Representation Theory of Disconnected Reductive Groups

Pramod N. Achar, William Hardesty, and Simon Riche

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Abstract. We study three fundamental topics in the representation theory of disconnected algebraic groups whose identity component is reductive: (i) the classification of irreducible representations; (ii) the existence and properties of Weyl and dual Weyl modules; and (iii) the decomposition map relating representations in characteristic 0 and those in characteristic p (for groups defined over discrete valuation rings of mixed characteristic). For each of these topics, we obtain natural generalizations of the well-known results for connected reductive groups.

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1 Introduction

Let G be a (possibly disconnected) affine algebraic group over an algebraically closed field $\Bbbk$, and let G° be its identity component. We call G a *(possibly) disconnected reductive group* if G° is reductive. The goal of this paper is to extend a number of well-known foundational facts about connected reductive groups to the disconnected case.

Such groups occur naturally, even when one is primarily interested in *connected* reductive groups. Namely, for a connected reductive group H, the stabilizer H^x of a nilpotent element in the Lie algebra of H may be disconnected. Let H^x_{unip} be its unipotent radical; then H^x/H^x_{unip} is a disconnected reductive group. The study of (the derived category of) coherent sheaves on the nilpotent cone $\mathcal{N}$ of H, and in particular of *perverse-coherent sheaves* on $\mathcal{N}$, leads naturally to

questions about representations of H^x/H^x_{unip}. See [AHR] for some questions of this form, and for some applications of the results of this paper.
The present paper contains three main results:

1. We classify the irreducible representations of G in terms of those of G°, via an adaptation of Clifford theory (Theorem 2.16).

2. Assuming that the characteristic of $\Bbbk$ does not divide $|G/G^\circ|$, we prove that the category of finite-dimensional G-modules has a natural structure of a highest-weight category (Theorem 3.7).

3. Starting from a disconnected reductive group scheme over a strictly Henselian discrete valuation ring of mixed characteristic, one obtains a "decomposition map" relating the Grothendieck groups of representations in characteristic 0 and in characteristic p. We prove that this map is an isomorphism.

These results are certainly not surprising, and some of them may be known to experts, but we are not aware of a reference that treats them in the detail and generality needed for the applications in [AHR].

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2 Classification of simple representations

In this section we consider (affine) algebraic groups over an arbitrary algebraically closed field $\Bbbk$. Our goal is to describe the representation theory of a disconnected algebraic group G whose neutral connected component G° is reductive in terms of the representation theory of G°, via a kind of Clifford theory.

2.1 Twist of a representation by an automorphism

Let G be an algebraic group, $\varphi : G \xrightarrow{\sim} G$ an automorphism, and let $\pi = (V, \varrho)$ be a representation of G. Then we define the representation ${}^\varphi\pi$ as the pair $(V, \varrho \circ \varphi^{-1})$. (Below, we will most of the time write V for π, and ${}^\varphi V$ for ${}^\varphi\pi$.) It is straightforward to check that if $\psi : G \xrightarrow{\sim} G$ is a second automorphism, then we have

$$ {}^\psi\big({}^\varphi\pi\big) = {}^{\psi\circ\varphi}\pi. \tag{1} $$

If $f : \pi \to \pi'$ is a morphism of G-representations, then the same linear map defines a morphism of G-representations ${}^{\varphi}\pi \to {}^{\varphi}\pi'$, which will sometimes be denoted ${}^{\varphi}f$.

Lemma 2.1. *Let $H \subset G$ be a subgroup, and (V, ϱ) be a representation of H. Then there exists a canonical isomorphism of G-modules*

$$ {}^{\varphi}\mathrm{Ind}_H^G(V, \varrho) \cong \mathrm{Ind}_{\varphi(H)}^G(V, \varrho \circ \varphi^{-1}). $$

Proof. By definition, we have

$$ \mathrm{Ind}_H^G(V, \varrho) = \{f : G \to V \mid \forall h \in H,\ f(gh) = \varrho(h^{-1})(f(g))\}, $$
$$ \mathrm{Ind}_{\varphi(H)}^G(V, \varrho \circ \varphi^{-1}) = \{f : G \to V \mid \forall h \in \varphi(H),\ f(gh) = \varrho \circ \varphi^{-1}(h^{-1})(f(g))\}. $$

Here, in both cases the functions are assumed to be algebraic, and the G-action is defined by $(g \cdot f)(h) = f(g^{-1}h)$. We have a natural isomorphism of vector spaces

$$ \mathrm{Ind}_H^G(V, \varrho) \xrightarrow{\sim} \mathrm{Ind}_{\varphi(H)}^G(V, \varrho \circ \varphi^{-1}) $$

sending f to $f \circ \varphi^{-1}$. It is straightforward to check that this morphism is an isomorphism of G-modules from ${}^{\varphi}\mathrm{Ind}_H^G(V, \varrho)$ to $\mathrm{Ind}_{\varphi(H)}^G(V, \varrho \circ \varphi^{-1})$. □

Remark 2.2. More generally, if G' is another algebraic group and $\varphi : G \xrightarrow{\sim} G'$ is an isomorphism, for any G-module π we can consider the G' module ${}^{\varphi}\pi$ defined as above. Then the same arguments as for Lemma 2.1 show that we have ${}^{\varphi}\mathrm{Ind}_H^G(\pi) \cong \mathrm{Ind}_{\varphi(H)}^{G'}({}^{\varphi}\pi)$.

In particular, assume that we are given an algebraic group G' and an embedding of G as a normal subgroup of G'. Then for any $g \in G'$, we have an automorphism $\mathrm{ad}(g)$ of G sending h to ghg^{-1}. In this setting, we will write ${}^{g}V$ for ${}^{\mathrm{ad}(g)}V$, and ${}^{g}f$ for ${}^{\mathrm{ad}(g)}f$. Then for $g, h \in G'$, since $\mathrm{ad}(g) \circ \mathrm{ad}(h) = \mathrm{ad}(gh)$, (1) translates to ${}^{g}\big({}^{h}V\big) = {}^{gh}V$.

The verification of the following lemma is straightforward.

Lemma 2.3. *Let (V, ϱ) be a representation of G. Then if $g \in G$, $\varrho(g^{-1})$ induces an isomorphism $V \xrightarrow{\sim} {}^{g}V$.*

2.2 Disconnected reductive groups

From now on we fix an algebraic group G whose identity component G° is reductive. We set $A := G/G^\circ$ (a finite group). The canonical quotient morphism $G \to A$ will be denoted ϖ.

Let T be the "universal maximal torus" of G°, i.e., the quotient $B/(B, B)$ for any Borel subgroup $B \subset G^\circ$. (Since all Borel subgroups in G° are G°-conjugate, and since $B = N_{G^\circ}(B)$ acts trivially on $B/(B, B)$, the quotient $B/(B, B)$ does not depend on B, up to canonical isomorphism.) Let $\mathbf{X} = X^*(T)$ be its weight lattice. If $T' \subset B$ is any maximal torus, then the composition $T' \hookrightarrow B \twoheadrightarrow T$ is an isomorphism, and this lets us identify $\mathbf{X}$ with $X^*(T')$. The image in $\mathbf{X}$

under this identification of the roots of (G, T'), and of the subset of positive roots (chosen as the opposite of the T'-weights on the Lie algebra of B), do not depend on the choice of T'; so they define the canonical root system $\Phi \subset \mathbf{X}$ and the subset $\Phi^+ \subset \Phi$ of positive roots. Similar comments apply to coroots, so that we can define the dominant weights $\mathbf{X}^+ \subset \mathbf{X}$. We denote by W the Weyl group of T. (This group is well defined because $N_B(T') = T'$ for a maximal torus T' contained in a Borel subgroup B.)

Given a weight $\lambda \in \mathbf{X}^+$, we denote by

$$L(\lambda), \qquad \Delta(\lambda), \qquad \nabla(\lambda)$$

the irreducible, Weyl, and dual Weyl G°-modules, respectively, corresponding to λ. Here $\nabla(\lambda)$ is defined as the induced module $\mathrm{Ind}_B^{G^\circ}(\Bbbk_B(\lambda))$ for some choice of Borel subgroup $B \subset G^\circ$, $L(\lambda)$ is the unique simple submodule of $\nabla(\lambda)$, and $\Delta(\lambda)$ is defined as $(\nabla(-w_0\lambda))^*$, where $w_0 \in W$ is the longest element. (These modules do not depend on the choice of B up to isomorphism thanks to Lemma 2.1 and Lemma 2.3.)

For any $g \in G$ and any Borel subgroup $B \subset G^\circ$, $\mathrm{ad}(g)$ induces an isomorphism $B/(B, B) \xrightarrow{\sim} gBg^{-1}/(gBg^{-1}, gBg^{-1})$. Since gBg^{-1} is also a Borel subgroup of G°, this defines an automorphism $\underline{\mathrm{ad}}(g)$ of T. Explicitly, we can choose an $h \in G^\circ$ such that $gBg^{-1} = hBh^{-1}$, and then for any element $b(B, B) \in B/(B, B) = T$, we set

$$\underline{\mathrm{ad}}(g)(b(B, B)) = h^{-1}gbg^{-1}h(B, B).$$

It is straightforward to check that the right-hand side is independent of h. The fact that T is well defined translates to the property that $\underline{\mathrm{ad}}(g) = \mathrm{id}$ if $g \in G^\circ$, so that $\underline{\mathrm{ad}}$ factors through a morphism $A \to \mathrm{Aut}(T)$, which we will also denote by $\underline{\mathrm{ad}}$.

For $a \in A$ and $\lambda \in \mathbf{X}$, we set

$$^{a}\lambda := \lambda \circ \underline{\mathrm{ad}}(a^{-1}). \tag{2}$$

This operation defines an action of A on $\mathbf{X}$. Now let $g \in \varpi^{-1}(a) \subset G$, and let $T' \subset B$ be a maximal torus. There is an $h \in G^\circ$ such that $gT'g^{-1} = hT'h^{-1}$ and $gBg^{-1} = hBh^{-1}$. Then $h^{-1}g$ normalizes B and (B, B). If x is a root vector for T' in the Lie algebra of (B, B), say with root $\lambda \in -\Phi^+$, then $\mathrm{Ad}(h^{-1}g)(x)$ is also a root vector with root ${}^{a}\lambda$. This shows that the action of A on $\mathbf{X}$ preserves Φ^+ and Φ. Similar reasoning shows that it preserves $\mathbf{X}^+$. Moreover, Lemma 2.1 implies that for any $\lambda \in \mathbf{X}^+$ and $g \in G$, we have canonical isomorphisms

$$^{g}\Delta(\lambda) \cong \Delta({}^{\varpi(g)}\lambda), \quad {}^{g}L(\lambda) \cong L({}^{\varpi(g)}\lambda), \quad {}^{g}\nabla(\lambda) \cong \nabla({}^{\varpi(g)}\lambda). \tag{3}$$

We will denote by $\mathrm{Irr}(G^\circ)$ the set of isomorphism classes of simple G°-modules. This set admits an action of G, where g acts via $[V] \mapsto [{}^{g}V]$. (Of course, this action factors through an action of A.) The constructions above provide a natural bijection $\mathbf{X}^+ \xrightarrow{\sim} \mathrm{Irr}(G^\circ)$ (sending λ to the isomorphism class of $L(\lambda)$), which is A-equivariant in view of (3).

Lemma 2.4. *Let V be an irreducible G-module. Then V is semisimple as a G°-module. All of its irreducible G°-submodules lie in a single G-orbit in* $\mathrm{Irr}(G^\circ)$.

Proof. Choose an irreducible G°-submodule $M \subset V$, and choose a set of coset representatives $g_1, \ldots, g_r$ for G° in G. The subspace

$$\sum_{i=1}^{r} g_i M \subset V$$

is stable under the action of G, so it must be all of V. Each summand $g_i M$ is stable under G°, so there is a surjective map of G°-representations

$$\bigoplus_{i=1}^{r} g_i M \to \sum_{i=1}^{r} g_i M = V.$$

Now, $g_i M$ is isomorphic as a G°-module to ${}^{g_i}M$; in particular, each $g_i M$ is an irreducible G°-module, and $\bigoplus_i g_i M$ is semisimple. Thus, as a G°-module, V is a quotient of a semisimple module, all of whose summands lie in a single G-orbit of $\mathrm{Irr}(G^\circ)$, so the same holds for V itself. ☐

2.3 The component group and induced representations

For each $a \in A = G/G^\circ$, let us choose, once and for all, a representative $\iota(a) \in G$. In the special case $a = 1_A$, we require that

$$\iota(1_A) = 1_G.$$

Given $a, b \in A$, the representative $\iota(ab)$ need not be equal to $\iota(a)\iota(b)$; but these elements lie in the same coset of G°. Explicitly, there is a unique element $\gamma(a, b) \in G^\circ$ such that

$$\iota(a)\iota(b) = \iota(ab)\gamma(a, b).$$

Our assumption on $\iota(1_A)$ implies that for any $a \in A$, we have

$$\gamma(1_A, a) = \gamma(a, 1_A) = 1_G.$$

By expanding $\iota(abc)$ in two ways, one finds that

$$\gamma(ab, c) \cdot \mathrm{ad}(\iota(c)^{-1})(\gamma(a, b)) = \gamma(a, bc)\gamma(b, c). \tag{4}$$

Now let V be a G°-module. By Lemma 2.3, for any $a, b \in A$ the action of $\gamma(a, b)$ defines an isomorphism of G°-modules

$${}^{\gamma(a,b)}V \xrightarrow{\sim} V.$$

Twisting by $\iota(ab)$ we deduce an isomorphism

$$\phi_{a,b} : {}^{\iota(a)\iota(b)}V \xrightarrow{\sim} {}^{\iota(ab)}V.$$

We can use the maps ι and γ to explicitly describe representations of G that are induced from G°, as follows. Let us denote by $\Bbbk[A]$ the group algebra of A over $\Bbbk$. Let V be a G°-module, and consider the vector space

$$\tilde{V} = \Bbbk[A] \otimes V = \bigoplus_{f \in A} \Bbbk f \otimes V. \tag{5}$$

We now explain how to make $\tilde{V}$ into a G-module. Note that every element of G can be written uniquely as $\iota(a)g$ for some $a \in A$ and $g \in G^\circ$. We put

$$\iota(a)g \cdot (f \otimes v) = af \otimes \gamma(a, f) \cdot \mathrm{ad}(\iota(f)^{-1})(g) \cdot v. \tag{6}$$

Using (4) one can check that this does indeed define an action of G on $\tilde{V}$.

Lemma 2.5. *The map*

$$f \mapsto \sum_{a \in A} a \otimes f(\iota(a))$$

defines an isomorphism of G-modules $\mathrm{Ind}_{G^\circ}^G(V) \xrightarrow{\sim} \tilde{V}$.

Proof. It is clear that our map is an isomorphism of vector spaces, and that its inverse sends $a \otimes v$ to the function $f : G \to V$ such that $f(\iota(a)g) = g^{-1} \cdot v$ for $g \in G^\circ$ and $f(\iota(b)g) = 0$ for $g \in G^\circ$ and $b \in A \smallsetminus \{a\}$. It is not difficult to check that this inverse map respects the G-actions, proving the proposition. □

In view of Lemma 2.5, it is clear that as G°-modules, we have

$$\mathrm{Ind}_{G^\circ}^G(V) \cong \bigoplus_{f \in A} {}^{\iota(f)}V, \tag{7}$$

as expected.

2.4 A twisted group algebra of a stabilizer

Let $\lambda \in \mathbf{X}^+$, and let $A^\lambda = \{a \in A \mid {}^a\lambda = \lambda\}$ be its stabilizer. We also set $G^\lambda := \varpi^{-1}(A^\lambda)$. In view of (3), we have

$$G^\lambda = \{g \in G \mid {}^gL(\lambda) \cong L(\lambda)\}. \tag{8}$$

We fix a representative for the simple G°-module $L(\lambda)$ and, for each $a \in A^\lambda$, an isomorphism of G°-modules

$$\theta_a : L(\lambda) \xrightarrow{\sim} {}^{\iota(a)}L(\lambda).$$

In the special case that $a = 1_A$, we require that

$$\theta_{1_A} = \mathrm{id}_{L(\lambda)}.$$

Explicitly, these maps have the property that for any $g \in G^\circ$ and $v \in L(\lambda)$, we have

$$\theta_a(g \cdot v) = \mathrm{ad}(\iota(a)^{-1})(g) \cdot \theta_a(v), \tag{9}$$

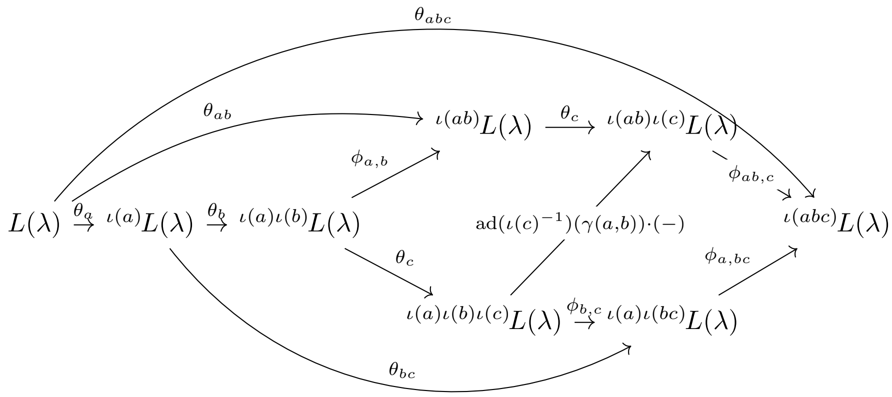
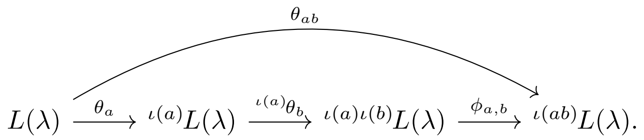


Figure 1: Isomorphisms of $L(\lambda)$ with ${}^{\iota(abc)}L(\lambda)$

where on the right-hand side we consider the given action of G° on $L(\lambda)$.
Now let $a, b \in A^\lambda$, and consider the diagram



This is *not* a commutative diagram. Rather, both θ_{ab} and $\phi_{a,b} \circ {}^{\iota(a)}\theta_b \circ \theta_a$ are isomorphisms of simple G°-modules, so they must be scalar multiples of one another. Let $\alpha(a, b) \in \Bbbk^\times$ be the scalar such that

$$\phi_{a,b}{}^{\iota(a)}\theta_b\theta_a = \alpha(a, b) \cdot \theta_{ab}.$$

Our assumptions on $\iota(1_A)$ and θ_{1_A} imply that for all $a \in A$, we have

$$\alpha(1_A, a) = \alpha(a, 1_A) = 1.$$

Given three elements $a, b, c \in A^\lambda$, we can form the diagram shown in Figure 1. The subdiagram consisting of straight arrows is commutative (by (4), (9) and the definitions), whereas each curved arrow introduces a scalar factor. Comparing the different scalars shows that

$$\alpha(a, b)\alpha(ab, c) = \alpha(a, bc)\alpha(b, c).$$

In other words, $\alpha : A^\lambda \times A^\lambda \to \Bbbk^\times$ is a 2-cocycle.

Let $\mathscr{A}^\lambda$ be the twisted group algebra of A^λ determined by this cocycle. Explicitly, we define $\mathscr{A}^\lambda$ to be the $\Bbbk$-vector space spanned by symbols $\{\rho_a : a \in A^\lambda\}$ with multiplication given by

$$\rho_a \rho_b = \alpha(a,b)\rho_{ab}.$$

This is a unital $\Bbbk$-algebra, with unit ρ_{1_A}.

The algebra $\mathscr{A}^\lambda$ can be described in more canonical terms as follows.

Proposition 2.6. *There exists a canonical isomorphism of* $\Bbbk$*-algebras*

$$\mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big) \cong (\mathscr{A}^\lambda)^{\mathrm{op}}.$$

Proof. We will work with the description of $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))$ from Lemma 2.5 (applied to the group G^λ): we identify it with $\Bbbk[A^\lambda] \otimes L(\lambda)$, where the action of G^λ is given by (6).

We begin by equipping $\Bbbk[A^\lambda] \otimes L(\lambda)$ with the structure of a right $\mathscr{A}^\lambda$-module as follows: given $a, f \in A^\lambda$ and $v \in L(\lambda)$, we put

$$(f \otimes v) \cdot \rho_a := (fa) \otimes \gamma(f,a) \cdot \theta_a(v). \tag{10}$$

Let us check that this is indeed a right $\mathscr{A}^\lambda$-module structure:

$$\begin{aligned}
((f \otimes v)\cdot \rho_a)\cdot \rho_b &= ((fa) \otimes \gamma(f,a)\cdot \theta_a(v))\cdot \rho_b \\
&= (fab) \otimes \gamma(fa,b) \cdot \theta_b(\gamma(f,a)\cdot \theta_a(v)) \\
&= (fab) \otimes \gamma(fa,b)\mathrm{ad}(\iota(b)^{-1})(\gamma(f,a)) \cdot \theta_b(\theta_a(v)) \\
&= (fab) \otimes \gamma(f,ab)\gamma(a,b) \cdot \theta_b(\theta_a(v)) \\
&= (fab) \otimes \alpha(a,b)(\gamma(f,ab)\cdot \theta_{ab}(v)) \\
&= (f \otimes v)\cdot (\alpha(a,b)\rho_{ab}).
\end{aligned}$$

(Here, the third equality relies on (9), and the fourth one on (4).) Next, we check that the right action of $\mathscr{A}^\lambda$ commutes with the left action of G:

$$\begin{aligned}
&\iota(a)g \cdot ((f\otimes v)\cdot \rho_b) \\
&\quad = \iota(a)g \cdot ((fb)\otimes \gamma(f,b)\cdot \theta_b(v)) \\
&\quad = (afb) \otimes \gamma(a,fb)\mathrm{ad}(\iota(fb)^{-1})(g)\gamma(f,b)\cdot \theta_b(v) \\
&\quad = (afb) \otimes \gamma(a,fb)\gamma(f,b)\mathrm{ad}((\iota(fb)\gamma(f,b))^{-1})(g)\cdot \theta_b(v) \\
&\quad = (afb) \otimes \gamma(af,b)\mathrm{ad}(\iota(b)^{-1})(\gamma(a,f))\mathrm{ad}((\iota(f)\iota(b))^{-1})(g)\cdot \theta_b(v) \\
&\quad = (afb) \otimes \gamma(af,b)\theta_b(\gamma(a,f)\mathrm{ad}(\iota(f)^{-1})(g)\cdot v) \\
&\quad = ((af)\otimes \gamma(a,f)\mathrm{ad}(\iota(f)^{-1})(g)\cdot v)\cdot \rho_b \\
&\quad = (\iota(a)g\cdot (f\otimes v))\cdot \rho_b.
\end{aligned}$$

As a consequence, the right $\mathscr{A}^\lambda$-action gives rise to an algebra homomorphism

$$\varphi : (\mathscr{A}^\lambda)^{\mathrm{op}} \to \mathrm{End}_{G^\lambda}(\Bbbk[A^\lambda] \otimes L(\lambda)).$$

For each $a \in A^\lambda$, the operator $\varphi(\rho_a)$ permutes the direct summands $\Bbbk f \otimes L(\lambda) \subset \Bbbk[A^\lambda] \otimes L(\lambda)$, as f runs over elements of A^λ. Moreover, distinct a's give rise to distinct permutations. It follows from this that the collection of linear operators $\{\varphi(\rho_a) : a \in A^\lambda\}$ is linearly independent. In other words, φ is injective.
On the other hand, by adjunction, we have

$$\dim \mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big) = \dim \mathrm{Hom}_{G^\circ}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda)), L(\lambda)\big). \tag{11}$$

Now, (7) implies that as a G°-module, $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))$ is isomorphic to a direct sum of $|A^\lambda|$ copies of $L(\lambda)$. So (11) shows that

$$\dim \mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big) = |A^\lambda| = \dim \mathscr{A}^\lambda.$$

Since φ is an injective map between $\Bbbk$-vector spaces of the same dimension, it is also surjective, and hence an isomorphism. ☐

Remark 2.7. 1. The G°-module $L(\lambda)$ is defined only up to isomorphism. But if $L'(\lambda)$ is another choice for this module, then an isomorphism $L(\lambda) \xrightarrow{\sim} L'(\lambda)$ is unique up to scalar (and exists). Hence the induced isomorphism $\mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big) \xrightarrow{\sim} \mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L'(\lambda))\big)$ does not depend on the choice of isomorphism. In other words, the algebra $\mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big)$ is completely canonical, i.e. does not depend on any choice.

2. Once the G°-module $L(\lambda)$ is fixed, our description of the $\Bbbk$-algebra $\mathscr{A}^\lambda$, and of its identification with $\mathrm{End}_{G^\lambda}(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda)))^{\mathrm{op}}$ in Proposition 2.6, depend on the choice of the isomorphisms θ_a for $a \in A \smallsetminus \{1\}$. However, if $\{\theta'_a : a \in A \smallsetminus \{1\}\}$ is another choice of such isomorphisms, and $\{\rho'_a : a \in A\}$ is the basis of the corresponding algebra $(\mathscr{A}')^\lambda$, then for any $a \in A$ there exists a unique $t_a \in \Bbbk^\times$ such that $\theta'_a = t_a\theta_a$. It is easy to check that the assignment $\rho'_a \mapsto t_a\rho_a$ defines an algebra isomorphism $(\mathscr{A}')^\lambda \xrightarrow{\sim} \mathscr{A}^\lambda$ which commutes with the identifications provided by Proposition 2.6.

3. If, instead of using Lemma 2.5 to describe the G^λ-module $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))$, we describe it in terms of algebraic functions $\phi : G^\lambda \to L(\lambda)$ satisfying $\phi(gh) = h^{-1} \cdot \phi(g)$ for $h \in G^\circ$, then the right action of $\mathscr{A}^\lambda$ on this module satisfies $(\phi \cdot \rho_a)(g) = \theta_a \circ \phi(g\iota(a)^{-1})$.

2.5 Simple G^λ-modules whose restriction to G° is a direct sum of copies of $L(\lambda)$

We continue with the setting of §2.4, and in particular with our fixed $\lambda \in \mathbf{X}^+$. If E is a finite-dimensional left $\mathscr{A}^\lambda$-module, we define a G^λ-action on the $\Bbbk$-vector space $E \otimes L(\lambda)$ by

$$\iota(a)g \cdot (u \otimes v) = \rho_a u \otimes \theta_a^{-1}(gv) \qquad \text{for } a \in A^\lambda \text{ and } g \in G^\circ. \tag{12}$$

Lemma 2.8. *The rule* (12) *defines a structure of* G^λ*-module on* $E \otimes L(\lambda)$.

Proof. Note that

$$\iota(a)g\iota(b)h = \iota(a)\iota(b)\mathrm{ad}(\iota(b)^{-1})(g)h = \iota(ab)(\gamma(a,b)\mathrm{ad}(\iota(b)^{-1})(g)h).$$

We now have

$$\begin{aligned}\iota(a)g \cdot (\iota(b)h \cdot (u \otimes v)) &= \iota(a)g \cdot (\rho_b u \otimes \theta_b^{-1}(hv)) \\ &= \rho_a\rho_b u \otimes \theta_a^{-1}(g\theta_b^{-1}(hv)) \\ &= \alpha(a,b)\rho_{ab}u \otimes (\theta_b \circ \theta_a)^{-1}(\mathrm{ad}(\iota(b)^{-1})(g)hv) \\ &= \rho_{ab}u \otimes \theta_{ab}^{-1}(\gamma(a,b)\mathrm{ad}(\iota(b)^{-1})(g)hv) \\ &= (\iota(a)g\iota(b)h) \cdot (u \otimes v),\end{aligned}$$

proving the desired formula. ☐

Proposition 2.9. *The assignment* $E \mapsto E \otimes L(\lambda)$ *defines a bijection between the set of isomorphism classes of simple* $\mathscr{A}^\lambda$*-modules and the set of isomorphism classes of simple* G^λ*-modules whose restriction to* G° *is a direct sum of copies of* $L(\lambda)$.

Proof. We will show that if V is a finite dimensional G^λ-module whose restriction to G° is a direct sum of copies of $L(\lambda)$, and if we set $E := \mathrm{Hom}_{G^\circ}(L(\lambda), V)$, then E has a natural structure of a left $\mathscr{A}^\lambda$-module, and there exists an isomorphism of G^λ-modules

$$\eta_{\lambda,E} : E \otimes L(\lambda) \xrightarrow{\sim} V.$$

We define the $\mathscr{A}^\lambda$-action on E by

$$(\rho_a \cdot f)(x) = \iota(a) \cdot f(\theta_a(x))$$

for $f \in E = \mathrm{Hom}_{G^\circ}(L(\lambda), V)$ and $x \in L(\lambda)$. (We leave it to the reader to check that $\rho_a \cdot f$ is a morphism of G°-modules.) To justify that this defines an $\mathscr{A}^\lambda$-module structure, we simply compute:

$$\begin{aligned}(\rho_a \cdot (\rho_b \cdot f))(x) &= \iota(a) \cdot (\rho_b \cdot f)(\theta_a(x)) \\ &= \iota(a) \cdot \iota(b) \cdot f(\theta_b \circ \theta_a(x)) \\ &= \iota(ab) \cdot \gamma(a,b) \cdot f(\theta_b \circ \theta_a(x)) \\ &= \iota(ab) \cdot f(\gamma(a,b) \cdot \theta_b \circ \theta_a(x)) \\ &= \alpha(a,b) \cdot \iota(ab) \cdot f(\theta_{ab}(x)) \\ &= ((\alpha(a,b)\rho_{ab}) \cdot f)(x).\end{aligned}$$

Now there exists a canonical isomorphism of G°-modules

$$\eta_{\lambda,E} : E \otimes L(\lambda) = \mathrm{Hom}_{G^\circ}(L(\lambda), V) \otimes L(\lambda) \xrightarrow{\sim} V,$$

defined by $\eta_{\lambda,E}(f\otimes v)=f(v)$. Let us check that this morphism also commutes with the action of $\iota(A)$. By definition we have

$$\iota(a)\cdot(f\otimes v)=(\rho_a\cdot f)\otimes\theta_a^{-1}(v)=\sigma(\iota(a))\circ f\circ\theta_a\otimes\theta_a^{-1}(v),$$

where $\sigma : G^\lambda\to\mathrm{GL}(V)$ is the morphism defining the G^λ-action. Hence

$$\eta_{\lambda,E}\big(\iota(a)\cdot(f\otimes v)\big)=\iota(a)\cdot f(v)=\iota(a)\cdot\eta_{\lambda,E}(f\otimes v),$$

proving that $\eta_{\lambda,E}$ is an isomorphism of G^λ-modules.
It is clear that the assignments

$$-\otimes L(\lambda): E\mapsto E\otimes L(\lambda)\quad\text{and}\quad\mathrm{Hom}_{G^\circ}(L(\lambda),-): V\mapsto\mathrm{Hom}_{G^\circ}(L(\lambda),V)$$

define functors from the category of finite-dimensional $\mathscr{A}^\lambda$-modules to the category of finite-dimensional G^λ-modules whose restriction to G° are isomorphic to a direct sum of copies of $L(\lambda)$, and from the category of finite-dimensional G^λ-modules whose restriction to G° are isomorphic to a direct sum of copies of $L(\lambda)$ to the category of finite-dimensional $\mathscr{A}^\lambda$-modules respectively. It is straightforward to construct an isomorphism of functors $\mathrm{Hom}_{G^\circ}(L(\lambda),-)\circ(-\otimes L(\lambda))\xrightarrow{\sim}\mathrm{id}$, as well as an isomorphism $(-\otimes L(\lambda))\circ\mathrm{Hom}_{G^\circ}(L(\lambda),-)\xrightarrow{\sim}\mathrm{id}$ defined by $\eta_{\lambda,-}$. Our functors are thus equivalences of categories, quasi-inverse to each other; hence they define bijections between the sets of isomorphism classes of simple objects in these categories. □

Remark 2.10. As in Remark 2.7, it can be easily checked that the assignment $E\mapsto E\otimes L(\lambda)$ does not depend on the choice of the isomorphisms $\{\theta_a : a\in A\}$, in the sense that if $\{\theta'_a : a\in A\}$ is another choice of such isomorphisms, and if $(\mathscr{A}')^\lambda$ is the corresponding algebra, then the identification $(\mathscr{A}')^\lambda\xrightarrow{\sim}\mathscr{A}^\lambda$ considered in Remark 2.7 defines a bijection between isomorphism classes of simple $(\mathscr{A}')^\lambda$-modules and $\mathscr{A}^\lambda$-modules, which commutes with the operations $-\otimes L(\lambda)$. Of course, these constructions do not depend on the choice of $L(\lambda)$ in its isomorphism class either.

2.6 Induction from G^λ to G

We continue with the setting of §§2.4–2.5. If E is a finite-dimensional $\mathscr{A}^\lambda$-module, we now consider the G-module

$$L(\lambda,E):=\mathrm{Ind}_{G^\lambda}^G(E\otimes L(\lambda)).$$

Lemma 2.11. *If E is a simple $\mathscr{A}^\lambda$-module, then $L(\lambda,E)$ is a simple G-module.*

Proof. Let $V\subset L(\lambda,E)$ be a simple G-submodule. For any simple G°-module L, let $[V:L]_{G^\circ}$ denote the multiplicity of L as a composition factor of V, regarded as a G°-module. The image of the embedding $V\hookrightarrow L(\lambda,E)$ under the isomorphism

$$\mathrm{Hom}_G\big(V,L(\lambda,E)\big)=\mathrm{Hom}_G\big(V,\mathrm{Ind}_{G^\lambda}^G(E\otimes L(\lambda))\big)\cong\mathrm{Hom}_{G^\lambda}(V,E\otimes L(\lambda))$$

given by Frobenius reciprocity provides a nonzero morphism of G^λ-modules $V \to E \otimes L(\lambda)$, which must be surjective since $E \otimes L(\lambda)$ is simple by Proposition 2.9. It follows that $[V : L(\lambda)]_{G^\circ} \geq \dim(E)$. Now, as in (7), if $g_1, \ldots, g_r$ are representatives in G of the cosets in G/G^λ, then as G°-modules we have

$$L(\lambda, E) = \mathrm{Ind}_{G^\lambda}^{G}(E \otimes L(\lambda)) \cong \bigoplus_{i=1}^{r} {}^{g_i}L(\lambda)^{\oplus \dim(E)}.$$

Since V is stable under the G-action, we have $[V : L(\lambda)]_{G^\circ} = [V : {}^{g_i}L(\lambda)]_{G^\circ}$ for all i (see Lemma 2.3), and hence $[V : {}^{g_i}L(\lambda)]_{G^\circ} \geq \dim(E)$ for all i. This implies that $\dim(V) \geq \dim(\mathrm{Ind}_{G^\lambda}^{G}(E \otimes L(\lambda)))$, so in fact $V = \mathrm{Ind}_{G^\lambda}^{G}(E \otimes L(\lambda))$, as desired. □

2.7 Simple G-modules

We come back to the general setting of §2.2. (In particular, the dominant weight λ is not fixed anymore.) We can now prove that the procedure explained in §§2.4–2.6 allows us to construct *all* simple G-modules (up to isomorphism).

Lemma 2.12. *Let V be a simple G-module. Then there exists $\lambda \in \mathbf{X}^+$, a simple $\mathscr{A}^\lambda$-module E, and an isomorphism of G-modules*

$$V \xrightarrow{\sim} L(\lambda, E).$$

Proof. Certainly there exists $\lambda \in \mathbf{X}^+$ and a surjection of G°-modules $V \twoheadrightarrow L(\lambda)$. By Frobenius reciprocity we deduce a nonzero (hence injective) morphism of G-modules $V \hookrightarrow \mathrm{Ind}_{G^\circ}^{G}(L(\lambda))$. So to conclude, it suffices to prove that all composition factors of $\mathrm{Ind}_{G^\circ}^{G}(L(\lambda))$ are of the form $L(\lambda, E)$ (with E a simple $\mathscr{A}^\lambda$-module). However, we have

$$\mathrm{Ind}_{G^\circ}^{G}(L(\lambda)) \cong \mathrm{Ind}_{G^\lambda}^{G}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big).$$

The restriction of $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))$ to G° is a direct sum of copies of $L(\lambda)$ by (7) applied to G^λ. Therefore, all of its composition factors are of the form $E \otimes L(\lambda)$ with E a simple $\mathscr{A}^\lambda$-module by Proposition 2.9. Since the functor $\mathrm{Ind}_{G^\lambda}^{G}$ is exact (by Lemma 2.5, or by [Ja, Corollary I.5.13]) and sends simple G^λ-modules of the form $E \otimes L(\lambda)$ to simple G-modules by Lemma 2.11, the claim follows. □

2.8 Conjugation

It now remains to understand when two modules of the form $L(\lambda, E)$ are isomorphic. For this, we need to analyze the relation between this construction applied to a dominant weight, and to a twist of this dominant weight by an element of A.

So, let $\lambda \in \mathbf{X}^+$, and $a \in A$. Then we have

$$A^{{}^a\lambda} = aA^\lambda a^{-1}, \quad G^{{}^a\lambda} = \iota(a) G^\lambda \iota(a)^{-1},$$

and we can choose as $L({}^a\lambda)$ the module ${}^{\iota(a)}L(\lambda)$, cf. (3).
Let us choose isomorphisms $\theta_b : L(\lambda) \xrightarrow{\sim} {}^{\iota(b)}L(\lambda)$ for all $b \in A^\lambda$. Again for $b \in A^\lambda$, we can consider the isomorphism

$$\begin{aligned}\tilde{\theta}_{aba^{-1}} : L({}^a\lambda) = {}^{\iota(a)}L(\lambda) &\xrightarrow{\theta_b} {}^{\iota(a)\iota(b)}L(\lambda) = {}^{\iota(a)\iota(b)\iota(a)^{-1}}\big({}^{\iota(a)}L(\lambda)\big)\\ &= {}^{\iota(a)\iota(b)\iota(a)^{-1}}\big(L({}^a\lambda)\big) \xrightarrow[\sim]{\iota(aba^{-1})^{-1}\iota(a)\iota(b)\iota(a)^{-1}\cdot(-)} {}^{\iota(aba^{-1})}L({}^a\lambda).\end{aligned}$$

(Here, the last isomorphism means the action of $\iota(aba^{-1})^{-1}\iota(a)\iota(b)\iota(a)^{-1}$ on $L({}^a\lambda)$, or in other words the action of $\iota(a)^{-1}\iota(aba^{-1})^{-1}\iota(a)\iota(b)$ on $L(\lambda)$.)
The following claim can be checked directly from the definitions.

Lemma 2.13. *For any $b, c \in A^\lambda$, we have*

$$\gamma(aca^{-1}, aba^{-1}) \circ \tilde{\theta}_{aba^{-1}} \circ \tilde{\theta}_{aca^{-1}} = \alpha(c, b) \cdot \tilde{\theta}_{acba^{-1}}$$

(where here $\gamma(aca^{-1}, aba^{-1})$ means the action of this element on $L({}^a\lambda)$).

If $\mathscr{A}^\lambda$ and its basis $\{\rho_b : b \in A^\lambda\}$ are defined in terms of the isomorphisms $\{\theta_b : b \in A^\lambda\}$ and if $\mathscr{A}^{{}^a\lambda}$ and its basis $\{\tilde{\rho}_b : b \in A^{{}^a\lambda}\}$ are defined in terms of the isomorphisms $\{\tilde{\theta}_a : a \in A^{{}^a\lambda}\}$, then Lemma 2.13 allows us to compare the cocycles that arise in the definitions of $\mathscr{A}^\lambda$ and $\mathscr{A}^{{}^a\lambda}$. More precisely, this lemma shows that the assignment $\rho_b \mapsto \tilde{\rho}_{aba^{-1}}$ defines an algebra isomorphism $\xi^a_\lambda : \mathscr{A}^\lambda \xrightarrow{\sim} \mathscr{A}^{{}^a\lambda}$.
The isomorphism ξ^a_λ can be described more canonically as follows. Recall that Proposition 2.6 provides canonical identifications

$$(\mathscr{A}^\lambda)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big), \quad (\mathscr{A}^{{}^a\lambda})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L({}^a\lambda))\big).$$

One can check that under these identifications, the automorphism ξ^a_λ is given by the isomorphism

$$\mathrm{End}_{G^\lambda}\big(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big) = \mathrm{End}_{G^{{}^a\lambda}}\big({}^{\iota(a)}\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))\big) \xrightarrow{\sim} \mathrm{End}_{G^{{}^a\lambda}}\big(\mathrm{Ind}_{G^\circ}^{G^{{}^a\lambda}}(L({}^a\lambda))\big)$$

(where we use the notation of Remark 2.2).
The properties of these isomorphisms that we will need below are summarized in the following lemma.

Lemma 2.14. *Let $\lambda \in \mathbf{X}^+$.*

1. *If $a, b \in A$, then we have $\xi^{ab}_\lambda = \xi^a_{{}^b\lambda} \circ \xi^b_\lambda$.*
2. *If $a \in A^\lambda$, then ξ^a_λ is an inner automorphism of $\mathscr{A}^\lambda$.*

Proof. (1) To simplify notation, we set $\mu := {}^{ab}\lambda$. Note that the simple G°-modules of highest weight μ used in the definitions of ξ^{ab}_λ and $\xi^a_{{}^b\lambda} \circ \xi^b_\lambda$ are different: for the former we use the module $L_1(\mu) := {}^{\iota(ab)}L(\lambda)$, while for the latter we use the module $L_2(\mu) := {}^{\iota(a)\iota(b)}L(\lambda)$. There exists a canonical isomorphism

$$L_1(\mu) \xrightarrow{\sim} L_2(\mu), \tag{13}$$

given by the action of $\gamma(a,b)^{-1}$ on $L(\lambda)$ (i.e. the inverse of the isomorphism denoted $\phi_{a,b}$ in §2.3).

Our algebras are all defined as endomorphisms of some induced module, which can be described in terms of functions with values in the vector space underlying the representation $L(\lambda)$. From this point of view, $\xi^a_{{}^b\lambda} \circ \xi^b_\lambda$ is conjugation by the isomorphism of vector spaces $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda)) \xrightarrow{\sim} \mathrm{Ind}_{G^\circ}^{G^\mu}(L_2(\mu))$ sending functions $G^\lambda \to L(\lambda)$ to functions $G^\mu \to L(\lambda)$ and given by $\phi \mapsto \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b))$, while ξ^{ab}_λ is conjugation by the isomorphism $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda)) \xrightarrow{\sim} \mathrm{Ind}_{G^\circ}^{G^\mu}(L_1(\mu))$ given by $\phi \mapsto \phi(\iota(ab)(-)\iota(ab)^{-1})$. Taking into account the isomorphism (13), we have to check that conjugation by the isomorphism given by

$$\phi \mapsto \gamma(a,b) \cdot \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b)) \tag{14}$$

(where $\gamma(a,b) \cdot (-)$ means the action of $\gamma(a,b) \in G^\circ$ on $L(\lambda)$) coincides with conjugation by the isomorphism given by

$$\phi \mapsto \phi(\iota(ab)(-)\iota(ab)^{-1}). \tag{15}$$

However, since $\gamma(a,b)$ belongs to G°, the functions ϕ we consider satisfy

$$\begin{aligned}\gamma(a,b) \cdot \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b)) &= \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(a)\iota(b)\gamma(a,b)^{-1}) \\ &= \phi(\iota(b)^{-1}\iota(a)^{-1}(-)\iota(ab)) = \phi(\gamma(a,b)^{-1}\iota(ab)^{-1}(-)\iota(ab)).\end{aligned}$$

Thus, the isomorphisms (14) and (15) do *not* coincide, but they differ only by the action of an element of G^λ (which, in fact, even belongs to G°) on $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))$. Therefore, conjugation by either (14) or (15) induces the *same* isomorphism of algebras $\mathrm{End}_{G^\lambda}(\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda))) \xrightarrow{\sim} \mathrm{End}_{G^\mu}(\mathrm{Ind}_{G^\circ}^{G^\mu}(L_1(\mu)))$.

(2) By the comments preceding the statement, ξ^a_λ is conjugation by an isomorphism $\mathrm{Ind}_{G^\circ}^{G^\lambda}(L(\lambda)) \xrightarrow{\sim} \mathrm{Ind}_{G^\circ}^{G^{{}^a\lambda}}(L({}^a\lambda))$. If $a \in A^\lambda$ then this isomorphism defines an invertible element of $\mathscr{A}^\lambda$, so that ξ^a_λ is indeed an inner automorphism. □

Given $a \in A$ and $\lambda \in \mathbf{X}^+$, the isomorphism ξ^a_λ defines a bijection between the set of isomorphism classes of simple $\mathscr{A}^\lambda$-modules and the set of isomorphism classes of simple $\mathscr{A}^{{}^a\lambda}$-modules. From Lemma 2.14(1) we see that this operation defines an action of the group A on the set of pairs (λ, E) where $\lambda \in \mathbf{X}^+$ and E is a simple $\mathscr{A}^\lambda$-module. Moreover, it follows from Lemma 2.14(2) that the induced action of A^λ on the set of isomorphism classes of simple $\mathscr{A}^\lambda$-modules is trivial.

Lemma 2.15. *Let $\lambda \in \mathbf{X}^+$, and let E be a simple $\mathscr{A}^\lambda$-module. Let $a \in A$, and let E' be the simple $\mathscr{A}^{{}^a\lambda}$-module deduced from E via the isomorphism $\xi^a_\lambda : \mathscr{A}^\lambda \xrightarrow{\sim} \mathscr{A}^{{}^a\lambda}$. Then there exists an isomorphism of G-modules*

$$L(\lambda, E) \xrightarrow{\sim} L({}^a\lambda, E').$$

Proof. As above we choose for our simple G°-module of highest weight ${}^a\lambda$ the module ${}^{\iota(a)}L(\lambda)$. Then conjugation by $\iota(a)$ induces an isomorphism $G^\lambda \xrightarrow{\sim} G^{{}^a\lambda}$, and using the notation of Remark 2.2 we have as $G^{{}^a\lambda}$-modules

$${}^{\iota(a)}(E \otimes L(\lambda)) = E' \otimes L({}^a\lambda).$$

In view of Lemma 2.1 we deduce an isomorphism of G-modules

$${}^{\iota(a)}\mathrm{Ind}_{G^\lambda}^G(E \otimes L(\lambda)) \xrightarrow{\sim} \mathrm{Ind}_{G^{{}^a\lambda}}^G(E' \otimes L({}^a\lambda)).$$

Now by Lemma 2.3 the left-hand side is isomorphic to $L(\lambda, E)$, and the claim follows. □

2.9 Classification of simple G-modules

We denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of simple G-modules. Now we can finally state the main result of this section.

Theorem 2.16. *The assignment $(\lambda, E) \mapsto L(\lambda, E)$ induces a bijection*

$$\left\{ (\lambda, E) \;\middle|\; \begin{array}{c} \lambda \in \mathbf{X}^+ \textit{ and } E \textit{ an isom. class} \\ \textit{of simple left } \mathscr{A}^\lambda\textit{-modules} \end{array} \right\} \Big/ A \longleftrightarrow \mathrm{Irr}(G).$$

Proof. From Lemma 2.11, we see that the assignment $(\lambda, E) \mapsto L(\lambda, E)$ defines a map from the set of pairs (λ, E) as in the statement to the set $\mathrm{Irr}(G)$. By Lemma 2.15 this map factors through a map

$$\left\{ (\lambda, E) \;\middle|\; \begin{array}{c} \lambda \in \mathbf{X}^+ \text{ and } E \text{ an isom. class} \\ \text{of simple left } \mathscr{A}^\lambda\text{-modules} \end{array} \right\} \Big/ A \to \mathrm{Irr}(G).$$

By Lemma 2.12, this latter map is surjective. Hence, all that remains is to prove that it is injective.

Let (λ, E) and (λ', E') be pairs as above. Let $V = L(\lambda, E)$ and $V' = L(\lambda', E')$, and assume that $V \cong V'$. As a G°-representation, V is isomorphic to a direct sum of twists of $L(\lambda)$, and V' is isomorphic to a direct sum of twists of $L(\lambda')$ (see the proof of Lemma 2.11). Hence $L(\lambda)$ and $L(\lambda')$ are twists of each other, which implies that λ and λ' are in the same A-orbit. Therefore, we can (and shall) assume that $\lambda = \lambda'$. Fix some isomorphism $V \xrightarrow{\sim} V'$, and consider the morphism of G^λ-modules $f : V \to E' \otimes L(\lambda)$ deduced by Frobenius reciprocity. If $g_1, \dots, g_r$ are representatives of the cosets in G/G^λ, with $g_1 = 1_G$, then we have an isomorphism of G°-modules

$$\mathrm{Ind}_{G^\lambda}^G(E \otimes L(\lambda)) \cong \bigoplus_{i=1}^r {}^{g_i}L(\lambda) \otimes E.$$

If $i \neq 1$, then ${}^{g_i}L(\lambda)$ is not isomorphic to $L(\lambda)$. Hence f is zero on the corresponding summand of $\mathrm{Ind}_{G^\lambda}^G(E \otimes L(\lambda))$. We deduce that the composition

$$E \otimes L(\lambda) \hookrightarrow \mathrm{Ind}_{G^\lambda}^G(E \otimes L(\lambda)) \xrightarrow{f} E' \otimes L(\lambda),$$

where the first morphism is again deduced from Frobenius reciprocity, is nonzero. But this morphism is a morphism of G^λ-modules. Since $L(\lambda, E)$ and $L(\lambda, E')$ are simple, it must be an isomorphism, and by Proposition 2.9 this implies that $E \cong E'$ as $\mathscr{A}^\lambda$-modules. □

Remark 2.17. 1. As explained above Lemma 2.15, for any $\lambda \in \mathbf{X}^+$ the action of A^λ on the set of isomorphism classes of irreducible $\mathscr{A}^\lambda$-modules is trivial. Hence if $\Lambda \subset \mathbf{X}^+$ is a set of representatives of the A-orbits in $\mathbf{X}^+$, the quotient considered in the statement of Theorem 2.16 can be described more explicitly as the set of pairs (λ, E) where $\lambda \in \Lambda$ and E is an isomorphism class of simple $\mathscr{A}^\lambda$-modules.

2. Assume that ι is a group morphism (so that G is isomorphic to the semidirect product $A \ltimes G^\circ$) and that moreover there exists a Borel subgroup $B \subset G^\circ$ such that $\iota(a)B\iota(a)^{-1} = B$ for any $a \in A$. Then if we define the standard and costandard G°-modules using this Borel subgroup, the isomorphisms

$$^{\iota(a)}\Delta(\lambda) \cong \Delta(^a\lambda), \quad ^{\iota(a)}\nabla(\lambda) \cong \nabla(^a\lambda)$$

(see (3)) can be chosen in a canonical way. In fact, our assumptions imply that there exist unique B-stable lines in $^{\iota(a)}\Delta(\lambda)$ and $\Delta(^a\lambda)$, and moreover that these lines coincide. Hence there exists a unique isomorphism of G-modules $^{\iota(a)}\Delta(\lambda) \xrightarrow{\sim} \Delta(^a\lambda)$ which restricts to the identity on these B-stable lines. Similar comments apply to $^{\iota(a)}\nabla(\lambda)$ and $\nabla(^a\lambda)$.

In particular, the isomorphisms θ_a of §2.4 can be chosen in a canonical way. Then the cocycle α will be trivial, so that in this case $\mathscr{A}^\lambda$ is canonically isomorphic to the group algebra $\Bbbk[A^\lambda]$.

2.10 Semisimplicity

We finish this section with a criterion ensuring that the algebra $\mathscr{A}^\lambda$ is semisimple unless p is small.

LEMMA 2.18. *Assume that $p \nmid |A|$. If V is a simple G°-module, then $\mathrm{Ind}_{G^\circ}^G(V)$ is a semisimple G-module.*

Proof. Let M be a G-submodule of $\mathrm{Ind}_{G^\circ}^G(V)$, and let $N = \mathrm{Ind}_{G^\circ}^G(V)/M$. We will show that the image c of the exact sequence $M \hookrightarrow \mathrm{Ind}_{G^\circ}^G(V) \twoheadrightarrow N$ in $\mathrm{Ext}^1_G(N, M)$ vanishes.

First we remark that for any two algebraic G-modules X, Y, the forgetful functor from $\mathrm{Rep}(G)$ to $\mathrm{Rep}(G^\circ)$ induces an isomorphism

$$\mathrm{Hom}_G(X,Y) \xrightarrow{\sim} \big(\mathrm{Hom}_{G^\circ}(X,Y)\big)^A.$$

Under our assumptions the functor $(-)^A$ is exact. On the other hand, it is easily checked that the restriction of any injective G-module to G° is injective. Hence this isomorphism induces an isomorphism

$$\mathrm{Ext}^n_G(X,Y) \xrightarrow{\sim} \big(\mathrm{Ext}^n_{G^\circ}(X,Y)\big)^A$$

for any $n \geq 0$. We deduce in particular that the forgetful functor induces an injection

$$\mathrm{Ext}^1_G(N,M) \hookrightarrow \mathrm{Ext}^1_{G^\circ}(N,M).$$

Hence to prove that $c = 0$ it suffices to prove that the sequence $M \hookrightarrow \mathrm{Ind}_{G^\circ}^G(V) \twoheadrightarrow N$, considered as an exact sequence of G°-modules, splits. This fact is clear since $\mathrm{Ind}_{G^\circ}^G(V)$ is semisimple as a G°-module, see (7). □

From this lemma (applied to the group A^λ) and Proposition 2.6 we deduce the following.

Lemma 2.19. *If $p \nmid |A^\lambda|$, then the algebra $\mathscr{A}^\lambda$ is semisimple (and in fact isomorphic to a product of matrix algebras).*

3 Highest weight structure

Our goal in this section is to prove that if $p \nmid |A|$, then the category $\mathrm{Rep}(G)$ of finite-dimensional G-modules admits a natural structure of a highest weight category.

For the beginning of the section, we continue with the setting of §2.2 (not imposing any further assumption).

3.1 The order

If (λ, E) is a pair as in Theorem 2.16, we denote by $[\lambda, E]$ the corresponding A-orbit. We define a relation $<$ on the set of such orbits as follows:

$$[\lambda, E] < [\lambda', E'] \qquad \text{if} \qquad \text{for some } a \in A, \text{ we have } {}^a\lambda < \lambda'. \tag{16}$$

(Here, the order on $\mathbf{X}$ is the standard one, where $\lambda \leq \mu$ iff $\mu - \lambda$ is a sum of positive roots.)

Lemma 3.1. *The relation $<$ is a partial order.*

Proof. Using the fact that for $a \in A$ and $\lambda, \mu \in \mathbf{X}$ such that $\lambda \leq \mu$ we have ${}^a\lambda \leq {}^a\mu$ (because the A-action is linear and preserves positive roots), one can easily check that this relation is transitive. What remains to be seen is that there cannot exist classes $[\lambda, E]$, $[\lambda', E']$ such that

$$[\lambda, E] < [\lambda', E'] < [\lambda, E].$$

However, in this case we have ${}^a\lambda < \lambda$ for some $a \in A$. Since a permutes the positive coroots of G°, then if we denote by $2\rho^\vee$ the sum of these coroots we must have $\langle {}^a\lambda, 2\rho^\vee \rangle = \langle \lambda, 2\rho^\vee \rangle$, hence $\langle \lambda - {}^a\lambda, 2\rho^\vee \rangle = 0$. On the other hand, by assumption $\lambda - {}^a\lambda$ is a nonzero sum of positive roots, so that its pairing with $2\rho^\vee$ cannot vanish. This provides the desired contradiction. □

3.2 Standard G-modules

Let $\lambda \in \mathbf{X}^+$. We will work in the setting of §§2.3–2.4, including, in particular, fixing a G°-module $L(\lambda)$, and notation such as ι, γ, θ, and α. We also fix a representative $\Delta(\lambda)$ for the Weyl module surjecting to $L(\lambda)$, and a surjection $\pi^\lambda : \Delta(\lambda) \to L(\lambda)$.

Since $\mathrm{End}_{G^\circ}(\Delta(\lambda)) = \Bbbk \cdot \mathrm{id}$, from (3) we see that for each $a \in A^\lambda$, there exists a unique isomorphism $\theta_a^\Delta : \Delta(\lambda) \xrightarrow{\sim} {}^{\iota(a)}\Delta(\lambda)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Delta(\lambda) & \xrightarrow{\theta_a^\Delta} & {}^{\iota(a)}\Delta(\lambda) \\ \downarrow & & \downarrow \\ L(\lambda) & \xrightarrow{\theta_a} & {}^{\iota(a)}L(\lambda). \end{array}$$

Moreover, this uniqueness implies that for any $a, b \in A^\lambda$, if we define $\phi_{a,b}^\Delta : \Delta(\lambda) \to \Delta(\lambda)$ as the action of $\gamma(a, b)$, then we have

$$\phi_{a,b}^\Delta \theta_b^\Delta \theta_a^\Delta = \alpha(a, b) \theta_{ab}^\Delta. \tag{17}$$

Remark 3.2. These considerations show that the subgroup $A^\lambda \subset A$ can be equivalently defined as consisting of the elements $a \in A$ such that ${}^{\iota(a)}\Delta(\lambda) \cong \Delta(\lambda)$. The twisted group algebra $\mathscr{A}^\lambda$ can also be defined in terms of a choice of isomorphisms $(\theta_a^\Delta : a \in A^\lambda)$ instead of isomorphisms $(\theta_a : a \in A^\lambda)$.

Lemma 3.3. *Let E be a finite-dimensional left $\mathscr{A}^\lambda$-module. The following rule defines the structure of a G^λ-module on the vector space $E \otimes \Delta(\lambda)$:*

$$\iota(a)g \cdot (u \otimes v) = \rho_a u \otimes (\theta_a^\Delta)^{-1}(gv) \qquad \text{for any } a \in A^\lambda \text{ and } g \in G^\circ.$$

If E is simple, this G^λ-module has $E \otimes L(\lambda)$ as its unique irreducible quotient. Moreover, all the G°-composition factors of the kernel of the quotient map $E \otimes \Delta(\lambda) \to E \otimes L(\lambda)$ are of the form $L(\mu)$ with $\mu < \lambda$.

Proof. We begin by noting that thanks to (17), the calculation from Lemma 2.8 can be repeated to show that the formula above does, indeed, define the structure of a G^λ-module on $E \otimes \Delta(\lambda)$. Moreover, the quotient map $\pi^\lambda : \Delta(\lambda) \to L(\lambda)$ induces a surjective map of G^λ-modules

$$\pi_E^\lambda := \mathrm{id}_E \otimes \pi : E \otimes \Delta(\lambda) \to E \otimes L(\lambda).$$

Now, assume that E is simple. If we forget the G^λ-module structure and regard $E \otimes \Delta(\lambda)$ as just a G°-module, then it is clear that its unique maximal semisimple quotient can be identified with $E \otimes L(\lambda)$, and that the highest weights of the kernel of π_E^λ are $< \lambda$. Let M be the head of $E \otimes \Delta(\lambda)$ as a G^λ-module. Since M must remain semisimple as a G°-module (by Lemma 2.4), it cannot be larger than $E \otimes L(\lambda)$. In other words, $E \otimes L(\lambda)$ is the unique simple quotient of $E \otimes \Delta(\lambda)$. □

Proposition 3.4. *Let E be a simple $\mathscr{A}^\lambda$-module. The G-module*

$$\Delta(\lambda, E) := \mathrm{Ind}_{G^\lambda}^{G}(E \otimes \Delta(\lambda))$$

admits $L(\lambda, E)$ as its unique irreducible quotient. Moreover, all the composition factors of the kernel of the quotient map $\Delta(\lambda, E) \to L(\lambda, E)$ are of the form $L(\mu, E')$ with $[\mu, E'] < [\lambda, E]$.

Proof. The surjection $E \otimes \Delta(\lambda) \to E \otimes L(\lambda)$ from Lemma 3.3 induces a surjection $\Delta(\lambda, E) \to L(\lambda, E)$ since the functor $\mathrm{Ind}_{G^\lambda}^{G}$ is exact (see the proof of Lemma 2.12). If $g_1, \cdots, g_r$ are representatives of the cosets in G/G^λ, then as G°-modules we have

$$\Delta(\lambda, E) \cong \bigoplus_{i=1}^{r} E \otimes {}^{g_i}\Delta(\lambda), \quad L(\lambda, E) \cong \bigoplus_{i=1}^{r} E \otimes {}^{g_i}L(\lambda). \tag{18}$$

Therefore, as in the proof of Lemma 2.12, $L(\lambda, E)$ is the head of $\Delta(\lambda, E)$ as a G°-module, hence also as a G-module.
If $L(\mu, E')$ is a G-composition factor of the kernel of the surjection $\Delta(\lambda, E) \to L(\lambda, E)$, then some twist of $L(\mu)$ must be a G°-composition factor of the surjection ${}^{g_i}\Delta(\lambda) \to {}^{g_i}L(\lambda)$ for some i. Therefore μ is smaller than some twist of λ, and we deduce that $[\mu, E'] < [\lambda, E]$. □

3.3 Ext^1-vanishing

The same proof as for Lemma 2.15 shows that, up to isomorphism, $\Delta(\lambda, E)$ only depends on the orbit $[\lambda, E]$. The following lemma shows that this module is a "partial projective cover" of $L(\lambda, E)$ (under the assumption that $p \nmid |A|$).

Lemma 3.5. *Assume that $p \nmid |A|$. For any two pairs (λ, E) and (μ, E'), we have*

$$\mathrm{Ext}^1_G\big(\Delta(\lambda, E), L(\mu, E')\big) \neq 0 \quad \Rightarrow \quad [\mu, E'] > [\lambda, E].$$

Proof. As in the proof of Lemma 2.18, we have a canonical isomorphism

$$\mathrm{Ext}^1_G\big(\Delta(\lambda, E), L(\mu, E')\big) \cong \Big(\mathrm{Ext}^1_{G^\circ}\big(\Delta(\lambda, E), L(\mu, E')\big)\Big)^A.$$

If we assume that $\mathrm{Ext}^1_G\big(\Delta(\lambda, E), L(\mu, E')\big) \neq 0$, then this isomorphism shows that we must also have $\mathrm{Ext}^1_{G^\circ}\big(\Delta(\lambda, E), L(\mu, E')\big) \neq 0$. Using (18), we deduce that for some $g, h \in G$ we have

$$\mathrm{Ext}^1_{G^\circ}({}^{g}\Delta(\lambda), {}^{h}L(\mu)) \neq 0.$$

This implies that ${}^{\varpi(h)}\mu > {}^{\varpi(g)}\lambda$, hence that $[\mu, E'] > [\lambda, E]$. ☐

3.4 Costandard G-modules

Fix again $\lambda \in \mathbf{X}^+$ and a simple $\mathscr{A}^\lambda$-module E. Then after fixing a costandard module $\nabla(\lambda)$ with socle $L(\lambda)$ and an embedding $L(\lambda) \hookrightarrow \nabla(\lambda)$, as in §3.2 the isomorphisms θ_a can be "lifted" to isomorphisms $\theta_a^\nabla : \nabla(\lambda) \xrightarrow{\sim} {}^{\iota(a)}\nabla(\lambda)$, which satisfy the appropriate analogue of (17). Using these isomorphisms one can define a G^λ-module structure on $E \otimes \nabla(\lambda)$ by the same procedure as in Lemma 3.3. Then the same arguments as for Proposition 3.4 show that $\nabla(\lambda, E) := \mathrm{Ind}_{G^\lambda}^G(E \otimes \nabla(\lambda))$ admits $L(\lambda, E)$ as its unique simple submodule, and that all the composition factors of the injection $L(\lambda, E) \hookrightarrow \nabla(\lambda, E)$ are of the form $L(\mu, E')$ with $[\mu, E'] < [\lambda, E]$. Moreover, as in Lemma 3.5, if $p \nmid |A|$ we have

$$\mathrm{Ext}^1_G\big(L(\mu, E'), \nabla(\lambda, E)\big) \neq 0 \quad \Rightarrow \quad [\mu, E'] > [\lambda, E].$$

Lemma 3.6. *Assume that $p \nmid |A|$, and let (λ, E) and (μ, E') be pairs as above. Then for any $i > 0$ we have*

$$\mathrm{Ext}^i_G(\Delta(\lambda, E), \nabla(\mu, E')) = 0.$$

Moreover

$$\mathrm{Hom}_G(\Delta(\lambda, E), \nabla(\mu, E')) = 0$$

unless $[\lambda, E] = [\mu, E']$, in which case this space is 1-dimensional.

Proof. As in the proof of Lemma 2.18, for any $i > 0$ we have

$$\mathrm{Ext}^i_G(\Delta(\lambda, E), \nabla(\mu, E')) \cong \big(\mathrm{Ext}^i_{G^\circ}(\Delta(\lambda, E), \nabla(\mu, E'))\big)^A.$$

As G°-modules $\Delta(\lambda, E)$ is isomorphic to a direct sum of Weyl modules, and $\nabla(\mu, E')$ is isomorphic to a direct sum of induced modules. Hence, the right-hand side vanishes unless $i = 0$, which proves the first claim.
For the second claim we remark that if $\mathrm{Hom}_G(\Delta(\lambda, E), \nabla(\mu, E')) \neq 0$, then $L(\lambda, E)$ is a composition factor of $\nabla(\mu, E')$, so that $[\lambda, E] \leq [\mu, E']$, and $L(\mu, E')$ is a composition factor of $\Delta(\lambda, E)$, so that $[\mu, E'] \leq [\lambda, E]$. We deduce that $[\mu, E'] = [\lambda, E]$. Moreover, in this case any nonzero morphism in this space must be a multiple of the composition

$$\Delta(\lambda, E) \twoheadrightarrow L(\lambda, E) \hookrightarrow \nabla(\lambda, E),$$

which concludes the proof. ☐

3.5 Highest weight structure

Let $\mathscr{C}$ be a finite-length $\Bbbk$-linear abelian category such that $\mathrm{Hom}_{\mathscr{C}}(M, N)$ is finite-dimensional for any M, N in $\mathscr{C}$. Let $\mathscr{S}$ be the set of isomorphism classes of irreducible objects of $\mathscr{C}$. Assume that $\mathscr{S}$ is equipped with a partial order $\leq$, and that for each $s \in \mathscr{S}$ we have a fixed representative of the simple object L_s. Assume also we are given, for any $s \in \mathscr{S}$, objects Δ_s and ∇_s, and morphisms $\Delta_s \to L_s$ and $L_s \to \nabla_s$. For $\mathscr{T} \subset \mathscr{S}$, we denote by $\mathscr{C}_{\mathscr{T}}$ the Serre subcategory of $\mathscr{C}$ generated by the objects L_t for $t \in \mathscr{T}$. We write $\mathscr{C}_{\leq s}$ for $\mathscr{C}_{\{t \in \mathscr{S} \mid t \leq s\}}$, and similarly for $\mathscr{C}_{<s}$. Finally, recall that an *ideal* of $\mathscr{S}$ is a subset $\mathscr{T} \subset \mathscr{S}$ such that if $t \in \mathscr{T}$ and $s \in \mathscr{S}$ are such that $s \leq t$, then $s \in \mathscr{T}$.

Recall that the category $\mathscr{C}$ (together with the above data) is said to be a *highest weight category* if the following conditions hold:

1. for any $s \in \mathscr{S}$, the set $\{t \in \mathscr{S} \mid t \leq s\}$ is finite;
2. for each $s \in \mathscr{S}$, we have $\mathrm{End}_{\mathscr{C}}(L_s) = \Bbbk$;
3. for any $s \in \mathscr{S}$ and any ideal $\mathscr{T} \subset \mathscr{S}$ such that $s \in \mathscr{T}$ is maximal, $\Delta_s \to L_s$ is a projective cover in $\mathscr{C}_{\mathscr{T}}$ and $L_s \to \nabla_s$ is an injective envelope in $\mathscr{C}_{\mathscr{T}}$;
4. the kernel of $\Delta_s \to L_s$ and the cokernel of $L_s \to \nabla_s$ belong to $\mathscr{C}_{<s}$;
5. we have $\mathrm{Ext}^2_{\mathscr{C}}(\Delta_s, \nabla_t) = 0$ for all $s, t \in \mathscr{S}$.

In this case, the poset $(\mathscr{S}, \leq)$ is called the *weight poset* of $\mathscr{C}$.

See [Ri, §7] for the basic properties of highest weight categories (following Cline–Parshall–Scott and Beĭlinson–Ginzburg–Soergel).

We can finally state the main result of this section.

Theorem 3.7. *Assume that $p \nmid |A|$. The category* $\mathrm{Rep}(G)$*, equipped with the poset*

$$\left\{ (\lambda, E) \;\middle|\; \begin{array}{c} \lambda \in \mathbf{X}^+ \textit{ and } E \textit{ an isom. class} \\ \textit{of simple } \mathscr{A}^\lambda\textit{-modules} \end{array} \right\} \Big/ A$$

(with the order defined in (16)*) and the objects* $\Delta(\lambda, E)$*,* $L(\lambda, E)$*,* $\nabla(\lambda, E)$*, is a highest weight category.*

Proof. The desired properties are verified in Theorem 2.16, Proposition 3.4 and Lemma 3.5, their variants for costandard objects (see §3.4), and Lemma 3.6. □

4 Grothendieck groups

Our goal in this section is to prove a generalization of a result of Serre [Se] providing a description of the Grothendieck group of any split connected reductive group over a strictly Henselian discrete valuation ring of mixed characteristic. (In [Se], the author considers more general coefficients, but we will restrict to a setting which is sufficient for the application we have in mind; see [AHR].)

4.1 Setting

We will denote by $\mathbb{O}$ a strictly Henselian discrete valuation ring. We denote its residue field by $\mathbb{F}$, and its fraction field by $\mathbb{K}$. Recall that $\mathbb{F}$ is separably closed by definition. We also let $\overline{\mathbb{F}}$ and $\overline{\mathbb{K}}$ be algebraic closures of $\mathbb{F}$ and $\mathbb{K}$, respectively. We will assume that $\mathbb{K}$ has characteristic 0, and that $\mathbb{F}$ has characteristic $p > 0$.

Lemma 4.1. *Any reductive group scheme over $\mathbb{O}$ (in the sense of [SGA3.3]) is split.*

Proof. [1] According to [SGA3.3, Exp. XXII, Corollaire 2.3], any reductive group scheme over $\mathbb{O}$ splits after base change along a suitable étale extension $\mathbb{O} \to \mathbb{O}'$. But because $\mathbb{O}$ is strictly Henselian, [EGA4.4, Proposition 18.8.1(c)] tells us that $\mathbb{O} \to \mathbb{O}'$ admits a section. It follows that any reductive group scheme over $\mathbb{O}$ is split. □

In this section we will consider an affine $\mathbb{O}$-group scheme G, a closed normal subgroup $G^\circ \subset G$, and we will denote by A the factor group of G by G° in the sense of [Ja, §I.6.1] (i.e. of [DG, III, §3, no. 3]). We will make the following assumptions:

1. G° is a reductive group scheme over $\mathbb{O}$ (which is automatically split by Lemma 4.1);
2. A is the constant group scheme associated with a finite group $\mathbf{A}$ (in the sense of [Ja, §I.8.5(a)]), and moreover p does not divide $|\mathbf{A}|$.

These assumptions have the following consequence.

Lemma 4.2. *The $\mathbb{O}$-group scheme G is flat and of finite type.*

Proof. By [DG, III, §3, Proposition 2.5] (see also [Ja, §I.5.7]), the morphism $G \to A$ is flat and of finite type. Since A is clearly flat and of finite type over $\mathbb{O}$, we deduce the same properties for G. □

If $\Bbbk$ is one of $\mathbb{F}$, $\overline{\mathbb{F}}$, $\mathbb{K}$ or $\overline{\mathbb{K}}$, we set

$$G_\Bbbk := \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{O})} G, \quad G^\circ_\Bbbk := \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{O})} G^\circ.$$

Then by [Ja, Equation I.5.5(4)], the quotient $G_\Bbbk / G^\circ_\Bbbk$ is the constant $\Bbbk$-group scheme associated with $\mathbf{A}$; in other words $G_\Bbbk$ is an extension of the constant (hence smooth) $\Bbbk$-group scheme associated with $\mathbf{A}$ by the smooth group scheme $G^\circ_\Bbbk$. In view of [Mi, Proposition 8.1] it follows that $G_\Bbbk$ is smooth, and then that G itself is smooth (see [SP, Tag 01V8]).
In particular, the groups $G_{\overline{\mathbb{F}}}$ and $G_{\overline{\mathbb{K}}}$ are algebraic groups (over $\overline{\mathbb{F}}$ and $\overline{\mathbb{K}}$) in the usual "naive" sense. Since $G^\circ_{\overline{\mathbb{F}}}$ is connected and $\mathbf{A}$ is finite, the latter group identifies with the group of components of $G_{\overline{\mathbb{F}}}$, and $G^\circ_{\overline{\mathbb{F}}}$ with the identity component of $G_{\overline{\mathbb{F}}}$ (which justifies our notation). Similarly, $\mathbf{A}$ also identifies with the group of components of $G_{\overline{\mathbb{K}}}$, and $G^\circ_{\overline{\mathbb{K}}}$ is the identity component of $G_{\overline{\mathbb{K}}}$.

[1] This proof, which was communicated to us by Torsten Wedhorn, replaces a more complicated proof that appeared in a previous draft of this paper.

Lemma 4.3. *The morphism $G(\mathbb{O}) \to \mathbf{A}$ induced by ϖ is surjective.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} G(\mathbb{O}) & \longrightarrow & A(\mathbb{O}) \\ \downarrow & & \| \\ G(\mathbb{F}) & \longrightarrow & A(\mathbb{F}) \end{array}$$

where the horizontal maps are induced by ϖ and the vertical ones by the quotient morphism $\mathbb{O} \to \mathbb{F}$. Here since $\mathbb{O}$ and $\mathbb{F}$ are integral domains the two groups in the right-hand column identify with $\mathbf{A}$, and the right-hand vertical arrow is an isomorphism. On the other hand, the left-hand vertical arrow is surjective by [EGA4.4, Théorème 18.5.17].
To finish the proof, it remains to show that $G(\mathbb{F}) \to A(\mathbb{F})$ is surjective. To do this, we consider the diagram

$$\begin{array}{ccc} G(\mathbb{F}) & \longrightarrow & A(\mathbb{F}) \\ \downarrow & & \| \\ G(\overline{\mathbb{F}}) & \longrightarrow & A(\overline{\mathbb{F}}). \end{array}$$

Here the bottom horizontal arrow is surjective, and the right-hand vertical arrow is again an isomorphism. All the arrows commute with the Frobenius endomorphism, denoted by Fr. Since A is a constant group scheme, its Frobenius endomorphism is the identity map. Since $\overline{\mathbb{F}}$ is a purely inseparable extension of $\mathbb{F}$, for any $g \in G(\overline{\mathbb{F}})$, there exists an integer $n \geq 1$ such that $\mathrm{Fr}^n(g)$ lies in the image of $G(\mathbb{F})$. The result follows. □

Thanks to Lemma 4.3, we can (and will) choose a section $\iota : \mathbf{A} \to G(\mathbb{O})$ of the projection induced by ϖ. Of course, we cannot assume that ι is a group morphism in general; but we will at least assume that $\iota(1) = 1$. For simplicity, we will also denote by $\iota : A \to G$ the morphism of $\mathbb{O}$-group schemes defined by ι, i.e. the $\mathbb{O}$-scheme morphism associated with the algebra morphism $\mathcal{O}(G) \to \mathcal{O}(A) = \mathrm{Fun}(\mathbf{A}, \mathbb{O})$ (where $\mathrm{Fun}(\mathbf{A}, \mathbb{O})$ denotes the algebra of functions from $\mathbf{A}$ to $\mathbb{O}$) sending f to the map $a \mapsto \iota(a)(f)$. Base-changing to $\overline{\mathbb{F}}$ and to $\overline{\mathbb{K}}$, ι provides sections of the projections of $G_{\overline{\mathbb{F}}}$ and $G_{\overline{\mathbb{K}}}$ onto their respective groups of components.

Lemma 4.4. *The morphism*

$$A \times G^\circ \to G$$

defined by $(a, g) \mapsto \iota(a) \cdot g$ is an isomorphism of $\mathbb{O}$-schemes.

Proof. Consider the algebra morphism $\varphi : \mathcal{O}(G) \to \prod_{a \in \mathbf{A}} \mathcal{O}(G^\circ)$ induced by our morphism; what we have to prove is that φ is an isomorphism. From the remarks preceding Lemma 4.3, we know that the algebra morphisms $\overline{\mathbb{F}} \otimes_{\mathbb{O}} \varphi$

and $\overline{\mathbb{K}} \otimes_{\mathbb{O}} \varphi$ are isomorphisms; hence so are the morphisms $\mathbb{F} \otimes_{\mathbb{O}} \varphi$ and $\mathbb{K} \otimes_{\mathbb{O}} \varphi$. From the invertibility of $\mathbb{K} \otimes_{\mathbb{O}} \varphi$ and the fact that $\mathcal{O}(G)$ is flat (hence torsion free) we deduce that φ is injective.

Now we denote by C the cokernel of φ. To prove that $C = 0$ it suffices to prove that for any maximal ideal $\mathfrak{m} \subset \mathcal{O}(G)$ the localization $C_{\mathfrak{m}}$ vanishes. Then, since $\prod_{a \in \mathbf{A}} \mathcal{O}(G^{\circ})$ is finitely generated as an $\mathcal{O}(G)$-module (because $\mathcal{O}(G^{\circ})$ is), so is C, so that by Nakayama's lemma it suffices to prove that $C_{\mathfrak{m}}/\mathfrak{m}C_{\mathfrak{m}} = C/\mathfrak{m}C$ vanishes. Now the kernel of the composition $\mathbb{O} \to \mathcal{O}(G)/\mathfrak{m}$ is either $\{0\}$ or the unique maximal ideal $\mathfrak{p}$ in $\mathbb{O}$. In the latter case $C/\mathfrak{m}C$ is a quotient of $C/\mathfrak{p}C = C \otimes_{\mathbb{O}} \mathbb{F}$, which vanishes since $\mathbb{F} \otimes_{\mathbb{O}} \varphi$ is an isomorphism. In the former case, the morphism $\mathbb{O} \to \mathcal{O}(G)/\mathfrak{m}$ factors through a morphism $\mathbb{K} \to \mathcal{O}(G)/\mathfrak{m}$, and then $C/\mathfrak{m}C = C \otimes_{\mathcal{O}(G)} \mathcal{O}(G)/\mathfrak{m}$ is a quotient of

$$C \otimes_{\mathbb{O}} \mathcal{O}(G)/\mathfrak{m} = (C \otimes_{\mathbb{O}} \mathbb{K}) \otimes_{\mathbb{K}} \mathcal{O}(G)/\mathfrak{m},$$

which vanishes since $\mathbb{K} \otimes_{\mathbb{O}} \varphi$ is invertible. □

4.2 Statement

Let us consider the Grothendieck groups

$$\mathsf{K}(G), \qquad \mathsf{K}(G_{\mathbb{K}}), \qquad \mathsf{K}(G_{\mathbb{F}})$$

of the categories of (algebraic) G-modules of finite type over $\mathbb{O}$, of finite-dimensional (algebraic) $G_{\mathbb{K}}$-modules, and of finite-dimensional (algebraic) $G_{\mathbb{F}}$-modules, respectively. We will also denote by $\mathsf{K}_{\mathrm{pr}}(G)$ the Grothendieck group of the exact category of G-modules which are free of finite rank over $\mathbb{O}$. Following [Se] we consider the commutative diagram of natural morphisms of abelian groups

$$\begin{array}{ccccc} \mathsf{K}_{\mathrm{pr}}(G) & \xrightarrow{\sim} & \mathsf{K}(G) & \twoheadrightarrow & \mathsf{K}(G_{\mathbb{K}}) \\ & \searrow & & \swarrow{\scriptstyle d_G} & \\ & & \mathsf{K}(G_{\mathbb{F}}). & & \end{array} \tag{19}$$

Here, on the upper line, the left horizontal map (which is induced by the natural inclusion of categories) is an isomorphism by [Se, Proposition 4]. The right horizontal map (induced by the exact functor $\mathbb{K} \otimes_{\mathbb{O}} (-)$) is surjective by [Se, Théorème 1]. The map from the top left-hand corner to the group on the bottom line is induced by the (exact) functor $\mathbb{F} \otimes_{\mathbb{O}} (-)$. Finally, the map d_G is the "decomposition" morphism from [Se, Théorème 2].

The main result of this section is the following.

Theorem 4.5. *All the maps in* (19) *are isomorphisms.*

According to [Se, Théorème 3], if d_G is surjective, then the right-hand morphism on the upper line is automatically an isomorphism. Thus, to prove Theorem 4.5, it is enough to prove that d_G is an isomorphism. This will be accomplished in §4.5 below.

4.3 Lattices

Our starting point will be [Se, Théorème 5], which is applicable here thanks to Lemma 4.1. This result asserts that if we consider the diagram

$$\begin{array}{ccccc} \mathsf{K}_{\mathrm{pr}}(G^\circ) & \xrightarrow{\sim} & \mathsf{K}(G^\circ) & \twoheadrightarrow & \mathsf{K}(G^\circ_{\mathbb{K}}) \\ & \searrow & & \swarrow_{d_{G^\circ}} & \\ & & \mathsf{K}(G^\circ_{\mathbb{F}}) & & \end{array} \tag{20}$$

similar to (19) but for the group G°, then the decomposition morphism d_{G° is an isomorphism, so that all the maps in (20) are isomorphisms. The main idea of the argument is as follows: first fix a split torus $T \subset G^\circ$ and set $\mathbf{X} := X^*(T)$. Then both $\mathsf{K}(G^\circ_{\mathbb{K}})$ and $\mathsf{K}(G^\circ_{\mathbb{F}})$ can be embedded in $\mathbb{Z}[\mathbf{X}]$ by taking characters, and d_{G° is characterized by the property that it preserves characters.

Let us delve a bit further into the details of the behavior of d_{G°. Choosing a system of positive roots in the root system of (G°, T), we obtain a Borel subgroup $B \subset G^\circ$ containing T (chosen such that B is the negative Borel subgroup), and a subset $\mathbf{X}^+ \subset \mathbf{X}$ of dominant weights.

By the well-known representation theory of connected reductive groups over algebraically closed fields, both the set of isomorphism classes of simple $G^\circ_{\overline{\mathbb{K}}}$-modules and the set of isomorphism classes of simple $G^\circ_{\overline{\mathbb{F}}}$-modules are in bijection with $\mathbf{X}^+$. More concretely, if $\lambda \in \mathbf{X}^+$ and if $L_{\overline{\mathbb{K}}}(\lambda)$ is a simple $G^\circ_{\overline{\mathbb{K}}}$-module of highest weight λ, then there exists a simple $G^\circ_{\mathbb{K}}$-module $L_{\mathbb{K}}(\lambda)$ and an isomorphism $\overline{\mathbb{K}} \otimes_{\mathbb{K}} L_{\mathbb{K}}(\lambda) \cong L_{\overline{\mathbb{K}}}(\lambda)$. Moreover, $L_{\mathbb{K}}(\lambda)$ is unique up to isomorphism (which justifies the notation), and every simple $G^\circ_{\mathbb{K}}$-module is of this form. (See e.g. [Se, §3.6] or [Ja, Corollary II.2.9] for details.)

There is a similar description of simple $G^\circ_{\mathbb{F}}$-modules. Note also that the Weyl and dual Weyl $G^\circ_{\overline{\mathbb{F}}}$-modules have obvious $\mathbb{F}$-versions, that will be denoted $\Delta_{\mathbb{F}}(\lambda)$ and $\nabla_{\mathbb{F}}(\lambda)$ respectively.

Next, if $V_{\mathbb{O}} \subset L_{\mathbb{K}}(\lambda)$ is a G°-stable $\mathbb{O}$-lattice, then the class of $\mathbb{F} \otimes_{\mathbb{O}} V_{\mathbb{O}}$ in $\mathsf{K}(G^\circ_{\mathbb{F}})$ coincides with the class of the Weyl module $\Delta_{\mathbb{F}}(\lambda)$ of highest weight λ (because $L_{\overline{\mathbb{K}}}(\lambda)$ and $\Delta_{\overline{\mathbb{F}}}(\lambda)$ have the same character). In fact, it is well known that the lattice $V_{\mathbb{O}}$ can be chosen in such a way that $\mathbb{F} \otimes_{\mathbb{O}} V_{\mathbb{O}} \cong \Delta_{\mathbb{F}}(\lambda)$ as $G^\circ_{\mathbb{F}}$-modules. For each λ we will fix such a lattice, and denote it by $L_{\mathbb{O}}(\lambda)$. To summarize, we have

$$d_{G^\circ}([L_{\mathbb{K}}(\lambda)]) = [\Delta_{\mathbb{F}}(\lambda)].$$

In the present setting, $\mathbf{A}$ is the group of components both of $G_{\overline{\mathbb{F}}}$ and of $G_{\overline{\mathbb{K}}}$. Identifying $T_{\overline{\mathbb{F}}}$ and $T_{\overline{\mathbb{K}}}$ with the universal maximal tori of $G^\circ_{\overline{\mathbb{F}}}$ and $G^\circ_{\overline{\mathbb{K}}}$ respectively (via the choice of Borel subgroups obtained from B by base change), we obtain two actions of $\mathbf{A}$ on $\mathbf{X} = X^*(T_{\overline{\mathbb{F}}}) = X^*(T_{\overline{\mathbb{K}}})$, see §2.2. The description of this action involves the property that Borel subgroups are conjugate, which is not true over $\mathbb{O}$; so it is not clear from the definition that they must coincide. In the next lemma we will show that they do at least coincide on $\mathbf{X}^+$.

Lemma 4.6. *The two actions of* $\mathbf{A}$ *on* $\mathbf{X}$ *agree on* $\mathbf{X}^+$.

Proof. Let us provisionally denote the two actions of $\mathbf{A}$ on $\mathbf{X}$ by $\cdot_{\overline{\mathbb{F}}}$ and $\cdot_{\overline{\mathbb{K}}}$. Since $\mathbf{A}$ acts by algebraic group automorphisms on G°, this group acts on all the Grothendieck groups in (20), and all the maps in this diagram are obviously $\mathbf{A}$-equivariant. Now for $\lambda \in \mathbf{X}^+$ we have $a \cdot [L_{\mathbb{K}}(\lambda)] = [L_{\mathbb{K}}(a \cdot_{\overline{\mathbb{K}}} \lambda)]$, hence

$$d_{G^\circ}(a \cdot [L_{\mathbb{K}}(\lambda)]) = [\Delta_{\mathbb{F}}(a \cdot_{\overline{\mathbb{K}}} \lambda)].$$

On the other hand, we have $a \cdot [\Delta_{\mathbb{F}}(\lambda)] = [\Delta_{\mathbb{F}}(a \cdot_{\overline{\mathbb{F}}} \lambda)]$. Since d_{G° is $\mathbf{A}$-equivariant, it follows that

$$d_{G^\circ}(a \cdot [L_{\mathbb{K}}(\lambda)]) = a \cdot [\Delta_{\mathbb{F}}(\lambda)] = [\Delta_{\mathbb{F}}(a \cdot_{\overline{\mathbb{F}}} \lambda)]$$

(see (3)). We deduce that $[\Delta_{\mathbb{F}}(a \cdot_{\overline{\mathbb{K}}} \lambda)] = [\Delta_{\mathbb{F}}(a \cdot_{\overline{\mathbb{F}}} \lambda)]$, hence that $a \cdot_{\overline{\mathbb{K}}} \lambda = a \cdot_{\overline{\mathbb{F}}} \lambda$. ☐

From now on we fix $\lambda \in \mathbf{X}^+$. It follows in particular from Lemma 4.6 that the two possible definitions of the subgroup $\mathbf{A}^\lambda \subset \mathbf{A}$ (see §2.4) coincide.

Lemma 4.7. *1. We have* $\mathrm{End}_{G^\circ}(L_{\mathbb{O}}(\lambda)) = \mathbb{O}$.

2. For any $a \in \mathbf{A}^\lambda$, there exists an isomorphism of G°-modules

$${}^{\iota(a)}L_{\mathbb{O}}(\lambda) \cong L_{\mathbb{O}}(\lambda).$$

Proof. We only explain the proof of (2); the proof of (1) is similar. Consider the object

$$R\,\mathrm{Hom}_{G^\circ}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda))$$

of the derived category of $\mathbb{O}$-modules. By [Ja, Lemma II.B.5 and its proof], this complex has bounded cohomology, and each of its cohomology objects is finitely generated. This implies that it is isomorphic (in the derived category of $\mathbb{O}$-modules) to a finite direct sum of shifts of finitely generated $\mathbb{O}$-modules. It follows from [MR, Proposition A.6 and Proposition A.8] that we have

$$\overline{\mathbb{F}} \overset{L}{\otimes}_{\mathbb{O}} R\,\mathrm{Hom}_{G^\circ}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda)) \cong R\,\mathrm{Hom}_{G^\circ_{\overline{\mathbb{F}}}}(\Delta_{\overline{\mathbb{F}}}(\lambda), \Delta_{\overline{\mathbb{F}}}(\lambda)),$$

$$\overline{\mathbb{K}} \overset{L}{\otimes}_{\mathbb{O}} R\,\mathrm{Hom}_{G^\circ}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda)) \cong R\,\mathrm{Hom}_{G^\circ_{\overline{\mathbb{K}}}}(L_{\overline{\mathbb{K}}}(\lambda), L_{\overline{\mathbb{K}}}(\lambda)).$$

Now we have $R\,\mathrm{Hom}_{G^\circ_{\overline{\mathbb{K}}}}(L_{\overline{\mathbb{K}}}(\lambda), L_{\overline{\mathbb{K}}}(\lambda)) \cong \overline{\mathbb{K}}$, so that $\mathrm{Hom}_{G^\circ}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda))$ is a sum of $\mathbb{O}$ and a torsion module. But since $\mathrm{Hom}_{G^\circ_{\overline{\mathbb{F}}}}(\Delta_{\overline{\mathbb{F}}}(\lambda), \Delta_{\overline{\mathbb{F}}}(\lambda)) = \overline{\mathbb{F}}$, this torsion module is zero; in other words we have $\mathrm{Hom}_{G^\circ}(L_{\mathbb{O}}(\lambda), {}^{\iota(a)}L_{\mathbb{O}}(\lambda)) \cong \mathbb{O}$. If $f : L_{\mathbb{O}}(\lambda) \to {}^{\iota(a)}L_{\mathbb{O}}(\lambda)$ is a generator of this rank-1 $\mathbb{O}$-module, the $G_{\overline{\mathbb{F}}}$-module morphism $\overline{\mathbb{F}} \otimes_{\mathbb{O}} f$ is an isomorphism, so that f is also an isomorphism. ☐

4.4 Comparison of twisted group algebras

We continue with the setting of §4.3 (and in particular with our fixed $\lambda \in \mathbf{X}^+$). By Lemma 4.7 we can choose, for any $a \in \mathbf{A}^\lambda$, an isomorphism $\theta_a : L_{\mathbb{O}}(\lambda) \xrightarrow{\sim} {}^{\iota(a)}L_{\mathbb{O}}(\lambda)$. Then $\overline{\mathbb{K}} \otimes_{\mathbb{O}} \theta_a$ is an isomorphism from $L_{\overline{\mathbb{K}}}(\lambda)$ to ${}^{\iota(a)}L_{\overline{\mathbb{K}}}(\lambda)$, and for

$a, b \in \mathbf{A}^\lambda$ the scalar $\alpha(a,b) \in \overline{\mathbb{K}}$ defined in §2.4 using these isomorphisms in fact belongs to $\mathbb{O}^\times$. In particular, if $\mathscr{A}^\lambda_{\overline{\mathbb{K}}}$ is the associated twisted group algebra (over $\overline{\mathbb{K}}$), then the $\mathbb{O}$-lattice $\mathscr{A}^\lambda_{\mathbb{O}} := \bigoplus_{a \in A^\lambda} \mathbb{O} \cdot \rho_a$ is an $\mathbb{O}$-subalgebra in $\mathscr{A}^\lambda_{\overline{\mathbb{K}}}$. On the other hand, $\overline{\mathbb{F}} \otimes_{\mathbb{O}} \theta_a$ is an isomorphism from $\Delta_{\overline{\mathbb{F}}}(\lambda)$ to ${}^{\iota(a)}\Delta_{\overline{\mathbb{F}}}(\lambda)$, and by Remark 3.2 the algebra $\mathscr{A}^\lambda_{\overline{\mathbb{F}}}$ from §2.4 (now for the group $G_{\overline{\mathbb{F}}}$ and its simple module $L_{\overline{\mathbb{F}}}(\lambda)$) can be described as the twisted group algebra of $\mathbf{A}^\lambda$ defined by the cocyle sending (a,b) to the image of $\alpha(a,b)$ in $\overline{\mathbb{F}}$.
Summarizing, we have obtained an $\mathbb{O}$-algebra $\mathscr{A}^\lambda_{\mathbb{O}}$ which is free over $\mathbb{O}$ and such that

$$\overline{\mathbb{K}} \otimes_{\mathbb{O}} \mathscr{A}^\lambda_{\mathbb{O}} \cong \mathscr{A}^\lambda_{\overline{\mathbb{K}}}, \quad \overline{\mathbb{F}} \otimes_{\mathbb{O}} \mathscr{A}^\lambda_{\mathbb{O}} \cong \mathscr{A}^\lambda_{\overline{\mathbb{F}}}.$$

From Lemma 2.19 we know that $\mathscr{A}^\lambda_{\overline{\mathbb{F}}}$ and $\mathscr{A}^\lambda_{\overline{\mathbb{K}}}$ are products of matrix algebras (over $\overline{\mathbb{F}}$ and $\overline{\mathbb{K}}$ respectively). In fact, the same arguments show that $\mathscr{A}^\lambda_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{O}} \mathscr{A}^\lambda_{\mathbb{O}}$ and $\mathscr{A}^\lambda_{\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{O}} \mathscr{A}^\lambda_{\mathbb{O}}$ are also products of matrix algebras (over $\mathbb{F}$ and $\mathbb{K}$ respectively). Hence we are in the setting of Tits' deformation theorem (see e.g. [GP, Theorem 7.4.6]), and we deduce that we have a canonical bijection between the sets of isomorphism classes of simple $\mathscr{A}^\lambda_{\mathbb{K}}$-modules and isomorphism classes of simple $\mathscr{A}^\lambda_{\mathbb{F}}$-modules, which sends a simple module M to $\mathbb{F} \otimes_{\mathbb{O}} M_{\mathbb{O}}$, where $M_{\mathbb{O}}$ is any $\mathscr{A}^\lambda_{\mathbb{O}}$-stable $\mathbb{O}$-lattice in M.
If E be a finite-dimensional $\mathscr{A}^\lambda_{\mathbb{K}}$-module, the same procedure as in §2.5 allows us to define a $G^\lambda_{\mathbb{K}}$-module structure on $E \otimes_{\mathbb{K}} L_{\mathbb{K}}(\lambda)$, where $G^\lambda_{\mathbb{K}}$ is the inverse image of $\mathbf{A}^\lambda$ under the map $G_{\mathbb{K}} \to A$ induced by ϖ. Similarly, copying the definitions in Lemma 3.3 and Proposition 3.4, if E' be a finite-dimensional $\mathscr{A}^\lambda_{\mathbb{F}}$-module, then we can consider the $G_{\mathbb{F}}$-module $\Delta_{\mathbb{F}}(\lambda, E')$, which is an $\mathbb{F}$-form of $\Delta_{\overline{\mathbb{F}}}(\lambda, \overline{\mathbb{F}} \otimes_{\mathbb{F}} E')$.

Lemma 4.8. *Let E be a simple $\mathscr{A}^\lambda_{\mathbb{K}}$-module, and let $\tilde{E}$ be the simple $\mathscr{A}^\lambda_{\mathbb{F}}$-module corresponding to E under the bijection above. Then we have*

$$d_G([\mathrm{Ind}_{G^\lambda_{\mathbb{K}}}^{G_{\mathbb{K}}}(E \otimes_{\mathbb{K}} L_{\mathbb{K}}(\lambda))]) = [\Delta_{\mathbb{F}}(\lambda, \tilde{E})].$$

Proof. If $E_{\mathbb{O}} \subset E$ is an $\mathscr{A}^\lambda_{\mathbb{O}}$-stable $\mathbb{O}$-lattice in E, then $E_{\mathbb{O}} \otimes_{\mathbb{O}} L_{\mathbb{O}}(\lambda)$ has a natural structure of G^λ-module (where $G^\lambda = \varpi^{-1}(\mathbf{A}^\lambda)$), and is a G^λ-stable $\mathbb{O}$-lattice in $E \otimes_{\mathbb{K}} L_{\mathbb{K}}(\lambda)$. Inducing to G, we deduce that $\mathrm{Ind}_{G^\lambda}^{G}(E_{\mathbb{O}} \otimes_{\mathbb{O}} L_{\mathbb{O}}(\lambda))$ is a G-stable $\mathbb{O}$-lattice in $\mathrm{Ind}_{G^\lambda_{\mathbb{K}}}^{G_{\mathbb{K}}}(E \otimes_{\mathbb{K}} L_{\mathbb{K}}(\lambda))$, whose modular reduction is $\Delta_{\mathbb{F}}(\lambda, \tilde{E})$. □

4.5 Invertibility of d_G

We can now prove that d_G is an isomorphism, which will finish the proof of Theorem 4.5.
In fact, for any $\lambda \in \mathbf{X}^+$, since $\mathscr{A}^\lambda_{\mathbb{K}}$ is a product of matrix algebras the assignment $E \mapsto \overline{\mathbb{K}} \otimes_{\mathbb{K}} E$ induces a bijection between the sets of isomorphism classes of simple modules for the algebras $\mathscr{A}^\lambda_{\mathbb{K}}$ and $\mathscr{A}^\lambda_{\overline{\mathbb{K}}}$ from §4.4. Then, using Theorem 2.16 and arguing as in [Se, §3.6], we see that the similar operation induces

a bijection between the sets of isomorphism classes of simple $G_{\mathbb{K}}$-modules and of simple $G_{\overline{\mathbb{K}}}$-modules.
The same construction gives a bijection between the sets of isomorphism classes of simple $G_{\mathbb{F}}$-modules and of simple $G_{\overline{\mathbb{F}}}$-modules.
Let us now fix a subset $\Lambda \subset \mathbf{X}^+$ of representatives for the $\mathbf{A}$-orbits on $\mathbf{X}^+$. By the remarks above, the classes of the modules $\mathrm{Ind}_{G_{\mathbb{K}}^{\lambda}}^{G_{\mathbb{K}}}(E \otimes L_{\mathbb{K}}(\lambda))$, where (λ, E) runs over the pairs consisting of an element $\lambda \in \Lambda$ and a simple $\mathscr{A}_{\mathbb{K}}^{\lambda}$-module, form a basis of $\mathsf{K}(G_{\mathbb{K}})$ (see in particular Remark 2.17(1)). In view of Lemma 4.8, Theorem 3.7, and the preceding paragraph, the image of this basis under d_G is a basis of $\mathsf{K}(G_{\mathbb{F}})$. Hence, d_G is indeed an isomorphism.

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Pramod N. Achar
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
USA
pramod@math.lsu.edu

William Hardesty
School of Mathematics and Statistics F07
University of Sydney NSW 2006
Australia
william.hardesty@sydney.edu.au

Simon Riche
Université Clermont Auvergne
CNRS, LMBP
F-63000 Clermont-Ferrand
France
simon.riche@uca.fr