The Weak Tracial Rokhlin Property for Finite Group Actions on Simple C*-Algebras

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Abstract. We develop the concept of weak tracial Rokhlin property for finite group actions on simple (not necessarily unital) C*-algebras and study its properties systematically. In particular, we show that this property is stable under restriction to invariant hereditary C*-algebras, minimal tensor products, and direct limits of actions. Some of these results are new even in the unital case and answer open questions asked by N. C. Phillips in full generality. We present several examples of finite group actions with the weak tracial Rokhlin property on simple stably projectionless C*-algebras. We prove that if $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite group G on a simple C*-algebra A with tracial rank zero and α has the weak tracial Rokhlin property, then the crossed product $A \rtimes_\alpha G$ and the fixed point algebra A^α are simple with tracial rank zero. This extends a result of N. C. Phillips to the nonunital case. We use the machinery of Cuntz subequivalence to work in this nonunital setting.

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1 Introduction

The Rokhlin property for actions on C*-algebras appeared in [20, 8, 31, 32]. Izumi gave a modern definition of the Rokhlin property for finite group actions on unital C*-algebras [23, 24]. This property is useful to understand the structure of the crossed product of C*-algebras and properties passing from the underlying algebra to the crossed product [40]. However, actions with the Rokhlin property are rare and many C*-algebras admit no finite group actions with the Rokhlin property. Indeed, the Rokhlin property imposes severe K-theoretical obstructions on C*-algebras. Phillips introduced the tracial Rokhlin property for finite group actions on simple unital C*-algebras [43] with the purpose of proving that every simple higher dimensional noncommutative torus is an AT algebra [41], and proving that certain crossed products of such algebras by finite cyclic groups are AF algebras [13] (see [39] for $\mathbb{Z}$ actions with this property). The tracial Rokhlin property is generic in many cases (see [44] and [51, Chapter 4]), and also can be used to study properties passing from the underlying algebra to the crossed product [43, 13, 3].

Weak versions of the tracial Rokhlin property in which one uses orthogonal positive contractions instead of orthogonal projections were studied for actions on simple unital C*-algebras with few projections [44, 36, 21, 51, 16, 52] (see Definition 3.7). As an example, the flip action on the Jiang-Su algebra $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ has the weak tracial Rokhlin property but it does not have the tracial Rokhlin property [21].

The Rokhlin property was extended to the case of actions on nonunital C*-algebras [37, 49, 17], and there are actions with the Rokhlin property on stably projectionless C*-algebras, in particular on the Razak-Jacelon algebra $\mathcal{W}$ [37]. However, there has been no work on extending the (weak) tracial Rokhlin property to the simple nonunital case. (As far as we know, a suitable definition of the tracial Rokhlin property for actions on nonsimple C*-algebras is not known.) Moreover, actions on simple nonunital C*-algebras naturally appear, for instance, the restriction of an action on a simple unital C*-algebra to an invariant nonunital hereditary subalgebra (see Proposition 4.2). Also, there are many examples of finite group actions on simple nonunital C*-algebras without the Rokhlin property, which have the weak tracial Rokhlin property (see Example 3.12). In fact, the problem of finding the right definition of the tracial Rokhlin property for actions on simple nonunital C*-algebras was asked by Phillips [42]. This motivated us to investigate the weak tracial Rokhlin property for finite group actions on simple C*-algebras. We give the following definition:

Definition 1.1. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C*-algebra A. We say that α has the *weak tracial Rokhlin property* if for every $\varepsilon > 0$, every finite subset $F \subseteq A$, and all positive elements $x, y \in A$ with $\|x\| = 1$, there exists a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:

1. $\|f_g a - a f_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - yfy - \varepsilon)_+ \precsim_A x$;
4. $\|fxf\| > 1 - \varepsilon$.

We say that α has the *tracial Rokhlin property* if we can arrange $(f_g)_{g\in G}$ above to be mutually orthogonal projections.

It turns out that our definition of the tracial Rokhlin property extends Phillips's definition of the tracial Rokhlin property [43, Definition 1.2] to the nonunital simple case. We recall that in Phillips's definition of the tracial Rokhlin property for finite group actions on simple unital C*-algebras, Condition (3) is formulated as follows:

(3)′ $1 - f \precsim_A x$ (or equivalently, $1 - f$ is Murray von Neumann equivalent to a projection in the hereditary subalgebra generated by x),

where $(f_g)_{g\in G}$ is a family of orthogonal projections in A. Phillips in [42] asked for a correct analogue of Condition (3)′ in the simple nonunital case. Condition (3) in Definition 1.1 contains our main idea for a suitable notion of the weak tracial Rokhlin property (as well as tracial Rokhlin property) in the nonunital case. This condition—which may seem strange at the first glance—says that $1 - f$ is small with respect to the Cuntz subequivalence relation. The rationale behind this condition is that since $y \in A_+$ is arbitrary, we can take it to be arbitrarily large (that is, close to 1) and so $y^2 - yfy = y(1 - f)y$ is close to $1 - f$. The ε gap in this condition is a technical condition needed, for example, when applying a key lemma in the Cuntz semigroup (Lemma 2.2).
The following result can be considered as a generalization of [46, Theorem 1.9] to the nonunital case (see Theorem 3.11).

Theorem 1.2. *Let A be a simple C*-algebra with tracial rank zero, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. If α has the weak tracial Rokhlin property then it has the tracial Rokhlin property.*

Phillips in [42, Problem 3.2] asked whether there is a reasonable formulation of the tracial Rokhlin property for finite group actions on simple unital C*-algebras in terms of the central sequence algebra. We give an answer to this question in the *not necessarily unital* simple case. Indeed, it turns out that if moreover A is separable and one works with the central sequence algebra A_∞, then Condition (3) can be replaced by $y^2 - yfy \precsim_{A_\infty} x$ (see Proposition 3.10). We prove that an action with the weak tracial Rokhlin property is pointwise outer (Proposition 3.2), and hence the resulting crossed product is simple. Moreover, we prove several permanence properties for finite group actions with the weak tracial Rokhlin property on simple C*-algebras, for example, passing to restriction to invariant hereditary C*-algebras, minimal tensor products, and direct limits of actions. In particular, the following result concerning tensor products gives an affirmative answer to a question of Phillips [42, Problem 3.18] (see Theorems 4.5 and 4.6).

Theorem 1.3. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of a finite group G on simple C*-algebras A and B. If α has the weak tracial Rokhlin property then so does $\alpha \otimes \beta\colon G \to \mathrm{Aut}(A \otimes_{\min} B)$. If α has the tracial Rokhlin* *property then so does $\alpha\otimes\beta\colon G\to \mathrm{Aut}(A\otimes_{\min} B)$ whenever B^{β} has an approximate identity (not necessarily increasing) consisting of projections.*

Phillips proved that the crossed product of a simple unital C*-algebras with tracial rank zero by a finite group action with the tracial Rokhlin property, is again simple with tracial rank zero [43]. The following theorem generalizes this result to the nonunital case (see Theorem 5.2).

Theorem 1.4. *Let A be a simple C*-algebra with tracial rank zero and let α be an action of a finite group G on A with the weak tracial Rokhlin property. Then the crossed product $A\rtimes_{\alpha} G$ and the fixed point algebra A^{α} are simple C*-algebras with tracial rank zero.*

The preservation of some other classes of simple C*-algebras under taking crossed products by finite group actions with the (weak) tracial Rokhlin property is given in Section 5 (and in [1] and [19]).

To prove Theorem 1.4, we need to work with simple nonunital C*-algebras with tracial rank zero. Recall that Lin in [34] first gave the definition of tracial rank for unital C*-algebras and then he defined the tracial rank of a nonunital C*-algebra to be the tracial rank of its minimal unitization. However, working with the unitization of C*-algebras is not always convenient. Moreover, the unitization of a simple nonunital C*-algebra is not simple and so one can not use techniques which are applicable only to simple C*-algebras. To deal with this difficulty, we develop an approach which unifies the concept of tracial rank zero for both unital and nonunital simple C*-algebras; see Theorem A.6. This approach helps us to study crossed products of simple nonunital C*-algebras with tracial rank zero by finite group actions with the weak tracial Rokhlin property. We also need some results about simple nonunital C*-algebras with tracial rank zero, such as Morita invariance and having real rank zero and stable rank one (the last two results in the simple *unital* case are proved in [34]). We did not find any reference proving these results (in the nonunital case), however, they may be known to some researchers. So we prove them in Appendix A.

2 Cuntz subequivalence

In this section, we recall some results on Cuntz subequivalence and provide some lemmas which will be used in the subsequent sections. We refer the reader to [2] and [45] for more information about Cuntz subequivalence.

Notation 2.1. We use the following notation in this paper.

1. For a C*-algebra A, A_{+} denotes the positive cone of A. Also, A^{+} denotes the forced unitization of A (adding a new identity even if A is unital), while $A^{\sim}=A$ if A is unital and $A^{\sim}=A^{+}$ if A is nonunital.

2. If p and q are projections in a C*-algebra A, then we write $p\sim_{\mathrm{MvN}} q$ if p is Murray-von Neumann equivalent to q.

3. If E and F are subsets of a C*-algebra A and $\varepsilon > 0$, then we write $E \subseteq_\varepsilon F$ if for every $a \in E$ there is $b \in F$ such that $\|a - b\| < \varepsilon$.

4. We write $\mathcal{K} = K(\ell^2)$ and $M_n = M_n(\mathbb{C})$.

5. Let A be a C*-algebra. For $a, b \in A_+$, we say that a is *Cuntz subequivalent* to b in A and we write $a \precsim_A b$, if there is a sequence $(v_n)_{n\in\mathbb{N}}$ in A such that $\|a - v_n b v_n^*\| \to 0$. We write $a \sim_A b$ if both $a \precsim_A b$ and $b \precsim_A a$. If $a, b \in (A \otimes \mathcal{K})_+$, we write $a \precsim_A b$ if a is Cuntz subequivalent to b in $A \otimes \mathcal{K}$. $[a]$ stands for the Cuntz equivalence class of a.

6. Let a be a positive element in a C*-algebra A and let $\varepsilon > 0$. Let $f_\varepsilon \colon [0, \infty) \to [0, \infty)$ be defined by $f_\varepsilon = 0$ on $[0, \varepsilon]$ and $f_\varepsilon(\lambda) = \lambda - \varepsilon$ on (ε, ∞). We denote $(a - \varepsilon)_+ = f_\varepsilon(a)$.

7. We use the notation $\mathbb{Z}_n$ for the group $\mathbb{Z}/n\mathbb{Z}$. Moreover, $\mathbb{N}$ denotes the set of natural numbers not including zero.

The following key lemma will be used several times throughout the paper.

Lemma 2.2 ([29], Lemma 2.2). *Let A be a C*-algebra, let $a, b \in A_+$, and let $\varepsilon > 0$. If $\|a-b\| < \varepsilon$ then there is a contraction $d \in A$ such that $(a-\varepsilon)_+ = dbd^*$. In particular, $(a - \varepsilon)_+ \precsim_A b$.*

In the preceding lemma, if instead of $\|a - b\| < \varepsilon$ we assume that $\|a - b\| \leq \varepsilon$, then again we get $(a - \varepsilon)_+ \precsim_A b$. In fact, for any $\delta > 0$ we have $\|a - b\| < \varepsilon + \delta$ and so $(a - \varepsilon - \delta)_+ \precsim_A b$. Letting $\delta \to 0$, we get $(a - \varepsilon)_+ \precsim_A b$.
We need the following lemma to work with Condition (3) in Definition 3.1.

Lemma 2.3 ([1]). *Let A be a C*-algebra, let $x \in A$ be a nonzero element, and let $b \in A_+$. Then for any $\varepsilon > 0$,*

$$(xbx^* - \varepsilon)_+ \precsim_A x\left(b - \varepsilon/\|x\|^2\right)_+ x^*.$$

In particular, if $\|x\| \leq 1$ then $(xbx^* - \varepsilon)_+ \precsim_A x(b - \varepsilon)_+ x^* \precsim_A (b - \varepsilon)_+$.

Proof. We have

$$\|xbx^* - x\left(b - \varepsilon/\|x\|^2\right)_+ x^*\| \leq \|x\|^2 \|b - \left(b - \varepsilon/\|x\|^2\right)_+\| \leq \|x\|^2 \frac{\varepsilon}{\|x\|^2} = \varepsilon.$$

Using the remark following Lemma 2.2, we get

$$(xbx^* - \varepsilon)_+ \precsim_A x\left(b - \varepsilon/\|x\|^2\right)_+ x^*.$$

If $\|x\| \leq 1$ then $\frac{\varepsilon}{\|x\|^2} \geq \varepsilon$ and so $(xbx^* - \varepsilon)_+ \precsim_A x(b - \varepsilon)_+ x^* \precsim_A (b - \varepsilon)_+$. □

The following lemma is well-know. (It follows from [29, Lemmas 2.2 and 2.4(i)].)

Lemma 2.4. *Let A be a C^*-algebra, let $a, b \in A_+$, and let $\delta > 0$. If $a \precsim_A (b-\delta)_+$ then there exists a bounded sequence (v_n) in A such that $\|a - v_n b v_n^*\| \to 0$. We can take this sequence such that $\|v_n\| \leq \|a\|^{\frac{1}{2}} \delta^{-\frac{1}{2}}$ for every $n \in \mathbb{N}$.*

The following lemma is known and follows from [2, Lemmas 2.18 and 2.19].

Lemma 2.5. *Let A be a C^*-algebra, let $a \in A_+$, and let $p \in A$ be a projection. The following statements are equivalent:*

1. *$p \precsim_A a$;*
2. *there exists $v \in A$ such that $p = vav^*$;*
3. *$p \sim_{\mathrm{MvN}} q$ in A for some projection q in $\overline{aAa}$.*

Note that the statements in Lemma 2.5 are also equivalent to $[p] \leq [a]$ in the sense of [34, Definition 2.2] (see Appendix A).
In general, there is no upper bound for the norm of v in the previous lemma, unless there is a gap between p and a; see the following lemma (which may be considered as a special case of [29, Lemma 2.4]).

Lemma 2.6. *Let A be a C^*-algebra, let $a \in A_+$, let $\varepsilon > 0$, and let $p \in A$ be a projection. If $p \precsim_A (a - \varepsilon)_+$, then there exists $v \in A$ such that $p = vav^*$ and $\|v\| \leq \varepsilon^{-\frac{1}{2}}$.*

Proof. By Lemma 2.5, there exists $w \in A$ such that $p = w(a - \varepsilon)_+ w^*$. Then [29, Lemma 2.4(i)] implies that there is $v \in A$ such that $p = vav^*$ and $\|v\| \leq \varepsilon^{-\frac{1}{2}}$. □

Part (1) of the following lemma is a variant of [29, Lemma 2.4]. We shall use this lemma in the proof of Lemma 3.9.

Lemma 2.7. *Let A be a C^*-algebra, let $a, b \in A_+$, and let $\varepsilon > 0$.*

1. *If $a = x(b - \varepsilon)_+$ for some $x \in A$, then $a = yb$ for some $y \in A$ with $\|y\| \leq \varepsilon^{-1}\|a\|$.*
2. *If $a \in \overline{A(b - \varepsilon)_+}$ then there is a sequence (v_n) in A such that $\|a - v_n b\| \to 0$ and $\|v_n\| \leq \varepsilon^{-1}(\|a\| + \frac{1}{n})$ for all $n \in \mathbb{N}$.*

Proof. We define continuous functions $f_\varepsilon, g_\varepsilon \colon [0, \infty) \to [0, \infty)$ for $\varepsilon > 0$ as in [29, Lemma 2.4], that is,

$$f_\varepsilon(t) = \begin{cases} \sqrt{\dfrac{t-\varepsilon}{t}} & t \geq \varepsilon \\ 0 & t < \varepsilon \end{cases} \quad \text{and} \quad g_\varepsilon(t) = \begin{cases} \dfrac{1}{t} & t \geq \varepsilon \\ \varepsilon^{-2} t & t < \varepsilon. \end{cases}$$

Then $t f_\varepsilon(t)^2 = (t - \varepsilon)_+$ and $f_\varepsilon(t)^2 = (t - \varepsilon)_+ g_\varepsilon(t)$. Thus $b f_\varepsilon(b)^2 = (b - \varepsilon)_+$ and $f_\varepsilon(b)^2 = (b - \varepsilon)_+ g_\varepsilon(b)$. Note that $\|g_\varepsilon(b)\| \leq \varepsilon^{-1}$.

To prove (1), put $y = xf_\varepsilon(b)^2$. Then $yb = xf_\varepsilon(b)^2 b = x(b-\varepsilon)_+ = a$. Also,

$$yy^* = xf_\varepsilon(b)^4x^* = x(b-\varepsilon)_+^2 g_\varepsilon(b)^2 x^* \leq \|g_\varepsilon(b)^2\| x(b-\varepsilon)_+^2 x^* \leq \varepsilon^{-2} aa^*.$$

Thus $\|y\| \leq \varepsilon^{-1}\|a\|$.

For (2), let $a \in \overline{A(b-\varepsilon)_+}$ and fix $n \in \mathbb{N}$. Then there is $w_n \in A$ such that $\|a - w_n(b-\varepsilon)_+\| < \frac{1}{n}$. Put $a_n = w_n(b-\varepsilon)_+$. Thus $\|a_n\| \leq \|a\| + \frac{1}{n}$. By (1) there is $v_n \in A$ such that $a_n = v_n b$ and $\|v_n\| \leq \varepsilon^{-1}(\|a\| + \frac{1}{n})$. Then $\|a - v_n b\| = \|a - a_n\| < \frac{1}{n}$, and so $\|a - v_n b\| \to 0$. □

3 The weak tracial Rokhlin property

In this section, we define the weak tracial Rokhlin property (as well as the tracial Rokhlin property) for finite group actions on simple not necessarily unital C*-algebras. We show that the weak tracial Rokhlin property implies pointwise outerness and so the resulting crossed product is simple. Then we compare it with other notions of the weak tracial Rokhlin property for actions on simple unital C*-algebras. Moreover, we show that the Rokhlin property in the sense of [49, Definition 3.2] implies the weak tracial Rokhlin property for actions on simple C*-algebras.

Definition 3.1. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C*-algebra A. We say that α has the *weak tracial Rokhlin property* if for every $\varepsilon > 0$, every finite subset $F \subseteq A$, and all positive elements $x, y \in A$ with $\|x\| = 1$, there exists a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:

1. $\|f_g a - a f_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - yfy - \varepsilon)_+ \precsim_A x$;
4. $\|fxf\| > 1 - \varepsilon$.

We say that α has the *tracial Rokhlin property* if we can arrange $(f_g)_{g\in G}$ above to be mutually orthogonal projections.

An action $\alpha\colon G \to \mathrm{Aut}(A)$ is called *pointwise outer* if for any $g \in G \setminus \{1\}$, the automorphism α_g is outer, that is, it is not of the form $\mathrm{Ad}(u)$ for any unitary u in the multiplier algebra of A.

Proposition 3.2. *Let α be an action of a finite group G on a simple C*-algebra A. If α has the weak tracial Rokhlin property then α is pointwise outer.*

Proof. The idea of the proof is similar to that of [49, Proposition 3.2]. However, we need Condition (4) in Definition 3.1 instead of Condition (iii) in [49, Definition 3.2], and so we need more estimates. Suppose to the contrary that there are $g_0 \in G \setminus \{1\}$ and a unitary u in the multiplier algebra of A such that $\alpha_{g_0} = \mathrm{Ad}(u)$. Set $n = \mathrm{card}(G)$. Choose ε with $0 < \varepsilon < 1$ such that

$$\frac{\sqrt{1-\varepsilon} - n\varepsilon}{n} > 0 \quad \text{and} \quad \left(\frac{\sqrt{1-\varepsilon} - n\varepsilon}{n}\right)^2 - 4\varepsilon > 0.$$

By [33, Lemma 2.5.11] (with $f(t) = t^{\frac{1}{2}}$ there), there is $\delta > 0$ such that if $x, y \in A$ are positive contractions with $\|xy - yx\| < \delta$ then $\|x^{\frac{1}{2}}y - yx^{\frac{1}{2}}\| < \varepsilon$. We may assume that $\delta < \varepsilon$. Choose a positive element $b \in A^{\alpha}$ with $\|b\| = 1$. Applying Definition 3.1 with $F = \{b, bu^*\}$, with δ in place of ε, with b in place of x, and with $y = 0$, we obtain a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that

1. $\|f_g b - bf_g\| < \delta$ and $\|f_g bu^* - bu^* f_g\| < \delta$ for all $g \in G$;
2. $\|\alpha_g(f_h) - f_{gh}\| < \delta$ for all $g, h \in G$;
3. $\|fbf\| > 1 - \delta$ where $f = \sum_{g\in G} f_g$.

Using (1) and (2) we have (1 denotes the neutral element of G):

$$\begin{aligned}\Big\|fb^{\frac{1}{2}} - \sum_{g\in G}\alpha_g(f_1 b^{\frac{1}{2}})\Big\| &= \Big\|\sum_{g\in G} f_g b^{\frac{1}{2}} - \sum_{g\in G}\alpha_g(f_1)b^{\frac{1}{2}}\Big\| \\ &\le \sum_{g\in G}\|f_g - \alpha_g(f_1)\| < n\delta.\end{aligned}$$

Thus, using (3) at the third step we get

$$n\delta > \|fb^{\frac{1}{2}}\| - \sum_{g\in G}\|\alpha_g(f_1 b^{\frac{1}{2}})\| = \|fbf\|^{\frac{1}{2}} - n\|f_1 b^{\frac{1}{2}}\| > \sqrt{1-\delta} - n\|f_1 b^{\frac{1}{2}}\|.$$

Hence,

$$\|f_1 b^{\frac{1}{2}}\| > \frac{\sqrt{1-\delta} - n\delta}{n} > \frac{\sqrt{1-\varepsilon} - n\varepsilon}{n}. \tag{1}$$

By (1), $\|f_1 b - bf_1\| < \delta$ and so $\|f_1^{\frac{1}{2}} b - bf_1^{\frac{1}{2}}\| < \varepsilon$. Thus,

$$\|f_1 b - f_1^{\frac{1}{2}} bf_1^{\frac{1}{2}}\| \le \|f_1^{\frac{1}{2}} b - bf_1^{\frac{1}{2}}\| < \varepsilon. \tag{2}$$

Similarly, since $\|f_g b - bf_g\| < \delta$ we have

$$\|f_{g_0} b - f_{g_0}^{\frac{1}{2}} bf_{g_0}^{\frac{1}{2}}\| < \varepsilon. \tag{3}$$

Note that $f_{g_0}^{\frac{1}{2}} bf_{g_0}^{\frac{1}{2}} \perp f_1^{\frac{1}{2}} bf_1^{\frac{1}{2}}$ and thus by (1) we have

$$\|f_{g_0}^{\frac{1}{2}} bf_{g_0}^{\frac{1}{2}} - f_1^{\frac{1}{2}} bf_1^{\frac{1}{2}}\| \ge \|f_1^{\frac{1}{2}} bf_1^{\frac{1}{2}}\| = \|f_1^{\frac{1}{2}} b^{\frac{1}{2}}\|^2 \ge \|f_1 b^{\frac{1}{2}}\|^2 > \left(\frac{\sqrt{1-\varepsilon} - n\varepsilon}{n}\right)^2. \tag{4}$$

Moreover, using (1) we have

$$\begin{aligned}\|uf_1bu^* - bf_1\| &= \|uf_1bu^* - \alpha_{g_0}(b)f_1\| \\ &= \|uf_1bu^* - ubu^*f_1\| \\ &\le \|f_1bu^* - bu^*f_1\| < \delta.\end{aligned} \tag{5}$$

Finally, using (2), (2), (3), (4), and (5) we obtain

$$\begin{aligned}\|\alpha_{g_0}(f_1b) - uf_1bu^*\| &\ge \|f_{g_0}^{\frac{1}{2}}bf_{g_0}^{\frac{1}{2}} - f_1^{\frac{1}{2}}bf_1^{\frac{1}{2}}\| - \|f_{g_0}^{\frac{1}{2}}bf_{g_0}^{\frac{1}{2}} - f_{g_0}b\| \\ &\quad - \|f_{g_0}b - \alpha_{g_0}(f_1b)\| - \|uf_1bu^* - bf_1\| - \|bf_1 - f_1^{\frac{1}{2}}bf_1^{\frac{1}{2}}\| \\ &> \left(\frac{\sqrt{1-\varepsilon} - n\varepsilon}{n}\right)^2 - \varepsilon - 2\delta - \varepsilon \\ &> \left(\frac{\sqrt{1-\varepsilon} - n\varepsilon}{n}\right)^2 - 4\varepsilon > 0,\end{aligned}$$

which is a contradiction. This shows that α is pointwise outer. □

COROLLARY 3.3. *Let α be an action of a finite group G on a simple C^*-algebra A. If α has the weak tracial Rokhlin property, then $A \rtimes_\alpha G$ is simple, and hence the fixed point algebra A^α is isomorphic to a full corner of $A \rtimes_\alpha G$.*

Proof. It follows from [30, Theorem 3.1] and Proposition 3.2 that $A \rtimes_\alpha G$ is simple. By [48], there exists a projection p in the multiplier algebra of $A \rtimes_\alpha G$ such that $A^\alpha \cong p(A \rtimes_\alpha G)p$. Since $A \rtimes_\alpha G$ is simple, $p(A \rtimes_\alpha G)p$ is a full corner. □

The following lemma shows that if the property stated in Definition 3.1 holds for some $y \in A_+$ (and every x, F, ε there), then it also holds for any $z \in A_+$ which is "smaller" than y (that is, for any positive z in $\overline{yAy}$). (Note that $\overline{Ay} \cap A_+ = \overline{yA} \cap A_+ = \overline{yAy} \cap A_+$.)

LEMMA 3.4. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C^*-algebra A. Let $x \in A_+$ with $\|x\| = 1$. Suppose that a positive element $y \in A$ has the following property: for every $\varepsilon > 0$ and every finite subset $F \subseteq A$ there exists a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:*

1. *$\|f_g a - af_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;*
2. *$\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;*
3. *$(y^2 - yfy - \varepsilon)_+ \precsim_A x$;*
4. *$\|fxf\| > 1 - \varepsilon$.*

Then every positive element $z \in \overline{Ay}$ also has the same property. Moreover, the statement holds if we replace "orthogonal positive contractions" with "orthogonal projections."

Proof. The idea of the proof is similar to the argument given in the proof of [1, Lemma 3.5]. Let $z \in \overline{Ay}$ be a positive element, and let a finite subset $F \subseteq A$, $\varepsilon > 0$, and an element $x \in A_+$ with $\|x\| = 1$ be given. Let δ be such that $0 < \delta < \min\left\{1, \frac{\varepsilon}{4(2\|z\|+1)}\right\}$. Since $z \in \overline{Ay}$, there exists a nonzero element $w \in A$ such that $\|z - wy\| < \delta$. Choose $\eta > 0$ such that $\eta < \min\{\varepsilon, \frac{\varepsilon}{2\|w\|^2}\}$. By assumption, there exists a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:

5. $\|f_g a - a f_g\| < \eta$ for all $a \in F$ and all $g \in G$;
6. $\|\alpha_g(f_h) - f_{gh}\| < \eta$ for all $g, h \in G$;
7. $(y^2 - yfy - \eta)_+ \precsim_A x$;
8. $\|fxf\| > 1 - \eta$.

Since $\eta < \varepsilon$, (5), (6), and (8) also hold for ε in place of η. It remains to show that $(z^2 - zfz - \varepsilon)_+ \precsim_A x$. To see this, first by Lemma 2.3 at the first step and by (7) at the last step, we have

$$\begin{aligned}
\big(wy^2w^* - wyfyw^* - \varepsilon/2\big)_+ &\precsim_A w\left(y^2 - yfy - \frac{\varepsilon}{2\|w\|^2}\right)_+ w^* \\
&\precsim_A \left(y^2 - yfy - \frac{\varepsilon}{2\|w\|^2}\right)_+ \\
&\precsim_A (y^2 - yfy - \eta)_+ \precsim_A x.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\big\|z^2 - zfz - &\big(wy^2w^* - wyfyw^* - \tfrac{\varepsilon}{2}\big)_+\big\| \\
&\leq \|z^2 - zfz - (wy^2w^* - wyfyw^*)\| + \tfrac{\varepsilon}{2} \\
&\leq \|z^2 - wyz\| + \|wyz - wy^2w^*\| + \|zfz - wyfz\| \\
&\quad + \|wyfz - wyfyw^*\| + \tfrac{\varepsilon}{2} \\
&\leq \delta(2\|z\| + 2\|wy\|) + \tfrac{\varepsilon}{2} \\
&\leq \delta(4\|z\| + 2\delta) + \tfrac{\varepsilon}{2} \leq \delta(4\|z\| + 2) + \tfrac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Therefore, by Lemma 2.2, $(z^2 - zfz - \varepsilon)_+ \precsim_A \big(wy^2w^* - wyfyw^* - \frac{\varepsilon}{2}\big)_+ \precsim_A x$, as desired. This finishes the proof. □

Remark 3.5. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C*-algebra A.

1. If A is σ-unital then α has the (weak) tracial Rokhlin property if *some* strictly positive element y in A has the property stated in Definition 3.1. This follows from Lemma 3.4 and $A = \overline{yAy} = \overline{Ay}$.

2. In Definition 3.1, it is enough to take y in a norm dense subset of A_+. Moreover, if $(e_i)_{i\in I}$ is a approximate identity for A, it is enough to take y from the set $\{e_i : i \in I\}$. This follows from Lemma 3.4 and the fact that the set $\{y \in A_+ : y \in \overline{Ae_i} \text{ for some } i \in I\}$ is norm dense in A_+.

3. In Definition 3.1, if moreover A is purely infinite then Condition (3) is automatic. Also, if A is finite then Condition (4) is redundant (this is proved in [1], however, we do not need it here).

4. If moreover A is unital, then α has the tracial Rokhlin property (in the sense of Definition 3.1) if and only if the conditions of Definition 3.1 hold only for $y = 1$ (and every ε, F, x as in that definition). This implies that our definition of the tracial Rokhlin property and [43, Definition 1.2] are equivalent in the unital case.

The Rokhlin property for finite group actions on *arbitrary* C*-algebras has been introduced in [49].

Proposition 3.6. *Let A be a simple C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. If α has the Rokhlin property in the sense of Definition 3.2 of [49], then it has the weak tracial Rokhlin property.*

Proof. Let α have the Rokhlin property in the sense of Definition 3.2 of [49]. Let $x, y \in A_+$ with $\|x\| = 1$, let $F \subseteq A$ be a finite subset, and let $\varepsilon > 0$. We may assume that $y \neq 0$ and $x, y \in F$. Also, by Lemma 3.4, we may further assume that $\|y\| \leq 1$. Since α has the Rokhlin property, there exists a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, we have:

1. $\|\alpha_g(f_h) - f_{gh}\| < \frac{\varepsilon}{2}$ for all $g, h \in G$;

2. $\|f_g a - a f_g\| < \frac{\varepsilon}{2}$ for all $g \in G$ and all $a \in F$;

3. $\|fa - a\| < \frac{\varepsilon}{2}$ for all $a \in F$.

Then, Conditions (1) and (2) in Definition 3.1 are satisfied (by (1) and (2) above). Since $y \in F$, by (3) we have

$$\|y^2 - yfy\| \leq \|y - yf\| < \varepsilon/2 < \varepsilon.$$

Thus $(y^2 - yfy - \varepsilon)_+ = 0 \precsim_A x$. Hence, Condition (3) in Definition 3.1 is also satisfied. To prove Condition (4), by (3) and that $x \in F$ we have

$$\|fxf - x\| \leq \|fxf - xf\| + \|xf - x\| \leq \|fx - x\| + \|xf - x\| < \varepsilon.$$

Thus $\|fxf\| > \|x\| - \varepsilon = 1 - \varepsilon$, and so Condition (4) in Definition 3.1 holds. Therefore, α has the weak tracial Rokhlin property. □

There are several weaker versions of the tracial Rokhlin property for actions on simple unital C*-algebras. In the sequel, we compare them with our definition of the weak tracial Rokhlin property given in Definition 3.1. First we recall the following definition for the convenience of the reader. (See [16, Definition 2.2] for an equivalent definition.)

Definition 3.7 (see [15]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple unital C*-algebra A. Then α has the *weak tracial Rokhlin property* if for every $\varepsilon > 0$, every finite subset $F \subseteq A$, and every positive element $x \in A$ with $\|x\| = 1$, there exists a family of orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:

1. $\|f_g a - a f_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $1 - f \precsim_A x$;
4. $\|fxf\| > 1 - \varepsilon$.

Proposition 3.8. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple unital C*-algebra A. The following statements are equivalent:*

(a) α has the weak tracial Rokhlin property in the sense of Definition 3.7;

(b) α has the weak tracial Rokhlin property in the sense of Definition 3.1.

Proof. The implication (a)⇒(b) follows from Remark 3.5(1) (which implies that it is enough to take $y = 1$ in Definition 3.1) and the fact that $(1 - f - \varepsilon)_+ \precsim_A 1 - f$.

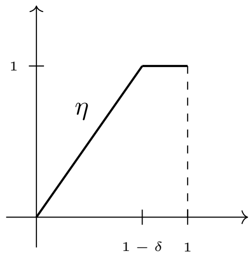
To show (b)⇒(a), one may take small cut-downs of f_g's in Definition 3.1 to get the desired Cuntz-subequivalence. We give an alternative proof using the functional calculus for order zero maps. Let F, x, and ε be as in Definition 3.7. We will find orthogonal positive contractions $(f_g)_{g\in G}$ in A satisfying (1)–(4) of Definition 3.7. We may assume that F is contained in the closed unit ball of A. Set $n = \mathrm{card}(G)$. Choose δ with $0 < \delta < \frac{\varepsilon}{2n+1}$. Applying Definition 3.1 with δ in place of ε, with $y = 1$, and with x, F as given, there are orthogonal positive contractions $(e_g)_{g\in G}$ in A such that, with $e = \sum_{g\in G} e_g$, the following hold:

1. $\|e_g a - a e_g\| < \delta$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(e_h) - e_{gh}\| < \delta$ for all $g, h \in G$;
3. $(1 - e - \delta)_+ \precsim_A x$;
4. $\|exe\| > 1 - \delta$.

We define a c.p.c. order zero map $\phi\colon C(G) \to A$ by $\phi(\xi) = \sum_{g\in G} \xi(g)e_g$. Then $\phi(1) = e$. Let $\eta\colon [0,1] \to [0,1]$ be the continuous function defined by

$$\eta(\lambda) = \begin{cases} (1-\delta)^{-1}\lambda & 0 \le \lambda \le 1-\delta, \\ 1 & 1-\delta < \lambda \le 1. \end{cases}$$

The graph of η is the following:



Using the functional calculus for c.p.c. order zero maps ([53, Corollary 4.2]), we define $\psi = \eta(\phi)$. Thus $\psi\colon C(G) \to A$ is a c.p.c. order zero map. Similar to the argument given in the proof of [4, Lemma 2.8], we see that $\|\psi(z)-\phi(z)\| \le \delta\|z\|$ for all $z \in C(G)$ with $\|z\| = 1$, and that

$$1 - \psi(1) = \tfrac{1}{1-\delta}\,(1-\phi(1)-\delta)_+ \sim_A (1-\phi(1)-\delta)_+ \precsim_A x.$$

For any $g \in G$, set $f_g = \psi(\chi_{\{g\}})$. Thus, $(f_g)_{g\in G}$ is a family of orthogonal positive contractions in A and we have

5. $\|f_g - e_g\| \le \delta$ for all $g \in G$.

Moreover, with $f = \sum_{g\in G} f_g$, we have $1 - f = 1 - \psi(1) \precsim_A x$, which is (3) in Definition 3.7. Using (5), it is easy to see that Conditions (1), (2), and (4) in Definition 3.7 follow from (1), (2), and (4) above, respectively. ☐

In the following lemma, we give a (seemingly) stronger equivalent definition of the (weak) tracial Rokhlin property for finite group actions on simple C*-algebras. This lemma says that we can take two different unknowns x, z in Conditions (3) and (4) of Definition 3.1 instead of x. The idea of the proof of this lemma will be used also in a number of places later.

Lemma 3.9. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C*-algebra A. Then α has the weak tracial Rokhlin property (respectively, tracial Rokhlin property) if and only if the following holds. For every $\varepsilon > 0$, every finite subset $F \subseteq A$, and all positive elements $x, y, z \in A$ with $x \neq 0$ and $\|z\| = 1$, there exists a family of orthogonal positive contractions (respectively, orthogonal projections) $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:*

1. *$\|f_g a - a f_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;*

2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ *for all* $g, h \in G$*;*
3. $(y^2 - yfy - \varepsilon)_+ \precsim_A x$*;*
4. $\|fzf\| > 1 - \varepsilon$.

Proof. We prove only the case of the weak tracial Rokhlin property since the proof for the tracial Rokhlin property is similar. The backward implication is obvious. For the forward implication, let α have the weak tracial Rokhlin property and let ε, F, x, y, z be as in the statement. We may assume that F is contained in the closed unit ball of A. Let $n = \mathrm{card}(G)$. Choose δ such that $0 < \delta < 1$ and

$$\left(\frac{\delta}{2-\delta}(1 - \tfrac{\varepsilon}{2})\right)^2 > 1 - \varepsilon.$$

Put $z_1 = (z^{1/2} - \delta)_+$. Since A is simple, [45, Lemma 2.6] implies that there is a positive element $d \in \overline{z_1 A z_1}$ such that $d \precsim_A x$ and $\|d\| = 1$. Applying Definition 3.1 with y and F as given, with $\frac{\varepsilon}{2}$ in place of ε, and with d in place of x, there exist orthogonal positive contractions $(f_g)_{g\in G}$ in A such that, with $f = \sum_{g\in G} f_g$, the following hold:

5. $\|f_g a - a f_g\| < \frac{\varepsilon}{2}$ for all $a \in F$ and all $g \in G$;
6. $\|\alpha_g(f_h) - f_{gh}\| < \frac{\varepsilon}{2}$ for all $g, h \in G$;
7. $(y^2 - yfy - \frac{\varepsilon}{2})_+ \precsim_A d$;
8. $\|fdf\| > 1 - \frac{\varepsilon}{2}$.

Clearly, (1), (2), and (3) follow from (5), (6), and (7), respectively. To see (4), first note that have $d \in \overline{z_1 A z_1} \subseteq \overline{A z_1} = \overline{A(z^{1/2} - \delta)_+}$. Thus by Lemma 2.7 there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in A such that $\|v_n z^{1/2} - d\| \to 0$ and $\|v_n\| \leq (\|d\| + \frac{1}{n})\delta^{-1} = (1 + \frac{1}{n})\delta^{-1}$. Then $\|f v_n z^{\frac{1}{2}} f - fdf\| \to 0$. Since $\|fdf\| > 1 - \frac{\varepsilon}{2}$ and $\delta < 1$, there is $n \in \mathbb{N}$ such that $\|f v_n x^{\frac{1}{2}} f\| > 1 - \frac{\varepsilon}{2}$ and $\frac{1}{n} < 1 - \delta$. Hence,

$$1 - \tfrac{\varepsilon}{2} < \|f v_n z^{\frac{1}{2}} f\| \leq \|z^{\frac{1}{2}} f\| \cdot \|v_n\| \leq \|z^{\frac{1}{2}} f\|(1 + \tfrac{1}{n})\delta^{-1} \leq \|z^{\frac{1}{2}} f\|(2 - \delta)\delta^{-1}.$$

Thus,

$$\|fzf\| = \|z^{\frac{1}{2}} f\|^2 > \left(\frac{\delta}{2-\delta}(1 - \tfrac{\varepsilon}{2})\right)^2 > 1 - \varepsilon.$$

This completes the proof. ☐

Phillips in [42, Problem 3.2] asked whether there is a reasonable formulation of the tracial Rokhlin property for finite group actions on simple unital C*-algebras in terms of the central sequence algebra. We give an answer to this question in the *not necessarily unital* simple case in the following proposition. For a C*-algebra A, we write

$$A_\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).$$

We consider the elements of A in A_∞ as the equivalence classes of constant sequences. We denote by $A_\infty \cap A'$ the relative commutant of A in A_∞. Also, $\pi_\infty \colon \ell^\infty(\mathbb{N}, A) \to A_\infty$ denotes the quotient map. If $\alpha \colon G \to \mathrm{Aut}(A)$ is an action, then we denote by α_∞ the induced action of G on A_∞.

PROPOSITION 3.10. *Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple separable C*-algebra A. Then α has the weak tracial Rokhlin property (respectively, tracial Rokhlin property) if and only if for every $x, y, z \in A_+$ with $x \neq 0$, there exists a family of orthogonal positive contractions (respectively, orthogonal projections) $(f_g)_{g \in G}$ in $A_\infty \cap A'$ such that, with $f = \sum_{g \in G} f_g$, the following hold:*

1. $(\alpha_\infty)_g(f_h) = f_{gh}$ *for all* $g, h \in G$;
2. $y^2 - yfy \precsim_{A_\infty} x$;
3. $\|fzf\| = \|z\|$.

If moreover A is unital, then Condition (2) can replaced by $1 - f \precsim_{A_\infty} x$.

Proof. We prove only the case of the weak tracial Rokhlin property since the proof for the tracial Rokhlin property is essentially the same.

Assume that α has the weak tracial Rokhlin property. Let $x, y, z \in A_+$ with $x \neq 0$. We may assume that $\|x\| = 1$ and $z \neq 0$. Let $\{a_1, a_2, \ldots\}$ be a norm dense countable subset of the closed unit ball of A. For $n \in \mathbb{N}$, set $F_n = \{a_1, \ldots, a_n\}$. Applying Lemma 3.9 with $(x - \frac{1}{2})_+$ in place of x, with $z/\|z\|$ in place of z, with F_n in place of F, and with $\frac{1}{n}$ in place of ε, we obtain mutually orthogonal positive contractions $(f_{(g,n)})_{g \in G}$ in A satisfying (1)–(4) in Lemma 3.9. Let $f_g \in A_\infty$ denote the equivalence class of $(f_{(g,n)})_{n \in \mathbb{N}}$. Then $(f_g)_{g \in G}$ is a family of orthogonal positive contractions in $A_\infty \cap A'$. It is easy to see that Conditions (1) and (3) in the statement hold. To see (2), put $h_n = \sum_{g \in G} f_{(g,n)}$. Then $f = \sum_{g \in G} f_g$ is the equivalence class of $(h_n)_{n \in \mathbb{N}}$ in A_∞ and $(y^2 - yh_ny - \frac{1}{n})_+ \precsim_A (x - \frac{1}{2})_+$. By Lemma 2.4, there is $v_n \in A$ such that $\|(y^2 - yh_ny - \frac{1}{n})_+ - v_nxv_n^*\| < \frac{1}{n}$ and $\|v_n\| \leq 2\|y\|$. Hence $\|y^2 - yh_ny - v_nxv_n^*\| < \frac{2}{n}$. Let v be the equivalence class of $(v_n)_{n \in \mathbb{N}}$ in A_∞. (Note that $(v_n)_{n \in \mathbb{N}}$ is a bounded sequence.) Then $y^2 - yfy = vxv^* \precsim_{A_\infty} x$.

Using Lemma 2.2 and [33, Lemma 2.5.12], the other implication follows.

If moreover A is unital and we replace Condition (2) by $1 - f \precsim_{A_\infty} x$, then the forward implication follows by taking $y = 1$ in the preceding argument. The backward implication follows from the previous case and the fact that $y^2 - yfy = y(1 - f)y \precsim_{A_\infty} 1 - f$. ☐

The following theorem says that for finite group actions on simple C*-algebras with tracial rank zero, the weak tracial Rokhlin property coincides with the tracial Rokhlin property. Similar results were proved in [46, Theorem 1.9] and [52, Theorem 2.7] for the unital case. In the proof of these results the tracial state space was used as an ingredient. However, in our nonunital setting, we

use techniques from Cuntz subequivalence in the proof of the following theorem instead of using traces.
We use the following fact (which is easy to prove) in the proof of the following theorem. If A is a C*-algebra with real rank zero and B is a finite dimensional C*-subalgebra of A, then both A_∞ and $A \cap B'$ have real rank zero (see [12, Theorem 4.10(iv)]).

Theorem 3.11. *Let A be a simple C*-algebra with tracial rank zero, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. If α has the weak tracial Rokhlin property then it has the tracial Rokhlin property.*

Proof. Suppose that α has the weak tracial Rokhlin property. We have to show that for every finite subset $F \subseteq A$, every $\varepsilon > 0$, and all positive elements $x, y \in A$ with $\|x\| = 1$, there exists a family of orthogonal projections $(p_g)_{g\in G}$ in A such that, with $p = \sum_{g\in G} p_g$, the following hold:

1. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - ypy - \varepsilon)_+ \precsim_A x$;
4. $\|pxp\| > 1 - \varepsilon$.

Set $n = \mathrm{card}(G)$. Choose $\delta > 0$ with $\delta < \min(\varepsilon/8, \varepsilon/2(n+1))$. Without loss of generality, we can assume that $x, y \in F$, that F is contained in the closed unit ball of A, and that $\alpha_g(F) = F$, for any $g \in G$. We may further assume that $y \in A^\alpha$ (by Remark 3.5(2) since A has an approximate identity contained in A^α). We claim that there exists $z \in A_+ \setminus \{0\}$ such that

5. $z \oplus \bigoplus_{g\in G} \alpha_g(z) \precsim_A x$.

In fact, by [45, Lemma 2.1], there is $z_1 \in A_+ \setminus \{0\}$ such that $z_1 \otimes 1_{n+1} \precsim_A x$. Then by [45, Lemma 2.4], there is $z \in A_+ \setminus \{0\}$ such that $z \precsim_A \alpha_g(z_1)$ for all $g \in G$. We may assume that $\|z\| = 1$. Hence, $z \oplus \bigoplus_{g\in G} \alpha_g(z) \precsim_A z_1 \otimes 1_{n+1} \precsim_A x$. This proves (5).
Since A has tracial rank zero, we can apply Theorem A.6 to find a finite dimensional C*-subalgebra B of A such that, with $q = 1_B$, the following hold:

6. $\|qa - aq\| < \delta$ for all $a \in F$;
7. $qAq \subseteq_\delta B$;
8. $(y^2 - yqy - \delta)_+ \precsim_A z$;
9. $\|qxq\| > 1 - \delta$.

Set $E = F \cup B$, and let B_0 be a finite subset of B with $\mathrm{span}(B_0) = B$. Since α has the weak tracial Rokhlin property, arguing as in the proof of Proposition 3.10 with z in place of x, with y as given, and with qxq in place of z (using

$F \cup B_0$ in place of F when applying Lemma 3.9 in the proof of Proposition 3.10), there are orthogonal positive contractions $(f_g)_{g\in G}$ in $A_\infty \cap (F\cup B_0)' = A_\infty \cap E'$ such that, with $f = \sum_{g\in G} f_g$, we have

10. $(\alpha_\infty)_g(f_h) = f_{gh}$ for all $g, h \in G$;
11. $y^2 - yfy \precsim_{A_\infty} z$;
12. $\|fqxqf\| = \|qxq\| > 1 - \delta$ (by (9)).

Note that $\|fqxqf\| = \max\{\|f_g qxq f_g\| \colon g \in G\|\}$. Hence, by (12), there exists $g_0 \in G$ such that

$$\|f_{g_0} qxq f_{g_0}\| > 1 - \delta. \tag{6}$$

Put $D = A_\infty \cap B'$. Note that $A_\infty \cap E' \subseteq D$ and that D has real rank zero by the remark preceding this theorem. In particular, the hereditary subalgebra $\overline{qf_{g_0} D f_{g_0} q}$ of D has real rank zero. Thus there is a projection $r_1 \in \overline{qf_{g_0} D f_{g_0} q}$ such that

$$\|r_1 q f_{g_0} - q f_{g_0}\| < \delta \quad \text{and} \quad \|r_1 q f_{g_0} r_1 - q f_{g_0}\| < \delta.$$

Since $r_1 \leq q$, we get

$$\|r_1 f_{g_0} - q f_{g_0}\| < \delta \quad \text{and} \quad \|r_1 f_{g_0} r_1 - q f_{g_0}\| < \delta. \tag{7}$$

Put $r_g = \alpha_g(r_1)$ for every $g \in G \setminus \{1\}$, and $r = \sum_{g\in G} r_g$. Thus $(r_g)_{g\in G}$ is a family of projections in A_∞. Since $r_1 \in \overline{f_{g_0} D f_{g_0}} \subseteq \overline{f_{g_0} A_\infty f_{g_0}}$, we have $r_g \in \overline{f_{gg_0} A_\infty f_{gg_0}}$, and so $r_g r_h = 0$ when $g \neq h$; that is, $(r_g)_{g\in G}$ is a family of orthogonal projections in A_∞. We show that

13. $\|r_g a - a r_g\| < \varepsilon/2$ for all $a \in F$ and all $g \in G$,
14. $(\alpha_\infty)_g(r_h) = r_{gh}$ for all $g, h \in G$,
15. $(y^2 - yry - \varepsilon/2)_+ \precsim_{A_\infty} x$, and
16. $\|rxr\| > 1 - \varepsilon/2$.

Observe that (14) follows from the definition of r_g. For (13), let $a \in F$. By (7), there is $b \in B$ such that $\|qaq - b\| < \delta$. Using this at the third step, and (6) at the fourth step, we get

$$\begin{aligned}
\|r_1 a - a r_1\| &= \|q r_1 q a - a q r_1 q\| \\
&\leq \|q r_1 q a - q r_1 q a q\| + \|q r_1 q a q - q a q r_1 q\| + \|q a q r_1 q - a q r_1 q\| \\
&\leq \|q r_1\| \cdot \|qa - qaq\| + 2\delta + \|qaq - aq\| \cdot \|r_1 q\| \\
&\leq 4\delta < \varepsilon/2.
\end{aligned}$$

Thus $\|r_g a - ar_g\| < \varepsilon/2$, for all $a \in F$ and all $g \in G$. Now, using (14) and that $\alpha_g(F) = F$, for all $g \in G$, we get (13). To see (15), first by (7) at the fifth step (also recall that $y \in A^\alpha$ and $f_g \in A_\infty \cap E'$), we get

$$\begin{aligned}
&\left\|(y^2 - yrfry) - \left[(y^2 - yfy) + \sum_{g\in G} f_{gg_0}^{1/2}\alpha_g((y^2 - yqy - \delta)_+)f_{gg_0}^{1/2}\right]\right\| \\
&\leq \left\|yfy - yrfry - \sum_{g\in G} f_{gg_0}^{1/2}\alpha_g(y^2 - yqy)f_{gg_0}^{1/2}\right\| + \delta \\
&\leq \left\|yfy - \sum_{g\in G} f_{gg_0}^{1/2}y^2 f_{gg_0}^{1/2}\right\| \\
&\quad + \left\|yrfry - \sum_{g\in G} yf_{gg_0}^{1/2}\alpha_g(q)f_{gg_0}^{1/2}y\right\| + \delta \qquad (8)\\
&= \left\|\sum_{g\in G} yr_g f_{gg_0} r_g y - \sum_{g\in G} y\alpha_g(q)f_{gg_0}y\right\| + \delta \\
&\leq \|y^2\| \cdot \left\|\sum_{g\in G} \alpha_g(r_1 f_{g_0} r_1) - \alpha_g(qf_{g_0})\right\| + \delta \\
&\leq n\delta + \delta = (n+1)\delta < \varepsilon/2.
\end{aligned}$$

On the other hand, $yrfry \leq yry$ and so $y^2 - yry \leq y^2 - yrfry$. Then [45, Lemma 1.7] implies that $(y^2 - yry - \varepsilon/2)_+ \precsim_{A_\infty} (y^2 - yrfry - \varepsilon/2)_+$. By this at the first step, by (8) at the second step, by (8) and (11) at the third step, and by (5) at the fifth step, we get

$$\begin{aligned}
(y^2 - yry - \varepsilon/2)_+ &\precsim_{A_\infty} (y^2 - yrfry - \varepsilon/2)_+ \\
&\precsim_{A_\infty} (y^2 - yfy) + \sum_{g\in G} f_{gg_0}^{1/2}\alpha_g((y^2 - yqy - \delta)_+)f_{gg_0}^{1/2} \\
&\precsim_{A_\infty} z \oplus \sum_{g\in G} f_{gg_0}^{1/2}\alpha_g(z)f_{gg_0}^{1/2} \\
&\precsim_{A_\infty} z \oplus \bigoplus_{g\in G} \alpha_g(z) \\
&\precsim_{A_\infty} x,
\end{aligned}$$

which is (15). To prove (16), by (7) at the sixth step and by (6) at the seventh step, we calculate

$$\begin{aligned}
\|rxr\| &= \|rx^{1/2}\|^2 = \|x^{1/2}rx^{1/2}\| \\
&\geq \|x^{1/2}r_1x^{1/2}\| = \|r_1xr_1\| \\
&\geq \|r_1f_{g_0}r_1xr_1f_{g_0}r_1\| \\
&\geq \|f_{g_0}qxqf_{g_0}\| - 2\delta \\
&> 1 - 3\delta > 1 - \varepsilon/2.
\end{aligned}$$

This completes the proof of (13)–(16).
For each $g \in G$, let $(r_{g,k})_{k\in\mathbb{N}} \in \ell^\infty(A)$ be a representing sequence for r_g, that is, $\pi_\infty(r_{g,1}, r_{g,2}, \cdots) = r_g$. Since r_g is a projection, we can assume that $r_{g,k}$ is also a projection for all $g \in G$ and all $k \in \mathbb{N}$.
Choose $\eta > 0$ such that if $a_1, \ldots, a_n$ are positive contractions in A where $\|a_i^2 - a_i\| < \eta$ and $\|a_i a_j\| < \eta$ for all $i, j = 1, \ldots, n$ with $i \neq j$, then there are mutually orthogonal projections $p_1, \ldots, p_n$ in A such that $\|a_i - p_i\| < \varepsilon/4n$ for all $i = 1, \ldots, n$.
It follows from (15) that there is $v \in A_\infty$ such that $\|(y^2 - yry - \varepsilon/2)_+ - vxv^*\| < \varepsilon/4$, and so

$$\|y^2 - yry - vxv^*\| < 3\varepsilon/4. \tag{9}$$

Let $(v_k)_{k\in\mathbb{N}}$ be a representing sequence for v. We can choose k large enough such that $\|r_{g,k} r_{h,k}\| < \eta$ for all $g, h \in G$ with $g \neq h$ (since $r_g r_h = 0$), and that the following hold (by (13), (14), (16), and (9)):

17. $\|ar_{g,k} - r_{g,k}a\| < \varepsilon/2$ for all $g \in G$ and all $a \in F$;
18. $\|\alpha_g(r_{h,k}) - r_{gh,k}\| < \varepsilon/2$ for all $g, h \in G$;
19. $\|y^2 - yr_k y - v_k x v_k^*\| < 3\varepsilon/4$, where $r_k = \sum_{g\in G} r_{g,k}$;
20. $\|r_k x r_k\| > 1 - \varepsilon/2$.

By the choice of η, there are orthogonal projections $(p_g)_{g\in G}$ in A such that $\|r_{g,k} - p_g\| < \varepsilon/4n$ for all $g \in G$. Since $\|r_{g,k} - p_g\| < \varepsilon/4$, (17) implies (1), and (18) implies (2). Put $p = \sum_{g\in G} p_g$. Then $\|r_k - p\| \leq \sum_{g\in G} \|r_{g,k} - p_g\| < \varepsilon/4$. Then by (19), we have $\|y^2 - ypy - v_k x v_k^*\| < 3\varepsilon/4 + \varepsilon/4 = \varepsilon$ and so $(y^2 - ypy - \varepsilon)_+ \precsim_A x$, which is (3). Finally, (20) implies that $\|pxp\| > 1 - \varepsilon$, which is (4). This finishes proof. □

We present a list of examples of finite group actions with the (weak) tracial Rokhlin property on simple nonunital C*-algebras. These examples are mainly based on the results of [1] (and Theorems 4.5 and 4.6 below). We refer the reader to [1] for the proofs.

Example 3.12. In the following, $\mathcal{W}$ denotes the Razak-Jacelon algebra [25] and S_m denotes the group of all permutations of $\{1, 2, \ldots, m\}$, for $m \in \mathbb{N}$.

1. Let $A = \bigotimes_{k=1}^\infty M_3$ be the UHF algebra of type 3^∞ and let B be a simple C*-algebra. Define $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ by

$$\alpha = \bigotimes_{k=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then the action $\alpha \otimes \mathrm{id}\colon \mathbb{Z}_2 \to \mathrm{Aut}(A \otimes B)$ has the weak tracial Rokhlin property, by Theorem 4.5 and the the fact α has the tracial Rokhlin property. (See [46, Proposition 2.5] and [18, Example 10.3.23 and Remark 10.4.9] for details about α.) In particular, if we take $B = \mathcal{W}$, then $M_{3^\infty} \otimes \mathcal{W} \cong \mathcal{W}$ is stably projectionless and $\alpha \otimes \mathrm{id}_{\mathcal{W}} \colon \mathbb{Z}_2 \to \mathrm{Aut}(M_{3^\infty} \otimes \mathcal{W})$ has the weak tracial Rokhlin property but not the tracial Rokhlin property since $M_{3^\infty} \otimes \mathcal{W}$ does not have any nonzero projections. (We do not know whether this action has the Rokhlin property.)

2. Let A be a nonelementary simple C*-algebra with tracial rank zero and let $m \in \mathbb{N} \setminus \{1\}$. Then the permutation action $\beta \colon S_m \to \mathrm{Aut}(A^{\otimes m})$ has the the tracial Rokhlin property [1]. Here, $A^{\otimes m}$ denotes the minimal tensor product of m copies of A. This result is similar to [21, Example 5.10] which states that the permutation action of S_m on $\mathcal{Z}^{\otimes m} \cong \mathcal{Z}$ has the generalized tracial Rokhlin property. It is not clear that the action β does not have the Rokhlin property. It may depend on A. For example, if either $K_0(A) = \mathbb{Z}$ or $K_1(A) = \mathbb{Z}$, then β does not have the Rokhlin property (by [37, Corollary 3.10]).

3. The flip action on $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$ has the weak tracial Rokhlin property [21, Example 5.10] but not the Rokhlin property. For a nonunital example, if we take $A = \mathcal{Z} \otimes \mathcal{K}$ then the flip action on $A \otimes A \cong A$ has the weak tracial Rokhlin property (by a result of [1] and that A is tracially $\mathcal{Z}$-absorbing). This action does not have the Rokhlin property, by [37, Corollary 3.10] since $K_0(A) = \mathbb{Z}$. We do not know whether this action has the tracial Rokhlin property.

4. Let A be a simple $\mathcal{Z}$-absorbing C*-algebra. Then for every finite nontrivial group G there is an action $\alpha \colon G \to \mathrm{Aut}(A)$ with the weak tracial Rokhlin property. This follows essentially from the fact that G embeds into some S_m and that the permutation action of S_m on $\mathcal{Z}^{\otimes m} \cong \mathcal{Z}$ has the weak tracial Rokhlin property. (One also needs Theorem 4.5 and Proposition 4.1). If moreover, A is separable with either $K_0(A) = \mathbb{Z}$ or $K_1(A) = \mathbb{Z}$, then α does not have the Rokhlin property [37, Corollary 3.10].

5. Let α be Blackadar's action of $\mathbb{Z}_2$ on M_{2^∞} (see [5, Section 5] and [46, Example 3.1]). Then α does not have the Rokhlin property but it has the tracial Rokhlin property [46, Proposition 3.4]. Now consider the action $\alpha \otimes \mathrm{id} \colon \mathbb{Z}_2 \to M_{2^\infty} \otimes \mathcal{K}$. Then $\alpha \otimes \mathrm{id}$ has the weak tracial Rokhlin property (by Theorem 4.5) and so the tracial Rokhlin property (by Theorem 3.11). However, $\alpha \otimes \mathrm{id}$ does not have the Rokhlin property (by [49, Theorem 3.2(ii)]).

4 Permanence properties

The purpose of this section is to study the behavior of the (weak) tracial Rokhlin property for finite group actions on simple C*-algebras under restriction to subgroups, invariant hereditary subalgebras, direct limits, and tensor products of actions. The following proposition is a nonunital version of [13, Lemma 5.6].

PROPOSITION 4.1. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C*-algebra A with the (weak) tracial Rokhlin property. If H is a subgroup of G, then the restriction of α to H also has the (weak) tracial Rokhlin property.*

Proof. The proof is similar to the unital case. The main idea is the following. Let T be a set of right coset representations for H in G. If $(f_g)_{g\in G}$ is a suitable family of Rokhlin elements satisfying Definition 3.1 for the action of G on A, then we set $e_h = \sum_{t\in T} f_{ht}$ for any $h \in H$. Then $(e_h)_{h\in H}$ is a family of Rokhlin elements for the action of H on A. □

Next, we show that the weak tracial Rokhlin property is preserved by passing to invariant hereditary subalgebras.

PROPOSITION 4.2. *Let A be a simple C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Let B be an α-invariant hereditary C*-subalgebra of A and let $\beta\colon G \to \mathrm{Aut}(B)$ be the restriction of α to B. If α has the weak tracial Rokhlin property then so does β. If α has the tracial Rokhlin property, then so does β whenever B have an approximate identity (not necessarily increasing) consisting of projections in the fixed point algebra.*

Proof. Suppose that α has the weak tracial Rokhlin property, and let we are give a finite set $F \subseteq B$, $\varepsilon > 0$, and $x, y \in B_+$ with $\|x\| = 1$. We can assume that $x, y \in F$ and that F is contained in the closed unit ball of B. Without loss of generality $\varepsilon < 1$. Set $n = \mathrm{card}(G)$. By [33, Lemma 2.5.12], there is $\delta > 0$ such that if $(e_g)_{g\in G}$ is a family of positive contractions in B satisfying $\|e_g e_h\| < 2\delta$ for all $g, h \in G$ with $g \neq h$, then there are orthogonal positive contractions $(f_g)_{g\in G}$ in B such that $\|f_g - e_g\| < \frac{\varepsilon}{4n}$ for any $g \in G$. We may assume that $\delta < \frac{\varepsilon}{28n}$.
Since any approximate identity for B^β is also an approximate identity for B, we can choose a positive contraction $b \in B^\beta$ such that

$$\|ab - a\| < \delta \quad\text{and}\quad \|ba - a\| < \delta, \tag{10}$$

for any $a \in F$. We set $z = bxb$ and $w = byb$. So by (10) we have

$$\|z - x\| < 2\delta \quad\text{and}\quad \|w - y\| < 2\delta. \tag{11}$$

Applying Definition 3.1 to α with $F \cup \{b\}$ in place of F, with $z/\|z\|$ in place of x, with w in place of y, and with δ in place of ε, we obtain orthogonal positive contractions $(r_g)_{g\in G}$ in A such that, with $r = \sum_{g\in G} r_g$, the following hold:

1. $\|r_g a - a r_g\| < \delta$ for all $a \in F \cup \{b\}$ and all $g \in G$;
2. $\|\alpha_g(r_h) - r_{gh}\| < \delta$ for all $g, h \in G$;
3. $(w^2 - wrw - \delta)_+ \precsim_A z$;

4. $\|rzr\| > \|z\|(1-\delta)$.

We put $e_g = br_g b$ for any $g \in G$, and $e = \sum_{g\in G} e_g$. For any $g, h \in G$ with $g \neq h$, using (1) and that $r_g r_h = 0$ at the third step, we have

$$\|e_g e_h\| = \|br_g b^2 r_h b\| \leq \|r_g b^2 r_h\| < 2\delta.$$

Hence by the choice of δ there are orthogonal positive contractions $(f_g)_{g\in G}$ in B such that $\|f_g - e_g\| < \frac{\varepsilon}{4n}$ for any $g \in G$. We set $f = \sum_{g\in G} f_g$. Then we have

$$\|f_g - e_g\| < \tfrac{\varepsilon}{4} \ \text{ and } \ \|f - e\| < \tfrac{\varepsilon}{4}. \tag{12}$$

In particular, $\|e\| < \|f\| + \frac{\varepsilon}{4} < 2$. We show that the family $(f_g)_{g\in G}$ satisfies the conditions of Definition 3.1 for the action $\beta\colon G \to \mathrm{Aut}(B)$, that is, we will show that the following hold:

5. $\|f_g a - af_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
6. $\|\beta_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
7. $(y^2 - yfy - \varepsilon)_+ \precsim_B x$;
8. $\|fxf\| > 1 - \varepsilon$.

To see (5), using (12), (10), and (1), for any $a \in F$ and any $g \in G$ we get

$$\begin{aligned}\|f_g a - af_g\| &\leq \|f_g a - e_g a\| + \|e_g a - br_g ab\| + \|br_g ab - bar_g b\| \\ &\quad + \|bar_g b - ae_g\| + \|ae_g - af_g\| \\ &< \tfrac{\varepsilon}{4} + 2\delta + \delta + 2\delta + \tfrac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

To prove (6), using (12) and (2), for any $g, h \in G$ we have

$$\begin{aligned}\|\beta_g(f_h) - f_{gh}\| &\leq \|\alpha_g(f_h) - \alpha_g(e_h)\| + \|b\alpha_g(r_h)b - br_{gh}b\| + \|e_{gh} - f_{gh}\| \\ &< \tfrac{\varepsilon}{4} + \delta + \tfrac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

To see (7), first using (11) at the second step and using (10) and (12) at the fifth step, we obtain

$$\begin{aligned}\big\|(y^2 - yfy) - (w^2 - wrw - \delta)_+\big\| &\leq \delta + \|y^2 - w^2\| + \|yfy - wrw\| \\ &< \delta + 4\delta + \|yfy - byeyb\| \\ &\leq 5\delta + \|yfy - byfy\| + \|byfy - byey\| \\ &\quad + \|byey - byeyb\| \\ &< 5\delta + \|y - by\| + \|f - e\| + \|e\| \cdot \|y - yb\| \\ &< 5\delta + \delta + \tfrac{\varepsilon}{4} + 2\delta = 8\delta + \tfrac{\varepsilon}{4} < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

Now, using Lemma 2.2 and (3), we get

$$(y^2 - yfy - \varepsilon)_+ \precsim_A (w^2 - wrw - \delta)_+ \precsim_A z \precsim_A x.$$

Since B is a hereditary subalgebra of A, we get $(y^2 - yfy - \varepsilon)_+ \precsim_B x$, which is (7). To show (8), first by (11) we have $\|z\| > 1 - 2\delta$, and so by (4) we get

$$\|rzr\| > \|z\|(1-\delta) > (1-2\delta)(1-\delta) > 1 - 3\delta. \tag{13}$$

Second, using (10) at the third step, using (1) at the fourth and fifth steps, and using (13) at the seventh step, we obtain

$$\begin{aligned} \|exe\| &= \|brbxbrb\| \\ &\geq \|brxrb\| - \|br(bxb - x)rb\| \\ &> \|brxrb\| - 2\delta \\ &> \|rbxrb\| - n\delta - 2\delta \\ &> \|rbxbr\| - 2n\delta - 2\delta \\ &= \|rzr\| - (2n+2)\delta \\ &> 1 - 3\delta - (2n+2)\delta \\ &= 1 - (2n+5)\delta \geq 1 - \tfrac{\varepsilon}{4}. \end{aligned} \tag{14}$$

Then using (12) at the third step and using (14) at the fourth step, we get

$$\begin{aligned} \|fxf\| &\geq \|exe\| - \|exe - exf\| - \|exf - fxf\| \\ &\geq \|exe\| - \|e\| \cdot \|e - f\| - \|e - f\| \\ &> \|exe\| - \tfrac{3\varepsilon}{4} \\ &> 1 - \tfrac{\varepsilon}{4} - \tfrac{3\varepsilon}{4} = 1 - \varepsilon, \end{aligned}$$

which is (8). This completes the proof of (5)–(8), and shows that $\beta\colon G \to \operatorname{Aut}(B)$ has the weak tracial Rokhlin property.

The proof of the second part of the statement about the tracial Rokhlin property is similar to the proof of the first part. ☐

Proposition 4.3. *Let G be a finite group. Let $\big((G, A_i, \alpha^{(i)})_{i\in I}, (\varphi_{j,i})_{i\leq j}\big)$ be a direct system of simple G-algebras. Let A be the direct limit of the A_i and let $\alpha\colon G \to \operatorname{Aut}(A)$ be the direct limit of the $\alpha^{(i)}$. If $\alpha^{(i)}$ has the (weak) tracial Rokhlin property for each i, then so does α.*

Proof. The statement follows essentially from the following fact. If $\alpha\colon G \to \operatorname{Aut}(A)$ is an action of G on a simple C*-algebra A such that for every finite set $F \subseteq A$ and every $\varepsilon > 0$ there is an α-invariant simple C*-subalgebra B of A such that $F \subseteq_\varepsilon B$ and the restriction of α to B has the (weak) tracial Rokhlin property, then $\alpha\colon G \to \operatorname{Aut}(A)$ has the (weak) tracial Rokhlin property. ☐

Phillips posed the following problem for actions on simple unital C*-algebras.

Problem 4.4 ([42], Problem 3.18). Let A and B be infinite dimensional simple unital C*-algebras, and let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action with the tracial Rokhlin property and $\beta\colon G \to \operatorname{Aut}(B)$ be an arbitrary action. Does it follow that $\alpha \otimes \beta\colon G \to \operatorname{Aut}(A \otimes_{\min} B)$ has the tracial Rokhlin property?

There are some partial solutions to this problem. Lemma 3.9 of [43] is the very special case $B = M_n$ and β is inner. If A has tracial rank zero and B has tracial rank at most one, then by [42, Proposition 3.19], $\alpha \otimes \beta$ has the tracial Rokhlin property. Moreover, it follows from [51, Proposition 2.4.6] that this problem has an affirmative answer provided that $A \otimes_{\min} B$ has Property (SP). The following two results are more general solutions to this problem in the *not necessarily unital* simple case.

Theorem 4.5. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of a finite group G on simple C^*-algebras A and B. If α has the weak tracial Rokhlin property then so does $\alpha \otimes \beta\colon G \to \mathrm{Aut}(A \otimes_{\min} B)$.*

Proof. Suppose that we are given a finite set $F \subseteq A \otimes_{\min} B$, $\varepsilon > 0$, and $x, y \in (A \otimes_{\min} B)_+$ with $\|x\| = 1$. Set $D = A \otimes_{\min} B$. We shall find a family of orthogonal positive contractions $(f_g)_{g\in G}$ in $A \otimes_{\min} B$ such that, with $f = \sum_{g\in G} f_g$, the following hold:

1. $\|f_g a - a f_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - yfy - \varepsilon)_+ \precsim_D x$;
4. $\|fxf\| > 1 - \varepsilon$.

We may assume that there exist $c_1, \ldots, c_m$ in A and $d_1, \ldots, d_m$ in B such that $F = \{c_i \otimes d_i \mid 1 \le i \le m\}$ and that $\|c_i\|, \|d_i\| \le 1$ for all $1 \le i \le m$. By Part (2) of Remark 3.5, we may assume that $y = y_1 \otimes y_2$ for some $y_1 \in A_+$ and $y_2 \in B_+$ with $\|y_1\|, \|y_2\| \le 1$. Choose δ such that $0 < \delta < \frac{\varepsilon}{3}$ and $\big((1-\delta)^2 - 4\delta - \delta^2\big) > 1 - \varepsilon$. There exists δ_1 such that $\frac{1}{2} < \delta_1 < 1$ and $\big((1-\delta)^2 - 4\delta - \delta^2\big)\,\delta_1^2 > 1 - \varepsilon$. Put $z = (x - \delta_1)_+$. It follows from Kirchberg's Slice Lemma ([47, Lemma 4.1.9]) that there are elements $x_1 \in A_+$ and $x_2 \in B_+$ such that $x_1 \otimes x_2 \precsim_D z$ and that $\|x_1\| = \|x_2\| = 1$. By Lemma 2.4, there exists $w \in A \otimes_{\min} B$ such that $\|wxw^* - x_1 \otimes x_2\| < \delta^2$ and $\|w\| \le \delta_1^{-1/2} < \delta_1^{-1}$. Thus there is $v \in A \otimes_{\min} B$ where $v = \sum_{i=1}^{k} v_i \otimes w_i$ for some $v_i \in A$ and $w_i \in B$, $i = 1, \ldots, k$, such that

5. $\|vxv^* - x_1 \otimes x_2\| < \delta^2$ and $\|v\| < \delta_1^{-1}$.

Put $E = \{c_i \mid 1 \le i \le m\} \cup \{v_i \mid 1 \le i \le k\}$. By [28, Proposition 2.7(v)], there is $n \in \mathbb{N}$ such that

6. $(y_2^2 - \delta)_+ \precsim_B x_2 \otimes 1_n$.

By Proposition 3.2, α is pointwise outer, and so A is not elementary. It follows from [7, Corollary IV.1.2.6] that A is not of Type I. Now, [45, Lemma 2.4] implies that there is a nonzero element $x_0 \in A_+$ such that

7. $x_0 \otimes 1_n \precsim_A x_1$.

Put $M = 1 + \sum_{i=1}^{k} \|v_i\| + \sum_{i=1}^{k} \|w_i\|$. Choose $\eta > 0$ such that $\eta < \frac{\delta}{2Mk\mathrm{card}(G)}$. Applying Lemma 3.9 to the action α with E in place of F, with η in place of ε, with x_0 in place of x, with y_1 in place of y, and with x_1 in place of z, we obtain a family of orthogonal positive contractions $(r_g)_{g\in G}$ in A such that, with $r = \sum_{g\in G} r_g$, the following hold:

8. $\|r_g c - c r_g\| < \eta$ for all $c \in E$ and all $g \in G$;
9. $\|\alpha_g(r_h) - r_{gh}\| < \eta$ for all $g, h \in G$;
10. $(y_1^2 - y_1 r y_1 - \eta)_+ \precsim_A x_0$;
11. $\|r x_1 r\| > 1 - \eta$.

On the other hand, since B has an approximate identity contained in B^β, we can choose a positive contraction $s \in B^\beta$ such that

12. $\|y_2 s y_2 - y_2^2\| < \eta$, $\|[s, d_i]\| < \eta$ for all $1 \le i \le m$, $\|[s, w_j]\| < \eta$ for all $1 \le j \le k$, and $\|s x_2 s\| > 1 - \eta$.

Put $f_g = r_g \otimes s$ for all $g \in G$, and put $f = \sum_{g\in G} f_g$. Then $(f_g)_{g\in G}$ is a family of mutually orthogonal positive contractions in $A \otimes_{\min} B$. We show that (1)–(4) hold. For (1), let $1 \le i \le m$. Then by (8) and (12) we have

$$\begin{aligned}
\|[f_g, c_i \otimes d_i]\| &= \|[r_g \otimes s, c_i \otimes d_i]\| \\
&= \|(r_g c_i) \otimes (s d_i) - (c_i r_g) \otimes (d_i s)\| \\
&\le \|[r_g, c_i] \otimes (s d_i)\| + \|(c_i r_g)[s, d_i]\| \\
&< \eta + \eta < 2\delta < \varepsilon.
\end{aligned}$$

Part (2) follows from (9). To prove (3), first using (12) at the third step we have

$$\begin{aligned}
&\| \left(y^2 - y f y\right) - \left(y_1^2 - y_1 r y_1 - \eta\right)_+ \otimes \left(y_2^2 - \delta\right)_+ \| \\
&\qquad \le \| \left(y_1^2 \otimes y_2^2\right) - (y_1 r y_1) \otimes (y_2 s y_2) - \left(y_1^2 - y_1 r y_1\right) \otimes y_2^2 \| + \eta + \delta \\
&\qquad = \| (y_1 r y_1) \otimes \left(y_2 s y_2 - y_2^2\right) \| + 2\delta \\
&\qquad < \delta + 2\delta < \varepsilon.
\end{aligned}$$

Then by Lemma 2.2 at the first step, by (6) and (10) at the second step, and by (7) at the fourth step, we get

$$\begin{aligned}
\left(y^2 - y f y - \varepsilon\right)_+ &\precsim_D \left(y_1^2 - y_1 r y_1 - \eta\right)_+ \otimes \left(y_2^2 - \delta\right)_+ \\
&\precsim_D x_0 \otimes (x_2 \otimes 1_n) \\
&\sim_D (x_0 \otimes 1_n) \otimes x_2 \\
&\precsim_D x_1 \otimes x_2 \precsim_D z \precsim_D x,
\end{aligned}$$

which is (3). To prove (4), first by (8) and (12) we have

$$\begin{aligned}\|fv - vf\| &= \left\|\sum_{i=1}^{k}(rv_i)\otimes(sw_i) - \sum_{i=1}^{k}(v_i r)\otimes(w_i s)\right\| \\ &\le \sum_{i=1}^{k}\|(rv_i - v_i r)\otimes(sw_i)\| + \sum_{i=1}^{k}\|(v_i r)\otimes(sw_i - w_i s)\| \\ &\le M\sum_{i=1}^{k}\sum_{g\in G}\|r_g v_i - v_i r_g\| + M\sum_{i=1}^{k}\|sw_i - w_i s\| \\ &< Mk\,\mathrm{card}(G)\eta + Mk\eta \le 2Mk\,\mathrm{card}(G)\eta < \delta.\end{aligned}$$

Also, by (11) and (12) we get

$$\begin{aligned}\|f(x_1\otimes x_2)f\| &= \|(rx_1r)\otimes(sx_2s)\| = \|rx_1r\|\cdot\|sx_2s\| \\ &> (1-\delta)(1-\eta) > (1-\delta)^2.\end{aligned}$$

Then by using these two latter inequalities and (5) we calculate

$$\begin{aligned}(1-\delta)^2 &< \|f(x_1\otimes x_2)f\| \\ &\le \|fvxv^*f\| + \|f(vxv^* - x_1\otimes x_2)f\| \\ &< \|vfxv^*f\| + \|(fv-vf)xv^*f\| + \delta^2 \\ &\le \|vfxfv^*\| + \|vfx(v^*f - fv^*)\| + \delta\|v\| + \delta^2 \\ &\le \|v\|^2\|fxf\| + \delta\|v\| + \delta\|v\| + \delta^2 \\ &\le \delta_1^{-2}\|fxf\| + 2\delta\delta_1^{-1} + \delta^2 \\ &< \delta_1^{-2}\|fxf\| + 4\delta + \delta^2.\end{aligned}$$

Therefore, by the choice of δ_1 we obtain

$$\|fxf\| > \left((1-\delta)^2 - 4\delta - \delta^2\right)\delta_1^2 > 1-\varepsilon$$

which is (4). This completes the proof. ☐

The proof of the following result is similar to that of the preceding theorem.

Theorem 4.6. *Let $\alpha\colon G\to\mathrm{Aut}(A)$ and $\beta\colon G\to\mathrm{Aut}(B)$ be actions of a finite group G on simple C^*-algebras A and B. Let α have the tracial Rokhlin property and let B^β have an approximate identity (not necessarily increasing) consisting of projections. Then the action $\alpha\otimes\beta\colon G\to\mathrm{Aut}(A\otimes_{\min}B)$ has the tracial Rokhlin property. In particular, Problem 4.4 has an affirmative answer.*

The following corollary follows from Theorems 4.5 and 4.6 by taking $B = M_n$ and β to be the trivial action.

Corollary 4.7. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G with the (weak) tracial Rokhlin property on a simple C*-algebra A. Then the induced action of G on $M_n(A)$ has the (weak) tracial Rokhlin property for any $n \in \mathbb{N}$.*

The following result gives a criterion for the nonunital tracial Rokhlin property in terms of the unital tracial Rokhlin property in the case that the underlying algebra has "enough" projections.

Proposition 4.8. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple C*-algebra A. Suppose that A has an approximate identity (not necessarily increasing) $(p_i)_{i\in I}$ consisting of projections such that each p_i is in A^α. Then α has the tracial Rokhlin property if and only if the restriction of α to p_iAp_i has the tracial Rokhlin property for every $i \in I$.*

Proof. The "if" part follows from Proposition 4.3, and the "only if" part follows from the second part of Proposition 4.2. □

5 Crossed products

The main goal of this section is to show that some classes of simple C*-algebras are closed under taking crossed products and fixed point algebras by actions of finite groups with the tracial Rokhlin property. In particular, this is true for the class of simple C*-algebras with tracial rank zero. This extends a result of Phillips ([43, Theorem 2.6]) to the nonunital case and is evidence that our definition of the (weak) tracial Rokhlin property on simple C*-algebras is the right one.

The following proposition is essential in the sequel.

Proposition 5.1. *Let G be a finite group and let $\mathcal{C}$ be a class of simple C*-algebras with the following properties:*

1. *if A is a simple C*-algebra and $p \in A$ is a nonzero projection, then $A \in \mathcal{C}$ if and only if $pAp \in \mathcal{C}$ (in particular, this is the case if $\mathcal{C}$ is closed under Morita equivalence);*
2. *if $A \in \mathcal{C}$ is unital and α is an action of G on A with the tracial Rokhlin property then $A \rtimes_\alpha G \in \mathcal{C}$;*
3. *if $A \in \mathcal{C}$ and B is a C*-algebra with $A \cong B$, then $B \in \mathcal{C}$.*

Then $\mathcal{C}$ is closed under taking crossed products and fixed point algebras by actions of G with the tracial Rokhlin property (and hence (2) above holds without the assumption that A is unital).

Proof. Let $\mathcal{C}$ be a class of simple C*-algebras as in the statement. Let $A \in \mathcal{C}$ and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the tracial Rokhlin property. We show that $A \rtimes_\alpha G \in \mathcal{C}$. We may assume that A is nonzero. First note that there exists a nonzero projection p in A^α. In fact, if $(p_g)_{g\in G}$ is a family of

Rokhlin projections for a given ε according to Definition 3.1 and if $q = \sum_{g\in G} p_g$, then $\|E(q) - q\| < \varepsilon \operatorname{card}(G)$, where $E\colon A \to A^\alpha$ is the canonical conditional expectation. Using functional calculus within A^α gives a projection $p \in A^\alpha$ arbitrarily close to $E(q)$. So, p is arbitrary close to q, and hence it is nonzero. Let $\alpha\colon G \to \operatorname{Aut}(pAp)$ be the restriction of α to pAp. Now, the second part of Proposition 4.2 implies that β has the tracial Rokhlin property. By Condition (1), $pAp \in \mathcal{C}$. Thus, by Condition (2), $pAp \rtimes_\beta G \in \mathcal{C}$. Observe that $pAp \rtimes_\beta G \cong p(A \rtimes_\alpha G)p$. In fact, the map $\varphi\colon pAp \rtimes_\beta G \to p(A \rtimes_\alpha G)p$ defined by $\varphi(\sum_{g\in G} b_g u_g) = \sum_{g\in G} b_g \delta_g$ where $b_g \in pAp$ for all $g \in G$, is easily seen to be a surjective $*$-isomorphism. Thus, by Condition (3), $p(A \rtimes_\alpha G)p \in \mathcal{C}$. Now Condition (1) implies that $A \rtimes_\alpha G \in \mathcal{C}$. Also, $A^\alpha \in \mathcal{C}$, by Condition (1). □

In the following theorem, we extend Phillips's result [43, Theorem 2.6] to the nonunital case.

Theorem 5.2. *Let A be a simple C*-algebra with tracial rank zero and let α be an action of a finite group G on A with the weak tracial Rokhlin property. Then the crossed product $A \rtimes_\alpha G$ and the fixed point algebra A^α are simple C*-algebras with tracial rank zero.*

Proof. It follows from Theorem 3.11 that α in fact has the tracial Rokhlin property. Let $\mathcal{C}$ denote the class of simple C*-algebras with tracial rank zero. By Theorem A.24, $\mathcal{C}$ satisfies Condition (1) in Proposition 5.1. Also, by [43, Theorem 2.6], $\mathcal{C}$ satisfies Condition (2) in Proposition 5.1. (Note that the assumption of separability is unnecessary in [43, Theorem 2.6].) Clearly, $\mathcal{C}$ satisfies Condition (3) in Proposition 5.1. Thus Proposition 5.1 yields the first part of the statement about the crossed product.
The second part of the statement about the fixed point algebra follows from Corollary 3.3 which says that A^α is isomorphic to a full corner of $A \rtimes_\alpha G$, and Theorem A.24 which implies that the tracial rank zero is passed to corners. □

The following corollary is immediate from Example 3.12(2) and Theorem 5.2.

Corollary 5.3. *Let A be a simple nonelementary C*-algebra with tracial rank zero and let $\beta\colon S_m \to A^{\otimes m}$ be the permutation action, where $m \geq 2$. Then the crossed product $A^{\otimes m} \rtimes_\beta S_m$ is a simple C*-algebra with tracial rank zero.*

Theorem 5.4. *The class of simple separable nuclear $\mathcal{Z}$-absorbing C*-algebras is preserved under taking crossed products and fixed point algebras by finite group actions with the tracial Rokhlin property.*

Proof. Let $\mathcal{C}$ denote the class of simple separable nuclear $\mathcal{Z}$-absorbing C*-algebras. By [50, Corollary 3.2], $\mathcal{Z}$-stability is preserved under Morita equivalence in the class of separable C*-algebras. Moreover, by [22, Theorem 3.15], nuclearity is preserved under Morita equivalence. Thus the class $\mathcal{C}$ satisfies (1) in Proposition 5.1. On the other hand, [21, Corollary 5.7] implies that the class $\mathcal{C}$ also satisfies (2) in Proposition 5.1. Therefore, by Proposition 5.1 the class $\mathcal{C}$ is preserved under taking crossed products by finite group actions of with the tracial Rokhlin property. The corresponding result about A^{α} holds by noting that A^{α} is a full corner of $A \rtimes_{\alpha} G$. □

Remark 5.5. There are some other classes of simple C*-algebras which are preserved under taking crossed products and fixed point algebras by finite group actions with the weak tracial Rokhlin property. For example, the class of simple purely infinite C*-algebras (by [26, Theorem 3]), the class of simple C*-algebras with Property (SP) (by the nonunital version of [27, Theorem 4.2]), and the class of simple tracially $\mathcal{Z}$-absorbing C*-algebras [1].

A C*-algebras with tracial rank zero

We begin this section with recalling the definition of tracial rank zero for C*-algebras from [34]. Then, we give a characterization of tracial rank zero which unifies the definitions for the simple unital and simple nonunital cases (Theorem A.6). The main advantage of this definition is to avoid working with the unitization of simple C*-algebras. In particular, we are able to show that having tracial rank zero is preserved under Morita equivalence in the class of simple C*-algebras.

A.1 Preliminaries

In this subsection we present some notation and results which will be used in the sequel. Also, the statement of the main theorem of this appendix (Theorem A.6) is given at the end of this subsection. The proof of this theorem will be given after Lemma A.19.

Notation A.1. We recall some notation from [34] for the convenience of the reader. We remark that, instead of notation $[a] \leq [b]$ used in [34], we will adopt the notation $a \precsim_s b$ from [38, Definition 2.1].

1. We denote by $\mathcal{I}^{(0)}$ the class of all finite dimensional C*-algebras.

2. Let σ_1, σ_2 be real numbers with $0 < \sigma_1 < \sigma_2 \leq 1$. Define a continuous function $f_{\sigma_1}^{\sigma_2} \colon [0,\infty) \to [0,1]$ by
$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 0 & 0 \leq t < \sigma_1, \\ \text{linear} & \sigma_1 \leq t < \sigma_2, \\ 1 & t \geq \sigma_2. \end{cases}$$

3. Let a and b be positive elements in a C*-algebra A. We say that a is *Blackadar subequivalent* to b and we write $a \precsim_s b$ if there exists $x \in A$ such that $x^*x = a$ and $xx^* \in \overline{bAb}$. Note that, $a \precsim_s b$ is equivalent to the relation $[a] \leq [b]$ which is used in [34, Definition 2.2] (see [38, Section 4]). Let n be a positive integer. We write $a \precsim_{s,n} b$ if there are n mutually orthogonal positive elements $b_1, \ldots, b_n \in \overline{bAb}$ such that $a \precsim_s b_i$ for all $i = 1, \ldots, n$.

Remark A.2. Observe that if p is a projection in a C*-algebra A and $a \in A_+$, then $p \precsim_s a$ if and only if $p \precsim_A a$ (see Lemma 2.5).

Definition A.3 (see [34], Definition 3.1). A unital C*-algebra A is said to have *tracial rank zero* if for any $\varepsilon > 0$, any finite subset $F \subseteq A$ containing a nonzero element $b \geq 0$, any $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, and any integer $n > 0$, there exist a nonzero projection $p \in A$ and a finite dimensional C*-subalgebra $E \subseteq A$ with $1_B = p$, such that

1. $\|pa - ap\| < \varepsilon$ for all $a \in F$;
2. $pFp \subseteq_\varepsilon B$;
3. $1 - p \precsim_{s,n} p$ and $f_{\sigma_1}^{\sigma_2}((1-p)b(1-p)) \precsim_{s,n} f_{\sigma_3}^{\sigma_4}(pbp)$.

If A has tracial rank zero, we will write $\mathrm{TR}(A) = 0$. A nonunital C*-algebra A is said to have $\mathrm{TR}(A) = 0$ if $\mathrm{TR}(A^\sim) = 0$.

Note that in [34], for any nonnegative integer k, the notion of a C*-algebra with tracial rank k is introduced. Lin also introduced a weaker version of the tracial rank zero as follows.

Definition A.4 (see [34], Definition 3.4). Let A be a unital C*-algebra. We write $\mathrm{TR}_w(A) = 0$ if for any $\varepsilon > 0$, any finite subset $F \subseteq A$ containing a nonzero element $b \geq 0$, any integer $n > 0$, and any full element $x \in A_+$, there exist a nonzero projection $p \in A$ and a finite dimensional C*-subalgebra $E \subseteq A$ with $1_B = p$, such that

(1) $\|pa - ap\| < \varepsilon$ for all $a \in F$;

(2) $pFp \subseteq_\varepsilon B$ and $\|pbp\| \geq \|b\| - \varepsilon$;

(3) $1 - p \precsim_{s,n} p$ and $1 - p \precsim_s x$.

If A is nonunital we write $\mathrm{TR}_w(A) = 0$ if $\mathrm{TR}_w(A^\sim) = 0$. Observe that for any C*-algebra A, $\mathrm{TR}_w(A) = 0$ if and only if A is TAF in the sense of [35].
Note that in Definition A.4, we may omit the assumption that b is positive. In fact, if b is not positive we may assume that $\|b\| = 1$ and then use b^*b instead of b.
The following theorem follows from [34, Theorem 6.13 and Remark 6.12].

Theorem A.5. *Let A be a simple unital C*-algebra. Then the following statements are equivalent:*

(a) $\mathrm{TR}(A) = 0$*;*

(b) $\mathrm{TR}_w(A) = 0$*;*

(c) for every finite set $F \subseteq A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there is a nonzero C-subalgebra $B \subseteq A$ with $B \in \mathcal{I}^{(0)}$ such that, with $p = 1_B$, the following hold:*

(1) $\|pa - ap\| < \varepsilon$ *for all* $a \in F$*;*

(2) $pFp \subseteq_{\varepsilon} B$*;*

(3) $1 - p \precsim_A x$*.*

Moreover, $\mathrm{TR}_w(A) = 0$ *if and only if* $\mathrm{TR}(A) = 0$*.*

The following is the main result of the appendix.

THEOREM A.6. *Let A be a simple C*-algebra. Then A has tracial rank zero if and only if it has an approximate identity (not necessarily increasing) consisting of projections and for any finite set $F \subseteq A$, any $x, y \in A_+$ with $x \neq 0$, and any $\varepsilon > 0$, there exists a finite dimensional C*-sublagebra $E \subseteq A$ such that, with $p = 1_E$, the following hold:*

(1) $\|pa - ap\| < \varepsilon$ *for all* $a \in F$*;*

(2) $pFp \subseteq_{\varepsilon} E$*;*

(3) $(y^2 - ypy - \varepsilon)_+ \precsim_A x$*;*

(4) $\|pxp\| > \|x\| - \varepsilon$*.*

The proof of this theorem needs some preparation and will be presented after Lemma A.19. Note that Theorem A.6 unifies the definitions of tracial rank zero for simple unital and simple nonunital cases.

A.2 C*-ALGEBRAS WITH PROPERTY (T_0)

To prove Theorem A.6, in this subsection we define Property (T_0) and study some of its properties.

DEFINITION A.7. Let A be a simple C*-algebra. We say that A has *Property* (T_0) if A has an approximate identity (not necessarily increasing) consisting of projections and for all positive elements $x, y \in A$ with $x \neq 0$, every finite set $F \subseteq A$, and every $\varepsilon > 0$, there is a finite dimensional C*-subalgebra $E \subseteq A$ such that, with $p = 1_E$, the following hold:

1. $\|pa - ap\| < \varepsilon$ for all $a \in F$;
2. $pFp \subseteq_{\varepsilon} E$;
3. $(y^2 - ypy - \varepsilon)_+ \precsim_A x$;
4. $\|pxp\| > \|x\| - \varepsilon$.

We need the following lemma in the sequel. The proof is very similar to that of Lemma 3.4 and so it is omitted.

Lemma A.8. *Let A be a C*-algebra and let $x \in A_+ \setminus \{0\}$. Suppose that $y \in A_+$ has the following property. For any finite set $F \subseteq A$ and any $\varepsilon > 0$ there exist a projection $p \in A$ and a finite dimensional C*-subalgebra $E \subseteq A$ with unit p such that the following hold:*

1. $\|pa - ap\| < \varepsilon$ *for all* $a \in F$;
2. $pFp \subseteq_\varepsilon E$;
3. $(y^2 - ypy - \varepsilon)_+ \precsim_A x$;
4. $\|pxp\| > \|x\| - \varepsilon$.

Then every positive element $z \in \overline{Ay}$ also has the same property.

The following proposition shows the relation between Property (T_0) and tracial rank zero for simple unital C*-algebras.

Proposition A.9. *Let A be a simple unital C*-algebra. The following statements are equivalent:*

1. *A is has Property* (T_0);
2. $\mathrm{TR}(A) = 0$;
3. $\mathrm{TR}_w(A) = 0$;
4. *for any x, y, ε, F as in Definition A.7 there is a nonzero finite dimensional C*-subalgebra $E \subseteq A$ such that (1), (2), and (3) in Definition A.7 hold.*

Proof. By Theorem A.5 we have (2)⇔(3). The implication (1)⇒(4) is obvious. Moreover, (4)⇒(3) follows from Theorem A.5 by applying (4) with $y = 1$ and using the fact that for any positive number $\varepsilon < 1$,

$$(1 - p - \varepsilon)_+ = (1 - \varepsilon)(1 - p) \sim_A (1 - p). \tag{15}$$

To see (3)⇒(1), note that (3) together with (15) imply that Definition A.7 is satisfied for $y = 1$. Now by Lemma A.8, Definition A.7 is satisfied for every $y \in A_+$. Therefore, (1) holds. □

We need the following lemma in the proof of Proposition A.11.

Lemma A.10. *Let A be a simple C*-algebra with Property* (T_0). *Then every unital hereditary C*-subalgebra of A also has Property* (T_0).

Proof. Let $B = qAq$ be a unital hereditary C*-subalgebra of A where q is a projection of A. Let $F \subseteq B$ be a finite subset, let $x, y \in B_+$ with $x \neq 0$, and let $\varepsilon > 0$. We may assume that $F \cup \{x, y\}$ is contained in the closed unit ball of B. Put $G = F \cup \{q\}$. Choose $\delta > 0$ with $\delta < \min\{\frac{1}{6}, \frac{\varepsilon}{43}\}$. Since A has Property (T_0), there is a subalgebra $E \subseteq A$ in $\mathcal{I}^{(0)}$ such that, with $p = 1_E$, the following hold:

1. $\|pa - ap\| < \delta$ for all $a \in F$;
2. $pGp \subseteq_\delta E$;
3. $(y^2 - ypy - \delta)_+ \precsim_A x$;
4. $\|pxp\| > \|x\| - \delta$.

Then by (1) we have

$$\|(qpq)^2 - qpq\| = \|qpqpq - qppq\| \leq \|qpq - pq\| < \delta.$$

Thus by [33, Lemma 2.5.5] (note that the assumption $\|a\| \geq \frac{1}{2}$ is unnecessary in the statement of that lemma), there is a projection $q_1 \in B$ such that:

5. $\|q_1 - qpq\| < 2\delta$.

By (2) there is $c \in E$ such that $\|qpq - c\| < \delta$. Then $\|q_1 - c\| < 3\delta$. Thus by [33, Lemma 2.5.4] (note that the assumption that a is self-adjoint is unnecessary in the statement of that lemma), there is a projection $e \in E$ such that:

6. $\|q_1 - e\| < 6\delta$.

Hence by [33, Lemma 2.5.1], there is a unitary $u \in A^\sim$ such that:

7. $u^*eu = q_1$ and $\|u - 1_{A^\sim}\| < 12\delta$.

Put $D = u^*eEeu$. Then D is in $\mathcal{I}^{(0)}$ and $D = q_1u^*Euq_1 \subseteq qAq = B$. Also, $1_D = u^*eu = q_1$. We show that:

8. $\|q_1a - aq_1\| < \varepsilon$ for all $a \in F$;
9. $q_1Fq_1 \subseteq_\varepsilon D$;
10. $(y^2 - yq_1y - \varepsilon)_+ \precsim_B x$;
11. $\|q_1xq_1\| > \|x\| - \varepsilon$.

By (1) and (5) we have $\|q_1 - pq\| \leq \|q_1 - qpq\| + \|qpq - pq\| < 3\delta$. Thus,

12. $\|q_1 - pq\| = \|q_1 - qp\| < 3\delta$.

To see (8), by (1) and (12) for all $a \in F$ we have

$$\|q_1a - aq_1\| \leq \|q_1a - pqa\| + \|pa - ap\| + \|aqp - aq_1\| < 7\delta < \varepsilon.$$

To show (9) let $a \in F$. By (2) there is $b \in E$ such that $\|pap - b\| < \delta$. Put $d = u^*ebeu \in D$. Then by (6), (7), and (12) we have:

$$\begin{aligned}\|q_1aq_1 - d\| &\leq \|q_1aq_1 - eq_1aq_1e\| + \|eq_1aq_1e - epqaqpe\| \\ &\quad + \|epape - ebe\| + \|ebe - d\| \\ &< 12\delta + 6\delta + \delta + 24\delta = 43\delta < \varepsilon.\end{aligned}$$

To see (11), by (12) at the first step and by (4) at the third step we get

$$\|q_1xq_1\| \geq \|pqxqp\| - 6\delta = \|pxp\| - 6\delta > \|x\| - 7\delta > \|x\| - \varepsilon.$$

To prove (10), first by (12) we have

$$\begin{aligned}\|(y^2 - ypy - \delta)_+ - (y^2 - yq_1y)\| &\leq \delta + \|(y^2 - ypy) - (y^2 - yq_1y)\| \\ &= \delta + \|ypqy - yq_1y\| < 4\delta < \varepsilon.\end{aligned}$$

Therefore, by Lemma 2.2, we get $(y^2 - yq_1y - \varepsilon)_+ \precsim_A (y^2 - ypy - \delta)_+ \precsim_A x$. Since B is hereditary in A, we obtain $(y^2 - yq_1y - \varepsilon)_+ \precsim_B x$. This completes the proof of (8)–(11), showing that B has Property (T_0). □

To compare Property (T_0) with tracial rank zero for simple not necessarily unital C*-algebras, we need the following result.

Proposition A.11. *Let A be a simple nonunital C*-algebra. Then A has Property* (T_0) *if and only if the following holds. For every $\varepsilon > 0$, every $n \in \mathbb{N}$, every nonzero positive element $x \in A^\sim$, every finite subset $F \subseteq A^\sim$ which contains a nonzero positive element x_1, and every σ_i, $1 \leq i \leq 4$, with $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, there exists a finite dimensional C*-subalgebra $E \subseteq A$ such that, with $p = 1_E$, the following hold:*

1. $\|pa - ap\| < \varepsilon$ *for all* $a \in F$;
2. $pFp \subseteq_\varepsilon E$ *and* $\|px_1p\| \geq \|x_1\| - \varepsilon$;
3. $1 - p \precsim_A x$ *and* $1 - p \precsim_{s,n} p$;
4. $f_{\sigma_1}^{\sigma_2}((1-p)x_1(1-p)) \precsim_{s,n} f_{\sigma_3}^{\sigma_4}(px_1p)$.

Proof. To prove the forward implication let A be a simple nonuintal C*-algebra with Property (T_0). By Definition A.7, there is a net $(p_i)_{i\in I}$ of projections in A which is a (not necessarily increasing) approximate identity for A. For any $y \in A^\sim$ we have

$$\|y\| = \lim_{i\to\infty} \|p_iyp_i\|. \tag{16}$$

In fact, write $y = \lambda + a$ where $\lambda \in \mathbb{C}$ and $a \in A$. Since A is not unital we have $\|y\| = \sup\{\|yb\| \colon b \in A \text{ with } \|b\| \leq 1\}$. Let $\delta > 0$. Then there is $b \in A$ with $\|b\| \leq 1$ such that $\|yb\| > \|y\| - \delta$. Note that $p_iyp_ib = \lambda p_ib + p_iap_ib$ which tends to $\lambda b + ab = yb$. Thus there is $j \in I$ such that $\|p_iyp_ib\| > \|y\| - \delta$ for all $i \geq j$. Then for every $i \geq j$ we have

$$\|y\| - \delta < \|p_iyp_ib\| \leq \|p_iyp_i\| \leq \|y\|,$$

and so (16) holds.

Next, let ε, n, F, x_1, and x be as in the statement. Write $F = \{x_1, \ldots, x_m\}$ and $x_j = \lambda_j + a_j$ where $\lambda_j \in \mathbb{C}$ and $a_j \in A$ for all $1 \leq j \leq m$. Choose d_3, d_4 with $\sigma_4 < d_3 < d_4 < \sigma_1$. By [34, Lemma 2.6], there exists $\eta > 0$ such that if $a, b \in A^\sim$ are positive elements with $\|a\|, \|b\| \leq \|x_1\|$ and $\|a - b\| < \eta$ then $[f_{d_3}^{d_4}(a)] \leq [f_{\sigma_3}^{\sigma_4}(b)]$. (Note that in [34, Lemma 2.6] it is assumed that $\|a\|, \|b\| \leq 1$ but the proof of this lemma works for any upper bound $M > 0$ instead of 1.) Choose δ with $0 < \delta < \min\{\frac{\varepsilon}{4}, \frac{\eta}{3}\}$. By the previous remark and that $(p_i)_{i\in I}$ is an approximate identity for A, there is $i \in I$ such that, with $a_j' = p_i a_j p_i$, the following hold:

5. $\|p_i x_1 p_i\| > \|x_1\| - \delta$;
6. $\|a_j p_i - a_j\| < \delta$ and $\|p_i a_j - a_j\| < \delta$ for all $1 \leq j \leq m$;
7. $\|a_j' - a_j\| < \delta$ for all $1 \leq j \leq m$;
8. $p_i x p_i \neq 0$.

Set $G = \{a_j' \mid 1 \leq j \leq m\}$. By Lemma A.10, $B = p_i A p_i$ has Property (T_0) and hence $\mathrm{TR}(B) \leq k$ by Proposition A.9. Then, by [34, Theorem 5.6], there is a C*-subalgebra $D \subseteq B$ with $D \in \mathcal{I}^{(0)}$ such that, with $q = 1_D$, the following hold:

9. $\|q a_j' - a_j' q\| < \delta$ for all $1 \leq j \leq m$;
10. $qGq \subseteq_\delta D$ and $\|q b_1 q\| \geq \|b_1\| - \delta$ where $b_1 = p_i x_1 p_i$;
11. $p_i - q \precsim_B p_i x p_i$, and $p_i - q \precsim_{s,n} q$.
12. $f_{\sigma_1}^{\sigma_2}((p_i - q) b_1 (p_i - q)) \precsim_{s,n} f_{d_3}^{d_4}(q b_1 q)$.

Put $E = \mathbb{C}(1 - p_i) + D$ and $p = 1 - p_i + q$ which is the unit of E (here 1 denotes the unit of $A^\sim$). Then $E \in \mathcal{I}^{(0)}$. Now we show that (1)–(4) in the statement hold. To see (1), by (7) and (9), for all $1 \leq j \leq m$ we have

$$\begin{aligned}\|p x_j - x_j p\| = \|p a_j - a_j p\| &\leq \|p a_j - p a_j'\| + \|p a_j' - a_j' p\| + \|a_j' p - a_j p\| \\ &< 2\delta + \|q a_j' - a_j' q\| < 3\delta < \varepsilon.\end{aligned}$$

To see (2), fix $1 \leq j \leq m$. By (10) there is $d \in D$ such that $\|q a_j' q - d\| < \delta$. Put $e = \lambda_j p + d \in E$. Then by (6) at the fifth step we have

$$\begin{aligned}\|p x_j p - e\| = \|p a_j p - d\| &\leq \|q a_j' q - d\| + \|p a_j p - q a_j' q\| \\ &< \delta + \|(1 - p_i) a_j (1 - p_i) + (1 - p_i) a_j q + q a_j (1 - p_i)\| \\ &< \delta + 2\|a_j - p_i a_j\| + \|a_j - a_j p_i\| < 4\delta < \varepsilon.\end{aligned}$$

For the second part of (2), by (6) at the third step, by (10) at the fifth step, and by (5) at the sixth step we get

$$\begin{aligned}\|p x_1 p\| &\geq \|(1 - p_i) x_1 (1 - p_i) + q x_1 q\| - \|(1 - p_i) x_1 q\| - \|q x_1 (1 - p_i)\| \\ &= \max\{\|(1 - p_i) x_1 (1 - p_i)\|, \|q x_1 q\|\} - \|(1 - p_i) a_1 q\| - \|q a_1 (1 - p_i)\| \\ &> \|q x_1 q\| - 2\delta = \|q p_i x_1 p_i q\| - 2\delta \\ &\geq \|p_i x_1 p_i\| - 3\delta > \|x_1\| - 4\delta > \|x_1\| - \varepsilon.\end{aligned}$$

To prove (3), note that $1-p=p_i-q$. Thus by (11), $1-p \precsim_{s,n} q \precsim_s p$. Also, we have $1-p=p_i-q \precsim_A p_i x p_i \precsim_A x$.
To see (4), first note that

$$(p_i-q)b_1(p_i-q)) = (p_i-q)x_1(p_i-q)) = (1-p)x_1(1-p)).$$

Thus by (12), $f_{\sigma_1}^{\sigma_2}((1-p)x_1(1-p)) \precsim_{s,n} f_{d_3}^{d_4}(qb_1q) = f_{d_3}^{d_4}(qx_1q)$. So to prove (4) it is enough to show that

$$f_{d_3}^{d_4}(qx_1q) \precsim_s f_{\sigma_3}^{\sigma_4}(px_1p). \tag{17}$$

For this, first by (6) we have (recall that $x_1=\lambda_1+a_1$):

$$\begin{aligned}
&\|px_1p-(qx_1q+\lambda_1(1-p_i))\| \\
&\qquad = \|(1-p_i)x_1q+qx_1(1-p_i)+(1-p_i)x_1(1-p_i)-\lambda_1(1-p_i)\| \\
&\qquad = \|(1-p_i)a_1q+qa_1(1-p_i)+(1-p_i)a_1(1-p_i)\| \\
&\qquad < 3\delta < \eta.
\end{aligned}$$

On the other hand, we have $\|px_1p\| \leq \|x_1\|$ and

$$\|qx_1q+\lambda_1(1-p_i)\| = \max\{\|qx_1q\|,\ \|\lambda_1(1-p_i)\|\} \leq \|x_1\|.$$

Thus, by the choice of η, we get

$$f_{d_3}^{d_4}(qx_1q+\lambda_1(1-p_i)) \precsim_s f_{\sigma_3}^{\sigma_4}(px_1p). \tag{18}$$

Also, since $qx_1q \perp \lambda_1(1-p_i)$ and $f_{d_3}^{d_4}(0)=0$ we have

$$f_{d_3}^{d_4}(qx_1q+\lambda_1(1-p_i)) = f_{d_3}^{d_4}(qx_1q) + f_{d_3}^{d_4}(\lambda_1(1-p_i)),$$

and hence,

$$f_{d_3}^{d_4}(qx_1q) \precsim_s f_{d_3}^{d_4}(qx_1q+\lambda_1(1-p_i)). \tag{19}$$

Combining (18) and (19), we get (17), and hence (4) follows.
Now we prove the backward implication. Suppose that the condition of the statement holds (we do not use (4) in the proof). We show that A has Property (T_0). Note that this condition is stronger than the definition of $\mathrm{TR}_w(A^\sim)=0$ (that is, $A^\sim$ is TAF), since it is not assumed that $x \in (A^\sim)_+$ is full. Observe that in [35, Proposition 2.7] the assumption that $a \in (A^\sim)_+$ is full is not used in the proof of both parts. Now let $\varepsilon>0$, let $x,y \in A_+$, and let $F \subseteq A$ be as in Definition A.7. By Lemma A.8 we may assume that $\|y\| \leq 1$. Then by (the proof of) [35, Proposition 2.7] with x in place of x_1, with $\mathcal{F}=F\cup\{x,y\}$, with $\frac{\varepsilon}{3}$ in place of ε, and with x in place of a, we obtain two orthogonal projections $p_1,p_2 \in A$ and a finite dimensional C*-subalgebra $E \subseteq A$ such that $p_1=1_E$ and that the following hold:

13. $\|p_ia-ap_i\| < \frac{\varepsilon}{3}$ for all $a \in \mathcal{F}$ and $i=1,2$;

14. $p_1\mathcal{F}p_1 \subseteq_{\frac{\varepsilon}{3}} E$, $\|p_1xp_1\| \geq \|x\| - \frac{\varepsilon}{3}$, and $\|(p_1+p_2)a - a\| < \frac{\varepsilon}{3}$ for all $a \in \mathcal{F}$;
15. $p_2 \precsim_A x$.

By (13) and (14), the finite dimensional C*-subalgebra $E \subseteq A$ with unit p_1 satisfies (1), (2), and (4) in Definition A.7. To see (3), first by (13) and (14) we get

$$\begin{aligned} \|y^2 - yp_1y - yp_2y\| &\leq \|y^2 - y^2p_1 - y^2p_2\| + \|y^2p_1 - yp_1y\| + \|y^2p_2 - yp_2y\| \\ &\leq \|y\|(\|y - y(p_1+p_2)\| + \|yp_1 - p_1y\| + \|yp_2 - p_2y\|) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then by (15) and Lemma 2.2 we have $(y^2 - yp_1y - \varepsilon)_+ \precsim_A yp_2y \precsim_A p_2 \precsim_A x$. Therefore, A has Property (T_0), as desired. (Note that (14) also implies that A has an approximate identity consisting of projections.) ☐

Observe that Proposition A.11 holds also for any simple *unital* C*-algebra A (note that in this case $A^\sim = A$ according to our convention). This follows from Proposition A.9 and [34, Theorem 5.6].

Remark A.12. Let A be a simple nonunital C*-algebra. Then A has Property (T_0) (equivalently, $\mathrm{TR}(A) = 0$ by Theorem A.6) if and only if Conditions (1)–(3) in Proposition A.11 hold (because Condition (4) is not used in the proof of the converse of Proposition A.11). Thus the only difference between the notion of having Property (T_0) (equivalently, $\mathrm{TR}(A) = 0$) and $\mathrm{TR}_w(A) = 0$ is that in the definition of $\mathrm{TR}_w(A) = 0$ (Definition A.4) it is required that the nonzero positive element $x \in A^\sim$ is full.

Now, we can prove one direction of Theorem A.6.

Proposition A.13. *Let A be a simple C*-algebra with Property* (T_0). *Then* $\mathrm{TR}(A) = 0$.

Proof. If A is a simple unital C*-algebra then A has Property (T_0) if and only if $\mathrm{TR}(A) = 0$, by Proposition A.9. Let A be a simple nonunital C*-algebra with Property (T_0). Then Proposition A.11 and Definition A.3 imply that $\mathrm{TR}(A) = 0$, as desired. ☐

A.3 Permanence properties

In this subsection we study some permanence properties of Property (T_0), and we give the proof of Theorem A.6. We begin with the following proposition which shows that if a simple C*-algebra has the local Property (T_0) then it has Property (T_0).

Proposition A.14. *Let A be a simple C*-algebra with the following property: for every $\varepsilon > 0$ and every finite subset $F \subseteq A$ there exists a simple C*-subalgebra B of A with Property* (T_0) *such that $F \subseteq_\varepsilon B$. Then A has Property* (T_0).

Proof. Let A be a simple C*-algebra with the property in the statement. Observe that A has an approximate identity (not necessarily increasing) consisting of projections. Let x, y, ε, and F be as in Definition A.7. We may assume that $\varepsilon < 1$ and $\|x\| = 1$. Also, by Lemma A.8 we may assume that $\|y\| < \frac{1}{2}$. Write $F = \{f_1, \ldots, f_m\}$. Choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{4}$ and $(2+\delta)\delta < \frac{\varepsilon}{12}$. Set $\tilde{F} = F \cup \{x^{\frac{1}{2}}, y^{\frac{1}{2}}\}$. By assumption there is a simple C*-subalgebra B of A with Property (T_0) such that $\tilde{F} \subseteq_\delta B$. Thus there is $b \in B$ such that $\|x^{\frac{1}{2}} - b\| < \delta$. Then

$$\begin{aligned} \|b^*b - x\| &\leq \|b^*b - b^*x^{\frac{1}{2}}\| + \|b^*x^{\frac{1}{2}} - x\| \\ &\leq \|b\|\delta + \|x^{\frac{1}{2}}\|\delta \\ &\leq (\|x^{\frac{1}{2}}\| + \delta + \|x^{\frac{1}{2}}\|)\delta < \tfrac{\varepsilon}{12}. \end{aligned} \tag{20}$$

Also, there exists $c \in B$ such that $\|y^{\frac{1}{2}} - c\| < \delta$. Similarly, we have $\|c^*c - y\| < \frac{\varepsilon}{12}$. Set $w = c^*c$. So $\|w\| < 1$. Also set $d = (b^*b - \frac{\varepsilon}{12})_+$. Note that $d \neq 0$ since $\|b^*b\| > 1 - \frac{\varepsilon}{12} > \frac{1}{2}$ and $\frac{\varepsilon}{12} < \frac{1}{2}$. Moreover, there exist $b_1, \ldots, b_m \in B$ such that $\|b_i - f_i\| < \delta$ for all $i = 1, \ldots, m$. Put $D = \{b_1, \ldots, b_m\}$. Since B has Property (T_0), by Definition A.7 there is a subalgebra $E \subseteq B$ in $\mathcal{I}^{(0)}$ such that, with $p = 1_E$, the following hold:

1. $\|pb_i - b_ip\| < \delta$ for all $i = 1, \ldots, m$;
2. $pDp \subseteq_\delta E$;
3. $(w^2 - wpw - \delta)_+ \precsim_B d$;
4. $\|pdp\| > \|d\| - \delta$.

Now we verify Conditions (1)–(4) in Definition A.7 for the given F, x, y, ε. For Condition (1), using (1) above and $\|b_i - f_i\| < \delta$ we have

$$\|pf_i - f_ip\| \leq \|pf_i - pb_i\| + \|pb_i - b_ip\| + \|b_ip - f_ip\| < 3\delta < \varepsilon.$$

To see Condition (2), fix $1 \leq i \leq m$. By (2) above there is $e \in E$ such that $\|pb_ip - e\| < \delta$. Then we have

$$\|pf_ip - e\| \leq \|pf_ip - pb_ip\| + \|pb_ip - e\| < 2\delta < \varepsilon.$$

To show Condition (4) in Definition A.7, using (4) at the second step and (20) at the fourth step we get

$$\begin{aligned} \|pxp\| &\geq \|pdp\| - \|p(x-d)p\| \\ &> \|d\| - \delta - \tfrac{\varepsilon}{12} \\ &> \|b^*b\| - \tfrac{\varepsilon}{12} - \tfrac{\varepsilon}{4} - \tfrac{\varepsilon}{12} \\ &> \|x\| - \tfrac{\varepsilon}{12} - \tfrac{\varepsilon}{2} > \|x\| - \varepsilon. \end{aligned}$$

To prove Condition (3) in Definition A.7, first using the inequalities $\|w-y\| < \frac{\varepsilon}{12}$ and $\|w\| < 1$, we obtain

$$\begin{aligned}
&\|(w^2 - wpw - \delta)_+ - (y^2 - ypy)\| \\
&\qquad \leq \|(w^2 - wpw - \delta)_+ - (w^2 - wpw)\| \\
&\qquad + \|(w^2 - wpw) - (y^2 - ypy)\| \\
&\qquad \leq \delta + \|w^2 - y^2\| + \|wpw - wpy\| + \|wpy - ypy\| \\
&\qquad \leq \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{6} + \tfrac{\varepsilon}{12} + \tfrac{\varepsilon}{12} < \varepsilon.
\end{aligned}$$

Therefore, by (3) and Lemma 2.2, $(y^2 - ypy - \varepsilon)_+ \precsim_A (w^2 - wpw - \delta)_+ \precsim_A d \precsim_A x$. This finishes the proof. □

The preceding proposition implies that the class of simple C*-algebras with Property (T_0) is closed under taking arbitrary inductive limits.
The following characterization of Property (T_0) is essential in the following.

Proposition A.15. *Let A be a simple C*-algebra. Then A has Property* (T_0) *if and only if there exists an approximate identity (not necessarily increasing) consisting of projections $(p_i)_{i\in I}$ for A such that $\mathrm{TR}(p_iAp_i) = 0$ for all $i \in I$.*

Proof. The forward implication follows from Definition A.7, Lemma A.10, and Proposition A.13. For the backward implication, let $(p_i)_{i\in I}$ be as in the statement. Then Proposition A.9 implies that each p_iAp_i is a simple C*-algebra with Property (T_0). Let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Since $(p_i)_{i\in I}$ is an approximate identity, there exists $i \in I$ such that $F \subseteq_\varepsilon p_iAp_i$. Applying Proposition A.14, we conclude that A has Property (T_0). □

With the preceding characterization of Property (T_0), we can obtain more properties of simple C*-algebras with Property (T_0).

Theorem A.16. *Let A be a simple C*-algebra with Property* (T_0)*. Then A has real rank zero and stable rank one.*

Proof. Let A be a nonzero simple C*-algebra with Property (T_0). Proposition A.15 implies the existence of a nonzero projection $p \in A$ such that $\mathrm{TR}(pAp) = 0$. Thus, by [34, Theorem 7.1], pAp has real rank zero. Since A is simple, pAp is a full corner of A and so pAp is Morita equivalent to A. Then by [11, Theorem 3.8], A has also real rank zero. To see that A has stable rank one, first note that [34, Theorem 6.9] and [34, Theorem 6.13] imply that $\mathrm{tsr}(pAp) = 1$. Moreover, by [6, Corollary 4.6], $\mathrm{tsr}(A) \leq \mathrm{tsr}(pAp)$. Hence, A has stable rank one. □

Proposition A.17. *Let A be a simple C*-algebra with Property* (T_0) *and let B be a hereditary C*-subalgebra of A. Then B has Property* (T_0) *if and only if it has an approximate identity (not necessarily increasing) consisting of projections.*

Proof. The forward implication follows from Definition A.7. For the backward implication let A be a simple C*-algebra with Property (T_0) and let B be a hereditary C*-subalgebra of A which contains an approximate identity consisting of projections $(p_i)_{i\in I}$. For each $i \in I$ we have $p_iBp_i = p_iAp_i$ which has Property (T_0) by Lemma A.10. Therefore, Proposition A.9 and Proposition A.15 imply that B has Property (T_0). □

Corollary A.18. *Let A be a simple C*-algebra. The following are equivalent:*

1. *A has Property (T_0);*
2. *$\overline{xAx}$ has Property (T_0) for all $x \in A_+$;*
3. *A has real rank zero and $\mathrm{TR}(pAp) = 0$ for all projections $p \in A$.*

Proof. (1)⇒(2): This follows from Proposition A.17 and the fact that $\overline{xAx}$ has real rank zero (by Theorem A.16). (2)⇒(3): Suppose that (2) holds. Then by Theorem A.16, $\overline{xAx}$ has real rank zero, for all $x \in A_+$. It follows that A has real rank zero. The second part of (3) follows from Proposition A.13. Finally, the implication (3)⇒(1) follows from Proposition A.15. □

Lemma A.19. *Let A be a simple C*-algebra with Property (T_0). Then $M_n(A)$ has Property (T_0) for all $n \in \mathbb{N}$.*

Proof. By Definition A.7, A has an approximate identity $(p_i)_{i\in I}$ consisting of projections. Put $q_i = \mathrm{diag}(p_i, \ldots, p_i) \in M_n(A)$. Then $(q_i)_{i\in I}$ is an approximate identity consisting of projections for $M_n(A)$. Lemma A.10 and Proposition A.9 yield that $\mathrm{TR}(p_iAp_i) = 0$. Thus $q_iM_n(A)q_i = M_n(p_iAp_i)$ has tracial rank zero, by [34, Theorem 5.8]. Hence, Proposition A.15 implies that $M_n(A)$ has Property (T_0). □

Now, we are in a position to prove Theorem A.6.

Proof of Theorem A.6. The backward implication follows from Proposition A.13. For the other direction, let A be a simple C*-algebra with $\mathrm{TR}(A) = 0$. We may assume that A is nonunital since the unital case follows from Proposition A.9. As $\mathrm{TR}(A) = 0$, [34, Corollary 5.7] implies that $\mathrm{TR}_w(A) = 0$ (recall that, by definition, $\mathrm{TR}(A) = \mathrm{TR}(A^\sim)$ and $\mathrm{TR}_w(A) = \mathrm{TR}_w(A^\sim)$). Thus, A is TAF in the sense of [35]. Now by [35, Corollary 2.8], A has an approximate identity $(p_i)_{i\in I}$ (not necessarily increasing) consisting of projections. (Note that the separability assumption is unnecessary in [35, Corollary 2.8].) Then it follows from [34, Theorem 5.3] that $\mathrm{TR}(p_iAp_i) = 0$ for all $i \in I$. Hence, Proposition A.15 implies that A has Property (T_0). □

Remark A.20. In view of Definition A.7, Theorem A.6 says that a simple C*-algebra A has tracial rank zero if and only if it has Property (T_0).

Theorem A.6 enables us to prove some permanence properties of simple C*-algebras of tracial rank zero which are not necessarily σ-unital.

Corollary A.21 (compare with [34], Proposition 4.8). *Let A be a simple C*-algebra which is an inductive limit of simple C*-algebras of tracial rank zero. Then A has tracial rank zero.*

Proof. This follows from Theorem A.6 and the remark after Proposition A.14. ☐

The following corollary was proved by Lin in the unital case. More precisely, Part (1) in the simple unital case follows from [34, Theorem 7.1] and [33, Theorem 3.6.11]. Part (2) is proved in [34, Theorem 5.8] in the unital not necessarily simple case. Part (3) in the case of a unital hereditary subalgebra follows from [34, Theorem 5.3]. We deal with the nonunital case.

Corollary A.22. *Let A be a simple C*-algebra with tracial rank zero. Then the following hold:*

1. *A has real rank zero and stable rank one;*
2. $\mathrm{TR}(M_n(A)) = 0$ *for all* $n \in \mathbb{N}$*;*
3. *if B is a hereditary C*-subalgebra of A then* $\mathrm{TR}(B) = 0$*.*

Proof. Part (1) follows from Theorems A.16 and A.6. Also, Part (2) follows from Lemma A.19 and Theorem A.6. Finally, Part (3) follows from Part (1), Proposition A.17, and Theorem A.6. ☐

A.4 Morita equivalence

In this subsection, we prove that the class of simple C*-algebras with tracial rank zero is closed under Morita equivalence (note that we do not assume any separability condition). This result was used in the proof of Theorem 5.2.

Proposition A.23. *Let A be a simple C*-algebra. Then A has Property* (T_0) *if and only if $A \otimes \mathcal{K}$ has Property* (T_0).

Proof. The forward implication follows from Lemma A.19, the remark after Proposition A.14, and the fact that $A \otimes \mathcal{K}$ is isomorphic to an inductive limit $\varinjlim M_n(A)$. The backward implication follows from Corollary A.22(3) and Theorem A.6. ☐

Theorem A.24. *Let A be a nonzero simple C*-algebra. The following statement are equivalent:*

1. $\mathrm{TR}(A) = 0$*;*
2. *A is Morita equivalent to a simple unital C*-algebra B with* $\mathrm{TR}(B) = 0$*;*

3. $\mathrm{TR}(pAp) = 0$ *for some (any) nonzero projection* $p \in A$.

*In particular, the class of simple C*algebras with tracial rank zero is closed under Morita equivalence.*

Proof. The implication (1)⇒(3) follows from Part (3) of Corollary A.22. Also, (3)⇒(2) is obvious. For (2)⇒(1), let B be a simple unital C*-algebra with $\mathrm{TR}(B) = 0$ such that B is Morita equivalent to A. By Part (1) of Corollary A.22 we have $\mathrm{RR}(B) = 0$. By [11, Theorem 3.8], having real rank zero is preserved under Morita equivalence, hence we get $\mathrm{RR}(A) = 0$. In particular, A has an approximate identity (not necessarily increasing) consisting of projections $(p_i)_{i\in I}$. For each $i \in I$, the simple unital C*-algebra p_iAp_i is Morita equivalent to A, and so it is Morita equivalent to B. Since both p_iAp_i and B are unital, they are stably isomorphic (by [10]). Thus by Proposition A.23, p_iAp_i has Property (T_0). Hence, Propositions A.9 and A.15 imply that A has Property (T_0). Now, Theorem A.6 yields that $\mathrm{TR}(A) = 0$.
The equivalence of Parts (1) and (2) implies that the class of simple C*algebras with tracial rank zero is closed under Morita equivalence. □

As an application of the preceding theorem, we give the following result.

Corollary A.25. *Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be an action of a second countable compact group* G *on a simple separable unital C*-algebra* A *with tracial rank zero. Suppose that* α *has the Rokhlin property in the sense of [14]. Then the crossed product* $A \rtimes_\alpha G$ *is a simple C*-algebra with tracial rank zero.*

The proof is mainly based on [14, Theorem 4.5] in which a similar result is obtained for the fixed point algebra. Note that the fixed point algebra is unital, and so the original definition of tracial rank for unital C*-algebras can be applied. However, when G is infinite, the crossed product is never unital. The Morita invariance of tracial rank zero for simple C*-algebras enables us to deal with this difficulty.

Proof of Corollary A.25. By [14, Theorem 4.5], the fixed point algebra A^α is a simple C*-algebra with tracial rank zero. Also, by [14, Proposition 2.7], the fixed point algebra and the crossed product are Morita equivalent. Thus, Theorem A.24 implies that the crossed product $A \rtimes_\alpha G$ is also simple with tracial rank zero. □

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REFERENCES

[1] M. AMINI, N. GOLESTANI, S. JAMALI, N. C. PHILLIPS, *Simple tracially $\mathcal{Z}$-absorbing C*-algebras*, preprint.

[2] P. ARA, F. PERERA, A. S. TOMS, *K-theory for operator algebras. Classification of C*-algebras*, 1–71 in: *Aspects of Operator Algebras and Applications*, P. Ara, F Lledó, and F. Perera (eds.), Contemporary Mathematics, vol. 534, Amer. Math. Soc., Providence RI, 2011.

[3] D. ARCHEY, *Crossed product C*-algebras by finite group actions with the tracial Rokhlin property,* Rocky Mountain J. Math. 41 (2011), no. 6, 1755–1768.

[4] D. ARCHEY, J. BUCK, N. C. PHILLIPS, *Centrally large subalgebras and tracial $\mathcal{Z}$-absorption,* Int. Math. Res. Not. IMRN 2018, no. 6, 1857–1877.

[5] B. BLACKADAR, *Symmetries of the CAR algebras*, Ann. of Math. (2) 131 (1990), no. 3, 589–623.

[6] B. BLACKADAR, *The stable rank of full corners in C*-algebras*, Proc. Amer. Math. Soc. 132 (2004), no. 10, 2945–2950.

[7] B. BLACKADAR, *Operator algebras: theory of C*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.

[8] O. BRATTELI, E. STØRMER, A. KISHIMOTO, M. RØRDAM, *The crossed product of a UHF algebra by a shift*, Ergodic Theory Dynam. Systems 13 (1993), no. 4, 615–626.

[9] L. G. BROWN, *Stable isomorphism of hereditary subalgebras of C*-algebras*, Pacific J. Math. 71 (1977), no. 2, 335–348.

[10] L. G. BROWN, P. GREEN, M. RIEFFEL, *Stable isomorphism and strong Morita equivalence of C*-algebras*, Pacific J. Math. 71 (1977), no. 2, 349–368.

[11] L. G. BROWN, G. K. PEDERSEN, *C*-algebras of real rank zero*, J. Funct. Anal. 99 (1991), no. 1, 131–149.

[12] M. DADARLAT, S. EILERS, *On the classification of nuclear C*-algebras*, Proc. London Math. Soc. 85 (2002), no. 3, 168–210.

[13] S. Echterhoff, W. Lück, N. C. Phillips, S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$,* J. Reine Angew. Math. 639 (2010), 173–221.

[14] E. Gardella, *Crossed product of compact group actions with the Rokhlin property*, J. Noncommut. Geom. 11(2017) 1593-1626.

[15] E. Gardella, I. Hirshberg, *Strongly outer actions of amenable groups on $\mathcal{Z}$-stable C*-algebras*, preprint 2020 (arXiv: 1811.00447v3 [math.OA]).

[16] E. Gardella, I. Hirshberg, L. Santiago, *Rokhlin dimension: duality, tracial properties and crossed products*, Ergod. Theory Dyn. Syst., Ergod. Theory Dyn. Syst. 41 (2021), no. 2, 408–460.

[17] E. Gardella, L. Santiago, *Equivariant $*$-homomorphisms, Rokhlin constraints and equivariant UHF-absorption*, J. Funct. Anal. 270 (2016), no. 7, 2543–2590.

[18] T. Giordano, D. Kerr, N. C. Phillips, A. S. Toms, *Crossed products of C*-algebras, topological dynamics, and classification*, Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Cham, 2018.

[19] N. Golestani, in preparation.

[20] R. Herman, A. Ocneanu, *Stability for integer actions on UHF C*-algebras,* J. Funct. Anal. 59 (1984), no. 1, 132–144.

[21] I. Hirshberg, J. Orovitz, *Tracially $\mathcal{Z}$-absorbing C*-algebras,* J. Funct. Anal. 265 (2013), no. 5, 765–785.

[22] A. A. Huef, I. Raeburn, D. P. Williams, *Properties preserved under Morita equivalence of C*-algebras*, Proc. Amer. Math. Soc. 135 (2007), no. 5, 1495–1503.

[23] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property, I,* Duke. Math. J. 122 (2004), no. 2, 233–280.

[24] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property, II,* Adv. Math. 184 (2004), no. 1, 119–160.

[25] J. Jacelon, *A simple, monotracial, stably projectionless C*-algebra*, J. Lond. Math. Soc. (2) 87 (2013), no. 2, 365–383.

[26] J. A. Jeong, *Purely infinite simple C*-crossed products*, Proc. Amer. Math. Soc. 123 (1995), no. 10, 3075–3078.

[27] J. A. Jeong, H. Osaka, *Extremely rich C*-crossed products and the cancellation property*, J. Austral. Math. Soc. (Series A) 64 (1998), 285–301.

[28] E. Kirchberg, M. Rørdam, *Non-simple purely infinite C*-algebras*, Amer. J. Math. 122 (2000), no. 3, 637–666.

[29] E. KIRCHBERG, M. RØRDAM, *Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Adv. Math. 167 (2002), no. 2, 195–264.

[30] A. KISHIMOTO, *Outer automorphisms and reduced crossed products of simple C*-algebras,* Comm. Math. Phys. 81 (1981), no. 3, 429–435.

[31] A. KISHIMOTO, *The Rohlin property for automorphisms of UHF algebras*, J. Reine Angew. Math. 465 (1995), 183–196.

[32] A. KISHIMOTO, *A Rohlin property for one-parameter automorphism groups*, Comm. Math. Phys. 179 (1996), no. 3, 599–622.

[33] H. LIN, *An introduction to the classification of amenable C*-algebras,* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[34] H. LIN, *The tracial topological rank of C*-algebras*, Proc. London Math. Soc. (3) 83 (2001), no. 1, 199–234.

[35] H. LIN, *Tracially AF C*-algebras*, Trans. Amer. Math. Soc. 353 (2001), no. 2, 693–722.

[36] H. MATUI, Y. SATO, *$\mathcal{Z}$-stability of crossed products by strongly outer actions*, Comm. Math. Phys. 314 (2012), no. 1, 193–228.

[37] N. NAWATA, *Finite group actions on certain stably projectionless C*-algebras with the Rohlin property*, Trans. Amer. Math. Soc. 368 (2016), no. 1, 471–493.

[38] E. ORTEGA, M. RØRDAM, H. THIEL, *The Cuntz semigroup and comparison of open projections*, J. Funct. Anal. 260 (2011), no. 12,3474–3493.

[39] H. OSAKA, N. C. PHILLIPS, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergod. Theory Dyn. Syst. 26 (2006), no. 5, 1579–1621.

[40] H. OSAKA, N. C. PHILLIPS, *Crossed products by finite group actions with the Rokhlin property*, Math. Z. 270 (2012), no. 1–2, 19–42.

[41] N. C. PHILLIPS, *Every simple higher dimensional noncommutative torus is an AT algebra*, preprint 2006 (arXiv: 0609783 [math.OA]).

[42] N. C. PHILLIPS, *Freeness of actions of finite groups on C*-algebras*, Operator structures and dynamical systems, 217–257, Contemp. Math., vol. 503, Amer. Math. Soc., Providence, RI, 2009.

[43] N. C. PHILLIPS, *The tracial Rokhlin property for actions of finite groups on C*-algebras*, Amer. J. Math. 133 (2011), no. 3, 581–636.

[44] N. C. PHILLIPS, *The tracial Rokhlin property is generic*, preprint 2012 (arXiv: 1209.3859 [math.OA]).

[45] N. C. Phillips, *Large subalgebras*, preprint 2014 (arXiv: 1408.5546v1 [math.OA]).

[46] N. C. Phillips, *Finite cyclic group actions with the tracial Rokhlin property*, Trans. Amer. Math. Soc. 367 (2015), no. 8, 5271–5300.

[47] M. Rørdam, *Classification of nuclear, simple C*-algebras*, 1–145 in: M. Rørdam and E. Størmer, *Classification of nuclear C*-algebras. Entropy in operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 126, Springer-Verlag, Berlin, 2002.

[48] J. Rosenberg, *Appendix to O. Bratteli's paper on "Crossed products of UHF-algebras"*, Duke Math. J. 46 (1979), no. 10, 25–26.

[49] L. Santiago, *Crossed products by actions of finite groups with the Rokhlin property*, Internat. J. Math. 26 (2015), no. 7, 1550042, 31 pp.

[50] A. S. Toms, W. Winter, *Strongly self-absorbing C*-algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029.

[51] Q. Wang, *Tracial Rokhlin property and non-commutative dimensions*, Thesis (Ph.D.)-Washington University in St. Louis. 2013.

[52] Q. Wang, *The tracial Rokhlin property for actions of amenable groups on C*-algebras*, Rocky Mountain J. Math. 48 (2018), no. 4, 1307–1344.

[53] W. Winter, J. Zacharias, *Completely positive maps of order zero*, Münster J. Math. 2 (2009), 311–324.

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