Generic Smooth Representations

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Abstract. Let F be a non-archimedean local field. In this paper we explore genericity of irreducible smooth representations of $GL_n(F)$ by restriction to a maximal compact subgroup K of $GL_n(F)$. Let (J, λ) be a Bushnell–Kutzko type for a Bernstein component Ω. The work of Schneider–Zink gives an irreducible K-representation $\sigma_{min}(\lambda)$, which appears with multiplicity one in $\mathrm{Ind}_J^K \lambda$. Let π be an irreducible smooth representation of $GL_n(F)$ in Ω. We prove that π is generic if and only if $\sigma_{min}(\lambda)$ is contained in π, in which case it occurs with multiplicity one.

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1 Introduction

We are concerned with the problem of understanding the genericity of irreducible smooth representations of a general linear group over a p-adic field.
Let G be a reductive p-adic group. Recall that a smooth irreducible representation π of G is called generic if π appears in $\mathrm{Ind}_U^G \psi$ (i.e. admits a Whittaker model), where Ind denotes induction and ψ is a nondegenerate character of a maximal unipotent subgroup U of G.
We will start by recalling a few facts about the category of smooth representations. Let C be an algebraically closed field of characteristic zero. Let $\mathcal{R}(G)$ be the category of all smooth C-representations of G. The Bernstein decomposition ([Ber84]) expresses the category of smooth C-valued representations

of G as the product of certain indecomposable full subcategories, called Bernstein components. Those components are parametrized by the inertial classes, whose definition we now recall. Consider the set of pairs (M, ρ), with M a Levi subgroup of G and ρ an irreducible supercuspidal representation of M. We say that two pairs (M_1, ρ_1) and (M_2, ρ_2) are inertially equivalent if and only if there are $g \in G$ and an unramified character χ of M_2 such that $M_2 = M_1^g$ and $\rho_2 \simeq \rho_1^g \otimes \chi$, where $M_1^g := g^{-1}M_1 g$ and $\rho_1^g(x) = \rho_1(gxg^{-1})$, for $x \in M_1^g$. The equivalence class of (M, ρ) will be denoted by $[M, \rho]_G$, and is called an *inertial class*. The set of inertial classes will be denoted by $\mathcal{B}(G)$.
We denote by $i_P^G : \mathcal{R}(M) \longrightarrow \mathcal{R}(G)$ the normalized parabolic induction functor, where $P = MN$ is a parabolic subgroup of G with Levi subgroup M. Let $\Omega := [M, \rho]_G$ be an inertial equivalence class, where ρ is a supercuspidal representation of M. To Ω we may associate a full subcategory $\mathcal{R}^\Omega(G)$ of $\mathcal{R}(G)$, such that the representation (π, V) is an object of $\mathcal{R}^\Omega(G)$ if and only if every irreducible G-subquotient π_0 of π appears as a composition factor of $i_P^G(\rho \otimes \omega)$ for ω some unramified character of M and P some parabolic subgroup of G with Levi factor M. The category $\mathcal{R}^\Omega(G)$ is called a Bernstein component of $\mathcal{R}(G)$. According to [Ber84], the Bernstein decomposition is written as, $\mathcal{R}(G) = \prod_{\Omega \in \mathcal{B}(G)} \mathcal{R}^\Omega(G)$. It follows that if we want to understand the category $\mathcal{R}(G)$, it is enough to restrict our attention to the Bernstein components. This can be done via the theory of types. This theory allows us to parametrize all the irreducible representations of G up to inertial equivalence using irreducible representations of compact open subgroups of G. Let J be a compact open subgroup of G and let λ be an irreducible representation of J. We say that (J, λ) is an Ω-type if, for (π, V) a representation of G, the representation (π, V) is an object of $\mathcal{R}^\Omega(G)$ if and only if V is generated by its λ-isotypical space V^λ as a G-representation.
Let F be a local non-archimedean field. For $G = GL_n(F)$, types can be constructed (cf. [BK93], [BK98] and [BK99]) for every Bernstein component. The simplest example of a type is $(I, 1)$, where I is the standard Iwahori subgroup of G and 1 is the trivial representation. In this case $\Omega = [T, 1]_G$, where T is the subgroup of diagonal matrices and 1 denotes the trivial representation of T. We will refer to this example as the Iwahori case.
Fix K a maximal compact subgroup of $G = GL_n(F)$. Given a Bushnell–Kutzko type (J, λ) with J contained in K, in [SZ99, section 6] (just above Proposition 2) the authors define irreducible K-representations $\sigma_{\mathcal{P}}(\lambda)$, the so called *tempered types*, where $\mathcal{P}$ belongs to some partially ordered set (cf. [SZ99, section 2]), with order $\leq$. One has the decomposition :

$$\mathrm{Ind}_J^K \lambda = \bigoplus_{\mathcal{P}} \sigma_{\mathcal{P}}(\lambda)^{\oplus m_{\mathcal{P},\lambda}}, \tag{1.1}$$

where the summation runs over the same partially ordered set as above. The integers $m_{\mathcal{P},\lambda}$ are finite and we call $m_{\mathcal{P},\lambda}$ the multiplicity of $\sigma_{\mathcal{P}}(\lambda)$. Let $\mathcal{P}_{max}$ be the maximal element and let $\mathcal{P}_{min}$ be the minimal one. Define $\sigma_{max}(\lambda) := \sigma_{\mathcal{P}_{max}}(\lambda)$ and $\sigma_{min}(\lambda) := \sigma_{\mathcal{P}_{min}}(\lambda)$. Both K-representations

$\sigma_{max}(\lambda)$ and $\sigma_{min}(\lambda)$ occur in $\mathrm{Ind}_J^K \lambda$ with multiplicity 1. In the Iwahori case those representations have a very simple description. Indeed, $\sigma_{min}(\lambda)$ is the inflation of the Steinberg representation of $GL_n(k_F)$ to K and $\sigma_{max}(\lambda)$ is the trivial representation.
Having introduced the main notation of this paper we may now state our main theorem:

Theorem 1.1. *Let π be an irreducible representation in $\mathcal{R}^{\Omega}(G)$ and let (J, λ) be the corresponding Bushnell–Kutzko type. We can associate to π an element $\mathcal{P}_\pi$ appearing in the decomposition (1.1). Then the following is equivalent:*

1. *π is generic.*
2. *π contains the tempered types $\sigma_{\mathcal{P}'}(\lambda)$ for all $\mathcal{P}' \leq \mathcal{P}_\pi$*
3. *π contains the minimal type $\sigma_{min}(\lambda)$.*

And if the equivalent conditions are fulfilled then $\sigma_{min}(\lambda)$ will occur with multiplicity 1.

Theorem 1.1 shows that the representation $\sigma_{min}(\lambda)$ has a very special role. One can wonder about other $\sigma_{\mathcal{P}}(\lambda)$'s. There is a recent result by Jack Shotton in that direction. He proves [Sho18, Thm.3.7] that by modifying the proof of [SZ99, Proposition 2 Section 6] and [BC09, Proposition 6.5.3] in the tempered case, one gets the same result in the generic case. In the author's thesis the result [Sho18, Thm.3.7] was proven independently but with a different method. First using the theory of types of Bushnell–Kutzko, we reduce the statement to the Iwahori case. Then, in the Iwahori case, we use the results of Rogawski [Rog85] on modules over Iwahori–Hecke algebra. In this case the proof relies on some easy combinatorics on partitions.
The multiplicity one statement can fail for other $\sigma_{\mathcal{P}}(\lambda)$'s. For example, consider the Iwahori case with $n = 3$, i.e. $G = GL_3(F)$. Take $\pi = i_B^G(1 \otimes \chi_1 \otimes \chi_2)$, where B is the subgroup of G of upper triangular matrices, 1 the trivial character and χ_1, χ_2 unramified characters such that $\chi_1.\chi_2^{-1} \neq |.|^{\pm 1}$. Then, writing $\sigma_{2,1}$ for the summand of $\mathrm{Ind}_I^K 1$ corresponding to the partition $(2, 1)$ (see section 2), one can easily verify that $\dim \mathrm{Hom}_K(\sigma_{2,1}, \pi) = 2$.
Let us observe that Theorem 1.1 can be also proven by considering Hecke algebras. First we use one of the main results of [BK99], which asserts that the Hecke algebra $\mathcal{H}(G, \lambda)$ is naturally isomorphic to a tensor product of affine Hecke algebras of type A. Moreover it is shown in [BK93] that any Hecke algebra of a simple type is isomorphic to an affine Hecke algebra of type A. In this manner we can reduce the statement about irreducible representations of general type to the Iwahori case.
Finally let us observe that to the best of our knowledge Theorem 1.1 and [Sho18, Thm.3.7] do not have an analogue for all reductive groups, because the crucial ingredient in the proofs is the tensor product decomposition of the Hecke algebra $\mathcal{H}(G, \lambda)$ and the existence of types, proven by Bushnell–Kutzko

in [BK99]. Indeed results of [BK93], [BK98] and [BK99] allow us to transfer the general situation to the Iwahori case, where the proofs are simpler. However we believe that those results should generalize easily to reductive groups with A_n root system. It would be interesting to investigate the case of other reductive groups.

Notation

For an arbitrary local non-archimedean field L, let $\mathcal{O}_L$ be its ring of integers and k_L the residue field. We also choose a uniformizer $\varpi_L \in \mathcal{O}_L$. From now on fix F a local non-archimedean field and $G = GL_n(F)$.
Recall that all the representations have their coefficients in an algebraically closed field C of characteristic zero. Assume that C has the same cardinality as the complex numbers $\mathbb{C}$. Fix an isomorphism $\iota : C \to \mathbb{C}$. Let $\tilde{G}$ be some p-adic group. A character $\chi : \tilde{G} \to C$ is defined by $\chi = \iota^{-1}(\iota \circ \chi)$, where $\iota \circ \chi$ is a character in a usual sense.
We are given an inertial class $\Omega = [M, \rho]_G$, where ρ is a supercuspidal representation of M and an Ω-type (J, λ) with $J \subset K$ a compact open subgroup of G. Write $\mathfrak{Z}_\Omega$ for the centre of the category $\mathcal{R}^\Omega(G)$. Recall that the centre of a category is the ring of endomorphisms of the identity functor. For example the centre of the category $\mathcal{H}(G, \lambda)$-Mod is $Z(\mathcal{H}(G, \lambda))$, where $Z(\mathcal{H}(G, \lambda))$ is the centre of the ring $\mathcal{H}(G, \lambda)$.
For an irreducible representation π of $GL_n(F)$ a *cuspidal support* of π is a pair (M, ρ) such that π is an irreducible subquotient of $i_P^G(\rho)$, where M is a Levi subgroup, $P = MN$ a parabolic subgroup, and ρ an irreducible supercuspidal representation of M. The pair (M, ρ) can be chosen such that $M = \prod_{i=1}^s GL_{r_i}(F)$ and $\rho = \omega_1 \otimes \ldots \otimes \omega_s$, because it has no impact on irreducible subquotients according to [BZ77, 4.1 and 4.7]. An inertial class $\Omega = [\prod_{i=1}^s GL_{r_i}(F), \omega_1 \otimes \ldots \otimes \omega_s]_G$ is called *simple* if $r_i = r$ and all factors ω_i belong to the same inertial equivalence class of supercuspidal representations. Otherwise Ω is called *semisimple*. From now on we will say that $(\omega_1, \ldots, \omega_s)$ is a *cuspidal support* of π. Furthermore, let us introduce the following notation:

$$\omega_1 \times \ldots \times \omega_s := i_P^G(\omega_1 \otimes \ldots \otimes \omega_s) \tag{1.2}$$

The representations of a Bernstein component can be seen as modules over a Hecke algebra. For any types (J, λ) let $\mathcal{R}_\lambda(G)$ be the full subcategory of $\mathcal{R}(G)$ such that (π, V) is an object of $\mathcal{R}_\lambda(G)$ if and only if V is generated by V^λ (the λ-isotypical component of V) as G-representation. Define $\mathcal{H}(G, \lambda) := \mathcal{H}(G, J, \lambda) := \mathrm{End}_G(\text{c–Ind}_J^G \lambda)$, the Hecke algebra of the type (J, λ). Then for any Ω-type (J, λ), by [BK98, Theorem 4.2 (ii)], the functor:

$$\begin{array}{cccc} \mathfrak{M}_\lambda & : & \mathcal{R}_\lambda(G) & \to & \mathcal{H}(G, \lambda)\text{-Mod} \\ & & \pi & \mapsto & \mathrm{Hom}_J(\lambda, \pi) = \mathrm{Hom}_G(\text{c–Ind}_J^G \lambda, \pi) \end{array}$$

is an equivalence of categories. Since (J, λ) is an Ω-type, we have $\mathcal{R}^\Omega(G) = \mathcal{R}_\lambda(G)$. The type (J, λ) is called *simple*(resp. *semisimple*) when the corresponding Ω is simple (resp. semisimple). The corresponding Hecke algebras $\mathcal{H}(G,\lambda)$ are then isomorphic to affine Hecke algebras of type A or tensor product of such Hecke algebras resp.([BK93], [BK99]).

A *partition* is a function $P:\mathbb{Z}_{\geq 1}\to\mathbb{Z}_{\geq 0}$ with finite support; we say that P is a partition of an integer $k:=\sum_{n=1}^{+\infty}P(n).n$. We may also represents a partition P of k as a sequence $(m_1,\ldots,m_l)$, with $m_1\geq\ldots\geq m_l\geq 0$ and $m_1+\ldots+m_l=k$, where one omits the zeroes from that list. Then $P(n)$ is the number of constituents $m_i=n$. Let $(m'_1,\ldots,m'_s)$, be the conjugate partition of $(m_1,\ldots,m_l)$, meaning that $m'_i:=\mid\{j:m_j\geq i\}\mid$. The integers m'_i are related to P as follows, $m'_s=P(s)$, $m'_{s-1}=P(s)+P(s-1)$,…, $m'_1=P(s)+\ldots+P(1)$.

We define a partial ordering on the set $\mathbb{P}$ of partitions as follows. Following the convention in [SZ99], we write $\lambda=(\lambda_1,\ldots,\lambda_l)\geqq\mu=(\mu_1,\ldots,\mu_l)$ if and only if $\sum_{i=1}^{j}\lambda_i\leq\sum_{i=1}^{j}\mu_i$ for all integers j. The smallest partition of k for this partial order is (k) and the biggest is $(1,\ldots,1)$ (k times 1). This order is the opposite of the usual dominance order on partitions ([Knu98, Chapter 5, Section 5.1.4]). For more information on partitions the reader may also consult[Ful97] and [Mac15].

As in [SZ99, section 2], let $\mathcal{C}$ be a system of representatives for the irreducible supercuspidal representations of any $GL_k(F)$ ($k\in\mathbb{Z}_{\geq 1}$) up to unramified twist. If $\omega\in\mathcal{C}$, is a representation of $GL_k(F)$, write $d(\omega):=k$.

A partition-valued function is a function $\mathcal{P}:\mathcal{C}\to\mathbb{P}$ with finite support. Let supp$\mathcal{P}$ be the support of $\mathcal{P}$. The set of partition-valued functions is partially ordered with respect to the partial ordering on partitions defined in the paragraph above by setting $\mathcal{P}\leq\mathcal{P}'$ if and only if $\mathcal{P}(\tau)\leq\mathcal{P}'(\tau)$, $\forall\tau\in\mathcal{C}$.

There is a natural map from the set of partition valued functions to the set of Bernstein components given by:

$$\mathcal{P}\mapsto\Omega(\mathcal{P}):=[\prod_{\omega\in\mathrm{supp}\mathcal{P}}GL_{d(\omega)}(F)^{\sum_{s\in\mathbb{Z}_{\geq 1}}s\mathcal{P}(\omega)(s)},\bigotimes_{\omega\in\mathrm{supp}\mathcal{P}}\omega^{\sum_{s\in\mathbb{Z}_{\geq 1}}s\mathcal{P}(\omega)(s)}]_G$$

If we are given a Bernstein component Ω and an Ω-type (J,λ) as above then, the decomposition (1.1) reads more precisely:

$$\mathrm{Ind}_J^K\lambda=\bigoplus_{\mathcal{P}:\Omega(\mathcal{P})=\Omega}\sigma_{\mathcal{P}}(\lambda)^{\oplus m_{\mathcal{P},\lambda}},$$

where the summation runs over partition-valued functions $\mathcal{P}$ such that $\Omega(\mathcal{P})=\Omega$. Among those partitions there is a unique minimal partition $\mathcal{P}^{min}$ and also a unique maximal partition $\mathcal{P}^{max}$, for the partial order $\leq$. In order to simplify the notation let $\sigma_{min}(\lambda):=\sigma_{\mathcal{P}^{min}}(\lambda)$. As an example consider the case when $\Omega=[G,\rho]_G$. In this case, λ is a representation of K, and the decomposition from above reads $\lambda=\lambda$, where there is only one partition valued function supported on ρ given by the unique partition of the number 1. The other extreme case is the Iwahori case, i.e $\Omega=[T,1]_G$ corresponding to $(J,\lambda)=(I,1)$.

2 Types

2.1 Representations of general linear groups over a finite field

In this section we will recall definitions from [SZ99, section 4]. Let $\overline{G}_n := GL_n(k_F)$ and let $\mathcal{R}(\overline{G})$ be the category of C-representations of $\overline{G}_n$. Similarly to the case of G-representations we have the notion of cuspidal support. Let $\mathcal{B}(\overline{G}_n)$ be the set of conjugacy classes $[\overline{M}, \overline{\sigma}]_n$ of pairs $(\overline{M}, \overline{\sigma})$, where $\overline{M}$ is a Levi subgroup of $\overline{G}_n$ and $\overline{\sigma}$ a cuspidal representation of $\overline{M}$. To $\nu \in \mathcal{B}(\overline{G}_n)$ we may associate a full subcategory $\mathcal{R}^\nu(\overline{G}_n)$ of $\mathcal{R}(\overline{G}_n)$, such that the representation (π, V) is an object of $\mathcal{R}^\nu(\overline{G}_n)$ if and only if every irreducible constituent π_0 of π appears as a composition factor of $\mathrm{Ind}_{\overline{P}}^{\overline{G}_n}(\overline{\sigma})$, where $(M, \overline{\sigma})$ represents an equivalence class ν and $\overline{P}$ some parabolic subgroup of $\overline{G}_n$ with Levi factor $\overline{M}$. Similarly to the Bernstein decomposition, we also have $\mathcal{R}(\overline{G}_n) = \prod_{\nu \in \mathcal{B}(\overline{G}_n)} \mathcal{R}^\nu(\overline{G}_n)$.
Let $\overline{\mathcal{C}}$ be a system of representatives of isomorphism classes of all the irreducible cuspidal representations of any $\overline{G}_k$, for k varying in the set of positive integers. If $\overline{\sigma}$ is a representation of $\overline{G}_d$, we write $d(\overline{\sigma}) = d$. For any $\overline{\sigma} \in \overline{\mathcal{C}}$ and $s \in \mathbb{Z}_{\geq 1}$, define:

$$\mathrm{st}(\overline{\sigma}, s)$$

to be the unique nondegenerate irreducible representation with cuspidal support $(\overline{\sigma}, \ldots, \overline{\sigma})$ (s-times). A partition-valued function is a function $\mathcal{P} : \overline{\mathcal{C}} \to \mathbb{P}$ with finite support, let $d(\mathcal{P}) := \sum_{(\overline{\sigma}, s) \in \overline{\mathcal{C}} \times \mathbb{Z}_{\geq 1}} \mathcal{P}(\overline{\sigma})(s) s d(\overline{\sigma})$. For any partition valued function $\mathcal{P}$, define the nondegenerate representation:

$$\mathrm{st}(\mathcal{P}) := \bigotimes_{(\overline{\sigma}, s) \in \overline{\mathcal{C}} \times \mathbb{Z}_{\geq 1}} \mathrm{st}(\overline{\sigma}, s)^{\otimes \mathcal{P}(\overline{\sigma})(s)}$$

of the Levi subgroup $M_{\mathcal{P}} = \prod_{(\overline{\sigma}, s) \in \overline{\mathcal{C}} \times \mathbb{Z}_{\geq 1}} \overline{G}_{s d(\overline{\sigma})}^{\times \mathcal{P}(\overline{\sigma})(s)}$ of $\overline{G}_{d(\mathcal{P})}$.
There is a natural map from the set of partition valued functions to the set of Bernstein components given by:

$$\mathcal{P} \mapsto \nu(\mathcal{P}) := [\prod_{\overline{\sigma} \in \mathrm{supp}\mathcal{P}} (\overline{G}_{d(\overline{\sigma})})^{\sum_{s \in \mathbb{Z}_{\geq 1}} s \mathcal{P}(\overline{\sigma})(s)}, \bigotimes_{\overline{\sigma} \in \mathrm{supp}\mathcal{P}} \overline{\sigma}^{\sum_{s \in \mathbb{Z}_{\geq 1}} s \mathcal{P}(\overline{\sigma})(s)}]_n$$

Let $\pi_{\mathcal{P}}$ denote the $\overline{G}_{d(\mathcal{P})}$-representation obtained by parabolic induction from $\mathrm{st}(\mathcal{P})$, i.e. $\pi_{\mathcal{P}} = \mathrm{Ind}_{\overline{P}}^{\overline{G}_{d(\mathcal{P})}}(\mathrm{st}(\mathcal{P}))$, where $\overline{P}$ is a standard parabolic with Levi subgroup $M_{\mathcal{P}}$. Let $\nu \in \mathcal{B}(\overline{G}_n)$ and consider the set of partition valued functions $\mathcal{P}$ such that $\nu(\mathcal{P}) = \nu$. Among them there is a unique minimal element $\mathcal{P}_{min}$ and there is a unique maximal element $\mathcal{P}_{max}$. As a consequence of [SZ99, Proposition, section 4], observe that:

1. For each $\mathcal{P}$ there exists a uniquely determined irreducible representation $\sigma_{\mathcal{P}}$ of $\overline{G}_{d(\mathcal{P})}$ which occurs with multiplicity one in $\pi_{\mathcal{P}}$, but does not occur in $\pi_{\mathcal{P}'}$, for $\mathcal{P}' < \mathcal{P}$. In particular $\sigma_{\mathcal{P}}$ has the cuspidal support $\nu(\sigma_{\mathcal{P}}) := \nu(\mathcal{P})$.

2. All irreducible representations with cuspidal support ν are of the form $\sigma_{\mathcal{P}}$, with $\nu(\sigma_{\mathcal{P}}) = \nu$, for some partition valued function $\mathcal{P}$.

3. $\sigma_{\mathcal{P}}$ occurs in $\pi_{\mathcal{P}'}$ if and only $\mathcal{P} \leq \mathcal{P}'$.

4. If $\sigma_{\mathcal{P}_{min}}$ occurs in $\pi_{\mathcal{P}}$, then $\sigma_{\mathcal{P}_{min}}$ has multiplicity one.

5. The definition of $\leq$ implies that

$$\pi_{\mathcal{P}_{max}} = \mathrm{Ind}_{\overline{P}}^{\overline{G}_n} \bigotimes_{\overline{\sigma} \in \mathrm{supp} \mathcal{P}_{max}} \overline{\sigma}^{\sum_{s \in \mathbb{Z}_{\geq 1}} s \mathcal{P}_{max}(\overline{\sigma})(s)},$$

hence $\pi_{\mathcal{P}_{max}}$ contains all irreducible representations with cuspidal support ν, and

$$\mathrm{Ind}_{\overline{P}}^{\overline{G}_n} \bigotimes_{\overline{\sigma} \in \mathrm{supp} \mathcal{P}_{max}} \overline{\sigma}^{\sum_{s \in \mathbb{Z}_{\geq 1}} s \mathcal{P}_{max}(\overline{\sigma})(s)} = \bigoplus_{\mathcal{P}: \nu(\mathcal{P}) = \nu} \sigma_{\mathcal{P}}^{\oplus m_{\mathcal{P}}}, \tag{2.1}$$

which is a finite field analogue of (1.1).

2.2 Representations of general linear groups over a non-archimedean local field

The Bernstein decomposition of the category of smooth representations allows us to restrict our attention to one Ω. From now on we will assume that $\Omega = [M, \rho]_G$ has been fixed, and all the smooth representations will be considered as objects in $\mathcal{R}^{\Omega}(G)$.

Let us recall some facts about smooth representations of $G = GL_n(F)$. We know that any irreducible representation π is a Langlands quotient of the form $L(\Delta_1, \ldots, \Delta_r)$ (cf. [Zel80, Theorem 6.1]) such that, for $i < j$, the segment Δ_i does not precede Δ_j (cf. [Zel80, section 4]).

According to [Zel80, Theorem 9.7] $\pi = L(\Delta_1, \ldots, \Delta_r)$ is generic if and only if no two segments Δ_i are linked. In which case $L(\Delta_1, \ldots, \Delta_r) = L(\Delta_1) \times \ldots \times L(\Delta_r)$ (notation (1.2)). In particular the essentially discrete series representations $L(\Delta)$ are generic.

We will now recall the definition of tempered types following the paper [SZ99]. First consider the case when $\Omega = [M, \rho]_G$ is simple and correspondingly (J, λ) is a simple type. Let $M_n(F)$ be all the $n \times n$ matrices with coefficients in F, and let $E = F[\beta]$ be a finite field extension of F of degree dividing n, hence $E \hookrightarrow M_n(F)$. Define $R = n/[E : F]$. The type (J, λ) has the following form: J is a compact open subgroup in G and $\lambda = \kappa \otimes \sigma$ with κ a β-extension and σ the inflation of $\tau \otimes \ldots \otimes \tau$ (e-times), where τ a cuspidal representation of $GL_f(k_E)$, and we have $R = ef$.

Let B be the centralizer of β in $M_n(F)$. Fix a pair $(\mathfrak{B}_{min}, \mathfrak{B}_{max})$ of hereditary $\mathcal{O}_E$-orders in B, such that $\mathfrak{B}_{min} \subseteq \mathfrak{B}_{max}$, where $\mathfrak{B}_{min}$ is minimal and $\mathfrak{B}_{max}$ is maximal. With every hereditary $\mathcal{O}_E$ order $\mathfrak{B}$ ($\mathfrak{B}_{min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{max}$) Bushnell and Kutzko associate via [BK93, (3.1.14)] a compact open subgroup $J(\mathfrak{B})$ of

$\mathfrak{A}(\mathfrak{B})^\times$, where $\mathfrak{A}(\mathfrak{B})$ is the unique hereditary $\mathcal{O}_F$ order in $M_n(F)$ such that $E^\times$ normalizes $\mathfrak{A}(\mathfrak{B})$ and $\mathfrak{A}(\mathfrak{B}) \cap B = \mathfrak{B}$ and they also associate via [BK93, (5.2)] an irreducible representation $\kappa(\mathfrak{B})$ of the group $J(\mathfrak{B})$. Let $J_{max} := J(\mathfrak{B}_{max})$ and $\kappa_{max} = \kappa(\mathfrak{B}_{max})$. Let $\mathrm{rad}(\mathfrak{A}(\mathfrak{B}))$ be the Jacobson radical of $\mathfrak{A}(\mathfrak{B})$ and let $U^1(\mathfrak{A}(\mathfrak{B})) := 1 + \mathrm{rad}(\mathfrak{A}(\mathfrak{B}))$.
Define $J^1_{max} := U^1(\mathfrak{A}(\mathfrak{B})) \cap J_{max}$, then $J_{max}/J^1_{max} \simeq GL_R(k_E)$, and the functor

$$\begin{array}{rcccc} \mathrm{Ind}^K_{J_{max}}(\kappa_{max} \otimes \cdot) & : & \mathcal{R}^\nu(GL_R(k_E)) & \to & \mathcal{R}_\lambda(K) \\ & & \sigma & \mapsto & \mathrm{Ind}^K_{J_{max}}(\kappa_{max} \otimes \sigma) \end{array},$$

is an equivalence of categories according the discussion above Proposition 11 in Section 5 [SZ99], where $\mathcal{R}^\nu(GL_R(k_E))$ the full subcategory of all $GL_R(k_E)$-representations whose irreducible constituents all have cuspidal support $\nu = (\tau, \dots, \tau)$, e-times as above.
Then in the simple type case define $\sigma_{\mathcal{P}}(\lambda) := \mathrm{Ind}^K_{J_{max}}(\kappa_{max} \otimes \sigma_{\mathcal{P}})$ and $\pi_{\mathcal{P}}(\lambda) := \mathrm{Ind}^K_{J_{max}}(\kappa_{max} \otimes \pi_{\mathcal{P}})$, where $\sigma_{\mathcal{P}}$ and $\pi_{\mathcal{P}}$ have been defined in section 2.1. Since we are in the simple type case, the partition valued function $\mathcal{P}$, which is supported on $\tau \in \overline{\mathcal{C}}$, is naturally identified with a partition of e.
Let now V be a representation in the Bernstein Ω, then $\mathrm{Hom}_{J^1_{max}}(\kappa_{max}, V)$ is naturally a J_{max}/J^1_{max}-module. This observation allows us to define the following functor:

$$\begin{array}{rcc} \mathcal{R}^\Omega(G) = \mathcal{R}_\lambda(G) & \to & \mathcal{R}^\nu(GL_R(k_E)) \\ V & \mapsto & V(\kappa_{max}) := \mathrm{Hom}_{J^1_{max}}(\kappa_{max}, V) \end{array}.$$

More generally in the semisimple type case we have similar definitions. Let (M, ρ) be a representative of Ω. Without loss of generality we may write $M = \prod_{i=1}^s GL_{r_i}(F)^{\times e_i}$ and $\rho = \rho_1 \otimes \dots \otimes \rho_s$, with supersupidals $\rho_i = \omega_i \otimes \dots \otimes \omega_i$ (e_i times), where ω_i is a supercuspidal representation of $GL_{r_i}(F)$, and $\sum_{i=1}^s r_i e_i = n$. Let $\tilde{M} = \prod_{i=1}^s GL_{e_i r_i}(F)$. Let $\Omega_i = [GL_{r_i}(F)^{\times e_i}, \rho_i]_{GL_{e_i r_i}(F)}$. The component Ω_i in the category $\mathcal{R}(GL_{e_i r_i}(F))$ has a simple type $(J^{(i)}, \lambda^{(i)})$. Following the discussion at the beginning of Section 6 in [SZ99], we may choose a pair $(J_0^{(i)}, \lambda_0^{(i)})$, satisfying $J_0^{(i)} \subseteq J^{(i)}$ and $\mathrm{Ind}^{J^{(i)}}_{J_0^{(i)}} \lambda_0^{(i)} = \lambda^{(i)}$, in such a way that $(\prod_{i=1}^s J_0^{(i)}, \bigotimes_{i=1}^s \lambda_0^{(i)})$ is a type of some Bernstein component in $\mathcal{R}(\tilde{M})$. Let (J, λ) be a G-cover of $(\prod_{i=1}^s J_0^{(i)}, \bigotimes_{i=1}^s \lambda_0^{(i)})$. Then (J, λ) is the corresponding Bushnell–Kutzko type to Ω. Let K be a maximal compact subgroup of G containing J. We have the following exact functor:

$$\begin{array}{rcccc} T_{K,\lambda} & : & \mathcal{R}^\Omega(G) = \mathcal{R}_\lambda(G) & \to & \mathcal{R}_\lambda(K) \\ & & \pi & \mapsto & K \cdot \pi^\lambda \end{array}.$$

Let $\mathcal{P}$ be a partition valued function, and define $\mathcal{P}_i$ another partition valued function supported on the equivalence class of ω_i such that $\mathcal{P}_i(\omega_i) = \mathcal{P}(\omega_i)$.
For any parabolic $\tilde{Q}$ with Levi factor $\tilde{M}$ the parabolic induction functor $i^G_{\tilde{Q}} : \mathcal{R}^{\Omega_1 \times \dots \Omega_s}(\tilde{M}) \to \mathcal{R}^\Omega(G)$ is an equivalence of categories and it induces

the equivalence of categories $\mathscr{I}_{\tilde{Q}} : \mathcal{R}_{\bigotimes_{i=1}^{s} \lambda_0^{(i)}}(K \cap \tilde{M}) \to \mathcal{R}_\lambda(K)$. Via the equivalence of categories $\mathscr{I}_{\tilde{Q}}$, define $\sigma_{\mathcal{P}}(\lambda) := \mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^{s} \sigma_{\mathcal{P}_i}(\lambda^{(i)}))$ and $\pi_{\mathcal{P}}(\lambda) := \mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^{s} \pi_{\mathcal{P}_i}(\lambda^{(i)}))$.

By construction we have the decomposition $\tilde{M} \cap K = \prod_{i=1}^{s} K^{(i)}$, with $K^{(i)}$ maximal compact subgroup of $GL_{e_i r_i}(F)$. There is the obvious product functor $\prod_{i=1}^{s} \mathcal{R}_{\lambda_0^{(i)}}(K^{(i)}) \xrightarrow{\otimes \ldots \otimes} \mathcal{R}_{\bigotimes_{i=1}^{s} \lambda_0^{(i)}}(K \cap \tilde{M})$. The functor $\mathcal{R}_{\bigotimes_{i=1}^{s} \lambda_0^{(i)}}(K \cap \tilde{M}) \xrightarrow{\mathscr{I}_{\tilde{Q}}} \mathcal{R}_\lambda(K)$, induces an isomorphism

$$\mathrm{Hom}_K(\sigma_{\mathcal{P}'}(\lambda), \pi_{\mathcal{P}}(\lambda)) \cong \bigotimes_{i=1}^{s} \mathrm{Hom}_{K^{(i)}}(\sigma_{\mathcal{P}_i'}(\lambda_0^{(i)}), \pi_{\mathcal{P}_i}(\lambda_0^{(i)})),$$

where the partition $\mathcal{P}_i$, $\mathcal{P}_i'$ refers to the support $\omega_i \in \mathcal{C}$. In particular we see here that $\sigma_{\mathcal{P}'}(\lambda)$ occurs in $\pi_{\mathcal{P}}(\lambda)$ with multiplicity one if and only if this is true with respect to all the $\mathcal{P}_i$, $\mathcal{P}_i'$.

With $\pi = L(\Delta_1, \ldots, \Delta_r)$ we can associate the partition valued function $\mathcal{P}_\pi = \mathcal{P}(\Delta_1, \ldots, \Delta_r)$, such that $\mathcal{P}(\omega_i)$ is the partition of e_i collecting the lengths of all the segments Δ_j which are contained in the inertial class of ω_i.

Lemma 2.1. *Let K be a maximal compact subgroup of G containing J, then $T_{K,\lambda}(L(\Delta_1) \times \ldots \times L(\Delta_r)) \cong \pi_{\mathcal{P}(\Delta_1, \ldots, \Delta_r)}(\lambda)$.*

Proof. We will begin by giving a proof in the simple type case. It follows from [SZ99, Proposition 5.9], that $(L(\Delta_1) \times \ldots \times L(\Delta_r))(\kappa_{max}) \cong \pi_{\mathcal{P}(\Delta_1, \ldots, \Delta_r)}$, where $\pi_{\mathcal{P}(\Delta_1, \ldots, \Delta_r)}$ is an object of $\mathcal{R}^\nu(GL_R(k_E))$. By computation [SZ99, p. 185], we have $\mathrm{Ind}_{J_{max}}^{K}(\kappa_{max} \otimes V(\kappa_{max})) = T_{K,\lambda}(V)$, then $\pi_{\mathcal{P}(\Delta_1, \ldots, \Delta_r)}(\lambda) = \mathrm{Ind}_{J_{max}}^{K}(\kappa_{max} \otimes \pi_{\mathcal{P}(\Delta_1, \ldots, \Delta_r)}) \cong \mathrm{Ind}_{J_{max}}^{K}(\kappa_{max} \otimes (L(\Delta_1) \times \ldots \times L(\Delta_r))(\kappa_{max})) = T_{K,\lambda}(L(\Delta_1) \times \ldots \times L(\Delta_r))$.

For the semisimple type case use previous computation and [SZ99, Proposition 6.1]. We may always group together the segments that have the same supercuspidal representation ω_i. Consider the representation $\pi = L(\Delta_1) \times \ldots \times L(\Delta_r)$ and write it as $\pi = L(\Delta_{1,1}) \times \ldots \times L(\Delta_{1,r_1}) \times \ldots \times L(\Delta_{s,1}) \times \ldots \times L(\Delta_{s,r_s})$, where $\Delta_{i,j} = (\omega_i \otimes \chi_{i,j}) \otimes \ldots \otimes (\omega_i \otimes \chi_{i,j} \otimes |\det|^{k_{i,j}-1})$ $(1 \leq j \leq r_i)$ are the segments, $\chi_{i,j}$ are some unramified characters and $k_{i,j}$ are positive integers, and ω_i $(1 \leq i \leq s)$ are pairwise non-isomorphic supercuspidal representations. Let $\pi_i := L(\Delta_{i,1}) \times \ldots \times L(\Delta_{i,r_i})$, then $\pi = \pi_1 \times \ldots \times \pi_s$. Recall that we have the decomposition $\tilde{M} \cap K = \prod_{i=1}^{s} K^{(i)}$. Then by [SZ99, Proposition 6.1], we have $T_{K,\lambda}(\pi) = \mathscr{I}_{\tilde{Q}}(T_{K \cap \tilde{M}, \bigotimes_{i=1}^{s} \lambda_0^{(i)}}(\pi_1 \otimes \ldots \otimes \pi_s)) = \mathscr{I}_{\tilde{Q}}(T_{K^{(1)}, \lambda_0^{(1)}}(\pi_1) \otimes \ldots \otimes T_{K^{(s)}, \lambda_0^{(s)}}(\pi_s)) \cong \mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^{s} \pi_{\mathcal{P}_i}(\lambda^{(i)})) = \pi_{\mathcal{P}_\pi}(\lambda)$, where $\mathcal{P}_\pi = \mathcal{P}(\Delta_{1,1}, \ldots, \Delta_{1,r_1}, \ldots, \Delta_{s,1}, \ldots, \Delta_{s,r_s}) = \mathcal{P}(\Delta_1, \ldots, \Delta_r)$. □

The following commutative diagram follows from [SZ99, Proposition 6.1]:

$$\begin{array}{ccccc} \prod_{i=1}^s \mathcal{R}^{\Omega_i}(GL_{e_i r_i}(F)) & \xrightarrow{\otimes\ldots\otimes} & \mathcal{R}^{\Omega_1\times\ldots\Omega_s}(\tilde{M}) & \xrightarrow{i_{\tilde{Q}}^G} & \mathcal{R}^{\Omega}(G) \\ \downarrow{\scriptstyle \prod_{i=1}^s T_{K^{(i)},\lambda_0^{(i)}}} & & \downarrow{\scriptstyle T_{K\cap\tilde{M},\otimes_{i=1}^s \lambda_0^{(i)}}} & & \downarrow{\scriptstyle T_{K,\lambda}} \\ \prod_{i=1}^s \mathcal{R}_{\lambda_0^{(i)}}(K^{(i)}) & \xrightarrow[\otimes\ldots\otimes]{} & \mathcal{R}_{\bigotimes_{i=1}^s \lambda_0^{(i)}}(K\cap\tilde{M}) & \xrightarrow{\mathscr{J}_{\tilde{Q}}} & \mathcal{R}_\lambda(K), \end{array} \qquad \text{(D)}$$

where $i_{\tilde{Q}}^G$ and $\mathscr{J}_{\tilde{Q}}$ are equivalences of categories.

Lemma 2.2. *$\sigma_{\mathcal{P}'}(\lambda)$ occurs in $\pi_{\mathcal{P}}(\lambda)$ if and only if $\mathcal{P}' \leq \mathcal{P}$, and $\sigma_{\mathcal{P}}(\lambda)$ occurs in $\pi_{\mathcal{P}}(\lambda)$ always with multiplicity one. Moreover if $\sigma_{min}(\lambda)$ occurs in $\pi_{\mathcal{P}}(\lambda)$, then it always occurs with multiplicity one.*

Proof. The first assertion follows from [Sho18, Corollary 6.22, Corollary 6.10] and [SZ99, Proposition, section 4]. Indeed, [Sho18, Corollary 6.22, Corollary 6.10] show that $\dim \mathrm{Hom}_K(\sigma_{\mathcal{P}'}(\lambda), \pi_{\mathcal{P}}(\lambda))$ is equal to the multiplicity of $\sigma_{\mathcal{P}'}$ in $\pi_{\mathcal{P}}$. Then by [SZ99, Proposition, section 4], $\dim \mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi_{\mathcal{P}}(\lambda)) = 1$, and $\sigma_{\mathcal{P}'}(\lambda)$ occurs in $\pi_{\mathcal{P}}(\lambda)$ if and only if $\mathcal{P}' \leq \mathcal{P}$. Furthermore, these results allow us to compute $\dim \mathrm{Hom}_K(\sigma_{\mathcal{P}'}(\lambda), \pi_{\mathcal{P}}(\lambda))$ as a product of the usual Kostka numbers $K_{\lambda\mu}$, where λ and μ are two partitions. For the definition of Kostka numbers $K_{\lambda\mu}$, we refer the reader to [Mac15, (6.4)] and [Sho18, Definition 6.2] with a comment on Kostka numbers below it. In particular, we have:

$$\dim \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi_{\mathcal{P}}(\lambda)) = \prod_{\omega\in\mathrm{supp}\mathcal{P}} K_{\mathcal{P}_{min}(\omega)\mathcal{P}(\omega)} = \prod_{\omega\in\mathrm{supp}\mathcal{P}_{min}} K_{\mathcal{P}_{min}(\omega)\mathcal{P}(\omega)}.$$

Therefore, in order to prove multiplicity one statement about $\sigma_{min}(\lambda)$, it is enough to convince the reader that $K_{(n)(m_1,\ldots,m_k)} = 1$, where $m_1+\ldots+m_k = n$. But, $K_{(n)(m_1,\ldots,m_k)}$ is by definition the number of ways to fill n boxes displayed in one rows with m_i copies of the integer i in the increasing order, such that each integer goes only into one box. □

Lemma 2.3. *We have $\pi_{\mathcal{P}_{max}}(\lambda) \cong \mathrm{Ind}_J^K \lambda$. In particular the multiplicity of $\sigma_{\mathcal{P}}(\lambda)$ in $\mathrm{Ind}_J^K \lambda$ can be computed in terms of Kostka numbers, i.e.* $\dim \mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \mathrm{Ind}_J^K \lambda) = \prod_{\omega\in\mathrm{supp}\mathcal{P}} K_{\mathcal{P}(\omega)\mathcal{P}_{max}(\omega)}$.

Proof. In the Iwahori case, we have $\mathrm{Ind}_I^K 1 = (\text{inflation of } \mathrm{Ind}_{\overline{B}}^{\overline{G}_n} 1) = \pi_{\mathcal{P}_{max}}(1)$, where $\overline{B}$ is a Borel subgroup of $\overline{G}_n$.

Now consider the simple type case. Let $\overline{P}$ be a parabolic subgroup of $GL_R(k_E)$, such that $\pi_{\mathcal{P}_{max}} = \mathrm{Ind}_{\overline{P}}^{GL_R(k_E)} \tau\otimes\ldots\otimes\tau$. First observe that the inflation of $\mathrm{Ind}_{\overline{P}}^{GL_R(k_E)} \tau\otimes\ldots\otimes\tau$ is $\mathrm{Ind}_J^{J_{max}}\sigma$. Recall that $\lambda = \kappa\otimes\sigma$, then we have $\mathrm{Ind}_J^K(\kappa\otimes\sigma) = \mathrm{Ind}_{J_{max}}^K \mathrm{Ind}_J^{J_{max}}(\kappa\otimes\sigma) = \mathrm{Ind}_{J_{max}}^K \mathrm{Ind}_J^{J_{max}}(\kappa_{max}|J\otimes\sigma) = \mathrm{Ind}_{J_{max}}^K(\kappa_{max}\otimes$

$(\mathrm{Ind}_J^{J_{max}}\sigma)) = \mathrm{Ind}_{J_{max}}^K(\kappa_{max} \otimes (\mathrm{Ind}_{P}^{GL_R(k_E)}\tau \otimes \ldots \otimes \tau)) = \pi_{\mathcal{P}_{max}}(\lambda)$, where in the third equality we have used the projection formula.
Finally in the semisimple type case we have the decomposition $\tilde{M} \cap K = \prod_{i=1}^s K^{(i)}$. It follows that: $\pi_{\mathcal{P}_{max}}(\lambda) = \mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^s \pi_{\mathcal{P}_{max,i}}(\lambda^{(i)})) = \mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^s \mathrm{Ind}_{J^{(i)}}^{K^{(i)}} \lambda^{(i)}) = \mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^s \mathrm{Ind}_{J^{(i)}}^{K^{(i)}} \mathrm{Ind}_{J_0^{(i)}}^{J^{(i)}} \lambda_0^{(i)}) =$
$\mathscr{I}_{\tilde{Q}}(\bigotimes_{i=1}^s \mathrm{Ind}_{J_0^{(i)}}^{K^{(i)}} \lambda_0^{(i)}) \cong \mathscr{I}_{\tilde{Q}}(\mathrm{Ind}_{\prod_{i=1}^s J_0^{(i)}}^{\tilde{M}\cap K} \bigotimes_{i=1}^s \lambda_0^{(i)}) = \mathrm{Ind}_J^K \lambda.$
The assertion about multiplicities follows from the proof of previous Lemma. □

3 Generic representations

In this section we will use the results proven above to deduce our main theorem. Let (M,ρ) be a representative of Ω, where $M = \prod_{i=1}^s GL_{r_i}(F)^{\times e_i}$ and $\rho = \rho_1 \otimes \ldots \otimes \rho_s$, with supersupidals $\rho_i = \omega_i \otimes \ldots \otimes \omega_i$ (e_i times), where ω_i is a supercuspidal representation of $GL_{r_i}(F)$, and $\sum_{i=1}^s r_i e_i = n$.

Theorem 3.1. *Let $\pi = L(\Delta_1, \ldots, \Delta_r)$ be an irreducible representation of G belonging to the Bernstein component Ω and let (J,λ) be the corresponding Bushnell–Kutzko type. Then the following is equivalent:*

1. *π is generic.*
2. *π contains the tempered types $\sigma_{\mathcal{P}'}(\lambda)$ for all $\mathcal{P}' \leq \mathcal{P} = \mathcal{P}(\Delta_1, \ldots, \Delta_r) = \mathcal{P}_\pi$.*
3. *π contains the minimal type $\sigma_{min}(\lambda)$.*

And if the equivalent conditions are fulfilled then $\sigma_{min}(\lambda)$ will occur with multiplicity 1.

Proof. Assume that π is generic. Then we have $\pi = L(\Delta_1, \ldots, \Delta_r) = L(\Delta_1) \times \ldots \times L(\Delta_r)$ and therefore from Lemma 2.1 we see:

$$K \cdot \pi^\lambda = T_{K,\lambda}(L(\Delta_1) \times \ldots \times L(\Delta_r)) \cong \pi_{\mathcal{P}}(\lambda)$$

and according to Lemma 2.2 this contains all the $\sigma_{\mathcal{P}'}(\lambda)$ for all $\mathcal{P}' \leq \mathcal{P}$.
Conversely consider any irreducible $\pi = L(\Delta_1, \ldots, \Delta_r)$ where $\Omega = [\prod_{i=1}^s GL_{r_i}(F)^{\times e_i}, \rho_1 \otimes \ldots \otimes \rho_s]_G$, and $\rho_i = \omega_i \otimes \ldots \otimes \omega_i$ (e_i times). The cuspidal support of π consists of e_i representations ω_{ij}, $1 \leq j \leq e_i$, which are members of the inertial class of ω_i, where $1 \leq i \leq s$. Then π is an irreducible subquotient of $\omega_{11} \times \ldots \times \omega_{1e_1} \times \ldots \times \omega_{s1} \times \ldots \times \omega_{se_s}$, where each supercuspidal ω_{ij} is considered as segment of length 1. Now we apply, the exact functor $T_{K,\lambda}$. Then we see that $T_{K,\lambda}(\pi)$ occurs in $T_{K,\lambda}(\omega_{11} \times \ldots \times \omega_{1e_1} \times \ldots \times \omega_{s1} \times \ldots \times \omega_{se_s}) \cong \pi_{\mathcal{P}_{max}}(\lambda)$, where the last equality follows from Lemma 2.1. But the irreducible representation $\sigma_{min}(\lambda)$ occurs here with multiplicity 1, and we know already that it is contained in the generic subquotient. Therefore if $\sigma_{min}(\lambda)$ occurs in $T_{K,\lambda}(\pi)$ then π must be generic. □

The following lemma has been shown already in the Theorem 3.1, we only keep it as a reference for [Pyv18]:

Lemma 3.2. *We have* $\dim \operatorname{Hom}_K(\sigma_{min}(\lambda), \pi) = 1$*, for* π *an irreducible generic representation of* G *in* Ω*.*

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