \in l_2'$ such that $||Q - \sum_{i=1}^p \varphi_i^n|| < \epsilon$. Note that $\sum_{i=1}^p (\varphi_i \otimes e_1)^n = \sum_{i=1}^p \varphi_i^n \otimes e_1 \in P_f(^n l_2, l_1)$. Then $||R - \sum_{i=1}^p (\varphi_i \otimes e_1)^n|| < \epsilon$ and hence $R \in \overline{P_f(^n l_2, l_1)} \setminus I\!P_f(^n l_2, l_1)$.

We remark that $\overline{I\!P_f(^n l_2, l_1)} = \overline{P_f(^n l_2, l_1)}$ for all $n \in \mathbb{N}$. This equality is true in the following general situation:

Proposition 2.5. *If E is a Banach space and F is a commutative Banach algebra such that $L(E, F) = \mathcal{K}(E, F)$, then $\overline{I\!P_f(^n E, F)} \subset P_{wu}(^n E, F)$ $\forall n \in I\!N$. If, in additon, F has the r_n-property and E' has the approximation property, we have $\overline{P_f(^n E, F)} = \overline{I\!P_f(^n E, F)}$.*

Proof. Let $P \in I\!P_f(^n E, F)$. For every bounded subset B of E let (x_α) be a net in B such that (x_α) converges weakly to $x \in B$. Since $L(E, F) = \mathcal{K}(E, F) = L_{wu}(E; F)$, we have that $(T(x_\alpha))$ converges to $T(x)$ for all $T \in L(E, F)$. As, by definition of $I\!P_f(^n E, F)$, there exist $T_1, T_2, \ldots, T_p \in L(E, F)$ such that $P = \sum\limits_{i=1}^{p} T_i^n$, it is clear that $(P(x_\alpha))$ converges to $P(x)$. So, $I\!P_f(^n E, F) \subset P_{wu}(^n E, F)$ and since $P_{wu}(^n E; F)$ is closed it follows that $\overline{I\!P_f(^n E, F)} \subset P_{wu}(^n E, F)$.

If, in addition, F has the r_n-property and E' has the approximation property $P_f(^n E, F) \subset I\!P_f(^n E, F)$ by Proposition 2.2 and $P_{wu}(^n E, F) = \overline{P_f(^n E, F)}$ by Proposition 2.7 of [1] and so $\overline{I\!P_f(^n E, F)} = \overline{P_f(^n E, F)}$. ☐

§3. The Space $I\!P(^n E, F)$

Let $I\!P(^n E, F) = \{P \in P(^n E, F) : P = \sum_{i=1}^\infty \varphi_i^n$ $(\varphi \in L(E, F))$ and $\sum_{i=1}^\infty \|\varphi_i\|^n < \infty\}$ endowed with $|||P||| := \inf\{\sum_{i=1}^\infty \|\varphi_i\|^n : P = \sum_{i=1}^\infty \varphi_i^n\}$

where the infimum are taken over all possible representations of P. It is clear that $|||\cdot|||$ is a norm and $||P|| \leq |||P|||$ for all $P \in I\!P({}^nE,F)$.

If $P = \varphi^n$ for some $\varphi \in L(E,F)$, we have that φ^n is a representation of P such that $||\varphi||^n < \infty$ and so $|||\varphi^n||| \leq ||\varphi||^n$. Standard arguments show that the completion of $(I\!P_f({}^nE,F),|||\ |||)$ is $I\!P({}^nE,F)$ and so we have $\overline{(I\!P_f({}^nE,F),|||\ |||)} = I\!P({}^nE,F)$.

Proposition 3.1. *The mapping* $\beta : I\!P({}^nE,F)' \longrightarrow \{Q \in P({}^nL(E,F))\colon \sum_{i=1}^{\infty} Q(\varphi_i) = 0$ *if* $\sum_{i=1}^{\infty}\varphi_i^n = 0\}$ *defined by* $\beta(T)(\varphi) := T(\varphi^n)$ *for every* $\varphi \in L(E,F)$ *establishes an isometric isomorphism between the two spaces. Under this isomorphism the equicontinuous subsets of* $I\!P({}^nE,F)'$ *correspond to the locally bounded subsets of* $\{Q \in P({}^nL(E,F)) : \sum_{i=1}^{\infty} Q(\varphi_i) = 0$ *if* $\sum_{i=1}^{\infty}\varphi_i^n = 0\}$*.*

Proof. Given $T \in I\!P_f({}^nE,F)'$, for every $\sum_{i=1}^{p}\varphi_i^n = 0$ we have $\sum_{i=1}^{p}\beta T(\varphi_i) = \sum_{i=1}^{p} T(\varphi_i^n) = T(\sum_{i=1}^{p}\varphi_i^n)$. So, β is well defined. Clearly β is linear. Also for every $\varphi \in L(E,F), |\beta T(\varphi)| = |T(\varphi^n)| \leq ||T||\cdot|||\varphi^n||| \leq ||T||\cdot||\varphi||^n$ so $||\beta T|| \leq ||T||$. Conversely, given an arbitrary $P \in I\!P_f({}^nE,F)$, for all representation $P = \sum_{i=1}^{m}\varphi_i^n$ of P we have $|T(P)| = |T(\sum_{i=1}^{m}\varphi_i^n)| \leq \sum_{i=1}^{m}|T(\varphi_i^n)| = \sum_{i=1}^{m}|\beta T(\varphi_i)| \leq ||\beta T||\cdot\sum_{i=1}^{m}||\varphi_i||^n$ so that $|T(P)| \leq ||\beta T||\cdot|||P|||$ and consequently $||T|| \leq ||\beta T||\quad \forall T \in I\!P_f({}^nE,F)'$. Since $I\!P_f({}^nE,F)' = I\!P({}^nE,F)'$ we have $||\beta T|| = ||T||\quad \forall T \in I\!P({}^nE,F)$ and so β is an isometry and hence 1-1. Let $Q \in P({}^nL(E,F))$ such that $\sum_{i=1}^{\infty} Q(\varphi_i) = 0$ whenever $\sum_{i=1}^{\infty}\varphi_i^n = 0$. In particular, $\sum_{i=1}^{m} Q(\varphi_i) = 0$ whenever $\sum_{i=1}^{m}\varphi_i^n = 0$ (where $m \in \mathbb{N}$ is arbitrary). We may define $T_Q : I\!P_f({}^nE,F) \longrightarrow \mathbb{C}$ by $T_Q(P) = \sum_{i=1}^{m} Q(\varphi_i)$ where $P = \sum_{i=1}^{m}\varphi_i^n$ is a representation of P. If $\sum_{i=1}^{m}\varphi_i^n = \sum_{i=1}^{p}\psi_i^n$ we have $\sum_{i=1}^{m}\varphi_i^n + \sum_{i=1}^{p}(\lambda_i\psi_i)^n = 0$ (where $\lambda_i \in \mathbb{C}$ is such that $\lambda_i^n = -1$).

Since $\sum_{i=1}^{m} Q(\varphi_i) + \sum_{i=1}^{p} Q(\lambda_i\cdot\psi_i) = 0$ we have $T_Q\left(\sum_{i=1}^{m}\varphi_i^n\right) - T_Q\left(\sum_{i=1}^{p}\psi_i^n\right) = \sum_{i=1}^{m} Q(\varphi_i) - \sum_{i=1}^{p} Q(\psi_i) = \sum_{i=1}^{m} Q(\varphi_i) + \sum_{i=1}^{p}\lambda_i^n Q(\psi_i) = 0$. This means that T_Q is well defined. It is clear that T_Q is linear and $|T_Q\left(\sum_{i=1}^{m}\varphi_i^n\right)| = |\sum_{i=1}^{m} Q(\varphi_i)| \leq \sum_{i=1}^{m}|Q(\varphi_i)| \leq ||Q||\cdot\sum_{i=1}^{m}||\varphi_i||^n$ for every representation $\sum_{i=1}^{m}\varphi_i^n$ of P. So, $|T_Q(P)| \leq ||Q||\cdot|||P|||$ for every $P \in I\!P_f({}^nE,F)$. Accordingly T_Q defines a continuous linear function on $I\!P_f({}^nE,F)$ which can be extended uniquely to a continuous linear function $\tilde{T_Q}$ on $I\!P({}^nE,F)$ such that $\beta\tilde{T_Q}(\varphi) = T_Q(\varphi^n) = Q(\varphi)$ for every $\varphi \in L(E,F)$. Hence β establishes an isometric isomorphism between $I\!P({}^nE,F)'$ and $\{Q \in P({}^nL(E,F)) : \sum_{i=1}^{\infty} Q(\varphi_i) = 0$ if $\sum_{i=1}^{\infty}\varphi_i^n = 0\}$.

Let $\aleph$ be an equicontinuous subset of $I\!\!P(^nE, F)'$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $\sup_{T\in\aleph} |T(P)| < \epsilon$ whenever $|||P||| < \delta$.

Let $r \in \mathbb{R}$ such that $r^n = \delta$. If $L = \{\varphi \in L(E, F) : ||\varphi|| < r\}$, we have $\sup_{\varphi\in L} |\beta T(\varphi)| = \sup_{\varphi\in L} |T(\varphi^n)| < \epsilon, \forall T \in \aleph$ since $|||\varphi^n||| \leq ||\varphi||^n < \delta$. So, $\{\beta T : T \in \aleph\}$ is locally bounded. □

Remark 3.1. For each compact subset K of E we consider the seminorm p_K in $I\!\!P(^nE, F)$ defined by $p_K(P) := \inf\{\sum_{i=1}^{\infty} ||\varphi_i||_K^n : P = \sum_{i=1}^{\infty} \varphi_i^n\}$ where the infimum is taken over all possible representations of P. Let τ_0 be the locally convex topology generated by the family $\{p_K : K \subset E$ compact $\}$. We denote by $L_0(E, F)$ the space $L(E, F)$ endowed with the compact open topology. A slight modification of the arguments from Proposition 3.1 shows that the mapping defined by $\beta(T)(\varphi) := T(\varphi^n)$ for every $\varphi \in L(E, F)$, establishes a continuous isomorphism from $(I\!\!P(^nE, F), \tau_0)'$ onto $\{Q \in P(^nL(E, F)) : \sum_{i=1}^{\infty} Q(\varphi_i) = 0$ if $\sum_{i=1}^{\infty} \varphi_i^n = 0\}$ and transforms the equicontinuous subsets of $(I\!\!P(^nE, F), \tau_0)'$ onto equicontinuous subsets of $\{Q \in P(^nL_0(E, F)) : \sum_{i=1}^{\infty} Q(\varphi_i) = 0$ if $\sum_{i=1}^{\infty} \varphi_i^n = 0\}$.

Example 3.1. Let $T_k : c_0 \longrightarrow c_0$ be defined by $T_k((x_l)) = (y_l)$ where $y_l = 0 \quad \forall l \neq k$ and $y_k = x_k/\sqrt[n]{k}$. Let $P_m = \sum_{k=1}^{m} T_k^n \in I\!\!P_f(^nc_0, c_0)$ for all $m \in \mathbb{N}$. It is easy to show that $(P_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $I\!\!P_f(^nc_0, c_0)$ and $(P_m)_{m\in\mathbb{N}}$ converges to $P = \sum_{k=1}^{\infty} T_k^n$.

Now, $\sum_{k=1}^{\infty} ||T_k||^n$ diverges since $||T_k||^n = 1/k$ for all $k \in \mathbb{N}$. So, $P \in \overline{I\!\!P_f(^nc_0, c_0)}^{||\,||} \setminus I\!\!P(^nc_0, c_0)$ i.e., $\overline{I\!\!P_f(^nc_0, c_0)}^{|||\,|||} \subsetneqq \overline{I\!\!P_f(^nc_0, c_0)}^{||\,||}$

Remark 3.2. Let $P_N(^nE, F)$ denote the space of all nuclear n-homogeneous polynomials from E into F, i.e, of all $P \in P(^nE, F)$ such that $P(x) = \sum_{k=1}^{\infty} \varphi_k^n(x)b_k$ for every $x \in E$ where $(\varphi_k)_{k\in\mathbb{N}} \subset E'$ and $(b_k)_{k\in\mathbb{N}} \subset F$ are sequences satisfying $\sum_{k=1}^{\infty} ||\varphi_k||^n||b_k|| < \infty$. We consider $P_N(^nE, F)$ endowed with the nuclear norm $||P||_N = \inf \sum_{k=1}^{\infty} ||\varphi_k||^n \cdot ||b_k||$ where the infimum is taken over all sequences (φ_k) and (b_k) that satisfy the definition. For $n = 1$, we always have $P_N(^1E, F) \subset I\!\!P(^1E, F)$, but it is not clear if this inclusion remains true in case $n \geq 2$ for all Banach space E and all Banach algebra F.

References

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