# Comparison Theory of Distance Spheres along Geodesics

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Abstract. Estimates for the principal curvature of distance spheres in Riemannian manifolds with sectional curvature bounded from below are well known. The same holds for the mean curvature of distance spheres in Riemannian manifolds with Ricci curvature bounded from below.

In this article we present new estimates for convexity properties of the distance function of a point under different assumptions, for example for manifolds with lower bounds on the conjugate or on the focal radius in addition to these curvature conditions.

The main idea is to introduce a new tensor field describing the differential of the exponential map and verifying a Riccati equation. This technique allows us to get new estimates for the volume form and for Jacobi fields in this context but also to gain new insights into well-known comparison theorems.

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[1]The author would like to mention that this paper is based on his PhD work ([6]) and that Uwe Abresch was his doctoral advisor.

## 1 Introduction and main results

In Riemannian geometry one aims to compare the local geometry of complete Riemannian manifolds $M$ with those of reference manifolds, usually with manifolds of constant curvature $M^n_\lambda$. Convexity properties of distance functions $d_p = d(p,.)$ of a point $p \in M$ are often useful for getting information about the global structure of $M$. Therefore we consider the Hessian $A = \operatorname{Hess} d_p$. $A$ can be seen as the second fundamental tensor or Weingarten map of the distance spheres centered at $p$, and $\frac{\operatorname{tr} A}{n-1} = \frac{\Delta d_p}{n-1}$ is the mean curvature of those spheres. The research question of this article is: What are necessary and sufficient conditions to get estimates for $\operatorname{Hess} d_p$ or $\Delta d_p$?
The task of finding estimates for $A$ can be reduced to an ordinary differential equation problem. $A$ is a symmetric tensor field along geodesics $c$ starting at $p$, the integral curves of $\operatorname{grad} d_p$. The changing of $A$ is described by the Riccati equation $\frac{\nabla}{dt} A_{c(t)} + A^2_{c(t)} + R = 0$, where $R = R_{\dot c} = R(.,\dot c)\dot c$ is the curvature tensor in the direction of $\dot c$. Firstly, this follows from the general equation $(\nabla_N \operatorname{Hess} f)(X) + (\operatorname{Hess} f)^2(X) + R_N(X) = \nabla_X \nabla_N N$ for functions $f : M \to \mathbb{R}$ with $N = \operatorname{grad} f$, $X$ a vector field and $R_N(X) = R(X,N)N$ the curvature tensor in direction $N$. Secondly, $N$ is autoparallel because $\|\operatorname{grad} f\| = 1$ for distance functions. $\operatorname{Hess} d_{p|q}$ at a point $q$ except at $p$ and the cut locus of $p$ is therefore entirely determined by the Riccati equation and the curvature tensor $R$ along the minimal geodesic from $p$ to $q$. $\operatorname{grad} d_{p|q}$ is an eigenvector of $\operatorname{Hess} d_p$ to the eigenvalue 0. Since distance spheres are increasingly convex with decreasing radius, $A$ is developing in $t = 0$ a pole of order 1 as in the Euclidean space. In general, the Laurent expansion of $A$ starts as $A(t) = A_{c(t)} = \frac{1}{t} I - \frac{R(0)}{3} t + O(t^2)$ as $t \to 0$.
In addition, for Jacobi fields $J$ orthogonal along geodesics $c$ starting in $p$ with $J(0) = 0$ and $\langle \frac{\nabla}{dt} J(0), \dot c(0)\rangle = 0$, we have $\frac{\nabla}{dt} J(t) = A_{c(t)} J(t)$. This relation shows that the solution of the Riccati equation holds up to the first conjugate point. In fact, it is natural to suppose that the geodesic segment is only of minimal length in all variations with fixed end points. $A$ can be interpreted as the shape operator of the Euclidean spheres centered in $0_p$ in the tangent space $T_pM$ with the pullback metric ${\exp_p}^* g(v,w) = g(\exp_* v, \exp_* w)$, which is positive definite outside critical points of $\exp_p$. The images of these spheres under the exponential map form locally a parallel hypersurface family $(S_t)_{t>0}$ along geodesics starting at $p$ up to the first conjugate point. These hypersurfaces $S_t$ are the levels of $d_p$ up to the cut locus of $p$ and the levels of a local distance function $d : \exp_p(U) \to \mathbb{R}$ given by $d(q) = \|(\exp_{p|U})^{-1}(q)\|$ for an open neighborhood $U$ of a non-critical point of $\exp_p$.
$A$ develops a singularity along $c$ in the first conjugate point $c(c_0)$, $c_0 > 0$. To be precise, $A$ is of the form $\frac{1}{t-c_0} P + B(t)$ near $c_0$. If $k > 0$ is the multiplicity of the conjugate point and $J_1, ..., J_k$ are the Jacobi fields with $J_i(0) = 0$ and $J_i(c_0) = 0$, so $P$ is the orthogonal projection on the linear subspace spanned by $J_i'(c_0)$. $B$ is differentiable in $c_0$ and vanishes on the image of $P$, i.e. $B(c_0)P = 0$ (cf. [11, Remark 2] and Section 6.2). In particular, the minimal eigenvalue of

$A$ goes to $-\infty$ at $c_0$.
To compare $A$ with the geometry of the model spaces $M^n_\lambda$ of constant curvature $\lambda \in \mathbb{R}$, we denote by $\mathrm{sn}_\lambda$ and $\mathrm{cs}_\lambda$ the solutions of the Jacobi equation $y''+\lambda y = 0$ with $\mathrm{sn}_\lambda(0) = 0$, $\mathrm{sn}'_\lambda(0) = 1$, $\mathrm{cs}_\lambda(0) = 1$ and $\mathrm{cs}'_\lambda(0) = 0$. We set $\mathrm{ct}_\lambda = \frac{\mathrm{sn}'_\lambda}{\mathrm{sn}_\lambda} = \frac{\mathrm{cs}_\lambda}{\mathrm{sn}_\lambda}$. If the curvature along $c$ is controlled by $\lambda \leq R \leq \Lambda$, the upper and lower bounds for $A$ are of type $\frac{1}{t} + O(t)$ as $t \to 0$. More precisely, the Riccati comparison (cf. for example [11]) yields $\mathrm{ct}_\Lambda I \leq A \leq \mathrm{ct}_\lambda I$. If the Ricci curvature along $c$ is bounded from below by $\mathrm{Ric}(\dot c) \geq (n-1)\lambda$, then the Riccati comparison gives $\mathrm{tr}\, A \leq (n-1)\,\mathrm{ct}_\lambda$.
These relations are used to prove, for example, the comparison theorems like Rauch's comparison theorems, the Bishop-Gromov volume comparison, Toponogov's theorem, Myers' theorem, Cheng's maximal diameter sphere therorem or Cheeger-Gromoll's splitting theorem (cf. [16], [20]). The key to this technique along a geodesic is that the second-order Jacobi equation $J'' = -RJ$ is split into two first-order equations

$$J' = AJ \quad \text{and} \quad A' + A^2 + R = 0 \tag{1.1}$$

and that one can get estimates of the solution of the non-linear Riccati equation which leads to Jacobi field estimates. These estimates are important because many proofs use geodesic variations. An alternative technique is the use of the index form (cf. [13, Section 4.5]).
In Theorem 1 we give new lower bounds for $\mathrm{Hess}\, d_p$ and $\Delta d_p$ of type $\frac{1}{t} + O(1)$ as $t \to 0$.

Theorem 1 (Convexity properties of the distance function of a point, cf. [6], Theorem 3.12). *Let $(M^n, g)$ be a complete $n$-dimensional Riemannian manifold, $c : [-l_1, l_2] \to M$ a normal, minimal geodesic, $l_1, l_2 > 0$ and $\delta(t) = \mathrm{ct}_\lambda(t + l_1) + \mathrm{ct}_\lambda(l_2 - t)$. Then for the distance function $d_{c(0)} : M \to \mathbb{R}$ of $c(0)$ we have along $c$ the following estimates:*

1. *If the curvature tensor $R = R(., \dot c)\dot c$ along $c$ is bounded from below by $\lambda \in \mathbb{R}$ on $[-l_1, l_2]$ we have for $t \in (0, l_2)$ and $v \in T_{c(t)}M$, $\|v\| = 1$, $\langle v, \dot c(t)\rangle = 0$*
$$-\delta(t) \leq \langle \mathrm{Hess}\, d_{c(0)|c(t)} v, v\rangle - \mathrm{ct}_\lambda(t) \leq 0 \tag{1.2}$$

2. *If the Ricci curvature along $c$ is bounded from below by $(n-1)\lambda$, $\lambda \in \mathbb{R}$ on $[-l_1, l_2]$, i.e. $\mathrm{Ric}(\dot c(t)) \geq (n-1)\lambda$, we have for $t \in (0, l_2)$*
$$-\delta(t) \leq \frac{\Delta d_{c(0)|c(t)}}{n-1} - \mathrm{ct}_\lambda(t) \leq 0 \tag{1.3}$$

Remark 1. *For the lower bound in (1.2) and (1.3) we have*

$$\delta(t) = \mathrm{ct}_\lambda(t + l_1) + \mathrm{ct}_\lambda(l_2 - t) = \frac{\mathrm{sn}_\lambda(l_1 + l_2)}{\mathrm{sn}_\lambda(t + l_1)\,\mathrm{sn}_\lambda(l_2 - t)}.$$

*$\delta \cdot I$ is the Hessian of triangle length excess function of the points $c_\lambda(-l_1)$ and $c_\lambda(l_2)$ of a normal comparison geodesic $c_\lambda$ restrained to the orthogonal complement of $\dot c_\lambda$ in the model space $M^n_\lambda$.*

Remark 2. *Theorem 1 is a generalization of a well-known estimate of the Hessian of distance spheres which can be obtained by considering the triangle length excess function $e_{0,l_2}(q) = d_{c(0)}(q) + d_{c(l_2)}(q) - l_2$ (cf. [1, remark to Lemma 1.4]). This identity is the special case $l_1 = 0$ of (1.5) (see also (1.9)). $e_{0,l_2}$ is convex on $c_{[0,l_2]}$, i.e. $0 \le \operatorname{Hess} e_{0,l_2|c(t)} \le \Delta e_{0,l_2|c(t)}$ or equivalent $\operatorname{Hess} d_{c(0)|c(t)} \ge -\operatorname{Hess} d_{c(l_2)|c(t)}$. So upper bounds for $\operatorname{Hess} d_{c(l_2)|c(t)}$ can be converted to lower bounds for $\operatorname{Hess} d_{c(0)|c(t)}$. However, these lower bounds do not describe the pole of $\operatorname{Hess} d_{c(0)}$ in $c(0)$ as Theorem 1 does.*

Remark 3. *In Theorem 1 and 2 we assume for regularity reasons of the distance function that the geodesic $c$ is minimal. This guarantees the existence of $\operatorname{Hess} d_p$ but also of the shape operator of the distance spheres of $c(-l_1)$ and $c(l_2)$ along $c$ on $(-l_1, l_2)$. As a matter of fact, the estimate also holds for the shape operator $A$ of the distance spheres of $c(0)$ along $c$ if the geodesic segment $c_{[-l_1,l_2]}$ has no conjugate points. This condition replaces the commonly used stronger assumption on the upper sectional curvature bound (see also Theorem 8).*

Remark 4. *It is also possible to have non-constant lower sectional curvature or Ricci curvature bounds. The non-constant curvature bounds are interesting, for example, in the examination of almost flat manifolds (cf. [27]). In [17] they proved a generalized Toponogov's theorem for this geometry. A complete analytic proof of Theorem 1 is given after the proof of Theorem 2 in Section 3.*

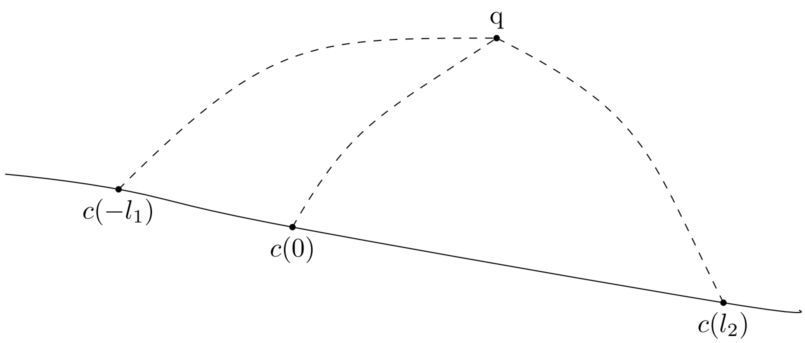
*Geometric Proof.* The upper bound in (1.2) and (1.3) follows from the Riccati comparison. Let $d_{c(-l_1)} : M \to \mathbb{R}$ be the distance function of $c(-l_1)$ and $d_{c(l_2)} : M \to \mathbb{R}$ the distance function of $c(l_2)$. The idea is to compare the distance spheres of $c(0)$ with those of $c(-l_1)$ and $c(l_2)$.
Therefore we consider the triangle length excess functions

$$e_{-l_1,l_2}(q) = d_{c(-l_1)}(q) + d_{c(l_2)}(q) - (l_1 + l_2)$$

and

$$e(q) = l_1 + d_{c(0)}(q) - d_{c(-l_1)}(q) \tag{1.4}$$

(see Figure 1). $e_{-l_1,l_2}(q)$ measures the length of the detour from $c(-l_1)$ to $c(l_2)$ going via $q$. $e(q)$ is the length of the detour from $c(-l_1)$ to $q$ always going via $c(0)$. The latter is a kind of reverse triangle inequality for the triangle $c(-l_1)$, $c(0)$ and $q$. The segment $c_{[-l_1,0]}$ is always greater than the difference of the other two sides. Because $c : [-l_1, l_2] \to M$ is minimal, we conclude from the triangle inequality that $e_{-l_1,l_2} \ge 0$, $e_{-l_1,l_2}(c(t)) = 0$ for $t \in [-l_1, l_2]$, $e \ge 0$ and $e(c(t)) = 0$ for $t \in [0, l_2]$. That is why $e_{-l_1,l_2}$ and $e$ are convex on




triangle inequality: $d(c(-l_1), c(l_2)) = l_1 + l_2 \leq d_{c(-l_1)}(q) + d_{c(l_2)}(q)$
reverse triangle inequality: $|d_{c(-l_1)}(q) - d_{c(0)}(q)| \leq l_1 = d(c(-l_1), c(0))$

Figure 1: triangle length excess functions

$c_{|(0,l_2)}$ (see Figure 2). $\operatorname{Hess} e_{-l_1,l_2} \geq 0$ and $\operatorname{Hess} e \geq 0$ means that the principal curvature of the spheres of $c(0)$ are bounded from the bottom by those of $c(-l_1)$ which for its part is bounded from the bottom by upper bounds of distance spheres of $c(l_2)$. We have

$$\begin{aligned} d_{c(0)}(q) = l_1 &+ d_{c(0)}(q) - d_{c(-l_1)}(q) \\ &+ d_{c(-l_1)}(q) + d_{c(l_2)}(q) - (l_1 + l_2) \\ &- d_{c(l_2)}(q) + l_2 \\ = e&(q) + e_{-l_1,l_2}(q) - d_{c(l_2)}(q) + l_2. \end{aligned} \tag{1.5}$$

By the Riccati comparison, the Hessian and the Laplacian of $d_{c(l_2)}$ in $c(t)$ are bounded from above by $\operatorname{ct}_\lambda(l_2 - t)$ and $(n-1)\operatorname{ct}_\lambda(l_2 - t)$, respectively. The lower bound now follows from this and from the lower convexity bound for $e$ in Theorem 2. $\square$

Corollary 1 (cf. [4], Theorem 5.6). *Let $(M^n, g)$ be a complete n-dimensional Riemannian manifold, $c : \mathbb{R} \to M$ a normal geodesic. Choose $i_0 > 0$ such that $c_{|[s,s+i_0]}$ is minimal for all $s \in [-\frac{i_0}{2}, 0]$. Then in Theorem 1 the lower bound can be defined by*

$$\delta(t) = \begin{cases} 2\operatorname{ct}_\lambda(\frac{i_0}{2}) & \text{for } 0 \leq t \leq \frac{i_0}{2} \\ \operatorname{ct}_\lambda(t) + \operatorname{ct}_\lambda(i_0 - t) & \text{for } \frac{i_0}{2} \leq t < i_0. \end{cases} \tag{1.6}$$

*In particular,* $\operatorname{Hess} d_{c(0)|\{\dot c\}^\perp}$ *is strictly convex or* $\Delta d_{c(0)} > 0$ *along* $c_{|(0,f_0)}$ *with* $f_0 > 0$ *such that* $\operatorname{ct}_\lambda(f_0) = 2\operatorname{ct}_\lambda(\frac{i_0}{2})$.

*Proof.* As the injectivity radius is a continuous function, $i_0$ with this property exists. For a given $t \in (0, \frac{i_0}{2})$ the function $\delta$ of Theorem 1 is minimal when $t$

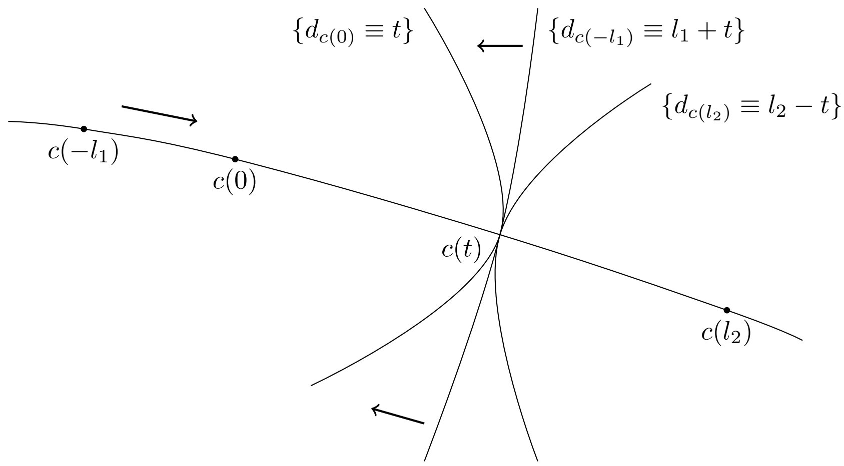



Figure 2: comparison of distance spheres

is in the middle of $[-l_1, l_2]$, that is $l_1 = \frac{i_0}{2} - t$ and $l_2 = t + \frac{i_0}{2}$. For a given $t \in [\frac{i_0}{2}, i_0)$ choose $l_1 = 0$ and $l_2 = i_0$. $\square$

In Section 6 we discuss the results and apply them to get new Jacobi field estimates (see Section 6.3), some rigidity properties (see Section 6.4) and volume estimates (see Section 6.5) in this context. The analytic examples in Section 7 show the sharpness of the results and allow us a better understanding. It ends with a geometric example constructed by a sequence of surfaces of revolution. The key argument of the proof of Theorem 1 is the following Theorem 2.

Theorem 2 (Comparison theorem for the convexity of a reverse triangle length excess function). *Let $(M^n, g)$ be a complete $n$-dimensional Riemannian manifold, $c : [-l_1, l_2] \to M$ a normal minimal geodesic, $l_1, l_2 > 0$ and $\epsilon(t) = \mathrm{ct}_\lambda(t) - \mathrm{ct}_\lambda(t + l_1)$. Then the reverse triangle length excess function $e : M \to \mathbb{R}$ defined by $e(q) = l_1 + d_{c(0)}(q) - d_{c(-l_1)}(q)$ is convex along $c_{|(0,l_2)}$.*

1. *If the curvature tensor $R(., \dot{c})\dot{c}$ along $c$ is bounded from below by $\lambda \in \mathbb{R}$ on $[-l_1, l_2]$ we have for $t \in (0, l_2)$ and $v \in T_{c(t)}M$, $\|v\| = 1$, $< v, \dot{c}(t) >= 0$*

$$\langle \mathrm{Hess}\, e_{|c(t)} v, v\rangle \geq \epsilon(t) > 0 \tag{1.7}$$

2. *If the Ricci curvature along $c$ is bounded from below by $(n-1)\lambda$ on $[-l_1, l_2]$, $\lambda \in \mathbb{R}$, i.e.* $\mathrm{Ric}(\dot{c}(t)) \geq (n-1)\lambda$, *we have for $t \in (0, l_2)$*

$$\frac{\Delta e_{|c(t)}}{n-1} \geq \epsilon(t) > 0 \tag{1.8}$$

Remark 5. *$\epsilon(t)$ are the non-zero eigenvalues of the Hessian of the corresponding excess function $e$ in the model space $M^n_\lambda$. It is*

$$\epsilon(t) = \mathrm{ct}_\lambda(t) - \mathrm{ct}_\lambda(t+l_1) = \frac{\mathrm{sn}_\lambda(l_1)}{\mathrm{sn}_\lambda(t)\,\mathrm{sn}_\lambda(l_1+t)}.$$

*So the Laplacian and the Hessian of $e$ are bounded from below by the geometry of the model space.*

Remark 6. *We consider $e_{0,l_2} = d_{c(0)} + d_{c(l_2)} - l_2$, the excess function of the points $c(0)$ and $c(l_2)$. As $e_{0,l_2}(q) = e(q) + e_{-l_1,l_2}(q)$ (see (1.5)), we have on the orthogonal complement of $\dot c$*

$$0 < \mathrm{Hess}\, e_{|c(t)} \le \mathrm{Hess}\, e_{0,l_2|c(t)} \le \Delta e_{0,l_2|c(t)} = \Delta d_{c(0)|c(t)} + \Delta d_{c(l_2)|c(t)}. \quad (1.9)$$

*Under the assumptions of Theorem 2 this gives a lower bound for the Hessian or the Laplacian of $e_{0,l_2}$ along the minimal geodesic connecting $c(0)$ and $c(l_2)$. $\mathrm{Hess}\, e_{0,l_2|c(t)}$ is only positive semidefinite when $c(l_2)$ is a conjugate point of $c(0)$, as for an orthogonal Jacobi field $J$ with $J(0) = 0$, $J(l_2) = 0$ we have $\mathrm{Hess}\, e_{0,l_2|c(t)} J(t) = 0$. In this case the inequalities (1.7) and (1.8) are not applicable.*
*The Riccati comparison gives upper bounds for the Hessian or Laplacian of $e_{0,l_2}$ and $e$. For other estimates of $e_{0,l_2}$ in the case of lower Ricci curvature bounds, we refer to [2, Proposition 2.3].*

Remark 7. *From the statement in Theorem 2 we deduce either*

$$\left(\tfrac{\partial}{\partial s}_{|s=0} \mathrm{Hess}\, d_{c(s)}\right)_{|c(t)} \ge \tfrac{1}{\mathrm{sn}^2_\lambda(t)} \quad \textit{or} \quad \left(\tfrac{\partial}{\partial s}_{|s=0} \tfrac{\Delta\, d_{c(s)}}{n-1}\right)_{|c(t)} \ge \tfrac{1}{\mathrm{sn}^2_\lambda(t)}\ ,$$

*(cf. [6, Theorem 1.2]) which means that the principal curvature or the mean curvature of the distance spheres at a given point $c(t)$ increases at least as much as in the model space when changing the center of the spheres along $c$. (see Figure 2). In Section 3 we show that this inequality holds when $R \ge \lambda$ or $\mathrm{Ric}(\dot c) \ge (n-1)\lambda$ on $c_{|[0,t]}$ (see (3.13) and (3.17), respectively). This comparison is the main argument to prove Theorem 2.*

The proof of Theorem 2 given in subsection 3 uses only elementary analysis techniques along a geodesic. The new idea is to introduce a regular tensor field $B(t) = A(t) - \frac{1}{t}I$ which can be geometrically interpreted as the Hessian of an excess function for right triangles (see Section 2.1). $B$ fulfills, like $A$, a Riccati equation (see (2.2)). In Section 6 we will see that Jacobi fields and the volume form can be expressed in terms of $B$ so that estimates of $B$ are the most interesting.
The convexity of the length excess function (1.9) can be seen as a comparison of $\mathrm{Hess}\, d_{c(0)}$ with a background shape operator $A_{l_2}$ of the distance spheres $c(l_2)$. The same holds for the reverse length excess function with the background shape operator $A_{-l_1}$ of the distance spheres $c(-l_1)$. The Riccati equation (1.1)

gives a $L^1$-barrier for these background shape operators on $[0,l_2)$ under the assumption of a lower Ricci curvature bound. Considering the inverse tensor field of $\operatorname{Hess} e$ and $\operatorname{Hess} e_{0,l_2}$ along $c$ allows us to express bounds of the Hessian of these excess functions in terms of this $L^1$-barrier (see Section 4). This improves the upper and lower bounds for the Laplacian of both length excess functions of Theorem 2.

Theorem 3 (Triangle length excess functions and Ricci curvature). *Let $(M^n,g)$ be a complete $n$-dimensional Riemannian manifold, $c:[-l_1,l_2]\to M$ a normal minimal geodesic, $l_1,l_2>0$. Let $e:M\to\mathbb{R}$ be the reverse triangle length excess function of the points $c(0)$ and $c(-l_1)$ defined by $e(q)=l_1+d_{c(0)}(q)-d_{c(-l_1)}(q)$ and $e_{0,l_2}:M\to\mathbb{R}$ be the triangle length excess function of the points $c(0)$ and $c(l_2)$ defined by $e_{0,l_2}(q)=d_{c(0)}(q)+d_{c(l_2)}(q)-l_2$. Suppose that the Ricci curvature $\operatorname{Ric}(\dot c)$ along $c$ is bounded from below by $(n-1)\lambda$, $\lambda\in\mathbb{R}$ on $[-l_1,l_2]$. Then the comparison with the model space gives for $t\in(0,l_2)$ and $v\in T_{c(t)}M$, $\|v\|=1$, $\langle v,\dot c(t)\rangle=0$*

$$e^{-2\gamma_\lambda(t)}\le\frac{\langle\operatorname{Hess} e_{|c(t)}v,v\rangle}{\operatorname{ct}_\lambda(t)-\operatorname{ct}_\lambda(t+l_1)}\le e^{2\gamma_\lambda(t)}\tag{1.10}$$

*and*

$$e^{-2\gamma_\lambda(t)}\le\frac{\langle\operatorname{Hess} e_{0,l_2|c(t)}v,v\rangle}{\operatorname{ct}_\lambda(t)+\operatorname{ct}_\lambda(l_2-t)}\le e^{2\gamma_\lambda(t)}\tag{1.11}$$

*with $\gamma_\lambda(t)=\sqrt{n-1}\sqrt{\delta_\lambda(t)}\sqrt{\operatorname{sn}_\lambda(t)}$ and $\delta_\lambda(t)=\frac{\operatorname{sn}_\lambda(l_1+l_2)}{\operatorname{sn}_\lambda(l_1)\operatorname{sn}_\lambda(l_2-t)}$.*
*Another possible estimate is*

$$0<\tfrac{1}{t}e^{-2\gamma_e(t)}\le\langle\operatorname{Hess} e_{|c(t)}v,v\rangle\le\langle\operatorname{Hess} e_{0,l_2|c(t)}v,v\rangle\le\tfrac{1}{t}e^{2\gamma_e(t)}$$

*for $0<t<l_2$ with $\gamma_e(t)=\sqrt{(n-1)}\sqrt{\delta_e(t)}\sqrt{t}$ and $\delta_e(t)=\operatorname{ct}_\lambda(l_1)+\operatorname{ct}_\lambda(l_2-t)-\lambda t$.*

The examples in Section 7 show that under a lower Ricci curvature bound one cannot expect that $B(t)=A(t)-\frac{1}{t}I$ is controlled along $c$ by a bounded function as in (1.2). However, from Theorem 3 a barrier for the $L^1$-norm of order $O(\sqrt{t})$ as $t\to0$, which is mostly sufficient for Jacobi field estimates as explained in Section 6.3, is possible.

Theorem 4 (Convexity, conjugate radius and Ricci curvature, (cf. [5], Theorem 2)). *Let $(M^n,g)$ be a complete $n$-dimensional Riemannian manifold, $c:[-l_1,l_2]\to M$ a normal, geodesic, $l_1,l_2>0$, without conjugate points and such that for the Ricci curvature along $c$ we have $\operatorname{Ric}(\dot c)\ge(n-1)\lambda$, $\lambda\in\mathbb{R}$. Furthermore let $A$ be the second fundamental tensor of the distance spheres of $c(0)$ along $c$. Then we have the following estimate*

$$\int_\tau^t\|A_{c(x)}-\operatorname{ct}_\lambda(x)I\|\,dx\le\sqrt{n-1}\sqrt{\delta_\lambda(t)}\left(1+4\,e^{2\gamma_\lambda(t)}\right)\sqrt{\operatorname{sn}_\lambda(t-\tau)}\tag{1.12}$$

*for all* $0 \leq \tau \leq t < l_2$ *with* $\gamma_\lambda(t)$ *and* $\delta_\lambda(t)$ *as in Theorem 3. The comparison with the Euclidean geometry gives*

$$\int_\tau^t \|A_{c(x)} - \tfrac{1}{x} I\| \, dx \leq \sqrt{n-1}\sqrt{\delta_e(t)} \left(1 + 4\, e^{2\gamma_e(t)}\right) \sqrt{t-\tau} \tag{1.13}$$

*for all* $0 \leq \tau \leq t < l_2$ *with* $\gamma_e(t)$ *and* $\delta_e(t)$ *as in Theorem 3.*

Convexity of the distance function of a point is directly related to convexity of metric balls. An open metric ball $B(p,r)$ of radius $r$ around $p \in M$ is meant to be convex in the sense that for two points $p_1$, $p_2 \in B(p,r)$ there exists exactly one minimal geodesic connecting $p_1$ and $p_2$ and that this geodesic lies in $B(p,r)$. A lower bound on the convexity radius $\operatorname{convexRad}(M) \geq c_0 > 0$ implies, for the injectivity radius, $\operatorname{injRad}(M) \geq 2\operatorname{convexRad}(M) \geq 2c_0 > 0$, and that the distance spheres are convex for a radius smaller than $c_0$, i.e. $\operatorname{Hess} d_p \geq 0$ on $B(p,c_0)$. On the other hand, if $\operatorname{Hess} d_p \geq 0$ on $B(p,c_0)$ for all $p \in M$, $c_0 > 0$ then $\operatorname{convexRad}(M) \geq \min\{c_0, \operatorname{injRad}(M)/2\}$. This strong assumption on the convexity radius leads, in the case of a lower bound on the Ricci curvature $\operatorname{Ric}(M) \geq (n-1)\lambda$, $\lambda \in \mathbb{R}$, to bounds for the Hessian of distance spheres. It is

$$0 \leq \operatorname{Hess} d_{p|q} \leq \Delta d_{p|q} \leq (n-1)\operatorname{ct}_\lambda(d_p(q)) \quad \text{for } q \in B(p,c_0)\backslash\{p\}. \tag{1.14}$$

For $q \in B(p,2c_0)\backslash B(p,c_0)$ we have, with the unique minimal geodesic $c$ from $p$ to $q$ and using the convexity of the length excess function,

$$\begin{aligned} \operatorname{Hess} d_{p|q} &= \operatorname{Hess} d_{p|q} + \operatorname{Hess} d_{c(2c_0)|q} - \operatorname{Hess} d_{c(2c_0)|q} \\ &\leq \Delta d_{p|q} + \Delta d_{c(2c_0)|q} \\ &\leq (n-1)\left(\operatorname{ct}_\lambda(d_p(q)) + \operatorname{ct}_\lambda(2c_0 - d_p(q))\right) \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} \operatorname{Hess} d_{p|q} &\geq -\operatorname{Hess} d_{c(2c_0)|q} \\ &\geq -\Delta d_{c(2c_0)|q} \\ &\geq -(n-1)\operatorname{ct}_\lambda(2c_0 - d_p(q)). \end{aligned} \tag{1.16}$$

For these manifolds we constructed a $C^{1,1}$-atlas (cf. [6, Chapter 8]) by defining, for a $p \in M$ and a unit vector $e_i \in T_pM$, a component $\varphi_i$ of a chart $\varphi$ on $B(p,c_0)$ by $\varphi_i(q) = c_0 - d_{\exp_p(c_0 e_i)}(q)$. Observe that $\varphi_i(p) = 0$, $\operatorname{grad}\varphi_{i|p} = e_i$ and $\|\operatorname{Hess}\varphi_i\| \leq (n-1)(\operatorname{ct}_\lambda(c_0 - r) + \operatorname{ct}_\lambda(c_0 + r))$ on $B(p,r)$. Using the volume comparison (see Theorem 7), this allows us to get a $C^1$-compactness theorem for the space of manifolds with $\operatorname{convexRad}(M) \geq c_0 > 0$, $\operatorname{Ric} \geq (n-1)\lambda$, $\lambda \in \mathbb{R}$ and $\operatorname{Vol}(M) \leq V$. Compared to the Anderson - Cheeger's $C^\alpha$-compactness theorem ([1]), the stronger assumption on the convexity radius allows us to get better regularity by only using ordinary differential equation techniques.
With regard to the Riccati equation it is natural to consider only explicit or implicit properties of the curvature tensor $R_{\dot c}$ along $c$. Implicit properties are,

for example, the conjugate or the focal radius of $c$, because they are determined by Jacobi fields. An upper curvature bound $R \leq \Lambda$ along a geodesic $c$ implies a lower bound on the conjugate radius of $c$ ($\text{conjRad}(c) \equiv \infty$ if $\Lambda \leq 0$, $\text{conjRad}(c) \geq \frac{\pi}{\sqrt{\Lambda}}$ if $\Lambda > 0$) and on the focal radius of $c$ ($\text{focalRad}(c) \equiv \infty$ if $\Lambda \leq 0$, $\text{focalRad}(c) \geq \frac{\pi}{2\sqrt{\Lambda}}$ if $\Lambda > 0$). For a geodesic, a lower bound on the focal radius $\text{focalRad}(c) \geq f_0 > 0$ means that if $c(s)$ is a focal point of $c(t)$, $s > t$, i.e. there exists a Jacobi field $J \neq 0$ with $\frac{\nabla}{dt} J(t) = 0$ and $J(s) = 0$, we have $s - t \geq f_0$.

The correlation between the radius function is expressed through $\text{convexRad}(\mathrm{M}) = \min\{\text{focalRad}(M), \frac{\text{injRad}(\mathrm{M})}{2}\}$ and $2\,\text{focalRad}(M) \leq \text{conjRad}(M)$. The relation between the focal radius and convexity of the distance function of a point from Lemma 2 (see Section 4.2) implies that a geodesic $c$ with conjugate radius ($\text{conjRad}(c) \geq c_0$) and sectional curvature bounded below ($R \geq \lambda$) also has a lower focal radius bound $f_0$. Indeed, from Corollary 1 it follows that all distance spheres are convex on $(0, f_0)$ with $0 < f_0 < \frac{c_0}{2}$ such that $\text{ct}_\lambda(f_0) = 2\,\text{ct}_\lambda(\frac{c_0}{2})$. The analytic example in Section 7 shows that this is not true when only the Ricci curvature along a geodesic is bounded below. But this behavior along a geodesic does not imply that manifolds with injectivity radius and Ricci curvature bounded below do not admit a universal lower focal radius bound. If this is true, the proof of the Anderson - Cheeger's compactness theorem could be simplified. The examples show only that an approach by arguing along a geodesic will be impossible.

The following Theorem 5 shows that a lower focal radius barrier implies estimates of the second fundamental tensor of distance spheres and that this represents a strong assumption on the local geometry of a manifold. The bounds from (1.14) can be improved in the way that we get estimates of type $\frac{1}{d_p} + O(1)$ as $d_p \to 0$ for $\text{Hess}\, d_p$.

Theorem 5 (Convexity, focal radius and Ricci curvature, (cf. [6], Chapter 6)). *Let $(M^n, g)$ be a complete $n$-dimensional Riemannian manifold, $c : \mathbb{R} \to M$ a normal geodesic. We suppose that the focal radius of $c$ and of the inverse geodesic $c^-(t) := c(-t)$ are bounded below, i.e., that there is a $f_0$ with* $\min\{\text{focalRad}(c), \text{focalRad}(c^-)\} \geq f_0 > 0$. *Then for the conjugate radius of $c$ we have*

$$\text{conjRad}(c) = \text{conjRad}(c^-) \geq \text{focalRad}(c) + \text{focalRad}(c^-) \geq 2f_0$$

*and the second fundamental tensor $A$ of the distance spheres of $c(0)$ along $c$ is strictly convex on $(0, f_0)$.*

*If, in addition, we have for the Ricci curvature* $\text{Ric}(\dot{c}(t)) \geq (n-1)\lambda$, $\lambda \in \mathbb{R}$, $\lambda f_0^2 \leq \frac{\pi^2}{4}$, *then*

$$0 < A_{c(t)} \leq (n-1)\,\text{ct}_\lambda(t) \quad \text{for } t \in (0, f_0) \tag{1.17}$$

*and we have for $t \in (f_0, 2f_0)$*

$$-(n-1)\,\text{ct}_\lambda(2f_0 - t) \leq A_{c(t)} \leq (n-1)\big(\text{ct}_\lambda(t) + \text{ct}_\lambda(2f_0 - t)\big). \tag{1.18}$$

*Moreover, there exists a continuous function* $\beta : [0, 2f_0) \to \mathbb{R}$ *only depending on* $n$, $\lambda$ *and* $f_0$ *with* $\beta(0) = (n-1)\operatorname{ct}_\lambda(f_0)$ *such that*

$$\|A_{c(t)} - \tfrac{1}{t} I\| \le \beta(t) \quad \text{for } t \in [0, 2f_0). \tag{1.19}$$

REMARK 8. *To be explicit in* (1.19)*, we have for* $0 \le t < f_0$

$$-(n-1)\operatorname{ct}_\lambda(f_0 - t) \le A(t) - \tfrac{1}{t} I \le (n-1)\operatorname{ct}_\lambda(f_0) \tag{1.20}$$

*which is much better than* (1.17) *for small* $t$*. On the whole interval* $(0, 2f_0]$ *another possible upper estimate is given by*

$$A(t) - \tfrac{1}{t} I \le (n-1)\tfrac{14}{9}\operatorname{ct}_\lambda(f_0 - \tfrac{t}{3}) - (n-1)\lambda\tfrac{t}{3}$$

*for* $0 \le t \le 2f_0$ *and a lower estimate by*

$$\tfrac{1}{t} I - A(t) \le (n-1)\tfrac{14}{9}\operatorname{ct}_\lambda(f_0 - \tfrac{t}{3}) - (n-1)\lambda\tfrac{t}{3} + \delta(t)$$

*for* $0 \le t < 2f_0$ *with* $\delta(t)$ *as in* (1.6) *and* $i_0 = 2f_0$ *(see* (4.15), (4.16)*).*

REMARK 9. *The hypothesis of a lower focal radius bound* $f_0$ *can also be applied in the context of a lower curvature bound* $\lambda \in \mathbb{R}$*. The technique in Theorem 5 gives then (cf. [6, Lemma 6.7] and* (4.6)*)*

$$-\operatorname{ct}_\lambda(f_0) \le A(t) - \operatorname{ct}_\lambda(t) \le 0 \qquad \text{for } 0 < t \le f_0 \tag{1.21}$$

*which is better than the estimates in Corollary 1 with* $i_0 = 2f_0$ *by a factor of* 2*. This estimate is sharp, as shown in Section 7.*

In Section 4.2 we give two different proofs. Both techniques are interesting because they can be used locally in different situations. Starting point of the proofs are the bounds of $A$ in (1.14), (1.15) and (1.16). These estimates also hold for other second fundamental tensors $A_l$ of distance spheres of other points $c(l)$ along $c$. The first approach uses the same technique as in Theorem 3 with these bounded background fields. The second approach (see Section 5) uses the Riccati equation $R_{\dot c} = -A_l' - A_l^2$. Bounds for the background field $A_l$ allow us to get a barrier function for a kind of integral curvature tensor on $[0, 2f_0]$ (see (1.22)). We can prove that this barrier function has a bounded distance to $A_{c(t)} - \frac{1}{t} I$ due to Theorem 1 (cf. [4, Theorem 5.7]).

To answer the research question we summarize the sufficient conditions yielding new estimates for $\operatorname{Hess} d_p$ and $\Delta d_p$ in the double entry Table 1. In the top horizontal row there are implicit curvature conditions guaranteeing the existence of $A$ along a geodesic of a certain interval. This replaces the usually explicit upper curvature condition. The vertical columns on the left side are explicit lower curvature conditions. Sufficient conditions for getting bounds for

Table 1: Overview of the new estimates (as $t \to 0$)

| | $\mathrm{focalRad}(c) \geq f_0$<br>$\mathrm{focalRad}(c^-) \geq f_0$ | $\mathrm{conjRad}(c) \geq c_0$ |
|---|---|---|
| $R_{\dot c} \geq \lambda$ | $A(t) - \frac{1}{t} I \in \mathrm{ct}_\lambda(f_0) + O(t)$ | $A(t) - \frac{1}{t} I \in 2\,\mathrm{ct}_\lambda(\frac{c_0}{2}) + O(t)$ |
| $\frac{\mathrm{Ric}(\dot c)}{n-1} \geq \lambda$ | $A(t) - \frac{1}{t} I$<br>$\in (n-1)\,\mathrm{ct}_\lambda(f_0) + O(t)$ | $\frac{\mathrm{tr}\, A(t)}{n-1} - \frac{1}{t} \in 2\,\mathrm{ct}_\lambda(\frac{c_0}{2}) + O(t)$<br>$\int_0^t \lVert A(x) - \frac{1}{x} I \rVert \, dx \in O(\sqrt{t})$ |

$B$ are: firstly, bounded curvature by the Riccati comparison, secondly, curvature and conjugate radius (Theorem 1) and thirdly, Ricci curvature and focal radius (Theorem 5) bounded below (see also Table 2).
The Laurent expansion of $A$ can also be written as (see (3.10))

$$A(t) = \tfrac{1}{t} I - \int_0^t (\tfrac{\tau}{t})^2 R(t)\, d\tau - t^3 W(t) \tag{1.22}$$

with a positive semi-definite $W$ and $W(0) = \frac{1}{45} R^2(0)$. The following Theorem 6 gives a sufficient and necessary condition only in terms of this integral curvature tensor expression.

Theorem 6 (cf. [4], Theorem 7.2). *Let $M^n$ be a complete $n$-dimensional Riemannian manifold and $c : \mathbb{R} \to M$ a normal geodesic. Let $A$ be the shape operator of the distance spheres of $c(0)$ and $R_{\dot c} = R(\cdot, \dot c, \dot c)$ the curvature tensor along $c$.*

1. *Suppose that there is a continuous function $\beta : [0, l_2) \to \mathbb{R}$ such that $\lVert A(t) - \frac{1}{t} I \rVert \leq \beta(t)$ on $[0, l_2)$.*
*Set $c_0 = \sup\{0 < t < l_2 \mid \tau\beta(\tau) < 1$ for all $0 \leq \tau \leq t\}$. Then we have*
   - *(a) the geodesic segment $c_{|[0,l_2)}$ has no conjugate points*
   - *(b) the distance spheres are strictly convex on $(0, c_0)$, i.e. $A(t) > 0$.*
   - *(c) No segment $c^-_{|[\alpha,\beta]}$ with $[\alpha, \beta] \subset (0, c_0)$ of the inverse geodesic $c^- : \mathbb{R} \to M$ with $c^-(t) = c(c_0 - t)$ has a focal point.*
   - *(d)*
$$\left\lVert \int_0^t \left(\tfrac{\tau}{t}\right)^2 R_{\dot c(\tau)}\, d\tau \right\rVert \leq \beta(t) + \int_0^t \left(\tfrac{\tau}{t}\right)^2 \beta^2(\tau)\, d\tau \quad \text{on } [0, l_2).$$
2. *Suppose that there exists a continuous function $\Lambda : \mathbb{R} \to \mathbb{R}$ such that*
$$\left\lVert \int_0^t \left(\tfrac{\tau}{t}\right)^2 R_{\dot c(\tau)}\, d\tau \right\rVert \leq \Lambda(t) \quad \text{on } [0, \infty).$$
*Let $l_2 > 0$ such that $\beta : [0, l_2) \to \mathbb{R}$ is the maximal solution of $\beta' = (\Lambda + \beta)^2$ with $\beta(0) = 0$. Then we have $\lVert A(t) - \frac{1}{t} I \rVert \leq \Lambda(t) + \beta(t)$ on $[0, l_2)$.*

The proof is given in Section 5 and uses the Riccati equation (2.2) for $B(t) = A_{c(t)} - \frac{1}{t}I$.

Remark 10. *Suppose that* $\Lambda$ *in Theorem 6 is a differentiable function with* $\Lambda' \geq 0$. *If* $\Lambda(0) = 0$ *we suppose furthermore* $\Lambda'(0) > 0$. *Then we have* $\beta(t) \leq \frac{t\Lambda^2(t)}{1-t\Lambda(t)}$ *and hence* $\|A(t) - \frac{1}{t}I\| \leq \frac{\Lambda(t)}{1-t\Lambda(t)}$ *on* $[0, t_0)$ *where* $t_0$ *is the unique value with* $t_0\Lambda(t_0) = 1$.

Remark 11. *With Theorem 6 we retrieve asymptotically the bounds from the Riccati comparison. If the curvature is bounded, e.g.* $\|R\| \leq \kappa$, $\kappa > 0$, *we can put* $\Lambda(t) = \frac{\kappa}{3}t$. $\beta(t) = \sqrt{\frac{\kappa}{3}}\tan(\sqrt{\frac{\kappa}{3}}t) - \frac{\kappa}{3}t$ *is a solution of* $\beta' = (\Lambda + \beta)^2$ *with* $\beta(0) = 0$ *on* $[0, \sqrt{\frac{3}{\kappa}}\frac{\pi}{2})$. *This gives* $\|A(t) - \frac{1}{t}I\| \leq \sqrt{\frac{\kappa}{3}}\tan(\sqrt{\frac{\kappa}{3}}t) = \frac{\kappa}{3}t + O(t^3)$ *as* $t \to 0$.

Remark 12. *Suppose that there is only a bounded barrier for the integral curvature tensor with* $\Lambda(t) = \kappa$, $\kappa > 0$. *Then* $\beta(t) = \frac{\kappa^2 t}{1-\kappa t}$ *is a solution of* $\beta' = (\Lambda + \beta)^2$ *with* $\beta(0) = 0$ *on* $[0, \frac{1}{\kappa})$. *This gives* $\|A(t) - \frac{1}{t}I\| \leq \frac{\kappa}{1-\kappa t} = \kappa + \kappa^2 t + O(t^2)$ *as* $t \to 0$. *This integral barrier can be obtained under the assumption of Theorem 1 and 5. Hence Theorem 6 confirms approximately these results for small* $t$.

One considers normally the focal radius in the context of the Rauch's second comparison theorem. For a geodesic $c$ the hypersurface $H_0$ defined by $\exp_{c(0)}(\{\dot{c}(0)\}^\perp)$ is totally geodesic in $c(0)$ and orthogonal to $\dot{c}(0)$. A signed distance function to $H_0$ defines an equidistant family $H_t$ of hypersurfaces. The shape operator $H$ of this family along $c$ develops a singularity in the focal point. A lower bound on the focal radius guarantees the existence of this family on a certain interval. In Section 6.6 we apply our techniques to $H$. In particular, this allows us to better understand the geometric impact of a lower focal radius bound and the relation between $H$ and the shape operators $A$ of distances spheres. It also gives another proof of Theorem 5 and shows that in Theorem 6 one can also take the integral curvature expression $\int_0^t R(\tau)\,d\tau$.

## 2 Geometric motivation

### 2.1 Excess function for right triangles and its convexity properties

For a better understanding of the analytic proof of Theorem 2 using only ordinary differential equation techniques along a geodesic, it will be helpful to consider some geometric aspects first. This leads to a geometric proof of this theorem for manifolds with sectional curvature bounded from below using Toponogov's theorem.

Let $c : \mathbb{R} \to M$ be a geodesic in $M$. For a fixed $t > 0$ let $v \in T_{c(t)}M$ be a unit vector orthogonal to $\dot{c}(t)$. We consider the total geodesic hypersurface $H_t$ orthogonal to $\dot{c}(t)$ given by $H_t := \exp_{c(t)}\{\dot{c}(t)^\perp\}$ and the normal geodesic

$\gamma(s) = \exp_{c(t)} sv$, which lies in $H_t$. The idea is to express $H_t$ in the model space $M^n_\lambda$ using distance functions. As the angle between $\gamma$ and $c$ is a right angle, the distance between $c(0)$ and $\gamma(s)$ can be calculated in the model spaces $M^n_\lambda$ of constant curvature. In $\mathbb{R}^n$, it follows from the Pythagorean theorem that we have

$$\begin{aligned} d^2_{c(0)}(\gamma(s)) &= d^2(c(0), c(t)) + d^2_{c(t)}(\gamma(s)) \\ &= t^2 + s^2. \end{aligned}$$

We conclude from Napier's rules for right spherical triangles in $S^n$

$$\cos(d_{c(0)}(\gamma(s))) = \cos(t)\cos(s)$$

and the hyperbolic law of cosines for hyperbolic triangles gives

$$\cosh(d_{c(0)}(\gamma(s))) = \cosh(t)\cosh(s).$$

For the reason that $\operatorname{cs}_\lambda(a+b) = \operatorname{cs}_\lambda(a)\operatorname{cs}_\lambda(b) - \lambda \operatorname{sn}_\lambda(a)\operatorname{sn}_\lambda(b)$, we have $1 = \operatorname{cs}^2_\lambda + \lambda \operatorname{sn}^2_\lambda$ and therefore $\operatorname{cs}_\lambda(a) = 1 - 2\lambda \operatorname{sn}^2_\lambda(\frac{a}{2})$. This gives for $\lambda \neq 0$

$$\begin{aligned} 2\lambda \operatorname{sn}^2_\lambda(\tfrac{1}{2} d_{c(0)}(\gamma(s))) &= 1 - \operatorname{cs}_\lambda(d_{c(0)}(\gamma(s))) \\ &= 1 - \operatorname{cs}_\lambda(t)\operatorname{cs}_\lambda(s) \\ &= 1 - \operatorname{cs}_\lambda(t)\left(1 - 2\lambda \operatorname{sn}^2_\lambda(\tfrac{s}{2})\right) \\ &= 2\lambda \operatorname{sn}^2_\lambda\left(\tfrac{t}{2}\right) + \operatorname{cs}_\lambda(t) 2\lambda \operatorname{sn}^2_\lambda(\tfrac{s}{2}). \end{aligned}$$

So the Euclidean Pythagorean relationship for right triangles can be generalized for all model spaces of constant curvature by

$$\operatorname{sn}^2_\lambda\left(\tfrac{1}{2} d_{c(0)}(\gamma(s))\right) = \operatorname{sn}^2_\lambda(\tfrac{t}{2}) + \operatorname{cs}_\lambda(t)\operatorname{sn}^2_\lambda\left(\tfrac{1}{2} d_{c(t)}(\gamma(s))\right). \tag{2.1}$$

The distance function is often rescaled to get a smooth modified distance function. In the model space, this modified function has equal eigenvalues, also in normal direction. This is done by introducing the function $\operatorname{md}_\lambda(r) = \int_0^r \operatorname{sn}_\lambda(t)\,dt = 2\operatorname{sn}^2_\lambda(\frac{r}{2})$ (see 1.4.3 in [16], [20] or the estimate for distance functions in [7]). Note that $\operatorname{md}_\lambda$ is even. Motivated by (2.1), we define for fixed $t > 0$ and $c$ minimal on $[0, t]$ the excess function for right triangles of $c(0)$ and $c(t)$ with right angle in $c(t)$ by

$$e_t(q) = \operatorname{md}_\lambda(d_{c(0)}(q)) - \operatorname{md}_\lambda(t) - \operatorname{cs}_\lambda(t)\operatorname{md}_\lambda(d_{c(t)}(q)).$$

It is $e_t(c(\tau)) = \operatorname{sn}_\lambda(t)\operatorname{sn}_\lambda(\tau - t)$. For the gradient of $e_t$ we have

$$\operatorname{grad} e_{t|q} \quad = \operatorname{sn}_\lambda(d_{c(0)}(q)) \cdot \operatorname{grad} d_{c(0)|q} - \operatorname{cs}_\lambda(t)\operatorname{sn}_\lambda(d_{c(t)}(q)) \cdot \operatorname{grad} d_{c(t)|q}$$

for $q \neq c(t)$ and $q \neq c(0)$. As $\operatorname{grad}(\operatorname{md}_\lambda \circ d_{c(t)})_{|c(t)} = 0$ it follows

$$\operatorname{grad} e_{t|c(\tau)} = \operatorname{sn}_\lambda(t)\operatorname{cs}_\lambda(\tau - t) \cdot \dot{c}(\tau) \quad \text{for } 0 \leq \tau \leq t.$$

The Hessian of $e_t$ in $q$ with $v \in T_qM$ is given by

$$\begin{aligned}\operatorname{Hess} e_{t|q} v = {} & \operatorname{cs}_\lambda(d_{c(0)}(q)) \langle \operatorname{grad} d_{c(0)|q}, v\rangle \cdot \operatorname{grad} d_{c(0)|q} \\ & + \operatorname{sn}_\lambda(d_{c(0)}(q)) \cdot \operatorname{Hess} d_{c(0)|q} v \\ & - \operatorname{cs}_\lambda(t) \operatorname{cs}_\lambda(d_{c(t)}(q)) \langle \operatorname{grad} d_{c(t)|q}, v\rangle \cdot \operatorname{grad} d_{c(t)|q} \\ & - \operatorname{cs}_\lambda(t) \operatorname{sn}_\lambda(d_{c(t)}(q)) \cdot \operatorname{Hess} d_{c(t)|q} v.\end{aligned}$$

So we have for $v$ orthogonal to $\dot c(\tau)$, $\tau \neq t$, $\tau \neq 0$,

$$\operatorname{Hess} e_{t|c(\tau)} v = \operatorname{sn}_\lambda(\tau) \cdot \operatorname{Hess} d_{c(0)|c(\tau)} v - \operatorname{cs}_\lambda(t) \operatorname{sn}_\lambda(t-\tau) \cdot \operatorname{Hess} d_{c(t)|c(\tau)} v.$$

As $\operatorname{Hess}(\operatorname{md}_\lambda \circ d_{c(t)})_{|c(t)} = \operatorname{id}$ it follows that for $v$ orthogonal to $\dot c(t)$

$$\operatorname{Hess} e_{t|c(t)} v = \operatorname{sn}_\lambda(t) \operatorname{Hess} d_{c(0)|c(t)} v - \operatorname{cs}_\lambda(t) v.$$

The hypersurface $E_t := \{e_t \equiv 0\}$ through $c(t)$ is orthogonal to $\dot c(t)$. $E_t$ corresponds in the model space to the total geodesic hypersurface $H_t$. The shape operator $B_\lambda(t)$ in $c(t)$ of $E_t$ is given by

$$\begin{aligned}B_\lambda(t) v &= \nabla_v \tfrac{\operatorname{grad} e_t}{\|\operatorname{grad} e_t\|} \\ &= \tfrac{1}{\|\operatorname{grad} e_t\|}\Big(\operatorname{Hess} e_t(v) - \tfrac{1}{\|\operatorname{grad} e_t\|^2}\langle \operatorname{Hess} e_t(v), \operatorname{grad} e_t\rangle \operatorname{grad} e_t\Big) \\ &= \tfrac{1}{\|\operatorname{grad} e_t\|}\left(\operatorname{Hess} e_t(v)\right)^\top = A_{c(t)} v - \operatorname{ct}_\lambda(t) v.\end{aligned}$$

$B_\lambda$ measures the difference between the principal curvature of the distance spheres in $M$ and those in $M_\lambda$. We will see that $B_\lambda$ plays an important role in the comparison theory along a geodesic and can be used to get new estimates for $A$.

If the curvature $K$ of $M$ is bounded from below by $\lambda \in \mathbb{R}$ it follows from Toponogov's theorem that $e_t(\gamma(s)) \leq 0$ and that $e_t$ therefore has a maximum in $s = 0$. We conclude

$$B_\lambda(t) \leq 0 \quad \text{if } K \geq \lambda, \text{ that is } E_t \text{ is concave (see Figure 3).}$$

This is equivalent to the upper bound for $A = \operatorname{Hess} d_p$ given by the Riccati comparison.

Another geometric property of $B_\lambda$ under curvature conditions is a monotony while changing $c(0)$ along $c$. Let

$$e_t^{-l_1} = \operatorname{md}_\lambda(d_{c(-l_1)}(q)) - \operatorname{md}_\lambda(t + l_1) - \operatorname{cs}_\lambda(t + l_1) \operatorname{md}_\lambda(d_{c(t)}(q))$$

be the excess function for right triangles of the points $c(-l_1)$ and $c(t)$ with right angle in $c(t)$, and suppose that $c$ is minimal on $[-l_1, t]$. We conclude from Toponogov's theorem that for a point $q$ with $e_t^{-l_1}(q) = 0$, we have $e_t(q) \geq 0$ if $K \geq \lambda$ (see Figure 3). Let $B_\lambda^{-l_1}(t)$ be the shape operator of $E_t^{-l_1} := \{e_t^{-l_1} \equiv 0\}$

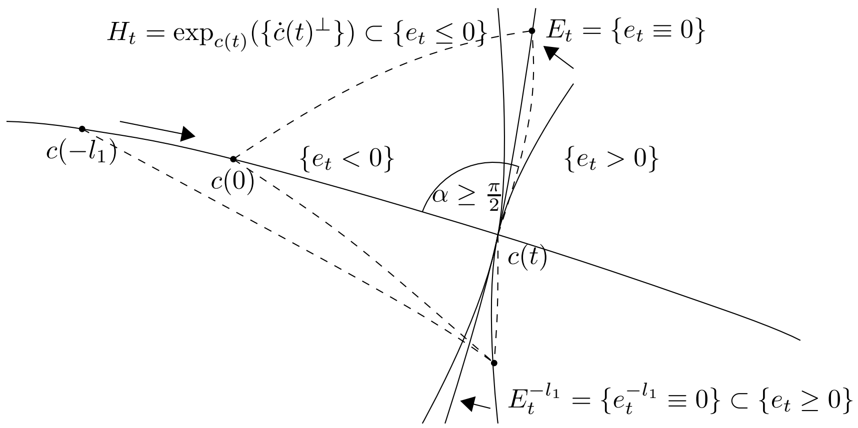



Figure 3: case $K \geq \lambda$: $E_t$ is concave and the principal curvature of $E_t$ is monotone while changing the center $c(0)$.

in $c(t)$ and $v \in T_{c(t)}M$ orthogonal to $\dot c(t)$. Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $e_t^{-l_1}(\gamma(s)) = 0$, $\gamma(0) = c(t)$ and $\dot\gamma(0) = v$. Then we have

$$\begin{aligned}
&\langle B_\lambda(t)v, v\rangle - \langle B_\lambda^{-l_1}(t)v, v\rangle \\
&\qquad= \frac{1}{\mathrm{sn}_\lambda(t)}\langle \operatorname{Hess} e_t v, v\rangle - \frac{1}{\mathrm{sn}_\lambda(t+l_1)}\langle \operatorname{Hess} e_t^{-l_1} v, v\rangle \\
&\qquad= \frac{1}{\mathrm{sn}_\lambda(t)}\left((e_t \circ \gamma)''(0) - \langle \operatorname{grad} e_t, \nabla_D \dot\gamma_{|s=0}\rangle\right) \\
&\qquad\quad - \frac{1}{\mathrm{sn}_\lambda(t+l_1)}\left((e_t^{-l_1} \circ \gamma)''(0) - \langle \operatorname{grad} e_t^{-l_1}, \nabla_D \dot\gamma_{|s=0}\rangle\right) \\
&\qquad= \frac{1}{\mathrm{sn}_\lambda(t)}(e_t \circ \gamma)''(0) \geq 0.
\end{aligned}$$

$0 \geq B_\lambda(t) \geq B_\lambda^{-l_1}(t)$ is in fact equivalent to (1.7) of Theorem 2.
The excess function for right triangles can also be used for the construction of almost linear functions in [15]. Indeed, $l(q) = \mathrm{md}_\lambda(d_{c(0)}(q)) - \mathrm{md}_\lambda(d_{c(2t)}(q)) = e_t(q) - e_t^{2t}(q)$ used for this purpose is also the difference of $e_t$ and the excess function for right triangles $e_t^{2t}$ of the points $c(2t)$ and $c(t)$ with right angle in $c(t)$.

## 2.2 Exponential map and Jacobi fields

In this section we will see that $B_\lambda$ will also appear naturally in the description of the differential of the exponential map $\exp_*$. To compare the geometry of $M$ around $p \in M$ with those of manifolds $M_\lambda$ with constant curvature $\lambda \in \mathbb{R}$ around $p_\lambda \in M_\lambda$, we identify $T_{p_\lambda}M_\lambda$ with $T_pM$ via a linear isometry $I : T_pM \to T_{p_\lambda}M_\lambda$ and consider the map

$\mathrm{Exp}_\lambda = \exp_p \circ I^{-1} \circ \exp_{p_\lambda}^{-1} : M_\lambda \to M$ (see Figure 4). If $\lambda > 0$ then $\mathrm{Exp}_\lambda$ is defined on $M_\lambda$ except for the cut locus, i.e. the antipodal point of $p_\lambda$. We can consider $\mathrm{Exp}_\lambda$ to be a generalized normal parametrization of $M$.

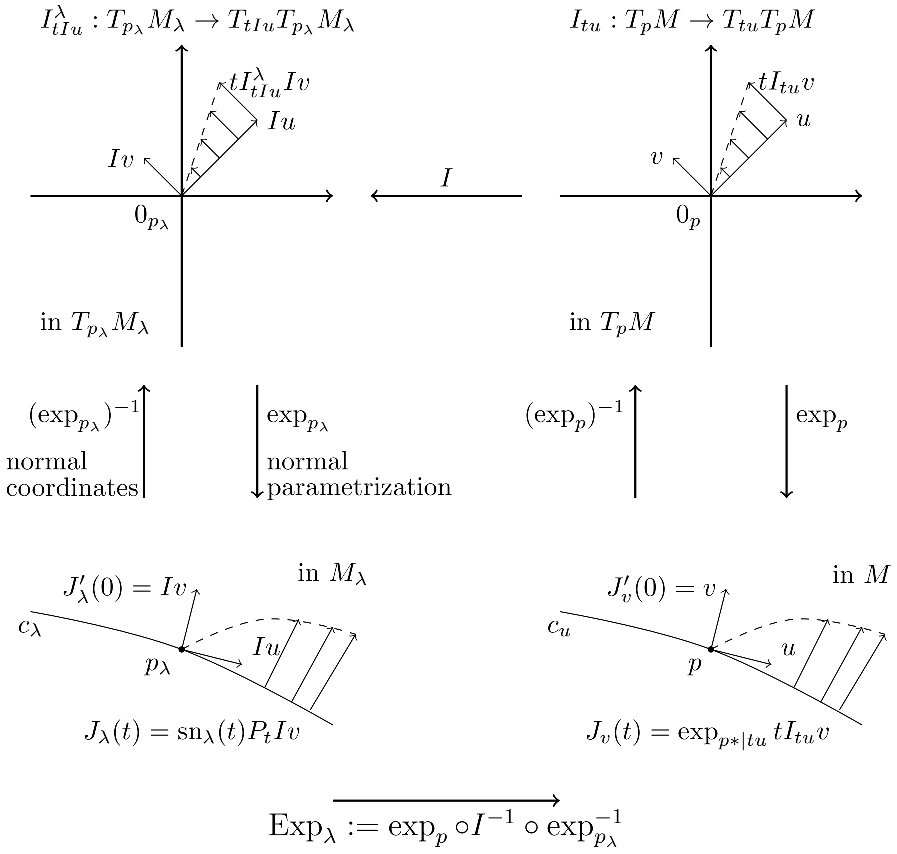



Figure 4: generalized normal parametrization - $\mathrm{Exp}_\lambda$

Jacobi fields describe the differential $\exp_*$ of the exponential map. For $p \in M$ and $u, v \in T_pM$, $\|u\| = 1$ and $\langle u, v\rangle = 0$, we have $\exp_{p*|tu} tI_{tu}v = J_v(t)$ with the canonical identification $I_{tu} : T_pM \to T_{tu}T_pM$ and $J_v$ the Jacobi field along the normal geodesic $c_u(t) = \exp_p(tu)$ with $J_v(0) = 0$ and $\frac{\nabla}{dt}J_v(0) = v$. $\exp_{p_*}$ maps the Euclidian Jacobi field $J_v^{eucl} = tI_{tu}v$ with $J_v^{eucl}(0) = 0$ and $\frac{d}{dt}J_v^{eucl}(0) = v$ onto the corresponding Jacobi field in $M$. Therefore the ratio $\frac{1}{t}J_v(t)$ between the Euclidean Jacobi field and the corresponding one in $M$ describes the differential of the exponential map $\exp_{p*|tu} I_{tu}v = \frac{1}{t}J_v(t)$.
$c_\lambda(t) = \exp_{p_\lambda} tIu$ is the corresponding normal comparison geodesic. The Jacobi field $J_\lambda$ along $c_\lambda$ with $J_\lambda(0) = 0$ and $\frac{\nabla}{dt}J_\lambda(0) = Iv$ is given by $\mathrm{sn}_\lambda(t)P_tIv$, where

$P_t : T_{p_\lambda} M_\lambda \to T_{c_\lambda(t)} M_\lambda$ is the parallel transport along $c_\lambda$. With the natural identification $I^\lambda_{tIu} : T_{p_\lambda} M_\lambda \to T_{tIu} T_{p_\lambda} M_\lambda$ we have $\exp_{p_\lambda * |tIu} I^\lambda_{tIu} Iv = \frac{1}{t} J_\lambda(t) = \frac{\mathrm{sn}_\lambda(t)}{t} P_t Iv$. Hence, $\mathrm{Exp}_{\lambda *}$ is described by the following relations

$$\mathrm{Exp}_{\lambda * | c_\lambda(t)} P_t Iv = \frac{J_v(t)}{\mathrm{sn}_\lambda(t)}, \quad \mathrm{Exp}_{\lambda * | c_\lambda(t)} \dot{c}_\lambda(t) = \dot{c}(t) \quad \text{and} \quad \mathrm{Exp}_{\lambda * | p_\lambda} Iv = v.$$

$\mathrm{Exp}_{\lambda *}$ maps the Jacobi field in the model space onto the corresponding one in $M$. It maps the tangent field of the comparison geodesic onto the tangent field of $c$. In $p_\lambda$ $\mathrm{Exp}_{\lambda *}$ is the identity.
We conclude from Rauch's first comparison theorem that $\mathrm{Exp}_\lambda$ contracts length, i.e. $\|\mathrm{Exp}_{\lambda *}\| \leq 1$, if for the sectional curvature we have $K \geq \lambda$. Similarly, $\mathrm{Exp}_\lambda$ dilates length if $K \leq \lambda$. Furthermore, $\mathrm{Exp}_\lambda$ contracts volumes, i.e. $|\det(\mathrm{Exp}_{\lambda *})| \leq 1$, if the Ricci curvature satisfies $\mathrm{Ric} \geq (n-1)\lambda$.
The ratio $E^v_\lambda(t) := \frac{J_v(t)}{\mathrm{sn}_\lambda(t)}$ of the Jacobi field in $M$ and the corresponding one in $M_\lambda$ is a vector field along $c$, which describes the differential of $\mathrm{Exp}_\lambda$ in orthogonal direction. For the covariant derivative of this vector field along $c$, we calculate

$$\begin{aligned}
\frac{\nabla}{dt} E^v_\lambda(t) &= \frac{1}{\mathrm{sn}_\lambda(t)} \frac{\nabla}{dt} J_v(t) - \frac{\mathrm{cs}_\lambda(t)}{\mathrm{sn}^2_\lambda(t)} J_v(t) \\
&= A_{c(t)} E^v_\lambda(t) - \mathrm{ct}_\lambda(t) E^v_\lambda(t) \\
&= B_\lambda(t) E^v_\lambda(t).
\end{aligned}$$

The shape operator $B_\lambda(t)$ of the level $E_t$ is therefore determined by these vector fields. Via this equation it defines a symmetric tensor field $B_\lambda(t)$ along $c$. $B_\lambda(t)$ can be extended to $t = 0$ smoothly and is defined on the orthogonal complement of $\dot{c}$. It fulfills, on the one hand, the equation

$$\begin{aligned}
\frac{\nabla^2}{dt^2} E^v_\lambda(t) &= -\frac{\mathrm{cs}_\lambda(t)}{\mathrm{sn}^2_\lambda(t)} \frac{\nabla}{dt} J_v(t) + \frac{1}{\mathrm{sn}_\lambda(t)} \frac{\nabla^2}{dt^2} J_v(t) \\
&\qquad - \left( \frac{-\lambda\, \mathrm{sn}_\lambda(t)}{\mathrm{sn}^2_\lambda(t)} - 2 \frac{\mathrm{cs}^2_\lambda(t)}{\mathrm{sn}^3_\lambda(t)} \right) J_v(t) - \frac{\mathrm{cs}_\lambda(t)}{\mathrm{sn}^2_\lambda(t)} \frac{\nabla}{dt} J_v(t) \\
&= -\, \mathrm{ct}_\lambda(t) A_{c(t)} E^v_\lambda(t) - R_{\dot{c}(t)} E^v_\lambda(t) \\
&\qquad + \lambda E^v_\lambda(t) + 2\, \mathrm{ct}^2_\lambda(t) E^v_\lambda(t) - \mathrm{ct}_\lambda(t) A_{c(t)} E^v_\lambda(t) \\
&= -\left( R_{c(t)} - \lambda I + 2\, \mathrm{ct}_\lambda(t) B_\lambda(t) \right) E^\lambda_v(t)
\end{aligned}$$

and, on the other hand,

$$\frac{\nabla^2}{dt^2} E^v_\lambda(t) = \left( \frac{\nabla}{dt} B_\lambda(t) + B^2_\lambda(t) \right) E^\lambda_v(t).$$

So we get two decoupled equations

$$\begin{aligned}
&\frac{\nabla}{dt} E^v_\lambda = B_\lambda E^v_\lambda && E^v_\lambda(0) = v, \quad E^v_\lambda = \frac{J_v}{\mathrm{sn}_\lambda} \\
&\frac{\nabla}{dt} B_\lambda = -2\, \mathrm{ct}_\lambda B_\lambda - B^2_\lambda - (R_{c(t)} - \lambda I) && B_\lambda(0) = 0.
\end{aligned} \tag{2.2}$$

$B_\lambda$ satisfies, like $A$, a Riccati equation along the geodesic $c$. These equations have a calculational advantage over the equations (1.1) because $B_\lambda$ is regular in $t = 0$, whereas $A$ develops a pole. The relation with the exponential map leads to another approach to Jacobi field and volume form estimates as explained in Section 6.

The construction of the family of hypersurfaces $E_t = e_t^{-1}(0)$ is based on a parametrized family of functions $e_t$, unlike the distance spheres of a point that are the level sets of a distance function. More generally, for a function $f : M \to \mathbb{R}$ and $p \in M$ with $\operatorname{grad} f_{|p} \neq 0$, the shape operator $A$ of the level set $f^{-1}(f(p))$ in a neighborhood of $p$ is given by $A(X) = \nabla_X \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|} = \frac{1}{\|\operatorname{grad} f\|} \operatorname{Hess} f(X) - \frac{1}{\|\operatorname{grad} f\|^3} \langle \operatorname{Hess} f(\operatorname{grad} f), X\rangle \operatorname{grad} f$, $X \perp \operatorname{grad} f$. For the covariant derivation in normal direction $T = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}$ we have the Riccati equation

$$(\nabla_T A)(X) + \Phi A(X) + A^2(X) + R_T(X) - \Lambda(X) = 0$$

with a vector field $X$, the curvature tensor $R_T(X) = R(X, T)T$, a function $\Phi = \frac{1}{\|\operatorname{grad} f\|^3} \langle \operatorname{Hess} f(\operatorname{grad} f), \operatorname{grad} f\rangle$ and a tensor field $\Lambda$ defined by

$$\begin{aligned}\Lambda(X) = \frac{3}{\|\operatorname{grad} f\|^6} &\langle \operatorname{Hess} f(\operatorname{grad} f), X\rangle \operatorname{grad} f \\ &- \frac{1}{\|\operatorname{grad} f\|^4} \langle \nabla_X (\operatorname{Hess} f(\operatorname{grad} f)), \operatorname{grad} f\rangle \operatorname{grad} f \\ &- \frac{1}{\|\operatorname{grad} f\|^4} \langle \operatorname{Hess} f(\operatorname{grad} f), \operatorname{Hess} f(X)\rangle \operatorname{grad} f \\ &- \frac{2}{\|\operatorname{grad} f\|^4} \langle \operatorname{Hess} f(\operatorname{grad} f), X\rangle \operatorname{Hess} f(\operatorname{grad} f) \\ &+ \frac{1}{\|\operatorname{grad} f\|^2} \nabla_X (\operatorname{Hess} f(\operatorname{grad} f)).\end{aligned}$$

The Riccati equation (2.2) for $B_\lambda$ has the same structure. Along an integral curve $\gamma$ of $\operatorname{grad} f$ we have $\Phi \circ \gamma = -\frac{d}{dt}\frac{1}{\|\dot\gamma\|}$. Thus (2.2) cannot come from such a construction. Note that for a distance function $f$, i.e. $\|\operatorname{grad} f\| = 1$, this equation is reduced to $(\nabla_T A)(X) + A^2(X) + R_T(X) = 0$.

## 3 On the analysis of the Riccati equation

The proof of Theorem 1 and 2 will be given at the end of this section. First we will reduce the problem to an elementary, purely analytical one, introduce the notations for the comparison geometry, and prove some properties of the second fundamental tensor of the distance spheres and related geometric objects. The definitions are motivated by the preceding Section 2.

We suppose that $c : \mathbb{R} \to M$ is a normal geodesic without conjugate points in the interval $[-l_1, l_2]$, $l_1, l_2 > 0$. Let us choose an orthonormal basis $(X_i)_{2 \le i \le n}$ of parallel vector fields along $c$ and orthogonal to $\dot c$. A vector field $X$ orthogonal to $\dot c$ can be considered a curve $\langle X, X_i\rangle_{2 \le i \le n}$ in $\mathbb{R}^{n-1}$. A symmetric tensor field $T$ along $c$ defined on the orthogonal complement of $\dot c$ is also a curve $\langle X_i, T(X_j)\rangle_{2 \le i,j \le n}$ in the space of symmetric $(n-1) \times (n-1)$ - matrices $\operatorname{Sym}(n-1, \mathbb{R})$. The vector field $T(X)$ is then the product of a matrix and a

vector. The covariant derivation $\left(\frac{\nabla}{dt}T\right)(X) := \frac{\nabla}{dt}T(X) - T(\frac{\nabla}{dt}X)$ is the componentwise derivation of the curve of matrices.
Let $R : \mathbb{R} \to \mathrm{Sym}(n-1, \mathbb{R})$ be the curvature tensor $R(\cdot, \dot{c})\dot{c}$ along $c$ expressed in this basis and $S : \mathbb{R} \to M(n-1, \mathbb{R})$ and $C : \mathbb{R} \to M(n-1, \mathbb{R})$ be the unique solution of the Jacobi equation $Y'' + RY = 0$ with $S(0) = 0$, $S'(0) = I$, $C(0) = I$ and $C'(0) = 0$. The columns of $S$ form a basis of Jacobi fields orthogonal to $\dot{c}$ vanishing in 0. $S(t)$ is invertible unless $c(t)$ and $c(0)$ are conjugated along $c$. The columns of $C$ form a basis of Jacobi fields $J$ orthogonal to $\dot{c}$ with $J'(0) = 0$. $C(t)$ is invertible unless $c(t)$ is a focal point of $c(0)$. $S$ and $C$ build a basis for all matrix solutions of the Jacobi equation. So for fixed $s$,

$$S(t, s) := S(t) \cdot {}^tC(s) - C(t) \cdot {}^tS(s) = -{}^tS(s, t) \tag{3.1}$$

and

$$C(t, s) := C(t) \cdot {}^tS'(s) - S(t) \cdot {}^tC'(s) = \frac{\partial {}^tS}{\partial t}(s, t) = -\frac{\partial S}{\partial s}(t, s) \tag{3.2}$$

are solutions of the Jacobi equation. We have $S(s, s) = 0$, $\frac{\partial S}{\partial t}(s, s) = I$, $C(s, s) = I$ and $\frac{\partial C}{\partial t}(s, s) = 0$. This follows from the fact that for two solutions $V$ and $W$ of the Jacobi equation, the term ${}^tV'(t)W(t) - {}^tV(t)W'(t)$ is constant and therefore

$$\begin{pmatrix} {}^tS'(s) & -{}^tS(s) \\ -{}^tC'(s) & {}^tC(s) \end{pmatrix}$$

is not only left inverse but also right inverse of

$$\begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}.$$

We define

$$\begin{aligned} E(t, s) := \frac{S(t, s)}{t - s} &= \int_0^1 \frac{\partial S}{\partial t}(s + \tau(t - s), s)\, d\tau \\ &= -\int_0^1 {}^tC(s, s + \tau(t - s))\, d\tau, \end{aligned}$$

the differential of the exponential map. $E$ is differentiable with

$$\frac{\partial^k E}{\partial t^k}(s, s) = \frac{1}{k+1}\frac{\partial^{k+1} S}{\partial t^{k+1}}(s, s) \quad \text{for } k \geq 0.$$

This gives

$$E(s, s) = I, \quad \frac{\partial E}{\partial t}(s, s) = 0, \quad \frac{\partial^2 E}{\partial t^2}(s, s) = -\frac{1}{3}R(s)$$
$$\text{and} \quad \frac{\partial^3 E}{\partial t^3}(s, s) = -\frac{1}{2}R'(s). \tag{3.3}$$

$E(t,s)$ is invertible if $c(t)$ is not conjugated to $c(s)$. We define furthermore

$$A(t,s) := \frac{\partial S}{\partial t}(t,s)S^{-1}(t,s) = {}^t\!C(s,t)S^{-1}(t,s), \tag{3.4}$$

which is the shape operator of the distance spheres along c. The shape operator $B$ of the hypersurface, which is defined by the excess function for right triangles, is given by

$$\begin{aligned} B(t,s) &:= \frac{\partial E}{\partial t}(t,s)E^{-1}(t,s) \\ &= \left(\frac{1}{t-s}\frac{\partial S}{\partial t}(t,s) - \frac{1}{(t-s)^2}S(t,s)\right)(t-s)S^{-1}(t,s) \\ &= A(t,s) - \frac{1}{t-s}I. \end{aligned}$$

$A$ and so also $B$ are symmetric, since for all solutions $V$ of the Jacobi equation we have

$$\begin{aligned} V'(t)V^{-1}(t) &- {}^t\!V^{-1}(t){}^t\!V'(t) \\ &= {}^t\!V^{-1}(t)\left({}^t\!V(t)V'(t) - {}^t\!V'(t)V(t)\right)V^{-1}(t) \\ &= {}^t\!V^{-1}(t)\left(\int_0^t {}^t\!V(\tau)({}^t\!R(\tau) - R(\tau))V(\tau)d\tau\right)V^{-1}(t) \\ &\quad + {}^t\!V^{-1}(t)\left({}^t\!V(0)V'(0) - {}^t\!V'(0)V(0)\right)V^{-1}(t) \\ &= {}^t\!V^{-1}(t)\left({}^t\!V(0)V'(0) - {}^t\!V'(0)V(0)\right)V^{-1}(t) \\ &= 0 \quad \text{if} \quad V(0) = 0 \text{ or } V'(0) = 0. \end{aligned} \tag{3.5}$$

One verifies with (3.3)

$$B(s,s) = 0, \quad \frac{\partial B}{\partial t}(s,s) = -\frac{1}{3}R(s) \quad \text{and} \quad \frac{\partial^2 B}{\partial t^2}(s,s) = -\frac{1}{2}R'(s).$$

We note that $A$ and $B$ fulfill a Riccati equation

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}S^{-1}\right) = \frac{\partial^2 S}{\partial^2 t}S^{-1} - \frac{\partial S}{\partial t}S^{-1}\frac{\partial S}{\partial t}S^{-1} = -R - A^2 \tag{3.6}$$

$$\frac{\partial B}{\partial t} = -R - A^2 + \frac{1}{(t-s)^2}I = -R - \frac{2}{(t-s)}B - B^2. \tag{3.7}$$

The key idea for the proof is the study of the dependence of $A$ on $s$. By the triangle inequality we have for $s < s'$ that the balls $B(c(s'), t-s') \subset B(c(s), t-s)$. This implies that the function $s \to A(t,s)$ is increasing. In fact, we have for all $t \neq s,\ t, s \in [-l_1, l_2]$

$$\begin{aligned}
\frac{\partial A}{\partial s} &= \frac{\partial}{\partial s}\left(\frac{\partial S}{\partial t}S^{-1}\right)\\
&= \frac{\partial^2 S}{\partial s \partial t}S^{-1} - \frac{\partial S}{\partial t}S^{-1}\frac{\partial S}{\partial s}S^{-1}\\
&= -\frac{\partial C}{\partial t}S^{-1} + {}^tS^{-1}\frac{\partial\, {}^tS}{\partial t}CS^{-1} \quad \text{as } \frac{\partial S}{\partial s} = -C,\ \frac{\partial S}{\partial t}S^{-1} = A = {}^tA \qquad (3.8)\\
&= {}^tS^{-1}\left(\frac{\partial\, {}^tS}{\partial t}C - {}^tS\frac{\partial C}{\partial t}\right)S^{-1} \quad \text{as } \frac{\partial\, {}^tS}{\partial t}C - {}^tS\frac{\partial C}{\partial t} \text{ is constant}\\
&= {}^tS^{-1}S^{-1} > 0.
\end{aligned}$$

This yields $\frac{\partial B}{\partial s}(s,s) = \frac{1}{3}R(s)$.

We now introduce the notations for the comparison geometry. We suppose that there is a manifold $(M^n_\lambda, g)$ with a geodesic $c_\lambda : \mathbb{R} \to M^n_\lambda$ without conjugate points in $[-l_1, l_2]$ and a curvature tensor along $c_\lambda$ given by $R_{\dot{c_\lambda}} = \lambda I$ with a function $\lambda : \mathbb{R} \to \mathbb{R}$. Analogous to the matrix solutions $S$ and $C$, we define $s_\lambda, c_\lambda : \mathbb{R} \to \mathbb{R}$ to be the solutions of the linear differential equation $y'' + \lambda y = 0$ with $s_\lambda(0) = 0$, $s'_\lambda(0) = 1$, $c_\lambda(0) = 1$ and $c'_\lambda(0) = 0$. Then $s_\lambda(t,s) := s_\lambda(t)\cdot c_\lambda(s) - c_\lambda(t)\cdot s_\lambda(s)$ and $c_\lambda(t,s) := c_\lambda(t)\cdot s'_\lambda(s) - s_\lambda(t)\cdot c'_\lambda(s)$ are the solutions of the Jacobi equation with $s_\lambda(s,s) = 0$, $\frac{\partial s_\lambda}{\partial t}(s,s) = 1$, $c_\lambda(s,s) = 1$ and $\frac{\partial c_\lambda}{\partial t}(s,s) = 0$ for fixed $s \in \mathbb{R}$. We define $e_\lambda(t,s) = s_\lambda(t,s)/(t-s)$ and $b_\lambda(t,s) = \frac{\partial e_\lambda}{\partial t}(t,s)/e_\lambda(t,s)$ for all $(t,s) \in [-l_1, l_2] \times [-l_1, l_2]$. For $t \neq s$, let $a_\lambda(t,s) = \frac{\partial s_\lambda}{\partial t}(t,s)/s_\lambda(t,s)$. To compare the geometry of $M$ with those of $M_\lambda$, we consider the differential of the generalized normal parametrization

$$E_\lambda(t,s) := \frac{E(t,s)}{e_\lambda(t,s)} = \frac{S(t,s)}{s_\lambda(t,s)} = {}^tE_\lambda(s,t),$$

and we introduce

$$\begin{aligned}
B_\lambda(t,s) :&= \frac{\partial E_\lambda}{\partial t}(t,s)E_\lambda^{-1}(t,s)\\
&= \left(\frac{1}{s_\lambda(t,s)}\frac{\partial S}{\partial t}(t,s) - \frac{\frac{\partial s_\lambda}{\partial t}(t,s)}{s^2_\lambda(t,s)}S(t,s)\right)\left(\frac{S(t,s)}{s_\lambda(t,s)}\right)^{-1}\\
&= A(t,s) - a_\lambda(t,s)I\\
&= B(t,s) - b_\lambda(t,s)I.
\end{aligned}$$

It is easy to check that $B_\lambda(s,s) = 0$ and $\frac{\partial B_\lambda}{\partial t}(s,s) = -\frac{1}{3}(R(s) - \lambda(s)I)$. From

(3.6) and $\frac{\partial a_\lambda}{\partial t}(t,s) = -\lambda(t) - a^2(t,s)$ we have

$$\begin{aligned}\frac{\partial}{\partial t}(s_\lambda^2 B_\lambda) &= 2s_\lambda \frac{\partial s_\lambda}{\partial t} B_\lambda + s_\lambda^2 \frac{\partial B_\lambda}{\partial t} \\ &= 2s_\lambda \frac{\partial s_\lambda}{\partial t} B_\lambda + s_\lambda^2(-R - A^2 + \lambda I + a_\lambda^2 I) \\ &= -s_\lambda^2(R - \lambda I) - s_\lambda^2(-2a_\lambda B_\lambda - a_\lambda^2 I + (B_\lambda + a_\lambda I)^2) \\ &= -s_\lambda^2(R - \lambda I) - s_\lambda^2 B_\lambda^2.\end{aligned}$$

It follows

$$\frac{\partial B_\lambda}{\partial t} = -2a_\lambda B_\lambda - B_\lambda^2 - (R - \lambda I) \tag{3.9}$$

and

$$B_\lambda(t,s) \;=\; -\int_s^t \frac{s_\lambda^2(\tau,s)}{s_\lambda^2(t,s)}(R(\tau) - \lambda(\tau))\,d\tau - \int_s^t \frac{s_\lambda^2(\tau,s)}{s_\lambda^2(t,s)} B_\lambda^2(\tau,s)\,d\tau. \tag{3.10}$$

With (3.8) we also have $\frac{\partial a_\lambda}{\partial s} = \frac{1}{s_\lambda^2}$ and therefore

$$\frac{\partial A}{\partial s} = {}^tS^{-1}S^{-1} = \frac{\partial a_\lambda}{\partial s} {}^tE_\lambda^{-1}E_\lambda^{-1}. \tag{3.11}$$

This gives, using $(\langle E_\lambda w, E_\lambda w\rangle)' = 2\langle B_\lambda E_\lambda w, E_\lambda w\rangle$ and $(\ln\|E_\lambda w\|^2)' = 2\langle B_\lambda \frac{E_\lambda w}{\|E_\lambda w\|}, \frac{E_\lambda w}{\|E_\lambda w\|}\rangle$,

$$\begin{aligned}\langle \frac{\partial A}{\partial s}(t,s)v, v\rangle &= \frac{\partial a_\lambda}{\partial s}(t,s)\langle {}^tE_\lambda^{-1}(t,s)E_\lambda^{-1}(t,s)v, v\rangle \\ &= \frac{\partial a_\lambda}{\partial s}(t,s)\|v\|^2 \exp\left(\ln\left\|E_\lambda^{-1}(t,s)\tfrac{v}{\|v\|}\right\|^2\right) \\ &= \frac{\partial a_\lambda}{\partial s}(t,s)\|v\|^2 \\ &\qquad \times \exp\left(-2\int_s^t \langle B_\lambda(\tau,s)Y(\tau,t,s), Y(\tau,t,s)\rangle d\tau\right)\end{aligned} \tag{3.12}$$

with $Y(\tau,t,s) = \frac{E_\lambda(\tau,s)E_\lambda^{-1}(t,s)v}{\|E_\lambda(\tau,s)E_\lambda^{-1}(t,s)v\|} = \frac{S(\tau,s)S^{-1}(t,s)v}{\|S(\tau,s)S^{-1}(t,s)v\|} = \frac{J_v(\tau)}{\|J_v(\tau)\|}$ and the Jacobi field $J_v$ defined by $J_v(s) = 0$ and $J_v(t) = v$. We note $\frac{\partial B_\lambda}{\partial s}(s,s) = \frac{1}{3}(R(s) - \lambda(s)I)$.

We can conclude now some first comparison results. Supposing that $R(t) \geq \lambda(t)$ on $[-l_1, l_2]$ we get from (3.10) and consequently from (3.12)

$$A(t,s) \leq a_\lambda(t,s) \quad \text{and} \quad \frac{\partial A}{\partial s}(t,s) \geq \frac{\partial a_\lambda}{\partial s}(t,s) \quad \text{for } s < t \tag{3.13}$$

or

$$(t-s)B_\lambda(t,s) \leq 0 \quad \text{and} \quad \frac{\partial B_\lambda}{\partial s}(t,s) \geq 0 \quad \text{for all } t,s \in [-l_1, l_2]. \tag{3.14}$$

The first inequality is the upper bound (1.2) in Theorem 1.
The second comparison result involves traces. From (3.10) we get

$$\begin{aligned} \operatorname{tr} B_\lambda(t,s) = &-\int_s^t \frac{s_\lambda^2(\tau,s)}{s_\lambda^2(t,s)}(\operatorname{tr} R(\tau)-(n-1)\lambda(\tau))\,d\tau \\ &-\int_s^t \frac{s_\lambda^2(\tau,s)}{s_\lambda^2(t,s)}\operatorname{tr} B_\lambda^2(\tau,s)\,d\tau. \end{aligned} \tag{3.15}$$

From the inequality of arithmetic and geometric means we have, for a $(n-1)\times(n-1)$-Matrix $M$, the inequality $\frac{\operatorname{tr}({}^tMM)}{n-1} \geq \sqrt[n-1]{\det({}^tMM)}$. We conclude with (3.11) and $(\ln\det E_\lambda)' = \operatorname{tr}(E_\lambda' E_\lambda^{-1}) = \operatorname{tr} B_\lambda$

$$\begin{aligned} \frac{\operatorname{tr}\frac{\partial A_\lambda}{\partial s}(t,s)}{n-1} &= \frac{\partial a_\lambda}{\partial s}(t,s)\frac{\operatorname{tr}({}^tE_\lambda^{-1}(t,s)E_\lambda^{-1}(t,s))}{n-1} \\ &\geq \frac{\partial a_\lambda}{\partial s}(t,s)\sqrt[n-1]{\det({}^tE_\lambda^{-1}(t,s)E_\lambda^{-1}(t,s))} = \sqrt[n-1]{\det\frac{\partial A_\lambda}{\partial s}(t,s)} \\ &= \frac{\partial a_\lambda}{\partial s}(t,s)(\det E_\lambda)^{-\frac{2}{n-1}}(t,s) \\ &= \frac{\partial a_\lambda}{\partial s}(t,s)\exp\left(-\frac{2}{n-1}\ln\det E_\lambda(t,s)\right) \\ &= \frac{\partial a_\lambda}{\partial s}(t,s)\exp\left(-2\int_s^t \frac{\operatorname{tr} B_\lambda(\tau,s)}{n-1}\,d\tau\right). \end{aligned} \tag{3.16}$$

If $\operatorname{tr} R \geq (n-1)\lambda$ on $[-l_1,l_2]$, we get from (3.15) and (3.16) the following relation with the comparison geometry

$$\frac{\operatorname{tr} A(t,s)}{n-1} \leq a_\lambda(t,s) \quad \text{and} \quad \frac{\frac{\partial A}{\partial s}(t,s)}{n-1} \geq \frac{\partial a_\lambda}{\partial s}(t,s) \quad \text{for } s<t \tag{3.17}$$

or

$$(t-s)\operatorname{tr} B_\lambda(t,s) \leq 0 \quad \text{and} \quad \operatorname{tr}\tfrac{\partial B_\lambda}{\partial s}(t,s) \geq 0 \quad \text{for all } t,s\in[-l_1,l_2]. \tag{3.18}$$

### 3.1 Proof of Theorem 1 and 2: Convexity and conjugate radius

*Proof of Theorem 2.* The Hessian of the reverse triangle length excess function $e$ in $c(t)$ restraint to the orthogonal complement of $\dot c(t)$ is analytically given by $A(t)-A(t,-l_1)$. With $\epsilon(t) = \int_{-l_1}^0 \frac{\partial a_\lambda}{\partial s}(t,\sigma)\,d\sigma = a_\lambda(t,0)-a_\lambda(t,-l_1) = \frac{s_\lambda(0,-l_1)}{s_\lambda(t)s_\lambda(t,-l_1)}$, the corresponding values in the model space, we have

$$\begin{aligned} A(t)-A(t,-l_1) &= \int_{-l_1}^0 \frac{\partial A}{\partial s}(t,\sigma)\,d\sigma \\ &= \int_{-l_1}^0 \frac{\partial B_\lambda}{\partial s}(t,\sigma)\,d\sigma + \epsilon(t)I \\ &= B_\lambda(t,0)-B_\lambda(t,-l_1)+\epsilon(t)I. \end{aligned}$$

The proof follows from (3.14) and (3.18). $\square$

On the one hand, the Riccati comparison is used to prove $\frac{\partial B_\lambda}{\partial s} \geq 0$, which is equivalent to Theorem 2 (see Remark 7). On the other hand, the Riccati comparison follows again from this inequality because of the equation

$$A(t) - a_\lambda(t) = B_\lambda(t) - B_\lambda(t,t) = -\int_0^t \tfrac{\partial B_\lambda}{\partial s}(t,\sigma)\, d\sigma.$$

*Proof of Theorem 1.* With the notations and equations above we can give a complete analytic proof of Theorem 1 and can extend the statement to a geodesic segment without conjugate points and non-constant curvature bounds. We start with the discussion of the Hessian of the length excess function of the points $c(-l_1)$ and $c(l_2)$ and show that it is positive-definite.
For two solutions $U$ and $V$ of the Jacobi equation $Y'' + RY = 0$, the expressions $C_0 = {}^tU'U - {}^tUU'$ and $C_1 = {}^tU'V - {}^tUV'$ are constant. If $U$ and $V$ are invertible on a same interval, we can consider the difference $U'U^{-1} - V'V^{-1} = {}^tU^{-1}({}^tUU' - {}^tU'U)U^{-1} + {}^tU^{-1}({}^tU'V - {}^tUV')V^{-1} = {}^tU^{-1}C_0U^{-1} + {}^tU^{-1}C_1V^{-1}$. With $U(t) = S(t,-l_1)$ and $V(t) = S(t,l_2)$ we have $C_0 = 0$, $C_1 = S(-l_1,l_2) = -{}^tS(l_2,-l_1)$ and

$$\begin{aligned} A(t,-l_1) - A(t,l_2) &= \frac{\partial S}{\partial t}(t,-l_1)S^{-1}(t,-l_1) - \frac{\partial S}{\partial t}(t,l_2)S^{-1}(t,l_2) \\ &= {}^tS^{-1}(t,-l_1)S(-l_1,l_2)S^{-1}(t,l_2) \\ &= -{}^tS^{-1}(t,-l_1){}^tS(l_2,-l_1)S^{-1}(t,l_2). \\ &= \delta(t)\cdot {}^tE_\lambda^{-1}(t,-l_1)E_\lambda(-l_1,l_2)E_\lambda^{-1}(t,l_2) \\ &= B_\lambda(t,-l_1) - B_\lambda(t,l_2) + \delta(t)I > 0 \end{aligned} \tag{3.19}$$

where we set $\delta(t) = a_\lambda(t,-l_1) - a_\lambda(t,l_2) = \frac{s_\lambda(-l_1,l_2)}{s_\lambda(t,-l_1)s_\lambda(t,l_2)}$ the non-zero eigenvalues of the Hessian of the corresponding length excess function in the model space. As the geodesic has no conjugate points, $A(t,-l_1) - A(t,l_2)$ is invertible. It behaves like $\frac{1}{t+l_1}I$ near $-l_1$ and like $\frac{1}{l_2-t}I$ near $l_2$. Thus $A(t,-l_1) - A(t,l_2)$ is positive definite on $-l_1 < t < l_2$. This means that the excess function of $c(-l_1)$ and $c(l_2)$ is strictly convex along $c$. Theorem 1 follows now from

$$\begin{aligned} B_\lambda(t) &= B_\lambda(t,0) - B_\lambda(t,-l_1) + B_\lambda(t,-l_1) \\ &= \int_{-l_1}^0 \frac{\partial B_\lambda}{\partial s}(t,\sigma)\, d\sigma + A(t,-l_1) - A(t,l_2) + B_\lambda(t,l_2) - \delta(t)I. \end{aligned}$$

If $R \geq \lambda$ we get from (3.19) and (3.14) $0 \geq B_\lambda(t) > -\delta(t)$. If $\operatorname{tr} R \geq (n-1)\lambda$ we conclude from (3.18) $0 \geq \frac{\operatorname{tr} B_\lambda(t)}{n-1} > -\delta(t)$. $\square$

For an alternative proof of Theorem 1 in the case of a lower curvature bound, see Remark 14 in Section 4.

## 4 Comparison of the shape operator of distance spheres with a background second fundamental tensor

The convexity of the length excess function and the reverse length excess function can be seen as a comparison of the second fundamental tensor of distance spheres with another background tensor field fulfilling the Riccati equation. If the principal curvature of this tensor field is bounded, then the following Lemma 1 already gives an estimate for the eigenvalues of the Weingarten map of distance spheres which is able to describe the pole.

Lemma 1 (Estimates for the shape operator of distance spheres using a comparison second fundamental tensor). *Let $M^n$ be an $n$-dimensional Riemannian manifold and $c : \mathbb{R} \to M$ a normal geodesic. Let $A$ denote the second fundamental tensor of the distance spheres of $c(0)$ along $c$ defined on $(0, l]$. Let $X$ be a comparison tensor field along $c$ on $[0, l]$ defined on the orthogonal complement of $\dot{c}$ and verifying the Riccati equation $\frac{\nabla}{dt}X + X^2 + R_{\dot{c}} = 0$, where $R_{\dot{c}}$ is the curvature tensor along $c$. Suppose that $f : [0, l] \to \mathbb{R}$ is a differentiable positive comparison function. Set*

$$g(t) := \int_0^t \frac{d\tau}{f^2(\tau)}.$$

*1) In the case of bounded $X$ we have the following comparison relation:*

$$\textit{If } X(t) \leq \frac{f'(t)}{f(t)}, \textit{ on } [0, l] \textit{ then } A(t) - X(t) \geq \frac{g'(t)}{g(t)} > 0 \quad \textit{on } (0, l]. \tag{4.1}$$

$$\textit{If } X(t) \geq \frac{f'(t)}{f(t)}, \textit{ on } [0, l] \textit{ then } 0 < A(t) - X(t) \leq \frac{g'(t)}{g(t)} \quad \textit{on } (0, l]. \tag{4.2}$$

*2) In the case of bounded $\operatorname{tr} X$ and Ricci curvature bounded from below, we have with $X_f := X - \frac{f'}{f} I$ and*

$$\delta_f(t) = f^2(0)\frac{\operatorname{tr} X_f(0)}{n-1} - f^2(t)\frac{\operatorname{tr} X_f(t)}{n-1} - \int_0^t f^2(x)\left(\frac{\operatorname{tr} R(x)}{n-1} + \frac{f''(x)}{f(x)}\right) dx$$

*the inequalities*

$$0 < \frac{g'(t)}{g(t)} e^{-2\sqrt{(n-1)\delta_f(t)g(t)}} \leq A(t) - X(t) \leq \frac{g'(t)}{g(t)} e^{2\sqrt{(n-1)\delta_f(t)g(t)}} \quad \textit{on } (0, l] \tag{4.3}$$

*and the $L^1$-bound*

$$\begin{aligned} &\int_\tau^t \|A(x) - (\tfrac{f'(x)}{f(x)} + \tfrac{g'(x)}{g(x)})I\| \, dx \\ &\qquad \leq \sqrt{n-1}\sqrt{\delta_f(t)}\left(1 + 4\, e^{2\sqrt{n-1}\sqrt{\delta_f(t)}\sqrt{g(t)}}\right)\sqrt{g(t) - g(\tau)} \\ &\qquad \in O(\sqrt{t-\tau}) \end{aligned} \tag{4.4}$$

*for $0 \leq \tau \leq t \leq l$.*

REMARK 13. *The asymptotic behavior of* $\frac{g'}{g}$ *is*

$$\frac{g'(t)}{g(t)} = \left(\int_0^t \left(\frac{f(t)}{f(\tau)}\right)^2 d\tau\right)^{-1} = \frac{1}{t} - \frac{f'(0)}{f(0)} + O(t) \quad \text{as } t \to 0.$$

*So the estimates can describe the pole of* $A$ *in* $t = 0$. *For the comparison function* $f(t) = s_\lambda(t, -l_1)$, $l_1 > 0$ *we have* $\frac{g'(t)}{g(t)} = a_\lambda(t) - a_\lambda(t, -l_1)$ *and for* $f(t) = -s_\lambda(t, l_2)$, $l_2 > 0$ *we obtain* $\frac{g'(t)}{g(t)} = a_\lambda(t) - a_\lambda(t, l_2)$.

REMARK 14. *Let* $c : \mathbb{R} \to M$ *a geodesic and* $l_1, l_2 > 0$ *such that the segment* $c_{[-l_1,l_2]}$ *has no conjugate points. Assume that* $R(t) \geq \lambda(t)$. *Then the Riccati comparison gives, with the notation of the proof of Theorem 1,* $A(t, -l_1) \leq a_\lambda(t, -l_1)$ *and* $A(t, l_2) \geq a_\lambda(t, l_2)$. *We conclude from the convexity of the length excess function that* $A(t, -l_1) - A(t, l_2) \geq 0$. *Consequently there are for both comparison fields* $A(t, -l_1)$ *and* $A(t, l_2)$ *the same bounds* $a_\lambda(t, l_2) \leq A(t, l_2) \leq A(t, -l_1) \leq a_\lambda(t, -l_1)$. *With* $f(t) = s_\lambda(t, -l_1)$ *and* $f(t) = -s_\lambda(t, l_2)$, *Lemma 1 gives bounded estimates for the Hessian of both length excess functions*

$$\begin{aligned} 0 < a_\lambda(t) - a_\lambda(t, -l_1) &\leq A(t) - A(t, -l_1) \\ &\leq A(t) - A(t, l_2) \leq a_\lambda(t) - a_\lambda(t, l_2). \end{aligned} \tag{4.5}$$

*The left inequality is the lower estimate of the Hessian of the reverse triangle excess function of the points* $c(0)$ *and* $c(-l_1)$ *along* $c$ *of Theorem 2 in the case of* $R \geq \lambda$, *whereas the right inequality is the upper bound of the Hessian of the length excess function of the points* $c(0)$ *and* $c(l_2)$ *due to the Riccati comparison. Theorem 3 means that this is also true under a lower Ricci curvature bound due to* (4.3). (4.5) *gives another proof of Theorem 1 in the case of a lower curvature bound as*

$$0 \geq A(t) - a_\lambda(t)I \geq A(t, l_2) - a_\lambda(t, -l_1) \geq a_\lambda(t, l_2) - a_\lambda(t, -l_1).$$

*If, in addition,* $l_1$ *is so small such that* $A(t, -l_1) \geq 0$ *on* $(0, l_2)$, *then* (4.5) *gives this better lower bound (see Remark 9)*

$$0 \geq A(t) - a_\lambda(t)I \geq A(t, -l_1) - a_\lambda(t, -l_1) \geq -a_\lambda(t, -l_1). \tag{4.6}$$

*The example in Section 7 shows that this estimate is sharp.*

*Proof Lemma 1.* We begin by proving that $A - X > 0$ on $(0, l)$. Let $t_0 \in (0, l)$ and let $V$ be the solution of the linear differential equation $V' = \frac{1}{2}(A + X) \cdot V$ with $V(t_0) = I$. Then $V$ is invertible, as the inverse of $V$ is the solution $W$ of the linear differential equation $W' = -W \cdot \frac{1}{2}(A + X)$ with $W(t_0) = I$. To see this we observe that $(WV)' = 0$. From $({}^tV(A - X)V)' = 0$ we conclude $A(t) - X(t) = {}^tV^{-1}(t)(A(t_0) - X(t_0))V^{-1}(t)$. As $A$ develops a pole in $t = 0$, $A - X > 0$ and is thus invertible.

Furthermore, let $U$ be the solution of the linear differential equation $U' = XU$ on $[0,l]$ with $U(0) = I$. We next claim that $A - X$ can be expressed in terms of $U$. $U$ fulfills the Jacobi equation as $U'' = X'U + XU' = -RU$ and can thus be extended smoothly on $\mathbb{R}$. The inverse of $U$ is given by the solution of the linear differential equation $W' = -WX$ with $W(0) = I$. So $X$ can be written on $[0,l]$ as $X = U'U^{-1}$. The columns of $U$ form a basis of Jacobi fields along c on $[0,l]$. The idea is to express the solution $S$ of the Jacobi field equation $S'' + RS = 0$ with $S(0) = 0$ and $S'(0) = I$ in this basis. As $U'U^{-1}$ is symmetric and ${}^tU'(\tau)\, S(\tau) - {}^tU(\tau)\, S'(\tau)$ is constant, we deduce

$$\begin{aligned} S(t) &= U(t)\, U^{-1}(t)\, S(t) \\ &= U(t) \int_0^t U^{-1}(\tau)\, S'(\tau) - U^{-1}(\tau)\, U'(\tau)\, U^{-1}(\tau)\, S(\tau)\, d\tau \\ &= U(t) \int_0^t U^{-1}(\tau)\, {}^tU^{-1}(\tau) \left( {}^tU(\tau)\, S'(\tau) - {}^tU'(\tau)\, S(\tau) \right) d\tau \\ &= U(t) \int_0^t U^{-1}(\tau)\, {}^tU^{-1}(\tau)\, d\tau. \end{aligned}$$

This gives

$$\begin{aligned} A(t) &= S'(t)\, S^{-1}(t) \\ &= \left( U'(t) \int_0^t U^{-1}(\tau)\, {}^tU^{-1}(\tau)\, d\tau + U(t)\, U^{-1}(t)\, {}^tU^{-1}(t) \right) \\ &\quad \times \left( \int_0^t U^{-1}(\tau)\, {}^tU^{-1}(\tau)\, d\tau \right)^{-1} U^{-1}(t) \\ &= U'(t)\, U^{-1}(t) + {}^tU^{-1}(t) \left( \int_0^t U^{-1}(\tau)\, {}^tU^{-1}(\tau)\, d\tau \right)^{-1} U^{-1}(t) \\ &= X(t) + \left( \int_0^t U(t)\, U^{-1}(\tau)\, {}^tU^{-1}(\tau)\, {}^tU(t)\, d\tau \right)^{-1}. \end{aligned} \tag{4.7}$$

We set $D(\tau,t) := {}^tU^{-1}(\tau)\ {}^tU(t)$. Note that $D(t,t) = I$ and $\frac{\partial D}{\partial \tau}(\tau,t) = -X(\tau)D(\tau,t)$. For $v \in \mathbb{R}^{n-1}$ we have

$$0 < \langle (A(t) - X(t))^{-1}\, v, v \rangle = \int_0^t \|D(\tau,t)v\|^2\, d\tau. \tag{4.8}$$

As

$$\tfrac{\partial}{\partial \tau} \ln \|D(\tau,t)v\|^2 = -2 \langle X(\tau) \tfrac{D(\tau,t)v}{\|D(\tau,t)v\|}, \tfrac{D(\tau,t)v}{\|D(\tau,t)v\|} \rangle$$

we conclude

$$\begin{aligned}\|D(\tau,t)v\|^2 &= \|v\|^2 \exp\left(2\int_\tau^t \langle X(x)\tfrac{D(x,t)v}{\|D(x,t)v\|}, \tfrac{D(x,t)v}{\|D(x,t)v\|}\rangle \, dx\right) \\ &= \|v\|^2 \tfrac{f^2(t)}{f^2(\tau)} \exp\left(2\int_\tau^t \langle (X(x) - \tfrac{f'(x)}{f(x)} I)\tfrac{D(x,t)v}{\|D(x,t)v\|}, \tfrac{D(x,t)v}{\|D(x,t)v\|}\rangle \, dx\right).\end{aligned} \tag{4.9}$$

If $X \le \frac{f'}{f}$ we have $\|D(\tau,t)v\|^2 \le \|v\|^2 \frac{f^2(t)}{f^2(\tau)} = \|v\|^2 \frac{g'(\tau)}{g'(t)}$ and so $0 < (A-X)^{-1} \le \frac{g}{g'}$. We conclude similarly that $0 < \frac{g}{g'} \le (A-X)^{-1}$ if $X \ge \frac{f'}{f}$. This shows (4.1) and (4.2).
The Cauchy-Schwarz inequality, $\|X_f\|^2 \le \operatorname{tr} X_f^2$ and the equation $f^2 X_f^2 = -(f^2 X_f)' - f^2(R + \frac{f''}{f} I)$ allows us to obtain for $0 \le \tau \le t \le l$

$$\begin{aligned}x_f(\tau,t) := \int_\tau^t \|X_f(x)\| \, dx &= \int_\tau^t \tfrac{1}{f(x)} f(x)\|X_f(x)\| \, dx \\ &\le \left(\int_\tau^t \tfrac{1}{f^2(x)} \, dx\right)^{1/2} \cdot \left(\int_\tau^t f^2(x)\|X_f(x)\|^2 \, dx\right)^{1/2} \\ &\le \sqrt{n-1}\sqrt{g(t)-g(\tau)} \cdot \left(\tfrac{1}{n-1}\int_0^t f^2(x) \operatorname{tr} X_f^2(x) \, dx\right)^{1/2} \\ &= \sqrt{n-1}\sqrt{\delta_f(t)}\sqrt{g(t)-g(\tau)}.\end{aligned}$$

From (4.8) and (4.9) we get for $v \in \mathbb{R}^{n-1}$ that

$$0 < \|v\|^2 f^2(t) g(t) e^{-2x_f(0,t)} \le \langle (A(t)-X(t))^{-1}v, v\rangle \le \|v\|^2 f^2(t) g(t) e^{2x_f(0,t)}.$$

We have thus proved (4.3)

$$\tfrac{g'(t)}{g(t)} e^{-2x_f(0,t)} \le A(t) - X(t) \le \tfrac{g'(t)}{g(t)} e^{2x_f(0,t)}.$$

We conclude that

$$\begin{aligned}-\tfrac{g'(t)}{g(t)}\left(e^{2x_f(0,t)}-1\right) &\le \tfrac{g'(t)}{g(t)}\left(e^{-2x_f(0,t)}-1\right) \\ &\le A(t) - X(t) - \tfrac{g'(t)}{g(t)} I \le \tfrac{g'(t)}{g(t)}\left(e^{2x_f(0,t)}-1\right).\end{aligned} \tag{4.10}$$

We deduce (cf.[5, Theorem 1])

$$\begin{aligned}&\left|\|A(t) - (\tfrac{f'(t)}{f(t)} + \tfrac{g'(t)}{g(t)})I\| - \|X_f(t)\|\right| \\ &\qquad \le \|A(t) - X(t) - \tfrac{g'(t)}{g(t)} I\| \\ &\qquad \le \tfrac{g'(t)}{g(t)}\left(e^{2\sqrt{n-1}\sqrt{\delta_f(t)}\sqrt{g(t)}} - 1\right) \quad \text{using (4.10)} \\ &\qquad \le 4\sqrt{n-1}\sqrt{\delta_f(t)} e^{2\sqrt{n-1}\sqrt{\delta_f(t)}\sqrt{g(t)}} \tfrac{g'(t)}{2\sqrt{g(t)}}\end{aligned}$$

where the last inequality follows from $e^x - 1 \leq xe^x$ for $x \geq 0$. We obtain (4.4)

$$\begin{aligned}\int_\tau^t &\|A(x) - (\tfrac{f'(x)}{f(x)} + \tfrac{g'(x)}{g(x)})I\| \, dx \\ &\leq \sqrt{n-1}\sqrt{\delta_f(t)} \\ &\quad \times \left(\sqrt{g(t)-g(\tau)} + 4\, e^{2\sqrt{n-1}\sqrt{\delta_f(t)}\sqrt{g(t)}} \left(\sqrt{g(t)} - \sqrt{g(\tau)}\right)\right) \\ &\leq \sqrt{n-1}\sqrt{\delta_f(t)} \left(1 + 4\, e^{2\sqrt{n-1}\sqrt{\delta_f(t)}\sqrt{g(t)}}\right) \sqrt{g(t)-g(\tau)}.\end{aligned}$$

□

Considering the tensor field $F := (A - X)^{-1}$ gives a more abstract point of view of Lemma 1. $F$ is the solution of the linear differential equation $F' = I + XF + FX$ with $F(0) = 0$. (4.7) is also a consequence of $(U^{-1}\, F\, {}^tU^{-1})' = U^{-1}\, {}^tU^{-1}$. In dimension two there is an explicit solution to this differential equation. This was used in [4] to prove Theorem 1 for dimension $n = 2$. Using the comparison functions we find out that $F_g := g'F - gI$, $F_g(0) = 0$, fulfills the linear differential equation $F_g' = 2gX_f + X_f F_g + F_g X_f$ with $X_f := X - \frac{f'}{f} I$. Defining $U_f(t) = \frac{f(0)}{f(t)} U(t)$ we have $(U_f^{-1}\ F_g\ {}^tU_f^{-1})' = 2g\ U_f^{-1}\ \ X_f\ {}^tU_f^{-1}$ which shows that $X_f \geq 0$ implies $F_g \geq 0$.

In [4, Chapter 5] and [5, Theorem 2] we used Grönwall's lemma for a barrier function for $F_g$ and the Neumann series

$$A - X = F^{-1} = \tfrac{g'}{g}\left(I + \tfrac{1}{g}F_g\right)^{-1} = \tfrac{g'}{g} I - \tfrac{g'}{g}\tfrac{1}{g}F_g \sum_{n=0}^{\infty} (-1)^n (\tfrac{1}{g}F_g)^n$$

for a similar barrier function for the $L^1$-norm of $A(t) - \frac{1}{t}I$ as in (4.4). This estimate for $A - \frac{g'}{g}I - X$ is also the key argument for the lower mean curvature estimates in the case of a lower Ricci curvature bound as in (1.3). In [4] we injected the background field $X$ in the integral equation (3.10) for $B$ which eliminates the curvature tensor

$$\begin{aligned}B(t) = X(t) - \int_0^t (\tfrac{\tau}{t})^2 \left(X(\tau)(B(\tau) - X(\tau)) + (B(\tau) - X(\tau))X(\tau)\right)\, d\tau \\ - \int_0^t (\tfrac{\tau}{t})^2 (B(\tau) - X(\tau))^2 \, d\tau - \int_0^t \tfrac{2\tau}{t^2} X(\tau)\, d\tau.\end{aligned}$$

Taking the trace gives a bounded lower estimate for $\operatorname{tr} B$ similar to Theorem 1.

### 4.1 Proof of Theorem 3 and Theorem 4: Ricci curvature, Excess functions and $L^1$ bounds for the principal curvature of distance spheres

We consider for the proof of Theorem 3 and Theorem 4 a non-constant lower Ricci curvature bound $\operatorname{Ric}(\dot c(t)) \geq (n-1)\lambda(t)$ and assume that the segment

$c_{[-l_1,l_2]}$ has no conjugate points.

*Proof of Theorem 3.* We conclude from the convexity of the excess function of the points $c(-l_1)$ and $c(l_2)$ (see (3.19)) and the Riccati comparison (3.17) that

$$(n-1)\,a_\lambda(t,l_2) \le \operatorname{tr} A(t,l_2) \le \operatorname{tr} A(t,-l_1) \le (n-1)\,a_\lambda(t,-l_1)$$

for $0 \le t < l_2$. To prove (1.10) we apply (4.3) of Lemma 1 with the comparison tensor field $X(t) := A(t,-l_1)$ and the comparison function $f(t) := s_\lambda(t,-l_1)$. We calculate $g(t) = \frac{s_\lambda(t)}{s_\lambda(0,-l_1)s_\lambda(t,-l_1)}$. The logarithmic derivation $\frac{g'}{g}$ are the eigenvalues of the excess functions in the model space $\frac{g'(t)}{g(t)} = a_\lambda(t) - a_\lambda(t,-l_1)$. It is

$$\begin{aligned}\delta_f(t) &\le f^2(t)(a_\lambda(t,-l_1) - a_\lambda(t,l_2)) = f^2(t)\tfrac{s_\lambda(-l_1,l_2)}{s_\lambda(t,-l_1)s_\lambda(t,l_2)}\\ &= s_\lambda(t,-l_1)\tfrac{s_\lambda(l_2,-l_1)}{s_\lambda(l_2,t)}\end{aligned}$$

and therefore $\delta_f(t)g(t) \le \frac{s_\lambda(l_2,-l_1)}{s_\lambda(0,-l_1)s_\lambda(l_2,t)}s_\lambda(t)$.
Applying Lemma 1 to $X(t) = A(t,l_2)$ and $f(t) = -s_\lambda(t,l_2)$ we get $g(t) = \frac{s_\lambda(t)}{s_\lambda(0,l_2)s_\lambda(t,l_2)}$. The logarithmic derivation $\frac{g'(t)}{g(t)} = a_\lambda(t) - a_\lambda(t,l_2)$ are the non-zero eigenvalues of the excess function in the model space. It is

$$\begin{aligned}\delta_f(t) &\le f^2(0)(a_\lambda(0,-l_1) - a_\lambda(0,l_2)) = f^2(0)\tfrac{s_\lambda(-l_1,l_2)}{s_\lambda(0,-l_1)s_\lambda(0,l_2)}\\ &= -s_\lambda(0,l_2)\tfrac{s_\lambda(l_2,-l_1)}{s_\lambda(0,-l_1)}\end{aligned}$$

and therefore $\delta_f(t)g(t) \le \frac{s_\lambda(l_2,-l_1)}{s_\lambda(0,-l_1)s_\lambda(l_2,t)}s_\lambda(t)$. This is the same upper barrier function as for the other comparison tensor field. (1.11) follows again from (4.3).
For the comparison with the Euclidean geometry we use $f(t) = 1$ as a comparison function. We have $g(t) = t$. Independently from $X(t) = A(t,-l_1)$ or $X(t) = A(t,l_2)$ it follows that

$$\delta_f(t) \le \tfrac{\operatorname{tr} X(0)}{n-1} - \tfrac{\operatorname{tr} X(t)}{n-1} - \int_0^t \lambda(\tau)\,d\tau \le a_\lambda(0,-l_1) - a_\lambda(t,l_2) - \int_0^t \lambda(\tau)\,d\tau.$$

□

*Proof of Theorem 4.* We choose $X(t) = A(t,-l_1)$ as the comparison tensor field to apply Lemma 1. For (1.12) set $f(t) = s_\lambda(t,-l_1)$ and for (1.13) set $f(t) = 1$ as the comparison function. The upper integral bound follows now from (4.4). □

X. Dai, Z. Shen and G. Wei pointed out (see [9], [10]) that a $L^1$-bound for $B$ can be expressed in terms of a bound for $\operatorname{tr} B$. So Theorem 4 is also a consequence of Theorem 1. Indeed, the Cauchy-Schwarz inequality and $\|B\|^2 \le \operatorname{tr} B^2$ give

$$\int_\tau^t \|B(x)\|\,dx \le \left(\int_\tau^t x^{-1/2}\,dx\right)^{1/2} \cdot \left(\int_\tau^t x^{1/2}\ \operatorname{tr} B^2(x)\,dx\right)^{1/2}. \tag{4.11}$$

We obtain $\sqrt{t}\operatorname{tr} B^2(t) = -\frac{2}{\sqrt{t}}\operatorname{tr} B(t) - \sqrt{t}\operatorname{tr} B'(t) - \sqrt{t}\operatorname{tr} R(t)$ using the Riccati equation (3.7) for $B$. Theorem 1 gives us bounded barrier functions depending on $\delta$ and $\lambda$ of the right term in (4.11), which leads to $L^1$-bounds for $B$ of order $O((t-\tau)^{1/4})$. Similarly, one gets $L^1$-estimates for $B_f$ using the differential equation $B_f' + 2\frac{f'}{f}B_f + B_f^2 + R + \frac{f''}{f}I = 0$ and integrating $\sqrt{f}\operatorname{tr} B_f^2$ (cf. [6, Theorm 5.5]) or simply by $\|B_f\| \leq \|B\| + |\frac{1}{t} - \frac{f'}{f}|$.

### 4.2 Proof of Theorem 5: Convexity radius, focal radius, Ricci curvature and bounded principal curvature of distance spheres

We first describe the relation between focal points and convexity of the distance function of a point.

Lemma 2 (focal points and convexity). *Let $c : [0,l] \to M$ be a normal geodesic in a Riemannian manifold $M$. The following two statements are equivalent*

1. *For every $\alpha \in [0,l)$ the geodesic segment $c_{|[\alpha,l]}$ has no focal points.*
2. *The shape operator $A_l$ of the distance spheres of $c(l)$ along $c^-(t) := c(l-t)$ is strictly convex on $(0,l]$, i.e. $A_l > 0$.*

*It follows that a lower bound $f_0$ on the focal radius of $c$ implies that all distance spheres of points on $c$ along $c^-$ exist and are convex at least on $(0,f_0)$. The reverse is also true.*

*Proof.* Suppose that there exists a focal point along $c$, i.e. there exists a Jacobi field $J \neq 0$ with $J'(\alpha) = 0$ and $J(\beta) = 0$ with $0 \leq \alpha < \beta \leq l$. Relation (1.1) implies a zero principal curvature of a distance sphere of $c(\beta)$ along $c^-$ in $c(\alpha)$ and vise versa. The strict convexity (3.8) completes the proof. □

The relation in Lemma 2 motivates the introduction of a radius function $\operatorname{convexARad} : T^1M \to \overline{\mathbb{R}}$ (see [6, Chapter 7]) from the unit tangent bundle to the compactification of $\mathbb{R}$ by

$$\begin{aligned}\operatorname{convexARad}(v) &= \sup\{r > 0 \mid A_v > 0 \text{ on } (0,r)\}\\ &= \sup\{r > 0 \mid S_v' \text{ is invertible on } [0,r)\}\end{aligned}$$

for a unit tangent vector $v \in T^1M$, $A_v$ the second fundamental tensor of the distance spheres along the normal geodesic $c_v(t) = \exp tv$ and $S_v$ the solution of the Jacobi equation along $c_v$ with $S_v(0) = 0$ and $S_v'(0) = I$. $(0, \operatorname{convexARad}(v))$ is the largest interval such that $A_v$ is strictly convex along $c_v$. convexARad is implicitly determined by the curvature tensor along $c_v$. Lemma 2 means that $\operatorname{focalRad}(\dot c(\alpha)) > l - \alpha$, $0 \leq \alpha < l$, is equivalent to $\operatorname{convexARad}(-\dot c(l)) > l$. In addition, the proof of Lemma 2 shows that

$$\operatorname{convexARad}(-\dot c(r)) \leq r \qquad \text{with } r = \operatorname{focalRad}(\dot c(0)) < \infty$$

and

$$\mathrm{focalRad}(\dot{c}(l-r)) = r \qquad \text{with } r = \mathrm{convexRad}(-\dot{c}(l)) < \infty.$$

convecARad is more convenient for expressing the assumptions of Theorem 5 but less known than the focal or conjugate radius function. Besides, the focal and conjugate raduis functions are locally Lipschitz continuous whereas convexARad is only lower semi-continuous (see [6, Chapter 7]).

*Proof of Theorem 5.* The idea is to apply Lemma 1 with a bounded comparison tensor field $X$. We start showing that the conjugate radius of $c$ is bounded below by $2f_0$. Suppose that $l_2 > 0$ is the first conjugate point of $c(0)$ of a normal geodesic $c : \mathbb{R} \to M$. We use the notation of Section 3. From (3.19) we conclude $A(t) - A(t, l_2) = \lim_{l \nearrow l_2} -{}^tS^{-1}(t)^tS(l)S^{-1}(t,l) \geq 0$. The dimension of the kernel of $A(t) - A(t, l_2)$ is equal to the multiplicity of the conjugate point $c(l_2)$ because $S(t) : \ker(S(l_2)) \to \ker(A(t) - A(t, l_2))$ is a linear isomorphism. To prove this we define for $u \in \ker(S(l_2))$ the Jacobi field $J_0(t) = S(t)u$ and $J_1(t) = S(t, l_2)S'(l_2)u$. These fields are equal because they have the same initial values in $t = l_2$. We have $(A(t) - A(t, l_2))S(t)u = J_0'(t) - J_1'(t) = 0$, i.e. $S(t)u \in \ker(A(t) - A(t, l_2))$. Conversely, if $v \in \ker(A(t) - A(t, l_2))$ we have $S(\tau)S^{-1}(t)v = S(\tau, l_2)S^{-1}(t, l_2)v$, as both sides define a Jacobi field with the same initial values in $\tau = t$. For $\tau = l_2$ we get $S^{-1}(t)v \in \ker S(l_2)$.
As $\mathrm{focalRad}(c) \geq f_0$, we conclude with Lemma 2 that $-A(t, l_2) > 0$ on $(l_2 - f_0, l_2)$. $\mathrm{focalRad}(c^-) \geq f_0$ implies $A > 0$ on $(0, f_0)$. These intervals cannot overlap because the kernel of $A(t) - A(t, l_2)$ is not empty. This implies that $l_2 \geq 2f_0$ and so $\mathrm{conjRad}(c) \geq 2f_0$.
$A$ is therefore defined at least on $(0, 2f_0)$ and all second fundamental tensors are convex, i.e. $A(t, l) > 0$ on $(l, l + f_0)$ and $A(t, l) < 0$ on $(l - f_0, l)$. As the Ricci curvature is bounded below, the Riccati comparison gives

$$0 < A(t, l) \leq \operatorname{tr} A(t, l) \leq (n-1)a_\lambda(t, l) \qquad \text{on } (l, l + f_0)$$

and

$$0 > A(t, l) \geq \operatorname{tr} A(t, l) \geq (n-1)a_\lambda(t, l) \qquad \text{on } (l - f_0, l).$$

For $l = 0$ this means that $A$ is bounded on $(0, f_0)$. From the convexity of the length excess function of the points $c(0)$ and $c(2f_0)$ we conclude that on $(f_0, 2f_0)$

$$\operatorname{tr} A(t, 2f_0) \leq A(t, 2f_0) \leq A(t) \leq A(t) - A(t, 2f_0) \leq \operatorname{tr}(A(t) - A(t, 2f_0)).$$

$A$ is therefore also bounded on $(f_0, 2f_0)$ with

$$(n-1)a_\lambda(t, 2f_0) \leq A(t) \leq (n-1)(a_\lambda(t) - a_\lambda(t, 2f_0)).$$

These elementary estimates do not describe the pole of A in $t = 0$. As there are bounded comparison tensor fields on $[0, f_0)$, Lemma 1 implies bounds for $A - \frac{1}{t}I$

on $(0, f_0)$. More precisely, for $t \in (0, f_0)$ we can choose $X(\tau) = A(\tau, t - f_0) > 0$. For an upper bound any constant comparison function $f$ leads to

$$A(t) \leq \tfrac{1}{t} I + A(t, t - f_0) \leq \tfrac{1}{t} + \operatorname{tr} A(t, t - f_0) \leq \tfrac{1}{t} + (n-1)\, a_\lambda(t, t - f_0).$$

Using the comparison function $f(\tau) := s_\lambda^{n-1}(\tau, t - f_0)$ Lemma 1 gives a lower bound

$$0 < \tfrac{g'(t)}{g(t)} = \left( \int_0^t \left( \tfrac{s_\lambda(t, t-f_0)}{s_\lambda(\tau, t-f_0)} \right)^{2(n-1)} \, d\tau \right)^{-1} \leq A(t)$$

also of type $\frac{1}{t} - (n-1) a_\lambda(0, -f_0) + O(t)$ as $t \to 0$. Using $X(\tau) = A(\tau, f_0)$ as the comparison field, estimates of the same type holds

$$(n-1) a_\lambda(t, f_0) + \tfrac{1}{t} \leq A(t) \leq \left( \int_0^t \left( \tfrac{s_\lambda(t, f_0)}{s_\lambda(\tau, f_0)} \right)^{2(n-1)} \, d\tau \right)^{-1}.$$

For constant $\lambda$ we find the inequalities (1.20). □

In the proof of Theorem 5 we get only piecewise continuous bounds for $A$ on $(0, 2f_0)$. For continuous estimates on $(0, 2f_0)$ one can extend Lemma 1 to piecewise differentiable barrier functions and use the elementary estimates from (1.17) and (1.18). Another possible technique for estimating $B$ is to use the integral equation (3.10) which gives

$$\operatorname{tr} B(t) + \int_0^t (\tfrac{\tau}{t})^2 \operatorname{Ric}(\dot c(\tau)) \, d\tau \leq B(t) + \int_0^t (\tfrac{\tau}{t})^2 R(\tau) \leq 0. \tag{4.12}$$

Using Corollary 1 we get that the distance between $B$ and the integral curvature expression is bounded (see [4, Theorem 5.7]). The basic idea is to get new estimates for the integral curvature expression. For this, we divide equidistantly the interval $[0, t]$ in sub-intervals of length $\frac{t}{k} < f_0$, $k \in \mathbb{N}$, $k > 0$. Under the assumptions of Theorem 5 we can consider $X_i(\tau) := A(\tau, t_i + f_0)$ on each sub-interval $[t_i, t_{i+1}]$, $0 \leq i \leq k-1$, with $t_i := \frac{i}{k} t$. It is $0 \leq -X_i(\tau) \leq -\operatorname{tr} X_i(\tau) \leq -(n-1)\, a_\lambda(\tau, t_i + f_0) = (n-1) \operatorname{ct}_\lambda(f_0 + t_i - \tau) \leq (n-1) \operatorname{ct}_\lambda(f_0 - \frac{t}{k})$ for $\tau \in [t_i, t_{i+1}]$. It follows

$$\begin{aligned}
\int_0^t \tfrac{\tau^2}{t^2} R(\tau)\, d\tau &= -\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \tfrac{\tau^2}{t^2} (X_i'(\tau) + X_i^2(\tau))\, d\tau \\
&\leq \sum_{i=0}^{k-1} \left( \tfrac{t_i^2}{t^2} X_i(t_i) - \tfrac{t_{i+1}^2}{t^2} X_i(t_{i+1}) + \int_{t_i}^{t_i+1} \tfrac{2\tau}{t^2} X_i(\tau)\, d\tau \right) \\
&\leq -\sum_{i=0}^{k-1} \tfrac{t_{i+1}^2}{t^2} X_i(t_{i+1}) \leq (n-1) \operatorname{ct}_\lambda(f_0 - \tfrac{t}{k}) \tfrac{1}{k^2} \sum_{i=1}^{k} i^2 \\
&\leq (n-1) \tfrac{(k+1)(2k+1)}{6k} \operatorname{ct}_\lambda(f_0 - \tfrac{t}{k}).
\end{aligned} \tag{4.13}$$

A lower integral curvature bound can be obtained using $\|X_i\|^2 \leq \operatorname{tr} X_i^2$

$$\begin{aligned}
-\int_0^t \tfrac{\tau^2}{t^2} R(\tau)\, d\tau &= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \tfrac{\tau^2}{t^2}(X_i'(\tau) + X_i^2(\tau))\, d\tau \\
&\leq \sum_{i=0}^{k-1} \left( -\tfrac{t_i^2}{t^2} \operatorname{tr} X_i(t_i) - \int_{t_i}^{t_i+1} \tfrac{2\tau}{t^2} \operatorname{tr} X_i(\tau)\, d\tau \right) \\
&\quad - \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \tfrac{\tau^2}{t^2}(\operatorname{tr} X_i'(\tau) + \operatorname{tr} R(\tau))\, d\tau \qquad (4.14)\\
&= -\sum_{i=0}^{k-1} \tfrac{t_{i+1}^2}{t^2} \operatorname{tr} X_i(t_{i+1}) - \int_0^t \tfrac{\tau^2}{t^2} \operatorname{tr} R(\tau)\, d\tau \\
&\leq (n-1)\tfrac{(k+1)(2k+1)}{6k} \operatorname{ct}_\lambda(f_0 - \tfrac{t}{k}) - (n-1)\lambda\tfrac{t}{3}.
\end{aligned}$$

With (4.12) we conclude

$$-B(t) = \tfrac{1}{t} I - A(t) \leq (n-1)\tfrac{(k+1)(2k+1)}{6k} \operatorname{ct}_\lambda(f_0 - \tfrac{t}{k}) - (n-1)\lambda\tfrac{t}{3} - \operatorname{tr} B(t) \quad (4.15)$$

and

$$B(t) = A(t) - \tfrac{1}{t} I \leq (n-1)\tfrac{(k+1)(2k+1)}{6k} \operatorname{ct}_\lambda(f_0 - \tfrac{t}{k}) - (n-1)\lambda\tfrac{t}{3}. \quad (4.16)$$

For $k \geq 3$ this upper barrier is better than the estimate in (1.18) which is not bounded in $t = 2f_0$. The inequalities in Remark 8 are the special case $k = 3$. This gives a second proof of Theorem 5.

## 5 Proof of Theorem 6: Integral curvature bounds and convexity

*Proof.* We use the notation of Section 3. Statement (1) is straightforward: to get the barrier for the integral curvature tensor we use the integral equation (3.10) with $\lambda \equiv 0$. The properties of the focal points follow from the relation $A(t) < A(t,s) = {}^tC(s,t)S^{-1}(t,s)$ for $0 < s < t < c_0$ (see (3.4) and (3.8)). Indeed, if $A(t,s)v = 0$ then $J(\tau) = S(\tau,s)S^{-1}(t,s)v$ is a Jacobi field with $J(s) = 0$ and $J'(t) = 0$ (see Lemma 2).

For the second statement we use again (3.10), which gives $\|A(t) - \frac{1}{t}I\| \leq \Lambda(t) + \int_0^t \|A(\tau) - \frac{1}{\tau}I\|^2\, d\tau$. Let $\beta_1(t) := \int_0^t \|A(\tau) - \frac{1}{\tau}I\|^2\, d\tau$. Then $\beta_1'(t) = \|A(t) - \frac{1}{t}I\|^2 \leq (\Lambda(t) + \beta_1(t))^2$. Define

$$\Delta(t) = (\beta(t) - \beta_1(t)) \exp\left( -\int_0^t 2\Lambda(\tau) + \beta_1(\tau) + \beta(\tau)\, d\tau \right).$$

Then $\Delta(0) = 0$ and $\Delta'(t) \geq 0$. This gives $\Delta \geq 0$ and proves the inequality. As $A$ is a solution of an ordinary differential equation, $A$ must exist at least on $(0, l_2)$. ∎

With (4.13) and (4.14) Theorem 6 gives another proof of Theorem 5.

## 6 Core applications

We have seen in Section 2.2 that the tensor field $B_\lambda$ naturally describes the exponential map. We will see that the use of $B_\lambda$ simplifies the proof of the known core comparison results and allows us to get new ones.

### 6.1 Convexity and upper curvature bounds

We start considering the general Riccati comparison and give a proof using $B$. Suppose that $\tilde{R}$ is a curve in $\mathrm{Sym}(n-1,\mathbb{R})$ and $\tilde{S}$ is the solution of $\tilde{S}''+\tilde{R}\tilde{S}=0$ with $\tilde{S}(0)=0$ and $\tilde{S}'(0)=I$. Put $\tilde{A}=\tilde{S}'\tilde{S}^{-1}$ and $\tilde{B}=\tilde{A}-\frac{1}{t}I$. Let $U$ be the solution of $U'=\frac{1}{2}(B+\tilde{B})\cdot U$ with $U(0)=I$ and $V$ the solution of $V'=-V\cdot\frac{1}{2}(B+\tilde{B})$ with $V(0)=I$. Then $U$ and $V$ are the inverse of each other as $(VU)'=V'U+VU'=-V\frac{1}{2}(B+\tilde{B})U+V\frac{1}{2}(B+\tilde{B})U=0$. We have

$$\begin{aligned} t^2(B(t)-\tilde{B}(t)) &= t^2(A(t)-\tilde{A}(t)) \\ &= -{}^t U^{-1}(t)\cdot\int_0^t \tau^2\,{}^tU(\tau)(R(\tau)-\tilde{R}(\tau))U(\tau)\,d\tau\cdot U^{-1}(t) \\ &= -\int_0^t \tau^2\,{}^tX(\tau,t)(R(\tau)-\tilde{R}(\tau))X(\tau,t)\,d\tau \end{aligned} \tag{6.1}$$

with $X(\tau,t)=U(\tau)U^{-1}(t)=U(\tau)V(t)$ since the left and the right side of the equation fulfill the linear differential equation

$$Y'=-t^2(R-\tilde{R})-\tfrac{1}{2}(B+\tilde{B})\cdot Y-Y\cdot\tfrac{1}{2}(B+\tilde{B})$$

with $Y(0)=0$. Equation (6.1) expresses the general Riccati comparison, i.e. $R\le\tilde{R}$ implies $A\ge\tilde{A}$ (cf. [11, Theorem 1]). If $R\le\Lambda$ we have $B_\Lambda\ge 0$. As a consequence of (3.12) we have $\frac{\partial B_\Lambda}{\partial s}\le 0$. These inequalities can be interpreted geometrically as in Section 2.1.

### 6.2 Behavior of the shape operator of distance spheres at the first conjugate point

We introduced $B$ to deal with the singularity of $A$ in $t=0$. We can generalize this approach to analyze the behavior at the first conjugate point $c(l)$, $l>0$. This is similar to the way in [11]. From the convexity of the triangle length excess functions it follows that $B(t,l)+\frac{1}{t-l}I\le A(t)\le A(t,s)$ for $s\in(0,l)$ and $s<t<l$. This gives a rough impression of the pole. Let $Q$ be the orthogonal projection on $\ker S(l)\neq\{0\}$ ($S(l)Q=0$) and $D(t)=\frac{1}{t-l}Q+I-Q$ for $t\neq l$. $D$ scales $\mathrm{im}\,Q=\ker S(l)$ and leaves $\ker Q=\mathrm{im}(I-Q)=\ker S(l)^\perp$ invariant. $k=\dim\ker S(l)$ is the index of the conjugate point $c(l)$. If $k=n-1$ we have $S(l)=0$, $Q=I$, $S(t,l)=S(t)\,{}^tC(l)-C(t)\,{}^tS(l)=S(t)\,{}^tC(l)$ and $S(t)=S(t,l)S'(l)+C(t,l)S(l)=S(t,l)S'(l)$. Thus $A(t)=A(t,l)=\frac{1}{t-l}I+B(t,l)$

describes the singularity in $t = l$. We define

$$E(t) = S(t)D(t) = \tfrac{S(t)Q - S(l)Q}{t-l} + S(t)(I-Q) = \int_0^1 S'(l+(t-l)\tau)Q\,d\tau + S(t)(I-Q).$$

$E$ is smooth in $t = l$ with $E(l) = S'(l)Q + S(l)(I-Q)$. $E(l)$ is invertible because $S$ is a Lagrange tensor (${}^tS(t)S'(t) - {}^tS'(t)S(t) = 0$) and non-degenerated ($\ker S'(l) \cap \ker S(l) = \{0\}$). In a neighborhood of $t = l$ we can consider (note that $D^{-1}(t) = (t-l)Q + I - Q$)

$$E'(t)E^{-1}(t) = A(t) - \tfrac{1}{t-l}E(t)QE^{-1}(t) = A(t) - \tfrac{1}{t-l}P(t)$$

with $P(t) = E(t)QE^{-1}(t) = S(t)QS^{-1}(t)$. $P(t)$ is a projection with $\operatorname{im} P(t) = S(t)(\operatorname{im} Q)$ and $\ker P(t) = S(t)(\ker Q)$ for $t \neq l$. Let

$$P := P(l) = E(l)QE^{-1}(l) = \lim_{t\to l}(t-l)A(t) = \lim_{t\to l}(t-l){}^tA(t) = {}^tP.$$

$P$ is an orthogonal projection with $\operatorname{im} P = S'(l)(\operatorname{im} Q)$ and $\ker P = S(l)(\ker Q)$. Define

$$B(t) := A(t) - \tfrac{1}{t-l}P = E'(t)E^{-1}(t) + \int_0^1 P'(l+(t-l)\tau)\,d\tau.$$

$B$ is a smooth curve in $\mathrm{Sym}(\text{n-1}, \mathbb{R})$ fulfilling the Riccati equation

$$B'(t) + \tfrac{1}{t-l}(B(t)P + PB(t)) + B^2(t) + R(t) = 0.$$

Since $B$ is smooth we obtain $B(l)P + PB(l) = 0$. This gives $B(l)P = 0$ and consequently $B(l) = (I-P)B(l)(I-P)$. Observe that $B'(l) + B'(l)P + PB'(l) + B^2(l) + R(l) = 0$. We conclude that the behavior of $A$ in $t = l$ is

$$\begin{aligned} A(t) &= \tfrac{1}{t-l}I + A(t) - A(t,l) + B(t,l) \\ &= \tfrac{1}{t-l}P + (I-P)B(l)(I-P) + O(t-l). \end{aligned}$$

As $(A(t) - A(t,l))S(t)Q = 0$ we have also $A(t)P(t) = \frac{1}{t-l}P(t) + B(t,l)P(t) = A(t,l)P(t)$.

### 6.3 Convexity and Jacobi field estimates

Another advantage of the operator $B_\lambda$ is that Jacobi fields $J$ with $J(0) = 0$ and $\langle J'(0), \dot c(0)\rangle = 0$ along a geodesic $c$ without conjugate points in $[0, l_2]$ can be expressed in terms of $B_\lambda$. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(0) = 0$, $f'(0) = 1$ and positive on $(0, l_2]$. The idea is to compare $A$ with the logarithmic derivative of $f$. We put $E_f = \frac{S}{f}$, $B_f = A - \frac{f'}{f}I$ and $J_f = \frac{J}{f} = E_f J'(0)$. Then $E_f(0) = I$, $J_f(0) = J'(0)$, $E_f' = B_f E_f$, $J_f' = B_f J_f$ and $B_f(0) = -\frac{1}{2}f''(0)I$.

The estimates in this section also provide estimates for the differential $E_f$ of the map $\mathrm{Exp}_\lambda$, which will not be mentioned explicitly. We have (cf. [16, 1.8])

$$\begin{aligned}
\frac{\|J(t)\|}{f(t)} &= \|J_f(t)\| \\
&= \|J_f(0)\| \cdot \exp\left(\frac{1}{2}\ln\|J_f(t)\|^2 - \frac{1}{2}\ln\|J_f(0)\|^2\right) \\
&= \|J_f(0)\| \cdot \exp\left(\int_0^t \frac{\langle J_f'(\tau), J_f(\tau)\rangle}{\|J_f(\tau)\|^2}\, d\tau\right) \\
&= \|J'(0)\| \cdot \exp\left(\int_0^t \langle B_f(\tau)\frac{J(\tau)}{\|J(\tau)\|}, \frac{J(\tau)}{\|J(\tau)\|}\rangle\, d\tau\right) \qquad (6.2)\\
&= \frac{\|J(t_0)\|}{f(t_0)} \cdot \exp\left(-\int_t^{t_0} \langle B_f(\tau)\frac{J(\tau)}{\|J(\tau)\|}, \frac{J(\tau)}{\|J(\tau)\|}\rangle\, d\tau\right). \qquad (6.3)
\end{aligned}$$

If $B_f \leq 0$ we conclude from (6.2) and (6.3) that $\|J\|/f$ is decreasing with

$$\frac{f(t)}{f(t_0)}\|J(t_0)\| \leq \|J(t)\| \leq f(t)\|J'(0)\| \quad \text{for} \quad 0 \leq t \leq t_0.$$

If $B_f \geq 0$ we get that $\|J\|/f$ is increasing with

$$f(t)\|J'(0)\| \leq \|J(t)\| \leq \frac{f(t)}{f(t_0)}\|J(t_0)\| \quad \text{for} \quad 0 \leq t \leq t_0.$$

For other Jacobi field estimates a bound on the $L^1$-norm for $B_f$ is sufficient (cf. [5, Theorem 3], [6, chapter 5.4], [9] and [10])). We define the integral barrier function of $B_f$ by $b_f(t_0, t_1) := \int_{t_0}^{t_1} \|B_f(\tau)\|\, d\tau$ for $0 \leq t_0 \leq t_1 \leq l_2$ and set $b_f(t) := b_f(0, t)$. From equation (6.2) we get an upper and a lower bound for the orthogonal Jacobi fields with $J(0) = 0$ in terms of this integral barrier function. We have

$$f(t) \cdot \|J'(0)\| \cdot \exp\left(-b_f(t)\right) \leq \|J(t)\| \leq f(t) \cdot \|J'(0)\| \cdot \exp\left(b_f(t)\right) \qquad (6.4)$$

and

$$\frac{f(t)}{f(t_0)} \cdot \|J(t_0)\| \cdot \exp\left(-b_f(t, t_0)\right) \leq \|J(t)\| \leq \frac{f(t)}{f(t_0)} \cdot \|J(t_0)\| \cdot \exp\left(b_f(t, t_0)\right). \qquad (6.5)$$

The comparison of a Jacobi field with the associated affine parallel vector field gives

$$\begin{aligned}
&\|J(t) - f(t)J'(0)\| \\
&\qquad = f(t) \cdot \|J_f(t) - J_f(0)\| \\
&\qquad = f(t) \cdot \left\| \int_0^t B_f(\tau) J_f(\tau)\, d\tau \right\| \\
&\qquad \le f(t) \cdot \int_0^t \|B_f(\tau)\|\, \|J_f(\tau)\|\, d\tau \\
&\qquad \le f(t) \cdot \|J'(0)\| \cdot \int_0^t \|B_f(\tau)\| \exp\left( \int_0^\tau \|B_f(\vartheta)\|\, d\vartheta \right) d\tau \quad \text{using (6.2)} \\
&\qquad = f(t) \cdot \|J'(0)\| \cdot \left( \exp\left( \int_0^t \|B_f(\tau)\|\, d\tau \right) - 1 \right) \\
&\qquad = f(t) \cdot \|J'(0)\| \cdot (\exp(b_f(t)) - 1)\,.
\end{aligned} \tag{6.6}$$

For the angular velocity of a Jacobi field we have to consider

$$\begin{aligned}
\left(\tfrac{J}{\|J\|}\right)' &= \left(\tfrac{J_f}{\|J_f\|}\right)' \quad \text{for all } f \text{ as above} \\
&= \tfrac{J_f'}{\|J_f\|} - \tfrac{\langle J_f', J_f\rangle}{\|J_f\|^2} \tfrac{J_f}{\|J_f\|} \\
&= \left( B_f - \langle B_f \tfrac{J}{\|J\|}, \tfrac{J}{\|J\|} \rangle I \right) \tfrac{J}{\|J\|} \\
&= \left( A - \langle A \tfrac{J}{\|J\|}, \tfrac{J}{\|J\|} \rangle I \right) \tfrac{J}{\|J\|}. \\
&= B_y \tfrac{J}{\|J\|}
\end{aligned}$$

with $y = \|J\| / \|J'(0)\|$. This follows from $\frac{y'}{y} = \langle A \frac{J}{\|J\|}, \frac{J}{\|J\|} \rangle$. We conclude

$$\begin{aligned}
\left\| \left(\tfrac{J}{\|J\|}\right)' \right\| &= \left\| B_y \tfrac{J}{\|J\|} \right\| \\
&= \sqrt{ \left\| B_f \tfrac{J}{\|J\|} \right\|^2 - \langle B_f \tfrac{J}{\|J\|}, \tfrac{J}{\|J\|} \rangle^2 } \\
&= \sqrt{ \left\| B_f \tfrac{J}{\|J\|} \right\|^2 - \left( \tfrac{y'}{y} - \tfrac{f'}{f} \right)^2 } \\
&\le \|B_f\|\,.
\end{aligned} \tag{6.7}$$

From the law of cosines we get for the angle between the Jacobi field and the corresponding parallel vector field $X$ with $X(0) = J'(0)$

$$\begin{aligned}2\sin\left(\frac{1}{2}\angle(J(t),J'(0))\right) &= \sqrt{2-2\cos\angle(J(t),J'(0))}\\ &= \left\|\frac{J(t)}{\|J(t)\|}-\frac{J'(0)}{\|J'(0)\|}\right\|\\ &= \left\|\int_0^t \frac{d}{d\tau}\frac{J(\tau)}{\|J(\tau)\|}\,d\tau\right\|\\ &\le \int_0^t \|B_f(\tau)\|\,d\tau = b_f(t) \quad \text{for all } f \text{ as above.}\end{aligned}$$

For a geodesic $c : [0,l] \to M$ without conjugate points the estimate from (6.2) holds as long as $B_f$ is defined. This may not be the entire interval $[0,l]$. One can obtain estimates on $[0,l]$ using $A(t,l)$ along $c$. To explain the idea, we simplify and take Euclidean geometry as comparison geometry. We have as long as $J \neq 0$

$$\tfrac{\|J(t)\|}{\|J'(0)\|} = t\cdot\exp\left(\int_0^t \tfrac{\langle J'(\tau),J(\tau)\rangle}{\langle J(\tau),J(\tau)\rangle} - \tfrac{1}{\tau}\,d\tau\right).$$

For $t_0 \in (0,l)$ one can use the equations

$$\tfrac{\langle J'(\tau),J(\tau)\rangle}{\langle J(\tau),J(\tau)\rangle} - \tfrac{1}{\tau} = \langle B(\tau)\tfrac{J(\tau)}{\|J(\tau)\|}, \tfrac{J(\tau)}{\|J(\tau)\|}\rangle \qquad \text{for } 0 \le \tau \le t_0$$

and

$$\begin{aligned}\tfrac{\langle J'(\tau),J(\tau)\rangle}{\langle J(\tau),J(\tau)\rangle} - \tfrac{1}{\tau} &= \langle (A(\tau)-A(\tau,l))\tfrac{J(\tau)}{\|J(\tau)\|}, \tfrac{J(\tau)}{\|J(\tau)\|}\rangle\\ &\quad + \langle B(\tau,l)\tfrac{J(\tau)}{\|J(\tau)\|}, \tfrac{J(\tau)}{\|J(\tau)\|}\rangle + \tfrac{1}{\tau-l} - \tfrac{1}{\tau} \qquad \text{for } t_0 \le \tau \le l\end{aligned}$$

to get lower estimates for $\|J(t)\|$ on $[t_0,l]$.

If $J(l_2) = 0$, $l_2 > l$, we have $J(t) = S(t)J'(0) = S(t,l_2)J'(l_2)$ because both terms solve the Jacobi equation with the same initial values at $t = l_2$. It follows that for all $t$, for which $A(t)$ and $A(t,l_2)$ are defined, we have $J'(t) = A(t)J(t) = A(t,l_2)J(t)$. If $c_{|[t_0,l_2]}$ has no conjugate points we obtain

$$\tfrac{\langle J'(\tau),J(\tau)\rangle}{\langle J(\tau),J(\tau)\rangle} - \tfrac{1}{\tau} = \langle B(\tau,l_2)\tfrac{J(\tau)}{\|J(\tau)\|}, \tfrac{J(\tau)}{\|J(\tau)\|}\rangle + \tfrac{1}{\tau-l_2} - \tfrac{1}{\tau} \qquad \text{for } t_0 \le \tau \le l_2.$$

This results in bounds for $\|J(t)\|$ on the entire interval $[0,l_2]$. $l_2$ can be the first conjugate point or may lie behind it.

We finish with an overview and a comparison of the possible estimates for $B_f$ under different assumptions due to the results in the previous sections.

Rauch's first comparison theorem is a direct consequence of the first and second row in Table 2. The comparison results due to the third row are also well known. The last four rows allow us to get new estimates for Jacobi fields.

Table 2: Overview of comparison results for $B_f$

| CONDITIONS | ESTIMATES AND COMPARISON FUNCTIONS |
|---|---|
| $R \leq \Lambda$ on $[0, l_2]$ | $B_f \geq 0$ with $f = s_\Lambda$ |
| $R \geq \lambda$ on $[0, l_2]$ | $B_f \leq 0$ with $f = s_\lambda$ |
| $\lambda \leq R \leq \Lambda$ on $[0, l_2]$ | $\|B_f\| \leq \frac{1}{2}(a_\lambda - a_\Lambda) \in O(t)$,<br>$b_f(t) \leq \frac{1}{2}\ln\left(\frac{s_\lambda(t)}{s_\Lambda(t)}\right) \in O(t^2)$<br>as $t \to 0$ with $f = \sqrt{s_\lambda s_\Lambda}$ |
| $R \geq \lambda$ on $[-l_1, f_0]$ and $c^-_{\|[-l_1,f_0]}$ without focal points | $B_f \geq 0$ with $f(t) = \frac{s_\lambda(0,-l_1)}{s_\lambda(t,-l_1)} s_\lambda(t)$ |
| $R \geq \lambda$ on $[-l_1, l_2]$, $c_{\|[-l_1,l_2]}$ without conjugate points | $B_f \geq 0$ with $f(t) = s_\lambda(t) \cdot \exp\left(-\int_0^t \delta(\tau)\, d\tau\right)$ |
| | $\|B_f\| \leq \frac{1}{2}\delta \in O(1)$,<br>$b_f(t) \leq \frac{1}{2}\int_0^t \delta(\tau)\, d\tau \in O(t)$ as $t \to 0$<br>with $f(t) = s_\lambda(t) \cdot \exp\left(-\frac{1}{2}\int_0^t \delta(\tau)\, d\tau\right)$ |
| $\operatorname{tr} R \geq (n-1)\lambda$ on $[-l_1, l]$ and $c^-_{\|[-l_1,l]}$ without focal points | $B_f \leq 0$ with $f(t) = t\left(\frac{s_\lambda(t,-l_1)}{s_\lambda(0,-l_1)}\right)^{n-1}$<br>$B_f \geq 0$ with $f(t) = \int_0^t \left(\frac{s_\lambda(0,-l_1)}{s_\lambda(\tau,-l_1)}\right)^{2(n-1)} d\tau$ |
| $\operatorname{tr} R \geq (n-1)\lambda$ on $[-l_1, l_2]$ and $c_{\|[-l_1,l_2]}$ without conjugate points | $b_f(t) \leq (1 + 4e^{2\beta_\lambda(t)})\beta_\lambda(t) \in O(\sqrt{t})$ as $t \to 0$<br>with $f(t) = s_\lambda(t)$<br>and $\beta_\lambda(t) = \sqrt{n-1}\sqrt{\delta(t)\frac{s_\lambda(t,-l_1)}{s_\lambda(0,-l_1)}}\sqrt{s_\lambda(t)}$ |
| with $\delta(t) = a_\lambda(t,-l_1) - a_\lambda(t,l_2) = \frac{\partial}{\partial t}\ln\left(\frac{s_\lambda(t,-l_1)}{-s_\lambda(t,l_2)}\right) = \frac{s_\lambda(-l_1,l_2)}{s_\lambda(t,-l_1)s_\lambda(t,l_2)}$ | |

The $L^1$ estimates from Theorem 4 of $B$ of type $O(\sqrt{t})$ in the last row as well as the application to Jacobi field estimates due to (6.4) and (6.6) were announced in [5]. This was applied in [9], [10], [21], [22], [23], [24] and [25].

To complete this section, we would like to mention that there are other approaches that can be more suitable for Jacobi field estimates. Let $X : \mathbb{R} \to M$ be a vector field along a geodesic $c$ with $X \neq 0$ on $(0, l)$, $l > 0$. Define $y = \|X\|$, $N_X = \frac{X}{\|X\|}$, $r := -\frac{\langle X'', X\rangle}{\langle X, X\rangle}$ and $a := \frac{\langle X', X\rangle}{\langle X, X\rangle} = \frac{\|X'\|}{\|X\|}\cos(\angle(X', X))$. Then for the angular velocity $N_X'$ we have $\|N_X'\|^2 = \frac{\|X'\|^2}{\|X\|^2}\sin^2(\angle(X', X))$ and for $y$ the differential equations $y' = ay$ and $y'' + (r - \|N_X'\|^2)y = 0$ on $(0, l)$.

These equations are used in [3, Theorem 5.6] to obtain lower estimates on $\|J\|$ for a Jacobi field $J$, $J(0) = 0$, depending on both boundary values $\|J'(0)\|$ and $\|J(t_0)\|$ under bounded curvature assumption. They use inequalities on the angular velocity of the normal field $N_J = \frac{J}{\|J\|}$ in combination with a maximum principal. This estimate of the angular velocity goes beyond the one in (6.7). As pointed out in [3], $\langle RN_J, N_J\rangle - \|N_J'\|^2$ can be interpreted as the intrinsic sectional curvature of a ruled surface defined by $N_J$ in $M$.

In [28] the virtual Jacobi field $X(t) := {}^tS^{-1}(t)v$ and

$$z(t) = \frac{\|v\|^2}{y(t)} = \frac{\|v\|^2}{\|X(t)\|} = \frac{\langle S^{-1}(t)S(t)v,v\rangle}{\|X\|} = \langle S(t)v, N_X(t)\rangle \le \|S(t)v\|$$

are discussed. From Section 6.2 we obtain $X(t) = \frac{1}{t-l}{}^tE^{-1}(t)Qv + (I-Q)v$ in a neighborhood of the first conjugate point $t = l$. Hence $z$ is in $t = l$ continuous. Furthermore, the transverse Jacobi field equation in [29] as well as the original techniques for Jacobi field estimates, the index form or the Sturm comparison theorem, are useful for getting bounds under different hypotheses (see e.g. [17], [18, Lemma 5.5], [26, Lemma 1.2]).

In [19, Lemma 2.1] the expression $fJ' - f'J = f(AJ - \frac{f'}{f}J) = fB_fJ = f^2B_fJ_f = f^2J_f'$ is of main interest with the comparison function $f(t) = t$ in the context of bounded curvature $\|R\| \le \kappa$. The curvature tensor appears explicitly in their estimates. They use $\frac{\|J(\tau)\|}{\|J'(0)\|} \le \frac{1}{\sqrt{\kappa}}$ for $0 \le \tau < \frac{1}{2\sqrt{\kappa}}$ and

$$\begin{aligned}\left|1 - \tfrac{\|J'(t)\|}{\|J'(0)\|}\right| \le \tfrac{\|J'(0)-J'(t)\|}{\|J'(0)\|} &= \tfrac{1}{\|J'(0)\|}\left\|-\int_0^t J''(\tau)\,d\tau\right\| \\ &\le \int_0^t \|R(\tau)\|\tfrac{\|J(\tau)\|}{\|J'(0)\|}\,d\tau < 1 \quad \text{for small } t\end{aligned}$$

to get

$$\begin{aligned}\|J(t) - tJ'(t)\| &= \left\|-\int_0^t \tau J''(\tau)\,d\tau\right\| \le \int_0^t \tau\|R(\tau)\|\|J(\tau)\|\,d\tau \\ &\le \|J'(t)\|\int_0^t \tau\|R(\tau)\|\tfrac{\|J(\tau)\|}{\|J'(0)\|}\,d\tau\left(1 - \int_0^t \|R(\tau)\|\tfrac{\|J(\tau)\|}{\|J'(0)\|}\,d\tau\right)^{-1}.\end{aligned}$$

From Theorem 6 we have with (6.4) a bound $\|J(t) - tJ'(t)\| = t\|B(t)J(t)\| \le \frac{\kappa}{3}t^3\|J'(t)\|(1 + O(t))$ or an estimate of order $O(t^2)$ with only $\int_0^t \|R(\tau)\|\,d\tau$ bounded.

### 6.4 Ricci curvature and rigidity

We conclude from Theorem 1 some rigidity properties in the case of a given lower bound on the Ricci curvature along a geodesic. The Riccati comparison for the mean curvature follows directly from (3.15).

$$\begin{aligned}\frac{\operatorname{tr} B_\lambda(t)}{n-1} &= \frac{\operatorname{tr} A(t)}{n-1} - a_\lambda(t) \\ &= -\int_0^t \left(\frac{s_\lambda(\tau)}{s_\lambda(t)}\right)^2\left(\frac{\operatorname{tr} R(\tau)}{n-1} - \lambda(\tau)\right)\,d\tau \\ &\quad - \frac{1}{n-1}\int_0^t \left(\frac{s_\lambda(\tau)}{s_\lambda(t)}\right)^2 \operatorname{tr} B_\lambda^2(\tau)\,d\tau \qquad (6.8)\\ &\le 0 \quad \text{if } \operatorname{tr} R \ge (n-1)\lambda. \qquad (6.9)\end{aligned}$$

Equation (6.9) means that the first conjugate point of $c$ appears before the first conjugate point of the comparison curve $c_\lambda$ (Myers' theorem). (6.8) has another consequence. If $\mathrm{Ric}(c(t)) \geq (n-1)\lambda(t)$ and if the mean curvature of the distance spheres of $c(0)$ along $c$ is identical to those along the comparison curve $c_\lambda$, i.e. $\operatorname{tr} B_\lambda = \operatorname{tr} A - (n-1)a_\lambda = 0$, it follows that $\operatorname{tr}(B_\lambda^2) = 0$. We conclude $A = a_\lambda I$ and so $R = -A' - A^2 = \lambda I$.
There are two well-known situations in Theorem 1 where this happens (cf. [12, Theorem 1 and Theorem 2]). If $\lambda(t) = 0$ and $c$ without conjugate points we have

$$0 \geq \frac{\operatorname{tr} B_\lambda(t)}{n-1} \geq a_\lambda(t, l_2) - a_\lambda(t, -l_1) = \frac{1}{t-l_2} - \frac{1}{t+l_1} \xrightarrow{l_1, l_2 \to \infty} 0.$$

We deduce that, if $\mathrm{Ric}(\dot{c}) \geq 0$ and $c$ is without conjugate points then the curvature tensor along $c$ vanishes, i.e. $R_{\dot{c}} = 0$. This result fits well with Cheeger-Gromoll's splitting theorem.
If $c_\lambda(-l_1)$ and $c_\lambda(l_2)$ are conjugate along the comparison geodesic, Theorem 1 gives in the case of $\mathrm{ric}(\dot{c}) \geq \lambda > 0$, $\lambda \in \mathbb{R}$ and $l_1 + l_2 = \frac{\pi}{\sqrt{\lambda}}$

$$0 \geq \frac{\operatorname{tr} B_\lambda(t)}{n-1} \geq \frac{s_\lambda(-l_1, l_2)}{s_\lambda(t, -l_1) s_\lambda(t, l_2)} = \frac{\operatorname{sn}_\lambda(l_1 + l_2)}{\operatorname{sn}_\lambda(t + l_1) \operatorname{sn}_\lambda(l_2 - t)} = 0.$$

We conclude that if $\mathrm{Ric}(\dot{c}) \geq \lambda > 0$ and $c : [0, \frac{\pi}{\sqrt{\lambda}}] \to M$ is a geodesic segment that has, as with the comparison geodesic $c_\lambda$, the first conjugate point at $\frac{\pi}{\sqrt{\lambda}}$, then the curvature tensor along $c_{[-l_1, l_2]}$ is given by $R_{\dot{c}} = \lambda I$. This result fits well with Cheng's maximal diameter sphere theorem.

## 6.5 Ricci curvature and volume estimates

The volume form $\omega$ along a geodesic in polar coordinates can be expressed in terms of $B_f$. We have

$$\begin{aligned} \frac{\omega(t)}{f^{n-1}(t)} &= \frac{\sqrt{\det({}^t S(t) S(t))}}{f^{n-1}(t)} = \frac{\det S(t)}{f^{n-1}(t)} = \det E_f(t) \\ &= \exp\left(\int_0^t \frac{d}{d\tau}(\ln \det E_f)(\tau)\, d\tau\right) \\ &= \exp\left((n-1)\int_0^t \frac{\operatorname{tr} B_f(\tau)}{n-1}\, d\tau\right). \end{aligned} \tag{6.10}$$

Bishop-Gromov's volume comparison follows directly from (6.9). In the situation (2) of Theorem 1 we obtain a lower bound for $\frac{\operatorname{tr} A}{n-1}$ of type $\mathrm{ct}_\lambda - \delta$ with a smooth non-negative function $\delta : [-l_1, l_2] \to \mathbb{R}$. Put $f(t) = \operatorname{sn}_\lambda(t) \exp\left(-\int_0^t \delta(\tau)\, d\tau\right)$. Then $\frac{\operatorname{tr} B_f}{n-1} \geq 0$ and we get an increasing quotient $\omega / f^{n-1}$. Using Corollary 1 this gives a lower volume estimate for balls.

Theorem 7 (volume comparison, cf. [4]/Chapter 6, [6]/Chapter 5.2). *Let $M^n$ be a complete Riemannian manifold with* $\mathrm{Ric}(M) \geq (n-1)\lambda$, $\lambda \in \mathbb{R}$ *and* $0 < i_0 \leq \mathrm{injRad}(M)$. *Then for* $p \in M$ *the quotient*

$$R(r) = \frac{\mathrm{Vol}(B(p,r) \subset M)}{\mathrm{Vol}(\mathbb{S}^{n-1}) \int_0^r f^{n-1}(\tau)\, d\tau} \text{ is increasing on } [0, i_0] \text{ with } R(0) = 1.$$

*$f$ is defined by*

$$f(t) := \begin{cases} \mathrm{sn}_\lambda(t) \cdot \exp\big( -2\, \mathrm{ct}_\lambda(\tfrac{i_0}{2})\, t \big) & \text{for } 0 \leq t \leq \tfrac{i_0}{2} \\ \mathrm{sn}_\lambda(i_0 - t) \cdot \exp\big( -\mathrm{ct}_\lambda(\tfrac{i_0}{2})\, i_0 \big) & \text{for } \tfrac{i_0}{2} \leq t \leq i_0. \end{cases}$$

This lower bound is not surprising because in $n$-dimensional Riemannian manifolds $M$ with injectivity radius bounded from below there exists a lower bound for the volume of all balls of type $\mathrm{Vol}(B(p,r)) \geq c(n)r^n$, $p \in M$, $0 \leq r \leq \mathrm{injRad}(M)$ and a constant $c(n)$ depending only on $n$ and not on any curvature assumptions (cf. [8]).

## 6.6 Focal radius and convexity

The levels of an oriented distance function to a total geodesic hypersurface orthogonal to a geodesic $c$ form another hypersurface family $H_t$ along $c$. The focal radius used in Theorem 5 to get estimates for the principal curvature of distance spheres normally ensures the existence of $H_t$ in this context. In this section we outline how the techniques developed in this article can be applied to $H_t$. We also discuss the relations between the Weingarten map of $H_t$ and that of distance spheres (cf. [6, paragraphs 2.3, 3.3, 6.4]).

So let $c : [-f_1, f_2] \to M$ be a normal geodesic and $H_0$ the total geodesic hypersurface through $c(0)$ orthogonal to $\dot{c}(0)$, i.e. $H_0 := \exp_{c(0)}\{w \in T_{c(0)}M | \langle w, \dot{c}(0)\rangle = 0, \|w\| < \epsilon\}$ with $\epsilon > 0$ small enough. Let $N$ be the normal vector field along $H_0$ with $N_{|c(0)} = \dot{c}(0)$. We suppose that there are no focal points of $c(0)$ along $c$ and along the inverse geodesic $c^-$. Therefore there exists an $\epsilon > 0$ such that $\mathrm{Exp} : H_0 \times [-f_1, f_2] \to M$ defined by $\mathrm{Exp}(p, \delta) := \exp_p(\delta N_p)$ is a local diffeomorphism and defines an oriented distance function $\delta$ in a neighborhood of $c$. The levels of $\delta$ define a family of hypersurfaces $H_t := \{\exp(tN_p) \mid p \in H_0\} = \{\delta \equiv t\}$. The Weingarten map $H$ of these levels along $c$ are described by the Jacobi fields along $c$ with $J'(0) = 0$ and $\langle J(0), \dot{c}(0)\rangle = 0$ by the equation $J'(t) = H_{c(t)}J(t)$. By assumption, these Jacobi fields form a basis along $c$ on $[-f_1, f_2]$. We start listing properties of the shape operator $H$. Using the notation of Section 3 we introduce $H := C'C^{-1}$, $h_\lambda := \frac{c'_\lambda}{c_\lambda}$ and $H_\lambda := H - h_\lambda I$. $h_\lambda = -\lambda \frac{\mathrm{sn}_\lambda}{\mathrm{cs}_\lambda}$ in the model space $M^n_\lambda$ of constant curvature $\lambda$. We have

$$H_\lambda(t) = -\int_0^t \frac{c_\lambda^2(\tau)}{c_\lambda^2(t)} \left(R(\tau) - \lambda(\tau)I\right)\, d\tau - \int_0^t \frac{c_\lambda^2(\tau)}{c_\lambda^2(t)} H_\lambda^2(\tau)\, d\tau. \tag{6.11}$$

The Taylor series of $H$ is given by

$$H(t) = -R(0)t - \tfrac{1}{2}R'(0)t^2 + O(t^3) = -\int_0^t R(\tau)\,d\tau - t^3 W(t)$$

with a positive semi-definite $W$ and $W(0) = \frac{1}{3}R^2(0)$.

Table 3: Overview of comparison results for $H_f = H - \frac{f'}{f}I$, $f(0) = 1$, $h_f(t) = \int_0^t \|H_f(\tau)\|\,d\tau$

| CONDITIONS | ESTIMATES AND COMPARISON FUNCTIONS |
|---|---|
| $R \le \Lambda$ on $[0, f_2]$ | $H_f \ge 0$ with $f = c_\Lambda$ |
| $R \ge \lambda$ on $[0, f_2]$ | $H_f \le 0$ with $f = c_\lambda$ |
| $\lambda \le R \le \Lambda$ on $[0, f_2]$ | $\|H_f\| \le \frac{1}{2}(h_\lambda - h_\Lambda) \in O(t)$,<br>$h_f(t) \le \frac{1}{2}\ln\left(\frac{c_\lambda(t)}{c_\Lambda(t)}\right) \in O(t^2)$<br>with $f = \sqrt{c_\lambda c_\Lambda}$ as $t \to 0$ |
| $\operatorname{tr} R \ge (n-1)\lambda$ on $[0, f_2]$ | $\operatorname{tr} H_f \le 0$ with $f = c_\lambda$ |
| $R \ge \lambda$ on $[0, f_2]$ and $c(0)$ without focal point in $[0, f_2]$ | $H_f \ge 0$ with $f(t) = \frac{s_\lambda(t,f_2)}{s_\lambda(0,f_2)}$ |
| $\operatorname{tr} R \ge (n-1)\lambda$ on $[0, f_2]$ and $c(0)$ without focal point in $[0, f_2]$ | $\operatorname{tr} H_f \ge 0$ with $f = \frac{s_\lambda(t,f_2)}{s_\lambda(0,f_2)}$<br>$h_f(t) \le \sqrt{n-1}\sqrt{\frac{c_\lambda(f_2)}{s_\lambda(f_2,t)}}\sqrt{s_\lambda(t)}$<br>with $f = c_\lambda$ |
| $\operatorname{tr} R \ge (n-1)\lambda$ on $[0, f_2]$ and without any focal points in $c_{\mid[0,f_2]}$ | $H_f \ge 0$ with $f(t) = \left(\frac{s_\lambda(t,f_2)}{s_\lambda(0,f_2)}\right)^{n-1}$<br>$H_f \le 0$ with $f(t) = \left(\frac{s_\lambda(0,f_2)c_\lambda(t)}{s_\lambda(t,f_2)}\right)^{n-1}$ |
| $\left\|\int_0^t R(\tau)\,d\tau\right\| \le \Lambda(t)$, $\Lambda' \ge 0$, if $\Lambda(0) = 0$ suppose $\Lambda'(0) > 0$ | $\|H\| \le \frac{\Lambda(t)}{1-t\Lambda(t)}$ for all $t \in [0, t_0)$ with the unique $t_0 > 0$ such that $t_0\Lambda(t_0) = 1$ |

For the proof of the Riccati comparison we suppose that $\tilde{R}$ is a curve in $\operatorname{Sym}(n-1, \mathbb{R})$ and that $\tilde{C}$ is the solution of $\tilde{C}'' + \tilde{R}\tilde{C} = 0$ with $\tilde{C}(0) = I$ and $\tilde{C}'(0) = 0$. Put $\tilde{H} = \tilde{C}'\tilde{C}^{-1}$. Let $U$ be the solution of $U' = \frac{1}{2}(H + \tilde{H}) \cdot U$ with $U(0) = I$ and $V$ the solution of $V' = -V \cdot \frac{1}{2}(H + \tilde{H})$ with $V(0) = I$. Then $U$ and $V$ are the inverse of each other because $(VU)' = V'U + VU' = -V\frac{1}{2}(H + \tilde{H})U + V\frac{1}{2}(H + \tilde{H})U = 0$. From $({}^tU(H - \tilde{H})U)' = -{}^tU(R - \tilde{R})U$ we get

$$H(t) - \tilde{H}(t) = -\int_0^t {}^tX(\tau, t)(R(\tau) - \tilde{R}(\tau))X(\tau, t)\,d\tau$$

with $X(\tau, t) = U(\tau)U^{-1}(t) = U(\tau)V(t)$. This integral equation expresses the Riccati comparison (cf. [11, Theorem 1] ). If $R \le \Lambda$ this gives $H \ge h_\Lambda$ and the focal point comes later than in the model space. If $R \geq \lambda$ we have $H \leq h_\lambda$. Taking the trace in (6.11) we get $\operatorname{tr} H \leq (n-1)h_\lambda$ if $\operatorname{tr} R \geq (n-1)\lambda$. It follows under both lower curvature conditions that the focal point comes earlier than in the model space. These are the first four well-known estimates for $H$ in Table 3.

The excess functions $e_{-f_1}(p) = d_{c(-f_1)}(p) - \delta(p) - f_1 \geq 0$ and $e_{f_2}(p) = \delta(p) + d_{c(f_2)}(p) - f_2 \geq 0$ are convex. It follows for $t \in (-f_1, f_2)$

$$A_{-f_1|c(t)} \geq H_{c(t)} \geq -A_{f_2|c(t)}. \tag{6.12}$$

This means that bounds for $H$ can be converted to bounds for $A$ and vice versa. In the model space of constant positive curvature $\lambda$ these inequalities are sharp. Here we can choose $f_1 = f_2 = \frac{\pi}{2\sqrt{\lambda}}$ and the distance spheres of the pole and those of the great circle match. In the Euclidean space we also have sharpness $(f_1, f_2 \to \infty)$ because the horospheres are flat hypersurfaces. In the hyperbolic case, $H_t$ is only bounded by the horospheres. From (6.12) we can deduce some inequalities for $H$ using estimates for $A(t, f_2)$ on $(-f_1, f_2)$. We get (see Table 3, row 5) with $H(t) - A(t, f_2) \geq 0$ on $[-f_1, f_2)$ from the Riccati comparison the lower bound

$$H(t) \geq A(t, f_2) \geq a_\lambda(t, f_2) \qquad \text{if } R \geq \lambda$$

and (see Table 3, row 6)

$$\operatorname{tr} H(t) \geq \operatorname{tr} A(t, f_2) \geq (n-1)a_\lambda(t, f_2) \qquad \text{if } \operatorname{tr} R \geq (n-1)\lambda.$$

Similar examples as in Section 7 (couple $\Lambda_n \delta_n = h_\lambda(t) - a_\lambda(t, f_2) > 0$ or $\Lambda_n^2 \delta_n^3 = \frac{3}{2}(h_\lambda(t) - a_\lambda(t, f_2)) > 0$) show that these two inequalities are sharp in this context. In the case of a lower Ricci curvature assumption one cannot expect bounded principal curvature but integral bounds for $H$ as for $B$ in Theorem 4. Here the Cauchy-Schwarz inequality and the Riccati equation for $H$ give for $0 \leq \tau \leq t < f_2$ (see Table 3, row 6)

$$\begin{aligned}
\int_\tau^t \|H_\lambda(x)\|\, dx &\leq \left(\int_\tau^t \tfrac{1}{c_\lambda^2(x)}\, dx\right)^{1/2} \left(\int_\tau^t c_\lambda^2(x) \operatorname{tr} H_\lambda^2(x)\, dx\right)^{1/2} \\
&= \sqrt{\tfrac{s_\lambda(t,\tau)}{c_\lambda(t) c_\lambda(\tau)}} \Bigg( c_\lambda^2(\tau) \operatorname{tr} H_\lambda(\tau) - c_\lambda^2(t) \operatorname{tr} H_\lambda(t) \\
&\qquad\qquad - \int_\tau^t c_\lambda^2(x)(\operatorname{tr} R(x) - (n-1)\lambda(x))\, dx \Bigg)^{1/2} \\
&\leq \left(\tfrac{c_\lambda(t)}{c_\lambda(\tau)}\right)^{1/2} \sqrt{-\operatorname{tr} H_\lambda(t)} \sqrt{s_\lambda(t,\tau)} \\
&\leq \left(\tfrac{c_\lambda(t)}{c_\lambda(\tau)}\right)^{1/2} \sqrt{n-1} \sqrt{h_\lambda(t) - a_\lambda(t, f_2)} \sqrt{s_\lambda(t,\tau)} \\
&= \sqrt{n-1} \sqrt{\tfrac{c_\lambda(f_2)}{c_\lambda(\tau) s_\lambda(f_2, t)}} \sqrt{s_\lambda(t,\tau)}.
\end{aligned} \tag{6.13}$$

(6.13) is used in Jacobi field estimates. As in Section 6.4 this inequality also implies two rigidity cases. If $\mathrm{Ric}(\dot{c}) \geq (n-1)\lambda > 0$, $\lambda \in \mathbb{R}$ and the focal point of $c$ comes at the point $f_2 = \frac{\pi}{2\sqrt{\lambda}}$, then the curvature tensor along $c_{[0,f_2]}$ is given by $R = \lambda I$. If $\mathrm{Ric}(\dot{c}) \geq 0$ and $c$ is a ray without a focal point, we have $R_{[0,\infty)} = 0$ (cf. [12, Lemma 6]).

If for every $\alpha \in [0, f_2]$ no geodesic segment $c_{[\alpha,f_2]}$ has a focal point, then by Lemma 2 we have $A(t, f_2) < 0$ on $[0, f_2)$. With a lower Ricci curvature bound this gives for the principal curvature of the parallel hypersurface family $H_t$ (see Table 3, row 7)

$$(n-1)a_\lambda(t, f_2) \leq H(t) \leq (n-1)(h_\lambda(t) - a_\lambda(t, f_2)) \qquad \text{on } [0, f_2) \tag{6.14}$$

because

$$0 < H(t) - A(t, f_2) \leq \operatorname{tr} H(t) - \operatorname{tr} A(t, f_2) \leq (n-1)(h_\lambda(t) - a_\lambda(t, f_2))$$

and

$$(n-1)a_\lambda(t, f_2) \leq \operatorname{tr} A(t, f_2) \leq A(t, f_2) < 0.$$

Under the assumptions of Theorem 5 the left side of (6.14) can be improved by comparing $H$ with $A(t, t - f_0)$. More precisely, we have $-(n-1)\operatorname{ct}_\lambda(f_0 - t) \leq A(t, f_0) \leq H(t) \leq A(t, t - f_0) \leq (n-1)\operatorname{ct}_\lambda(f_0)$. This means that it is rather normal to control the principal curvature and not only the mean curvature in case of a lower focal radius and Ricci curvature bound. With (6.11) one gets an analogous statement like in Theorem 6 with the integral curvature tensor $\int_0^t R(\tau)\, d\tau$ (see Table 3, row 8). For an orthonormal basis $(X_i)_{2\leq i\leq n}$ of parallel vector fields along $c$ and orthogonal to $\dot{c}$, the index form $I_t$ of the segement $c_{[0,t]}$ is given by $-\int_0^t r_{i,j}(\tau)\, d\tau$. This means that for a variation $v(\tau, s) := \exp(sX_i(\tau))$ the length function $l(s) = \int_0^t \|\dot{v}(\tau, s)\|\, d\tau$ not only has a minimum in $s = 0$, i.e. $l'(0) = 0$, but $l''(0) = I(X_i, X_i)$ is bounded under this integral curvature condition. From a geometric point of view it is not surprising that $H$ is bounded in this case. From (6.12) we obtain, with Theorem 6, that bounds for the integral curvature tensor $\int_0^t R(\tau)\, d\tau$ are equivalent to bounds for the integral curvature tensor $\int_0^t (\frac{\tau}{t})^2 R(\tau)\, d\tau$.

We now apply bounds for $H$ to estimate Jacobi fields. On the one hand, Jacobi fields $J \neq 0$ orthogonal to $\dot{c}$ with $J'(0) = 0$ describe $H$ by the equation $J' = HJ$. On the other hand, $J$ can be expressed in terms of $H$ by the relation

$$\begin{aligned} \|J(t)\| &= \|C(t)J(0)\| \\ &= \|J(0)\| \exp\left(\tfrac{1}{2} \ln \|C(t)\tfrac{J(0)}{\|J(0)\|}\|^2\right) \\ &= \|J(0)\| \exp\left(\int_0^t \langle H(\tau)\tfrac{J(\tau)}{\|J(\tau)\|}, \tfrac{J(\tau)}{\|J(\tau)\|}\rangle\, d\tau\right) \\ &= \|J(t_0)\| \exp\left(-\int_t^{t_0} \langle H(\tau)\tfrac{J(\tau)}{\|J(\tau)\|}, \tfrac{J(\tau)}{\|J(\tau)\|}\rangle\, d\tau\right). \end{aligned} \tag{6.15}$$

Rauch's second comparison theorem is a consequence of (6.15) and the Riccati comparison for $H$. Equivalent estimates for these Jacobi fields like in Section 6.3 are possible. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function with $f(0)=1$ and positive as long as $J\neq 0$. Put $H_f = H - \frac{f'}{f}I$ and $J_f=\frac{J}{f}$. Then $\|J_f\|$ is decreasing if $H_f\leq 0$ and increasing if $H_f\geq 0$.

Define the integral barrier function of $H_f$ by $h_f(t_0,t_1) := \int_{t_0}^{t_1}\|H_f(\tau)\|\,d\tau$ for $0\leq t_0\leq t_1< f_2$ and set $h_f(t):=h_f(0,t)$. We have for $0\leq t_0\leq t_1<f_2$

$$\|J_f(t_0)\|\cdot\exp\left(-h_f(t_1,t_0)\right)\leq \|J_f(t_1)\|\leq \|J_f(t_0)\|\cdot\exp\left(h_f(t_1,t_0)\right). \tag{6.16}$$

The comparison of a Jacobi field with the associated affine parallel vector field gives

$$\|J(t)-f(t)J(0)\|\leq f(t)\cdot\|J(0)\|\cdot\left(\exp(h_f(t))-1\right).$$

For the angular velocity of this Jacobi field we have

$$\left\|\left(\frac{J}{\|J\|}\right)'\right\|\leq\|H_f\|.$$

and for the angle between the Jacobi field and the corresponding parallel vector field $X$ with $X(0)=J(0)$

$$2\sin\left(\frac{1}{2}\angle(J(t),J(0))\right)\leq h_f(t)$$

for all $f$ as above.

We continue regarding the shape operator $H$ of other points $c(s)$ along $c$. We define $H(t,s) := \frac{\partial}{\partial t}C(t,s)C^{-1}(t,s)$ for all $(t,s)$ where $C(t,s)$ is invertible. Using the symmetry of $H$ and $\frac{\partial C}{\partial s}(t,s) = -C(t)^t S(s)R(s)+S(t)^t C(s)R(s) = S(t,s)R(s)$ we have

$$\begin{aligned}\frac{\partial H}{\partial s}(t,s) &= \frac{\partial^2 C}{\partial s\partial t}(t,s)\,C^{-1}(t,s) - \frac{\partial C}{\partial t}(t,s)\,C^{-1}\frac{\partial C}{\partial s}(t,s)\,C^{-1}(t,s)\\ &= {}^tC^{-1}(t,s)\left({}^t\,C(t,s)\frac{\partial^2 C}{\partial s\partial t}(t,s) - \frac{\partial^t C}{\partial t}(t,s)\frac{\partial C}{\partial s}(t,s)\right)C^{-1}(t,s)\\ &= {}^tC^{-1}(t,s)\left({}^tC(t,s)\frac{\partial S}{\partial t}(t,s)\,R(s) - \frac{\partial^t C}{\partial t}(t,s)S(t,s)\,R(s)\right)C^{-1}(t,s)\\ &= {}^tC^{-1}(t,s)R(s)C^{-1}(t,s).\end{aligned} \tag{6.17}$$

It is not surprising that $s\mapsto H(t,s)$ is not increasing in general as for distance spheres (see (3.8)). But for non-negative sectional curvature $K\geq 0$ this result can be interpreted geometrically as in Section 2.1. Indeed, Rauch's second comparison theorem shows that $\delta(\exp_{c(t)}(w))\leq t$ where $w\perp\dot c(t)$ and small enough. This means $H(t)\leq 0$. Let $\delta_{-f_1}$ be the oriented distance function of the total geodesic hypersurface orthogonal to $\dot c(-f_1)$. Then again with Rauch's second comparison theorem we have $\delta_{-f_1}\leq f_1+\delta$, which means $H(t,-f_1)\leq H(t)$.

Another application of the results of Table 3 are estimates for the eigenvalues of the Hessian of the excess functions $e_{-f_1}$ and $e_{f_2}$. As $C$ is invertible on $[-f_1, f_2]$, one can express $S(t,f)$, $-f_1 < f < f_2$, in terms of $C$ by

$$S(t,f) = -C(t) \int_t^f C^{-1}(\tau){}^tC^{-1}(\tau)\, d\tau \; {}^tC(f).$$

This gives, with $D(\tau,t) = {}^tC^{-1}(\tau)\, {}^tC(t)$ for $t \in (-f_1, f_2)$ like in Lemma 1

$$H(t) - A(t,f) = \left( \int_t^f {}^tD(\tau,t) D(\tau,t)\, d\tau \right)^{-1} > 0$$

Since we have

$$\|D(\tau,t)v\|^2 = \|v\|^2 \exp\left( -2 \int_t^\tau \langle H(x) \tfrac{D(x,t)v}{\|D(x,t)v\|}, \tfrac{D(x,t)v}{\|D(x,t)v\|} \rangle \, dx \right)$$

for $v \in \mathbb{R}^{n-1}$, estimates follow from Table 3. For example, if $R \geq \lambda$ on $[-f_1, f_2]$ then for $t \in (-f_1, f_2)$ we have $H(t) > A(t, f_2) \geq a_\lambda(t, f_2)$ and therefore for the Hessian of the excess function $H(t) - A(t,f) \geq a_\lambda(t, f_2) - a_\lambda(t,f) > 0$ for $0 \leq t < f < f_2$.

We finish with some remarks about the interaction of $H$ and $A$. As long as $A$ is convex, the duality of $A$ and $H$ is given by the formulas (see [6, Lemma 6.4])

$$\begin{aligned} A^{-1}(t) &= \int_0^t C^{-1}(x,t){}^tC^{-1}(x,t)\; dx \\ &= -\int_0^t \tfrac{\partial}{\partial s} A^{-1}(t,s)\; ds = A^{-1}(t,0) - A^{-1}(t,t) \end{aligned}$$

and

$$\begin{aligned} \langle A^{-1}(t)v, v\rangle &= \int_0^t \exp\left( 2\int_\tau^t \langle H(x,t) \tfrac{{}^tC^{-1}(x,t)v}{\|{}^tC^{-1}(x,t)v\|}, \tfrac{{}^tC^{-1}(x,t)v}{\|{}^tC^{-1}(x,t)v\|} \rangle \; dx \right)\; d\tau \\ &= \int_0^t \tfrac{1}{c_\lambda^2(\tau,t)} \exp\left( 2\int_\tau^t \langle H_\lambda(x,t) \tfrac{{}^tC^{-1}(x,t)v}{\|{}^tC^{-1}(x,t)v\|}, \tfrac{{}^tC^{-1}(x,t)v}{\|{}^tC^{-1}(x,t)v\|} \rangle \; dx \right)\; d\tau \end{aligned}$$

with $v \in \mathbb{R}^{n-1}$, $\|v\| = 1$. This can be seen as in the proof of Lemma 1. Note that $A(t) = A(t) - H(t,t)$. Thus $L^1$-barriers for $H$ already give bounds for the principal curvature of distance spheres

$$0 < a_\lambda(t) \cdot e^{-2\int_0^t \|H_\lambda(x,t)\|\; dx} \leq A(t) \leq a_\lambda(t) \cdot e^{2\int_0^t \|H_\lambda(x,t)\|\; dx}. \tag{6.18}$$

With the estimate of type $O(\sqrt{t})$ as $t \to 0$ from (6.13), these inequalities (6.18) do not describe the pole of $A$ in $t = 0$ but are better than the elementary estimates in (1.17). Nevertheless, we again obtain bounded comparison tensor fields necessary for the proof of Theorem 5. Then (6.12) also implies bounded principal curvature for $H$. Applying these bounds again in (6.18) it follows that we have

$$\begin{aligned}\|A(t)-a_\lambda(t)I\| &\le 2a_\lambda(t)\int_0^t \|H_\lambda(x,t)\|\,dx\cdot\exp\left(2\int_0^t\|H_\lambda(x,t)\|\,dx\right)\\ &= 2ta_\lambda(t)\int_0^1 \|H_\lambda(xt,t)\|\,dx\cdot\exp\left(2\,t\int_0^1\|H_\lambda(xt,t)\|\,dx\right)\\ &\in O(1)\quad \text{as } t\to 0\end{aligned}$$

as long as A is strictly convex. This is the idea of the proof of Theorem 5 given in [6].

Finally, we want to mention that the upper bound in (6.16) was also developed for $\lambda = 0$ in [14] and was used there to get a local splitting theorem.

## 7 Sharpness of the results

The following three examples show that the results presented in Section 1 are optimal. The first two analytic examples show also the potentials and the limitations of the Riccati equation and motivate a geometric example in Section 7.3.

Assume that $l_1, l_2 \in \mathbb{R}$ are positive and that $\lambda : [-l_1, l_2] \to \mathbb{R}$ is a continuous function. We use the same notation as in Section 3 for the comparison functions. The motivation for these examples is the following interaction: on the one hand, high curvature will decrease the principal curvature of distance spheres and on the other hand, this will lead earlier to conjugate or focal points. It is like linear optics. For the first example one adds a biconvex lens at a point on the ray and for the second example one puts in addition a biconcave lens behind it.

### 7.1 One-dimensional analytic example of the Riccati equation

We start with the case of a lower curvature bound in a two-dimensional Riemannian manifold and treat the lower bound on the Ricci curvature separately in Section 7.2. We have thus only a one-dimensional Jacobi or Riccati equation. Choose $t \in (0, l_2)$. Let $\delta_n > 0$ be strictly monotonically decreasing and $\Lambda_n$ a strictly monotonically increasing sequence with $\lim_{n\to\infty} \delta_n = 0$ and $\lim_{n\to\infty} \Lambda_n = \infty$. We define a sequence of curvature tensors $r_n : [-l_1, l_2] \to \mathbb{R}$ by

$$r_n(\tau) := \begin{cases}\Lambda_n & \text{for } \tau\in[t-\delta_n, t],\\ \lambda(t) & \text{else}\end{cases}$$

and assume that $r_n \ge \lambda$. The idea is to increase the curvature only in a small interval before $t$. For clarity reasons we define $r_n$ only piecewise continuously. It is clear that a smooth example which has approximately the same properties can also be constructed.

The geodesic segment with the disturbed curvature $r_n$ should be without conjugate points. This condition can be translated by $a_\lambda(t - \delta_n, -l_1) = a_n(t-\delta_n, -l_1) > a_n(t-\delta_n, t) = -\operatorname{ct}_{\Lambda_n}(\delta_n)$ and $a_n(t, -l_1) \geq a_n(t, l_2) = a_\lambda(t, l_2)$. This follows from the Riccati comparison, which ensures that $a_n(\tau, -l_1) > a_n(\tau, t) = -\operatorname{ct}_{\Lambda_n}(t-\tau)$ on $[t-\delta_n, t]$ and $a_n(\tau, -l_1) \geq a_\lambda(\tau, l_2)$ on $[t, l_2]$. So there will be no singularity of $a_n(t, -l_1)$ in $(-l_1, l_2)$. If we couple $\delta_n$ and $\Lambda_n$ in the way that $\delta_n \Lambda_n = a_\lambda(t, -l_1) - a_\lambda(t, l_2) > 0$, Lemma 3 implies

$$\begin{aligned} a_n(t, -l_1) &= \frac{a_\lambda(t-\delta_n, -l_1) - \Lambda_n \delta_n \operatorname{tg}(\Lambda_n \delta_n^2)}{1 + a_\lambda(t-\delta_n, -l_1)\delta_n \operatorname{tg}(\Lambda_n \delta_n^2)} \\ &\xrightarrow[n\to\infty]{} a_\lambda(t, -l_1) - (a_\lambda(t, -l_1) - a_\lambda(t, l_2)) = a_\lambda(t, l_2). \end{aligned}$$

This and $\lim_{n\to\infty}(-\operatorname{ct}_{\Lambda_n}(\delta_n)) = -\infty$ mean that asymptotically this condition is fulfilled. We conclude that a geodesic segment without conjugate points and a lower curvature bound does not allow us to get an upper curvature bound. Note that $-l_1$ is conjugated to $l_2$ along the geodesic with modified curvature $r_n$ if and only if $a_n(t, -l_1) = a_n(t, l_2) = a_\lambda(t, l_2)$, which is approximately the case. Furthermore, we have

$$a_n(t) = \frac{a_\lambda(t-\delta_n) - \Lambda_n \delta_n \operatorname{tg}(\Lambda_n \delta_n^2)}{1 + a_\lambda(t-\delta_n)\delta_n \operatorname{tg}(\Lambda_n \delta_n^2)} \xrightarrow[n\to\infty]{} a_\lambda(t) - (a_\lambda(t, -l_1) - a_\lambda(t, l_2)).$$

This means that the estimate in Theorem 1 is sharp.
Along the inverse geodesic the point $t$ has a focal point at $f_n < t$ if $a_n(t, f_n) = 0$. This is equivalent to $a_\lambda(t-\delta_n, f_n) = \Lambda_n \operatorname{tg}_{\Lambda_n}(\delta_n) = \Lambda_n \delta_n \operatorname{tg}(\Lambda_n \delta_n^2)$. So $a_\lambda(t, f_n) \approx a_\lambda(t, -l_1) - a_\lambda(t, l_2)$. If there are conjugate points, a unique $f_n$ exists because $s \to a_\lambda(t, s)$ is increasing. $t - f_n$ is then the distance to the focal point. This confirms that a lower conjugate radius bound in combination with a lower sectional curvature bound also implies a lower focal radius bound (see Corollary 1) and that (1.21) is sharp.
We now consider the case of a convex background field. Supposing that $a_\lambda(t, -l_1) > 0$ on $(-l_1, t]$, we also have convexity for the distance spheres for the geodesic with modified curvature if $a_n(t-\delta_n, -l_1) = a_\lambda(t-\delta_n, -l_1) \geq a_n(t-\delta_n, t) = \Lambda_n \operatorname{tg}_\Lambda(\delta_n) > 0$. To fulfill this condition we couple $\delta_n$ and $\Lambda_n$ such that $\delta_n \Lambda_n = a_\lambda(t, -l_1) > 0$. This gives $a_n(t) \xrightarrow[n\to\infty]{} a_\lambda(t) - a_\lambda(t, -l_1)$ and $a_n(t, -l_1) \xrightarrow[n\to\infty]{} 0$. This shows that the comparison with a convex background field in (4.6) is also optimal.

Lemma 3. *Let $\kappa \in \mathbb{R}$. Then $\operatorname{tg}_\kappa(t) = \frac{\operatorname{sn}_\kappa(t)}{\operatorname{cs}_\kappa(t)} = t \operatorname{tg}(\kappa t^2)$ with a meromorphic function $\operatorname{tg}: \mathbb{C} \setminus \{\pi^2(\frac{1}{2}+k)^2, k \in \mathbb{N}\} \to \mathbb{C}$ that has for $|z| < \frac{\pi^2}{4}$ the power series*

$$\operatorname{tg}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} z^{n-1} = 1 + \frac{z}{3} + \frac{2}{15} z^2 + \dots$$

*$B_n$ are the Bernoulli numbers.*

*The function* $a_\kappa = \frac{-\kappa\,\mathrm{sn}_\kappa + a_0\,\mathrm{cs}_\kappa}{\mathrm{cs}_\kappa + a_0\,\mathrm{sn}_\kappa}$ *is a solution of the Riccati equation* $a' + a^2 + \kappa = 0$ *with* $a_\kappa(0) = a_0$. *For* $|\kappa| t^2 < \frac{\pi^2}{4}$ *we have*

$$a_\kappa(t) = \frac{a_0 - \kappa\,\mathrm{tg}_\kappa(t)}{1 + a_0\,\mathrm{tg}_\kappa(t)} = \frac{a_0 - \kappa\, t\,\mathrm{tg}(\kappa t^2)}{1 + a_0\, t\,\mathrm{tg}(\kappa t^2)} = \frac{a_0 - \kappa\, t\, - \frac{\kappa^2 t^3}{3} - \frac{2}{15}\kappa^3 t^5\, g(\kappa\, t^2)}{1 + a_0\, t\,\mathrm{tg}(\kappa t^2)}$$

*with a function* $g$ *defined by the relation* $\mathrm{tg}(z) = 1 + \frac{z}{3} + \frac{2}{15} z^2 g(z)$, $g(0) = 1$. *For* $\lambda \in \mathbb{R}$ *the solution* $a_\lambda$ *of* $a' + a^2 + \lambda = 0$ *with* $a_\lambda(\delta) = a_\kappa(\delta)$, $\delta > 0$, $|\kappa|\delta^2 < \frac{\pi^2}{4}$ *is given by*

$$\begin{aligned} a_\lambda(\delta + \tau) &= \frac{a_0 - \kappa\,\mathrm{tg}_\kappa(\delta) - \lambda\,\mathrm{tg}_\lambda(\tau) - a_0 \lambda\,\mathrm{tg}_\lambda(\tau)\,\mathrm{tg}_\kappa(\delta)}{1 + a_0(\mathrm{tg}_\kappa(\delta) + \mathrm{tg}_\lambda(\tau)) - \kappa\,\mathrm{tg}_\kappa(\delta)\,\mathrm{tg}_\lambda(\tau)} \\ &= \frac{a_0 - \kappa\delta - \lambda\tau - \frac{1}{3}\kappa^2\delta^3 - \frac{1}{3}\lambda^2\tau^3}{1 + a_0(\delta\,\mathrm{tg}(\kappa\delta^2) + \tau\,\mathrm{tg}(\lambda\tau^2)) - \kappa\delta\tau\,\mathrm{tg}(\kappa\delta^2)\,\mathrm{tg}(\lambda\tau^2)} \\ &\quad + \frac{-\frac{2}{15}\kappa^3\delta^5 g(\kappa\delta^2) - \frac{2}{15}\lambda^3\tau^5 g(\lambda\tau^2) - a_0\lambda\tau\delta\,\mathrm{tg}(\lambda\tau^2)\,\mathrm{tg}(\kappa\delta^2)}{1 + a_0(\delta\,\mathrm{tg}(\kappa\delta^2) + \tau\,\mathrm{tg}(\lambda\tau^2)) - \kappa\delta\tau\,\mathrm{tg}(\kappa\delta^2)\,\mathrm{tg}(\lambda\tau^2)} \end{aligned}$$

*with* $\tau \geq 0$, $|\lambda|\tau^2 < \frac{\pi^2}{4}$.

*Proof.* The proof is straightforward and the power series follows from the Taylor series of the tangent and hyperbolic tangent functions. It is $\tan(z) = -i\tanh(iz) = z\,\mathrm{tg}(z^2)$ and $h_\kappa(t) = \frac{\mathrm{cs}'_\kappa(t)}{\mathrm{cs}_\kappa(t)} = -\kappa\frac{\mathrm{sn}_\kappa(t)}{\mathrm{cs}_\kappa(t)} = -\kappa t\,\mathrm{tg}(\kappa t^2)$. □

### 7.2 Two-dimensional analytic example of the Riccati equation

For the example with a lower Ricci curvature bound, we consider only three-dimensional Riemannian manifolds. This leads to a two-dimensional Jacobi or Riccati equation along a geodesic. Let $\lambda_n$, $\Lambda_n$ and $\delta_n$ strictly monotone sequences with $\lim_{n\to\infty} \lambda_n = -\infty$, $\lim_{n\to\infty} \Lambda_n = \infty$ and $\lim_{n\to\infty} \delta_n = 0$. For fixed $t \in (0, l_2)$ we define

$$R_n(\tau) = \begin{cases} \begin{pmatrix} \Lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix} & \text{for } \tau \in [t - 2\delta_n, t - \delta_n] \\ \begin{pmatrix} \lambda_n & 0 \\ 0 & \Lambda_n \end{pmatrix} & \text{for } \tau \in (t - \delta_n, t] \\ \begin{pmatrix} \lambda(\tau) & 0 \\ 0 & \lambda(\tau) \end{pmatrix} & \text{else.} \end{cases}$$

The objective is to construct a sequence of diagonal piecewise continuous curvature tensors $R_n \in \mathrm{Sym}(2, \mathbb{R})$ with $\mathrm{tr}\, R_n \geq 2\lambda$. Therefore the sequences should be taken such that $\Lambda_n + \lambda_n \geq 2\lambda(\tau)$ for all $\tau \in [t - 2\delta_n, t]$. Furthermore, $\Lambda_n + \lambda_n$ should be bounded from above. For example, we can couple $\Lambda_n + \lambda_n = 2 \cdot \max\{\lambda(\tau) | \tau \in [t - 2\delta_0, t]\}$. It is important that the two intervals

$[t-2\delta_n, t-\delta_n]$ and $[t-\delta_n, t]$ are of the same size. So in one direction, we first increase on half of an interval before $t$ the curvature to infinity and then on the second half to minus infinity. In the other direction we do it the other way around. The idea is that the negative curvature compensates the positive curvature to avoid conjugate points. In contrast to the one-dimensional example this will lead to unbounded principal curvature.

To guarantee that there are not any conjugate points on $[-l_1, l_2]$ we need the following requirements: $a_\lambda(t-2\delta_n, -l_1)I = A(t-2\delta_n, -l_1) = A_n(t-2\delta_n, -l_1) > A_n(t-2\delta_n, t-\delta_n)$, $A_n(t-\delta_n, -l_1) > A_n(t-\delta_n, t)$ and $A_n(t, -l_1) \geq A_n(t, l_2) = A(t, l_2) = a_\lambda(t, l_2)I$.

Coupling $\Lambda_n^2\delta_n^3 = \frac{3}{2}(a_\lambda(t, -l_1) - a_\lambda(t, l_2)) > 0$ will give the required properties. We see that along a geodesic, a lower bound on the Ricci curvature and on the conjugate radius does not imply bounded sectional curvature. Realize that $\lim_{n\to\infty} \lambda_n^2\delta_n^3 = \frac{3}{2}(a_\lambda(t, -l_1) - a_\lambda(t, l_2))$, $\lim_{n\to\infty} \Lambda_n\delta_n^2 = \lim_{n\to\infty} \lambda_n\delta_n^2 = \lim_{n\to\infty} \Lambda_n^3\delta_n^5 = \lim_{n\to\infty} \lambda_n^3\delta_n^5 = 0$, $\lim_{n\to\infty}(\Lambda_n + \lambda_n)\delta_n = 0$, $\lim_{n\to\infty} \Lambda_n\delta_n = \infty$ and $\lim_{n\to\infty} \lambda_n\delta_n = -\infty$. Using Lemma 3 we find $\lim_{n\to\infty} A_n(t, -l_1) = a_\lambda(t, l_2)I$ and $\lim_{n\to\infty} A_n(t) = a_\lambda(t)I - (a_\lambda(t, -l_1) - a_\lambda(t, l_2))I$. Therefore the lower bound in Theorem 1 is optimal.

This example shows additionally that we cannot expect to control the principal curvature in this case. In fact, we have for the non-zero components $a_n^{00}$ and $a_n^{11}$ of $A_n$

$$a_n^{00}(t-\delta_n, 0) = \frac{a_\lambda(t-2\delta_n) - \Lambda_n\delta_n - \frac{\Lambda_n^2\delta_n^3}{3} - \frac{2}{15}\Lambda_n^3\delta_n^5 g(\Lambda_n\delta_n^2)}{1 + a_\lambda(t-2\delta_n)\delta_n \operatorname{tg}(\Lambda_n\delta_n^2)}$$

and

$$a_n^{11}(t-\delta_n, 0) = \frac{a_\lambda(t-2\delta_n) - \lambda_n\delta_n - \frac{\lambda_n^2\delta_n^3}{3} - \frac{2}{15}\lambda_n^3\delta_n^5 g(\lambda_n\delta_n^2)}{1 + a_\lambda(t-2\delta_n)\delta_n \operatorname{tg}(\lambda_n\delta_n^2)}.$$

We see that neither $a_n^{00}(t-\delta_n, 0)$ nor $a_n^{11}(t-\delta_n, 0)$ is bounded. We get any principal curvature as low in the first direction and as high in the second direction as we want. However, the requirements to avoid conjugate points are always satisfied.

Concerning the focal points in the first dimension we observe that $f_1^n < t-2\delta_n$ is a focal point of $t-\delta_n$ along the inverse geodesic if and only if $a_n(t-\delta_n, f_1^n) = 0$. This is equivalent to $a_n(t-2\delta_n, f_1^n) = a_\lambda(t-2\delta_n, f_1^n) = \Lambda_n \operatorname{tg}_{\Lambda_n}(\delta_n) = \Lambda_n\delta_n \operatorname{tg}(\Lambda_n\delta_n^2)$. For $n \to \infty$ the unique solution $f_1^n$ comes ever closer to $t-2\delta_n$ so that the focal radius of the inverse geodesic goes to 0. The same holds for the second dimension along the geodesic for the points where the focal point $f_2^n > t$ of $t-\delta_n$ is more and more closer to $t$. This means that a lower bound on the Ricci curvature and on the conjugate radius along one geodesic does not imply a lower focal radius bound.

Similarly to the one-dimensional example, for a pair $f_1$, $f_2$ with $f_1 < t < f_2$ and $a_\lambda(t, f_1) = -a_\lambda(t, f_2) > 0$, coupling $\Lambda_n\delta_n = a_\lambda(t, f_1)$ implies that the focal radius (approximately $\min\{t - f_1, f_2 - t\}$) is bounded from below. In this case we also get bounded principal curvature as stated in Theorem 5.

The idea for the two previous examples comes from the examples in [6, Chapter 4]. There the comparison geometry was the Euclidean space ($\lambda \equiv 0$). $r_n$ is then in the first analytic example an even function and in the second example an odd function. Using these properties we determined the conjugate radius for $r_n$ as in the following example in Section 7.3.

### 7.3 A sequence of surfaces of revolution

In this subsection we construct a sequence of surfaces of revolution embedded in $\mathbb{R}^3$ (cf. [4]) for which we can show some global geometric properties as well. This geometric example has the analytic behavior of the one-dimensional example of subsection 7.1 in the case of non-negative curvature. It gives a better understanding of the interactions between curvature, injectivity radius, conjugate radius or focal radius, and principal curvature of distance spheres.

The idea is to take a right circular cone and replace the apex with a spherical cap (see Figure 5). The sequence can now be constructed by reducing the radius $r$ of the spherical part and by increasing the aperture $2\alpha$ of the flat cone such that the sequence converges to the Euclidean plane. In this way,

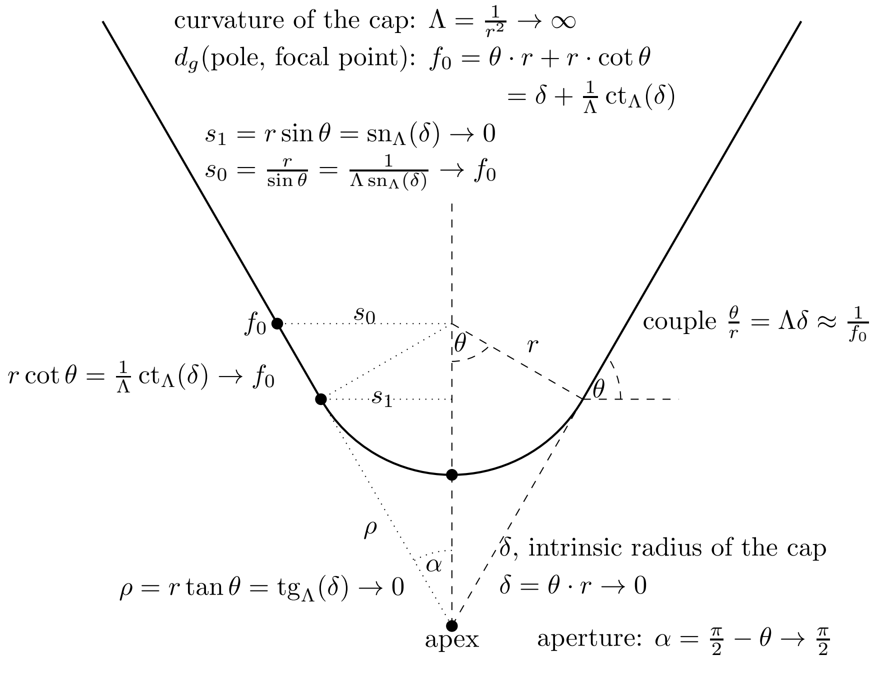



Figure 5: surface of revolution

the constant positive curvature of the spherical cap $\Lambda = \frac{1}{r^2}$ will run to infinity and the inscribed angle $2\theta = \pi - 2\alpha$ and the intrinsic diameter of the sphere $2\delta = 2\theta r$ will run to zero. Without the cap, the cone's lateral surface rolled out in the plane is an unlimited circular sector of central angle $2\pi \cos\theta$ where a circular sector of the same central angle and radius $r\tan\theta$ is cut out. The border circle of the cap has the length $2\pi r \sin\theta$ (see Figure 6). The idea

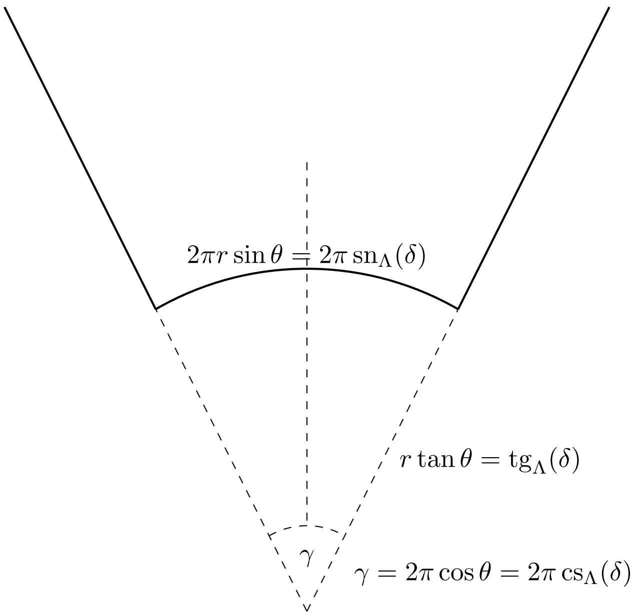



Figure 6: circular sector, the flat part of the surface of revolution rolled out in the plane

is to couple $\theta$ and $r$ such that the distance to the focal point of the apex $f_0 = \theta r + r\cot\theta = \delta + \frac{1}{\Lambda}\,\mathrm{ct}_\Lambda\,\theta$ of a meridian, a geodesic through the pole, will be constant. We will see that in this case the injectivity radius and the conjugate radius of all surfaces of revolution are equal to $2f_0$, that the focal radius equals $\frac{f_0}{2}$, and that the sequence converges in the Lipschitz distance to the flat plane. As a consequence:

Theorem 8. *The class of all complete Riemannian manifolds $M$ with non-negative curvature and injectivity radius* $\mathrm{injRad}(M)$ *(or conjugate radius* $\mathrm{conjRad}(M)$*) bounded from below by a fixed positive constant does not permit a universal upper curvature bound. Furthermore, the lower bound for the Hessian of the distance function $d_p$ of a point $p \in M$ from Corollary 1 is optimal. More precisely, we have*

$$\frac{1}{d_p(q)} - \frac{4}{\mathrm{injRad}(M)} \leq \left\langle \mathrm{Hess}\, d_{p|q} v, v \right\rangle \leq \frac{1}{d_p(q)}$$

*for all* $q \in M$ *with* $d_p(q) \leq \frac{\operatorname{injRad}(M)}{2}$, $v \in T_qM$, $\|v\| = 1$ *and* $v$ *orthogonal to* $\operatorname{grad} d_{p|q}$. *In addition, the lower bound for the focal radius* $\operatorname{focalRad}(M) \geq \frac{\operatorname{conjRad}(M)}{4}$ *from* (1.6) *is also optimal and the inequalities* $\frac{1}{4}\operatorname{injRad}(M) \leq \operatorname{convexRad}(M) \leq \frac{1}{2}\operatorname{injRad}(M)$ *are sharp. Moreover,* injRad, conjRad *and* focalRad *are not continuous functions of the metric space of Riemannian manifolds with the Lipschitz distance.*

Just as in [17] or [18] we start defining a metric in $\mathbb{R}^2$. Then we deduce the Levi-Civita connection and the curvature. We continue with properties of geodesics and calculate the conjugate, the injectivity, and the focal radius. Finally, we show the isometry to an embedded surface of revolution in $\mathbb{R}^3$ and prove the convergence in the Lipschitz topology to the flat plane.
Let $\kappa : \mathbb{R} \to \mathbb{R}$ be an even and a monotonically decreasing function on $[0, \infty)$ and a smooth approximation of

$$\kappa_{\Lambda,\delta}(t) := \begin{cases} \Lambda & |t| \leq \delta \\ 0 & \text{else} \end{cases}$$

with $\Lambda > 0$, $\delta > 0$, $\sqrt{\Lambda}\delta < \frac{\pi}{2}$ such that $\kappa_{\Lambda,\delta-\delta^2} \leq \kappa \leq \kappa_{\Lambda,\delta}$. Moreover let $s_\kappa$ be the solution of $s_\kappa'' + \kappa s_\kappa = 0$ with $s_\kappa(0) = 0$ and $s_\kappa'(0) = 1$. $s_\kappa$ is an odd function. By Rauch's first comparison theorem, $s_\kappa$ behaves like the solutions of the Jacobi equation with the piecewise continuous curvature $\kappa_{\Lambda,\delta}$. $s_\kappa$ is positive on $(0, \infty)$. As $\kappa \geq 0$, $\frac{s_\kappa(t)}{t}$ and $s_\kappa'$ are decreasing with $1 \geq s_\kappa'(t) \geq s_\kappa'(\delta) \approx \operatorname{cs}_\kappa(\delta) > 0$.
In the Euclidean plane $\mathbb{R}^2$ a chart

$$\psi = (r, \varphi) : \mathbb{R}^2 \backslash \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ and } x \leq 0\} \to (0, \infty) \times (-\pi, \pi) \subset \mathbb{R}^2$$

defined by the relation $(x, y) = r(x, y)\big(\cos\varphi(x, y), \sin\varphi(x, y)\big)$, the polar coordinates, leads to a basis $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi})$. $\frac{\partial}{\partial r}$ is the radial vector field

$$\frac{\partial}{\partial r}_{|(x,y)} = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x}_{|(x,y)} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y}_{|(x,y)}$$

which can be extended to $\mathbb{R}^2 \backslash \{0\}$. $\frac{\partial}{\partial \varphi}$ is the Killing vector field of the counter-clockwise rotation about the origin

$$\frac{\partial}{\partial \varphi}_{|(x,y)} = -y \frac{\partial}{\partial x}_{|(x,y)} + x \frac{\partial}{\partial y}_{|(x,y)}$$

which can be extended to $\mathbb{R}^2$. We define the metric $g$ by the following relations

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1, \qquad g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\right) = 0, \qquad g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right) = s_\kappa^2(r),$$

i.e. $g = dr^2 + s_\kappa^2\, d\varphi^2$. $g$ can be extended to a $C^2$-tensor field in $0 \in \mathbb{R}^2$ by $g_{|0} = \langle , \rangle_{eucl}$.

This means that $\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}$ is an orthogonal basis. For the Lie bracket we have $[\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}] = 0$ because these vector fields are partial derivatives of a chart. The rotations are isometries for $g$, and $\frac{\partial}{\partial \varphi}$ is the associated Killing vector field with $\|\frac{\partial}{\partial \varphi}\|_g = s_\kappa(r)$. The reflection at a line through zero is also an isometry for $g$. The rays starting at zero are minimal and so the function $r$ is the distance function of the origin. The integral curves of $\mathrm{grad}_g(r) = \frac{\partial}{\partial r}$, the meridians, are therefore geodesics and $\frac{\partial}{\partial r}$ is autoparallel for g. Hence, $0 \in \mathbb{R}^2$ is a pole. By the Hopf-Rinow theorem the manifold is geodesically complete. As a consequence, we get for the Levi-Civita connection the following defining relations

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0, \qquad \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \varphi} = \nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial r} = \frac{s'_\kappa(r)}{s_\kappa(r)} \frac{\partial}{\partial \varphi},$$
$$\nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} = -s'_\kappa(r) s_\kappa(r) \frac{\partial}{\partial r}.$$

In addition, we have $\frac{\partial}{\partial \varphi}_{|(x,y)} = s_\kappa^2(r(x,y))\, \mathrm{grad}_g\, \varphi_{|(x,y)}$. We see that along a meridian $\frac{1}{s_\kappa(r)} \frac{\partial}{\partial \varphi}$ is a parallel vector field orthogonal to that meridian. We deduce for the curvature $K_{|(x,y)} = \langle R(\frac{1}{s_\kappa(r)} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial r}) \frac{\partial}{\partial r}, \frac{1}{s_\kappa(r)} \frac{\partial}{\partial \varphi} \rangle_{|(x,y)} = \kappa(r(x,y))$. Furthermore $\mathrm{Hess}_g\, r(v) = \frac{s'_\kappa}{s_\kappa} v$ where $v$ is orthogonal to $\frac{\partial}{\partial r}$. The distance spheres of the pole with principal curvature $a_\kappa(r) = \frac{s'_\kappa}{s_\kappa}$ are always convex.
For a curve $c : \mathbb{R} \to (\mathbb{R}^2 \backslash \{0\}, g)$ we have for the velocity $\dot{c} = (r \circ c)' \frac{\partial}{\partial r}_{|c} + \omega \frac{\partial}{\partial \varphi}_{|c}$ with the radial velocity $(r \circ c)' = g(\dot{c}, \frac{\partial}{\partial r}_{|c})$ and the angular velocity $\omega = (\varphi \circ c)' = g(\mathrm{grad}_g\, \varphi_{|c}), \dot{c}) = \frac{1}{s_\kappa^2(r \circ c)} g(\dot{c}, \frac{\partial}{\partial \varphi}_{|c})$. $c$ is a normal geodesic if and only if we have

$$1 = ((r \circ c)')^2 + s_\kappa^2(r \circ c)\,((\varphi \circ c)')^2, \qquad (\|\dot{c}\|_g = 1)$$
$$(r \circ c)'' = s'_\kappa(r \circ c) s_\kappa(r \circ c)\,((\varphi \circ c)')^2 \qquad (g(\nabla_D \dot{c}, \tfrac{\partial}{\partial r}) = 0)$$

and (Clairaut's theorem) the angular momentum $(g(\nabla_D \dot{c}, \frac{\partial}{\partial \varphi}) = 0)$

$$g(\dot{c}, \tfrac{\partial}{\partial \varphi}_{|c}) = s_\kappa(r \circ c) \cos(\angle_g(\dot{c}, \tfrac{\partial}{\partial \varphi}_{|c})) = s_\kappa^2(r \circ c)(\varphi \circ c)' = const$$

is constant. As $s_\kappa > 0$ and $s'_\kappa > 0$ we have that $r \circ c$ is strictly convex and hence has some minimum $r_0$. In particular, no circle of latitude is a geodesic. $c$ is a meridian if and only if the angular momentum is 0. Otherwise we can suppose that $r(c(t)) \geq r(c(0)) = r_0 > 0$. $c$ is then up to a parameter transformation identical to a geodesic starting tangentially at a circle of latitude with radius $r_0$ and has an angular momentum of $s_\kappa(r_0)$.
To calculate the conjugate radius one only has to consider a meridian. To deduce this, we define a variation of geodesics orthogonal to a meridian. So let $c : \to \mathbb{R}^2$ be the meridian $c(s) = (0, s)$. The vector field $\frac{1}{s_\kappa(s)} \frac{\partial}{\partial \varphi}_{|c(s)}$ is parallel along $c$ and orthogonal to $\dot{c}$. Put $v_s(t) := v(t, s) := \exp_{c(s)} \left( \frac{t}{s_\kappa(s)} \frac{\partial}{\partial \varphi}_{|c(s)} \right)$.

$v_0(t) = (-t,0)$ is a meridian. For the variation vector field $J_s$ along each geodesic $v_s$ we have $J_s(0) = \frac{\partial}{\partial r}_{|(0,s)}$ and $J_s'(0) = 0$. As the curvature is bounded from above by $\Lambda$ it follows from Rauch's second comparison theorem that $\|J_s(t)\|_g \geq \mathrm{cs}_\Lambda(t)$ for $0 \leq t \leq \delta < \frac{\pi}{2\sqrt{\Lambda}}$. Furthermore, the angle between $\frac{\partial}{\partial r}$ and $J_s$ is the same as the angle between $\dot{v}_s$ and $\frac{\partial}{\partial \varphi}$. For that reason, and using Clairaut's theorem we get

$$\begin{aligned} r(v_s(t)) - r(v_0(t)) &= \int_0^s \tfrac{\partial}{\partial s}(r \circ v)(t,\sigma)\, d\sigma \\ &= \int_0^s g\left(\tfrac{\partial}{\partial r}_{|v(t,\sigma)}, J_\sigma(t)\right) d\sigma \\ &= \int_0^s \|J_\sigma(t)\|_g \cdot g\left(\dot{v}_s(t,\sigma), \tfrac{1}{s_\kappa(r(v(t,\sigma)))}\tfrac{\partial}{\partial \varphi}_{|v(t,\sigma)}\right) d\sigma \\ &= \int_0^s \|J_\sigma(t)\|_g \tfrac{s_\kappa(s)}{s_\kappa(r(v(t,\sigma)))}\, d\sigma \\ &\geq \tfrac{s_\kappa(s)}{s_\kappa(t+s)} d(v_s(t), c(t)) > 0. \end{aligned}$$

As the curvature is decreasing and flat for $r \geq \delta$ we have $\kappa(c(t)) \geq \kappa(v_s(t))$. From Rauch's first comparison theorem we conclude that the conjugate radius of the surface of revolution equals the conjugate radius of a meridian. Let $c_\kappa$ be the solution of $c_\kappa'' + \kappa c_\kappa = 0$ with $c_\kappa(0) = 1$ and $c_\kappa'(0) = 0$. The first zero of $c_\kappa$ is the distance $f_0$ between the pole and its focal point along a meridian. As $\kappa$ is an even function, we have $c_\kappa(-f_0) = c_\kappa(f_0) = 0$. This means that along a meridian $c$ the points $c(-f_0)$ and $c(f_0)$ are conjugate points. Since $\kappa \equiv 0$ for $t \geq \delta$ we have that $f_0 = \delta - \frac{c_\kappa(\delta)}{c_\kappa'(\delta)} \approx \delta + \frac{1}{\Lambda}\,\mathrm{ct}_\Lambda(\delta)$. This is also the minimal distance between two conjugate points. To prove this we consider $s_\kappa(t,s) := c_\kappa(s)s_\kappa(t) - s_\kappa(s)c_\kappa(t)$ the solution of the Jabobi field equation with $s_\kappa(s,s) = 0$ and $\frac{\partial s_\kappa}{\partial t}(s,s) = 1$. As $s_\kappa$ is an odd function and positive on $(0,\infty)$ we can write

$$s_\kappa(t,s) = s_\kappa(t)s_\kappa(s)(a_\kappa(0,t) - a_\kappa(0,s)).$$

$a(s) = a_\kappa(0,s) = -\frac{c_\kappa(s)}{s_\kappa(s)}$ is the principal curvature of the distance spheres of $c(s)$ in the pole. We are looking for a pair $(s,t)$ such that the principal curvature of the distance spheres of $c(s)$ and $c(t)$ in the pole fit together. $a$ is an odd function, $a' = \frac{1}{s_\kappa^2}$ and $a'' = -2\frac{s_\kappa'}{s_\kappa^3}$. Therefore $a$ is strongly concave on $(0,\infty)$ and strongly convex on the interval $(-\infty,0)$. For $s < 0$ there is at most one positive conjugate point. For the distance $\mathrm{conjRad}(s)$ between two conjugate points we have

$$\mathrm{conjRad}(s) = a^{-1}_{|(0,\infty)}(a(s)) - s.$$

This gives

$$\mathrm{conjRad}'(s) = \frac{s_\kappa^2(s + \mathrm{conjRad}(s))}{s_\kappa^2(s)} - 1.$$

Therefore the focal point of the pole is a minimum for the conjugate radius, i.e. $\operatorname{conjRad}(\mathbb{R}^2, g) = 2f_0$.
To calculate the injectivity radius we define the open subsets $U := \{(x,y) \in \mathbb{R}^2 \mid r(x,y) > \delta, \varphi(x,y) \in (-\pi,\pi)\} \subset (\mathbb{R}^2, g)$ and $V := \{(x,y) \in \mathbb{R}^2 \mid r(x,y) > \rho, \varphi(x,y) \in (-\alpha,\alpha)\} \subset \big(\mathbb{R}^2, \langle,\rangle_{eucl}\big)$ with $\rho = \frac{s_\kappa(\delta)}{s'_\kappa(\delta)}$, $\alpha = \pi s'_\kappa(\delta)$ and observe that

$$\begin{gathered}\phi : (U, g) \to (V, \langle,\rangle_{eucl}) \\ \phi(x,y) = (\rho - \delta + r(x,y))\big(\cos(\tfrac{\alpha}{\pi}\varphi(x,y)), \sin(\tfrac{\alpha}{\pi}\varphi(x,y))\big)\end{gathered}$$

is an isometry. This expresses the fact that the surface can be rolled out in the plane as a circular sector where a circular sector of the same angle $\alpha$ and finite radius $\rho$ is cut out. We conclude that the injectivity radius is at least as great as the radius of the largest circle which fits in the circular sector. As the injectivity radius is a continuous function there is a $p \in (\mathbb{R}^2, g)$ with $\operatorname{injRad}(p) = \operatorname{injRad}(\mathbb{R}^2, g)$. As there are no periodic geodesics we have $\operatorname{injRad}(\mathbb{R}^2, g) = \operatorname{conjRad}(\mathbb{R}^2, g) = 2f_0$.
The minimal distance to a focal point along the meridian is given, when starting at distance $\delta$ from the pole going through the pole. The focal radius is given by $\operatorname{focalRad}(M) = 2\delta - \frac{c_\kappa(2\delta)}{c'_\kappa(2\delta)} \approx 2\delta + \frac{1}{\Lambda}\operatorname{ct}_\Lambda(2\delta) \approx \frac{f_0}{2}$.
It is straightforward to verify that $\chi : \mathbb{R}^2 \to \mathbb{R}^3$ with $\chi(0) = 0$ and

$$\chi(x,y) = \left(\frac{s_\kappa(r(x,y))}{r(x,y)}x, \frac{s_\kappa(r(x,y))}{r(x,y)}y, \int_0^{r(x,y)} \sqrt{1 - (s'_\kappa)^2(\tau)\; d\tau}\right)$$

is an isometric embedding of $(\mathbb{R}^2, g)$ in the Euclidean space $\mathbb{R}^3$. $\chi(\mathbb{R}^2, g)$ is therefore obtained by rotating the curve

$$\begin{aligned} t \mapsto &\left(s_\kappa(t), 0, \int_0^t \sqrt{1 - (s'_\kappa)^2(\tau))}\; d\tau\right) \\ &\approx \begin{cases} \left(\operatorname{sn}_\Lambda(t), 0, \frac{1-\operatorname{cs}_\Lambda(t)}{\sqrt{\Lambda}}\right) & \text{for } 0 \le t \le \delta \\ \left(\operatorname{sn}_\Lambda(\delta) + \operatorname{cs}_\Lambda(\delta)(t-\delta), 0, \frac{1-\operatorname{cs}_\Lambda(t)}{\sqrt{\Lambda}} + \sqrt{\Lambda}\operatorname{sn}_\Lambda(\delta)(t-\delta)\right) & \text{for } \delta \le t \end{cases}\end{aligned}$$

around the z-axes. Finally we calculate the Lipschitz distance between $(\mathbb{R}^2, g)$ and the Euclidean plan. Therefore we note that for a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ not passing through the origin we have

$$\begin{aligned} g(\dot\gamma(t), \dot\gamma(t)) &= \left\langle \dot\gamma(t), \frac{\partial}{\partial r}_{|\gamma(t)}\right\rangle_{eucl}^2 + \frac{s_\kappa^2\big(r(\gamma(t))\big)}{r^4(\gamma(t))}\left\langle \dot\gamma(t), \frac{\partial}{\partial \varphi}_{|\gamma(t)}\right\rangle_{eucl}^2 \\ &\le \left\langle \dot\gamma(t), \frac{\partial}{\partial r}_{|\gamma(t)}\right\rangle_{eucl}^2 + \frac{1}{r^2(\gamma(t))}\left\langle \dot\gamma(t), \frac{\partial}{\partial \varphi}_{|\gamma(t)}\right\rangle_{eucl}^2 \\ &= \langle \dot\gamma(t), \dot\gamma(t)\rangle_{eucl} \end{aligned}$$

and

$$g(\dot\gamma(t),\dot\gamma(t)) \geq \left\langle \dot\gamma(t), \frac{\partial}{\partial r}_{|\gamma(t)} \right\rangle^2_{eucl} + \frac{s_\kappa'(\delta)}{r^2(\gamma(t))} \left\langle \dot\gamma(t), \frac{\partial}{\partial \varphi}_{|\gamma(t)} \right\rangle^2_{eucl}$$
$$\geq s_\kappa'\big(r(\gamma(t))\big) \langle \dot\gamma(t), \dot\gamma(t)\rangle_{eucl}.$$

The identity $id : (\mathbb{R}^2, g) \to (\mathbb{R}^2, \langle,\rangle_{eucl})$ is therefore a Lipschitz function with

$$s_\kappa'(\delta)\|x-y\| \leq d_g(x,y) \leq \|x-y\|.$$

For the Lipschitz distance we get

$$d_L((\mathbb{R}^2, g), (\mathbb{R}^2, \langle,\rangle_{eucl})) \leq -\ln s_\kappa'(\delta) \leq -\ln \mathrm{cs}_\Lambda(\delta).$$

*Proof.* (of Theorem 8) For a given focal radius $f_0$ it is sufficient to couple $\Lambda$ and $\delta$ such that $\delta\Lambda = \frac{1}{f_0}$. The theorem is now a consequence of the analytic example in Section 7.1. □

If one couples $\Lambda_n$ and $\delta_n$ such that $\Lambda_n\delta_n \to 0$ then the radius functions go to infinity like in the Euclidean space. If $\Lambda_n\delta_n \to \infty$ then the radius functions are not bounded from below.

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