

MONODROMY FOR SOME RANK TWO GALOIS REPRESENTATIONS  
OVER CM FIELDS

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Received: November 15, 2019

Revised: June 24, 2020

Communicated by Don Blasius

ABSTRACT. We investigate local-global compatibility for cuspidal automorphic representations  $\pi$  for  $\mathrm{GL}_2$  over CM fields that are regular algebraic of weight 0. We prove that for a Dirichlet density one set of primes  $l$  and any  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ , the  $l$ -adic Galois representation attached to  $\pi$  and  $\iota$  has nontrivial monodromy at any  $v \nmid l$  in  $F$  at which  $\pi$  is special.

2020 Mathematics Subject Classification: 11F80, 11R39

Keywords and Phrases: Galois representations, automorphic forms, local-global compatibility, monodromy operator

## 1 INTRODUCTION

Let  $\pi$  be a regular algebraic cuspidal automorphic representation of  $\mathrm{GL}_n$  over a CM field  $F$ . Choose a prime  $l$  and an isomorphism  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ . If  $\pi$  is polarizable, then for any finite place  $v$  of  $F$ , the Galois representation  $r_\iota(\pi)$  attached to  $\pi$  and  $\iota$  satisfies local-global compatibility at  $v$  [BLGGT14a, Car12, Car14, HT01, Shi11, TY07]. The most subtle part is identifying the monodromy operator, the proofs of which rely on finding a base change of  $r_\iota(\pi)$  or its tensor square in the cohomology of a Shimura variety.

When  $\pi$  is not polarizable, it should not be possible to find  $r_\iota(\pi)$  itself in the cohomology of a Shimura variety (for precise statements, see [JT20]). One can hope to access the direct sum of  $r_\iota(\pi)$  and a twist of its conjugate dual via the cohomology of Shimura varieties, which is a basic starting point for the construction of  $r_\iota(\pi)$  by Harris–Lan–Talar–Thorne [HLTT16] as well as the alternate construction by Scholze [Sch15]. These constructions use  $l$ -adic interpolation, so are well suited to keeping track of characteristic polynomials,

and local-global compatibility was proved up to semisimplification by Varma [Var14] for all  $v \nmid l$ . But it doesn't seem possible to understand the monodromy operator in this way. We overcome this problem in almost all cases of rank 2 and weight 0:

**THEOREM 1.1.** *Suppose that  $F$  is a CM field and that  $\pi$  is a regular algebraic cuspidal automorphic representation of  $\mathrm{GL}_2(\mathbf{A}_F)$  of weight 0. There is a set of primes  $l$  of Dirichlet density one such that for any  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ , the  $l$ -adic Galois representation  $r_\iota(\pi) : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_l)$  attached to  $\pi$  and  $\iota$  satisfies*

$$\iota \mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{\mathrm{F}\text{-ss}} \cong \mathrm{rec}_{F_v}(\pi_v|\mathrm{det}|^{-1/2}),$$

for all finite places  $v \nmid l$  in  $F$ .

We prove a more technical result, Theorem 4.1 below, that applies to any prime  $l$  and  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$  to which we can apply an automorphy lifting theorem. The hypotheses necessary to apply the automorphy lifting theorem are known to hold for a density one set of primes (see Lemma 2.9), but should hold for all but finitely many (see Remark 2.10).

#### METHOD OF PROOF

In light of Varma's results, we need to prove that if  $v \nmid l$  is a finite place of  $F$  at which  $\pi$  is special, then  $r_\iota(\pi)|_{G_{F_v}}$  has nontrivial monodromy. Results of a similar spirit were proved by one of us (J.N.) [New15] in the context of Hilbert modular forms of partial weight one. In this situation, the Galois representations are also constructed by congruences, and one cannot realize these Hecke eigensystems in the Betti cohomology of a Shimura variety. The proof relies on a  $p$ -adic version of Mazur's principle [New13].

Another approach was developed by Luu [Luu15], relying on automorphy lifting theorems. The basic idea in the context of  $\mathrm{GL}_2$  is as follows. Assume that  $\pi$  is a twist of the Steinberg representation at  $v$ . After a base change, we can assume that  $\pi$  is an unramified twist of the Steinberg representation at  $v$ . Now assume that the  $l$ -adic Galois representation  $r_\iota(\pi)$  is unramified at  $v$ . Then so is its residual representation  $\overline{r}_\iota(\pi)$ , so we can hope to find a congruence to an automorphic representation  $\pi_1$  such that  $\pi_1$  is unramified at  $v$ . One can then apply automorphy lifting with the place  $v$  left out of the ramification set to prove that  $r_\iota(\pi) \cong r_\iota(\pi_2)$  for some automorphic representation  $\pi_2$  that is unramified at  $v$ . This contradicts strong multiplicity one.

The main ingredient needed to execute this strategy in the present context is an automorphy lifting theorem of [ACC<sup>+</sup>18], recalled in Theorem 2.1 below. However, there is still a subtlety that needs to be overcome: we need to produce a congruence to the automorphic representation  $\pi_1$  that is unramified at  $v$ . In the situations where the Galois representation in question does not appear in the Betti cohomology of a Shimura variety, these congruences don't always exist, see [CV19, §7.4.1, §7.4.2] for examples of level lowering congruences to torsion classes which do not have a characteristic 0 lift at the lower level. To

get around this problem here, we use Taylor’s potential automorphy method to first prove (see Theorem 3.9 for a more precise statement):

**THEOREM 1.2.** *Suppose that  $F$  is a CM field and  $l$  is an odd prime unramified in  $F$ . Let  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(k)$  be a continuous representation with  $k/\mathbf{F}_l$  finite such that  $\det(\bar{\rho}) = \bar{\epsilon}_l^{-1}$  and such that for each  $w|l$ ,  $\bar{\rho}|_{G_{F_w}}$  admits a crystalline lift with all labelled Hodge–Tate weights equal to  $\{0, 1\}$*

*Then there is a CM extension  $F'/F$  such that  $\bar{\rho}|_{G_{F'}}$  arises from a regular algebraic weight 0 cuspidal automorphic representation  $\pi_1$  of  $\mathrm{GL}_2(\mathbf{A}_{F'})$  that we can assume is unramified above our fixed  $v \nmid l$  in  $F$ .*

Applying the automorphy lifting theorem, we deduce that  $r_l(\pi)|_{G_{F'}}$  arises from a cuspidal automorphic representation  $\pi_2$  of  $\mathrm{GL}_2(\mathbf{A}_{F'})$  that is unramified at all places above  $v$ . We can no longer use multiplicity one, as this would require knowing the base change of  $\pi$  to  $F'$  exists, and  $F'/F$  may not be solvable. However, by Varma’s results, we know the Frobenius eigenvalues of  $r_l(\pi)$  at  $v$  and thus at any place above  $v$  in  $F'$ . This together with the fact that  $\pi_2$  is unramified above  $v$  contradicts the genericity of  $\pi_2$ .

Naturally, the automorphy lifting theorem we use contains several technical assumptions, reflected in the statement of Theorem 4.1. These technical assumptions should hold for all but finitely many primes  $l$ , but we do not know how to prove this. Using results of Larsen [Lar95], one can show that they hold on a density one set (see Lemma 2.9), resulting in Theorem 1.1.

Let us finally remark on the restriction to rank 2 and weight 0 in Theorems 1.1 and 4.1. These restrictions come from the combination of automorphy lifting theorems and potential automorphy theorems available to us in the non-polarizable case over CM fields. (There is also a simplification in rank 2 that there are only two possible conjugacy classes for the monodromy operator, but we do not believe that this is a serious issue.) Common to any potential automorphy theorem is some moduli space of motives in which one realizes a fixed mod  $l$  Galois representation. In [ACC+18, §7.2.5], a moduli space of elliptic curves is used, and this could be used to prove a version Theorem 4.1 with the additional assumption that the prime  $l$  splits completely in the coefficient field  $M_\pi$  of  $\pi$ . This in turn would yield a version of Theorem 1.1 valid for a positive density set of primes  $l$ . In order to allow more general primes  $l$  in Theorem 4.1 and obtain the density one set in Theorem 1.1, we use a moduli space of Hilbert–Blumenthal abelian varieties, as in Taylor’s original work on the subject [Tay03]. In either case, we are confined to rank 2, and since the Fontaine–Laffaille automorphy lifting theorem that we use does not allow a change of weight, we are also confined to weight 0.

Still working in rank 2, it seems possible to remove the weight 0 assumption in Theorem 4.1 at the cost of imposing an ordinarity assumption on  $\pi$  at the prime  $l$  by using the ordinary automorphy lifting theorem of [ACC+18], which does allow change of weight. This is currently being investigated by Yuji Yang. In higher rank, there are robust potential automorphy theorems in the polarizable case [HSBT10, BLGHT11, BLGGT14b] using the so-called Dwork family

of motives. If theorems similar to these were proved in the non-polarizable case, then we believe that the methods of this paper could be used to prove versions of Theorem 1.1 and 4.1 in higher rank.

#### NOTATION

For a field  $F$ , we let  $\overline{F}$  denote a separable closure and  $G_F = \text{Gal}(\overline{F}/F)$  the absolute Galois group. If  $F$  is a CM number field, then we write  $F^+$  for its maximal totally real subfield. CM number fields are always assumed to be imaginary.

Let  $F$  be a number field. If  $v$  a finite place of  $F$ ,  $l$  is a prime, and  $r : G_{F_v} \rightarrow \text{GL}_2(\overline{\mathbf{Q}}_l)$  is a continuous representation, we let  $\text{WD}(r)^{\text{F-ss}}$  be the associated Frobenius semisimple Weil–Deligne representation. If  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$  is an isomorphism of fields, we let  ${}^\iota\text{WD}(r)^{\text{F-ss}}$  denote its extension of scalars to  $\mathbf{C}$  via  $\iota$ . We write  $\text{rec}_{F_v}$  for the local Langlands correspondence of [HT01]. Let  $\pi$  be a regular algebraic cuspidal automorphic representation of  $\text{GL}_2(\mathbf{A}_F)$ . We say that  $\pi$  has *weight* 0 if it has the same infinitesimal character as the trivial (algebraic) representation of  $\text{Res}_{F/\mathbf{Q}} \text{GL}_2$ . We let  $M_\pi \subset \mathbf{C}$  denote the coefficient field of  $\pi$ ; it is the fixed field of  $\{\sigma \in \text{Aut}(\mathbf{C}) : \sigma\pi^\infty \cong \pi^\infty\}$ . If  $l$  is a prime and  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$  is an isomorphism of fields, we let  $r_\iota(\pi) : G_F \rightarrow \text{GL}_2(\overline{\mathbf{Q}}_l)$  be the  $l$ -adic Galois representation attached to  $\pi$  and  $\iota$  by Harris–Lan–Taylor–Thorne [HLTT16]. It is characterized by the property that if  $p \neq l$  is a prime above which  $\pi$  and  $F$  are unramified and  $v|p$  in  $F$ , then

$${}^\iota\text{WD}(r_\iota(\pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}_{F_v}(\pi_v|\det|^{-1/2}).$$

The isomorphism  $\iota$  induces a prime  $\lambda|l$  in  $M_\pi$  and an algebraic closure  $\overline{M}_{\pi,\lambda} = \overline{\mathbf{Q}}_l$  of the completion  $M_{\pi,\lambda}$ , and we also write  $r_{\pi,\lambda} : G_F \rightarrow \text{GL}_2(\overline{M}_\lambda)$  for  $r_\iota(\pi)$  in this case. Conversely, given  $\lambda|l$  in  $M_\pi$ , an algebraic closure  $\overline{M}_{\pi,\lambda}$  of  $M_{\pi,\lambda}$ , and an isomorphism  $\iota : \overline{M}_{\pi,\lambda} \xrightarrow{\sim} \mathbf{C}$ , we obtain  $r_{\pi,\lambda} = r_\iota(\pi)$  by identifying  $\overline{M}_{\pi,\lambda}$  with  $\overline{\mathbf{Q}}_l$ .

We let  $\epsilon_l$  denote the  $l$ -adic cyclotomic character. We normalize our Hodge–Tate weights so that  $\epsilon_l$  has all labelled Hodge–Tate weights equal to  $-1$ . We let  $\zeta_l$  denote a primitive  $l$ th root of unity.

Let  $F$  and  $M$  be number fields. If  $A$  is an abelian variety over  $F$  equipped with an embedding of rings  $\mathcal{O}_M \hookrightarrow \text{End}(A)$  and  $\mathfrak{l}$  is a prime of  $M$ , then we let  $r_{A,\mathfrak{l}}$  denote the representation of  $G_F$  on  $T_{\mathfrak{l}}(A) \otimes_{\mathcal{O}_{M,\mathfrak{l}}} \overline{M}_{\mathfrak{l}}$ .

#### ACKNOWLEDGEMENTS

We would like to thank Ariel Weiss, Yuji Yang, and the anonymous referees for comments on earlier versions of this paper. We would like to thank Frank Calegari, Ana Caraiani, David Helm, Toby Gee, Bao Le Hung, Peter Scholze, Richard Taylor, and Jack Thorne for many helpful conversations. This project grew out of our joint work [ACC<sup>+</sup>18], and the intellectual debt this work owes

to [ACC<sup>+</sup>18] will be clear to the reader. P.A. was supported in part by Simons Foundation Collaboration Grant for Mathematicians 527275.

## 2 AUTOMORPHY OF COMPATIBLE SYSTEMS

The crucial ingredient we need for the results of this paper is the following automorphy lifting theorem over CM fields, which is a special case of [ACC<sup>+</sup>18, Theorem 6.1.1] (the notions of enormous and decomposed generic will be recalled after the statement of the theorem):

**THEOREM 2.1.** *Let  $F$  be a CM field and let  $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_l)$  be a continuous representation satisfying the following conditions:*

1.  $\rho$  is unramified almost everywhere.
2. For each place  $v|l$  of  $F$ , the representation  $\rho|_{G_{F_v}}$  is crystalline with labelled Hodge–Tate weights all equal to  $\{0, 1\}$ . The prime  $l$  is unramified in  $F$ .
3.  $\bar{\rho}$  is decomposed generic and  $\bar{\rho}|_{G_{F(\zeta_l)}}$  is absolutely irreducible with enormous image. There exists  $\sigma \in G_F - G_{F(\zeta_l)}$  such that  $\bar{\rho}(\sigma)$  is a scalar. We have  $l \geq 5$ .
4. There exists a cuspidal automorphic representation  $\pi$  of  $\mathrm{GL}_2(\mathbf{A}_F)$  satisfying the following conditions:
  - (a)  $\pi$  is regular algebraic of weight 0.
  - (b) There exists an isomorphism  $\iota : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$  such that  $\bar{\rho} \cong \bar{r}_l(\pi)$ .
  - (c) If  $v|l$  is a place of  $F$ , then  $\pi_v$  is unramified.

Then  $\rho$  is automorphic: there exists a cuspidal automorphic representation  $\Pi$  of  $\mathrm{GL}_2(\mathbf{A}_F)$  of weight 0 such that  $\rho \cong r_l(\Pi)$ . Moreover, if  $v$  is a finite place of  $F$  and either  $v|l$  or both  $\rho$  and  $\pi$  are unramified at  $v$ , then  $\Pi_v$  is unramified.

Let  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_l)$  be a continuous representation and let  $\mathrm{ad}^0$  denote the set of trace zero matrices in  $M_{2 \times 2}(\overline{\mathbf{F}}_l)$ . The image  $H = \bar{\rho}(G_F)$  of  $\bar{\rho}$  is called *enormous* if it satisfies the following (c.f. [ACC<sup>+</sup>18, Definition 6.2.28 and Lemma 6.2.29]):

1.  $H$  has no nontrivial  $l$ -power order quotient.
2.  $H^0(H, \mathrm{ad}^0) = H^1(H, \mathrm{ad}^0) = 0$ .
3. For any simple  $\overline{\mathbf{F}}_l[H]$ -submodule  $W \subseteq \mathrm{ad}^0$ , there is a regular semisimple  $h \in H$  such that  $W^h \neq 0$ .

We say a prime  $p \neq l$  is *decomposed generic* for  $\bar{\rho}$  if it splits completely in  $F$  and for any  $v|p$  in  $F$ ,  $\bar{\rho}$  is unramified at  $v$  and the eigenvalues  $\alpha_v, \beta_v$  of  $\bar{\rho}(\mathrm{Frob}_v)$  satisfy  $\alpha_v \beta_v^{-1} \notin \{1, p, p^{-1}\}$  (c.f. [ACC<sup>+</sup>18, Definition 2.2.4]). We say that  $\bar{\rho}$  is *decomposed generic* if there is a prime  $p \neq l$  that is decomposed generic for  $\bar{\rho}$ .

LEMMA 2.2. *Let  $F$  be a number field and let  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_l)$  be a continuous representation with  $l > 5$ . If  $\bar{\rho}(G_F) \supseteq \mathrm{SL}_2(\mathbf{F}_l)$ , then  $\bar{\rho}(G_{F(\zeta_l)})$  is enormous.*

*Proof.* By [GN16, Lemma 3.2.3], and since we have assumed that  $l > 5$ , it suffices to show that  $\bar{\rho}|_{G_{F(\zeta_l)}}$  is absolutely irreducible. This follows from the assumption that  $\bar{\rho}(G_F) \supseteq \mathrm{SL}_2(\mathbf{F}_l)$  and the fact that  $\mathrm{SL}_2(\mathbf{F}_l)$  is perfect when  $l > 3$ .  $\square$

LEMMA 2.3. *Let  $F/\mathbf{Q}$  be a finite Galois extension and let  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_l)$  be a continuous representation with  $l > 3$ . If  $\bar{\rho}(G_F) \supseteq \mathrm{SL}_2(\mathbf{F}_l)$ , then  $\bar{\rho}$  is decomposed generic.*

*Proof.* This is contained in the proof of [ACC<sup>+</sup>18, Lemma 7.1.5]. For the convenience of the reader, we give the details. It suffices to prove  $\bar{\rho}$  is decomposed generic after replacing  $F$  with some finite extension. Conjugating  $\bar{\rho}$  if necessary, [DDT97, Theorem 2.47(b)] implies that there is an extension  $F'/F$  of degree at most 2 such that  $\bar{\rho}|_{G_{F'}}$  has projective image  $\mathrm{PSL}_2(k)$ , for some finite  $k/\mathbf{F}_l$ , which is simple since  $l > 3$ . If  $\widetilde{F}'$  is the Galois closure of  $F'/\mathbf{Q}$ , then  $\widetilde{F}'/F$  is abelian since  $F/\mathbf{Q}$  is Galois and  $F'/F$  is abelian. We can thus replace  $F$  with  $\widetilde{F}'$ .

Let  $H/F$  be the extension cut out by the projective image of  $\bar{\rho}$  and let  $\widetilde{H}/H$  be the Galois closure of  $H/\mathbf{Q}$ . Since  $\mathrm{PSL}_2(k)$  is simple, Goursat's lemma implies that  $\mathrm{Gal}(\widetilde{H}/F) \cong \mathrm{PSL}_2(k)^n$  for some  $n \geq 1$ . Fix a non-identity semisimple  $g \in \mathrm{PSL}_2(k)$ . By Chebotarev density, we can find a prime  $w$  of  $F$  such that  $\mathrm{Frob}_w$  in  $\mathrm{Gal}(\widetilde{H}/F)$  is  $(g, g, \dots, g)$ . Moreover, we can assume that the residue field at  $w$  is  $\mathbf{F}_p$  with  $p \neq l$  unramified in  $F$ , since such primes have Dirichlet density one in  $F$ . Since  $F/\mathbf{Q}$  is Galois,  $p$  is totally split in  $F$ . Take  $v|p$  in  $F$ . Since  $\mathrm{PSL}_2(k)$  is simple, the only normal subgroups of  $\mathrm{PSL}_2(k)^n$  are of the form  $\mathrm{PSL}_2(k)^I$  for  $I \subset \{1, \dots, n\}$ , so  $\mathrm{Aut}(\mathrm{PSL}_2(k)^n) = \mathrm{Aut}(\mathrm{PSL}_2(k))^n \rtimes S_n$ , and  $\mathrm{Aut}(\mathrm{PSL}_2(k)) = \mathrm{PGL}_2(k) \rtimes \mathrm{Gal}(k/\mathbf{F}_l)$  by [Die80]. So the image of  $\bar{\rho}(\mathrm{Frob}_v)$  in  $\mathrm{PSL}_2(k)$  is  $\tau(g)$  for some  $\tau \in \mathrm{PGL}_2(k) \rtimes \mathrm{Gal}(k/\mathbf{F}_l)$ . In particular, it is semisimple and  $\neq 1$ , so  $\bar{\rho}(\mathrm{Frob}_v)$  has distinct eigenvalues. As  $\zeta_l \in F$ ,  $\bar{\rho}$  is decomposed generic.  $\square$

LEMMA 2.4. *Suppose  $l > 3$ , let  $F$  be a number field in which  $l$  is unramified and let  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_l)$  be a continuous representation such that  $\bar{\rho}(G_F) \supseteq \mathrm{SL}_2(\mathbf{F}_l)$ . Then there exists  $\sigma \in G_F - G_{F(\zeta_l)}$  such that  $\bar{\rho}(\sigma)$  is a scalar.*

*Proof.* This is again contained in the proof of [ACC<sup>+</sup>18, Lemma 7.1.5], but we give the details. By [DDT97, Theorem 2.47(b)], the image of  $\bar{\rho}(G_F)$  in  $\mathrm{PGL}_2(\overline{\mathbf{F}}_l)$  is conjugate to  $\mathrm{PSL}_2(k)$  or  $\mathrm{PGL}_2(k)$  for some finite subfield  $k \subset \overline{\mathbf{F}}_l$ . This projective image is isomorphic to the image of  $\mathrm{ad} \bar{\rho}$ . Since  $l$  is unramified in  $F$ , we have  $\mathrm{Gal}(F(\zeta_l)/F) \cong (\mathbf{Z}/l\mathbf{Z})^\times$ . If  $l > 3$ , neither  $\mathrm{PSL}_2(k)$  nor  $\mathrm{PGL}_2(k)$  admits  $(\mathbf{Z}/l\mathbf{Z})^\times$  as a quotient, so we deduce that  $F(\zeta_l)$  is not contained in  $\overline{F}^{\ker \mathrm{ad} \bar{\rho}}$ . This amounts to the existence of the desired element  $\sigma$ .  $\square$

## 2.5 COMPATIBLE SYSTEMS

We follow the terminology of [ACC<sup>+</sup>18, §7.1]. For  $F$  a number field, a *rank  $n$  extremely weakly compatible system of Galois representations* is a tuple

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\})$$

where

- $M$  is a number field;
- $S$  is a finite set of primes of  $F$ ;
- for each prime  $v \notin S$  of  $F$ ,  $Q_v(X) \in M[X]$  is a monic polynomial of degree  $n$ ;
- for each prime  $\lambda$  of  $M$ ,  $r_\lambda : G_F \rightarrow \mathrm{GL}_n(\overline{M}_\lambda)$  is a continuous semisimple representation such that for every prime  $v$  of  $F$  with  $v \notin S$  and not dividing the residual characteristic of  $\lambda$ ,  $r_\lambda$  is unramified at  $v$  and  $r_\lambda(\mathrm{Frob}_v)$  has characteristic polynomial  $Q_v(X)$ .

There are obvious notions of direct sums, duals, tensor products, inductions, etc. for extremely weakly compatible systems. In particular, we have a rank one extremely weakly compatible system  $\det \mathcal{R}$  obtained by taking determinants of the  $r_\lambda$ . By [Hen82],  $\det(r_\lambda)$  is de Rham for each  $\lambda$ , and for any embedding  $\tau : F \hookrightarrow \overline{M}_\lambda$ ,  $\mathrm{HT}_\tau(\det(r_\lambda))$  is independent of  $\lambda$ .

We say  $\mathcal{R}$  is *irreducible* if every  $r_\lambda$  is irreducible (in the case of rank two, this is equivalent to any one  $r_\lambda$  being irreducible, see [ACC<sup>+</sup>18, Lemma 7.1.1]). We say that  $\mathcal{R}$  is *strongly irreducible* if  $\mathcal{R}|_{G_{F'}}$  is irreducible for any finite extension  $F'/F$ . The following lemma is contained in [ACC<sup>+</sup>18, Lemma 7.1.3] (which relies on a result of Larsen [Lar95]).

LEMMA 2.6. *Let  $F$  be a number field and let*

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\})$$

*be a strongly irreducible rank two extremely weakly compatible system. Then there is a set  $\mathcal{L}$  of rational primes with Dirichlet density one such that for all  $l \in \mathcal{L}$  and  $\lambda|l$  in  $M$ , the image of  $\overline{r}_\lambda$  contains a conjugate of  $\mathrm{SL}_2(\mathbf{F}_l)$ .*

By the main theorem of [HLTT16] (together with [Var14], to get local–global compatibility at all places where  $\pi$  is unramified), if  $\pi$  is a regular algebraic cuspidal automorphic representation of  $\mathrm{GL}_2(\mathbf{A}_F)$  with  $F$  a CM field, then we have a rank two extremely weakly compatible system

$$\mathcal{R}_\pi = (M_\pi, S_\pi, \{Q_{\pi,v}(X)\}, \{r_{\pi,\lambda}\})$$

with

- $M_\pi \subset \mathbf{C}$  the coefficient field of  $\pi$ ;

- $S_\pi$  the set of primes of  $F$  at which  $\pi$  is ramified;
- $Q_{\pi,v}(X)$  is the characteristic polynomial of  $\text{rec}_{F_v}(\pi_v|\det|^{-1/2})(\text{Frob}_v)$ .

LEMMA 2.7. *Let  $F$  be a CM field and let  $\pi$  be a regular algebraic cuspidal automorphic representation of  $\text{GL}_2(\mathbf{A}_F)$  of weight 0. If  $\pi$  is a twist of Steinberg at some finite place of  $F$ , then  $\mathcal{R}_\pi$  is strongly irreducible.*

*Proof.* It is well known that  $\mathcal{R}_\pi$  is irreducible. Since  $\pi$  has weight 0, there is a finite order character  $\chi : G_F \rightarrow M_\pi^\times$  such that  $\det(r_{\pi,\lambda}) = \chi\epsilon_l^{-1}$  for any  $l$  and  $\lambda|l$ . It then follows from [ACC+18, Lemma 7.1.2] that either  $\mathcal{R}_\pi$  is strongly irreducible or there is a quadratic extension  $K/F$  and an extremely weakly compatible system  $\mathcal{X}$  of characters of  $G_K$  such that  $\mathcal{R} = \text{Ind}_{G_K}^{G_F} \mathcal{X}$ . In the latter case, the system  $\mathcal{X}$  is the extremely weakly compatible system associated to a Hecke character  $\psi : K^\times \backslash \mathbf{A}_K^\times \rightarrow \mathbf{C}^\times$ , and we deduce that  $\pi$  is the automorphic induction of  $\psi$ . Such a  $\pi$  cannot be a twist of the Steinberg representation at any finite place.  $\square$

THEOREM 2.8. *Let  $F$  be a CM field and let  $\pi$  be a regular algebraic cuspidal automorphic representation of  $\text{GL}_2(\mathbf{A}_F)$  of weight 0. Let  $\lambda|l$  be a prime of the coefficient field  $M_\pi \subset \mathbf{C}$  of  $\pi$ , and let  $r_{\pi,\lambda} : G_F \rightarrow \text{GL}_2(\overline{M}_{\pi,\lambda})$  be the  $\lambda$ -adic Galois representation attached to  $\pi$ . Assume that the residual representation  $\overline{r}_{\pi,\lambda}$  is absolutely irreducible and decomposed generic. Assume also that  $l$  is unramified in  $F$  and lies under no prime at which  $\pi$  is ramified. Then for any  $v|l$  in  $F$ ,  $r_{\pi,\lambda}|_{G_{F_v}}$  is crystalline with all labelled Hodge–Tate weights equal to  $\{0, 1\}$ .*

*Proof.* The deduction of the theorem from [ACC+18, Theorem 4.5.1], is contained in [ACC+18, Lemma 7.1.8]. We give a sketch. Fix  $v|l$  in  $F$ . We can replace  $F$  with a finite solvable extension in which  $v$  splits completely. Doing so, we may assume the following:

- $F = F^+F_0$  with  $F^+$  totally real and  $F_0$  an imaginary quadratic field in which  $l$  splits.
- There are at least three places above  $l$  in  $F^+$ , and letting  $\overline{v}$  be the place of  $F^+$  below  $v$ , there is  $\overline{v}' \neq \overline{v}$  dividing  $l$  in  $F^+$  such that

$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F_{\overline{v}''}^+ : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}_p].$$

Then  $r_{\pi,\lambda}$  is a  $\overline{M}_\lambda$ -point of the Hecke algebra  $\mathbf{T}^S(K, 0)$  of [ACC+18, Theorem 4.5.1] for appropriate choices of a finite set of primes  $S$  of  $F$  and a level subgroup  $K \subset \text{GL}_2(\mathbf{A}_F^\infty)$ , from which the theorem follows.  $\square$

LEMMA 2.9. *Let  $F$  be a CM field, let  $\pi$  be a regular algebraic cuspidal automorphic representation of  $\text{GL}_2(\mathbf{A}_F)$  of weight 0, and let  $M_\pi \subset \mathbf{C}$  be its coefficient*



field. Assume that  $\pi_v$  is a twist of the Steinberg representation at some finite place  $v$  of  $F$ . Then there is a set  $\mathcal{L}$  of rational primes with Dirichlet density one such that for all  $l \in \mathcal{L}$  and all  $\lambda|l$  in  $M_\pi$ , the Galois representation  $r_{\pi,\lambda} : G_F \rightarrow \mathrm{GL}_2(\overline{M}_\lambda)$  satisfies the following:

1. For each place  $v|l$  of  $F$ , the representation  $r_{\pi,\lambda}|_{G_{F_v}}$  is crystalline with labelled Hodge–Tate weights all equal to  $\{0, 1\}$ . The prime  $l$  is unramified in  $F$ .
2.  $\overline{r}_{\pi,\lambda}$  is absolutely irreducible and decomposed generic. The image of  $\overline{r}_{\pi,\lambda}|_{G_{F(\zeta_l)}}$  is enormous. There exists  $\sigma \in G_F - G_{F(\zeta_l)}$  such that  $\overline{r}_{\pi,\lambda}(\sigma)$  is a scalar. We have  $l \geq 5$ .

*Proof.* Let  $\mathcal{R} = (M_\pi, S_\pi, \{Q_{\pi,v}\}, \{r_{\pi,\lambda}\})$  be the rank 2 extremely weakly compatible system attached to  $\pi$ . By Lemma 2.7,  $\mathcal{R}$  is strongly irreducible. Let  $\widetilde{F}$  be the Galois closure of  $F/\mathbf{Q}$ . By Lemmas 2.2, 2.3, 2.4, and Theorem 2.8, it suffices to show that there is a Dirichlet density one set  $\mathcal{L}$  of primes  $l$ , unramified in  $F$ , such that for all  $l \in \mathcal{L}$  and  $\lambda|l$  in  $M_\pi$ , the following hold:

- (a)  $\overline{r}_{\pi,\lambda}(G_{\widetilde{F}})$  contains a conjugate of  $\mathrm{SL}_2(\mathbf{F}_l)$ .
- (b)  $l > 5$  and lies under no prime at which  $\pi$  is ramified.

The restriction  $\mathcal{R}|_{G_{\widetilde{F}}}$  is again strongly irreducible, so Lemma 2.6 implies that there is a Dirichlet density one set  $\mathcal{L}'$  of primes  $l$  such that  $\overline{r}_{\pi,\lambda}(G_{\widetilde{F}})$  contains a conjugate of  $\mathrm{SL}_2(\mathbf{F}_l)$  for any  $l \in \mathcal{L}'$  and  $\lambda|l$  in  $M_\pi$ . We obtain  $\mathcal{L}$  by removing from  $\mathcal{L}'$  the finite set of primes  $l$  satisfying either  $l \leq 5$ ,  $l$  ramifies in  $F$ , or  $l$  lies under a place at which  $\pi$  is ramified.  $\square$

*Remark 2.10.* It should be apparent from the proof of Lemma 2.9 that the statement could be improved from “Dirichlet density one” to “all but finitely many,” provided one could prove that the image of  $\overline{r}_{\pi,\lambda}$  contains a conjugate of  $\mathrm{SL}_2(\mathbf{F}_l)$  for all but finitely many places  $\lambda$ . This would imply a similar strengthening of Theorem 1.1. In fact, it would suffice to know that  $\overline{r}_{\pi,\lambda}|_{G_{F(\zeta_l)}}$  is absolutely irreducible and decomposed generic for all but finitely many  $\lambda$ . Using the purity of  $r_{\pi,\lambda}$  (see [ACC<sup>+</sup>18, Corollary 7.1.12]), it is not hard to see that the decomposed generic condition holds for all but finitely many  $\lambda$ , so the main obstruction is showing residual irreducibility. This can be shown (see, for example, the main theorem of [HL16]), provided we know that the representations  $r_{\pi,\lambda}$  are crystalline with the correct Hodge–Tate weights (without assuming residual irreducibility as in Theorem 2.8). Such a crystallinity result has been proven by Mok [Mok14], under some technical hypotheses and using Arthur’s classification for automorphic representations of  $\mathrm{GSp}_4$  (see [Art04, GT19]).

### 3 POTENTIAL AUTOMORPHY

We begin by recalling a theorem of Moret–Bailly [MB89]:

PROPOSITION 3.1. *Let  $L$  be a totally real number field and let  $S_1 \amalg S_2$  be a finite set of finite places of  $L$ . Suppose that  $X/L$  is a smooth, geometrically connected variety. Suppose also that  $X(L_v) \neq \emptyset$  for all real places  $v$  of  $L$ , that  $\Omega_v \subset X(L_v^{\text{nr}})$  is a non-empty open (for the  $v$ -topology)  $\text{Gal}(L_v^{\text{nr}}/L_v)$ -invariant subset for the places  $v \in S_1$  and that  $\Omega_v \subset X(\overline{L}_v)$  is a non-empty open  $\text{Gal}(\overline{L}_v/L_v)$ -invariant subset for the places  $v \in S_2$ . Suppose finally that  $L^{\text{avoid}}/L$  is a finite extension.*

*Then there is a finite Galois totally real extension  $L_1/L$  and a point  $P \in X(L_1)$  such that*

- $L_1/L$  is linearly disjoint from  $L^{\text{avoid}}/L$
- every place  $v \in S_1$  is unramified in  $L_1$  and if  $w$  is a prime of  $L_1$  above  $v$  then  $P \in \Omega_v \cap X(L_{1,w})$
- if  $w$  is a prime of  $L_1$  above  $v \in S_2$  then  $P \in \Omega_v \cap X(L_{1,w})$ .

*Proof.* Our precise statement is a special case of [HSBT10, Prop. 2.1].  $\square$

We also recall a result on potential modularity of elliptic curves which is essentially contained in [Tay06]:

PROPOSITION 3.2. *Suppose that  $E/\mathbf{Q}$  is a non-CM elliptic curve, and that  $\mathcal{L}$  is a finite set of rational primes at which  $E$  has good reduction. Suppose also that  $L_1^{\text{avoid}}/\mathbf{Q}$  is a finite extension.*

*Then we can find*

- a finite Galois extension  $L_2^{\text{avoid}}/\mathbf{Q}$  linearly disjoint from  $L_1^{\text{avoid}}$  over  $\mathbf{Q}$  and
- a finite totally real Galois extension  $L^{\text{suffices}}/\mathbf{Q}$ , unramified above  $\mathcal{L}$  such that  $L^{\text{suffices}}$  is linearly disjoint from  $L_1^{\text{avoid}}L_2^{\text{avoid}}$  over  $\mathbf{Q}$

*such that for any finite totally real extension  $L_2/L^{\text{suffices}}$  which is linearly disjoint from  $L_2^{\text{avoid}}$  over  $\mathbf{Q}$ , there is a regular algebraic cuspidal automorphic representation  $\pi$  of  $\text{GL}_2(\mathbf{A}_{L_2})$  of weight 0 such that for every rational prime  $l$  and any  $\iota : \overline{\mathbf{Q}}_l \cong \mathbf{C}$  we have*

$$r_\iota(\pi) \cong r_{E,l}^\vee|_{G_{L_2}}.$$

*Moreover,  $\pi$  is unramified above any prime where  $E$  has good reduction.*

*Proof.* Our precise statement is a special case of [ACC<sup>+</sup>18, Corollary 7.2.4].  $\square$

### 3.3 HBAVs

Let  $M$  be a totally real number field and let  $S$  be a scheme. For an Abelian scheme  $A/S$  equipped with a ring embedding  $\iota : \mathcal{O}_M \hookrightarrow \text{End}(A/S)$  we denote by  $(\mathcal{M}_A, \mathcal{M}_A^+)$  the module  $\mathcal{M}_A$  of  $\mathcal{O}_M$ -linear, symmetric homomorphisms from  $A$  to  $A^\vee$ , with its positive cone  $\mathcal{M}_A^+$  of polarizations.

DEFINITION 3.4. An  $M$ -HBAV over  $S$  is a pair  $(A, \iota)$  as above, such that the natural map  $A \otimes_{\mathcal{O}_M} \mathcal{M}_A \rightarrow A^\vee$  is an isomorphism.

For a non-zero fractional ideal  $\mathfrak{c} \subset M$ , a  $\mathfrak{c}$ -polarization of an  $M$ -HBAV  $A$  is an isomorphism  $j : \mathfrak{c} \xrightarrow{\sim} \mathcal{M}_A$  of  $\mathcal{O}_M$ -modules with  $j(\mathfrak{c}^+) = \mathcal{M}_A^+$ , where  $\mathfrak{c}^+$  denotes the totally positive elements of  $\mathfrak{c}$ .

Remark 3.5. When the fractional ideal  $\mathfrak{c}$  contains  $\mathcal{O}_M$  the map  $x \mapsto x \otimes 1$  induces an isomorphism  $A/A[\mathfrak{c}^{-1}] \cong A \otimes_{\mathcal{O}_M} \mathfrak{c}$ . It follows that specifying a  $\mathfrak{c}$ -polarization  $j$  of an  $M$ -HBAV  $A$  is equivalent to specifying a single  $\mathcal{O}_M$ -linear polarization  $\lambda : A \rightarrow A^\vee$  with kernel  $A[\mathfrak{c}^{-1}]$ . The polarization  $\lambda$  corresponds to  $j(1)$ . See [DP94, §2.6].

If  $(A, \iota, j)$  is an  $M$ -HBAV equipped with a  $\mathfrak{c}$ -polarization and  $\mathfrak{a}$  is an ideal in  $\mathcal{O}_M$  we obtain perfect  $\mathcal{O}_M/\mathfrak{a}$ -bilinear alternating pairings (see [DP94, §2.12])

$$A[\mathfrak{a}] \times A[\mathfrak{a}] \rightarrow (\mathfrak{c}^{-1}\mathcal{D}^{-1} \otimes \mu_{N\mathfrak{a}})[\mathfrak{a}], \tag{3.5.1}$$

with  $\mathcal{D} \subseteq \mathcal{O}_M$  the different of  $M/\mathbf{Q}$  and  $N\mathfrak{a} = |\mathcal{O}_M/\mathfrak{a}|$  the norm of  $\mathfrak{a}$ , and isomorphisms

$$A[\mathfrak{a}] \otimes_{\mathcal{O}_M} \mathfrak{c} \rightarrow A[\mathfrak{a}]^\vee.$$

We refer to these alternating pairings as the Weil pairing for an HBAV with a fixed  $\mathfrak{c}$ -polarization.

We will also use the following related construction. Let  $K$  be either  $F$  or its completion at some prime. Let  $\mathfrak{l}$  be a prime of  $M$  of residual characteristic  $l$ , and let  $\bar{\tau} : G_K \rightarrow \mathrm{GL}_2(k_{\mathfrak{l}})$  be a continuous representation with  $\det \bar{\tau} = \bar{\epsilon}_{\mathfrak{l}}$ . Letting  $V_{\bar{\tau}}$  be the étale  $k_{\mathfrak{l}}$ -vector space scheme over  $K$  defined by  $\bar{\tau}$ , the standard symplectic pairing on  $V_{\bar{\tau}}$  is an  $\mathcal{O}_M$ -bilinear perfect pairing

$$V_{\bar{\tau}} \times V_{\bar{\tau}} \rightarrow (\mathcal{O}_M \otimes \mu_l)[\mathfrak{l}] \tag{3.5.2}$$

that induces an isomorphism  $V_{\bar{\tau}} \otimes \mathcal{D}^{-1} \cong V_{\bar{\tau}}^\vee$ , with  $V_{\bar{\tau}}^\vee$  the Cartier dual  $\mathrm{Hom}(V_{\bar{\tau}}, \mathcal{G}_m)$  of the group scheme  $V_{\bar{\tau}}$  (which naturally inherits the structure of  $k_{\mathfrak{l}}$ -vector space scheme). If further  $\mathfrak{l}$  is unramified in  $M$ , then  $x \mapsto x \otimes 1$  defines an isomorphism  $V_{\bar{\tau}} \cong V_{\bar{\tau}} \otimes \mathcal{D}^{-1}$  and the symplectic pairing defines an canonical isomorphism  $V_{\bar{\tau}} \cong V_{\bar{\tau}}^\vee$ .

If  $\mathfrak{l}$  is a prime of  $\mathcal{O}_M$  of residue characteristic  $l$ , then a *divisible*  $\mathcal{O}_{M,\mathfrak{l}}$ -module over a scheme  $S$  will mean a  $l$ -divisible group  $\mathcal{G}/S$  equipped with a ring homomorphism  $\mathcal{O}_{M,\mathfrak{l}} \rightarrow \mathrm{End}_S \mathcal{G}$ .

PROPOSITION 3.6. Let  $k$  be algebraically closed of characteristic  $l$ , let  $\mathfrak{l}$  be a prime of  $M$  lying over  $l$ . Let  $(\mathcal{G}/k, \lambda)$  be a divisible  $\mathcal{O}_{M,\mathfrak{l}}$ -module of height  $2[M_{\mathfrak{l}} : \mathbf{Q}_{\mathfrak{l}}]$  equipped with an  $\mathcal{O}_{M,\mathfrak{l}}$ -linear symmetric isomorphism (i.e. a principal quasi-polarization)  $\lambda : \mathcal{G} \cong \mathcal{G}^\vee$ .

Let  $\mathfrak{c} \supseteq \mathcal{O}$  be a fractional ideal of  $M$  such that  $l$  is coprime to  $\mathfrak{c}^{-1}$ . Then there exists an  $M$ -HBAV over  $k$  equipped with  $\mathfrak{c}$ -polarization  $(A, \iota, j)$  and an isomorphism  $i : A[\mathfrak{l}^\infty] \cong \mathcal{G}$  compatible with the  $\mathcal{O}_{M,\mathfrak{l}}$  actions on both sides such that  $i^\vee \circ \lambda \circ i = j(1)$ .

*Proof.* The statement is essentially [Yu03, Thm. 7.4], we just need to take a little care to ensure that our  $M$ -HBAV has a  $\mathfrak{c}$ -polarization. By [Yu03, Thm. 7.3(1)] (see also [GO00, Cor. 5.4.12] and [Gor01] for the case with  $l$  unramified in  $M$ , which will suffice for our applications) there is an  $M$ -HBAV with  $\mathfrak{c}'$ -polarization (for some fractional ideal  $\mathfrak{c}'$  of  $M$ )  $(A_0, \iota_0, j_0)$  such that the isocrystals of  $A_0[l^\infty]$  and  $\mathcal{G}$  (ignoring the polarization and the  $\mathcal{O}_{M,\iota}$ -action) are isomorphic. We let  $\mathfrak{b}$  be an (integral) ideal of  $\mathcal{O}_M$  with  $[\mathfrak{b}\mathfrak{c}'] = [\mathfrak{c}]$  in the narrow class group of  $M$ . Replacing  $A_0$  with  $A_0/A_0[\mathfrak{b}]$  we obtain a  $\mathfrak{c}$ -polarized  $M$ -HBAV  $A_0$  such that the isocrystals of  $A_0[l^\infty]$  and  $\mathcal{G}$  are isomorphic.

It follows from [Yu03, Cor. 3.7] that the quasi-polarized isocrystals with  $\mathcal{O}_{M,\iota}$ -action arising from the two quasi-polarised divisible  $\mathcal{O}_{M,\iota}$ -modules  $(A_0[l^\infty], \iota_0, j_0(1))$ ,  $(\mathcal{G}, \lambda)$  are isomorphic. We can also fix choices of principally quasi-polarized divisible  $\mathcal{O}_{M,\iota'}$ -modules for places  $\iota' \neq \iota|l$  and demand that  $A_0[l'^\infty]$  has quasi-polarized isocrystal isomorphic to these.

By Dieudonné theory, we have an  $\mathcal{O}_M$ -linear isogeny  $A_0 \xrightarrow{\pi} A$  with kernel contained in  $A_0[l^n]$  for some  $n$  and a symmetric  $\mathcal{O}_M$ -linear isogeny  $\lambda_A : A \rightarrow A^\vee$  with degree prime to  $l$  (since we ensure it induces our principal quasi-polarizations on our fixed divisible  $\mathcal{O}_{M,\iota'}$ -modules for all places  $\iota'|l$ ) such that  $\pi^\vee \circ \lambda_A \circ \pi = l^{2n}j_0(1)$ , together with an isomorphism  $i : A[l^\infty] \rightarrow \mathcal{G}$  such that  $i^\vee \circ \lambda \circ i = \lambda_A$ . Since  $\lambda_A$  has degree prime to  $l$  and  $\pi$  has  $l$ -power degree, it follows from the equation  $\pi^\vee \circ \lambda_A \circ \pi = l^{2n}j_0(1)$  that  $\lambda_A$  is a polarization with kernel  $A[\mathfrak{c}^{-1}]$ . Applying Remark 3.5, we obtain the desired  $M$ -HBAV  $A$  equipped with a  $\mathfrak{c}$ -polarization.  $\square$

*Remark 3.7.* If we have  $\mathfrak{c}$  and  $\mathcal{G}$  as in the above Proposition, the map  $x \mapsto x \otimes 1$  induces an isomorphism  $\mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_{M,\iota}} \mathfrak{c}_\iota$  (it induces an isomorphism from  $\mathcal{G}/\mathcal{G}[\mathfrak{c}_\iota^{-1}]$  to  $\mathcal{G} \otimes_{\mathcal{O}_{M,\iota}} \mathfrak{c}_\iota$ , and  $\mathfrak{c}^{-1}$  is coprime to  $l$ ). The quasi-polarization  $\lambda : \mathcal{G} \cong \mathcal{G}^\vee$  therefore corresponds to an isomorphism  $j_{\mathcal{G}} : \mathcal{G} \otimes_{\mathcal{O}_{M,\iota}} \mathfrak{c}_\iota \cong \mathcal{G}^\vee$ . The condition that  $j(1) = i^\vee \circ \lambda \circ i$  implies that, under the isomorphism  $i : A[l^\infty] \cong \mathcal{G}$ ,  $j_{\mathcal{G}}$  is induced by the  $\mathfrak{c}$ -polarization  $j$  on  $A$ .

**LEMMA 3.8.** *Let  $l$  be an odd prime and let  $v$  and  $\iota$  be primes of  $F$  and  $M$ , respectively, unramified over  $l$ . Let  $\bar{r} : G_{F_v} \rightarrow \mathrm{GL}_2(k_\iota)$  be a continuous representation such that:*

- $\det \bar{r} = \bar{\epsilon}_\iota$ ,
- *there is a crystalline lift  $r : G_{F_v} \rightarrow \mathrm{GL}_2(\mathcal{O})$  (for a finite extension  $\mathcal{O}/\mathcal{O}_{M,\iota}$ ) with labelled Hodge–Tate weights all equal to  $\{-1, 0\}$ .*

*Let  $V_{\bar{r}}$  be the  $k_\iota$ -vector space scheme over  $F_v$  underlying  $\bar{r}$ . Then we can find a divisible  $\mathcal{O}_{M,\iota}$ -module  $\mathcal{G}$  defined over  $\mathcal{O}_{F_v}$  of height  $2[\mathcal{O}_{M,\iota} : \mathbf{Z}_l]$  equipped with a  $\mathcal{O}_{M,\iota}$ -linear symmetric isomorphism  $\lambda : \mathcal{G} \cong \mathcal{G}^\vee$ , and an isomorphism  $i : V_{\bar{r}} \cong \mathcal{G}[l]_{F_v}$  such that  $i^\vee \circ \lambda[l]_{F_v} \circ i$  is the isomorphism  $V_{\bar{r}} \cong V_{\bar{r}}^\vee$  induced by the standard symplectic pairing on  $V_{\bar{r}}$ .*

*Proof.* This follows from Fontaine–Laffaille theory [FL82]. First, since  $l$  is unramified in  $F_v$ , the crystalline lift assumption implies that  $\bar{r}$  is in the image of the Fontaine–Laffaille functor: using the notation of *loc. cit.*, there is a  $k_l$ -object  $\bar{M}$  of  $\text{MF}_{\text{tor}}^{f,2}$  such that the action of  $G_{F_v}$  on  $U_S(\bar{M})$  is isomorphic to  $\bar{r}$ . By [CHT08, Lemma 2.4.1], we can find a lift  $r' : G_{F_v} \rightarrow \text{GL}_2(\mathcal{O}_{M,l})$  of  $\bar{r}$  such that for each  $n \geq 1$ , there is an  $\mathcal{O}_{M,l}$ -object  $M_n \in \text{MF}_{\text{tor}}^{f,2}$  such that the action of  $G_{F_v}$  on  $U_S(M_n)$  is isomorphic to  $r' \bmod l^n$ . (*Loc. cit.* uses a covariant version of the functor  $U_S$ , but the proof shows that the Fontaine–Laffaille modules can be deformed through Artinian thickenings, so carries over unchanged.) Then  $r$  is crystalline with all labelled Hodge–Tate weights equal to  $\{-1, 0\}$ , so  $\det r'|_{I_K} = \epsilon_l$ . Since  $\det \bar{r} = \epsilon_l$  and  $l > 2$ , we can find an unramified character  $\psi : G_{F_v} \rightarrow 1 + l\mathcal{O}_{M,l}$  such that  $\psi^2 = (\det r')\epsilon_l^{-1}$  and  $r'' := r' \otimes \psi$  is a lift of  $\bar{r}$  with determinant  $\epsilon_l$ . Moreover, there are  $\mathcal{O}_{M,l}$ -objects  $N_n$  of  $\text{MF}_{\text{tor}}^{f,2}$  corresponding to  $\psi \bmod l^n$  for each  $n \geq 1$ , and the action of  $G_{F_v}$  on  $U_S(M_n \otimes N_n)$  is given by  $r' \otimes \psi \bmod l^n$ . Applying [FL82, §9.11 and Proposition 9.12] to the collection  $\{M_n \otimes N_n\}_{n \geq 1}$ , we obtain a divisible  $\mathcal{O}_{M,l}$ -module  $\mathcal{G}$  defined over  $\mathcal{O}_{F_v}$  such that the  $G_{F_v}$ -action on the Tate module  $T_l(\mathcal{G})$  is isomorphic to  $r''$ . In particular, we have an isomorphism  $i : V_{\bar{r}} \cong \mathcal{G}[l]_{F_v} = \mathcal{G}[l]_{F_v}$  of  $k_l$ -vector space schemes over  $F_v$ .

It remains to produce  $\lambda$ . Since  $\det r'' = \epsilon_l$ , letting  $T = \mathcal{O}_{M,l}^2$  with  $G_{F_v}$ -action by  $r''$ , the standard symplectic pairing on  $T$  composed with the trace pairing  $\mathcal{O}_{M,l} \otimes \mathcal{O}_{M,l} \rightarrow \mathbf{Z}_l$  gives an isomorphism  $T \cong \text{Hom}_{\mathbf{Z}_l}(T, \mathbf{Z}_l(1))$ . This implies  $T_l(\mathcal{G}) \cong T_l(\mathcal{G}^\vee)$  compatibly with the  $\mathcal{O}_{M,l}$ -module structure. By a theorem of Tate [Tat67, Theorem 4], we obtain a  $\mathcal{O}_{M,l}$ -linear symmetric isomorphism  $\lambda : \mathcal{G} \cong \mathcal{G}^\vee$  such that  $i^\vee \circ \lambda[l]_{F_v} \circ i$  is the isomorphism  $V_{\bar{r}} \cong V_{\bar{r}}^\vee$  induced by the standard symplectic pairing on  $V_{\bar{r}}$ .  $\square$

**THEOREM 3.9.** *Suppose  $F$  is a CM field,  $l$  is an odd prime which is unramified in  $F$  and we have a continuous absolutely irreducible representation*

$$\bar{\rho} : G_F \rightarrow \text{GL}_2(k)$$

with  $k/\mathbf{F}_l$  finite such that:

- $\det \bar{\rho} = \bar{\epsilon}_l^{-1}$
- For all  $v|l$ ,  $\bar{\rho}|_{G_{F_v}}$  has a crystalline lift  $\rho_v : G_{F_v} \rightarrow \text{GL}_2(\mathcal{O})$  (for a finite extension  $\mathcal{O}/W(k)$ ) with labelled Hodge–Tate weights all equal to  $\{0, 1\}$

Suppose moreover that  $F^{\text{avoid}}/F$  is a finite extension. Then we can find a finite CM extension  $F_1/F$ , linearly disjoint from  $F^{\text{avoid}}$  over  $F$  and with  $l$  unramified in  $F_1$ , a regular algebraic cuspidal automorphic representation  $\pi$  for  $\text{GL}_2(\mathbf{A}_{F_1})$  unramified at places above  $l$  and of weight 0, together with an isomorphism  $\iota : \bar{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$  such that (composing  $\bar{\rho}$  with some embedding  $k \hookrightarrow \bar{\mathbf{F}}_l$ )

$$\bar{r}_\iota(\pi) \cong \bar{\rho}|_{G_{F_1}}.$$

If  $\bar{v}_0 \nmid l$  is a finite place of  $F^+$ , then we can moreover find  $F_1$  and  $\pi$  as above with  $\pi$  unramified above  $\bar{v}_0$ .

*Proof.* We begin by choosing a totally real number field  $M$  together with a prime  $l|l$  of  $M$  such that  $l$  is unramified in  $M$  and  $k_l$  is isomorphic to  $k$ . We fix an isomorphism  $k \cong k_l$  and regard  $\bar{\rho}$  as a representation with coefficients in  $k_l$ . Choose a non-CM elliptic curve  $E/\mathbf{Q}$  with good reduction at  $l$  and the rational prime  $q$  under  $\bar{v}_0$ . Choose a rational prime  $p \neq l$  such that

- $p > 5$  splits completely in  $FM$ ,
- $\mathrm{SL}_2(\mathbf{F}_p) \subset \bar{\tau}_{E,p}(G_F)$ ,  $E$  has good reduction at  $p$  and  $\bar{\rho}$  is unramified at places dividing  $p$ .

The second condition is satisfied by all but finitely many primes and the first condition is satisfied by a positive density set of primes, so we can find such a  $p$ . We fix a prime  $\mathfrak{p}|p$  of  $M$ .

Let  $V_{\bar{\rho}}^{\vee}$  denote the  $k_l$ -vector space scheme over  $F$  underlying the dual representation  $\bar{\rho}^{\vee}$  and fix the standard symplectic pairing on it, which  $\bar{\rho}^{\vee}$  will preserve up to multiplier  $\bar{\epsilon}_l$ . We also have the  $k_{\mathfrak{p}} \cong \mathbf{F}_p$ -vector space scheme  $(E \otimes_{\mathbf{Z}} \mathcal{O}_M)[\mathfrak{p}] \cong E[p]$  over  $F$ , which comes equipped with the Weil pairing. Denoting the inverse different of  $M$  by  $\mathcal{D}^{-1}$ , we let  $Y$  be the scheme over  $F$  classifying tuples  $(A, j, \alpha_{\bar{\rho}}, \alpha_E)$  where:

- $A$  is an  $M$ -HBAV with  $\mathcal{D}^{-1}$ -polarization  $j$
- $\alpha_{\bar{\rho}} : A[l] \rightarrow V_{\bar{\rho}}^{\vee}$  and  $\alpha_E : A[\mathfrak{p}] \rightarrow E[p]$  are isomorphisms of vector space schemes compatible with our fixed symplectic pairings on the right hand sides and with the pairings (see (3.5.1))  $A[l] \times A[l] \rightarrow (\mathcal{O}_M/l)(1)$  and  $A[\mathfrak{p}] \times A[\mathfrak{p}] \rightarrow (\mathcal{O}_M/\mathfrak{p})(1)$  on the left hand sides.

As in [Tay06],  $Y/F$  is a smooth, geometrically connected variety. We let  $X$  be the restriction of scalars  $X = \mathrm{Res}_{F/F^+} Y$ , which is also smooth and geometrically connected.

Now we apply Proposition 3.2 with  $\mathcal{L} = \{l, p\}$  and  $L_1^{\mathrm{avoid}}$  the normal closure of  $F^{\mathrm{avoid}} \bar{F}^{\ker(\bar{\rho} \times \bar{\tau}_{E,p})}$  over  $\mathbf{Q}$ . We obtain a finite Galois extension  $L_2^{\mathrm{avoid}}/\mathbf{Q}$  linearly disjoint from  $L_1^{\mathrm{avoid}}$  over  $\mathbf{Q}$  and a finite totally real Galois extension  $L^{\mathrm{suffices}}/\mathbf{Q}$  which is unramified above  $p$  and  $l$  and linearly disjoint from  $L_1^{\mathrm{avoid}} L_2^{\mathrm{avoid}}$  over  $\mathbf{Q}$ . We are going to apply Proposition 3.1 to  $X$  with the following input data:

- $L = F^+$ ,  $S_1 = \{\bar{v}|lp\}$ ,  $S_2 = \{\bar{v}_0\}$ ,  $L^{\mathrm{avoid}} = L_1^{\mathrm{avoid}} L_2^{\mathrm{avoid}} L^{\mathrm{suffices}}$
- for  $\bar{v}|lp$ ,  $\Omega_{\bar{v}} \subset X((F_{\bar{v}}^+)^{\mathrm{nr}}) = \prod_{v|\bar{v}} Y(F_v^{\mathrm{nr}})$  is the subset given by Abelian varieties  $A$  with good reduction at  $v$
- $\Omega_{\bar{v}_0} \subset X(\overline{F_{\bar{v}_0}^+}) = \prod_{v_0|\bar{v}_0} Y(\overline{F_{v_0}})$  is the subset given by Abelian varieties  $A$  with good reduction at  $v_0$ .

We need to check that the various hypotheses of Proposition 3.1 are satisfied. It is clear that  $X(F_{\bar{v}}^+) = Y(F_v)$  is non-empty for the real places  $\bar{v}$  of  $F^+$  ( $v$  denotes the unique complex place of  $F$  extending  $\bar{v}$ ).

For  $v$  a place of  $F$  dividing  $p$ , we can find a positive integer  $f$  such that  $\bar{\rho}(\text{Frob}_v)^{-f}$  and  $\bar{\tau}_{E,l}(\text{Frob}_v)^f$  are trivial. We can then take  $A$  to be the base change of  $E \otimes_{\mathbf{Z}} \mathcal{O}_M$  to the unramified degree  $f$  extension of  $F_v$ ,  $j$  to be induced by the Weil pairing on  $E$ ,  $\alpha_E$  to be the canonical identification (recall that  $p$  splits completely in  $M$ ) and  $\alpha_{\bar{\rho}}$  to be an isomorphism compatible with the Weil pairing on  $A[\mathfrak{l}]$  and our fixed pairing on  $V_{\bar{\rho}}^{\vee}$ . This shows that for  $\bar{v}|p$ ,  $\Omega_{\bar{v}}$  is non-empty. A similar argument applies to  $\Omega_{\bar{v}_0}$ ; we can work over an extension which trivialises  $\bar{\rho}|_{G_{F_{v_0}}}$  for  $v_0|\bar{v}_0$ .

It remains to handle the case of  $v|l$ ; we set  $K = F_v$ . By Lemma 3.8, we have a divisible  $\mathcal{O}_{M,l}$ -module  $\mathcal{G}$  over  $\mathcal{O}_K$  equipped with a principal quasi-polarization  $\lambda : \mathcal{G} \cong \mathcal{G}^{\vee}$  such that the  $G_K$  action on  $\mathcal{G}[\mathfrak{l}]_K$  is isomorphic to  $\bar{\rho}^{\vee}$  and  $\lambda$  induces our fixed pairing on  $V_{\bar{\rho}}^{\vee}$ . We can work with an integral model  $\mathcal{Y}/\mathcal{O}_K$  for  $Y_K$ , classifying tuples  $(A, j, \alpha_{\bar{\rho}}, \alpha_E)$ , where now  $A/S$  ( $S$  an  $\mathcal{O}_K$ -scheme) is an  $M$ -HBAV with  $\mathcal{D}^{-1}$ -polarization  $j$  and  $\alpha_{\bar{\rho}} : A[\mathfrak{l}] \rightarrow \mathcal{G}[\mathfrak{l}]$  is an isomorphism of vector space schemes, compatible with the isomorphisms  $A[\mathfrak{l}] \cong A[\mathfrak{l}]^{\vee}$  induced by  $j$  (see Remark 3.7) and  $\lambda : \mathcal{G}[\mathfrak{l}] \cong \mathcal{G}[\mathfrak{l}]^{\vee}$  and similarly for  $\alpha_E$  ( $E$  has good reduction at  $l$ , so  $E[p]$  extends to a vector space scheme over  $\mathcal{O}_K$  equipped with a canonical isomorphism  $E[p] \cong E[p]^{\vee}$ ). Now it suffices to show that  $\mathcal{Y}(\mathcal{O}_K^{\text{nr}})$  is non-empty. In fact, by Greenberg’s approximation theorem [Gre66, Corollary 2], it suffices to show that  $\mathcal{Y}(\check{\mathcal{O}})$  is non-empty, where  $\check{\mathcal{O}}$  is the  $l$ -adic completion of  $\mathcal{O}_K^{\text{nr}}$ . It follows from Proposition 3.6 that we have a  $\mathcal{D}^{-1}$ -polarized  $M$ -HBAV  $(A_1, j)$  over  $k = \mathcal{O}_K^{\text{nr}}/l$  with  $l$ -divisible module isomorphic to  $\mathcal{G}_k$  and  $j(1)$  inducing our fixed quasi-polarization on  $\mathcal{G}_k$ . By Serre–Tate deformation theory, we can lift  $A_1$  to an Abelian scheme  $\tilde{A}_1$  over  $\check{\mathcal{O}}$  equipped with a  $\mathcal{D}^{-1}$ -polarization  $\tilde{j}$  and an isomorphism  $\tilde{A}_1[\mathfrak{l}^{\infty}] \xrightarrow{\sim} \mathcal{G}_{\check{\mathcal{O}}}$  under which  $\tilde{j}$  corresponds to  $\lambda$ . In particular, the induced isomorphism  $\alpha_{\bar{\rho}} : \tilde{A}_1[\mathfrak{l}] \xrightarrow{\sim} \mathcal{G}_{\check{\mathcal{O}}}[\mathfrak{l}]$  is compatible with the (quasi-)polarizations on both sides. This gives us the  $\tilde{A}_1, j$  and  $\alpha_{\bar{\rho}}$  we need. We let  $\alpha_E$  be an isomorphism (between two trivial vector space schemes) compatible with the polarizations on each side. Now we have described a point of  $\mathcal{Y}(\check{\mathcal{O}})$  as desired.

We have checked the hypotheses of Proposition 3.1. So we obtain a finite Galois totally real extension  $F_0^+/F^+$ , linearly disjoint from  $L_1^{\text{avoid}}L_2^{\text{avoid}}L^{\text{suffices}}$  over  $F^+$  (and in particular from  $F$ , so  $F_0 := F_0^+F$  is a totally imaginary quadratic extension of  $F_0^+$ ) and a point  $(A, j, \alpha_{\bar{\rho}}, \alpha_E)$  of  $X(F_0^+)$  such that  $A$  has good reduction above  $\bar{v}_0lp$ . Moreover,  $l$  and  $p$  are unramified in  $F_0$ .

Finally, we set  $F_1 = F_0^+L^{\text{suffices}}F$ , a CM extension of  $F$  which is unramified above  $p$  and  $l$ . Since  $F_1^+$  is linearly disjoint from  $L_2^{\text{avoid}}$  over  $\mathbf{Q}$  and contains  $L^{\text{suffices}}$ , Proposition 3.2 tells us that there is a regular algebraic conjugate self-dual cuspidal automorphic representation  $\sigma$  of  $\text{GL}_2(\mathbf{A}_{F_1})$  of weight 0 such that  $\bar{\tau}_{\iota}(\sigma) \cong \bar{\tau}_{E,p}^{\vee}|_{G_{F_1}}$  for all  $\iota : \bar{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ . Moreover we can assume that  $\sigma$  is unramified above  $\bar{v}_0lp$ . Since  $F_1$  is linearly disjoint from  $L_1^{\text{avoid}}$  over  $F$ , we have  $\text{SL}_2(\mathbf{F}_p) \subset \bar{\tau}_{E,p}(G_{F_1})$ .

Fixing a choice of  $\iota$  and applying Theorem 2.1, we deduce that we have a regular algebraic cuspidal automorphic representation  $\pi$  of  $\text{GL}_2(\mathbf{A}_{F_1})$ , unramified at

places above  $pl\bar{v}_0$  and of weight 0 such that  $r_{A,p}^\vee \cong r_\iota(\pi)$ . Our choice of  $\iota$  determines an embedding  $\tau : M \hookrightarrow \mathbf{C}$  by composing  $\iota$  with  $M \hookrightarrow M_p = \mathbf{Q}_p$ . We choose  $\iota_l : \bar{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$  so that the embedding  $\iota_l^{-1} \circ \tau$  induces the place  $\mathfrak{l}$ , and denote the induced embedding  $M_l \hookrightarrow \bar{\mathbf{Q}}_l$  by  $\iota_{M_l}$ . It follows that we have  $r_{\iota_l}(\pi) \cong \iota_{M_l} \circ r_{A,l}^\vee$ , and we deduce the statement of the theorem since we have an isomorphism  $\alpha_{\bar{p}} : A[\mathfrak{l}] \cong V_{\bar{p}}^\vee$ .  $\square$

4 LOCAL–GLOBAL COMPATIBILITY

**THEOREM 4.1.** *Suppose that  $F$  is a CM field and that  $\pi$  is a regular algebraic cuspidal automorphic representation of  $\mathrm{GL}_2(\mathbf{A}_F)$  of weight 0, and let  $M_\pi \subset \mathbf{C}$  be its coefficient field. Let  $\lambda|l$  be a prime of  $M_\pi$  such that:*

1.  $l \geq 5$ ,  $l$  is unramified in  $F$ , and lies under no prime at which  $\pi$  is ramified.
2.  $\bar{\tau}_{\pi,\lambda}$  is decomposed generic,  $\bar{\tau}_{\pi,\lambda}(G_{F(\zeta_l)})$  is enormous, and there is a  $\sigma \in G_F - G_{F(\zeta_l)}$  such that  $\bar{\tau}_{\pi,\lambda}(\sigma)$  is scalar.

Then, for any  $\iota : \bar{M}_{\pi,\lambda} \xrightarrow{\sim} \mathbf{C}$  and any finite  $v \nmid l$  in  $F$ , we have

$$\iota \mathrm{WD}(r_{\pi,\lambda}|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_{F_v}(\pi_v |\det|^{-1/2}).$$

*Proof.* Fix a prime  $p \neq l$  for which  $\bar{\tau}_{\pi,\lambda}$  is decomposed generic. By the main result of [Var14], to prove the theorem it suffices to show that if  $v \nmid l$  is a finite place at which  $\pi$  is special, then  $r_{\pi,\lambda}$  has nontrivial monodromy at  $v$ . Fix  $\iota : \bar{M}_{\pi,\lambda} \xrightarrow{\sim} \mathbf{C}$  and let  $N$  be the monodromy operator for  $\mathrm{WD}(r_{\pi,\lambda}|_{G_{F_v}})^{\mathrm{F-ss}}$ . To show  $N \neq 0$ , it suffices to do so after restriction to any finite extension. In particular, making a solvable base change that is disjoint from  $\bar{F}^{\ker(\bar{\tau}_{\pi,\lambda})}$  in which  $l$  is unramified and  $p$  is totally split, we may assume that

- $\pi_v$  is an unramified twist of the Steinberg representation,
- $\bar{\tau}_{\pi,\lambda}$  is unramified at  $v$  and  $v^c$ .

Now assume for a contradiction that  $N = 0$ . Then the main result of [Var14] implies that  $r_{\pi,\lambda}|_{G_{F_v}} \cong \chi \oplus \chi_{\epsilon_l}$  for an unramified character  $\chi : G_{F_v} \rightarrow \bar{M}_{\pi,\lambda}^\times$ . Now we apply Theorem 3.9 with  $F^{\mathrm{avoid}}$  equal to the Galois closure of  $\bar{F}^{\ker(\bar{\tau}_{\pi,\lambda})}(\zeta_l)/\mathbf{Q}$ , to obtain a CM Galois extension  $F_1/F$ , linearly disjoint from  $F^{\mathrm{avoid}}$  over  $F$  and with  $l$  unramified in  $F_1$ , such that  $\bar{\tau}_{\pi,\lambda}|_{G_{F_1}}$  is automorphic (coming from a weight 0, unramified above  $v$  and  $l$ , automorphic representation). We now wish to apply Theorem 2.1. By our choice of  $F^{\mathrm{avoid}}$ , it is easy to see that  $\bar{\tau}_{\pi,\lambda}(G_{F_1(\zeta_l)})$  is enormous and that there is  $\sigma \in G_{F_1} - G_{F_1(\zeta_l)}$  such that  $\bar{\tau}_{\pi,\lambda}(\sigma)$  is scalar. We claim that  $\bar{\tau}_{\pi,\lambda}|_{G_{F_1}}$  is also decomposed generic.

Let  $\tilde{F}$  and  $\tilde{F}_1$  be the Galois closures of  $F/\mathbf{Q}$  and  $F_1/\mathbf{Q}$ , respectively. Since  $F^{\mathrm{avoid}}/\mathbf{Q}$  is Galois and  $F^{\mathrm{avoid}} \cap F_1 = F$ , we have  $F^{\mathrm{avoid}} \cap \tilde{F}_1 = \tilde{F}$ . Since  $p$  is totally split in  $F$ , it is totally split in  $\tilde{F}$  and the conjugacy class of  $\mathrm{Frob}_p$



in  $\text{Gal}(F^{\text{avoid}}/\mathbf{Q})$  lies in  $\text{Gal}(F^{\text{avoid}}/\tilde{F})$ . Using Chebotarev density, we choose a prime  $q$  unramified in  $F^{\text{avoid}}\tilde{F}_1$  such that  $\text{Frob}_q \in \text{Gal}(F^{\text{avoid}}\tilde{F}_1/\mathbf{Q})$  lies in  $\text{Gal}(F^{\text{avoid}}\tilde{F}_1/\tilde{F})$  and corresponds to  $\text{Frob}_p \times 1$  under the isomorphism

$$\text{Gal}(F^{\text{avoid}}\tilde{F}_1/\tilde{F}) \cong \text{Gal}(F^{\text{avoid}}/\tilde{F}) \times \text{Gal}(\tilde{F}_1/\tilde{F}).$$

This  $q$  is decomposed generic for  $\bar{r}_{\pi,\lambda}|_{G_{F_1}}$ .

By Theorem 2.8,  $r_{\pi,\lambda}|_{G_{F_1}}$  is crystalline with all labelled Hodge–Tate weights equal to  $\{0, 1\}$  at all places above  $l$  in  $F_1$ . The representation  $r_{\pi,\lambda}|_{G_{F_1}}$  thus satisfies the assumptions of Theorem 2.1, and we obtain a regular algebraic cuspidal automorphic representation  $\Pi$  of  $\text{GL}_2(\mathbf{A}_{F_1})$  such that  $r_{\pi,\lambda}|_{G_{F_1}} \cong r_\ell(\Pi)$  and with  $\Pi_w$  unramified at all  $w|v$  in  $F_1$ . Then for any  $w|v$  in  $F_1$ ,  $r_\ell(\Pi)|_{G_{F_1,w}} \cong \chi|_{G_{F_1,w}} \oplus \chi|_{G_{F_1,w}} \epsilon_l$ , and  $\Pi_w$  is an unramified principal series. By local-global compatibility at unramified places [HLTT16, Var14], this contradicts the genericity of  $\Pi$ .  $\square$

*Proof of Theorem 1.1.* If  $\pi$  is everywhere potentially unramified, then this follows from the main result of [Var14], so we can assume that  $\pi$  is special at some finite place of  $F$ . Theorem 1.1 then follows at once from Theorem 4.1 and Lemma 2.9.  $\square$

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